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Discrete-Time versus Continuous-Time Models of Neural Networks*

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In mathematical modeling, very often discrete-time (DT) models are taken from, or can be viewed as numerical discretizations of, certain continuous-time (CT) models. In this paper, a general criterion, the asymptotic consistency criterion, for these DT models to inherit the dynamical behavior of their CT counterparts is derived. Detailed instances of this criterion are established for several classes of neural networks. © 1992 Academic Press, Inc.

1. INTRODUCTION

In modeling biological neural networks, various approaches differing in matters of biological and technical details can be distinguished (see, for example, [AR90, Arb89, Gro88]). One interesting distinction is between models where network evolution is a continuous process and those where the network state is updated only at discrete-time (DT) instants. Very often discrete-time models are taken from, or can be viewed as numerical discretizations of, certain continuous-time (CT) models. It is hoped that interesting properties of the CT models transcend the numerical discretization and pass to the DT models. In this paper, we investigate, from the dynamical system point of view [GH83, Dev86], conditions under which these DT models inherit the dynamical behavior of their CT counterparts.

To be concrete, we consider an autonomous neural network of semilinear type as a working example, although the approach we are taking applies more generally. One much-studied CT model of an n-neuron network [Cow68, Ama72, GC83, Hop84, Sej76] is given by a system of nonlinear differential equations

$$\tau_i \frac{dx_i}{dt} = -x_i + g_{\mu_i}\left(\sum_{j=1}^{n} w_{ij} x_j + J_i\right), \qquad 1 \leqslant i \leqslant n. \tag{1}$$

For obvious reasons we refer to system (1) as semilinear. A related discrete-time version [BW90, Gro88, Hop82, MW90] is specified by a system of nonlinear difference equations

$$x_i(t+1) = g_{\mu_i}\left(\sum_{j=1}^{n} w_{ij} x_j(t) + J_i\right), \qquad t = 0, 1, \ldots, \tag{2}$$

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where x_i represents the internal state of neuron i, $\mu_i(>0)$ is the neuron gain, $W=[w_{ij}]$ is the real-valued matrix of synaptic connection weights, $g_{\mu_i}(x_i)$ is the activation function which often takes a sigmoidal form (whose exact definition is given later), and J_i is a constant external input to neuron i. As is well known, the CT model in Eq. (1) can be cast in the equivalent form of electric circuit equations described by

$$C_i \frac{du_i(t)}{dt} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} \frac{w_{ij}}{R_i} g_{\mu_j}(u_j(t)) + \frac{J_i}{R_i},$$

by the transformation $u = Wx + J$ and assuming that the time constants $\tau_i = C_i R_i$ are equal for all i.

The CT model has been widely used for many purposes, such as in modeling of associative memories, combinatorial optimization, and pattern recognition. But it suffers some computational limitations in practice. For example, its digital computer simulation is a very time-consuming process because of the numerical integration of the differential equations, and its implementation with analog hardware can have stability problems [WS87].

Although the DT model (2) may bear less resemblance to biological neurons than the CT model (1), it turns out to be a very acceptable differentiable approximation to McCulloch–Pitts Boolean networks in a continuum state space when the gains μ_j are high enough [BW90] that the activation function almost reaches the saturated limit and is close to a Heaviside function [Hir91]. Also, because discretization is essential in simulating CT models on digital computers, some DT models may be better adapted to the applications cited above.

Certainly, it is desirable that a DT model, when derived as a numerical approximation to a CT model, preserve the CT model's dynamical features, such as asymptotic attractor points, global, and local stabilities. Once this is established, the DT model can be fully used without loss of any functional similarity to the CT model and preserving any biological reality that the CT model has. In many applications like associative memory, pattern recognition, and combinatorial optimization problems, it is the asymptotic stability and the attraction domains of fixed points (equilibrium states) that are of great interest. Hence, in this paper, we focus on the asymptotic stability of fixed points of corresponding CT and DT models.

Numerically, any discretization method of solving ODEs tends to generate spurious asymptotic points (fixed points, periodic points, quasi-periodic points, strange attractors, etc.) and the resulting difference equations may be less stable than the underlying ODEs [IPS91]. This is also true for Eqs. (1) and (2) above.

It is easy to see that the fixed points of (1) and (2) are identical. But when the asymptotic dynamics are taken into account, the two models may be different in many aspects. This is often the case. For example, when a network has symmetric connection weights, the Boolean DT model in [Hop82] has the same quasi-convergent property [Hir89] as the CT model in [Hop84], provided that asynchronous state updating is used (see also [Blu90]). In this case, all trajectories of the network approach local minima of a Liapunov function, which turn out to be the fixed points of the network. However, the corresponding synchronous analog DT model may have oscillations of period 2 (spurious periodic points) and has been shown in [MW89a] to be quasi-convergent when the connection matrix and the gain values satisfy a simple constraint:

$$W + M^{-1} \text{ is positive definite,} \tag{3}$$

where $M = \mathrm{diag}(\mu_i)$ is the diagonal matrix formed by μ_i's. (Note that the quasi-convergence means that every trajectory approaches the set of fixed points [Hir89].)

Also, the work done by Atiya and Baldi in [AB89] on the CT model (1) and the work done by Blum and Wang in [BW90, Wan91b] on the DT model (2) for various connection architectures of the networks show some dramatic differences of the two models in their asymptotic dynamics. For instance, by the Poincaré–Bendixson Theorem [HS74], the CT model (1) of a two-neuron network has a rather simple asymptotic dynamics: it either converges to a fixed point or approaches a limit cycle, but the DT model (2) of the same network, as a two-dimensional map, can display very complicated and even chaotic behavior [Wan91b].

Even in the quasi-convergent case for symmetric networks, it is not clear whether under the constraint (3) the DT model (2) is guaranteed to mimic the CT model (1) locally such that, for example, attractors in the CT model are inherited in the DT model. It is very likely that an attractor in the CT model (1) may be distorted in the DT model (2) into a saddle or even a repellor.

