Chapter 4

Multifractal scaling: general theory and approach by wavelets

4.1. Introduction and Summary

Fractal processes have a been successfully applied in various fields such as the theory of fully developed turbulence [MAN 74, FRI 85, BAC 93], stock market modeling[EVE 95, MAN 97, MAN 99], and more recently in the study of network data traffic [LEL 94, NOR 94]. In networking, models using fractional Brownian motion (fBm) have helped advance the field through their ability of capturing fractal features such as statistical self-similarity and long-range dependence (LRD). It has been recognized, however, that multifractal features need to be accounted for towards a better understanding of network traffic, but also of stock exchange [RIE 97a, RIE 99, RIE 00, FEL 98, MAN 97]. In short, there is a call for more versatile models which can, e.g., incorporate LRD and multifractal properties independently of each other.

Roughly speaking, a fractal entity is characterized by the inherent, ubiquitous occurrence of irregularities which governs its shape and complexity. The most prominent example is certainly fBm $B_H(t)$ [MAN 68]. Its paths are almost surely continuous but not differentiable. Indeed, the oscillation of fBm in any interval of size δ is of the order δ^H where $H \in (0,1)$ is the self-similarity parameter:

$$B_H(at) \stackrel{\text{fd}}{=} a^H B_H(t). \tag{4.1}$$

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Real world signals, on the other hand, often possess an erratically changing oscillation exponent, limiting the appropriateness of fBm as a model. Due to the various exponents being present in such signals, they have been termed *multifractals*.

This chapter's main objective is to present the framework for describing and detecting such a multifractal scaling structure. Doing so we survey local and global multifractal analysis and relate them via the multifractal formalism in a stochastic setting. Thereby, the importance of higher order statistics will become evident. It might be especially appealing to the reader to see wavelets put to novel use. We focus mainly on the analytical computation of the so-called multifractal spectra, and on their mutual relations, dwelling extensively on variations of binomial cascades. Statistical properties of estimators of multifractal quantities as well as modeling issues are addressed elsewhere (see [GON 98, ABR 00, GON 99] and [MAN 97, RIE 99, RIB 06]).

The remainder of this introduction provides a summary of the contents of the paper, following roughly its structure.

4.2. Singularity Exponents

For simplicity we consider processes Y over a probability space $(\Omega, \mathcal{F}, P_\Omega)$ and defined on a compact interval, which we assume without loss of generality to be $[0, 1]$. Generalization to higher dimensions is straigthforward and extending to processes defined on $\Re$ is simple and will be indicated.

4.2.1. *Hölder Continuity*

The erratic behavior or, more precisely, *degree of local Hölder regularity* of a continuous process $Y(t)$ at a *fixed* given time t can be characterized to a first approximation by comparison with an algebraic function: Y is said to be in C_t^h if there is a polynomial P_t such that $|Y(s) - P_t(s)| \leq C|s-t|^h$ for s sufficiently close to t. If P_t is a constant, i.e. $P_t(s) = Y(t)$ for all s, then Y is in C_t^h for all $h < \underline{h}(t)$ and not in C_t^h for all $h > \underline{h}(t)$ where

$$\underline{h}(t) := \liminf_{\varepsilon \to 0} \frac{1}{\log_2(2\varepsilon)} \log_2 \sup_{|s-t|<\varepsilon} |Y(s) - Y(t)|. \tag{4.2}$$

On the other hand, it easy to prove the following

LEMMA 4.1.– If $\underline{h}(t) \notin \mathbb{N}$ then P_t is a constant, and $\underline{h}(t) = \sup\{h : Y \in C_t^h\}$.

As the example $Y(s) = s^2 + s^{2.4}$ with $t = 0$ shows the conclusion does not necessarily hold when $\underline{h}(t) \in \mathbb{N}$. Here, $|Y(s) - Y(0)| \sim s^2$ for $s \sim 0$, thus $\underline{h}(0) = 2$, while $P_0(s) = s^2$, $Y(s) - P_0(s) = s^{2.4}$, and thus $\sup\{h : Y \in C_0^h\} = 2.4$.

PROOF.– Assume there is $h > \underline{h}(t)$ and $P_t(s)$ such that $Y \in C_t^h$. We will argue that $\underline{h}(t)$ must be an integer in this case. Note first that P_t is not constant by definition of $\underline{h}(t)$ and we may write $P_t(s) = Y(t) + (s-t)^m \cdot Q(s)$ for some integer $m \geq 1$ and some polynomial Q without zero at t. Assume first that $m < \underline{h}(t)$ and choose h' such that $m < h' < \underline{h}(t)$. Writing $Y(s) - P_t(s) = (Y(s) - Y(t)) - (P_t(s) - Y(t))$, the first term is smaller than $|s-t|^{h'}$ and the second term, decaying as $C|s-t|^m$, governs. Whence $h = m < \underline{h}(t)$, against the assumption. Assuming $m > \underline{h}(t)$ choose h' such that $m > h' > \underline{h}(t)$ and a sequence s_n such that $|Y(s_n) - Y(t)| \geq |s_n - t|^{h'}$, whence $|Y(s_n) - P_t(s_n)| \geq (1/2)|s_n - t|^{h'}$ for large n and $h \leq h'$. Letting $h' \to \underline{h}(t)$ we get again a contradiction. We conclude that $\underline{h}(t)$ equals m.

For reasons of symmetry we define

$$\overline{h}(t) := \limsup_{\varepsilon \to 0} \frac{1}{\log_2(2\varepsilon)} \log_2 \sup_{|s-t|<\varepsilon} |Y(s) - Y(t)|. \tag{4.3}$$

If $\underline{h}(t)$ and $\overline{h}(t)$ coincide we denote the common value by $h(t)$.

We note first that the continuous limit in (4.2) may be replaced by a discrete one. To this end we introduce $k_n(t) := \lfloor t2^n \rfloor$, an integer defined uniquely by

$$t \in I_{k_n}^n := [k_n 2^{-n}, (k_n+1)2^{-n}[. \tag{4.4}$$

As n increases the intervals I_k^n form a nested decreasing sequence (compare Figure 4.1). Provided n is chosen such that $2^{-n+1} \leq \varepsilon < 2^{-n+2}$ we have

$$[(k_n-1)2^{-n}, (k_n+2)2^{-n}[\subset [t+\varepsilon, t-\varepsilon[\subset [(k_{n-2}-1)2^{-n+2}, (k_{n-2}+2)2^{-n+2}[$$

from which it follows immediately that

$$\underline{h}(t) = \liminf_{n\to\infty} h_{k_n}^n \qquad \overline{h}(t) = \limsup_{n\to\infty} h_{k_n}^n$$

where

$$h_{k_n}^n := -\frac{1}{n} \log_2 \sup \big\{ |Y(s) - Y(t)| \, : \, s \in [(k_n-1)2^{-n}, (k_n+2)2^{-n}[\big\}. \tag{4.5}$$

It is essential to note that the countable set of numbers $h_{k_n}^n$ contains all the scaling information of interest to us. Being defined pathwise, they are random variables.

4.2.2. *Scaling of Wavelet Coefficients*

A convenient tool for scaling analysis is found in the wavelet transform, both the discrete and the continuous. The discrete transform, e.g., allows to represent a

1-d process $Y(t)$ in terms of shifted and dilated versions of a prototype bandpass wavelet function $\psi(t)$, and shifted versions of a low-pass scaling function $\phi(t)$ [DAU 92, VET 95]. While such representations exist also in the framework of continuous wavelet transforms we use the latter mainly as a "microscope" in this chapter. In the vocabulary of Hilbert spaces, the discrete wavelet and scaling functions

$$\psi_{j,k}(t) := 2^{j/2}\,\psi\big(2^j t-k\big)\,,\quad \phi_{j,k}(t) := 2^{j/2}\,\phi\big(2^j t-k\big)\,,\qquad j,k \text{ integer} \tag{4.6}$$

form an orthonormal basis and we have the representations [DAU 92, VET 95]

$$Y(t)=\sum_k D_{J_0,k}\,\phi_{J_0,k}(t)\;+\;\sum_{j=J_0}^{\infty}\sum_k C_{j,k}\,\psi_{j,k}(t), \tag{4.7}$$

with

$$C_{j,k} := \int Y(t)\,\psi^*_{j,k}(t)\,\mathrm{d}t,\qquad D_{j,k} := \int Y(t)\,\phi^*_{j,k}(t)\,\mathrm{d}t. \tag{4.8}$$

The *wavelet coefficient* $C_{j,k}$ measures the signal content around time $2^{-j}k$ and frequency $2^j f_0$, provided that the wavelet $\psi(t)$ is centered at time zero and frequency f_0. The *scaling* coefficient $D_{j,k}$ measures the local mean around time $2^{-j}k$. In the wavelet transform, j indexes the *scale* of analysis: J_0 can be chosen freely and indicates the coarsest scale or lowest resolution available in the representation.

The most simple example of an orthonormal wavelet basis are the Haar scaling and wavelet functions (see Figure 4.1(a)). Here, ϕ is the indicator function of the unit interval, while $\psi=\phi(2\cdot)-\phi(2\cdot-1)$. For a process supported on the unit interval a convenient choice is thus $J_0=0$. The supports of the fine-scale scaling functions nest inside the supports of those at coarser scales; this can be neatly represented by the binary tree structure of Figure 4.1(b). Row (scale) j of this scaling coefficient tree contains an approximation to $Y(t)$ of resolution 2^{-j}. Row j of the complementary wavelet coefficient tree (not shown) contains the details in scale $j+1$ of the scaling coefficient tree that are suppressed in scale j. In fact, for the Haar wavelet we have

$$\begin{aligned} D_{j,k} &= 2^{-1/2}(D_{j+1,2k}+D_{j+1,2k+1}),\\ C_{j,k} &= 2^{-1/2}(D_{j+1,2k}-D_{j+1,2k+1}). \end{aligned} \tag{4.9}$$

Wavelet decompositions contain considerable information on the singularity behavior of a process Y. Indeed, adapting the argument of [JAF 95, p. 291] and correcting for the L^2 wavelet normalization used here — as opposed to L^1 in [JAF 95] — it is easily shown that $|Y(s)-Y(t)|=O(|s-t|^h)$ implies that

$$2^{n/2}\,|C_{n,k_n}| = O\big(2^{-nh}\big)\,. \tag{4.10}$$

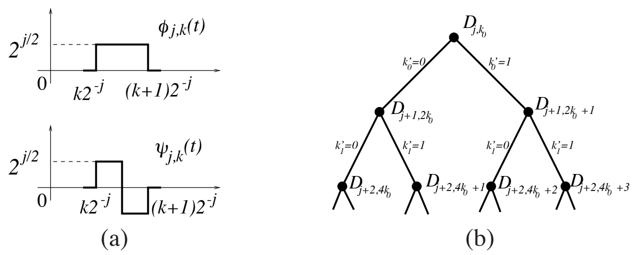

Figure 4.1. *(a) The Haar scaling and wavelet functions $\phi_{j,k}(t)$ and $\psi_{j,k}(t)$. (b) Binary tree of scaling coefficients from coarse to fine scales.*

This holds for any $h > 0$ and any compactly supported wavelet. As a matter of fact only $\int \psi = 0$ is needed to obtain this result since the Taylor polynomial of Y is implicitly assumed to be constant. If we are interested in an analysis only we may, thus, consider analyzing wavelets ψ such as derivatives of the Gaussian $\exp(-x^2)$ which don't necessarily form a basis. To distinguish them from the orthogonal wavelets we will address them as 'analyzing wavelets'. In order to invert (4.10), however, we need the representation (4.7) as well as some knowledge on the decay of the maximum of the wavelet coefficients in the vicinity of t and sufficient wavelet regularity. For a precise statement, see [JAF 95] and [DAU 92, Thm. 9.2].

All this suggests that replacing h_k^n (4.5) by the left hand side of (4.10) would produce an alternative description of the local behavior of Y. Consequently, we set

$$\underline{w}(t) := \liminf_{n\to\infty} w_{k_n}^n \qquad \overline{w}(t) := \limsup_{n\to\infty} w_{k_n}^n \tag{4.11}$$

where

$$w_{k_n}^n := -\frac{1}{n}\log_2 \left|2^{n/2} C_{n,k_n}\right|. \tag{4.12}$$

If $\underline{w}(t)$ and $\overline{w}(t)$ coincide we denote the common value by $w(t)$.

Using wavelets has the advantage of yielding an analysis which is largely unaffected by polynomial trends in Y due to vanishing moments $\int t^m \psi(t)\mathrm{d}t = 0$ which are typically built into wavelets [DAU 92]. In this context recall lemma 4.1. It has the disadvantage of complicating the analysis since maxima of wavelet coefficients have to be considered for a reliable estimation of true Hölder continuity [JAF 95, DAU 92, JAF 97, BAC 93]. In any case, the decay of wavelet coefficients is interesting in itself as it relates to LRD (compare [ABR 95]) and regularity spaces such as Besov spaces [RIE 99].

