Absolute Stability Conditions for Discrete-Time Recurrent Neural Networks

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***Abstract*—In this paper, an analysis of the absolute stability for a general class of discrete-time recurrent neural networks (RNN's) is presented. A discrete-time model of RNN's is represented by a set of nonlinear difference equations. Some sufficient conditions for the absolute stability are derived using Ostrowski's theorem and the similarity transformation approach. For a given RNN model, these conditions are determined by the synaptic weight matrix of the network. The results reported in this paper need fewer constraints on the weight matrix and the model than in previously published studies.**

I. INTRODUCTION

The dynamic behavior and stability of continuous-time recurrent neural networks (RNN's) described by a set of nonlinear differential equations have been widely studied during the past decade. A series of local and global stable conditions have been derived using different nonlinear analysis approaches by Cohen and Grossberg [3]; Guez, Protopopsecu, and Berhen [7]; and Kelly [17] for a general class of continuous-time RNN's as well as the Hopfield network [11], [12]. Recently, Matsuoka [22] improved the previous stability criteria using a novel Lyapunov function, so that the absolute stability conditions may be easily checked using only the synaptic connection weights of the network. For a RNN model described by a set of nonlinear difference equations, the brain-state-in-a-box (BSB) neural model with a symmetric connection weight matrix was conceived by Anderson and coworkers [1] and the stability and implementation of associative memories of it with symmetric or nonsymmetric weight matrices were recently discussed by Hui and Żak [14] and Michel *et al.* [23], respectively. The main difference between the BSB neural model and the usual nonlinear difference equation is that the former is defined on the closed hypercube by a *symmetric ramp function* [1], while the latter is defined on R^n. On the other hand, a global stability condition for a so-called iterated-map neural network with a symmetric weight matrix was proposed by Marcus and Westervelt [20] using Lyapunov's function method. The stability condition was used in the associative memory learning algorithms in [21] by Marcus, Waugh, and Westervelt. More recently, the stability and bifurcation properties of some neural networks were analyzed by Blum and Wang [2], [25], and the stability of the fixed points was studied for a class of discrete-time recurrent networks by Li [18] using the norm condition of a matrix. Changes in the stable region of the fixed points due to the changing of the neuron gain were also obtained.

The absolute stability of a general class of discrete-time RNN's is analyzed in this paper. In Section II, a model of discrete-time RNN's is represented by a set of nonlinear difference equations. The existence of the equilibrium points of the neural model is verified in Section III. Some sufficient conditions for absolute stability are derived in Sections IV and V by using Ostrowski's theorem [13], not a commonly used Gerschgorin's theorem [13], and the similarity transformation approach. For a given RNN model, these conditions are determined only by the synaptic weight matrix of the network. It is shown that these results need fewer constraints on the synaptic weight matrix and the model than required in other studies previously described [18].

II. DISCRETE-TIME MATHEMATICAL MODEL

Assume that a recurrent neural network with n neurons is described by a discrete-time nonlinear system of the form

$$x_i(k+1) = \alpha_i x_i(k) + \beta_i \sigma\left[\sum_{j=1}^{n} w_{i,j} x_j(k) + s_i\right] \qquad i = 1, 2, \cdots, n \tag{1}$$

where x_i represents the internal state of the ith *neuron*, $W = [w_{i,j}]_{n\times n}$ is the real-valued matrix of the synaptic connection weights, s_i is a constant external input to the ith neuron, $\sigma(\cdot)$ is the nonlinear activation function and the time constant α_i and the neural gain β_i are assumed to be $-1 \le \alpha_i \le 1$ and $\beta_i \ne 0$.

Model (1) is a general expression of discrete-time recurrent neural networks with state feedbacks. For the case $\alpha_i = 0, \beta_i = 1$ and a *symmetric ramp* activation function $\sigma(\cdot)$ defined by [1]

$$\sigma(x) = \begin{cases} \gamma, & \text{if } x \ge \gamma \\ x, & \text{if } |x| \le \gamma \\ -\gamma, & \text{if } x \le -\gamma \end{cases}$$

model (1) becomes a BSB neural model with a asymmetric connection weight matrix W [1], [14]. For the case $\alpha_i = (1 - h/\tau_i)$, $\beta_i = h/\tau_i$, and $x_i(t + kh) \equiv x_i(k)$, model (1) becomes Euler's approximation of a well-known continuous-time neural network model [6], [22]

$$\tau_i \frac{dx_i}{dt} = -x_i + \sigma\left(\sum_{j=1}^{n} w_{i,j} x_j + s_i\right) \qquad i = 1, 2, \cdots, n \tag{2}$$

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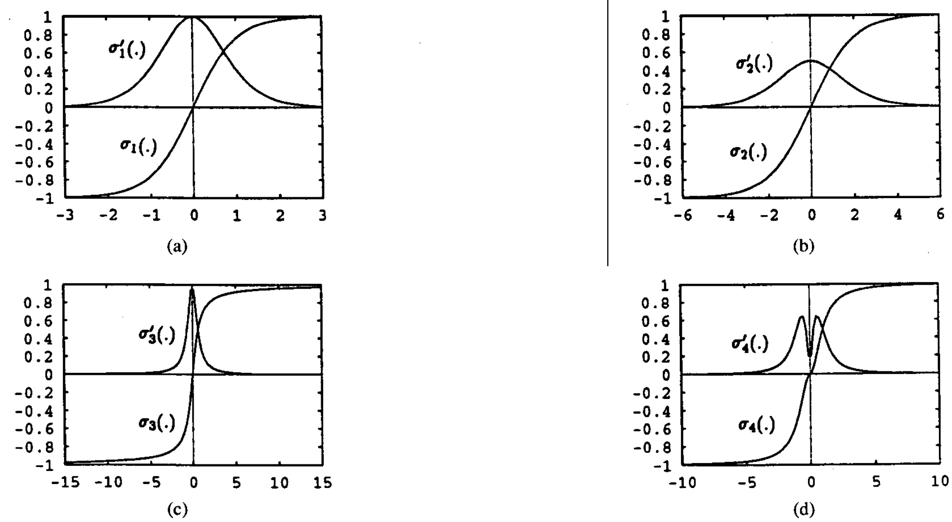

Fig. 1. The nonlinear neural activation functions and their derivatives.

where h is a discrete step and τ_i is the time constant associated with the ith neuron.

System (1) may be represented by the following vector difference equation form

$$\begin{aligned} x(k+1) &= Ax(k) + B\sigma[Wx(k) + s] \\ &\equiv f(x(k)) \end{aligned} \tag{3}$$

where $x = [x_1, x_2, \cdots, x_n]^T$ is the state vector of the network, $s = [s_1, s_2, \cdots, s_n]^T$ is the input vector, $A = \text{diag}[\alpha_1, \alpha_2, \cdots, \alpha_n], B = \text{diag}[\beta_1, \beta_2, \cdots, \beta_n]$, and $\sigma(\cdot)$ is a vector of a nonlinear activation function, $\sigma_i(\cdot) = \sigma\left(\sum_{j=1}^{n} w_{i,j}x_j + s_i\right)$.