In general, we would like a DT model to preclude the existence of spurious asymptotic points and to preserve the asymptotic behavior of a corresponding CT model. The theory of local asymptotic stability in dynamical systems provides conditions necessary (and sometimes sufficient) for achieving this goal. In this paper we develop a general criterion, called *the asymptotic consistency criterion*, which is a necessary and sufficient condition for a DT model resulting from the Euler method to have an asymptotic behavior consistent with the corresponding CT model locally at a fixed point, under the assumption [Hir89] that the fixed point is hyperbolic (see Section 2). Then we apply it to feed-forward, symmetric, asymmetric, and circulant neural networks of the semilinear type. For the feed-forward, symmetric, and anti-symmetric networks, we establish sufficient conditions (Theorems 3, 4, 5) for the DT model to be asymptotically consistent with the CT model at *all* fixed points, which only involve the connection matrix and the gain values. These sufficient conditions sometimes are also necessary, for example, when there are no external inputs and the origin is a fixed point. Finally, we show (Theorem 6) that the DT model (2) sometimes violates the criterion and fails to mimic the asymptotic behavior of the CT model (1).

Note that our results are based on local rather than global asymptotic stability analysis and therefore do not provide a rigorous guarantee of coincidence of the dynamics of both models in the global sense. But rather surprisingly, some of these results, when compared with the constraint (3) for global quasi-convergent dynamics, are also sufficient for the global results, for example, in symmetric networks. On a more profound level, this shows a relationship between our approach, which is mainly based on local stability analysis, and the approach taken by Grossberg and Cohen [GC83], Hopfield [Hop84], Hirsch [Hir89], Marcus and Westervelt [MW89a], etc., which is based on Liapunov stability theory. (Many of the results presented here are part of the Ph.D. thesis of Wang [Wan91a].)

2. A Theory of Asymptotic Consistency of CT and DT Models

Let ϕ_t be a flow [HS74] defined by an autonomous system of ODEs,

$$\frac{dx}{dt} = F(x), \qquad x \in \mathbb{R}^n. \tag{4}$$

Throughout we assume that F is a C^1 function on $\mathbb{R}^n$ to $\mathbb{R}^n$ and the set of fixed points (equilibrium states) $\mathrm{Fix}(F) = \{x \in \mathbb{R}^n \mid F(x) = 0\}$ of the flow is not empty. We call ϕ_t a CT *flow*.

The simplest one-step discretization of (4), *Euler's method*, with a constant time step $h > 0$, is defined by the difference equation,

$$x(t+h) = x(t) + hF(x(t)). \tag{5}$$

This gives rise to a map $f_h \colon \mathbb{R}^n \to \mathbb{R}^n$,

$$f_h(x) = x + hF(x) \tag{6}$$

and the iterated-map dynamics such that, for any $k = 0, 1, 2, \ldots$, $x_{t+kh} = f_h^k(x_t)$, where f_h^k is the kth iterate of f_h. We call (f_h^k) a DT *flow* corresponding to the CT flow ϕ_t.

It is obvious that, for any $h > 0$, the set of fixed points,

$$\mathrm{fix}(f_h) = \{x \in \mathbb{R}^n \mid f_h(x) = x\}$$

of the iterated-map f_h is identical to the set of fixed points $\mathrm{Fix}(F)$ of the flow ϕ_t. Thus, from now on we shall not distinguish the fixed points of the CT flow and the DT flow.

Let x^* be a hyperbolic fixed point of (4) and (6). It follows from the Stable Manifold Theorems for ODEs and maps [GH83] that the local asymptotic stability of x^* under the CT flow and DT flow is determined respectively by the linear ODE

$$\frac{dx}{dt} = DF(x^*) \cdot x \tag{7}$$

and the linear map

$$x \mapsto Df_h(x^*) \cdot x, \tag{8}$$

where $DF(x^*)$ and $Df_h(x^*)$ are the respective Jacobians of F and f_h at x^*.

We call an eigenvalue Λ of $DF(x^*)$ (or λ of $Df_h(x^*)$) *attracting*, *repelling*, or *neutral*, according to whether the real part of Λ satisfies $\mathrm{Re}(\Lambda) < 0$, $\mathrm{Re}(\Lambda) > 0$, or $\mathrm{Re}(\Lambda) = 0$ (or according to $|\lambda| < 1$, $|\lambda| > 1$, or $|\lambda| = 1$), respectively. Then the fixed point x^* is *hyperbolic* under the CT flow ϕ_t (or iterated map f_h) if every eigenvalue of $DF(x^*)$ ($Df_h(x^*)$) is not neutral. A fixed point x^* is *asymptotically stable* (a.s.), *asymptotically unstable* (a.u.), or a *saddle* under the flow ϕ_t (or map f_h) if the eigenvalues of $DF(x^*)$ (or $Df_h(x^*)$) are respectively all attracting, all repelling, or some attracting and others repelling. If x^* is a.s., it is called a (local) *attractor*. If x^* is a.u., it is a (local) *repellor*.

Clearly the relation between the two Jacobian matrices, $DF(x^*)$ and $Df_h(x^*)$ at x^*, is

$$Df_h(x^*) = I + hDF(x^*) \tag{9}$$

and the eigenvalues Λ of $DF(x^*)$ and λ of $Df_h(x^*)$ can be paired up such that

$$\lambda = 1 + h\Lambda. \tag{10}$$

Further, they have the same eigenvectors: for any nonzero v,

$$Df_h(x^*) \cdot v = \lambda v \qquad \text{if and only if} \quad DF(x^*) \cdot v = \Lambda v.$$

The subspaces spanned by the (generalized) eigenvectors of $DF(x^*)$ and $Df_h(x^*)$ are usually grouped into three classes: the stable, unstable, and center subspaces, spanned by the eigenvectors whose eigenvalues are attracting, repelling, and neutral, respectively. If the fixed point is hyperbolic, its center subspace is of zero dimension. The stable manifold theorem [GH83] for a CT flow and a DT flow at a hyperbolic fixed point states that there are local stable and unstable manifolds, tangent to the stable and unstable subspaces of the linearized flow at the corresponding fixed point and of the same dimensions as the corresponding subspaces. The comparison of local behavior of (4) and (5) near hyperbolic fixed points, therefore, can be based on the eigen analysis of their linearizations.

DEFINITION. The (DT flow of the) map f_h in (6) is said to be *asymptotically consistent* with (the CT flow defined by) F in (4) at a hyperbolic fixed point x^* if the linear map (8) has the same stable and unstable subspaces as the linear flow (7).

