Dynamics and Bifurcation of Neural Networks

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1 Introduction

While a feedforward neural network implements a function from its input layer to its output layer, a recurrent neural network (RNN) defines a dynamical system whose dynamics is utilized to carry out a computation task such as implementing associative memories, pattern recognition and optimization; The practical objective of studying RNNs is to design network dynamics for performing a given task by adjusting network parameters. This involves two major issues in dynamical system theory, namely, *dynamics* and *bifurcation*. Dynamics is concerned with the asymptotic behavior of the networks, which includes limit sets (e.g., fixed points, periodic orbits and chaotic invariant sets) and their asymptotic stability (e.g., stable, unstable and saddle), while bifurcation is concerned with how the dynamics changes as parameters are varied. In many applications of RNNs in information processing, such as associative memories, pattern recognition and classification, combinatorial optimization, segmentation and binding of objects, etc., limit sets of a network are used to represent the computational objects, namely, memories, temporal patterns, optimal solutions or visual objects, etc.; asymptotic stability of these limit sets is considered as associativity, fault-tolerance, generalization and robustness; and bifurcation of these asymptotic structures as parameters change is related to "learning" happening in the networks. A fundamental problem in recurrent networks is *to design attractors and their basins of attraction such that the dynamics of the networks* (*possibly driven by external inputs*) *leads to transitions among the attractors so that some computation is performed.*

Mathematical analysis of the dynamical behavior as well as bifurcation behavior of RNNs helps understanding intrinsic mechanism, capacity and limitation of RNNs, when they are used to process and generate temporal information. More generally, analysis of this type also helps understanding the fundamental principles of parallel and distributed information processing as complementary to the computational principles suggested by biological experiments.

2 Neural Networks as Dynamical Systems

In general, a *dynamical system* is a triple $\mathbb{D} = (X, T, \phi)$, consisting of a state space or phase space X, a temporal domain T, and a function $\phi : X \times T \to X$ such that (i) X is a nonempty set, (ii) either $T = \mathbb{R}_+$ (the set of non-negative real numbers) or $T = \mathbb{N}$ (the set of natural numbers) and (iii) for all $x \in X$ and all $t \in T$,

$$\phi(x, 0) = x, \quad \phi(\phi(x, t), s) = \phi(x, s + t).$$

Essentially, the function ϕ describes how the state of the system changes in time. The system $\mathbb{D}$ is called a *continuous-time* (CT) system when $T = \mathbb{R}_+$ and a *discrete-time* (DT) system when $T = \mathbb{N}$.

As all biological neurons have bounded activities, only bounded dynamical systems on the n-dimensional real vector space $\mathbb{R}^n$ will be considered in this chapter, that is, dynamical systems whose state spaces X are bounded subsets of $\mathbb{R}^n$.

Differential equations and maps are usually used to define dynamical systems. A differential equation of the form

$$\frac{dx(t)}{dt} = F(x(t)), \quad x(0) = x_0, \quad x(t) \in X \subseteq \mathbb{R}^n \tag{1}$$

defines, under certain conditions (see Hirsch and Smale, 1974), a CT dynamical system $D = (X, \mathbb{R}_+, \phi)$, where ϕ is the solution of (1),

$$\phi(x, t) = \int_0^t F(x(s))ds, \quad \phi(x, 0) = x$$

for $x \in X$ and $t \in \mathbb{R}_+$. A map $f : X \to X$ defines a DT dynamical system $D = (X, \mathbb{N}, \phi)$, where ϕ is given by iterates of f,

$$\phi(x, k) = f^k(x) \tag{2}$$

for $x \in X$ and $k \in \mathbb{N}$; Here $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$.