In (1), the neural activation function $\sigma(\cdot)$ may be chosen as the continuous and differentiable nonlinear sigmoidal function satisfying the following conditions:

i) $\sigma(x) \longrightarrow \pm 1$ as $x \longrightarrow \pm\infty$;
ii) $\sigma(x)$ is bounded with the upper bound 1 and the lower bound -1;
iii) $\sigma(x) = 0$ at a unique point $x = 0$;
iv) $\sigma'(x) > 0$ and $\sigma'(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$;
v) $\sigma'(x)$ takes a global maximal value $c \leq 1$ at a unique point $x = 0$.

Typical examples of such a function $\sigma(\cdot)$ are

$$\begin{aligned} \sigma_1(x) &= \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ \sigma_2(x) &= \frac{1 - e^{-x}}{1 + e^{-x}} \\ \sigma_3(x) &= \frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{2}x\right) \\ \sigma_4(x) &= \frac{x^2}{1+x^2}\,\text{sign}(x) \end{aligned}$$

where sign $(\cdot)$ is a signum function and all the above nonlinear activation functions are bounded, monotonic and nondecreasing functions as shown in Fig. 1.

III. Analysis of the Equilibrium Points

The basic goal of dynamical analysis is to understand the eventual or asymptotic behavior of a dynamical system. For a discrete process of the recurrent neural network, the goal is to understand the eventual behavior of the points $x \equiv f^0(x)$, $f(x) \equiv f^1(x)$, $f(f(x)) \equiv f^2(x), \cdots, f^n(x)$ as n becomes large. The discussion of the stability or instability of a point or a stored pattern in associative memory can be made only if it is an *equilibrium point*, a periodic, or a general invariant set of the system.

Definition 1: A point $x^* \in [a, b]^n$ is defined as an equilibrium point of the dynamical system (3) if $x^* = Ax^* + B\sigma(Wx^* + s)$.

The *fixed points* of the nonlinear function *f(x)* are points x that satisfy $f(x) = x$. The following application of the *Intermediate Value Theorem* gives an important criterion for the existence of a fixed point.

Lemma 1 (Brouwer's Fixed Point Theorem [4]): Let $H^n = [a, b]^n$ be a closed set of R^n and $f : H^n \longrightarrow H^n$ be a continuous vector-valued function. Then f has at least one fixed point in H^n.

For the discrete-time nonlinear system (3), the fixed points of the function $f(x) \equiv Ax + B\sigma(Wx + s)$ are the equilibrium points of the system (3) for a given input s and the connection weight matrices W. The existence of the equilibrium points of the system (3) is then obtained as follows using the above lemma.

Theorem 1: Let all $\alpha_i \neq 1$, $i = 1, 2, \cdots, n$ in (3), and $\beta = \max\{|\frac{\beta_i}{1-\alpha_i}| : i = 1, 2, \cdots, n\}$. For any given input s and the connection weight matrices W, then there exists at least one equilibrium point $x^* \in [-\beta, \beta]^n$ of the dynamical system (3); that is, $(I - A)x^* = B\sigma(Wx^* + s)$.

Proof: Since an equivalent form of the n-dimensional algebraic equation $x = f(x)$ is

$$x_i = \frac{\beta_i}{1 - \alpha_i}\sigma_i\left[\sum_{j=1}^{n} w_{i,j}x_j + s_i\right] \equiv g_i(x), \quad i = 1, 2, \cdots, n$$

the fixed points of the vector-valued function $g_1(x) = [g(x)_1, \cdots, g_n(x)]^T$ are also the fixed points of function $f(x)$. Note that for an arbitrary $x \in [-\beta, \beta]^n$, $g_i(x)$ satisfies

$$|g_i(x)| = \left|\frac{\beta_i}{1-\alpha_i}\right| \left|\sigma_i\left[\sum_{j=1}^{n} w_{i,j}x_j + s_i\right]\right|$$
$$\leq \left|\frac{\beta_i}{1-\alpha_i}\right| \in [-\beta, \beta]$$

that is, $g(x) \in [-\beta, \beta]^n$. This implies that for any given input s and the connection weight matrix $W, g(x)$ is a $[-\beta, \beta]^n \longrightarrow [-\beta, \beta]^n$ the continuous vector-valued function because σ is continuous. Thus, by the Brouwer's fixed point theorem f has a fixed point $x^* \in [-\beta, \beta]^n$; that is, x^* is an equilibrium point of the system (3). □

Theorem 2: Let all $\alpha_i = 1,\ i = 1, 2, \cdots, n$ in (3). Then, the dynamical system (3) has at least one equilibrium point $x^* \in R^n$ if and only if the given input s and the connection weight matrix satisfy

$$\text{rank}[W \vdots -s] = n. \tag{4}$$

Furthermore, system (3) has a unique equilibrium point if and only if the weight matrix W is invertible and this equilibrium point is $x^* = -W^{-1}s$.

Proof: In this case, the n-dimension algebraic equation $x = f(x)$ becomes

$$0 = B\sigma(Wx + s).$$

Since $\sigma(u) = 0$ only for $u = 0$, the fixed points of system (3) exist if and only if the n-dimension linear equation

$$Wx + s = 0$$

is solvable and the remaining part can easily be obtained. □

In the case when there are only some $\alpha_i = 1$ in (3), the existence of the equilibrium points of system (3) may be discussed using the results given in Theorems 1 and 2. Without loss of generality, let $\alpha_{i_1} \neq 1,\ i_1 = 1, \cdots, n_1$ and $\alpha_{i_2} = 1,\ i_2 = n_1 + 1, \cdots, n$ in (3). The state and input vectors may be divided then into $x_1, s_1 \in R^{n_1}$ and $x_2, s_2 \in R^{n-n_1}$, and the n-dimension algebraic equation $x = f(x)$ may be represented as

$$x_1 = A_1x_1 + B_1\sigma(W_{1,1}x_1 + W_{1,2}x_2 + s_1) \tag{5}$$
$$0 = \sigma(W_{2,1}x_1 + W_{2,2}x_2 + s_2) \tag{6}$$

where $A_1 = \text{diag}[\alpha_1, \cdots, \alpha_{n_1}]$, $B_1 = \text{diag}[\beta_1, \cdots, \beta_{n_1}]$, $W_{1,1} \in R^{n_1 \times n_1}$, $W_{1,2} \in R^{n_1 \times (n-n_1)}$, $W_{2,1} \in R^{(n-n_1) \times n_1}$ and $W_{2,2} \in R^{(n-n_1) \times (n-n_1)}$ with

$$W = \begin{bmatrix} W_{1,1} & \vdots & W_{1,2} \\ \cdots & \vdots & \cdots \\ W_{2,1} & \vdots & W_{2,2} \end{bmatrix}.$$

Equation (5) has at least one equilibrium point $x_1^* \in R^{n_1}$ for the arbitrary input s_1 and x_2 based on Theorem 1. Indeed, the solution of (6) may be discussed using Theorem 2. Therefore, the following results are obtained.

Corollary 1: Let $\alpha_{i_1} \neq 1,\ i_1 = 1, \cdots, n_1$ and $\alpha_{i_2} = 1,\ i_2 = n_1 + 1, \cdots, n$ in (3). If the submatrix $W_{2,2}$ is invertible, then the dynamical system (3) has at least one equilibrium point $[(x_1^*)^T, (x_2^*)^T]^T \in R^n$.

