An introduction to multifractal measurements

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Multifractals – why use them?

We suspect that a dataset is self-similar across many scales. We want to verify and quantify this similarity so as to yield:

- **a more compact, simplified representation;**
- **better predictions via pattern characterization;**
- **signal separation.**

We want to estimate the scaling properties of a broad statistical distribution.

- **We describe how algorithms scale according to deterministic “worst case” time complexity, ex $O(N^2)$. However, the data we operate on is often stochastic with a distribution that varies in time and space.**

Multifractals – why use them?

We are working with a process known to be multiplicative, and we want to understand how the system will behave after some number of iterations.

We want to understand the extent to which an event is "rare" in relation to (or with respect to) multiple scales.

Multifractals – why use them?

Multifractal analysis concerns the scaling behavior of a distribution of measures in a geometrical and statistical fashion.

- **Signal self-similarity**
- **Broad probability distributions**
- **Multiplicative processes**
- **Rare events**

Outline

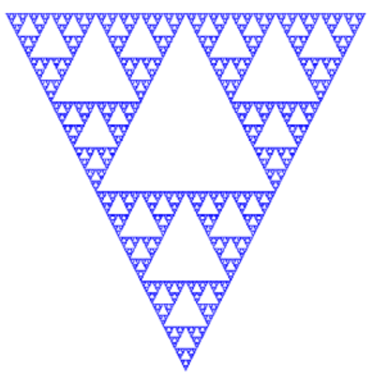
1. Theoretical background
   - Fractals and fractal dimension
   - Multifractal measures

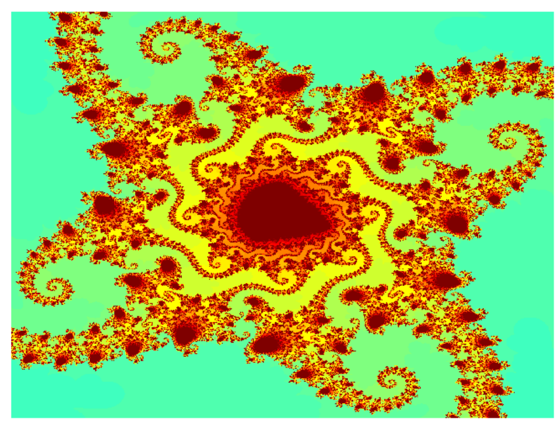
2. Computational techniques
   - Method of moments
     - Time series
     - Planar data
   - Wavelet Transform Modulus Maxima

3. Applications of multifractal measures

A fractal is loosely defined as a geometric shape having symmetry of scale

- A fractal is a set, which is (by definition) a collection of objects, in this case points.
- Each point is a member of the set, or it is not. The measure on each point is therefore drawn from {0,1}

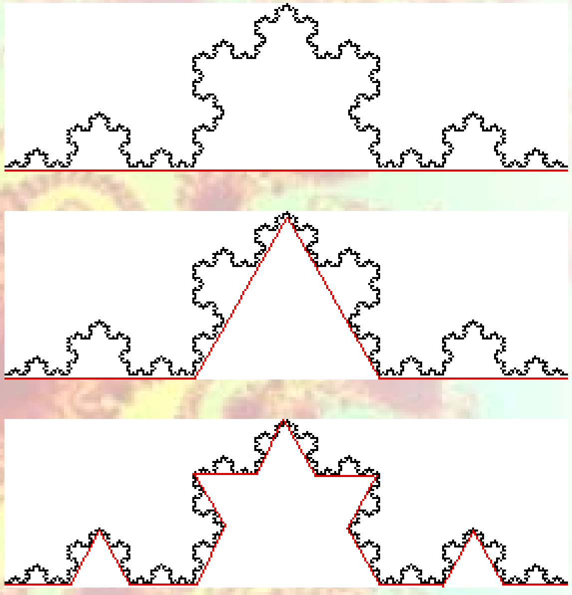




Fractal objects can't be measured by usual ways

How can we measure the length of the Koch curve?