To fix ideas, consider a network of n (≥ 1) neurons of the following type. Let $x_i(t)$ be the state of neuron i at time t, taking values in a real interval I (e.g., $I = [-1, 1]$ or $[0, 1]$), $f_i : \mathbb{R} \to I$ a neuron activation function, and J_i the external input to neuron i, and w_{ij} connection weights from neuron j to neuron i. A CT model of the network is given by

$$\frac{dx_i(t)}{dt} = -x_i(t) + f_i(\sum_{j=1}^{n} w_{ij}x_j(t) + J_i) \tag{3}$$

and a DT model of the network is given by

$$x_i(t+1) = f_i(\sum_{j=1}^{n} w_{ij} x_j(t) + J_i). \tag{4}$$

Usually the functions f_i take a *sigmoidal* form, namely, they are bounded and monotonically increasing functions such that there exists a unique value $z^* \in \mathbb{R}$ at which the derivative $f'(z)$ attains a global maximum $f'(z^*)$ at z^*. Two commonly used sigmoidal functions are $\tanh(z)$ with $I = [-1, 1]$ and $1/(1 + e^{-z})$ with $I = [0, 1]$, both having $z^* = 0$.

Denote $X = I^n$, $x(t) = (x_1(t), ..., x_n(t))$, $W = [w_{ij}]$, $J = (J_1, ..., J_n)$ and $f(z) = (f_1(z_1), ..., f_n(z_n))$. Then X is regarded as the state space of the network, $x(t)$ as the state at time t, W as the connection weight matrix, J as the input of the network and f as the network activation function. The models (3) and (4) have the following vector forms:

$$\frac{dx(t)}{dt} = -x(t) + f(Wx(t) + J) \tag{5}$$

and

$$x(t+1) = f(Wx(t) + J), \tag{6}$$

It is the main objective of RNNs to study behaviors of the CT dynamical system and the DT dynamical system (with the map $x \mapsto f(Wx + J)$) defined by these models.

Note that in the above models neuron states $x_i(t)$ are intended to represent neuron firing rates. Very similar models arise from considering membrane potentials as neuron states. Let u_i be the weighted sums $\sum_{j=1}^{n} w_{ij} x_j + J_i$ with $x_i = f_i(u_i)$, or in vector form $u = Wx + J$ with $x = f(u)$. Then the models (5) and (6) become

$$\frac{du(t)}{dt} = -u(t) + Wf(u(t)) + J \tag{7}$$

and

$$u(t+1) = Wf(u(t)) + J. \tag{8}$$

It turns out that if the matrix W is invertible, these two sets of neural network models have the same dynamical behavior (geometric structure and stability) under the transformation $u(t) = Wx(t) + J$. A mathematical merit of considering the former over the latter, however, is that the former are dynamical system on the hypercube I^n in $\mathbb{R}^n$, depending only on the range I of the activation function f, whereas the latter vary with the matrix W as well.

There are many other models of RNNs which have been developed and analyzed in the neural network literature. Some are described in forms of Boolean networks, partial differential equations and delayed differential equations, for which the reader is referred to (Hirsch, 1989; Hertz, Krough and Palmer, 1991) and references therein.

3 Preliminaries on Dynamical Systems

This section reviews some basic concepts and results of dynamical systems theory; for comprehensive treatments, see (Hirsch and Smale, 1974; Devaney, 1986; Ruelle, 1989b; Hale and Kocak, 1991).

3.1 Dynamics

Let $D = (X, T, \phi)$ be a dynamical system. Define, for each $x \in X$, a map $\phi_x : T \to X$ by $\phi_x(t) = \phi(x, t)$ and, for each $t \in T$, a map $\phi^t : X \to X$ by $\phi^t(x) = \phi(x, t)$. The map ϕ_x is called the *trajectory* of initial state x and the map ϕ^t is called the *flow* of the system over time t. The set of states $\Gamma(x) = \{\phi_x(t) \,|\, t \in T\}$ is called the *orbit* of x.

A *limit point* p of a trajectory $x(t) = \phi_x(t)$ is a point satisfying $p = \lim_{t_k} x(t_k)$ for some sequence $t_k \to \infty$. The set of all limit points of the trajectory ϕ_x is called the *limit set* of x and it is denoted by $\omega(x)$. As X is assumed to be bounded, the limit set $\omega(x)$ is a nonempty, closed, bounded set. Moreover, $\omega(x)$ is *invariant*, namely, any trajectory starting at a point y in $\omega(x)$ remains in $\omega(x)$.