When f_h is asymptotically consistent with F at a hyperbolic fixed point x^*, f_h and F have local manifolds of the same stability type along any eigenvector direction: either both stable or both unstable. As a result, the local stable manifolds at x^* of f_h and F have the same dimension and coordinates and so do the unstable manifolds.

Note that aspects of interest here are dynamical rather than numerical. In traditional numerical analysis [Blu72], a discrete-time trajectory x_t of (5) or other DT method is regarded as an approximation to the exact solution $x(t)$ of (4) and the main concern is the convergence and stability of the numerical methods with respect to the "discretization" error $|x(t)-x_t|$.

May [May74] discussed the similar problem of global asymptotically convergent behavior of CT and DT models in ecosystems. More recently in [IPS91], general numerical methods for the initial value problems of differential equations are viewed as dynamical systems in which the time step h plays the role of a bifurcation parameter, and a unified approach is presented to prevent generation of spurious asymptotic points.

We state the following lemma, leaving the proof to the reader.

LEMMA. *Let x^* be a fixed point of* (4) *and* (6). *Let Λ and λ be respective eigenvalues of $DF(x^*)$ and $Df_h(x^*)$ that are related as in* (10).

(i) *If λ is attracting for some h, then $\Lambda=(\lambda-1)/h$ must be attracting. It follows that any fixed point that is a.s. under some map f_h is a.s. under the CT flow. In particular, if the map is globally asymptotically convergent, then so is the CT flow.*

(ii) *If Λ is repelling, then $\lambda=1+h\Lambda$ must be repelling for all h. It follows that any fixed point that is a.u. under the CT flow is a.u. under the DT flow for any $h>0$.*

Consequently, the effect of the time step h upon local asymptotic stability is, at worst, to cause some stable eigen directions of $DF(x^*)$ to be unstable eigen directions of $Df_h(x^*)$. But h never turns any unstable eigen direction of $DF(x^*)$ into a stable one of $Df_h(x^*)$. This confirms the known fact that maps resulting from numerical discretizations tend to be less stable than their continuous-time flows [IPS91].

Now, we give conditions on h to ensure that the map f_h is asymptotically consistent with F.

THEOREM 1 (Asymptotic Consistency Criterion). *Let x^* be a hyperbolic fixed point for both F in* (4) *and f_h in* (6) *for a given h. The following are equivalent*:

(i) *f_h is asymptotically consistent with F at x^*;*

(ii) $|1+h\Lambda|<1$ *for all attracting eigenvalues Λ of the Jacobian $DF(x^*)$;*

(iii) $h<-2\,\mathrm{Re}(\Lambda)/|\Lambda|^2$ *for all attracting eigenvalues Λ of the Jacobian $DF(x^*)$.*

Proof. Use the Lemma and the Definition.

COROLLARY 1. (a) *A sufficient condition for f_h to be asymptotically consistent with F at x^* is*

$$h<\min\frac{2|\mathrm{Re}(\Lambda)|}{|\Lambda|^2}, \tag{11}$$

where the min *is over all eigenvalues Λ of $DF(x^*)$.*

(b) *When the eigenvalues of $DF(x^*)$ are all real and $\Lambda_{\min}$ is the minimal one, a sufficient condition for f_h to be asymptotically consistent with F at x^* is*

$$h < \frac{2}{|\Lambda_{\min}|}. \tag{12}$$

If, further, $\Lambda_{\min} < 0$ (i.e., it is attractive), then condition (12) is also necessary.

The continuous-time flow is the limit of the discrete-time flow of the map f_h as the time step h approaches zero. This is also the case for asymptotic consistency. By condition (ii) of Theorem 1, to ensure that λ attracting when Λ is, h must be chosen so that all attracting Λ lie inside a circle of radius $1/h$, centered at $-1/h$ on the real axis in the complex plane. As h approaches zero, the circle eventually contains all Λ with $\mathrm{Re}(\Lambda) < 0$. By Lemma (ii), this establishes asymptotic consistency as $h \to 0$.

Also, if h is fixed prescriptively (e.g., $h = 1$ as in (2)), then certain restrictions must be placed upon F in order that f_h and F behave the same locally around the fixed points. For example, if $h = 1$, then one requirement for F is that $|1 + \Lambda| < 1$ for all attracting eigenvalues Λ. A necessary condition for this is that $-2 < \mathrm{Re}(\Lambda) < 0$ for all attracting Λ's.

Therefore, as far as local asymptotic behavior around hyperbolic fixed points is concerned, the DT flow of the map f_h mimics the CT flow defined by F if f_h is asymptotically consistent with F at all such fixed points. Once this is established, any asymptotically stability analysis and simulation results of the discrete-time flow may in principle be taken over and applied to the continuous-time flow, and vice versa, for common hyperbolic fixed points.

3. Asymptotic Consistency of Neural Network Models

Now we apply the general theory of asymptotic consistency to specific models of neural networks. We consider the CT model in Eq. (1) with all $\tau_i = 1$ (for the purpose of simplicity), which can be rewritten in vector form (4) with

$$F(x) = -x + G(W \cdot x + J), \tag{13}$$

where $G(z) = [g_{\mu_1}(z_1), ..., g_{\mu_n}(z_n)]^{\mathrm{T}}$. The first-order numerical approximation (5) to this CT model gives rise to a DT model whose map is

$$f_h(x) = x + h \cdot F(x) = (1 - h)\, x + hG(W \cdot x + J). \tag{14}$$

It is easy to see that Eq. (2) is a special case of Eq. (14), where h is taken as 1.