4.2.3. *Other Scaling Exponents*

The 'classical' multifractal analysis of a singular measure μ on the line constitutes a study of the singularity structure of its primitive $\mathcal{M}$ given by

$$\mathcal{M}(t) = \int_0^t \mu(\mathrm{d}s) = \mu([0,t]), \tag{4.13}$$

Since $\mathcal{M}$ is an almost surely increasing process, the coarse exponents $h^n_{k_n}$ (see (4.5)) simplifies to $h^n_{k_n} = -\frac{1}{n}\log_2 |\mathcal{M}\left((k_n+2)2^{-n}\right) - \mathcal{M}((k_n-1)2^{-n})|$ one can be motivated (some would say "seduced") to study an even simpler notion of a coarse exponent:

$$\underline{\alpha}(t) := \liminf_{n\to\infty} \alpha^n_{k_n} \qquad \overline{\alpha}(t) := \limsup_{n\to\infty} \alpha^n_{k_n} \tag{4.14}$$

where

$$\alpha^n_{k_n} := -\frac{1}{n}\log_2 \left|\mathcal{M}\left((k_n+1)2^{-n}\right) - \mathcal{M}(k_n 2^{-n})\right| = -\frac{1}{n}\log_2 \mu\big(I^n_{k_n}\big). \tag{4.15}$$

If $\underline{\alpha}(t)$ and $\overline{\alpha}(t)$ coincide we denote the common value by $\alpha(t)$. This exponent $\alpha(t)$ has attracted considerable attention in the multifractal community, potential due to its simplicity. In [LV 98] various examples of more general exponents were introduced, all of which are so-called Choquet capacities, a notion which is not needed to develop the multifractal formalism.

As an interesting alternative, [PEY 98] considers an arbitrary function $\xi(I)$ from the space of all intervals to $\Re^+$ (instead of only the I^n_k) and develops a multifractal formalism similar to ours. There, it is suggested to consider the oscillations of Y around the mean, i.e.

$$\xi(I) := \int_I \left| Y(t) - \frac{\int_I Y(s)\mathrm{d}s}{|I|} \right| \mathrm{d}t \tag{4.16}$$

Proceeding as with $h^n(t)$ we are lead to the singularity exponent $-(1/n)\log_2(\xi(I^n_k))$ which is of particular interest since it can be used to define oscillation spaces such as Sobolev spaces and Besov spaces. Another useful choice consists in interpolating Y in the interval I by the linear function $a_I + b_I t$ and considering

$$\xi(I) := \left(\int_I \left(Y(t) - (a_I + b_I t)\right)^2 \mathrm{d}t \right)^{1/2}. \tag{4.17}$$

This exponent measures the variability of Y and is related to the dimension of the paths of Y. Deducting constant, resp. linear terms in the definitions (4.16) and (4.17) reminds one of the use of wavelets with one, resp. two vanishing moments.

4.3. Multifractal Analysis

Multifractal analysis has been discovered and developed in [MAN 74, FRI 85, KAH 76,GRA 83,HEN 83,HAL 86,CUT 86,CAW 92,BRO 92,BAC 93,MAN 90b, HOL 92, FAL 94, OLS 94, ARB 96, JAF 97, PES 97, RIE 95a, MAN 02, BAC 03, BAR 04, BAR 02, CHA 05, JAF 99] to give only a short list of some relevant work done in this area. The main insight consisted of the fact that local scaling exponents on fractals as measured by $h(t)$, $\alpha(t)$ or $w(t)$, is not uniform or continuous as a function of t, in general. In other words, $h(t)$, $\alpha(t)$ and $w(t)$ change typically in an erratic way as a function of t, thus imprinting a rich structure on the object of interest. This structure can be captured either in geometrical terms making use of the concept of dimensions, or in statistical terms based on sample moments. A useful connection between these two descriptions emerges from the *multifractal formalism*.

As we will see, as far as the multifractal formalism is concerned there is no restriction in choosing a singularity exponent which seem fit for describing scaling behavior of interest. To express this fact we consider in this section the arbitrary scaling exponent

$$\underline{s}(t) := \liminf_{n\to\infty} s^n_{k_n} \qquad \text{and} \qquad \overline{s}(t) := \limsup_{n\to\infty} s^n_{k_n}, \tag{4.18}$$

where s^n_k ($k = 0, ..., 2^n - 1$, $n \in \mathbb{N}$) is any sequence of random variables. To keep a connection with what was said before think of s^n_k as representing a coarse scaling exponent of Y over the dyadic interval I^n_k.

4.3.1. *Dimension based Spectra*

A geometric description of the erratic behavior of the scaling exponents of a multifractal can be achieved via a quantification of the prevalence of particular exponents in terms of fractal dimensions as follows: One considers the sets K_a which are defined pathwise in terms of limiting behavior of $s^n_{k_n}$ as $n \to \infty$, as

$$E_a := \{t \,:\, \underline{s}(t) = a\}, \qquad \overline{E}_a := \{t \,:\, \overline{s}(t) = a\}, \qquad K_a := \{t \,:\, s(t) = a\} \tag{4.19}$$

These sets K_a are typically "fractal" meaning loosely that they have a complicated geometric structure and more precisely that their dimensions are non-integer. A compact description of the singularity structure of Y is, therefore, in terms of the following so-called *Hausdorff spectrum*

$$d(a) := \dim(K_a), \tag{4.20}$$

where $\dim(E)$ denotes the Hausdorff dimension of the set E [TRI 82].

The sets E_a ($a \in \Re$) — and also $\overline{E}_a$ — form a *multifractal decomposition* of the support of Y. We will loosely address Y as a *multifractal* if this decomposition is rich, i.e. if the sets E_a ($a \in \Re$) are highly interwoven or even dense in the support of Y.

However, the study of singular measures (deterministic and random) has been often restricted to the simpler sets K_a and their spectrum $d(a)$ [KAH 76, CAW 92, FAL 94, ARB 96, OLS 94, RIE 98, RIE 95a, RIE 95b, BAR 97]. With the theory developed here (lemma 4.2) it becomes clear that most of these results extend to provide formulas for $\dim(E_a)$ and $\dim(\overline{E}_a)$ also. This aspect of multifractal analysis has found strong interests in the mathematical community.

4.3.2. *Grain based Spectra*

An alternative description of the prevalence of singularity exponents, statistical in nature due the counting involved, is

$$f(a) := \lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \frac{1}{n} \log_2 N^n(a, \varepsilon), \tag{4.21}$$

where[1]

$$N^n(a, \varepsilon) := \#\{k = 0, ..., 2^n - 1 \, : \, a - \varepsilon \leq s_k^n < a + \varepsilon\}. \tag{4.22}$$

This notion has grown out of the dilemma of any real world application that the computation of actual Hausdorff dimensions is often hard, if not impossible. Using a mesh of given grain size as in (4.22) instead of arbitrary coverings as in $\dim(K_a)$ leads generally to more simple notions. However, f should not be regarded as an auxiliary vehicle but recognize its own merit which will be become apparent in the remainder of this section.

Our first remark on $f(a)$ concerns the fact that the counting used in its definition, i.e. $N^n(a, \varepsilon)$ may be used to estimate box dimensions. Based on this fact it was shown in [RIE 98] that

$$\dim(K_a) \leq f(a). \tag{4.23}$$

Here, we state a slightly improved version:

LEMMA 4.2.–

$$\dim(E_a) \leq f(a) \qquad \dim(\overline{E}_a) \leq f(a) \tag{4.24}$$

1. More generally, using c-ary intervals in Euclidean space $\Re^d$ k_n will range from 0 to $c^{nd} - 1$. Logarithms will have to be taken to the base c since we seek the asymptotics of $N^n(a, \varepsilon)$ in terms of a powerlaw of resolution at stage n, i.e. $N^n(a, \varepsilon) \simeq c^{nf(a)}$. The maximum value of $f(a)$ will be d.

and

$$\dim(K_a) \leq \underline{f}(a) := \lim_{\varepsilon \downarrow 0} \liminf_{n\to\infty} \frac{1}{n} \log_2 N^n(a,\varepsilon). \tag{4.25}$$

It follows immediately that $\dim(K_a) \leq \dim(E_a) \leq f(a)$, but $\dim(E_a)$ is not necessarily smaller than $\underline{f}(a)$.

4.3.3. *Partition Function and Legendre Spectrum*

The second comment regarding the grain spectrum $f(a)$ concerns its interpretation as a Large Deviation Principle (LDP). We may consider $N^n(a,\varepsilon)/2^n$ to be the probability to find (for a fixed realization of Y) a number $k_n \in \kappa_n := \{0, ..., 2^n - 1\}$ such that $s^n_{k_n} \in [a-\varepsilon, a+\varepsilon]$. Typically, there will be one value of $s(t)$ that appears most frequently, denoted $\hat{a}$, and $f(a)$ will reach its maximum 1 at $a = \hat{a}$. However, by definition, for $a \neq \hat{a}$ the chance to observe coarse exponents $s^n_{k_n}$ which lie in $[a-\varepsilon, a+\varepsilon]$ will decrease exponentially fast with rate given by $f(a)$.

Appealing to the theory of LDP-s we consider the random variable $A_n = -ns^n_K \ln(2)$ where K is randomly picked from $\kappa_n = \{0, ..., 2^n - 1\}$ with uniform distribution U_n (recall that the we study one fixed realization or path of Y) and define its 'logarithmic moment generating function' or *partition function*

$$\tau(q) := \liminf_{n\to\infty} -\frac{1}{n} \log_2 S^n(q), \tag{4.26}$$

where

$$S^n(q) := \sum_{k=0}^{2^n-1} \exp\left(-qns^n_k \ln(2)\right) = \sum_{k=0}^{2^n-1} 2^{-nqs^n_k} = 2^n \, \mathrm{E}_n \left[2^{-nqs^n_k}\right]. \tag{4.27}$$

Here, E_n stands for expectation with respect to U_n. The theorem of Gärtner-Ellis [ELL 84] applies then to yield the following result (see [RIE 95a] for a slightly stronger version):

THEOREM 4.1.– If the limit

$$\tau(q) = \lim_{n\to\infty} -\frac{1}{n} \log_2 S^n(q) \tag{4.28}$$

exists and is finite for all $q \in \Re$, and if $\tau(q)$ is a differentiable function of q, then the double limit

$$f(a) = \lim_{\varepsilon \downarrow 0} \lim_{n\to\infty} \frac{1}{n} \log_2 N^n(a,\varepsilon) \tag{4.29}$$

exists, in particular $f(a) = \underline{f}(a)$, and

$$f(a) = \tau^*(a) := \inf_{q\in\Re}(qa - \tau(q)) \tag{4.30}$$

for all a.

PROOF.– Applying [ELL 84, Thm II] to our situation gives immediately

$$\limsup_{n\to\infty} \frac{1}{n}\log_2 N^n(a,\varepsilon) \le \limsup_{n\to\infty}\frac{1}{n}\log_2 \#\{k : |s_k^n - a| \le \varepsilon\} \le \sup_{|a'-a|\le\varepsilon}\tau^*(a')$$

and

$$\liminf_{n\to\infty} \frac{1}{n}\log_2 N^n(a,\varepsilon) \ge \liminf_{n\to\infty}\frac{1}{n}\log_2 \#\{k : |s_k^n - a| < \varepsilon\} \ge \sup_{|a'-a|<\varepsilon}\tau^*(a').$$

By continuity of $\tau^*(a)$ these two bounds coincide and (4.29) is established. Letting now $\varepsilon \to 0$ shows that $f(a) = \tau^*(a)$.

Sometimes, the assumptions on differentiability of this theorem are too restrictive. Before dwelling more on the relation between τ and f in section 4.4 let us note a simple fact, providing also a simple reason why the Legendre transform appears in this context.

LEMMA 4.3.– We have always

$$f(a) \le \tau^*(a). \tag{4.31}$$

PROOF.– Fix $q \in \Re$ and consider a with $f(a) > -\infty$. Let $\gamma < f(a)$ and $\varepsilon > 0$. Then, one finds arbitrarily large n such that $N^n(a,\varepsilon) \ge 2^{n\gamma}$. For such n we bound $S^n(q)$ by noting

$$\sum_{k=0}^{2^n-1} 2^{-nqs_k^n} \ge \sum_{|s_k^n - a|<\varepsilon} 2^{-nqs_k^n} \ge N^n(a,\varepsilon)2^{-n(qa+|q|\varepsilon)} \ge 2^{-n(qa-\gamma+|q|\varepsilon)} \tag{4.32}$$

and hence $\tau(q) \le qa - \gamma + |q|\varepsilon$. Letting $\varepsilon \to 0$ and $\gamma \to f(a)$, we find $\tau(q) \le qa - f(a)$. Since this is trivial if $f(a) = -\infty$ we find

$$\tau(q) \le qa - f(a) \quad \text{and} \quad f(a) \le qa - \tau(q) \quad \text{for all } a \text{ and } q. \tag{4.33}$$

From this it follows trivially that $\tau(q) \le f^*(q)$ and $f(a) \le \tau^*(a)$.