Let A be a closed invariant subset of X. A is called *asymptotically stable*, or equivalently A is an *attractor*, if there is a neighborhood U of A such that the limit set of any state x in U is a subset of A; that is, the trajectory ϕ_x approaches A as t tends to infinity. If, on the other hand, for any neighborhood U of A, the limit set of any state x in $U - A$ has empty intersection with A, the set A is called *asymptotically unstable*. In cases where A is neither asymptotically stable nor unstable, A is called *saddle*, which implies that in any neighborhood of A some of trajectories tend to A and others stay away from A. In general, the set of states whose limit sets are subsets of A is called the *basin of attraction* of A, or in other words, the basin of attraction of A is the union of the trajectories tending to A as t approaches infinity. The asymptotical stability of A can also be characterized by its basin of attraction $B(A)$. Clearly, if $B(A)$ contains some neighborhood of A, then A is asymptotically stable; in this case, if $B(A)$ is equal to the whole state space X, A is called *globally asymptotically stable*; otherwise, A is *locally asymptotically stable*. On the other extreme, if $B(A)$ contains only A itself, A is asymptotically unstable. Finally, if either of these cases holds, A is saddle.

There are just a few elementary types of attractor with simple geometry of points or smooth surfaces, in forms of equilibria (or fixed points), periodic and quasiperiodic trajectories. A trajectory $\phi_x(t)$ is *stationary* if $\phi_x(t) = x$ for all $t \in T$. The state x then is called an *equilibrium* or a *fixed point* of the system. A necessary and sufficient condition for a state x to be an equilibrium is that $F(x) = 0$ in CT dynamical systems defined by (1) and $f(x) = x$ in DT dynamical systems defined by (2). Often cited for showing existence of fixed points of functions is the Brouwer fixed point theorem, which says that if X is a compact (bounded and closed) subset of $\mathbb{R}^n$ and $f : X \to X$ is a continuous map, then f has at least one fixed point in X. This theorem can be directly applied to DT systems, whereas for CT systems it is applicable to the map $f(x) \stackrel{\text{def}}{=} F(x) + x$, because the fixed points of such defined f are the roots of $F(x) = 0$. However, the Brouwer theorem does not tell how many fixed points there are and where they are.

Two other simple types of attractors take forms of periodic and quasi-periodic trajectories. In the periodic case, $\phi_x(p) = x$ for a certain $p > 0$, and x is called a *periodic state* with period p if p is the smallest one satisfying $p > 0$ and $\phi_x(p) = x$. In the quasi-periodic case, $\phi_x(t)$ is not periodic but its limit set $\omega(x)$ is a smooth curve or surface in $\mathbb{R}^n$; More precisely, a trajectory $x(t) = \phi_x(t)$ is *quasi-periodic* if there exists a finite set of base periods $\{p_1, p_2, ..., p_k\}$, which are *linearly independent* (i.e., there is not a set of rational numbers $\alpha_1, ..., \alpha_k$ such that $p_i = \sum_{j \neq i} \alpha_j p_j$), such that $x(t) = \sum_i a_i \phi_i(t)$, where $\phi_i(t)$ are periodic functions of periods p_i. This happens especially when a rotation map on the unit circle in $\mathbb{R}^2$ has a rotating angle $\alpha\pi$ with α irrational: no point on the circle is periodic but the limit set of any trajectory is the whole circle. Often associated with periodic and quasi-periodic trajectories are *limit cycles*. A limit cycle is a periodic or quasi-periodic trajectory by itself and at the same time a limit set of some other trajectories.

Besides periodic and quasiperiodic ones, there are also complicated attractors with fractal structure; unlike fixed points, periodic and quasi-periodic attractors which are either isolated points or smooth, connected curves and surfaces that have integer-valued geometric dimensions such as $0, 1, 2$, etc., these attractors are normally characterized by non-integer (fractional) Hausdorff dimension (see Ruelle, 1989a) and references therein). Very often these attractors exist in nonlinear dynamical systems that possess *sensitive dependence on initial conditions*, which means that the distance between their trajectories $\phi_y(t)$ and $\phi_x(t)$ of two very closed initial states x and y increases exponentially in time t. Attractors of this kind are often loosely called *chaotic* or *strange attractors* and related systems are *chaotic systems*. The most distinguishable dynamical feature is their practically unpredictable dynamics, as any small error in measuring or computing initial conditions will become magnified for a long period of time. There are also some other dynamical and geometric structures associated with a chaotic attractor A; Important ones are topologically transitivity on A (i.e., A cannot be decomposed into two or more independent parts), and density of a set of system's periodic trajectories in A (i.e., "complex" dynamics on A interplays with "regular" dynamics on the periodic trajectories). At present, there exist some rigorous approaches and results concerning chaotic systems and attractors, most of which are for low-dimensional systems: characterization of dynamics of one-dimensional unimodal maps based on symbolic dynamics analysis, and presence of homoclinic and heteroclinic structures in two or higher dimensional dynamical systems, to name a few. Among others, the one-dimensional logistic map, the two-dimensional Hénon map and the three-dimensional Lorenz equation are often cited examples that have strange attractors.