n	Segment length	Number of segments	L_n	L_{1m}
0	1	1	1	1 m
1	1/3	4	4/3	1.33 m
2	$1/3^2$	4^2	$(4/3)^2$	1.77 m
3	$1/3^3$	4^3	$(4/3)^3$	2.370 m
…	…	…	…	…
24	$1/3^{24}$	4^{24}	$(4/3)^{24}$	996 m
…	…	…	…	…
128	$1/3^{128}$	4^{128}	$(4/3)^{128}$	9.82×10^{15} m
n	$1/3^n$	4^n	$(4/3)^n$	∞



http://classes.yale.edu/fractals/

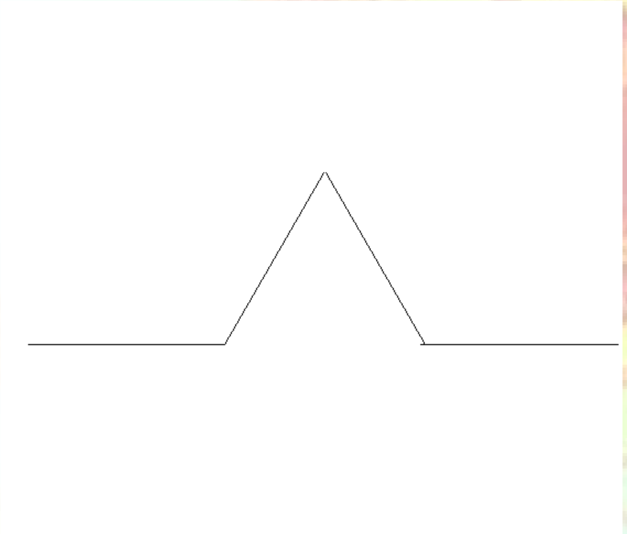
The concept of dimension is more appropriate to characterize fractal objects

There are three categories of fractal constructions

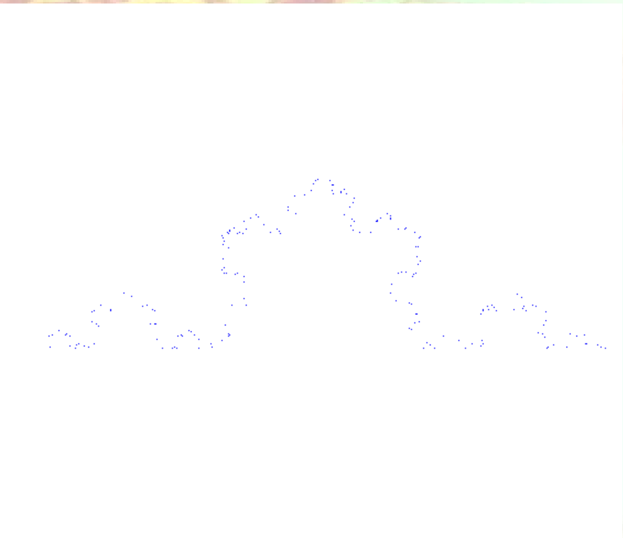
- Iterated function systems
- Escape time fractals
- Random walks