If the submatrix $W_{2,2}$ of W is invertible, the x_2^* is then solved from the following linear equation

$$W_{2,1}x_1 + W_{2,2}x_2 + s_2 = 0. \tag{7}$$

Therefore, the equilibrium points of system (3) are determined by the following two lower order algebraic equations

$$(I - A_1)x_1^* = B_1\sigma[(W_{1,1} - W_{1,2}W_{2,2}^{-1}W_{2,1})x_1^* - W_{1,2}W_{2,2}^{-1}s_2 + s_1] \tag{8}$$
$$x_2^* = -W_{2,2}^{-1}W_{2,1}x_1^* - W_{2,2}^{-1}s_2. \tag{9}$$

Since (8) may have multi-solutions, system (3) may thus have multi-equilibrium points.

In order to analyze the stability of the equilibrium points, the Jacobian of the function $f(x)$ is given by

$$\frac{df}{dx} = f'(x) = A + B\Sigma(x)W$$

where $\Sigma(x)$ is a diagonal matrix with $\Sigma_{i,i}(x) \equiv \sigma_i'(x) = \sigma'(\sum_{m=1}^{n} w_{i,m}x_m + s_i)$. The function $f(x)$ has only aymptotically stable equilibrium points if all the eigenvalues of the Jacobian are inside the unit circle for all the states x, and a given weight matrix W and the input s. In this case, system (3) has only attractors. Hence, for an arbitrary initial state, the state of system (3) will converge to one of the equilibrium points for a given weight matrix W and input s. For the local asymptotically stability of the equilibrium point x^*, the eigenvalues of the Jacobian at the equilibrium point x^* should be examined. If all the eigenvalues of the Jacobian at x^* are within a unit circle, x^* is then a local asymptotically stable equilibrium point of the system (3) and this equilibrium point is a *sink*. If all the eigenvalues of the Jacobian at x^* are outside the unit circle, it is a *source*. If some eigenvalues are inside and some are outside the unit circle, the equilibrium point is a *saddle* or unstable equilibrium point. Obviously, the equilibrium points of the neural system (3) depend on the input s for a given weight matrix W.

An important property of the neural network system will be introduced now using the asymptotical stability of system (3) for any input s.

Definition 2 [3], [22]: If the neural system (3) has only asymptotically stable equilibrium points for a given weight matrix W and any input s, then the neural system is said to be absolutely stable.

It is important to note the difference between the asymptotic stability and the absolute stability; the asymptotic stability may depend upon the input s, whereas the absolute stability does not depend upon the input s. Therefore, for an absolutely stable neural network, the system state will converge to one of the asymptotically stable equilibrium points regardless of the initial state and the input signal. This paper will concentrate on the absolute stability of system (3).

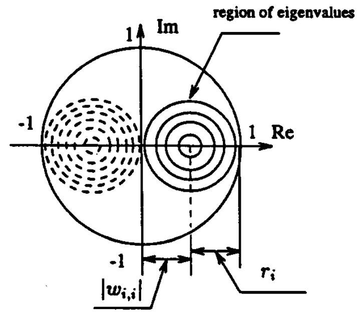

Fig. 2. The positions of all the eigenvalues of a matrix $W = [w_{i,j}]_{n\times n}$, where $r_i = R_i^\gamma C_i^{1-\gamma}$ is Ostrowski's radius.

IV. Absolute Stability Conditions for All $|\alpha_i| < 1$

If all $|\alpha_i| < 1$, then using Theorem 1, the existence of the fixed points of system (3) is guaranteed. To further discuss the positions of the eigenvalues of the Jacobian $f'(x)$ in the complex plane, the following Lemma is required.

Lemma 2 (Ostrowski's theorem [13]): Let $W = [w_{i,j}]_{n\times n}$ be a complex matrix, $\gamma \in [0,1]$ be given, and R_i and C_i denote the deleted row and deleted column sums of W as follows, respectively

$$\begin{cases} R_i = \sum\limits_{j=1,j\neq i}^{n} |w_{i,j}| \\ C_i = \sum\limits_{j=1,j\neq i}^{n} |w_{j,i}| \end{cases}.$$

All the eigenvalues of W are then located in the union of n closed discs in the complex plane with centers $w_{i,i}$ and radii $r_i = R_i^\gamma C_i^{1-\gamma}$, $i = 1,2,\cdots,n$.

Corollary 2: Let $W = [w_{i,j}]_{n\times n}$ be a complex matrix, $\gamma \in [0,1]$ be given, and R_i and C_i be defined by (6). If

$$|w_{i,i}| + R_i^\gamma C_i^{1-\gamma} < 1, \quad i = 1,2,\cdots,n$$

then all the eigenvalues of W are located inside the unit circle in the complex plane.

Corollary 2 is shown in Fig. 2. Using the results of Corollary 2, the eigenvalues of the Jacobian (5) of system (3) are examined in order to obtain the sufficient conditions for the absolute stability of the neural system (3).

Theorem 3: Let $-1 < \alpha_i < 1$; $i = 1,2,\cdots,n$ in (3), $\gamma \in [0,1]$ be given and define

$$\begin{cases} R_i = \sum\limits_{j=1,j\neq i}^{n} |w_{i,j}| \\ \bar{C}_i = \sum\limits_{j=1,j\neq i}^{n} |\beta_j||w_{j,i}| \end{cases}. \tag{10}$$

If the connection weight matrix $W = [w_{i,j}]_{n\times n}$ satisfies the inequalities

$$|w_{i,i}| + |\beta_i|^{\gamma-1} R_i^\gamma \bar{C}_i^{1-\gamma} < \frac{1}{|\beta_i| c}(1-|\alpha_i|) \equiv \delta_i, \quad i = 1,2,\cdots,n \tag{11}$$

where $\delta_i \equiv (1-|\alpha_i|)/(|\beta_i|c)$ and c is the maximum slope of the function $\sigma(\cdot)$, then system (3) is absolutely stable.

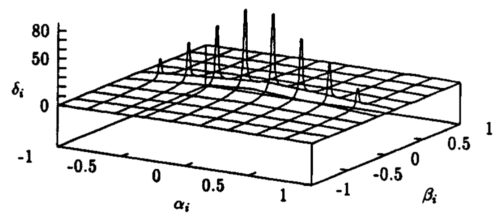

Fig. 3. The relationship between parameters α_i, β_i, and δ_i, where α_i is the time constant, β_i is the neural gain and δ_i is the radius of the absolutely stable region.