Each neuron activation function g_{μ_i} is assumed to take a form $g_{\mu_i}(z) = \psi(\mu_i z)$, where ψ is a C^1 function satifying: (i) $\psi(z) \to \pm 1$ as $z \to \pm\infty$; (ii) $\psi' > 0$; (iii) $\psi'(z) \to 0$ as $z \to \pm\infty$; and (iv) $\psi'(z)$ takes a local maximal value 1 at a unique value $z=0$, so that g'_{μ_i} achieves the maximum μ_i at the origin $z=0$; (v) $z=0$ is a fixed point of ψ. Typical examples of such a function $\psi(z)$ are

$$2\left(\sigma(z)-\frac{1}{2}\right) \text{ with } \sigma(z)=\frac{1}{1+e^{-z}}, \qquad \tanh(z)=\frac{e^z-e^{-z}}{e^z+e^{-z}}, \qquad \frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{2}z\right).$$

At any fixed point x^*, the respective Jacobian matrices of F and f_h are

$$DF(x^*) = -I + G'(W\cdot x^* + J)\,W \tag{15}$$

$$\begin{aligned} Df_h(x^*) &= I + h\,DF(x^*) \\ &= (1-h)\,I + hG'(W\cdot x^* + J)\,W, \end{aligned} \tag{16}$$

where I is the identity matrix and $G'(z)=\operatorname{diag}(g'_{\mu_i}(z_i))$ is the derivative of G at z. The eigenvalues λ of $Df_h(x^*)$ and the eigenvalues Λ of $DF(x^*)$ satisfy the simple relation (10). The asymptotic consistency criterion (Theorem 1) provides some necessary and sufficient conditions for the DT flow defined by (14) to be asymptotically consistent with the CT flow defined by (13). However, when the eigenvalues Λ and λ are all real, we have a more concrete result.

THEOREM 2 (Real Eigenvalue Case). *Assume x^* is a hyperbolic fixed point of the CT flow defined by F in* (13) *and the DT flow of* (14). *Suppose that all eigenvalues of $DF(x^*)$ are real. If x^* is not a repellor of F, then the DT flow is asymptotically consistent with the CT flow at x^* if and only if h is such that the matrix*

$$N(W,h) \equiv M_{x^*}W + \frac{2-h}{h}I$$

has only positive eigenvalues, where $M_{x^*}=G'(W\cdot x^*+J)=\operatorname{diag}(g'_{\mu_i}(\sum_{j=1}^{n} w_{ij}x_j^* + J_i))$.

Proof. If x^* is not a repellor of F, then some $\Lambda<0$. Hence, the minimal eigenvalue Λ_{min} of $DF(x^*)$ is negative. The asymptotic consistency criterion in this case reduces to

$$\Lambda_{\min} + 2/h > 0. \tag{17}$$

By (15), $DF(x^*) = M_{x^*}W - I$. Therefore, (17) holds if and only if all eigenvalues of the matrix $M_{x^*}W - I + (2/h)I$ are positive. ∎

Remark. If x^* is a repellor of the CT flow, it follows from Lemma (ii) that the DT flow for any h is asymptotically consistent with the CT flow at x^*.

Remark. If $h=1$ in the theorem, then

$$N(W,1) = M_{x^*}W + I$$

has positive eigenvalues for $\|W\|$ sufficiently small or for sufficiently small gains μ_i.

In the following, we show that, for feed-forward, symmetric, and anti-symmetric networks, explicit sufficient (and sometimes necessary) conditions can be obtained to ensure that the DT model is asymptotically consistent with the CT model at *all* hyperbolic fixed points. These conditions only depend on the gain parameters and the eigenvalues of the connection weight matrices. Incidentally, the condition in the symmetric case is identical to the sufficient condition given in [MW89a] for ensuring global convergent dynamics of symmetric networks. This suggests that, in the cases where the condition is also necessary for asymptotic consistency, it may be also necessary for global quasi-convergence of the DT model. Finally, we show by an example of circulant networks that the time step h in the DT model may need to be very small in order to maintain the same behavior as the CT model.

3.1. *Feed-Forward Networks*

The form of W for this kind of network is lower-triangular (possibly after a renaming of all neurons). Note that w_{ii} need not be 0. This includes the usual feed-forward layered networks where all $w_{ii}=0$. As a cascade of n subnetworks, each of which consists only of a single neuron, the CT model has been shown in [Hir89] to have global quasi-convergent dynamics.

Since the eigenvalues of W are $w_{11}, w_{22}, ..., w_{nn}$ (which are all real), the eigenvalues of $DF(x^*)$ at any fixed point x^* are real and equal to

$$g'_{u_i}\left(\sum_{j\leqslant i} w_{ij}x_j^* + J_i\right) w_{ii} - 1, \qquad i = 1, ..., n.$$

THEOREM 3 (Feed-Forward Networks). *For a feed-forward network, a sufficient condition for the* DT *flow defined by* (14) *with* $h<2$ *to be asymptotically consistent with the* CT *flow defined by* (13) *at all hyperbolic fixed points is that*

$$\mu_i w_{ii} + \frac{2-h}{h} > 0.$$

The condition is also necessary when all external inputs $J=0$.

Proof. The condition rephrases that the matrix

$$MW + \frac{2-h}{h} I$$

has only positive eigenvalues, where M is the matrix given in (3). As both M_{x^*} and M are diagonal and $M_{x^*} \leqslant M$ (entry-wise) for any fixed point x^*, the condition implies that matrix

$$M_{x^*} W + \frac{2-h}{h} I$$

has only positive eigenvalues. Then the result follows from Theorem 2.

When all external $J=0$, the origin is a fixed point and $M=M_0$. Therefore the condition is necessary. ∎

COROLLARY 2. *For $h=1$, the condition is*

$$\mu_i w_{ii} + 1 > 0$$

3.2. *Symmetric Networks*

Let W be symmetric ($w_{ij}=w_{ji}$). Its eigenvalues are all real. The symmetric networks include those with W's formed by an outer-product rule [Blu90, Hop82],

$$W = VV^{\top} - mI$$

and the pseudo-inverse rule [PD85, MW90],

$$W = V(V^{\top}V)^{-1}V^{\top},$$

where $V=[v_1, ..., v_m]$ is a set of vectors (patterns) in $[-1, 1]^n$.

THEOREM 4 (Symmetric Networks). *For a symmetric neural network, a sufficient condition for the DT flow defined by* (14) *to be asymptotically consistent with the CT flow defined by* (13) *at all hyperbolic fixed points is that the matrix*

$$W + \frac{2-h}{h} M^{-1}$$

is positive definite. When all external inputs $J=0$ and thus the origin is a fixed point, the condition is also necessary.

