Wikipedia

Sequence space

In functional analysis and related areas of mathematics, a **sequence space** is a vector space whose elements are infinite sequences of real or complex numbers. Equivalently, it is a function space whose elements are functions from the natural numbers to the field *K* of real or complex numbers. The set of all such functions is naturally identified with the set of all possible infinite sequences with elements in *K*, and can be turned into a vector space under the operations of pointwise addition of functions and pointwise scalar multiplication. All sequence spaces are linear subspaces of this space. Sequence spaces are typically equipped with a norm, or at least the structure of a topological vector space.

The most important sequence spaces in analysis are the ℓ^p spaces, consisting of the p-power summable sequences, with the p-norm. These are special cases of L^p spaces for the counting measure on the set of natural numbers. Other important classes of sequences like convergent sequences or null sequences form sequence spaces, respectively denoted c and c_0, with the sup norm. Any sequence space can also be equipped with the topology of pointwise convergence, under which it becomes a special kind of Fréchet space called FK-space.

Definition

A sequence $x_\bullet = (x_n)_{n\in\mathbb{N}}$ in a set X is just an X-valued map $x_\bullet : \mathbb{N} \to X$ whose value at $n \in \mathbb{N}$ is denoted by x_n instead of the usual parentheses notation $x(n)$.

Space of all sequences

Let $\mathbb{K}$ denote the field either of real or complex numbers. The set $\mathbb{K}^\mathbb{N}$ of all sequences of elements of $\mathbb{K}$ is a vector space for componentwise addition

$$(x_n)_{n\in\mathbb{N}} + (y_n)_{n\in\mathbb{N}} = (x_n + y_n)_{n\in\mathbb{N}},$$

and componentwise scalar multiplication

$$\alpha(x_n)_{n\in\mathbb{N}} = (\alpha x_n)_{n\in\mathbb{N}}.$$

A **sequence space** is any linear subspace of $\mathbb{K}^\mathbb{N}$.

As a topological space, $\mathbb{K}^\mathbb{N}$ is naturally endowed with the product topology. Under this topology, $\mathbb{K}^\mathbb{N}$ is Fréchet, meaning that it is a complete, metrizable, locally convex topological vector space (TVS). However, this topology is rather pathological: there are no continuous norms on $\mathbb{K}^\mathbb{N}$ (and thus the product topology cannot be defined by any norm).[1] Among Fréchet spaces, $\mathbb{K}^\mathbb{N}$ is minimal in having no continuous norms:

Theorem[1] — Let X be a Fréchet space over $\mathbb{K}$. Then the following are equivalent:

1. X admits no continuous norm (that is, any continuous seminorm on X has a nontrivial null space).
2. X contains a vector subspace TVS-isomorphic to $\mathbb{K}^{\mathbb{N}}$.
3. X contains a complemented vector subspace TVS-isomorphic to $\mathbb{K}^{\mathbb{N}}$.

But the product topology is also unavoidable: $\mathbb{K}^{\mathbb{N}}$ does not admit a strictly coarser Hausdorff, locally convex topology.[1] For that reason, the study of sequences begins by finding a strict linear subspace of interest, and endowing it with a topology *different* from the subspace topology.

ℓ^p spaces

For $0 < p < \infty$, ℓ^p is the subspace of $\mathbb{K}^{\mathbb{N}}$ consisting of all sequences $x_\bullet = (x_n)_{n\in\mathbb{N}}$ satisfying

$$\sum_n |x_n|^p < \infty.$$

If $p \geq 1$, then the real-valued function $\|\cdot\|_p$ on ℓ^p defined by

$$\|x\|_p \ = \ \left(\sum_n |x_n|^p\right)^{1/p} \qquad \text{for all } x \in \ell^p$$

defines a norm on ℓ^p. In fact, ℓ^p is a complete metric space with respect to this norm, and therefore is a Banach space.

If $p = 2$ then ℓ^2 is also a Hilbert space when endowed with its canonical inner product, called the **Euclidean inner product**, defined for all $x_\bullet, y_\bullet \in \ell^p$ by

$$\langle x_\bullet, y_\bullet \rangle \ = \ \sum_n \overline{x_n} y_n.$$

The canonical norm induced by this inner product is the usual ℓ^2-norm, meaning that $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for all $\mathbf{x} \in \ell^p$.