Iterated function systems are formed from the repetitive action of a rule

Deterministic



Probabilistic



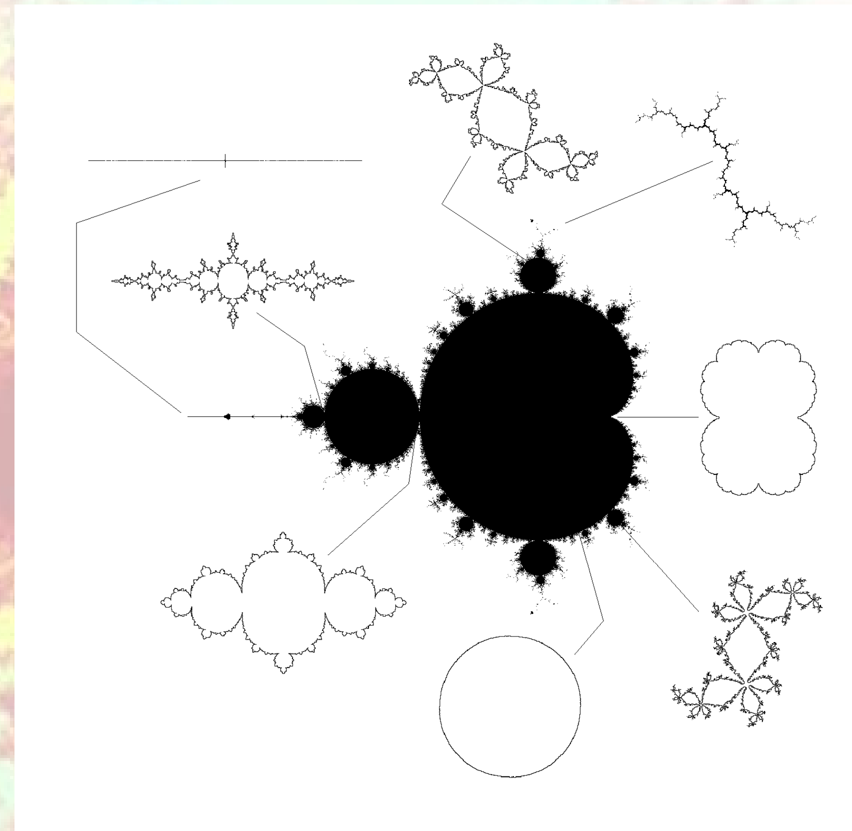
Escape-time fractals are formed from a recurrence relation at every point in space

The Mandelbrot set is a well-known example.

It is calculated by the iterative rule, with $z_0 = 0$ and c is on the complex plane

$$z_{n+1} = z_n^2 + c$$

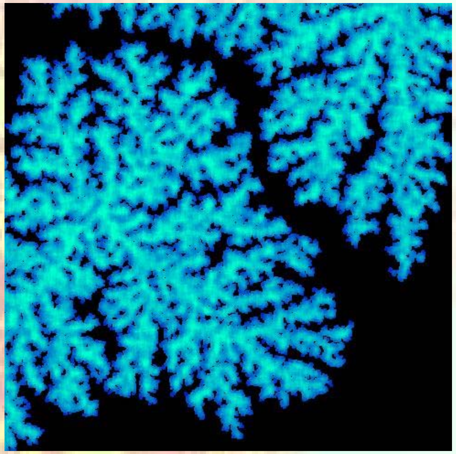
If z diverges, c in not in the Mandelbrot set



From http://hypertextbook.com/chaos/diagrams/con.2.02.gif

Random walks are governed by stochastic processes

- Brownian motion
- Levy flight
- Fractal landscape
- Brownian tree
- Diffusion-limited aggregation



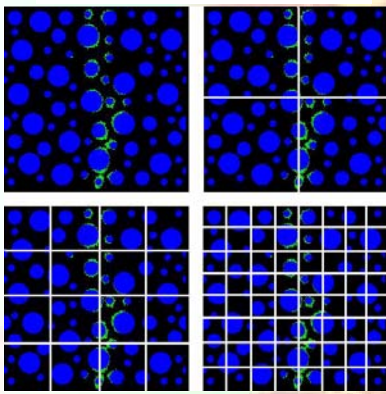
http://en.wikipedia.org/wiki/Brownian_tree

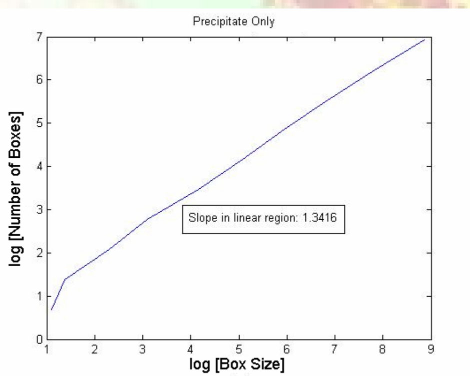
Fractal dimensions measure how a set fills space