Proof. Since

$$MW + \frac{2-h}{h} I = M^{1/2}\left(M^{1/2}\left(W + \frac{2-h}{h} M^{-1}\right)M^{1/2}\right)M^{-1/2},$$

the symmetric matrix

$$Q(W, M, h) = W + \frac{2-h}{h} M^{-1}$$

is positive definite if and only if

$$N(W, M, h) = MW + \frac{2-h}{h} I$$

has only positive eigenvalues.

The same applies to

$$Q(W, M_{x^*}, h) = W + \frac{2-h}{h} M_{x^*}^{-1}$$

and

$$N(W, M_{x^*}, h) = M_{x^*} W + \frac{2-h}{h} I.$$

A simple calculation shows that

$$v^\top Q(W, M_{x^*}, h) v \geqslant v^\top Q(W, M, h) v \tag{18}$$

for any v, since $M_{x^*}^{-1} \geqslant M^{-1}$. Hence, $Q(W, M_{x^*}, h)$ is positive definite. By Theorem 2, the DT flow is asymptotically consistent with the CT flow at x^*.

COROLLARY 3. *When* $h = 1$ *as in Eq.* (2), *the condition becomes*

$$W + M^{-1}$$

is positive definite.

Remark. First, the condition in the theorem holds for all sufficiently small positive h. Second, the condition in the corollary is the sufficient condition given in [MW89a, MW90] for the DT model to have globally convergent dynamics and to have fixed points as the only attractors. As pointed out in [MW89a], to ensure that $W + M^{-1}$ is positive definite, it suffices to require that

$$1/\mu_i > -\alpha_{\min} \qquad \text{for all } i,$$

where $\alpha_{\min}$ is the minimum eigenvalue of the symmetric connection matrix W and μ_i is the gain value for neuron i.

In [BW90], the case $h=1$ is studied in detail for a particular outer-product $W=vv^{\top}-I$, $v^{\top}=(1, 1, ..., 1)\in\mathbb{R}^n$, and $J=0$. The activation function is $\psi_\alpha(z)=2(\sigma(az)-\frac{1}{2})$ with $\sigma(z)=1/(1+e^{-z})$ and $a>0$. The case $n=2$ of two neurons is completely analyzed for fixed points and periodic orbits. For $a<2$, the vector 0 is a global a.s. fixed point of the map f_1 in (14). By Lemma (i), the CT flow for (4), (13) also has 0 as its global attractor. The eigenvalues of $Df_1(0)$ are $\pm a/2$. At $a=2$ there is a bifurcation. For $a>2$ the DT flow of f_1 has two local attractors $\pm x_a$, where $x_a=(c_a, c_a)^{\top}$ and $\psi_a(c_a)=c_a$. The fixed point 0 becomes a local repellor. By Lemma (i), $\pm x_a$ are also a.s. fixed points of the CT flow. But at 0, the matrix

$$W+M^{-1}=\begin{bmatrix} 2/a & 1 \\ 1 & 2/a \end{bmatrix}$$

has eigenvalues $2/a\pm 1$, one positive and the other negative, which shows, according to Corollary 3, that the DT model is not asymptotically consistent with the CT flow at 0. In fact,

$$DF(0)=\begin{bmatrix} -1 & a/2 \\ a/2 & -1 \end{bmatrix}$$

has eigenvalues $\Lambda_1=-1+a/2>0$ and $\Lambda_2=-1-a/2<0$. (See also (10) with $h=1$.) Therefore, 0 is a saddle of the CT flow rather than a repellor. At 0, the CT flow has the unstable subspace $V^u(0)$ spanned by $(1, 1)^{\top}$ and the stable subspace $V^s(0)$ spanned by $(1, -1)$. Further, for any $x\in V^u(0)$ we have $Wx=x$ and $G(Wx)=Kx$, where $K>1$ for $x<c_a$ and $K<1$ for $x>c_a$. Thus for $x\in V^u(0)$, the vector $F(x)=(K-1)x$ always points toward x_a in the first quadrant and toward $-x_a$ in the third. For $x\in V^s(0)$, $Wx=-x$ and $G(Wx)=-Kx$, where $K>0$. Thus, the vector $F(x)=-(K+1)x$ always points toward 0 for $x\in V^s(0)$. Finally, note that

$$DF(\pm x_a)=\begin{bmatrix} -1 & \psi'_\alpha(x_a) \\ \psi'_a(x_a) & -1 \end{bmatrix}$$

has the eigenvalue $-1+\psi'_a(x_a)$ with eigenvector $(1, 1)^{\top}$ and eigenvalue $-(1+\psi'_a))$ with eigenvector $(1, -1)^{\top}$. Again, we see that $\pm x_a$ are local attractors of the CT flow. (This also follows from Lemma (i).) So at $\pm x_a$ the DT flow of f_1 is asymptotically consistent with the CT flow. From these observations we can sketch the flow field of $F(x)$ for $a>2$ as follows:

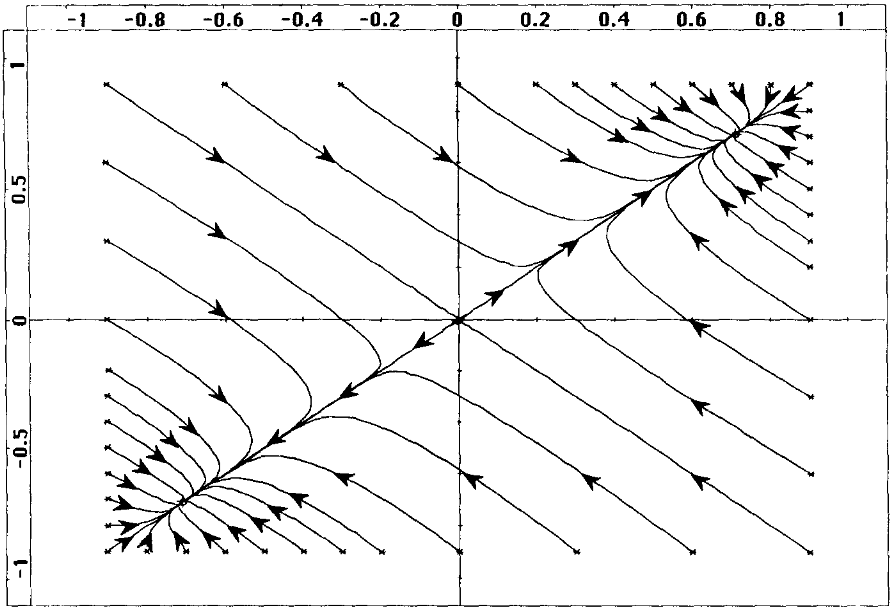

FIG. 1. View: y vs x.

$$dx/dt = -x + 2*(1/(1+\exp(-2.5*y)) - 0.5)$$
$$dy/dt = -y + 2*(1/(1+\exp(-2.5*x)) - 0.5).$$

The figure was produced, with kindly assistance of Professor Robert Sacker, by using a program Phase Portraits from Drexel University.