If $p = \infty$, then ℓ^∞ is defined to be the space of all bounded sequences endowed with the norm

$$\|x\|_\infty \ = \ \sup_n |x_n|,$$

ℓ^∞ is also a Banach space.

If $0 < p < 1$, then ℓ^p does not carry a norm, but rather a metric defined by

$$d(x,y) \;=\; \sum_n |x_n - y_n|^p.$$

c, c_0 and c_{00}

A *convergent sequence* is any sequence $x_\bullet \in \mathbb{K}^\mathbb{N}$ such that $\lim_{n\to\infty} x_n$ exists. The set c of all convergent sequences is a vector subspace of $\mathbb{K}^\mathbb{N}$ called the *space of convergent sequences*. Since every convergent sequence is bounded, c is a linear subspace of ℓ^∞. Moreover, this sequence space is a closed subspace of ℓ^∞ with respect to the supremum norm, and so it is a Banach space with respect to this norm.

A sequence that converges to 0 is called a *null sequence* and is said to *vanish*. The set of all sequences that converge to 0 is a closed vector subspace of c that when endowed with the supremum norm becomes a Banach space that is denoted by c_0 and is called the *space of null sequences* or the *space of vanishing sequences*.

The *space of eventually zero sequences*, c_{00}, is the subspace of c_0 consisting of all sequences which have only finitely many nonzero elements. This is not a closed subspace and therefore is not a Banach space with respect to the infinity norm. For example, the sequence $(x_{nk})_{k\in\mathbb{N}}$ where $x_{nk} = 1/k$ for the first n entries (for $k = 1,\ldots,n$) and is zero everywhere else (that is, $(x_{nk})_{k\in\mathbb{N}} = (1, 1/2, \ldots, 1/(n-1), 1/n, 0, 0, \ldots)$) is a Cauchy sequence but it does not converge to a sequence in c_{00}.

Space of all finite sequences

Let

$$\mathbb{K}^\infty = \left\{(x_1, x_2, \ldots) \in \mathbb{K}^\mathbb{N} : \text{all but finitely many } x_i \text{ equal } 0\right\},$$

denote the **space of finite sequences over** $\mathbb{K}$. As a vector space, $\mathbb{K}^\infty$ is equal to c_{00}, but $\mathbb{K}^\infty$ has a different topology.

For every natural number $n \in \mathbb{N}$, let $\mathbb{K}^n$ denote the usual Euclidean space endowed with the Euclidean topology and let $\operatorname{In}_{\mathbb{K}^n} : \mathbb{K}^n \to \mathbb{K}^\infty$ denote the canonical inclusion

$$\operatorname{In}_{\mathbb{K}^n}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, 0, \ldots).$$

The image of each inclusion is

$$\operatorname{Im}(\operatorname{In}_{\mathbb{K}^n}) = \{(x_1, \ldots, x_n, 0, 0, \ldots) : x_1, \ldots, x_n \in \mathbb{K}\} = \mathbb{K}^n \times \{(0, 0, \ldots)\}$$

and consequently,

$$\mathbb{K}^\infty = \bigcup_{n\in\mathbb{N}} \operatorname{Im}(\operatorname{In}_{\mathbb{K}^n}).$$

This family of inclusions gives $\mathbb{K}^\infty$ a final topology τ^∞, defined to be the finest topology on $\mathbb{K}^\infty$ such that all the inclusions are continuous (an example of a coherent topology). With this topology, $\mathbb{K}^\infty$ becomes a complete, Hausdorff, locally convex, sequential, topological vector space that is *not* Fréchet–Urysohn. The topology τ^∞ is also strictly finer than the subspace topology induced on $\mathbb{K}^\infty$ by $\mathbb{K}^\mathbb{N}$.

Convergence in τ^∞ has a natural description: if $v \in \mathbb{K}^\infty$ and $v_\bullet$ is a sequence in $\mathbb{K}^\infty$ then $v_\bullet \to v$ in τ^∞ if and only $v_\bullet$ is eventually contained in a single image $\mathbf{Im}(\mathbf{In}_{\mathbb{K}^n})$ and $v_\bullet \to v$ under the natural topology of that image.