With the special choice $s_k^n = \alpha_k^n$ for the distribution function $\mathcal{M}$ of a measure μ, $S^n(q)$ becomes

$$S^n{}_\alpha(q) = \sum_{k=0}^{2^n-1} \left|\mathcal{M}\left((k+1)2^{-n}\right) - \mathcal{M}(k2^{-n})\right|^q = \sum_{k=0}^{2^n-1} \left(\mu(I_k^n)\right)^q. \tag{4.34}$$

This is the original form in which $\tau(q)$ as been introduced in multifractal analysis [HAL 86, HEN 83, FRI 85, MAN 74]. Note that there is a close connection to the thermo-dynamical formalism [TEL 88].

4.3.4. *Deterministic Envelops*

An analytical approach is often useful in order to gain an intuition on the various spectra of a typical path of Y, or at least some estimate of it. To establish such an approach, we consider the position, i.e. t or k_n, as well as the path Y to be random *simultaneously*. Then, we apply the LDP to the according, larger probability space. More precisely, the exponents s_K^n are now random variables over $(\Omega \times \kappa_n, P_\Omega \times U_n)$. The 'deterministic partition function' corresponding to this setting reads as

$$T(q) := \liminf_{n\to\infty} -\frac{1}{n} \log_2 \mathrm{E}_\Omega[S^n(q)]. \tag{4.35}$$

NOTE 4.1.– **[Ergodic Processes]** So far, we have assumed in the definitions of $\tau(q)$ and $T(q)$ that Y is defined on a compact interval. Without loss of generality, this interval was assumed to be $[0,1]$. In order to allow for processes defined on $\Re$ we modify $S^n(q)$ to

$$S^n(q) := \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N2^n-1} 2^{-nqs_k^n}$$

and $N^n(a,\varepsilon)$ similarly. For *ergodic* processes this becomes $S^n(q) = 2^n\,\mathrm{E}_\Omega[2^{-nqs_k^n}]$ almost surely. Thus, $\mathrm{E}_\Omega[S^n(q)] = S^n(q)$ a.s. and

$$T(q) \overset{\text{a.s.}}{=} \tau(q,\omega). \tag{4.36}$$

We refer to (4.74) for an account on the extent to which marginal distributions may be reflected in multifractal spectra in general. For processes on $[0,1]$ we can not expect to have (4.36) in all generality. Nevertheless, we will point out scenarios where (4.36) holds. Notably $T(q)$ does always serve as a *deterministic envelop* of $\tau(q,\omega)$:

LEMMA 4.4.– With probability one[2]

$$\tau(q,\omega) \geq T(q) \qquad \text{for all } q \text{ with } T(q) < \infty. \tag{4.37}$$

PROOF.– Consider any q with finite $T(q)$ and let $\varepsilon > 0$. Let n_0 be such that $\mathrm{E}_\Omega[S^n(q)] \leq 2^{-n(T(q)-\varepsilon)}$ for all $n \geq n_0$. Then,

$$\mathrm{E}\left[\limsup_{n\to\infty} 2^{n(T(q)-2\varepsilon)} S_n(q,\omega)\right] \leq \mathrm{E} \sum_{n\geq n_0} 2^{n(T(q)-2\varepsilon)} S_n(q,\omega) \leq \sum_{n\geq n_0} 2^{-n\varepsilon} < \infty.$$

2. For clarity, we make the randomness of τ explicit here.

Thus, almost surely $\limsup_{n\to\infty} 2^{n(T(q)-2\varepsilon)} S_n(q,\omega) < \infty$, and $\tau(q) \geq T(q) - 2\varepsilon$. Consequently, this estimate holds with probability one simultaneously for all $\varepsilon = 1/m$ ($m \in \mathbb{N}$) and some countable, dense set of q values with $T(q) < \infty$. Since $\tau(q)$ and $T(q)$ are always concave due to corollary 4.2 below, they are continuous on open sets and the claim follows.

Along the same lines we may define the corresponding *deterministic grain spectrum*. By analogy, we will replace probability over $\kappa_n = \{0, ..., 2^n - 1\}$ in (4.21), i.e. $N^n(a,\varepsilon)$, by probability over $\Omega \times \kappa_n$, i.e.

$$\sum_{k=0}^{2^n-1} P_\Omega[a-\varepsilon \leq s_k^n < a+\varepsilon] = 2^n \,\mathrm{E}_{\Omega\times\kappa_n}\left[\mathbf{1}_{[a-\varepsilon,a+\varepsilon)}(s_K^n)\right] = \mathrm{E}_\Omega[N^n(a,\varepsilon)] \quad (4.38)$$

and define

$$F(a) := \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \frac{1}{n} \log_2 \mathrm{E}_\Omega[N^n(a,\varepsilon)] \quad (4.39)$$

Replacing $N^n(a,\varepsilon)$ by (4.38) in the proof of theorem 4.1 and taking expectations in (4.32) we find properties analogous to the pathwise spectra τ and f:

THEOREM 4.2.– For all a

$$F(a) \leq T^*(a). \quad (4.40)$$

Furthermore, under conditions on $T(q)$ analogous to $\tau(q)$ in theorem 4.1

$$F(a) = T^*(a) = \underline{F}(a) := \lim_{\varepsilon\downarrow 0} \liminf_{n\to\infty} \frac{1}{n} \log_2 \mathrm{E}_\Omega[N^n(a,\varepsilon)]. \quad (4.41)$$

It follows from lemma 4.4 that with probability one $\tau^*(a,\omega) \leq T^*(a)$ for all a. Similarly, the deterministic grain spectrum $F(a)$ is an upper bound to its pathwise defined random counterpart $f(a,\omega)$, however, only pointwise. On the other hand, we have here almost sure equality under certain conditions.

NOTE 4.2.–**[Negative Dimensions]** Defined through counting $f(a)$ is always positive— or $-\infty$. The envelops T^* and F, being defined through expectations of counts and sums, may assume negative values. Consequently, the negative values of T^* and F are not very useful in the estimation of f; however, they do contain further information and can be 'observed'. Negative $F(a)$ and $T^*(a)$ have been termed negative dimensions [MAN 90b]. They correspond to probabilities of observing a coarse Hölder exponent a which decay faster than the $2^n = \#\kappa_n$ 'samples' s_k^n available in one realization. Oversampling the process, i.e. analyzing several independent realizations will increase the number of samples and more 'rare' s_k^n may be observed. In loose terms, in $\exp(-n\ln(2)F(a))$ independent traces one has a fair chance to see at least one s_k^n of size $\simeq a$. Thereby, it is essential not to average the spectra $f(a)$ of the various realizations but the numbers $N^n(a,\varepsilon)$. This way, negative 'dimensions' $F(a)$ become visible.

4.4. The Multifractal Formalism

In the previous section, Various *multifractal spectra* have been introduced along with some simple relations between them. These can be summarized as follows:

COROLLARY 4.1.– **[Multifractal formalism]**
For every a

$$\dim(K_a) \leq \dim(E_a) \leq f(a) \leq \tau^*(a) \overset{\text{a.s.}}{\leq} T^*(a) \tag{4.42}$$

where the first relations hold pathwise and the last one with probability one. Similarly

$$\dim(K_a) \leq \underline{f}(a) \leq f(a) \overset{\text{a.s.}}{\leq} F(a) \leq T^*(a). \tag{4.43}$$

The spectra on the left end have stronger implications on the local scaling structure while the ones on the right end are more easy to estimate or calculate.

This set of inequalities could fairly be called the 'multifractal formalism'. However, in the mathematical community a slightly different terminology is already established which goes as 'the multifractal formalism holds' and means that for a particular process (or one of its paths, according to context) $\dim(K_a)$ can be calculated using some adequate partition function (such as $\tau(q)$) and taking its Legendre transform. Consequently, when 'the multifractal formalism holds' for a path or process, then we find often that *equality* holds between several or all spectra appearing in (4.42), depending on the context of the formalism that had been established.

This property (that the 'multifractal formalism holds') is a very strong one and suggests the presence of one single underlying multiplicative structure in Y. This intuition is supported by the fact that the multifractal formalism in known to 'hold' up to now only for objects with strong rescaling properties where multiplication is involved such as self-similar measures, products of processes and infinitely divisible cascades (see [CAW 92, FAL 94, ARB 96, RIE 95a, PES 97, HOL 92], respectively [MAN 02] and [BAC 03, BAR 04, BAR 02, CHA 05] as well as references therein). A notable exception of processes without injected multiplicative structure are the Lévy processes the multifractal propoerties of which are well understood due to [JAF 99].

Though we pointed out some conditions for equality between f, τ^* and T^* we must note that in general we may have strict inequality in some or all parts of (4.42). Such cases have been presented in [RIE 95a] and [RIE 98]. There is, however, one equality which holds under mild conditions and connects the two spectra in the center of (4.42).

THEOREM 4.3.– Consider a realization or path of Y. If the sequence s_k^n is bounded, then

$$\tau(q) = f^*(q) \qquad \text{for all } q \in \Re. \tag{4.44}$$

PROOF.– Note that $\tau(q) \leq f^*(q)$ from lemma 4.3. Now, to estimate $\tau(q)$ from below, choose $\overline{a}$ larger than $|s_k^n|$ for all n and k and group the terms in $S^n(q)$ conveniently, i.e.

$$S^n(q) \leq \sum_{i=-\lfloor \overline{a}/\varepsilon \rfloor}^{\lfloor \overline{a}/\varepsilon \rfloor} \sum_{(i-1)\varepsilon \leq s_k^n < (i+1)\varepsilon} 2^{-nqs_k^n} \leq \sum_{i=-\lfloor \overline{a}/\varepsilon \rfloor}^{\lfloor \overline{a}/\varepsilon \rfloor} N^n(i\varepsilon, \varepsilon) 2^{-n(qi\varepsilon - |q|\varepsilon)}. \tag{4.45}$$

Next, we need uniform estimates on $N^n(a, \varepsilon)$ for various a. Fix $q \in \Re$ and let $\eta > 0$. Then, for every $a \in [-\overline{a}, \overline{a}]$ there is $\varepsilon_0(a)$ and $n_0(a)$ such that $N^n(a, \varepsilon) \leq 2^{n(f(a)+\eta)}$ for all $\varepsilon < \varepsilon_0(a)$ and all $n > n_0(a)$. We would like to have ε_0 and n_0 independent from a for our uniform estimate. To this end note that $N^n(a', \varepsilon') \leq N^n(a, \varepsilon)$ for all $a' \in [a - \varepsilon/2, a + \varepsilon/2]$ and all $\varepsilon' < \varepsilon/2$. By compactness we may choose a finite set of a_j $(j = 1, ..., m)$ such that the collection $[a_j - \varepsilon_0(a_j)/2, a_j + \varepsilon_0(a_j)/2]$ covers $[-\overline{a}, \overline{a}]$. Set $\varepsilon_1 = (1/2) \min_{j=1,\ldots,m} \varepsilon_0(a_j)$ and $n_1 = \max_{j=1,\ldots,m} n_0(a_j)$. Then, for all $\varepsilon < \varepsilon_1$ and $n > n_1$, and for all $a \in [-\overline{a}, \overline{a}]$ we have $N^n(a, \varepsilon) \leq 2^{n(f(a)+\eta)}$ and, thus,

$$S^n(q) \leq \sum_{i=-\lfloor \overline{a}/\varepsilon \rfloor}^{\lfloor \overline{a}/\varepsilon \rfloor} 2^{-n(qi\varepsilon - f(i\varepsilon) - \eta - |q|\varepsilon)} \leq (2\lfloor \overline{a}/\varepsilon \rfloor + 1) \cdot 2^{-n(f^*(q) - \eta - |q|\varepsilon)}. \tag{4.46}$$

Letting $n \to \infty$ we find $\tau(q) \geq f^*(q) - \eta - |q|\varepsilon$ for all $\varepsilon < \varepsilon_1$. Now we let $\varepsilon \to 0$ and finally $\eta \to 0$ to find the desired inequality.