Proof: The Jacobian is $f'(x) = [f'_{i,j}(x)]_{n\times n} = A + B\Sigma(x)W$, where $\Sigma = \operatorname{diag}[\sigma_1(x), \sigma_2(x), \cdots, \sigma_n(x)]$. Therefore,

$$f'_{i,i}(x) = \alpha_i + \beta_i \sigma'_i(x) w_{i,i} \quad i = 1,2,\cdots,n \tag{12}$$

and

$$f'_{i,j}(x) = \beta_i \sigma'_i(x) w_{i,j}. \quad i,j = 1,2,\cdots,n; i \neq j \tag{13}$$

Then

$$\begin{aligned} &|f'_{i,i}(x)| + \left(\sum_{j=1,j\neq i}^{n} |f'_{i,j}(x)|\right)^\gamma \left(\sum_{j=1,j\neq i}^{n} |f'_{j,i}(x)|\right)^{1-\gamma} \\ &= |\alpha_i + \beta_i\sigma'_i(x) w_{i,i}| + |\beta_i \sigma'_i(x)|^\gamma \left(\sum_{j=1,j\neq i}^{n} |w_{i,j}|\right)^\gamma \\ &\quad \times \left(\sum_{j=1,j\neq i}^{n} |\sigma'_j(x)||\beta_j||w_{j,i}|\right)^{1-\gamma} \\ &< |\alpha_i| + |\beta_i||w_{i,i}| + |\beta_i|^\gamma R_i^\gamma \bar{C}_i^{\gamma-1} \\ &< 1. \end{aligned} \tag{14}$$

Therefore, using Corollary 2, it is proven that all the eigenvalues of the Jacobian $f'(x)$ are located inside the unit circle in the complex plane which implies that the system (3) is absolutely stable. □

The absolute stability condition (11) is independent of the signs of the connection weights and contains only the parameters and connection weights of the network. For the given parameters of a network, inequality (11) defines a solution space of the connection weights $w_{i,j}$ for which the absolute stability is ensured. This solution space for such connection weights is referred to as an absolutely stable region of system (3) described by the radius δ_i of system (3). In this case, the absolutely stable region becomes large as the absolute value of the neural gain β_i, or the time constant α_i, decreases. In the limiting case, $\beta_i \longrightarrow 0$, the radius of the absolutely stable region constant $\delta_i \longrightarrow \infty$, which implies that for a sufficiently small neural gain β_i regardless of the connection weight matrix W the system (3) has a unique global stable equilibrium point $x^* = 0$. Indeed, as shown in Fig. 3, changes of the absolutely stable region are more sensitive to the changes of the neural gain β_i than that of the time constant α_i.

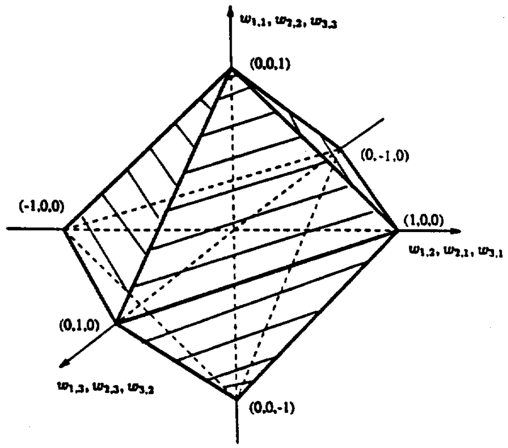

Fig. 4. The solution space of connection weights corresponding to inequality (15) for a three-neuron system.

Theorem 4: Let $-1 < \alpha_i < 1$; $i = 1, 2, \cdots, n$ in (3). If the connection weight matrix $W = [w_{i,j}]_{n \times n}$ satisfies the inequalities

$$\sum_{j=1}^{n} |w_{i,j}| < \delta_i \quad i = 1, 2, \cdots, n \tag{15}$$

or

$$|w_{j,j}| + \frac{1}{|\beta_j|} \sum_{i=1, i \neq j}^{n} |\beta_i w_{i,j}| < \delta_j \quad j = 1, 2, \cdots, n \tag{16}$$

then system (3) is absolutely stable.

Proof: The results of (15) and (16) are obtained by setting respectively $\gamma = 1$ and $\gamma = 0$ in Theorem 1. □

The results of Theorem 4 may also be obtained directly using Gerschgorin's theorem [13] which can be implied by setting $\gamma = 1$ and $\gamma = 0$ in Ostrowski's theorem. The solution space of the connection weights given by inequalities (15) and (16) forms n open convex hypercones in n-dimensional space. Fig. 4 shows a three-dimension convex cone for a three-neuron system with $\delta_i = 1$. Since $(R_i C_i)^{1/2} \leq (R_i + C_i)/2$; that is, when $\gamma = 1/2$, Ostrowski's radius r_i is located between the Gersgorin's radii R_i and C_i. Consequently, in case some of the R_i and $\bar{C}_i$ satisfy the conditions of Theorem 4 and others do not, Theorem 3 may give a better estimation of the absolutely stable region of system (3). This fact is illustrated in the following examples of two-neuron network systems.

Example 1: Consider a two-neuron network system described by

$$\begin{cases} x_1(k+1) &= (\frac{1}{2})x_1(k) + \tanh((\frac{1}{4})x_1(k) + (\frac{1}{8})x_2(k) + s_1) \\ x_2(k+1) &= -(\frac{1}{2})x_2(k) + \tanh((\frac{1}{3})x_1(k) - (\frac{1}{4})x_2(k) + s_2) \end{cases}. \tag{17}$$

One can easily obtain that $\delta_i = 1/2$, $i = 1, 2$ and

$$\begin{cases} R_1 = 1/8 \\ R_2 = 1/3 \end{cases}, \quad \begin{cases} C_1 = 1/3 \\ C_2 = 1/8 \end{cases}.$$

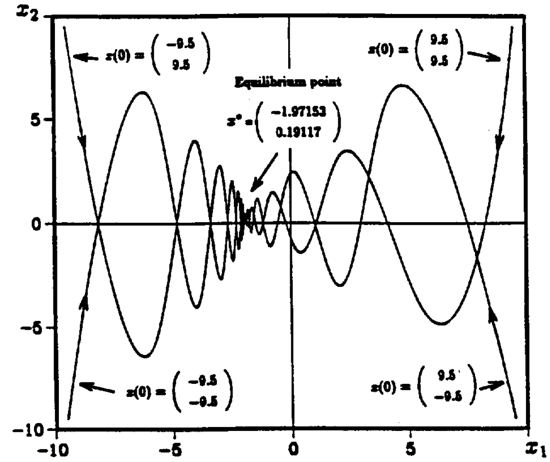

Fig. 5. The phase plane diagram of the states x_1 and x_2 of the neural network (17) with inputs $s_1 = -2$ and $s_2 = 1$. In this case, the unique absolutely stable equilibrium point is $x^* = (-1.97153, 0.19117)^T$.

Let $\gamma = 1/2$ in Theorem 3. Then

$$\begin{cases} |w_{1,1}| + R_1^{1/2} C_1^{1/2} = 1/4 + 1/(26)^{1/2} < 1/2 \\ |w_{2,2}| + R_2^{1/2} C_2^{1/2} = 1/4 + 1/(26)^{1/2} < 1/2 \end{cases}.$$

Therefore, using Theorem 3, the absolute stability of the neural network (17) is ensured. Now, the absolute stability of system (17) is examined using the results of Theorem 4 as follows

$$\begin{cases} |w_{1,1}| + |w_{1,2}| &= 3/8 < 1/2 \\ |w_{2,1}| + |w_{2,2}| &= 7/12 > 1/2 \end{cases}$$

and

$$\begin{cases} |w_{1,1}| + |w_{2,1}| &= 7/12 > 1/2 \\ |w_{1,2}| + |w_{2,2}| &= 3/8 < 1/2 \end{cases}$$

The absolute stability of system (17) can not be determined using Theorem 4. In fact, the absolute stability of system (17) is indicated by the simulation results as shown in Fig. 5, where the state $x = (x_1, x_2)^T$ converges to a unique equilibrium point $x^* = (-1.97153, 0.19117)^T$ regardless of the initial state $x(0)$ as shown in Fig. 5.