Often, each image $\mathbf{Im}(\mathbf{In}_{\mathbb{K}^n})$ is identified with the corresponding $\mathbb{K}^n$; explicitly, the elements $(x_1, \ldots, x_n) \in \mathbb{K}^n$ and $(x_1, \ldots, x_n, 0, 0, 0, \ldots)$ are identified. This is facilitated by the fact that the subspace topology on $\mathbf{Im}(\mathbf{In}_{\mathbb{K}^n})$, the quotient topology from the map $\mathbf{In}_{\mathbb{K}^n}$, and the Euclidean topology on $\mathbb{K}^n$ all coincide. With this identification, $\left((\mathbb{K}^\infty, \tau^\infty), (\mathbf{In}_{\mathbb{K}^n})_{n\in\mathbb{N}}\right)$ is the direct limit of the directed system $\left((\mathbb{K}^n)_{n\in\mathbb{N}}, (\mathbf{In}_{\mathbb{K}^m\to\mathbb{K}^n})_{m\leq n\in\mathbb{N}}, \mathbb{N}\right)$, where every inclusion adds trailing zeros:

$$\mathbf{In}_{\mathbb{K}^m\to\mathbb{K}^n}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

This shows $(\mathbb{K}^\infty, \tau^\infty)$ is an LB-space.

Other sequence spaces

The space of bounded series, denote by bs, is the space of sequences x for which

$$\sup_n \left| \sum_{i=0}^n x_i \right| < \infty.$$

This space, when equipped with the norm

$$\|x\|_{bs} = \sup_n \left| \sum_{i=0}^n x_i \right|,$$

is a Banach space isometrically isomorphic to ℓ^∞, via the linear mapping

$$(x_n)_{n\in\mathbb{N}} \mapsto \left(\sum_{i=0}^n x_i \right)_{n\in\mathbb{N}}.$$

The subspace *cs* consisting of all convergent series is a subspace that goes over to the space *c* under this isomorphism.

The space Φ or c_{00} is defined to be the space of all infinite sequences with only a finite number of non-zero terms (sequences with finite support). This set is dense in many sequence spaces.

Properties of ℓ^p spaces and the space c_0

The space ℓ^2 is the only ℓ^p space that is a Hilbert space, since any norm that is induced by an inner product should satisfy the parallelogram law

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2\|x\|_p^2 + 2\|y\|_p^2.$$

Substituting two distinct unit vectors for x and y directly shows that the identity is not true unless $p = 2$.

Each ℓ^p is distinct, in that ℓ^p is a strict subset of ℓ^s whenever $p < s$; furthermore, ℓ^p is not linearly isomorphic to ℓ^s when $p \neq s$. In fact, by Pitt's theorem (Pitt 1936), every bounded linear operator from ℓ^s to ℓ^p is compact when $p < s$. No such operator can be an isomorphism; and further, it cannot be an isomorphism on any infinite-dimensional subspace of ℓ^s, and is thus said to be strictly singular.

If $1 < p < \infty$, then the (continuous) dual space of ℓ^p is isometrically isomorphic to ℓ^q, where q is the Hölder conjugate of p: $1/p + 1/q = 1$. The specific isomorphism associates to an element x of ℓ^q the functional

$$L_x(y) = \sum_n x_n y_n$$

for y in ℓ^p. Hölder's inequality implies that L_x is a bounded linear functional on ℓ^p, and in fact

$$|L_x(y)| \leq \|x\|_q \, \|y\|_p$$

so that the operator norm satisfies

$$\|L_x\|_{(\ell^p)^*} \stackrel{\text{def}}{=} \sup_{y \in \ell^p, y \neq 0} \frac{|L_x(y)|}{\|y\|_p} \leq \|x\|_q.$$

In fact, taking y to be the element of ℓ^p with

$$y_n = \begin{cases} 0 & \text{if } x_n = 0 \\ x_n^{-1}|x_n|^q & \text{if } x_n \neq 0 \end{cases}$$

gives $L_x(y) = ||x||_q$, so that in fact

$$\|L_x\|_{(\ell^p)^*} = \|x\|_q.$$

Conversely, given a bounded linear functional L on ℓ^p, the sequence defined by $x_n = L(e_n)$ lies in ℓ^q. Thus the mapping $x \mapsto L_x$ gives an isometry

$$\kappa_q : \ell^q \to (\ell^p)^*.$$

The map

$$\ell^q \xrightarrow{\kappa_q} (\ell^p)^* \xrightarrow{(\kappa_q^*)^{-1}} (\ell^q)^{**}$$

obtained by composing κ_p with the inverse of its transpose coincides with the canonical injection of ℓ^q into its double dual. As a consequence ℓ^q is a reflexive space. By abuse of notation, it is typical to identify ℓ^q with the dual of ℓ^p: $(\ell^p)^* = \ell^q$. Then reflexivity is understood by the sequence of identifications $(\ell^p)^{**} = (\ell^q)^* = \ell^p$.