- **Box Counting Dimension**

Measure is the number of covers that contain a point of the set

$$\lim_{r \to 0} \frac{-\log M(r)}{\log r}$$





- **Information Dimension**

Measure is the sum of the probabilities of finding a point of the set in the k*th* cover

$$\lim_{r \to 0} \frac{-\sum p_k \log p_k}{\log r}$$

where $p_k \approx \frac{N_k}{N}$ and N_k is the number of points in the kth cover, and the summation is from $k = 1$ to $M(r)$

- **Correlation Dimension**

Measure is the number of pairs of points whose distance apart is less than r

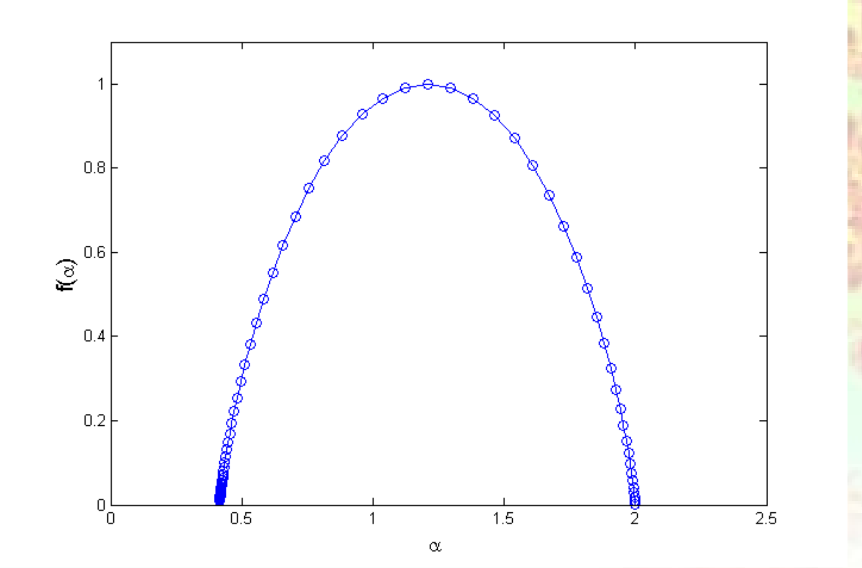
$$\lim_{r \to 0} \frac{-\log C(r)}{\log r} \text{ where } C(r) = \frac{1}{N^2} \sum_{i \neq j} \theta(r - |X_i - X_j|)$$

A Multifractal is a set composed of a multitude of interwoven subsets, each of differing fractal dimension.

Each point is associated with a measure, which is typically a non-negative (often normalized) real value.

The data in a multifractal set may not “appear” to be self-similar, because so many individual fractal subsets are present.

A multifractal set is typically represented by the spectrum of these scaling dimensions



The *f(α)* **singularity spectrum** and the **generalized fractal dimensions** are common multifractal measures

The *f(α)* singularity spectrum:

For a usual fractal set on which a measure μ is defined, the dimension D related the increase of μ with the size ε of a ball $B_x(\varepsilon)$ centered at *x:*

$$\mu(B_x(\varepsilon)) = \int_{B_x(\varepsilon)} d\mu(y) \sim \varepsilon^{-D}$$

But the measure can display different scaling from point to point → multifractal measure

The local scaling behavior becomes important

$$\mu(B_x(\varepsilon)) \sim \varepsilon^{\alpha(x)}$$

Where $\alpha(x)$ represents the singularity strength at point *x*

To look at the local scaling, we

1. cover the support of the measure with boxes of radius ε
2. Look at the scaling in each box
3. Count the number of boxes that scale like ε^{α} for a given α to get

$$N_{\alpha}(\varepsilon) \sim \varepsilon^{-f(\alpha)}$$

$N_{\alpha}(\varepsilon)$ can be seen as a histogram

$f(\alpha)$ describes how $N_{\alpha}(\varepsilon)$ varies when $\varepsilon \to 0$

$f(\alpha)$ is defined as the fractal dimension of the set of all points x such that $\alpha(x) = \alpha$