Example 2: In this example, it will be shown that if a suitable parameter γ is chosen, the stability condition presented in Theorem 3 is more relaxed than the norm stability condition which was recently proposed by Li [18] for a class of discrete-time neural networks. Consider a simple two-neuron system without linear feedback terms and external inputs with the following form

$$\begin{cases} x_1(k+1) &= \tanh((\frac{49}{100})x_1(k) + (\frac{1}{3})x_2(k)) \\ x_2(k+1) &= \tanh((\frac{3}{4})x_1(k) + (\frac{49}{100})x_2(k)) \end{cases} \tag{18}$$

where the 2×2 weight matrix is

$$W = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{49}{100} & \frac{1}{3} \\ \frac{3}{4} & \frac{49}{100} \end{pmatrix}.$$

In this case, $\delta_i = 1, i = 1, 2$, and

$$\begin{cases} R_1 = 1/3 \\ R_2 = 3/4 \end{cases}, \quad \begin{cases} C_1 = 3/4 \\ C_2 = 1/3 \end{cases}.$$

Let $\gamma = 1/2$ in Theorem 3. Then

$$|w_{i,i}| + R_i^{1/2}C_i^{1/2} = \frac{99}{100} < 1.$$

Hence, the absolute stability of the network (18) is guaranteed. The stability of the network (18) can now be tested using the norm stability condition [18]. It is easy to show that the norm stability condition for the above system can be expressed as $||W|| < 1$, where $||\cdot||$ is a matrix norm. In addition, the following results may be obtained as

$$||W||_1 = \max_j \sum_{i=1}^{2} |w_{i,j}| = \frac{31}{25} > 1$$
$$||W||_\infty = \max_i \sum_{j=1}^{2} |w_{i,j}| = \frac{31}{25} > 1$$
$$||W||_2 = \{\lambda_{\max}(W^TW)\}^{1/2} = 1.1516.$$

Unfortunately, based on the above choices of the matrix norms, the norm stability condition can not ensure the stability of the neural network (18). In fact, the norms $||W||_1$ and $||W||_\infty$ are Gerschgorin's radii of the weight matrix W.

V. Absolute Stability Conditions for $|\alpha_i| \le 1$

For an absolutely stable neural network (3) with all $|\alpha_i| < 1$, the arbitrary initial state will quickly converge to the equilibrium point. The dynamics of network (3), however, become more complicated when $|\alpha_i| = 1$ in (3). In this section, a discussion is given about the absolute stability conditions for the case of all $|\alpha_i| = 1$ in (3). As a natural extension, the absolute stability conditions for a general case of $|\alpha_i| \le 1$ are also derived in this section.

Lemma 3: [13] Let $W = [w_{i,j}]_{n\times n}$ be a complex matrix. If W is strictly diagonally dominant; that is,

$$|w_{i,i}| > \sum_{j=1,j\neq i}^{n} |w_{i,j}| \quad i = 1,2,\cdots,n$$

or

$$|w_{j,j}| > \sum_{i=1,i\neq j}^{n} |w_{i,j}|. \quad j = 1,2,\cdots,n$$

W is invertible.

Theorem 5: Let $|\alpha_i| = 1, i = 1,2,\cdots,n$ in (3). If the weight matrix $W = [w_{i,j}]_{n\times n}$ satisfies the inequalities

$$\begin{cases} -1/c \le sign(\alpha_i)\beta_i w_{i,i} < 0 \\ -sign(\alpha_i)\beta_i w_{i,i} > |\beta_i| \sum\limits_{j=1,j\neq i}^{n} |w_{i,j}| \quad i = 1,2,\cdots,n \end{cases} \tag{19}$$

or

$$\begin{cases} -2/c < \mathrm{sign}(\alpha_i)\beta_i w_{i,i} \le -1/c \\ -\mathrm{sign}(\alpha_i)\beta_i w_{i,i} + |\beta_i| \sum\limits_{j=1,j\neq i}^{n} |w_{i,j}| < 2/c; \quad i=1,2,\cdots,n \end{cases} \tag{20}$$

system (3) is absolutely stable.

Proof: When $\alpha_i = -1, i = 1,2,\cdots,n$ in (3), the existence of the fixed points are shown by Theorem 1. For $\alpha_i = 1, i = 1,2,\cdots,n$ in (3), it is easy to show that if the inequalities (19) or (20) are satisfied, the weight matrix W is strictly diagonally dominant; that is, W is invertible based on Lemma 3. In this case, system (3) has a unique equilibrium point $x^* = -W^{-1}s$. Since the centers of the n Gerschgorin's discs of the Jacobian $f'(x)$ are located inside the unit circle in the complex plane for absolute stability

$$-1 < |f'_{i,i}(x)| < 1$$

Therefore, for $|\alpha_i| = 1$, if

$$-1/c \le sign(\alpha_i)\beta_i w_{i,i} < 0$$

one may imply that

$$\begin{aligned} |f'_{i,i}(x)| &= |\alpha_i + \beta_i\sigma'_i(x)w_{i,i}| \\ &= \mathrm{sign}(\alpha_i)(\alpha_i + \beta_i\sigma'_i(x)w_{i,i}) \\ &= 1 + \mathrm{sign}(\alpha_i)\beta_i\sigma'_i(x)w_{i,i} \end{aligned}$$

Furthermore, if

$$-2/c < -\mathrm{sign}(\alpha_i)\beta_i w_{i,i} \le -1/c$$

then

$$\begin{aligned} |f'_{i,i}(x)| &= -\mathrm{sign}(\alpha_i)(\alpha_i + \beta_i\sigma'_i(x)w_{i,i}) \\ &= -1 - \mathrm{sign}(\alpha_i)\beta_i\sigma'_i(x)w_{i,i} \end{aligned}$$

Using Corollary 2 with $\gamma = 1/2$, the remaining part of the proof is the same as that of the proof of Theorem 3. □

Theorem 6: Let $|\alpha_i| = 1, i = 1,2,\cdots,n$ in (3). If the weight matrix $W = [w_{i,j}]_{n\times n}$ satisfies the inequalities

$$\begin{cases} -1/c \le \mathrm{sign}(\alpha_j)\beta_j w_{j,j} < 0 \\ -\mathrm{sign}(\alpha_j)\beta_j w_{j,j} > \sum\limits_{i=1,i\neq j}^{n} |\beta_i w_{i,j}|; \quad j = 1,2,\cdots,n \end{cases} \tag{21}$$

or

$$\begin{cases} -2/c < \mathrm{sign}(\alpha_j)\beta_j w_{j,j} \le -1/c \\ -\mathrm{sign}(\alpha_j)\beta_j w_{j,j} + \sum\limits_{i=1,i\neq j}^{n} |\beta_i w_{i,j}| < 2/c; \quad j=1,2,\cdots,n \end{cases} \tag{22}$$

system (3) is absolutely stable.