As a consequence of the Poincaré–Bendixson theorem [HS74], any closed orbit C of the CT flow must contain one of the fixed points $\pm x_a$ or 0 in its interior. From the flow field geometry, it is clear that such a curve C would intersect other trajectories, which is impossible for a differentiable flow of (4), (13). Hence, the CT flow has no periodic orbits, whereas [BW90] shows that the DT flow of f_1 has the stable orbit $((c_a, -c_a), (-c_a, c_a))$ of period 2 and two unstable orbits $((0, c_a), (c_a, 0))$ and $((0, -c_a), (-c_a, 0))$ of period 2. Although our general theory of asymptotic consistency as developed herein does not cover periodic orbits, for this case of $n=2$ and $a>2$ we can determine such orbits for f_h as $h \to 0$.

Since $\operatorname{fix}(f_h) = \operatorname{Fix}(F) = \operatorname{fix}(f_1)$, the fixed points for any f_h are 0 and $\pm x_a$. Further,

$$M_{\pm x_a} = Df_h(\pm x_a) = \begin{bmatrix} 1-h & h\psi_a'(x_a) \\ h\psi_a'(x_a) & 1-h \end{bmatrix}.$$

The eigenvalues of $Df_h(\pm x_a)$ are

$$\lambda_{1h} = 1 - h(1 + \psi_a'(c_a)) \qquad \text{and} \qquad \lambda_{2h} = 1 - h(1 - \psi_a'(c_a)).$$

Since $0<\psi'_a(c_a)<1$, for $0<h<1$ we have $|\lambda_{ih}|<1$. Hence, $\pm x_a$ are local attractor of the DT flow for any $h \leqslant 1$. Similarly,

$$M = Df_h(0) = \begin{bmatrix} 1-h & ha/2 \\ ha/2 & 1-h \end{bmatrix}$$

has eigenvalues $\lambda_1 = 1-h(1+a/2)$ and $\lambda_2 = 1-h(1-a/2)$. Clearly, $\lambda_2 > 1$ for $a/2>1$ and any $h>0$, whereas $|\lambda_1|<1$ for $h<4/(a+2)$. Thus, 0 becomes a saddle as $h \to 0$. For $h<4/(a+2)$ the DT flow of f_h is asymptotically consistent with CT flow at all fixed points (the flip bifurcation at $a=2$ disappears) and the DT trajectories follow (approximately) the flow field of $F(x)$ shown in the figure. In fact, a direct calculation for the point $y_a=(c_a, -c_a)$ shows that $f_h(y_a)=(1-h)\, y_a + h(-Ky_a) = (1-h(1+K))\, y_a$, where $0<K<a/2$. For $h<4/(a+2)$, $f_h(y_a)$ remains on the same side of 0 as y_a and similarly for $-y_a$.

For the homogeneous network of $n>2$ neurons with $h=1$, it is known [BW90] in general for the DT flow of f_1 that, for $a \leqslant 2/(n-1)$, the only fixed point is the origin 0 and it is an attractor, and, for $a>2/(n-1)$, 0 becomes an unstable fixed point and there are two a.s. fixed points, $\pm x_a = \pm(c_{(n-1)a}, ..., c_{(n-1)a})$. To ensure that the DT flow is asymptotically consistent with the CT flow, the condition in Corollary 3 requires that the two distinct eigenvalues of the matrix,

$$W + M^{-1} = \begin{bmatrix} 2/a & 1 & \cdots & 1 \\ 1 & 2/a & \cdots & 1 \\ \vdots & & \cdots & \\ 1 & 1 & \cdots & 2/a \end{bmatrix},$$

$(n-1)+2/a$ and $-1+2/a$ (with multiplicity $(n-1)$) be positive, or equivalently, $a<2$. Thus, when $0<a<2$, the fixed points 0 and $\pm x_a$ have the same asymptotic stability under the CT flow, especially when $n>2$ and $2/(n-1)<a<2$, 0 is a saddle fixed point and $\pm x_a$ are two a.s. fixed points. Note that the interval $(2/(n-1), 2)$ for the parameter a is the so-called *recall region* of the network when considered as a model of associative memories [MW90].

3.3. *Anti-Symmetric Networks*

If W is anti-symmetric ($w_{ij} = -w_{ij}$ and $w_{ii}=0$), its eigenvalues α are all purely imaginary. At any fixed point x^*, the matrix $M_{x^*}W$ also has only purely imaginary eigenvalues, since the matrix $M_{x^*}^{-1}(M_{x^*}W)M_{x^*}^{1/2} = M_{x^*}^{1/2}WM_{x^*}^{1/2}$ is also anti-symmetric. Hence all eigenvalues of $DF(x^*) = M_{x^*}W - I$ are attracting, which means that any fixed point x^* is an attractor of the CT flow.