Due to the properties of Legendre transforms[3] it follows:

COROLLARY 4.2.– **[Properties of the partition function]** If the sequence s_k^n is bounded, then the partition function $\tau(q)$ is concave and monotonous. Consequently, $\tau(q)$ is continuous on $\Re$, and differentiable in all but a countable number of exceptional points.

In order to efficiently invert theorem 4.3 we need:

LEMMA 4.5.– **[Lower semi-continuity of f and F]** Let a_m converge to a_*. Then

$$f(a_*) \geq \limsup_{m \to \infty} f(a_m) \tag{4.47}$$

3. For a tutorial on the Legendre transform see [RIE 99, App. A].

and analogous for F.

PROOF.– For all $\varepsilon > 0$ one can find m_0 such that $a_* - \varepsilon < a_m - \varepsilon/2 < a_m + \varepsilon/2 < a_* + \varepsilon$ for all $m > m_0$. Then, $N^n(a_*, \varepsilon) \geq N^n(a_m, \varepsilon/2)$ and $\mathrm{E}[N^n(a_*, \varepsilon)] \geq \mathrm{E}[N^n(a_m, \varepsilon/2)]$. We find

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 N^n(a_*, \varepsilon) \geq \limsup_{n\to\infty} \frac{1}{n} \log_2 N^n(a_m, \varepsilon/2) \geq f(a_m)$$

for any $m > m_0(\varepsilon)$ and similar for F. Now, let first $m \to \infty$ and then $\varepsilon \to 0$.

COROLLARY 4.3.– **[Central multifractal formalism]** We always have

$$f(a) \leq f^{**}(a) = \tau^*(a). \tag{4.48}$$

Further, denoting by $\tau'(q\pm)$ the right- resp. left-sided limits of derivatives we have,

$$f(a) = \tau^*(a) = q\tau'(q\pm) - \tau(q\pm) \qquad \text{at } a = \tau'(q\pm). \tag{4.49}$$

PROOF.– The graph of f^{**} is the concave hull of the graph of f which implies (4.48). It is an easy task to derive (4.49) under assumptions suitable to make the tools of calculus available such as continuous second derivatives. To prove it in general let us first assume that τ is differentiable at a fixed q. In particular, $\tau(q')$ is then finite for q' close to q.

Since $\tau(q) = f^*(q)$ there is a sequence a_m such that $\tau(q) = \lim_m qa_m - f(a_m)$. Since $\tau(q') \leq q'a - f(a)$ for all q' and a by (4.33), and since τ is differentiable at q this sequence a_m must converge to $a_* := \tau'(q)$. From the definition of a_m we conclude that $f(a_m)$ converges to $qa_* - \tau(q)$. Applying lemma 4.5 we find that $f(a_*) \geq qa_* - \tau(q)$. Recalling (4.33) implies the desired equality.

Now, for an arbitrary q the concave shape of τ implies that there is a sequence of numbers q_m larger than q in which τ is differentiable and which converges down to q. Consequently, $\tau'(q+) = \lim_m \tau'(q_m)$. The formula (4.49) being established at all q_m lemma 4.5 applies with $a_m = \tau'(q_m)$ and $a_* = \tau'(q+)$ to yield $f(\tau'(q+)) \geq q\tau'(q+) - \tau(q+)$. Again, (4.33) furnishes the opposite inequality. A similar argument applies to $\tau'(q-)$.

COROLLARY 4.4.– If $T(q)$ is finite for an open interval of q-values then $|s_k^n|$ is bounded for almost all paths, and

$$T(q) = F^*(q) \qquad \text{for all } q. \tag{4.50}$$

Moreover,

$$F(a) = T^*(a) = qT'(\pm q) - T(\pm q) \qquad \text{at } a = T'(\pm q). \tag{4.51}$$

PROOF.– Assume for a moment that s_k^n is unbounded from above with positive probability. Then, the grouping (4.45) requires an additional term collecting the $s_k^n > \overline{a}$. In fact, for any number $\overline{a}$ we can find arbitrarily large n such that $s_k^n > \overline{a}$ for some k. This implies that for any negative q we have $S^n(q) \geq 2^{-nq\overline{a}}$ and $\tau(q) \leq q\overline{a}$. Letting $\overline{a} \to \infty$ shows that $\tau(q) = -\infty$. By lemma 4.4 we must have $T(q) = -\infty$, a contradiction. Similarly, one shows that s_k^n is bounded from below. The remaining claims can be established analogously to the ones for $\tau(q)$ by taking expectations in (4.45).

NOTE 4.3.– **[Estimation and unbounded moments]** In order to apply corollary 4.4 in a real world situation, but also for the purpose of estimating $\tau(q)$, it is of great importance to possess a method to estimate the range of q-values for which the moments of a stationary process (such as the increments or the wavelet coefficients of Y) are finite. Such a procedure is proposed in [GON 05] (see also [RIE 04]).

4.5. Binomial Multifractals

The Binomial measure has a long standing tradition in serving as the paradigm of multifractal scaling [MAN 74, KAH 76, MAN 90a, CAW 92, HOL 92, Ben 87, RIE 95a, RIE 97b]. We present it here with an eye on possible generalizations of use in modeling.

4.5.1. *Construction*

To be consistent in notation we denote the binomial measure by μ_b and its distribution function by $\mathcal{M}_b(t) := \mu_b(]-\infty, t[)$. Note that μ_b is a measure or (probability) distribution, i.e. not a function in the usual sense, while $\mathcal{M}_b$ is a right-continuous and increasing function by definition.

In order to define μ_b we use again the notation (4.4): For any fixed t there is a unique sequence $k_1, k_2, \ldots$ such that the dyadic intervals $I_{k_n}^n = [k_n 2^{-n}, (k_n+1)2^{-n}[$ contain t for all integer n. So, the I_k^n form a decreasing sequence of half open intervals which shrink down to $\{t\}$. Moreover, $I_{2k_n}^{n+1}$ is the left subinterval of $I_{k_n}^n$ and $I_{2k_n+1}^{n+1}$ the right one (compare Figure 4.1). Note that the first n elements of such a sequence, i.e. $(k_1, k_2, \ldots, k_n)$ are identical for all points $t \in I_{k_n}^n$. We call this a *nested sequence* and it is uniquely defined by the value of k_n. We set

$$\mu_b(I_k^n) = \mathcal{M}_b((k_n+1)2^{-n}) - \mathcal{M}_b(k_n 2^{-n}) = M_{k_n}^n \cdot M_{k_{n-1}}^{n-1} \cdots M_{k_1}^1 \cdot M_0^0. \quad (4.52)$$

In words: the mass lying in $I_{k_n}^n$ is redistributed among its two dyadic subintervals $I_{2k_n}^{n+1}$ and $I_{2k_n+1}^{n+1}$ in the proportions $M_{2k_n}^{n+1}$ and $M_{2k_n+1}^{n+1}$. For consistency we require $M_{2k_n}^{n+1} + M_{2k_n+1}^{n+1} = 1$.

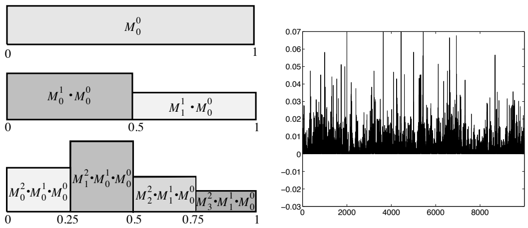

Figure 4.2. *Iterative construction of the binomial cascade.*

Having defined the mass of dyadic intervals we obtain the mass of any interval $]-\infty,t[$ by writing it as a disjoint union of dyadic intervals J^n and noting $\mathcal{M}_b(t) = \mu_b(]-\infty,t[= \sum_n \mu_b(J^n)$. Therefore, integrals (expectations) with respect to μ_b can be calculated as

$$\int g(t)\mu_b(\mathrm{d}t) \quad = \quad \lim_{n\to\infty} \sum_{k=0}^{2^n-1} g(k2^{-n})\mu_b(I_k^n) \tag{4.53}$$

$$= \int g(t)\mathrm{d}\mathcal{M}_b(t) \quad = \quad \lim_{n\to\infty} \sum_{k=0}^{2^n-1} g(k2^{-n}) \left(\mathcal{M}_b((k+1)2^{-n}) - \mathcal{M}_b(k2^{-n})\right) \tag{4.54}$$

Alternatively, the measure μ_b can be defined via ist distribution function $\mathcal{M}_b$. indeed, as a distribution function, $\mathcal{M}_b$ is monotone and continuous from the right. Since (4.52) defines $\mathcal{M}_b$ in all dyadic points it can be obtained in any other point as the right-sided limit. Note that $\mathcal{M}_b$ is continuous at a given point t unless $M^n_{k_n(t)} = 1$ for all n large.

To generate randomness in $\mathcal{M}_b$, we choose the various M_k^n to be random variables. The above properties hold then pathwise.

We will make the following assumptions on the distributions of the multipliers M_k^n:

i): Conservation of mass. Almost surely for all n and k M_k^n is positive and

$$M^{n+1}_{2k_n} + M^{n+1}_{2k_n+1} = 1. \tag{4.55}$$

As we have seen, this guarantees that $\mathcal{M}_b$ is well defined.

ii): Nested independence. All multipliers of a nested sequence are mutually independent. In analogy to (4.52) we have for any nested sequence

$$\mathrm{E}_\Omega[M^n_{k_n} \cdots M^0_0] = \mathrm{E}_\Omega[M^n_{k_n}] \cdots \mathrm{E}_\Omega[M^0_0] \tag{4.56}$$

and similar for other moments. This will allow for simple calculations in the sequel.

iii): Identical distributions For all n and k

$$M^n_k \overset{\mathrm{fd}}{=} \begin{cases} M_0 & \text{if } k \text{ is even,} \\ M_1 & \text{if } k \text{ is odd.} \end{cases} \tag{4.57}$$

A more general version of iii) was given in [RIE 99] to allow for more flexibility in model matching. The theory of cascades or, more properly, T-martingales[4] [KAH 76, Ben 87, HOL 92, BAR 97], provides a wealth of possible generalizations. Most importantly, it allows to soften the almost sure conservation condition i) to

i'): Conservation in the mean

$$\mathrm{E}_\Omega[M_0 + M_1] = 1. \tag{4.58}$$

In this case, $\mathcal{M}_b$ is well defined since (4.52) forms a martingale due to the nested independence (4.56). The main advantage of such an approach is that we can use unbounded multipliers M_0 and M_1 such as log-normal random variables. Then, the marginals of the increment process, i.e. $\mu_b(I^n_k)$ are exactly log-normal on all scales. For general binomials, always asusming ii) it can be argued that the marginals $\mu_b(I^n_k)$ are at least asymptocially log-normal by applying a Central Limit Theorem to the logarithm of (4.52).

4.5.2. *Wavelet Decomposition*

The scaling coefficients of μ_b using the *Haar wavelet* are simply

$$D_{n,k}(\mu_b) = \int \phi^*_{j,k}(t)\, \mu_b(\mathrm{d}t) = 2^{n/2} \int_{k2^{-n}}^{(k+1)2^{-n}} \mu_b(\mathrm{d}t) = 2^{n/2} \mu_b(I^n_k) \tag{4.59}$$

4. For any fixed t the sequence (4.52) forms a martingale due to the nested independence (4.56).

from (4.8) and (4.53). With (4.9) and (4.52) we get the explicit expression for the *Haar wavelet* coefficients:

$$2^{-n/2}C_{n,k_n}(\mu_b) = \mu_b(I_{2k_n}^{n+1}) - \mu_b(I_{2k_n+1}^{n+1}) = (M_{2k_n}^{n+1} - M_{2k_n+1}^{n+1}) \prod_{i=0}^{n} M_{k_i}^i. \quad (4.60)$$

Similar scaling properties hold when using *arbitrary, compactly supported wavelets*, provided the distributions of the multipliers are scale independent. This comes about from (4.52) and (4.53) which give the following rule for *substituting* $t' = 2^n t - k_n$

$$2^{-n/2}C_{n,k_n}(\mu_b) = \int_{I_{k_n}^n} \psi(2^n t - k_n)\mu_b(\mathrm{d}t) = M_{k_n}^n \cdots M_{k_1}^1 \cdot \int_0^1 \psi(t')\mu_b^{(n,k_n)}(\mathrm{d}t'). \quad (4.61)$$

Here $\mu_b^{(n,k_n)}$ is a binomial measure constructed with the same method as μ_b itself, however, with multipliers taken from the subtree which has as its root at the node k_n of level n of the original tree. More precisely, for any nested sequence $i_1, ..., i_m$

$$\mu_b^{(n,k_n)}(I_{i_m}^m) = M_{2k_n+i_1}^{n+1} \cdot M_{4k_n+i_2}^{n+2} \cdots M_{2^m k_n+i_m}^{n+m}.$$

From the nested independence (4.56) we infer that this measure $\mu_b^{(n,k_n)}$ is independent of $M_{k_i}^i$ $(i = 1, ..., n)$. Furthermore, the identical distributions of the multipliers iii) imply that for *arbitrary, compactly supported wavelets*

$$\int_0^1 \psi(t)\mu_b^{(n,k_n)}(\mathrm{d}t) \stackrel{\mathrm{d}}{=} C_{0,0}(\mu_b) = \int_0^1 \psi(t)\mu_b(\mathrm{d}t) \quad (4.62)$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution. In particular, for the *Haar wavelet* we have

$$\int_0^1 \psi_{\text{Haar}}(t)\mu_b^{(n,k_n)}(\mathrm{d}t) = M_{2k_n}^{n+1} - M_{2k_n+1}^{n+1} \stackrel{\mathrm{d}}{=} M_0 - M_1 = C_{0,0}^{\text{Haar}}(\mu_b) \quad (4.63)$$

(The deterministic analogue has also been observed in [BAC 93]). Finally, note that if ψ is supported on $[0,1]$, then $\psi(2^n(\cdot) - k)$ is supported on I_k^n. So, the tree of wavelet coefficients $C_{n,k}$ of μ_b possess a structure similar to the tree of increments of $\mathcal{M}_b$ (compare (4.52)).