The procedure for the proof is the same as that of Theorem 4. □

For the case $-1 \le \alpha \le 1$, if there are only some $|\alpha_i| = 1$ in (3), the absolute stability criteria of (3) are easily synthesized using Theorems 4 and 5. Let $|\alpha_{i_1}| \neq 1$, $i_1 = 1,\cdots,n_1$ and $|\alpha_{i_2}| = 1$, $i_2 = n_1+1,\cdots,n$ in (3), the absolute stability conditions are then given by (23) and (24) found at the bottom of the next page.

In fact, only the existence of the equilibrium points in this case are need to be proven. Without the loss of generality, let all $\alpha_{i_2} = 1$, $i_2 = n_1+1,\cdots,n$ in inequalities (23) and (24). Obviously, the last two conditions of inequality (23) imply that

$$|w_{i_2,i_2}| > \sum_{j=1,j\neq i_2}^{n} |w_{i_2,j}| \ge \sum_{j=n_1+1,j=i_2}^{n} |w_{i_2,j}| \tag{25}$$

that is, the submatrix $W_{2,2}$ is invertible using Lemma 3. Furthermore, it will be shown that the last two conditions of inequality (24) can also ensure the nonsingularity of the matrix $W_{2,2}$. Note that the following inequalities may be obtained from the second and third equations of (24)

$$|\beta_{i_2} w_{i_2,i_2}| > 1/c$$

and

$$|\beta_{i_2} w_{i_2,i_2}| + |\beta_{i_2}| \sum_{j=1,j\neq i_2}^{n} |w_{i_2,j}| < 2/c. \quad i_2 = n_2+1,\cdots,n$$

Hence,

$$|\beta_{i_2}| \sum_{j=n_2+1,j\neq i_2}^{n} |w_{i_2,j}| \leq |\beta_{i_2}| \sum_{j=1,j\neq i_2}^{n} |w_{i_2,j}| < 1/c$$

and

$$|w_{i_2,i_2}| > \sum_{j=n_1+1,j=i_2}^{n} |w_{i_2,j}| \tag{26}$$

that is, $W_{2,2}$ is invertible based on Lemma 3. The existence of the equilibrium point is thus demonstrated using Corollary 1. Consequently, the absolute stability conditions for the case of some $|\alpha_i| = 1$ and others $|\alpha_i| < 1$ in (3) are straightforward using the results obtained for the case of all $|\alpha_i| < 1$ and all $|\alpha_i| = 1$ in (3).

It is to be noted that when $|\alpha_i| = 1$, the signs of the recurrent connection weights $w_{i,i}$ are important for the absolute stability, though they do not appear in the case of all $|\alpha_i| < 1$. Theorems 5 and 6 show that if system (3) is absolutely stable for $\alpha_i = 1$ (or -1) and $\beta_i > 0$, the recurrency connection of the ith hidden neuron then has negative (or positive) feedback, and the recurrency weight of the neuron is greater than the absolute value of the sum of all the weights in the feedforward, feedback and intra-layer connections within all the neurons. In other words, in this case the recurrency functions of the neurons play a dominant role in the dynamic properties of the neural networks.

Example 3: Consider a three-neuron recurrent neural network whose nonlinear dynamics are described by the nonlinear difference equations (27) below where the parameters of the network and the connection weight matrix are given as

$$\begin{cases} \alpha_1 = -1, & \alpha_2 = -1, & \alpha_3 = -1 \\ \beta_1 = 1, & \beta_2 = 1/2, & \beta_3 = 1/2 \end{cases}$$

and

$$\begin{cases} w_{1,1} = 1/5, & w_{1,2} = 1/12, & w_{1,3} = 1/7 \\ w_{2,1} = 1/6, & w_{2,2} = 1/4, & w_{2,3} = 1/9 \\ w_{3,1} = 1/24, & w_{3,2} = 1/16, & w_{3,3} = 1/3 \end{cases}.$$

It is easy to show that

$$sign(\alpha_i)\beta_i w_{i,i} = \begin{cases} -1/5 > -1; & i=1 \\ -1/8 > -1; & i=2 \\ -1/6 > -1; & i=3 \end{cases}.$$

Hence, the sufficient conditions of (19) and (21) obtained in Theorems 5 and 6 may be used to verify the absolute stability of system (27). For the condition (19)

$$|\beta_i| \sum_{j=1,j\neq i}^{3} |w_{i,j}| = \begin{cases} 19/84 > 1/5; & i=1 \\ 1/9 < 1/8; & i=2 \\ 5/96 < 1/6; & i=3 \end{cases}.$$

Since the absolute stable condition (19) is not satisfied for $i=1$, Theorem 5 fails to prove the absolute stability of system (27). For the condition (21)

$$\sum_{i=1,i\neq j}^{3} |\beta_i w_{i,j}| = \begin{cases} 5/48 < 1/5; & j=1 \\ 11/96 < 1/8; & j=2 \\ 25/126 < 1/6; & j=3 \end{cases}.$$

System (27) is, therefore, absolutely stable for the arbitrary constant input s based on Theorem 6. For the input $s_1 = -2$, $s_2 = 1$ and $s_3 = 3$, the initial states $x(0) = (6,7,8)^T$, the unique absolutely stable equilibrium point is $x^* =$

$$\begin{cases} \sum_{j=1}^{n} |w_{i_1,j}| < \delta_{i_1}, & i_1 = 1,2,\cdots,n_1 \\ -1/c \leq sign(\alpha_{i_2})\beta_{i_2} w_{i_2,i_2} < 0, & i_2 = n_1+1,\cdots,n \\ -sign(\alpha_{i_2})\beta_{i_2} w_{i_2,i_2} > |\beta_{i_2}| \sum_{j=1,j\neq i_2}^{n} |w_{i_2,j}|; & i_2 = n_1+1,\cdots,n \end{cases} \tag{23}$$

$$\begin{cases} \sum_{j=1}^{n} |w_{i_1,j}| < \delta_{i_1}, & i_1 = 1,2,\cdots,n_1 \\ -2/c < sign(\alpha_{i_2})\beta_{i_2} w_{i_2,i_2} \leq -1/c, & i_2 = n_1+1,\cdots,n \\ -sign(\alpha_{i_2})\beta_{i_2} w_{i_2,i_2} + |\beta_{i_2}| \sum_{j=1,j\neq i_2}^{n} |w_{i_2,j}| < 2/c; & i_2 = n_1+1,\cdots,n \end{cases}. \tag{24}$$

$$\begin{cases} x_1(k+1) = -x_1(k) + \tanh[\frac{1}{5}x_1(k) + \frac{1}{12}x_2(k) + \frac{1}{7}x_3(k) + s_1] \\ x_2(k+1) = -x_2(k) + \frac{1}{2}\tanh[\frac{1}{6}x_1(k) + \frac{1}{4}x_2(k) + \frac{1}{9}x_3(k) + s_2] \\ x_3(k+1) = -x_2(k) + \frac{1}{2}\tanh[\frac{1}{24}x_1(k) + \frac{1}{16}x_2(k) + \frac{1}{3}x_3(k) + s_3] \end{cases}. \tag{27}$$