For determining the local asymptotic stability of fixed points, there are some specific criteria. Most notable is the linearized stability based on the Grobman-Hartman theorem. For a CT system defined by (1), denote $DF(x)$ as the Jacobian of F at x. If all the eigenvalues of $DF(x)$ are not pure imaginary, then x is called a *hyperbolic* fixed point. If all the eigenvalues have negative (respectively, positive, or some negative and some positive) real parts, x is *asymptotically stable* (*unstable* or *saddle*). For a DT system defined by (2), denote $Df(x)$ as the Jacobian of f at x. If all the eigenvalues of $Df(x)$ are not on the unit circle, then x is called a *hyperbolic* fixed point. If all the eigenvalues are inside (respectively, outside or some inside and some outside) the unit circle, x^* is *asymptotically stable* (*unstable*, or *saddle*).

3.2 Bifurcations

When a dynamical system depends on a set of parameters, study of bifurcations, i.e., quantitative changes of dynamic structure as the parameters are varied, becomes possible and necessary.

The bifurcation idea is normally formalized based upon the concepts of *topological equivalence*. Two dynamical systems $\mathbb{D} = (X, T, \phi)$ and $\mathbb{D}' = (Y, T, \psi)$ with a same temporal domain T are said to be *topologically conjugate* if there is a *homeomorphism* (i.e., a continuous one-to-one map) h from X into Y such that for any $x \in X$,

$$h(\phi(t, x)) = \psi(t, h(x)), \quad \text{for all} \;\; t \in T,$$

that is, the map h establishes a one-to-one correspondence in a continuous way between trajectories of the two systems.

Suppose that a dynamical system $\mathbb{D}_\mu$ on $\mathbb{R}^n$ depends on a set of parameters $\mu \in \mathbb{R}^r$, where $\mathbb{R}^r$ ($r \geq 1$) is the parameter space of the system. If in any small neighborhood U of a set of parameter values μ_0 there are two sets of values $\mu_1, \mu_2 \in U$ such that $\mathbb{D}_{\mu_1}$ and $\mathbb{D}_{\mu_2}$ are not topologically conjugate, then it is said that a *bifurcation* occurs at μ_0 and μ_0 is a *bifurcation point*. For instance, a bifurcation occurs when an attracting fixed point becomes asymptotically unstable or saddle and the related parameter values at which the fixed point changes its stability is the bifurcation point.

Current research in bifurcation theory is still restricted to systems with only one or two parameters and most results are obtained for either local bifurcations around fixed points that are caused by loss of hyperbolicity of fixed points or global bifurcations caused by changes of system's homoclinic and heteroclinic structures. Some examples of local bifurcations such as Hopf, pitchfolk, and period-doubling bifurcations will be illustrated in Section 5 in the context of neural networks. For details on bifurcations, see (Ruelle, 1989b; Hale and Kocak, 1991).

4 Dynamics of Neural Networks

4.1 Asymptotic Stability of Fixed Points

Notice that the detection of fixed points for a given RNN amounts to solving a certain equation such as $F(x) = 0$ for (1) or $f(x) = x$ for (2). For the CT (5) and DT (6) models, the equation is $x = f(Wx + J)$.