With little more effort we calculate the wavelet coefficients of $\mathcal{M}_b$ itself, provided ψ is admissible and supported on $[0,1]$. Indeed,

$$\mathcal{M}_b(t) - \mathcal{M}_b(k_n 2^{-n}) = \mu_b([k_n 2^{-n}, t]) = M_{k_n}^n \cdots M_{k_1}^1 \mathcal{M}_b^{(n,k_n)}(2^n t - k_n), \quad (4.64)$$

where $\mathcal{M}_b^{(n,k_n)}(t') := \mu_b^{(n,k_n)}([0,t'])$. Using $\int \psi = 0$ and substituting $t' = 2^n t - k_n$ this yields

$$\begin{aligned} 2^{-n/2}C_{n,k_n}(\mathcal{M}_b) &= \int_{I_{k_n}^n} \psi(2^n t - k_n)\big(\mathcal{M}_b(t) - \mathcal{M}_b(k_n 2^{-n})\big)\,\mathrm{d}t \\ &= 2^{-n} \cdot M_{k_n}^n \cdots M_{k_1}^1 \cdot \int_0^1 \psi(t')\mathcal{M}_b^{(n,k_n)}(t')\mathrm{d}t'. \end{aligned} \tag{4.65}$$

Again, we have

$$\int_0^1 \psi(t)\mathcal{M}_b^{(n,k_n)}(\mathrm{d}t) \stackrel{\mathrm{d}}{=} C_{0,0}(\mathcal{M}_b) = \int_0^1 \psi(t)\mathcal{M}_b(\mathrm{d}t) \tag{4.66}$$

LEMMA 4.6.– Let ψ be a wavelet supported on $[0,1]$. Let $\mathcal{M}_b$ be a Binomial with i)-iii). Then, $C_{n,k_n}(\mu_b)$ is given by (4.61), and if ψ is admissible then $C_{n,k_n}(\mathcal{M}_b)$ is given by (4.65). Further, (4.62) and (4.66) hold.

It is obvious that the dyadic structure present in both, the construction of the binomial measure as well as in the wavelet transform, are responsible for the simplicity of the computation above. It is, however, standard by now to extend the procedure to more general multinomial cascades such as $\mathcal{M}_c$ introduced in Section 4.5.5 (see [ARB 96, RIE 95a]).

4.5.3. *Multifractal Analysis of the Binomial Measure*

In the light of lemma 4.6 it becomes clear that the singularity exponent $\alpha(t)$ is most easily accessible for $\mathcal{M}_b$ while $w(t)$ is readily available for both, $\mathcal{M}_b$ and μ_b. On the other hand, increments as they appear in $\alpha(t)$ are not well defined for μ_b. Thus, it is natural to compute the spectra of both, $\mathcal{M}_b$ and μ_b, with appropriate singularity exponents, i.e. $f_{\alpha,\mathcal{M}_b}$, $f_{w,\mathcal{M}_b}$ and f_{w,μ_b}.

Now, lemma 4.6 indicates that the singularity structures of μ_b and $\mathcal{M}_b$ are closely related. Indeed, μ_b is the distributional derivative of $\mathcal{M}_b$ in the sense of (4.52) and (4.54). Since taking a derivative "should" simply reduce the scaling exponent by one, we would expect that their spectra are identical up to a shift in a by -1. Indeed, this is true for increasing processes such as the $\mathcal{M}_b$ as we will elaborate in Section 4.6.2. However, it has to be pointed out that this rule can not be correct for oscillating processes. This is well demonstrates by the example $t^a \cdot \sin(t^{-b})$ with $b > 0$. Though this example has the exponent a at zero, its derivative behaves like t^{a-b-1} there. This is caused by the strong oscillations, also called *chirp*, at zero. In order to deal with such situations the two-microlocalization has to be employed [JAF 91].

Let us first dwell on the well known multifractal analysis of $\mathcal{M}_b$ based on α_k^n. Recall that $\mathcal{M}_b((k_n+1)2^{-n}) - \mathcal{M}_b(k_n)$ is given by (4.52), and use the nested independence (4.56) and identical distributions (4.57) to obtain

$$\begin{aligned}
\mathrm{E}[S^n{}_{\alpha,\mathcal{M}_b}(q)] &= \sum_{k_n=0}^{2^n-1} \mathrm{E}\Big[\big(M_{k_n}^n\big)^q \cdots \big(M_{k_1}^1\big)^q \big(M_0^0\big)^q\Big] \\
&= \mathrm{E}\Big[\big(M_0^0\big)^q\Big] \cdot \sum_{i=0}^{n} \binom{n}{i} \mathrm{E}\,[M_0^q]^i\, \mathrm{E}\,[M_1^q]^{n-i} \\
&= \mathrm{E}\Big[\big(M_0^0\big)^q\Big] \cdot \left(\mathrm{E}[M_0^q] + \mathrm{E}[M_1^q]\right)^n . \qquad (4.67)
\end{aligned}$$

From this, it follows immediately that

$$T_{\alpha,\mathcal{M}_b}(q) \quad = \quad -\log_2 \mathrm{E}[(M_0)^q + (M_1)^q]\,. \qquad (4.68)$$

Note that this value may be $-\infty$ for some q.

THEOREM 4.4.– Assume that i'), ii) and iii) hold. Assume furthermore that M_0 and M_1 have at least some finite moment of negative order. Then, with probability one

$$\dim(K_a) = f(a) = \tau^*(a) = T^*{}_{\alpha,\mathcal{M}_b}(a) \qquad (4.69)$$

for all a such that $T^*{}_{\alpha,\mathcal{M}_b}(a) > 0$. Thereby, all the spectra are in terms of the singularity exponents α_k^n or h_k^n of $\mathcal{M}_b$.

NOTE 4.4.– **[Wavelet analysis]** In the sequel we will show that we obtain the same spectra for $\mathcal{M}_b$ replacing α_k^n by w_k^n for certain analyzing wavelets. We will also mention the changes which become necessary when studying distribution functions of measures with *fractal* support (see Section 4.5.5).

PROOF.– By inspection of [BAR 97] one finds that $\dim(K_a) = T^*(a)$ for α_k^n under the given assumptions. Earlier results such as [FAL 94, ARB 96] used more restrictive assumptions but are somewhat easier to read. Though weaker than [BAR 97] they are sufficient in some situtations.

4.5.4. *Examples*

Example 1 (βeta Binomial) Consider multipliers M_0 and M_1 that follow a β distribution, which has the density $c_p t^{p-1}(1-t)^{p-1}$ for $t \in [0,1]$ and 0 elsewhere.

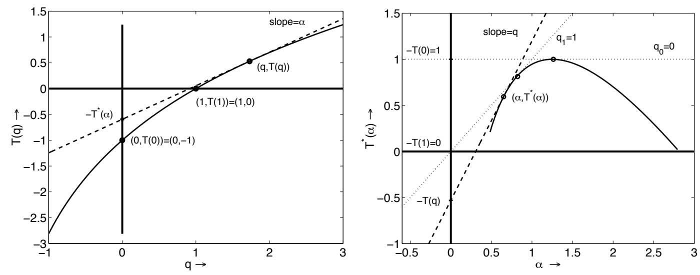

Figure 4.3. *The spectrum of a Binomial measure with β distributed multipliers with $p = 1.66$. Trivially, $T(0) = -1$, whence the maximum of T^* is 1. In addition, every positive increment process has $T(1) = 0$, whence T^* touches the bisector. Finally, the LRD parameter is $H_{\text{var}} = (T(2) + 1)/2 = 0.85$ (see (4.90) below).*

Hereby, $p > 0$ is a parameter and c_p is a normalization constant. Note that the conservation of mass i) imposes a symmetrical distribution since M_0 and M_1 are set to be equally distributed.

The β distribution has finite moments of order $q > -p$ which can be expressed explicitly using the Γ-function. We get

$$\beta\text{-Binomial:} \qquad T_\alpha(q) = -1 - \log_2 \frac{\Gamma(p+q)\Gamma(2p)}{\Gamma(2p+q)\Gamma(p)} \qquad (q > -p), \tag{4.70}$$

and $T(q) = -\infty$ for $q \leq -p$. For a typical shape of these spectra see Figure 4.3.

An application of the βeta Binomial for the modeling of data traffic on the internet can be found in [RIE 99]. ♠

Example 2 (Uniform Binomial) As a special case of the βeta Binomial we obtain uniform distributions for the multipliers when setting $p = 1$. The formula (4.70) simplifies to $T_\alpha(q) = -1 + \log_2(1+q)$ for $q > -1$. Applying the formula for the Legendre transform (4.51) yields the explicit expression

$$\text{uniform Binomial:} \qquad {T^*}_\alpha(a) = 1 - a + \log_2(e) + \log_2\left(\frac{a}{\log_2(e)}\right) \tag{4.71}$$

for $a > 0$ and ${T^*}_\alpha(a) = -\infty$ for $a \leq 0$. ♠

Example 3 (Log-normal Binomial) Another case of strong interest are log-normal distributions for the multipliers M_0 and M_1. Note that we have to replace i) by i') in this case since log-normal variables can be arbitrarily large, i.e. larger than 1. Recall that the log-normal binomial enjoyes the advantage of having *exactly log-normal marginals* $\mu_b(I_k^n)$ since the product of independent log-normal variables is again a log-normal variable. Having conservation of mass only in the mean, however, may cause problems in simulations since the *sample mean* of the process $\mu_b(I_k^n)$ $(k = 0, ..., 2^n - 1)$ is not M_0^0 as in case i), but depends on n. Indeed, the negative (virtual) a appearing in the spectrum of the log-normal Binomial reflect the possibility that the sample average my increase locally (compare [MAN 90a]).

The computation of its spectrum starts by observing that the exponential $M = e^G$ of a $\mathcal{N}(m, \sigma^2)$ variable G, i.e. a Gaussian with mean m and variance σ^2, has the q-th moment $\mathrm{E}[M^q] = \mathrm{E}[\exp(qG)] = \exp(qm + q^2\sigma^2/2)$. Assuming that M_0 and M_1 are equally distributed as M their mean must be $1/2$. Hence $m + \sigma^2 = -\ln(2)$, and

$$\text{log-normal Binomial:} \qquad T_\alpha(q) = (q-1)\left(1 - \frac{\sigma^2}{2\ln(2)}q\right) \tag{4.72}$$

for all $q \in \Re$ such that $\mathrm{E}[(\mathcal{M}_b(1))^q]$ is finite. Note that the parabola in (4.72) has two zeros: 1 and $q_{\text{crit}} = 2\ln(2)/\sigma^2$. It follows from [KAH 76] that $\mathrm{E}[(\mathcal{M}_b(1))^q] < \infty$ exactly for $q < q_{\text{crit}}$.

Since $T(q)$ is differentiable exactly for $q < q_{\text{crit}}$ we may obtain its Legendre transform implicitly from (4.51) for $a = T'(q)$ with $q < q_{\text{crit}}$, i.e., for all $a > a_{\text{crit}} = T'(q_{\text{crit}}) = \sigma^2/(2\ln(2)) - 1$. Eliminating q from (4.51) yields the explicit form

$$T^*{}_\alpha(a) = 1 - \frac{\ln(2)}{2\sigma^2}\left(a - 1 - \frac{\sigma^2}{2\ln(2)}\right)^2 \qquad (a \geq a_{\text{crit}}) \tag{4.73}$$

For $a \leq a_{\text{crit}}$ the Legendre transform yields $T^*(a) = a \cdot q_{\text{crit}}$. Thus, at a_{crit} the spectrum T^* crosses over from the parabola (4.73) to its tangent through the origin with slope q_{crit} (the other tangent through the origin is the bisector).