For **time series**, the singularities represent the cusps and steps in the data

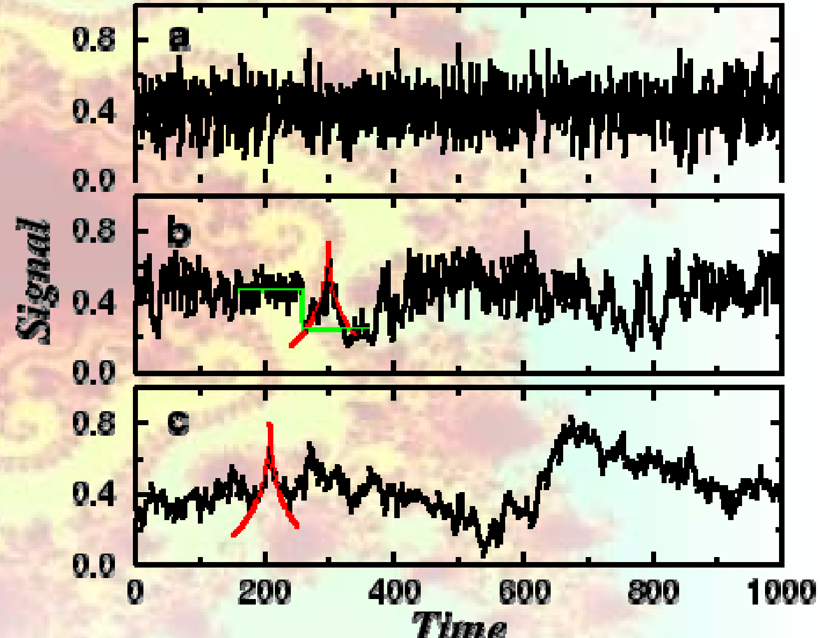
A typical Taylor expansion can't represent the series locally, and we must write

$$f(t) = a_0 + a_1(t - t_i) + a_2(t - t_i)^2 + a_3(t - t_i)^3 + \ldots + a_\alpha (t - t_i)^{\alpha_i}$$

or alternatively

$$\left| f(t) - P_n(t - t_i) \right| \le a_\alpha \left| t - t_i \right|^{\alpha_i}$$

and α_i is the largest exponent such that there exist a polynomial $P_n(x)$ satisfying the inequality



From http://www.physionet.org/tutorials/multifractal/behavior.htm

The **generalized fractal dimensions** provide another representation of the multifractal spectrum

The dimensions D_q correspond to scaling exponents for the q^{th} moments of the measure μ

As before, the support of the measure is covered with boxes $B_i(\varepsilon)$ of size ε, and the partition function is defined as

$$Z(q,\varepsilon) = \sum_{i=1}^{N(\varepsilon)} \mu_i^q(\varepsilon)$$

In the limit $\varepsilon \to 0^+$, $Z(q,\varepsilon)$ behaves as a power law

$$Z(q,\varepsilon) \sim \varepsilon^{\tau(q)}$$

And the spectrum is obtained from the relation

$$D_q = \tau(q)/(q-1)$$

Some **well known dimensions** commonly cited in the literature can be found as points on the multifractal spectrum

$$D_q = \lim_{\varepsilon \to 0} \left(\frac{1}{q-1} \frac{\ln Z(q,\varepsilon)}{\ln \varepsilon} \right)$$

When q = 0; the definition becomes the capacity dimension
When q = 1; the definition becomes the information dimension
When q = 2; the definition becomes the correlation dimension

Provides a selective characterization of the nonhomogeneity of the measure, positive q's accentuating the densest regions and negative q's the smoothest regions.