Determination of asymptotic stability of fixed points is the problem of calculating or estimating the eigenvalues of the Jacobian of the networks at the fixed points. For the CT networks (5), this is to calculate the eigenvalues λ of the matrix $-I + DW$, where $D = diag(f_i'(\sum_{j=1}^n w_{ij}x_j^* + J - i))$ and $x^* = (x_1^*, ..., x_n^*)$ is a fixed point. If all λ's have negative (positive, or some negative and some positive) real parts, then x^* is asymptotically stable (unstable, or saddle). For the DT networks (6), the asymptotic stability of a fixed point x^* relies on the eigenvalues λ of the matrix DW. If all λ's have moduli smaller (larger, or some smaller and some larger) than one, then x^* is asymptotically stable (unstable, or saddle).

4.2 Convergent Dynamics

Convergent dynamics is further classified in (Hirsch, 1989) into two types: *quasi-convergence* and *global convergence*; the former means that every trajectory approaches a set of fixed points, while the latter means that there is a unique fixed point to which every trajectory converges.

There are basically three approaches for showing that a RNN has convergent dynamics. They are Liapunov, contraction and monotonicity approaches.

4.2.1 Liapunov Approach

The Liapunov approach is the commonest way to guarantee convergence. It involves constructing an energy function E which is continuous on the state space X and non-increasing along trajectories. Such a function is constant on the set of limit points of a trajectory. If E is strict, meaning that E is strictly decreasing ($dE(x(t))/dt < 0$ in the CT case and $E(x(t+1)) - E(x(t)) < 0$ in the DT case) for all non-fixedpoints $x(t)$, then all limit points of any trajectory are fixed points (Hirsch and Smale, 1974).

Many conditions for convergence have been derived using the Liapunov approach (Cohen and Grossburg, 1983; Hopfield, 1984; Marcus and Westervelt, 1989). For example, some simple conditions on the activation functions f_i and weights w_{ij}, such as f_i being sigmoidal and W being zero-diagonal symmetry ($w_{ij} = w_{ji}$, $w_{ii} = 0$) (Hopfield, 1984), guarantee that the network (3) has an energy function

$$E = -\frac{1}{2}\sum_{i,j} w_{ij}x_ix_j + \sum_i x_ix_i.$$

For a more general class of RNNs of form

$$\frac{dx_i(t)}{dt} = a_i(x_i)[b_i(x_i) - \sum_{j=1}^{n} w_{ij}c_j(x_j)], \tag{9}$$

where a_i, b_i and d_i $(i = 1, ..., n)$ are functions, it is shown in (Cohen and Grossburg, 1983) that, when $a_i > 0$, $c_i' > 0$ and $[w_{ij}]$ is symmetric, the function

$$E = -\sum_i \int_0^{x_i} b_i(z)c_i'(z)dz + \frac{1}{2}\sum_{ij} w_{ij}c_i(x_i)c_j(x_j)$$

is a strict energy function and therefore the network (9) is quasi-convergent. RNNs described by (4) with symmetric weight matrices are shown in (Marcus and Westervelt, 1989) by considering a Liapunov function to have fixed points and periodic points of period 2 as only limit points. Notice that this result is different from the convergence result obtained in (Hopfield, 1982) where a type of asynchronous dynamics is considered; The asynchronous dynamics will be addressed later on in Section 6.

4.2.2 Contraction Approach

The function f in (5) and (6) is a *contractive mapping* if f has some norm $\| f \| < 1$. It has been shown by Kelly (see Hirsch, 1989 for the reference) that the network (5) is globally convergent to a unique fixed point x^* if $f(x)$ is a contraction, or equivalently, $\mu \| W \| < 1$, where μ is the maximum of the neuron gains $\mu_i = \max f_i'$. The proof uses the fact that the function $\| x - x^* \|$ is indeed a strict energy function.

Depending on which norm $\| \cdot \|$ is chosen, several sufficient conditions for the network (5) to converge globally have been obtained. For example, one is $\mu(w_{ii} + (\sum_{j\neq i} |w_{ij}| + |w_{ji}|))/2 < 1$ presented in (Hirsch, 1989). More detailed considerations on using the norm condition of the weight matrix W will lead to more sophisticated conditions.

It should be pointed out that the contraction condition is implied by having all the eigenvalues of the Jacobian of f in (5) and (6) inside the unit circle at all states x. Therefore, a careful analysis on the location of the eigenvalues will result in a sufficient condition for the global convergence. For example, theorems on location of eigenvalues of matrices will help to obtain some sufficient conditions for global convergence (i.e., absolute stability).