The space c_0 is defined as the space of all sequences converging to zero, with norm identical to $||x||_\infty$. It is a closed subspace of ℓ^∞, hence a Banach space. The dual of c_0 is ℓ^1; the dual of ℓ^1 is ℓ^∞. For the case of natural numbers index set, the ℓ^p and c_0 are separable, with the sole exception of ℓ^∞. The dual of ℓ^∞ is the ba space.

The spaces c_0 and ℓ^p (for $1 \leq p < \infty$) have a canonical unconditional Schauder basis $\{e_i \mid i = 1, 2, \ldots\}$, where e_i is the sequence which is zero but for a 1 in the i^{th} entry.

The space ℓ^1 has the Schur property: In ℓ^1, any sequence that is weakly convergent is also strongly convergent (Schur 1921). However, since the weak topology on infinite-dimensional spaces is strictly weaker than the strong topology, there are nets in ℓ^1 that are weak convergent but not strong convergent.

The ℓ^p spaces can be embedded into many Banach spaces. The question of whether every infinite-dimensional Banach space contains an isomorph of some ℓ^p or of c_0, was answered negatively by B. S. Tsirelson's construction of Tsirelson space in 1974. The dual statement, that every separable Banach space is linearly isometric to a quotient space of ℓ^1, was answered in the affirmative by Banach & Mazur (1933). That is, for every separable Banach space X, there exists a quotient map $Q : \ell^1 \to X$, so that X is isomorphic to $\ell^1 / \ker Q$. In general, $\ker Q$ is not complemented in ℓ^1, that is, there does not exist a subspace Y of ℓ^1 such that $\ell^1 = Y \oplus \ker Q$. In fact, ℓ^1 has uncountably many uncomplemented subspaces that are not isomorphic to one another (for example, take $X = \ell^p$; since there are uncountably many such X's, and since no ℓ^p is isomorphic to any other, there are thus uncountably many $\ker Q$'s).

Except for the trivial finite-dimensional case, an unusual feature of ℓ^p is that it is not polynomially reflexive.

ℓ^p spaces are increasing in p

For $p \in [1, \infty]$, the spaces ℓ^p are increasing in p, with the inclusion operator being continuous: for $1 \leq p < q \leq \infty$, one has $\|x\|_q \leq \|x\|_p$. Indeed, the inequality is homogeneous in the x_i, so it is sufficient to prove it under the assumption that $\|x\|_p = 1$. In this case, we need only show that $\sum |x_i|^q \leq 1$ for $q > p$. But if $\|x\|_p = 1$, then $|x_i| \leq 1$ for all i, and then $\sum |x_i|^q \leq \sum |x_i|^p = 1$.

ℓ^2 is isomorphic to all separable, infinite dimensional Hilbert spaces

Let H be a separable Hilbert space. Every orthogonal set in H is at most countable (i.e. has finite dimension or $\aleph_0$).[2] The following two items are related:

- If H is infinite dimensional, then it is isomorphic to ℓ^2

- If $\dim(H) = N$, then H is isomorphic to $\mathbb{C}^N$

Properties of ℓ^1 spaces

A sequence of elements in ℓ^1 converges in the space of complex sequences ℓ^1 if and only if it converges weakly in this space.[3] If K is a subset of this space, then the following are equivalent:[3]

1. *K* is compact;
2. *K* is weakly compact;
3. *K* is bounded, closed, and equismall at infinity.

Here K being **equismall at infinity** means that for every $\varepsilon > 0$, there exists a natural number $n_\varepsilon \geq 0$ such that $\sum_{n=n_\epsilon}^{\infty} |s_n| < \varepsilon$ for all $s = (s_n)_{n=1}^{\infty} \in K$.

See also

- L^p space
- Tsirelson space
- beta-dual space
- Orlicz sequence space
- Hilbert space

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3. Trèves 2006, pp. 451–458.

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