THEOREM 5 (Anti-Symmetric Networks). *For an anti-symmetric network, a sufficient condition for the DT flow defined by* (14) *to be asymptotically consistent* *with the CT flow defined by* (13) *at all hyperbolic fixed points is that any eigenvalue* $\beta_{\max}$ *of the matrix* MW *with the maximal modulus satisfies*

$$|\beta_{\max}|^2 < (2-h)/h. \tag{19}$$

When all external inputs $J=0$ *and therefore the origin is a fixed point, the condition is also necessary.*

Proof. Let x^* be a fixed point, $\beta^*_{\max}$ an eigenvalue of the matrix $M_{x^*}W$ with the maximal modulus and v a nonzero eigenvector of $M_{x^*}^{1/2}WM_{x^*}^{1/2}$ associated with $\beta^*_{\max}$. Similar to the proof of Theorem 4, we have

$$\begin{aligned}
|\beta^*_{\max}| &= \frac{|v^\top M_{x^*}^{1/2}WM_{x^*}^{1/2}v|}{|v^\top v|} \\
&\leqslant \frac{|v^\top M_{x^*}^{1/2}WM_{x^*}^{1/2}v|}{|v^\top M_{x^*}M^{-1}v|} \\
&= \frac{|(M^{-1/2}M_{x^*}^{1/2}v)^\top M^{1/2}WM^{1/2}(M^{-1/2}M_{x^*}^{1/2}v)|}{|(M^{-1/2}M_{x^*}^{1/2}v)^\top (M^{-1/2}M_{x^*}^{1/2}v)|} \\
&\leqslant |\beta_{\max}|.
\end{aligned}$$

Thus, from condition (19),

$$|\beta^*_{\max}|^2 < \frac{2-h}{h} \qquad \text{and} \qquad h < \frac{2}{|\beta^*_{\max}-1|^2} = \frac{-2Re(\Lambda^*_{\max})}{|\Lambda^*_{\max}|^2},$$

where $\Lambda^*_{\max}$ is an eigenvalue of $DF(x^*)$ with the maximal modulus. By Theorem 1, the DT flow is asymptotically consistent with the CT flow at x^*. ∎

In [AB89] Atiya and Baldi have shown, by constructing a Liapunov function, that the CT flow defined by a variation of (13), where

$$\bar{F}(x) = -x + WG(x) + J,$$

with an anti-symmetric weight matrix W and $J=0$ has a global convergent dynamics; that is, any trajectory converges to the origin, which is the unique fixed point. By a similar Liapunov function and the LaSalle invariance principle [LaS76], it is easily seen that the CT flow defined by (13) with ananti-symmetric W and $J=0$ also has global convergence toward the origin. Therefore, for the DT flow defined by (14) to be asymptotically consistent with the CT flow, it is necessary and sufficient that the origin is also a unique attracting fixed point in the DT flow (when $J=0$). For the case $n=2$ and $W=\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$, it is known from [BW90] that the DT flow with $g_a(z)=2(\sigma(az)-\frac{1}{2})$ and $h=1$ has the origin as a (unique) attracting fixed point provided $a<2$, which agrees with condition (19) where $\beta_{\max}$ is either one of the eigenvalues $\pm ia/2$ of $MW=\left[\begin{smallmatrix}0 & -a/2\\ a/2 & 0\end{smallmatrix}\right]$.

3.4. *Circulant Networks*

In a circulant network, W is a circulant matrix. By definition [Dav79], such W takes the general form

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{bmatrix} = \operatorname{circ}(c_1, c_2, ..., c_n). \tag{20}$$

Since W can be written as $c_1 I + c_2 \Pi + \cdots + c_n \Pi^{n-1}$, where $\Pi = \operatorname{circ}(0, 1, 0, ..., 0)$ is the right-cyclically shifting permutation matrix, the eigenvalues α_k, $k = 1, ..., n$, of W can be expressed as [Dav79]

$$\alpha_k = c_1 + c_2 \omega_k + c_3 \omega_k^2 + \cdots + c_n \omega_k^{n-1},$$

where $\omega_k = e^{2\pi(k-1)i/n}$ are the $(k-1)$th roots of the unity.

Typical examples of circulant networks [AB89, BW90, MW89b] are (i) $W = \operatorname{circ}(a, ..., a)$ for a fully connected homogeneous network; (ii) $W = \operatorname{circ}(0, a, ..., a)$ for a fully connected homogeneous network without self-connections; (iii) $W = \operatorname{circ}(a, b, 0, ..., 0)$ for a unidirectional homogeneous ring; and (iv) $W = \operatorname{circ}(a, b, 0, ..., 0, c)$ for a bidirectional homogeneous ring.

Here we only take an example of the unidirectional rings to illustrate a point; that is, in order to maintain the asymptotic consistency for the rings, the time step h must depend on the size n of the network, and h approaches zero as n approaches infinity. This indicates that the DT model defined by (14) with any fixed h, like $h = 1$ in (2), cannot be applied to networks of arbitrary connection topology and size. We leave asymptotic consistency conditions for the other cases as exercises for the reader to work out.

THEOREM 6 (Uni-directed Ring Networks). *For a unidirected ring network of n neurons with $W = \operatorname{circ}(1, 1, 0, ..., 0)$, all $\mu_i = \mu = 1$ and $J = 0$. The asymptotic consistency of the DT flow defined by* (14) *and CT flow defined by* (13) *at all hyperbolic fixed points requires that*

$$h < O(1/n).$$

Proof. The eigenvalues of W are $1 + \cos 2\pi k/n + i \sin 2\pi/n$, $k = 0, 1, ..., n-1$. If all external input $J = 0$, then the origin 0 is a fixed point. Hence an eigenvalue Λ of $DF(0)$ is attracting if $\operatorname{Re}(\Lambda) = \cos(2\pi k/n) < 0$. Since the minimum of real parts of all attracting eigenvalues goes to zero and the modulus of the corresponding eigenvalue remains one as the network size n grows to infinity, the time step h must tend to zero, as the asymptotic consistency criterion requires that $h < -2\operatorname{Re}(\Lambda)/|\Lambda|^2 = O(1/n)$ for all attracting Λ. ∎

This shows a situation where the time step h must be taken essentially infinitely small to let the DT model capture the same local behavior as the CT model for all n. This is related to A-stability in numerical analysis and to stiffness of ODE's [Blu72]. As we know, it can be overcome by using the backward Euler method, for example.

4. Conclusion

The Euler approximation DT model defined by (14) to the CT model defined by (13) depends on the choice of the time step h; the smaller h is, the more accurate the approximation. The given analysis shows that, in some specific cases like feed-forward, symmetric, and antisymmetric networks, the choices of h that ensure the DT model asymptotically consistent with the CT model at all fixed points can be determined directly from the architecture of the network, W, and the gain values of neurons, $M = \mathrm{diag}(\mu_i)$.