4.2.3 Monotonicity Approach

This approach is based on the results in (Hirsch, 1984) on convergence of monotone dynamical systems. Let $\preceq$ be a partial order on a set $X \subseteq \mathbb{R}^n$. For any $x, y \in X$, define $x \ll y$ when $x \preceq y$ and $x_i \neq y_i$ for all $i = 1, ..., n$. A function $f : X \to X$ is called *monotone* (respectively, *strongly monotone*) if $x \preceq y$ implies $f(x) \preceq f(y)$ ($f(x) \ll f(y)$). A dynamical system (X, T, ϕ) is (*strongly*) *monotone* (with respect to the order $\preceq$) if for each $t \in T$, map ϕ_t is (strongly) monotone. An important class of monotone CT dynamical systems defined by (1) are *cooperative systems*, namely, all the off-diagonal terms of its Jacobian matrices $Df = [\partial f_i / \partial x_j]$ are ≥ 0. If, in addition, the systems are *irreducible* (i.e., all the Jacobian matrices are *irreducible*), the systems are *strongly monotone*. It turns out that in a monotone system (1), if a trajectory $x(t) = \phi_x(t)$ satisfies $x \ll x(t_0)$ for some $t_0 > 0$, then $x(t)$ converges to a fixed point x^* with $x \ll x^*$, and in a strongly monotone system, almost all trajectories (except a set of trajectories of measure zero) converge to a set of equilibria; that is, the system has almost quasi-convergent dynamics.

An application of the convergence results on monotone dynamical systems is to obtain convergence theorems for excitatory neural networks (Hirsch, 1989). The network (3) with sigmoidal activation functions f_i is called *excitatory* if $w_{ij} \geq 0$ for all $i \neq j$. It is clear that excitatory networks (3) are cooperative. Hence, if, in addition, the matrix W is irreducible (in this case, the dynamical system is strongly monotone), then almost all states converge to a set of fixed points.

4.3 Oscillatory Dynamics

Oscillations are the main function of some networks, e.g., in spiking neuron models, pacemakers or central pattern generators, and oscillation in visual cortex.

Though there have been many studies of biological oscillations (Cronin, 1977; Glass and Mackey, 1988) the oscillatory dynamics of RNNs of reasonable size is in general still difficult to describe, mainly because mathematical tools for the analysis of periodic orbits in high dimensional state spaces are not sufficiently developed. A practical approach to studying oscillatory neural dynamics is *modular*, meaning to first study oscillatory dynamics of networks of small size and then the dynamics of collections of small networks through certain functional coupling structures. Typical examples are classes of models of coupled oscillators. Oscillatory dynamics of some special neural networks are analyzed regarding their stability and bifurcation properties.

4.4 Chaotic Dynamics

It has been realized that chaotic dynamics may play a significant computational role in biological information processing (Glass and Mackey, 1988; Yao and Freeman, 1990). Many researchers (see

Wang, 1991; Yao and Freeman, 1990 and references therein) have performed numerical simulations on both CT and DT RNNs, most of which are driven by external inputs and/or have time delays, and observed various bifurcations and chaos.

However, to prove rigorously that a RNN displays chaotic dynamics or has a strange attractor is not an easy matter. A practical approach is to see if the dynamics of a network within some parameter region can be indeed reduced through topological equivalence to the dynamics of some familiar models, such as the logistic map, Hénon map and Lorenz equation, which are known to be chaotic and have some strange attractors.

In (Wang, 1991), the DT network (4) is treated as a one-parameter family of maps with the neuron gain as the parameter. For a certain class of weight matrices of form

$$\begin{bmatrix} a & -a \\ b & -b \end{bmatrix},$$

the RNN of two neurons is analytically shown to be dynamically equivalent to a full family of S-unimodal maps on the interval $[0, 1]$. Essentially, an S-unimodal map is a mathematical generalization of the one-dimensional logistic map, and a full family of S-unimodal maps become chaotic through the period-doubling route as the parameter varies (Devaney, 1986). The period-doubling bifurcation is a scenario in which, at each bifurcation point, a previously stable periodic orbits loss their stability, giving arise to new stable periodic orbits with doubled periods, and the maps eventually become chaotic. Figure 1 shows the bifurcation diagram for the activity of one of the two neurons.