These two spectra can be related

For a scale ε, we postulate a distribution of α's in the form $\rho(\alpha)\varepsilon^{-f(\alpha)}$ and introduce in the partition function

$$Z(q,\varepsilon) \cong \int \rho(\alpha)\varepsilon^{q\alpha - f(\alpha)} d\alpha$$

In the limit $\varepsilon \to 0^+$, this sum is dominated by the term $\varepsilon^{\min(q\alpha - f(\alpha))}$

$$\tau(q) = \min_{\alpha}(q\alpha - f(\alpha))$$

The $\tau(q)$ spectrum, thus the D_q spectrum, is obtained by the Legendre transform of the $f(\alpha)$ singularity spectrum

Relation between $\tau(q)$ and $f(\alpha)$

Knowing that

$$\tau(q) = \min_{\alpha}(q\alpha - f(\alpha))$$

We have the extremal conditions

$$\left.\frac{d}{d\alpha}[q\alpha - f(\alpha)]\right|_{\alpha=\alpha(q)} = 0 \qquad \left.\frac{d^2}{d\alpha^2}[q\alpha - f(\alpha)]\right|_{\alpha=\alpha(q)} > 0$$

$$\begin{aligned} \tau(q) &= q\alpha - f(\alpha) \\ \frac{df}{d\alpha} &= q \\ \frac{d^2 f(\alpha)}{d\alpha^2} &< 0 \end{aligned}$$

Example: binomial cascade

Consider the following multiplicative process on the unit interval S = [0,1]:

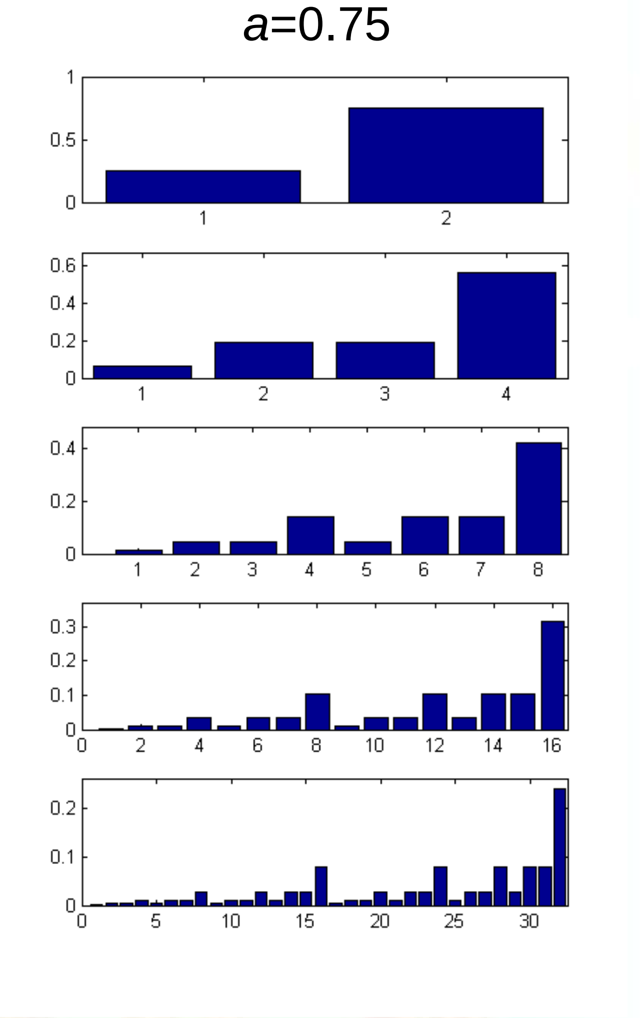
Split S in 2 parts of equal length $d=2^{-1}$