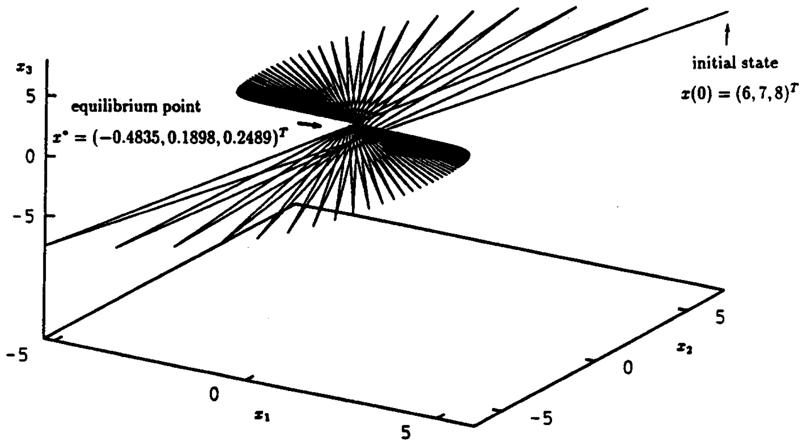

Fig. 6. The phase space trajectory of the states x_1, x_2 and x_3 of the recurrent neural network (27) with the input $s_1 = -2$, $s_2 = 1$, and $s_3 = 2$, and the initial state $x(0) = (6, 7, 8)^T$. In this case, the unique absolutely stable state equilibrium point is $x^* = (-0.4835, 0.1898, 0.2489)^T$.

$(-0.4835, 0.1898, 0.2489)^T$. The three-dimension phase space trajectory for the neural system (27) is plotted in Fig. 6.

VI. Similarity Transformation Based Stability Results

Since $P^{-1}WP$ has the same eigenvalues as W whenever P is a nonsingular $n \times n$ matrix, the estimation of the union of eigenvalues of $P^{-1}AP$ can be obtained applying Ostrowski's theorem. In fact, for some choices of P the bounds obtained may be sharper. A particularly convenient choice is $P = \mathrm{diag}[p_1, p_2, \cdots, p_n]$ with all $p_i > 0$. System (3) may then be represented using the new coordinate $x(k) = Pz(k)$ as follows

$$\begin{aligned} z(k+1) &= P^{-1}APz(k) + P^{-1}B\sigma[WPz(k) + s] \\ &\equiv \bar{f}(z(k)). \end{aligned} \tag{28}$$

The Jacobian is given as

$$\begin{aligned} \bar{f}'(z) &= [\bar{f}'_{i,j}(z)]_{n\times n} \\ &= P^{-1}AP + P^{-1}B\Sigma WP \\ &= P^{-1}f'(x)P \end{aligned} \tag{29}$$

where $\Sigma = \mathrm{diag}[\sigma_1(z), \sigma_2(z), \cdots, \sigma_n(z)]$. Therefore, the Jacobians $\bar{f}'(z)$ and $f'(x)$ have the same eigenvalues; that is, system (28) has the same stability as the original system (3).

Theorem 7: Let $-1 < \alpha_i < 1$; $i = 1, 2, \cdots, n$ and $p_1, p_2, \cdots, p_n$ be positive real numbers, $\gamma \in [0, 1]$ be given, and

$$\begin{cases} R_i^p &= \sum\limits_{j=1, j\neq i}^{n} p_j|w_{i,j}| \\ \bar{C}_i^p &= \sum\limits_{j=1, j\neq i}^{n} \frac{1}{p_j}|\beta_j||w_{j,i}| \end{cases}.$$

If the connection weight matrix $W = [w_{i,j}]_{n\times n}$ satisfies

$$|w_{i,i}| + \frac{|\beta_i|^{\gamma-1}}{p_i^{2\gamma-1}}(R_i^p)^{\gamma}(\bar{C}_i^p)^{1-\gamma} < \delta_i \quad i = 1, 2, \cdots, n \tag{30}$$

system (3) is absolutely stable.

Proof: Note that

$$\bar{f}'_{i,i}(z) = \alpha_i + \beta_i\sigma'_i(z)w_{i,i} \quad i = 1, 2, \cdots, n \tag{31}$$

and

$$\bar{f}'_{i,j}(z) = \frac{1}{p_i}\beta_i\sigma'_i(z)w_{i,j}p_j \quad i, j = 1, 2, \cdots, n; \quad i \neq j \tag{32}$$

then

$$\begin{aligned} &|\bar{f}'_{i,i}(z)| + \left(\sum_{j=1, j\neq i}^{n} |\bar{f}'_{i,j}(z)|\right)^{\gamma}\left(\sum_{j=1, j\neq i}^{n} |\bar{f}'_{j,i}(z)|\right)^{1-\gamma} \\ &= |\alpha_i + \beta_i\sigma'_i(z)w_{i,i}| + |\beta_i\sigma'_i(x)|^{\gamma}\left(\sum_{j=1, j\neq i}^{n} \frac{1}{p_i}|w_{i,j}|p_j\right)^{\gamma} \\ &\times \left(\sum_{j=1, j\neq i}^{n} |\sigma'_j(x)|\frac{1}{p_j}|\beta_j||w_{j,i}|p_i\right)^{1-\gamma} \\ &< |\alpha_i| + |\beta_i|c|w_{i,i}| + \frac{|\beta_i|^{\gamma}}{p_i^{2\gamma-1}}(R_i^p)^{\gamma}(\bar{C}_i^p)^{\gamma-1} < 1. \end{aligned} \tag{33}$$

Therefore, using Corollary 2, the absolutely stability of system (3) is assured. □

Furthermore, the results corresponding to Theorems 4–6 in the previous section are respectively given in the following theorems derived using the same proof procedures as that in Theorem 7.

Theorem 8: Let $-1 < \alpha_i < 1$; $i = 1, 2, \cdots, n$ and $p_1, p_2, \cdots, p_n$ be positive real numbers. If the connection weight matrix $W = [w_{i,j}]_{n\times n}$ satisfies

$$|w_{i,i}| + \frac{1}{p_i}\sum_{j=1, j\neq i}^{n} p_j|w_{i,j}| < \delta_i \quad i = 1, 2, \cdots, n \tag{34}$$

or

$$|w_{j,j}| + \frac{p_j}{|\beta_j|}\sum_{i=1, i\neq j}^{n} \frac{1}{p_i}|\beta_i w_{i,j}| < \delta_j \quad j = 1, 2, \cdots, n \tag{35}$$

system (3) is absolutely stable.

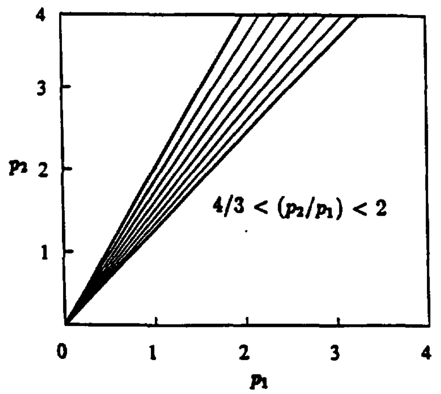

Fig. 7. The choice region of parameters p_1 and p_2.