Insert Figure 1 here

In (Yao and Freeman, 1990), a CT RNN of is constructed based on Lorenz-like strange attractors to understand the role of chaos in biological pattern recognition. Usually, regular (fixed-point, periodic and quasi-periodic) attractors are used to represent patterns or memories, and transitions towards and among these attractors are used to model pattern recognition procedures. In Yao and Freeman's model, the system maintains a global chaotic attractor that provides a basal activity of the resting system and allows transitions driven by external inputs between chaotic and other periodic attractors. It is believed that the chaos allows easy and rapid information retrieval.

Practically, several invariant statistic measurements, such as Liapunov (characteristic) exponents, entropy and information dimension, are used as indications for a dynamical system to be "chaotic" (Ruelle, 1989a). For instance, the Liapunov exponents of a given trajectory can be viewed as the long time average of the real parts of the eigenvalues of the fundamental solution matrix associated with the linearization of the system about the trajectory. They capture asymptotic information concerning local expansion and contraction of state space, in such a way that *positive* ones indicate eventual expansion and hence sensitive dependence on initial conditions, while *negative* ones indicate contraction along some directions.

5 Bifurcations

Bifurcation analysis provides a means to describe qualitatively the behavior of a network in different parameter regions and to predict changes in the networks dynamics as parameters are varied. In (Borisyuk and Kirillov, 1992), the Wilson and Cowan model on the dynamics of average activities of excitatory and inhibitory populations of neurons, which shows hysteresis phenomena and limit-cycle activity, is analyzed for its bifurcation behavior depending on two parameters. As a result, the parameter plane is partitioned into regions of topologically equivalent dynamics bounded by bifurcation curves.

Possible (local) bifurcations in neural networks can be illustrated using the DT RNN in (6) with the neuron gain as a bifurcation parameter. Consider a two-neuron DT network with no input ($J = 0$). Assume that all the neural activation functions have the same maximal slope (neuron gain) μ at the origin. Suppose the weight matrix W takes the following form,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for some real numbers a, b. The Jacobians at the origin, $Df(0,0) = \mu W$ has two eigenvalues $\lambda_{1,2} = \mu(a \pm ib)$. Hence, $\mu_0 = 1/\sqrt{a^2+b^2}$ is a bifurcation point because the eigenvalues satisfy $|\lambda_{1,2}| = 1$. By choosing parameters a, b, one can have the following bifurcations:

Hopf bifurcation (Figure 2). If a and b satisfy $[(a+ib)/\sqrt{a^2+b^2}]^k \neq 1$ for $k = 1, 2, 3, 4$, then as the parameter μ increases past the bifurcation point μ_0, the origin loses its asymptotic stability and an attracting limit-cycle surrounding the origin emerges.

Insert Figure 2 (a) and 2 (b) here

Pitchfork bifurcation (Figure 3). If $a > 0$ and $b = 0$, then as μ increases past $\mu_0 = 1/a$, the origin loses its asymptotical stability and two new attracting fixed points emerge.

Insert Figure 3 here

Period-doubling bifurcation (Figure 4). If $a < 0$ and $b = 0$, then as μ increases past $\mu_0 = 1/a$, the origin loses its asymptotical stability and a new pair of attracting periodic points of period 2 emerges.

Insert Figure 4 here

6 Asynchronous Dynamics

Neural networks as dynamical systems have one distinct feature: they are massively distributed computing systems. One of the major issues in parallel and distributed computation is *asynchronization* (Bertsekas and Tsitsiklis, 1989). The dynamics as described in models (5) and (6) requires that the neurons communicate their states to all others instantaneously and synchronize their dynamical evolution precisely all the time. Any mechanism used to enforce such synchronization may have an important effect on performance of the network when it is implemented or simulated over a real distributed system. Moreover, the biological manifestation of asynchrony is inherent, as it can be caused by delays in nerve signal propagation, variability of neuron parameters such as refractory periods and adaptive neuron gains. Therefore, asynchronous dynamics of neural networks is very important to analog implementation, distributed simulation as well as mathematical modeling of neural networks.