It should be remembered that only the positive part of this spectrum can be estimated from *one* realization of $\mathcal{M}_b$. The negative part corresponds to events so rare that they can only be observed in a large array of realizations (compare remark 4.2).

The log-normal framework allows also to calculate $F(a)$ explicitly, demonstrating which rescaling properties of the marginal distributions of the increment processes of $\mathcal{M}_b$ are captured in the multifractal spectra. Indeed, if all $\ln(M_k^n)$ are $\mathcal{N}(m, \sigma^2)$ then $-\ln(2) \cdot \alpha_k^n$ is $\mathcal{N}(m, \sigma^2/n)$. The mean value theorem of integration gives

$$\begin{aligned} P_\Omega[|\alpha_k^n - a| < \varepsilon] &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \int_{\ln(2)(-a-\varepsilon)}^{\ln(2)(-a+\varepsilon)} \exp\left(-\frac{(x-m)^2}{2\sigma^2/n}\right) \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \ln(2) \cdot 2\varepsilon \cdot \exp\left(-\frac{(-\ln(2)x_{a,n} - m)^2}{2\sigma^2/n}\right) \end{aligned}$$

with $x_{a,n} \in [a-\varepsilon, a+\varepsilon]$ for all n. Keeping only the exponential term in n and substituting $m = -\sigma^2 - \ln(2)$ we find

$$\frac{1}{n}\log_2\left(2^n P_\Omega[|\alpha_k^n - a| < \varepsilon]\right) \simeq 1 - \frac{\ln(2)}{2\sigma^2}\left(x_{a,n} - 1 - \frac{\sigma^2}{2\ln(2)}\right)^2. \tag{4.74}$$

Comparing with (4.73) we see that $T^*(a) = F(a)$, as stated in theorem 4.2. The above computation shows impressively how well adapted a multiplicative iteration with log-normal multipliers is to the multifractal analysis (or vice versa): F extracts, basically, the exponent of the Gaussian kernel.

Since the multifractal formalism holds for $\mathcal{M}_b$ these features can be measured or estimated via the *re-normalized histogram*, i.e. the grain based multifractal spectrum $f(a)$. This is a property which could be labeled with the term ergodicity. Note, however, that classical ergodic theory deals with observations along an orbit of increasing length while $f(a)$ is in terms of a sequence of orbits. ♠

4.5.5. *Beyond Dyadic Structure*

We elaborate here on generalizations of the binomial cascade.

Statistically self-similar measuresA natural generalization of the random binomial, denoted here by $\mathcal{M}_c$, is obtained by splitting intervals J_k^n iteratively into c subintervals $J_{ck}^{n+1}, ..., J_{ck+c-1}^{n+1}$ with length $|J_{ck+i}^{n+1}| = L_{ck+i}^{n+1}|J_k^n|$ and mass $\mu_c(J_{ck+i}^{n+1}) = M_{ck+i}^{n+1}\mu_c(J_{ck}^n)$. In the most simple case, one will require conservation of mass, i.e. $M_{ck}^{n+1} + ... + M_{ck+c-1}^{n+1} = 1$, but also $L_{ck}^{n+1} + ... + L_{ck+c-1}^{n+1} = 1$ which guarantees that μ_c lives everywhere. Assuming the analogous properties of ii) and iii) to hold for both, the length- as well as the mass-multipliers we find that $T_{\mathcal{M}_c}(q)$ is the unique solution of

$$\mathrm{E}\Big[(M_0)^q(L_0)^{-T(q)} + ... + (M_{c-1})^q(L_{c-1})^{-T(q)}\Big] = 1. \tag{4.75}$$

This formula of $T(q)$ can be derived rigorously by taking expectations where appropriate in the proof of [RIE 95a, Prop 14]. Doing so shows, moreover, that $T(q)$ assumes a limit in these examples.

Multifractal formalism: It is notable that the multifractal formalism 'holds' for the class of statistically self-similar measures described above in the sense of theorem 4.4 (see [ARB 96]).

However, if $L_{ck}^{n+1} + ... + L_{ck+c-1}^{n+1} = \lambda < 1$, e.g. choosing $L_k^n = (1/c')^n$ almost surely with $c' > c$, then the measure μ_c lives on a set of *fractal dimension* and its distribution function $\mathcal{M}_c(t) = \mu_c([0,t))$ is constant almost everywhere. In this case, equality in the multifractal formalism will fail: indeed, unless the scaling exponents s_k^n are modified to account for boundary effects caused by the fractal support, the partition function will be unbounded for negative q, e.g. $\tau_\alpha(q) = -\infty$ for $q < 0$ (see [RIE 95a]). As a consequence, $T_\alpha(q) = -\infty$ and (4.75) is no longer valid for $q < 0$. Interestingly, the fine spectrum $\dim(K_a)$ is still known, however, due to [ARB 96].

Stationary increments: However, an entirely different and novel way of introducing randomness in the geometry of multiplicative cascades which leads to perfectly *stationary increments* has been given recently in [MAN 02] and in [BAR 02,BAR 03,BAR 04,MUZ 02,BAC 03,CHA 02,CHA 05,RIE 07b,RIE 07a]. The description of these model is, unfortunately, beyond the scope of this paper.

Binomial in the wavelet domain: In concluding this section we should mention that, with regard to (4.61), one may choose to model directly the wavelet coefficients of a process in a multiplicative fashion in order to obtain a desired multifractal structure. First steps in this direction have been taken in [ARN 98].

4.6. Wavelet based Analysis

4.6.1. *The Binomial Revisited with Wavelets*

The deterministic envelop is the most simple of the wavelet-based spectra of μ_b to compute. Taking into account the normalization factors in (4.12) when using lemma 4.6, the calculation of (4.67) carries over to give

$$S^n{}_{w,\mu_b}(q) = 2^{nq}\,\mathrm{E}[|C_{0,0}|^q] \cdot \left(\mathrm{E}_\Omega[M_0^q] + \mathrm{E}_\Omega[M_1^q]\right)^n ,$$

and similar for $\mathcal{M}_b$. Provided $\mathrm{E}[|C_{0,0}|^q]$ is *finite* this gives immediately

$$\begin{aligned} T_{w,\mu_b}(q) + q &= T_{w,\mathcal{M}_b}(q) = T_{\alpha,\mathcal{M}_b}(q), &(4.76)\\ T^*{}_{w,\mu_b}(a-1) &= T^*{}_{w,\mathcal{M}_b}(a) = T^*{}_{\alpha,\mathcal{M}_b}(a). &(4.77)\end{aligned}$$

Imposing additional assumptions on the distributions of the multipliers we may also control $w_k^n(\mu_b)$ themselves and not only their moments. To this end, we should be able to guarantee that the wavelet coefficients don't decay too fast (compare (4.10)), i.e. the random factor (4.62) which appears in (4.61) doesn't become too small. Indeed, it is sufficient to assume that there is some $\varepsilon > 0$ such that $|C_{0,0}(\mu_b)| \geq \varepsilon$ almost surely. Then for all t, $(1/n)\log(\int \psi(t)\mu_b^{(n,k_n)}(\mathrm{d}t)) \to 0$, and with (4.61)

$$\underline{w}_{\mu_b}(t) = \liminf_{n\to\infty} -\frac{1}{n}\log_2\left(2^{n/2}|C_{n,k_n}|\right) = \underline{\alpha}_{\mathcal{M}_b}(t) - 1, \tag{4.78}$$

and similarly $\overline{w}_{\mu_b}(t) = \overline{\alpha}_{\mathcal{M}_b}(t) - 1$. Observe that this is precisely the relation we expect between the scaling exponents of a process and its (distributional) derivative – at least in nice cases — and that it is in agreement with (4.77). In summary (first observed for deterministic binomials in [BAC 93]):

COROLLARY 4.5.– Assume that μ_b is a random binomial measure satisfying i)-iii). Assume, that the random variables $|\int \psi(t)\mu_b^{(n,k)}(\mathrm{d}t)|$ resp. $|\int \psi(t)\mathcal{M}_b^{(n,k)}(t)\mathrm{d}t|$ are uniformly bounded away from 0. Then, the multifractal formalism 'holds' for the wavelet based spectra of μ_b, resp. $\mathcal{M}_b$, i.e.

$$\dim(E_a^{w,\mu_b}) \stackrel{\text{a.s.}}{=} f_{w,\mu_b}(a) \stackrel{\text{a.s.}}{=} \tau^*{}_{w,\mu_b}(a) \stackrel{\text{a.s.}}{=} T^*{}_{w,\mu_b}(a), \tag{4.79}$$

respectively

$$\dim(E_a^{w,\mathcal{M}_b}) \stackrel{\text{a.s.}}{=} f_{w,\mathcal{M}_b}(a) \stackrel{\text{a.s.}}{=} \tau^*{}_{w,\mathcal{M}_b}(a) \stackrel{\text{a.s.}}{=} T^*{}_{w,\mathcal{M}_b}(a). \tag{4.80}$$

Requiring that $|\int \psi(t)\mu_b^{(n,k)}(\mathrm{d}t)|$ resp. $|\int \psi(t)\mathcal{M}_b^{(n,k)}(t)\mathrm{d}t|$ should be bounded away from zero in order to insure (4.78), though satisfied in some simple cases, seems unrealisticly restrictive to be of practical use. A few remarks are in order, then.

First, this condition can be weakened to allow arbitrarily small values of these integrals, as long as all their negative moments exist. This can be shown by an argument using the Borel-Cantelli lemma.

Second, the condition may simplify in two ways. For i.i.d. multipliers we know that these integrals are equal in distribution to $C_{0,0}$, whence only $n = k = 0$ has to checked. Further, for the Haar wavelet and symmetrical multipliers, it becomes simply the condition that M_0 be uniformly bounded away from zero (compare (4.60)), or at least that $\mathrm{E}[|M_0 - 1/2|^q] < \infty$ for all negative q.

Third, if one drops iii) and allows the distributions of the multipliers to depend on scale (compare [RIE 99]), then $|\int \psi(t)\mu_b^{(n,k)}(\mathrm{d}t)|$ resp. $|\int \psi(t)\mathcal{M}_b^{(n,k)}(t)\mathrm{d}t|$ has to be bounded away from zero only for large n. In applications such as network traffic modeling one finds indeed that on fine scales $M_{2k}^{n+1} - M_{2k+1}^{n+1}$ is best modeled by *discrete* distributions on $[0, 1]$ with large variance, i.e. without mass around $1/2$.

Fourth, another way out is to avoid small wavelet coefficients at all in a multifractal analysis. More precisely, one would follow [BAC 93, JAF 97] and replace C_{n,k_n} in the definition of $w_{k_n}^n$ (4.12) by the maximum over certain wavelet coefficients 'close' to t. Of course, the multifractal formalism of Section 4.4 still holds. [JAF 97] gives conditions under which the spectrum $\tau^*{}_{w,\mu_b}(a)$ based on this modified w_k^n agrees with the 'Hölder ' spectrum $\dim(E_a)$ based on $h_k^n(\mathcal{M}_b)$.

4.6.2. *Multifractal Properties of the Derivative*

Corollary 4.5 establishes for the binomial what intuition suggests in general, i.e. that the multifractal spectra of processes and their derivative should be related in a simple fashion — at least for certain classes of processes. As we will show, increasing processes have this property at least for the wavelet based multifractal spectra.

However, the order of Hölder regularity in the sense of the spaces C_t^h (see lemma 4.1) might under differentiation decrease by an amount different from 1. This is particularly true in the presence of highly oscillatory behavior such as 'chirps', as the example $t^a \sin(1/t^2)$ demonstrates. In order to assess the proper space C_t^h a two-microlocalization has to be employed. For nice surveys see [JAF 95, JAF 91].

In order to establish a general result on derivatives we place ourselfes in the framework in which one cares less for a representation of a process in terms of wavelet coefficients and is interested purely in an analysis of osciallatory behavior. A typical example of an analysing mother wavelet ψ are the derivatives of the Gaussian kernel $\exp(-t^2/2)$ which were used to produce Figure 4.4.