Since important stability properties of CT models remain intact in corresponding DT models when the asymptotic consistency criterion is satisfied, a major conceptual obstacle between the CT and DT models has been eliminated. This suggests that the computational ability of the two types of models which depend on fixed points are not essentially due to whether they operate in continuous time or discrete time, but are consequences of the network structure underlying the models. Because the DT model resulting from the Euler method is very easy to simulate on a digital computer, it will often be more practical to develop ideas and simulations on that model even when use of the CT model is intended.

Here we have given results on the local asymptotic consistency between discrete-time flows and continuous-time flows with the emphasis on how the DT flows remain asymptotically consistent with the CT flows. We note that, with almost no effort, the results can be stated in the reversed direction, continuation of DT flows into CT flows, by letting

$$\frac{dx}{dt} = -x + f(x) \equiv F(x),$$

ensuring the resulting CT flows asymptotically consistent with the DT flows (without converting some unstable manifolds into stable ones of any fixed points).

Our analysis can be generalized to other models of neural networks and to more complicated numerical discretization methods [Wan91a]. For example, a similar analysis can be performed for linear multistep methods [Blu72, IPS91] to discretize a CT model of semilinear neural networks with a bounded time delay [MW89b].

References

[AB89] A. Atiya and P. Baldi, Oscillations and synchronizations in neural networks: An exploration of the labeling hypothesis, *Internat. J. Neural Networks* **1**, No. 2 (1989), 103–124.

[Ama72] S. Amari, Characteristics of random nets of analog neuron-like elements, *IEEE Trans. System Man Cybernetics* **SMC-2**, No. 5 (1972), 643–657.

[AR90] J. A. Anderson and E. Rosenfeld, Eds., "Neurocomputing: Foundations of Research," Vols. I, II, Wiley–Interscience, New York, 1990.

[Arb89] M. Arbib, "Metaphorial Brain 2: Neural Networks and Beyond," Wiley–Interscience, New York, 1989.

[Blu72] E. K. Blum, "Numerical Analysis and Computation: Theory and Practice," Addison–Wesley, Reading, MA, 1972.

[Blu90] E. K. Blum, Mathematical aspects of outer-product asynchronous content-addressable memories. *Biol. Cybernet.* **62** (1990), 337–348.

[BW90] E. K. Blum and Xin Wang, Stability of fixed-points and periodic orbits, and bifurcations in analog neural networks, *Neural Networks*, to appear.

[Cow68] J. D. Cowan, Statistical mechanics of neural nets, *in* "Neural Networks" (E.R. Caianiello, Ed.), pp. 181–188, Springer-Verlag, Berlin, 1968.

[Dav79] P. J. Davis, "Circulant Matrices," Wiley, New York, 1979.

[Dev86] R. L. Devaney, "An Introduction to Chaotic Dynamical Systems," Benjamin/Cummings, 1986.

[GC83] S. Grossberg and M. Cohen, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, *IEEE Trans. Systems Man Cybernet.* **SMC-13** (1983), 815–826.

[GH83] J. Guckenheimer and P. J. Holmes, "Non-linear Oscillations, Dynamical Systems and Bifurcation of Vector Fields," Vol. 42, Applied Mathematical Science, Springer Verlag, New York/Berlin, 1983.

[Gro88] S. Grossberg, Nonlinear neural networks: Principles, mechanisms, and architectures. *Neural Networks* **1**, No. 1 (1988), 17–61.

[Hir89] M. W. Hirsch, Convergent activation dynamics in continuous time networks, *Neural Networks* **2** (1989), 331–349.

[Hir91] M. W. Hirsch, Saturated outputs for high-gain, self-exciting networks. *Bifurcation Chaos Appl. Sci. Engrg.* **1**, No. 1 (1991), 211–218.

[HS74] M. W. Hirsch and S. Smale, "Differential Equations, Dynamical Systems, and Linear Algebra," Academic Press, New York, 1974.

[Hop82] J. J. Hopfield, Neural networks and physical systems with collective computational abilities, *Proc. Nat. Acad. Sci. USA* **79** (1982), 2554–2558.

[Hop84] J. J. Hopfield, Neurons with graded response have collective computational properties like those of two-state neurons. *Proc. Nat. Acad. Sci. USA* **81** (1984), 3088–3092.

[IPS91] A. Iserles, A. T. Peplow, and A. M. Stuart, A unified approach to spurious solutions introduced by time discretisation. Part i: Basic theory. *SIAM J. Math. Anal.*, to appear.

[LaS76] J. P. LaSalle, "The Stability of Dynamical Systems," SIAM Regional Conference Series in Applied Mathematics, Vol. 25, 1976.

[May74] R. M. May, "Stability and Complexity in Models Ecosystems," Princeton Univ. Press, Princeton, NJ, 1974.

[MW89a] C. M. Marcus and R. M. Westervelt, Dynamics of iterated-map neural networks, *Phys. Rev. A* **40**, No. 1 (1989), 501–504.

[MW89b] C. M. Marcus and R. M. Westervelt, Stability of analog neural networks with delay, *Phys. Rev. A* **39**, No. 1 (1989), 347–359.

[MW90] C. M. Marcus and R. M. Westervelt, Associative memory in an analog iterated-map neural network, *Phys. Rev. A* **41**, No. 6 (1990), 3355–3364.

[PGD85] L. Personnaz, I. Guyon, and G. Dreyfus, Information storage and retrieval in spin-glass like neural networks, *J. Phys. Lett. (Paris)* **46** (1985), L359–365.

[Sej76] T. J. Sejnowski, On the stochastic dynamics of neuronal interaction, *Biol. Cybernet.* **22** (1976), 203–211.

[Wan91a] Xin Wang, Discrete-time neural networks as dynamical systems, Ph.D. thesis, Mathematics Department, University of Southern California, 1991.

[Wan91b] XIN WANG, Period-doublings to chaos in a simple neural network: An analytic proof. *Complex Systems* **5**, No. 4 (1991), 425–441; also in "Proceedings of IJCNN-91-Seattle," Vol. II, pp. 333–339.

[WS87] J. L. WYATT AND D. L. STANLEY, A method for the design of stable lateral inhibition networks that is robust in the presence of circuit parasites, *in* "Neural Information Processing Systems" (D. Z. Anderson, Ed.), pp. 860–867, American Institute of Physics, New York, 1987.