Asynchronous dynamics has been thoroughly studied in the contexts of DT dynamical systems (Bertsekas and Tsitsiklis, 1989). Among others, some contractive maps on Banach spaces and continuous maps on partial ordered sets (see Wang, Li and Blum, 1993 for references) are *asynchronizable*, i.e., any asynchronous iterations of these maps will converge to the fixed points under synchronous iterations.

The asynchronization issue has also been addressed in the context of neural networks. For example, in the celebrated DT Hopfield model (Hopfield, 1982), only one randomly chosen neuron is allowed to update its state at each iterative step and the network with symmetric weight matrices is always convergent to a set of fixed points, governed by an energy function. The issue is also discussed for CT RNNs. The previous approaches are, however, to convert the CT model (3) into a DT version through the Euler discretization and then to apply the existing result for contractive mappings.

Recently a rigorous formulation of asynchronuous dynamics is presented in (Wang, Li and Blum, 1993) for CT dynamical systems, based on concepts of local time scales of individual components and communication time delays between the components. Asynchronous dynamics of general CT dynamical systems on R^n of the form

$$C\frac{dx}{dt} = -x + F(x) \tag{10}$$

(where $C = diag(c_1, c_2, ..., c_n)$ with $c_i > 0$ and $F = [f_i] \in C^1(R^n)$), is discussed in two situations: (i) F is a contractive map on R^n, and (ii) F is a monotone map on R^n. When the concept and results are applied to the neural networks (5), the contractive and monotone CT RNNs discussed in Sections 4.2.2 and 4.2.3 are, under certain conditions, asynchronizable.

References

D. P. Bertsekas and J. N. Tsitsiklis (1989). *Parallel and Distributed Computation: Numerical Methods.* Englewood Cliffs, NJ: Prentice Hall.

R. M. Borisyuk and A. B. Kirillov (1992). Bifurcation analysis of a neural network model. *Biological Cybernetics*, 66:319–325.

J. Cronin (1977). Some mathematics of biological oscillations. *SIAM Review*, 19(1):100–138.

R. L. Devaney (1986). *An Introduction to Chaotic Dynamical Systems.* The Benjamin/Cunnimgs Publishing Co. Inc.

L. Glass and M.C. Mackey (1988). *From clock to chaos: the rhythms of life.* Princeton University Press, Princeton.

J. K. Hale and H. Kocak (1991). *Dynamics and Bifurcations.* Springer-Verlag: New York.

J. Hertz, A. Krogh, and R. G. Palmer (1991). *Introduction to the Theory of Neural Computation.* Redwood, CA: Addison-Wesley Publishing Company.

M. W. Hirsch (1984). The dynamical systems approach to differential equations. *Bull. Amer. Math. Soc.*, 11:451–514.

M. W. Hirsch (1989). Convergent activation dynamics in continuous time networks. *Neural Networks*, 2:331–349.

M. W. Hirsch and S. Smale (1974). *Dynamical Systems and Linear Algebra.* Academic Press, New York.

J. Hopfield. (1982) Neural networks and physical systems with emergent computational abilities. *Proc. Nat. Acad. Sci. USA*, 79:2554–2558.

J. Hopfield. (1984) Neurons with graded response have collective computational properties like those of two-state neurons. *Proc. Nat. Acad. Sci. USA*, 81:3088-3092.

C. M. Marcus and R. M. Westervelt (1989). Dynamics of iterated-map neural networks. *Physical Review A*, 40(1):501–504.

D. Ruelle (1989a). *Chaotic and Strange Attractors: The Statistical Analysis of Time Series for Deterministic Nonlinear Systems.* Cambridge University Press, Cambridge, New York.

D. Ruelle (1989b). *Elements of Differentiable dynamics and Bifurcation Theory.* Academic Press, Inc.

X. Wang (1991). Period-doublings to chaos in a simple neural network: An analytical proof. *Complex Systems*, 5:425–441.

X. Wang, Q. Li, and E. K. Blum (1993). Asynchronous dynamics of continuous-time neural networks. In Cowan, J., Tesauro, G. and Alspector, J., Editors, *Neural Information Processing Systems*, 6, 493-500.

Y. Yao and W. J. Freeman (1990). Model of biological pattern recognition with spatially chaotic dynamics. *Neural Networks*, 3:153–170.