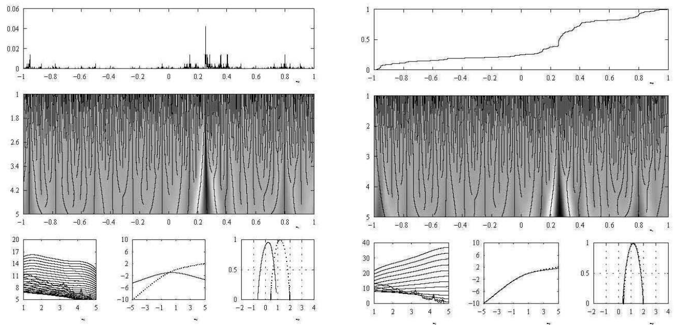

Figure 4.4. *Demonstration of the multifractal behavior of a binomial measure μ_b (left) and its distribution function $\mathcal{M}_b$ (right). On the top a numerical simulation, i.e. (4.52) on the left and $\mathcal{M}_b(k2^{-n})$ on the right for $n = 20$. In the middle the moduli of a continuous wavelet transform [DAU 92] where the second derivative of the Gaussian was taken as the analysing wavelet $\psi(t)$ for μ_b, resp. the third derivative ψ' for $\mathcal{M}_b$. The dark lines indicate the 'lines of maxima' [JAF 97, BAC 93], i.e. the locations where the modulus of $\int \psi(2^j t - s)\mu_b(\mathrm{d}t)$ has a local maximum as a function of s with j fixed. On the bottom a multifractal analysis in three steps. First, a plot of $\log S^n{}_w(q)$ against n tests for linear behavior for various q. Second, the partition function $\tau(q)$ is computed as the slopes of a least square linear fit of $\log S^n$. Finally, the Legendre transform $\tau^*(a)$ of $\tau(q)$ is computed following (4.49). Indicated with dashes in the plots of $\tau(q)$ and $\tau^*(a)$ of μ_b are the corresponding function for $\mathcal{M}_b$, providing empirical evidence for (4.76), (4.77), and (4.83).*

The idea is to use integration by parts. For a continuous measure μ on $[0,1]$ with distribution function $\mathcal{M}(t) = \mu([0,t))$ and a continuously differentiable function g this reads as

$$\begin{aligned}\int g(t)\mu(\mathrm{d}t) &= \lim_{n\to\infty}\sum_{k=0}^{2^n-1} g(k2^{-n})\big(\mathcal{M}((k+1)2^{-n}) - \mathcal{M}(k2^{-n})\big)\\ &= \lim_{n\to\infty}\sum_{k=0}^{2^n-1} \mathcal{M}(k2^{-n})\big(g((k-1)2^{-n}) - g(k2^{-n})\big)\\ &\qquad +\mathcal{M}(1)g(1-2^{-n}) - \mathcal{M}(0)g(-2^{-n})\\ &= \mathcal{M}(1)g(1) - \mathcal{M}(0)g(0) - \int \mathcal{M}(t)g'(t)\mathrm{d}t \end{aligned} \tag{4.81}$$

where we alluded to (4.53) and regrouped terms. As a matter of fact, $\mathcal{M}(0) = 0$ and $\mathcal{M}(1) = 1$. A similar computation can be done for a general, not neccessarily increasing process Y, provided it has a derivative Y', by replacing $\mu(\mathrm{d}t)$ by $Y'(t)\mathrm{d}t$. Now, setting $g(t) = 2^{n/2}\psi(2^n t - k)$ for a smooth analysing wavelet ψ one has $g'(t) = 2^{3n/2}\psi'(2^n t - k)$ and gets

$$C_{n,k}(\psi,\mu) = 2^{n/2}\psi(2^n - k) - 2^n \cdot C_{n,k}(\psi',\mathcal{M}). \tag{4.82}$$

Estimating $2^n - k_n = 2^n - \lfloor t2^n \rfloor \simeq (1-t)2^n$ and assuming exponential decay of $\psi(t)$ at infinity allows to conclude

$$\underline{w}(t)_{\psi,\mu} = -1 + \underline{w}(t)_{\psi',\mathcal{M}}, \tag{4.83}$$

and similar for $\overline{w}(t)$.

COROLLARY 4.6.–

$$f_{\psi,\mu}(a) = f_{\psi',\mathcal{M}}(a+1) \qquad\qquad {\tau^*}_{\psi,\mu}(a) = {\tau^*}_{\psi',\mathcal{M}}(a+1) \tag{4.84}$$

This is impressively demonstrated in Figure 4.4. We should note that ψ' has one more vanishing moment than ψ which is easily seen by integrating by parts. Thus, it is natural to analyze the integral of a process, here the distribution function $\mathcal{M}$ of the measure μ, using ψ' since the degree of the Taylor polynomials grow typically by 1 under integration.

NOTE 4.5.– **[Visibility of singularities and regularity of the wavelet]** It is remarkable that the Haar wavelet yields the *full* spectra of the binomial $\mathcal{M}_b$ (and also of its distributional derivative μ_b). This fact is in some discord with the folklore saying that a wavelet cannot detect degrees of regularity larger than its own. In other words, a signal will rarely be more regular than the basis elements it is composed of. To resolve the apparent paradox recall the peculiar property of multiplicative measures which is to have constant Taylor polynomials. So, it will reveal its scaling structure to any analysing wavelet with $\int \psi = 0$. No higher regularity, i.e. $\int t^k \psi(t) \mathrm{d}t = 0$ is required. The correct reading of the folklore is indeed, that wavelets are only *guaranteed* to detect singularities smaller than their own regularity.

4.7. Self-similarity and LRD

The statistical self-similarity as expressed in (4.1) makes fBm, or rather its increment process a paradigm of *long range dependence* (LRD). To be more explicit let δ denote some fixed lag and define *fractional Gaussian noise* (fGn) as

$$G(k) := B_H((k+1)\delta) - B_H(k\delta). \tag{4.85}$$

Possessing the LRD property means that the auto-correlation $r_G(k) := \mathrm{E}_\Omega[G(n+k)G(n)]$ decays so slowly that $\sum_k r_G(k) = \infty$. The presence of such strong dependence bears an important consequence on the aggregated processes

$$G^{(m)}(k) := \frac{1}{m} \sum_{i=km}^{(k+1)m-1} G(i). \tag{4.86}$$

They have a much higher variance, and variability, than would be the case for a short range dependent process. Indeed, if X is a process with i.i.d. values $X(k)$, then $X^{(m)}(k)$ has variance $(1/m^2)\,\mathrm{var}(X_0 + ... + X_{m-1}) = (1/m)\,\mathrm{var}(X)$. For G we find, due to (4.1) and $B_H(0) = 0$,

$$\mathrm{var}(G^{(m)}(0)) = \mathrm{var}\left(\frac{1}{m} B_H(m\delta)\right) = \mathrm{var}\left(\frac{m^H}{m} B_H(\delta)\right) = m^{2H-2}\,\mathrm{var}\left(B_H(\delta)\right). \tag{4.87}$$

For $H > 1/2$ this expression decays indeed much slower than $1/m$. As is shown in [COX 84] $\mathrm{var}(X^{(m)}) \simeq m^{2H-2}$ is equivalent to $r_X(k) \simeq k^{2H-2}$ and so, $G(k)$ is indeed LRD for $H > 1/2$.

Let us demonstrate with fGn how to relate LRD with multifractal analysis using only that it is a zero-mean processes, not (4.1). To this end let $\delta = 2^{-n}$ denote the finest resolution we will consider, and let 1 be the largest. For $m = 2^i$ $(0 \leq i \leq n)$ the process $mG^{(m)}(k)$ becomes simply $B_H((k+1)m\delta) - B_H(km\delta) = B_H((k+1)2^{i-n}) - B_H(k2^{i-n})$. But the second moment of this expression —which is also the variance— is exactly what determines $T_\alpha(2)$. More precisely, using stationarity of G and substituting $m = 2^i$, we get

$$2^{-(n-i)T_\alpha(2)} \simeq \mathrm{E}_\Omega \left[S^{n-i}{}_\alpha(2) \right] = \sum_{k=0}^{2^{n-i}-1} \mathrm{E}_\Omega \left[|mG^{(m)}(k)|^2 \right] = 2^{n-i} 2^{2i} \operatorname{var}\left(G^{(2^i)} \right). \tag{4.88}$$

This should be compared with the definition of the LRD-parameter H via

$$\operatorname{var}(G^{(m)}) \simeq m^{2H-2} \qquad \text{or} \qquad \operatorname{var}(G^{(2^i)}) \simeq 2^{i(2H-2)}. \tag{4.89}$$

At this point a conceptual difficulty arises. Multifractal analysis is formulated in the limit of small scales ($i \to -\infty$) while LRD is a property at large scales ($i \to \infty$). Thus, the two exponents H and $T_\alpha(2)$ can in theory only be related when assuming that the scaling they represent is actually exact at all scales, and not only asymptotically.

In any real world application, however, one will determine both, H and $T_\alpha(2)$, by finding a *scaling region* $\underline{i} \leq i \leq \overline{i}$ in which (4.88) and (4.89) hold up to satisfactory precision. Comparing the two scaling laws in i yields $T_\alpha(2) + 1 - 2 = 2H - 2$, or

$$H = \frac{T_\alpha(2) + 1}{2}. \tag{4.90}$$

This formula expresses most pointedly, how *multifractal analysis goes beyond second order statistics*: with $T(q)$ one captures the scaling of *all* moments. The relation (4.90), here derived for zero-mean processes, can be put on more solid grounds using wavelet estimators of the LRD parameter [ABR 95] which are more robust than the ones through variance. The same formula (4.90) reappears also for certain multifractals (see (4.100)).

In this context it is worthwhile pointing forward to (4.96) from which we conclude that $T_{B_H}(q) = qH - 1$ if $q > -1$. The fact tonote here is that fBm requires indeed only one parameter to capture its scaling while multifractal scaling, in principle, is described by an array of parameters $T(q)$.

4.8. Multifractal Processes

The most prominent examples where one finds coinciding, strictly concave multifractal spectra are the distribution functions of *cascade* measures [MAN 74, KAH 76, CAW 92, FAL 94, ARB 96, OLS 94, HOL 92, RIE 95a, RIE 97b, PES 97] for which $\dim(K_a)$ and $T^*(a)$ are equal and have the form of a $\cap$ (see Figure 4.3 and also 4.5 (e)). These cascades are constructed through some multiplicative iteration scheme such as the Binomial cascade, which is presented in detail in the paper with special emphasis on its wavelet decomposition. Having positive increments, this class of processes is, however, sometimes too restrictive. fBm, as noted, has the disadvantage of a poor multifractal structure and does not contribute to a larger pool of stochastic processes with multifractal characteristics.

It is also notable that the first 'natural', truly multifractal stochastic process to be identified was Lévy motion [JAF 99]. This example is particularly appealing since scaling is not injected into the model by an iterative construction (this is what we mean by the term natural). However, its spectrum is, though it shows a non-trivial range of scaling exponents $h(t)$, degenerated in the sense that it is linear.

4.8.1. *Construction and Simulation*

With the formalism presented here, the stage is set for constructing and studying new classes of truly multifractional processes. The idea, to speak in Mandelbrot's own words, is inevitable after the fact. The ingredients are simple: a multifractal 'time warp', i.e. an increasing function or process $\mathcal{M}(t)$ for which the multifractal formalism is known to hold, and a function or process V with strong mono-fractal scaling properties such as *fractional Brownian motion* (fBm), a Weierstrass process or self-similar martingales such as Lévy motion. One then forms the compound process

$$\mathcal{V}(t) := V(\mathcal{M}(t)). \tag{4.91}$$

To build an intuition let us recall the method of *midpoint displacement* which can be used to define simple Brownian motion $B_{1/2}$ which we will also call *Wiener motion* (WM) for a clear distinction from fBm. This method constructs $B_{1/2}$ iteratively at dyadic points. Having constructed $B_{1/2}(k2^{-n})$ and $B_{1/2}((k+1)2^{-n})$ one defines $B_{1/2}((2k+1)2^{-n-1})$ as $(B_{1/2}(k2^{-n}) + B_{1/2}((k+1)2^{-n}))/2 + X_{k,n}$. The off-sets $X_{k,n}$ are independent zero mean Gaussian variables with variance such as to satisfy (4.1) with $H = 1/2$. Thus the name of the method. One way to obtain *Wiener motion in multifractal time* WM(MF) is then to keep the off-set variables $X_{k,n}$ as they are but to apply them at the time instances $t_{k,n}$ defined by $t_{k,n} = \mathcal{M}^{-1}(k2^{-n})$, i.e. $\mathcal{M}(t_{k,n}) = k2^{-n}$:

$$\mathcal{B}_{1/2}(t_{2k+1,n+1}) := \frac{\mathcal{B}_{1/2}(t_{k,n}) + \mathcal{B}_{1/2}(t_{k+1,n})}{2} + X_{k,n}. \tag{4.92}$$

This amounts to a *randomly located random displacement*, the location being determined by $\mathcal{M}$. Indeed, (4.91) is nothing but a time warp.