Example 4: Consider again the two-neuron system given in Example 1. In Section IV, it has been shown that Theorem 4 fails to determine the absolute stability of the system. Now let $p_1, p_2 > 0$ and

$$\begin{cases} |w_{1,1}| + (p_2/p_1)|w_{1,2}| = 1/4 + (p_2/p_1)1/8 < 1/2 \\ |w_{2,2}| + (p_1/p_2)|w_{2,1}| = 1/4 + (p_1/p_2)1/3 < 1/2 \end{cases} \quad (36)$$

The solution of the above inequalities is a region described by $4/3 < (p_2/p_1) < 2$ in the p_1-p_2 plane, as shown in Fig. 7. Therefore, based on the Theorem 8, the absolutely stability of the system (17) is proven.

Theorem 9: Let $|\alpha_i| = 1, i = 1, 2, \cdots, n$ in (3), and $p_1, p_2, \cdots, p_n$ be positive real numbers. If the connection weight matrix $W = [w_{i,j}]_{n\times n}$ satisfies

$$\begin{cases} -1/c \leq sign(\alpha_i)\beta_i w_{i,i} < 0 \\ -sign(\alpha_i)\beta_i w_{i,i} > \frac{|\beta_i|}{p_i} \sum_{j=1, j\neq i}^{n} p_j |w_{i,j}|; \quad i = 1, 2, \cdots, n \end{cases} \quad (37)$$

or

$$\begin{cases} -2/c < sign(\alpha_i)\beta_i w_{i,i} \leq -1/c \\ -sign(\alpha_i)\beta_i w_{i,i} + \frac{|\beta_i|}{p_i} \sum_{j=1, j\neq i}^{n} p_j |w_{i,j}| < 2/c; \quad i = 1, 2, \cdots, n \end{cases} \quad (38)$$

system (3) is absolutely stable.

Theorem 10: Let $|\alpha_i| = 1$, $i = 1, 2, \cdots, n$ in (3), and $p_1, p_2, \cdots, p_n$ be positive real numbers. If the connection weight matrix $W = [w_{i,j}]_{n\times n}$ satisfies

$$\begin{cases} -1/c \leq \mathrm{sign}(\alpha_j)\beta_j w_{j,j} < 0 \\ -\mathrm{sign}(\alpha_j)\beta_j w_{j,j} > |p_j| \sum_{i=1, i\neq j}^{n} \frac{1}{p_i} |\beta_i w_{i,j}|; \quad j = 1, 2, \cdots, n \end{cases} \quad (39)$$

or

$$\begin{cases} -2/c < \mathrm{sign}(\alpha_j)\beta_j w_{j,j} \leq -1/c \\ \mathrm{sign}(\alpha_j)\beta_j w_{j,j} + |p_j| \sum_{i=1, i\neq j}^{n} \frac{1}{p_i} |\beta_i w_{i,j}| < 2/c; \quad j = 1, 2, \cdots, n \end{cases} \quad (40)$$

system (3) is absolutely stable.

Example 5: The absolute stability problem of the three-neurons system (27) will now be discussed using Theorem 9. Let $p_1, p_2, p_3 > 0$. Condition (37) for system (27) may be given as

$$\begin{cases} 1/p_1(p_2/12 + p_3/7) & < 1/5 \\ 1/p_2(p_1/6 + p_3/9) & < 1/4 \\ 1/p_3(p_1/24 + p_2/16) & < 1/3 \end{cases}. \quad (41)$$

The solution of above the set of the inequalities forms a convex set in R^3. If one of the three parameters p_1, p_2 and p_3 is fixed, the solution region of the other two parameters is then an open square in the plane. For instance, let $p_3 = 1$ in the inequalities (41), then p_1 and p_2 may be determined by the inequalities

$$\begin{cases} p_2/12 + 1/7 & < & p_1/5 \\ p_1/6 + 1/9 & < & p_2/4 \\ p_1/24 + p_2/16 & < & 1/3 \end{cases}.$$

The solution is obtained as

$$\frac{340}{2457} < p_1 < \frac{11}{3}$$
$$\frac{348}{273} < p_2 < \frac{612}{161}.$$

Obviously, there exist p_1, p_2 and p_3 which satisfy inequalities (41). Hence, the absolute stability of system (27) can be determined using Theorem 9.

For $-1 < \alpha_i < 1$; $i = 1, 2, \cdots, n$, consider a trivial network structure where the majority of the feedback connections are eliminated such that the network is a dynamic feedforward network. In this case, the weight matrix is a strict lower triangular matrix possibly after a renaming of all neurons; that is, $w_{ij} = 0$, for all $j \geq i$. The absolutely stable conditions (34) and (35) are then represented by the following simple forms

$$\frac{1}{p_i}\sum_{j>i}^{n} p_j |w_{i,j}| < \delta_i \quad i = 1, 2, \cdots, n-1 \quad (42)$$

and

$$\frac{p_j}{|\beta_j|}\sum_{i<j}^{n} \frac{1}{p_i}|\beta_i w_{i,j}| < \delta_j. \quad j = 2, \cdots, n. \quad (43)$$

If $p_i = \epsilon^i$, where $\epsilon > 0$ is a small real number, (39) and (40) are then modified as

$$\sum_{j>i}^{n} \epsilon^{j-i} |w_{i,j}| < \delta_i \quad i = 1, 2, \cdots, n-1 \quad (44)$$

and

$$\frac{1}{|\beta_j|}\sum_{i<j}^{n} \epsilon^{j-i} |\beta_i w_{i,j}| < \delta_j. \quad j = 2, \cdots, n \quad (45)$$

It is easy to see that the above conditions are satisfied for arbitrary connection weights $w_{i,j}$ $(j > i)$ by setting ϵ enough small . Hence, the dynamic neural network (3) without the majority of the recurrent connections $(w_{ij} = 0, \text{for } j \geq i)$ is inherently absolutely stable for the arbitrary connection weights.

VII. Conclusion

Some absolute stability conditions for a general class of discrete-time recurrent neural networks were derived using Ostrowski's theorem as well as the similarity transformation. For a given RNN model with bounded derivatives of the nonlinear activation functions, these sufficient conditions, which were determined only by the synaptic weight matrix of the network, were easy to check using simple algebraic manipulation about the connection weights. The analytical results showed that the changing region of the connection weights $w_{i,j}$ for ensuring the absolute stability of the network became large as the absolute value of the neural gain β_i, or time constant α_i, decreased. Some comparisons of the results were presented in this paper with the results of others in the literature. When the parameter γ in Ostrowski's theorem was appropriately chosen, the results presented in Theorems 3 and 7 gave better estimations of the stability than the stability conditions derived from well-known Gerschgorin's theorem. For the simple case when the time constant $\alpha_i = 0$ input $s_i = 0$; and $i = 1, 2, \cdots, n$, the absolutely stable conditions in Theorem 3 were found to be more relaxed than the norm condition given by Li [18].

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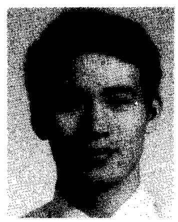

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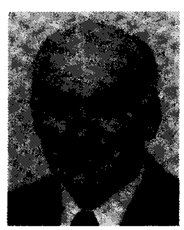

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