An alternative construction of 'warped Wiener motion' WM(MF) which yields an equally spaced sampling as opposed to the samples $\mathcal{B}_{1/2}(t_{k,n})$ provided by (4.92) is desirable. To this end, note first that the increments of WM(MF) become independent Gaussians once the path of $\mathcal{M}(t)$ is realized. To be more precise, fix n and let

$$\mathcal{G}(k) := \mathcal{B}((k+1)2^{-n}) - \mathcal{B}(k2^{-n}) = B_{1/2}(\mathcal{M}(k+1)2^{-n})) - B_{1/2}(\mathcal{M}(k2^{-n})).$$

(4.93)

For a sample path of $\mathcal{G}$ one starts by producing first the random variables $\mathcal{M}(k2^{-n})$. Once this is done, the $\mathcal{G}(k)$ simply are independent zero-mean Gaussian variables with variance $|\mathcal{M}(k+1)2^{-n})) - \mathcal{M}(k2^{-n})|$. This procedure has been used in Figure 4.5.

4.8.2. *Global analysis*

To compute the multifractal envelop $T(q)$ we need only to know that V is an H-sssi process, i.e. that the increment $V(t+u) - V(t)$ is equal in distribution to $u^H V(1)$ (compare (4.1)). Assuming independence between V and $\mathcal{M}$ a simple calculation reads as

$$\begin{aligned} &\mathrm{E}_\Omega \sum_{k=0}^{2^n-1} |\mathcal{V}((k+1)2^{-n}) - \mathcal{V}(k2^{-n})|^q \\ &= \sum_{k=0}^{2^n-1} \mathrm{E}\,\mathrm{E}\left[|V(\mathcal{M}((k+1)2^{-n})) - V(\mathcal{M}(k2^{-n}))|^q \,\middle|\, \mathcal{M}(k2^{-n}), \mathcal{M}((k+1)2^{-n})\right] \\ &= \sum_{k=0}^{2^n-1} \mathrm{E}\left[|\mathcal{M}((k+1)2^{-n}) - \mathcal{M}(k2^{-n})|^{qH}\right] \mathrm{E}\left[|V(1)|^q\right]. \end{aligned} \tag{4.94}$$

With little more effort the increments $|\mathcal{V}((k+1)2^{-n}) - \mathcal{V}(k2^{-n})|$ can be replaced by suprema, i.e. by $2^{-nh_k^n}$, or even certain wavelet coefficients under appropriate assumptions (see [RIE 88]). It follows that

$$\text{Warped } H\text{-sssi:} \qquad T_{\mathcal{V}}(q) = \begin{cases} T_{\mathcal{M}}(qH) & \text{if } \mathrm{E}_\Omega\left[|\sup_{0\le t\le 1} V(t)|^q\right] < \infty \\ -\infty & \text{else.} \end{cases} \tag{4.95}$$

Simple H-sssi process: When choosing the deterministic warp time $\mathcal{M}(t) = t$ we have $T_{\mathcal{M}}(q) = q - 1$ since $S^n{}_{\mathcal{M}}(q) = 2^n \cdot 2^{-nq}$ for all n. Also, $\mathcal{V} = V$. We obtain $T_{\mathcal{M}}(qH) = qH - 1$ which has to be inserted into (4.95) to obtain

$$\text{Simple } H\text{-sssi:} \qquad T_V(q) = \begin{cases} qH - 1 & \text{if } \mathrm{E}_\Omega\left[|\sup_{0\le t\le 1} V(t)|^q\right] < \infty \\ -\infty & \text{else.} \end{cases} \tag{4.96}$$

4.8.3. *Local analysis of warped fBm*

Let us now turn to the special case where V is fBm. Then, we use the term FB(MF) to abbreviate *fractional Brownian motion in multifractal time*: $\mathcal{B}(t) = B_H(\mathcal{M}(t))$. First, to obtain an intuition on what to expect from the spectra of $\mathcal{B}$ let us note that the moments appearing in (4.95) are finite for all $q > -1$ (see [RIE 88, Lem 7.4] for a detailed discussion). Applying the Legendre transform yields easily that

$$T^*{}_{\mathcal{B}}(a) = \inf_q (qa - T_{\mathcal{M}}(qH)) = T^*_{\mathcal{M}}(a/H). \tag{4.97}$$

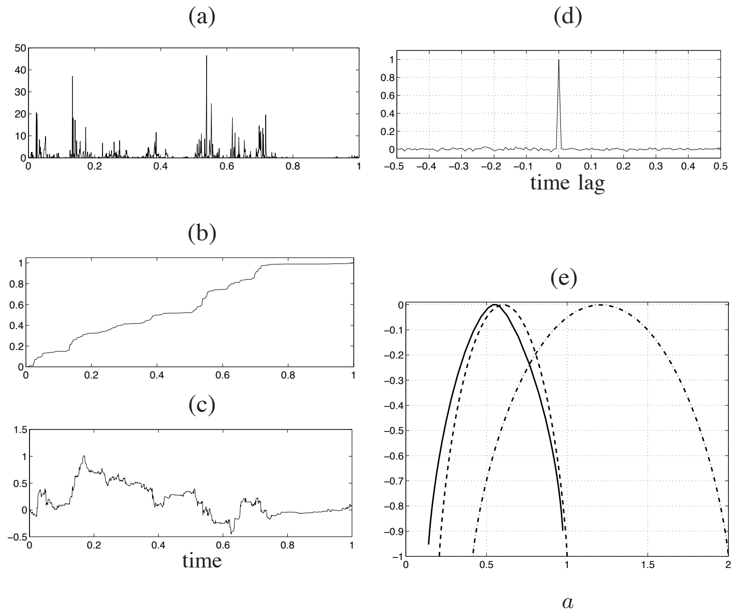

Figure 4.5. *Left: Simulation of Brownian motion in binomial time (a) Sampling of $\mathcal{M}_b((k+1)2^{-n}) - \mathcal{M}_b(k2^{-n})$ ($k = 0, ..., 2^n - 1$), indicating distortion of dyadic time intervals (b) $\mathcal{M}_b((k2^{-n}))$: the time warp (c) Brownian motion warped with (b): $\mathcal{B}(k2^{-n}) = B_{1/2}(\mathcal{M}_b(k2^{-n}))$*
Right: Estimation of $\dim E_a^{\mathcal{B}}$ via $\tau^*{}_{w,\mathcal{B}}$ (d) Empirical correlation of the Haar wavelet coefficients. (e) Dot-dashed: $T^*{}_{\mathcal{M}_b}$ (from theory), dashed: $T^*{}_{\mathcal{B}}(a) = T^*{}_{\mathcal{M}_b}(a/H)$ Solid: the estimator $\tau^*{}_{w,\mathcal{B}}$ obtained from (c). (Reproduced from [GON 99].)

Second, towards the local analysis we recall the uniform and strict Hölder continuity of the paths of fBm[5] which reads roughly as

$$\sup_{|u|\le\delta} |\mathcal{B}(t+u)-\mathcal{B}(t)| = \sup_{|u|\le\delta} |B_H(\mathcal{M}(t+u))-B_H(\mathcal{M}(t))| \simeq \sup_{|u|\le\delta} |\mathcal{M}(t+u)-\mathcal{M}(t)|^H.$$

This is the key to conclude that B_H simply squeezes the Hölder regularity exponents by a factor H. Thus, $\underline{h}_{\mathcal{B}}(t) = H \cdot \underline{h}_{\mathcal{M}}(t)$, etc, and

$$K^{\mathcal{M}}_{a/H} = K^{\mathcal{B}}_a,$$

and, consequently, analogous to (4.97),

$$d_{\mathcal{B}}(a) = d_{\mathcal{M}}(a/H).$$

Figure 4.5 (d)-(e) displays an estimation of $d_{\mathcal{B}}(a)$ using wavelets which agrees very closely with the form $d_{\mathcal{M}}(a/H)$ predicted by theory. (For statistics on this estimator see [GON 99, GON 98].)

In conclusion:

COROLLARY 4.7.– **[Fractional Brownian Motion in Multifractal Time]**
Let B_H denote fBm of Hurst parameter H. Let $\mathcal{M}(t)$ be of almost surely continuous paths and independent of B_H. Then, **the multifractal warp formalism**

$$\dim(K^{\mathcal{B}}_a) = f_{\mathcal{B}}(a) = \tau^*{}_{\mathcal{B}}(a) = T^*{}_{\mathcal{B}}(a) = T^*_{\mathcal{M}}(a/H) \tag{4.98}$$

holds for $\mathcal{B}(t) = B_H(\mathcal{M}(t))$ for any a such that the multifractal formalism holds for $\mathcal{M}$ at a/H, i.e., for which $\dim(K^{\mathcal{M}}_{a/H}) = T^*{}_{\mathcal{M}}(a/H)$. This means that the local, or fine, multifractal structure of $\mathcal{B}$ captured in $\dim(K^{\mathcal{B}}_a)$ on the left can be estimated through grain based, simpler and numerically more robust spectra on the right side, such as $\tau^*{}_{\mathcal{B}}(a)$ (compare Figure 4.5 (e)).

The 'warp formula' (4.98) is appealing since it allows to *separate* the LRD parameter of fBm and the multifractal spectrum of the time change $\mathcal{M}$. Indeed, provided that $\mathcal{M}$ is almost surely increasing one has $T_{\mathcal{M}}(1) = 0$ since $S^n(0) = \mathcal{M}(1)$ for all n. Thus, $T_{\mathcal{B}}(1/H) = 0$ exposes the value of H. Alternatively, the tangent at $T^*{}_{\mathcal{B}}$ through the origin has slope $1/H$. Once H is known $T^*{}_{\mathcal{M}}$ follows easily from $T^*{}_{\mathcal{B}}$.

Simple fBm: When choosing the deterministic warp time $\mathcal{M}(t) = t$ we have $\mathcal{B} = B_H$ and $T_{\mathcal{M}}(q) = q - 1$ since $S^n{}_{\mathcal{M}}(q) = 2^n \cdot 2^{-nq}$ for all n. We conclude that

$$T_{B_H}(q) = qH - 1 \tag{4.99}$$

for all $q > -1$. This confirms (4.90) for fGn. With (4.98) it shows that all spectra of fBm consist of the one point $(H, 1)$ only, making the mono-fractal character of this process most explicit.

5. For a precise statement see Adler [ADL 81] or [RIE 88, Thm 7.4].

4.8.4. *LRD and estimation of warped fBm*

Let $\mathcal{G}(k) := \mathcal{B}((k+1)2^{-n}) - \mathcal{B}(k2^{-n})$ be fGn in multifractal time (see (4.93) for the case $H = 1/2$). Calculating auto-correlations explicitly shows that $\mathcal{G}$ is *second order stationary* under mild conditions with

$$H_{\mathcal{G}} = \frac{T_{\mathcal{M}}(2H)+1}{2}. \tag{4.100}$$

Let us discuss some special cases. For a continuous, increasing warp time $\mathcal{M}$, e.g., we have always $T_{\mathcal{M}}(0) = -1$ and $T_{\mathcal{M}}(1) = 0$. Exploiting the concave shape of $T_{\mathcal{M}}$ we find that $H < H_{\mathcal{G}} < 1/2$ for $0 < H < 1/2$, and $1/2 < H_{\mathcal{G}} < H$ for $1/2 < H < 1$. Thus, multifractal warping can not create LRD and it seems to weaken the dependence as measured through second order statistics.

Especially in the case of $H = 1/2$ ('white noise in multifractal time') $\mathcal{G}(k)$ becomes *uncorrelated*. This follows from (4.100). Notably, this is a different statement than the observation that the $\mathcal{G}(k)$ are *independent conditioned* on $\mathcal{M}$ (compare Section 4.8.1). As a particular consequence, wavelet coefficients will decorrelate fast for the entire process $\mathcal{G}$, not only when conditioning on $\mathcal{M}$ (compare Figure 4.5 (d)). This is favorable for estimation purposes as it reduces the error variance.

Of larger importance, however, is the warning that the vanishing correlations should not make one conclude on independence of $\mathcal{G}(k)$. After all, $\mathcal{G}$ becomes Gaussian only when conditioning on knowing $\mathcal{M}$. A strong, higher order dependence in $\mathcal{G}$ is hidden in the dependence of the increments of $\mathcal{M}$ which determine the variance of $\mathcal{G}(k)$ as in (4.93). Indeed, Figure 4.5 (c) shows clear phases of monotony of $\mathcal{B}$ indicating positive dependence in its increments $\mathcal{G}$, despite vanishing correlations. Mandelbrot calls this the 'blind spot of spectral analysis'.

4.9. Bibliography

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