Left half is given a fraction a of the population

Right half is given a fraction $1-a$ of the population

Repeat for $d=2^{-2}$

…



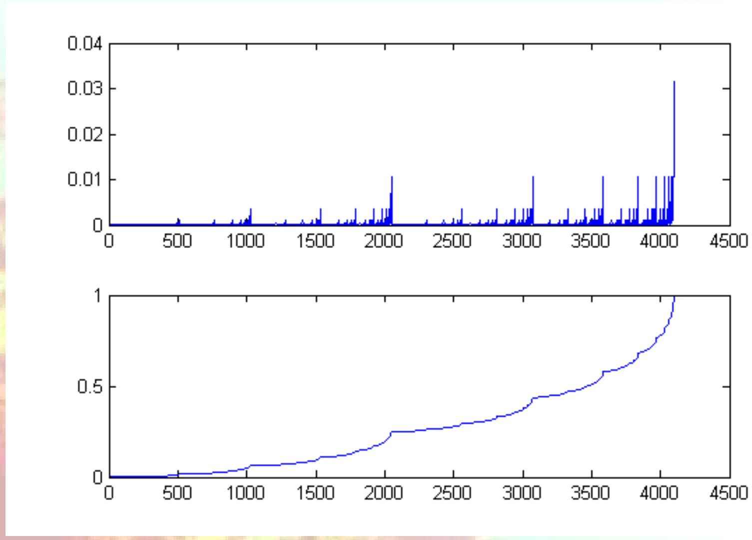
Example: binomial cascade

$M_1 = \mu_0, \mu_1$

$M_2 = \mu_0\mu_0, \mu_0\mu_1, \mu_1\mu_0, \mu_1\mu_1$

$M_3 = \mu_0\mu_0\mu_0, \mu_0\mu_0\mu_1, \mu_0\mu_1\mu_0, \mu_0\mu_1\mu_1, \mu_1\mu_0\mu_0,$
$\mu_1\mu_0\mu_1, \mu_1\mu_1\mu_0, \mu_1\mu_1\mu_1$

…



Binomial cascade has the same structure as binary numbers.

The cascade can be represented by

$$x_k = a^{n(k-1)}(1-a)^{n_{\max}-n(k-1)}$$

Where n is the number of 1 in binary representation and the length of the series is $N=2^{nmax}$

Decimal	Binary	Bit count
1	1	1
2	10	1
3	11	2
4	100	1
5	101	2
6	110	2

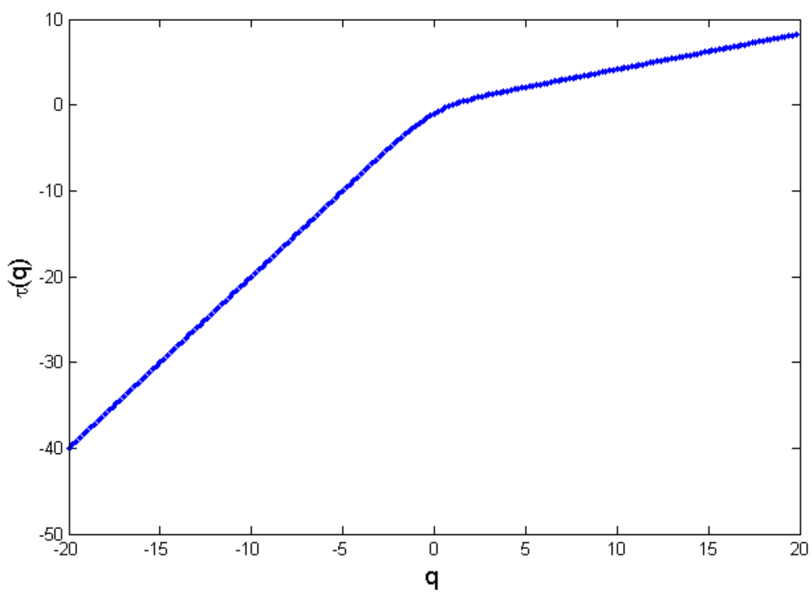
The sequence of mass exponents

Scaling of the partition function: $Z_q(\varepsilon) = \sum_i \mu_i^q \sim \varepsilon^{\tau(q)}$

The mass exponent is given by: $\tau(q) = \lim_{\varepsilon \to 0} \frac{\ln Z_q(\varepsilon)}{\ln \varepsilon}$

The partition function is equivalent to a weighted box counting

The exponents describe how the measures (probabilities) μ_i scale with ε



$q \to -\infty$: $\tau(q) \to q\alpha_{\min}$

$q = 0$: $\tau(q) = -D$

$q = 1$: $\tau(q) = 0$

$q \to \infty$: $\tau(q) \to q\alpha_{max}$

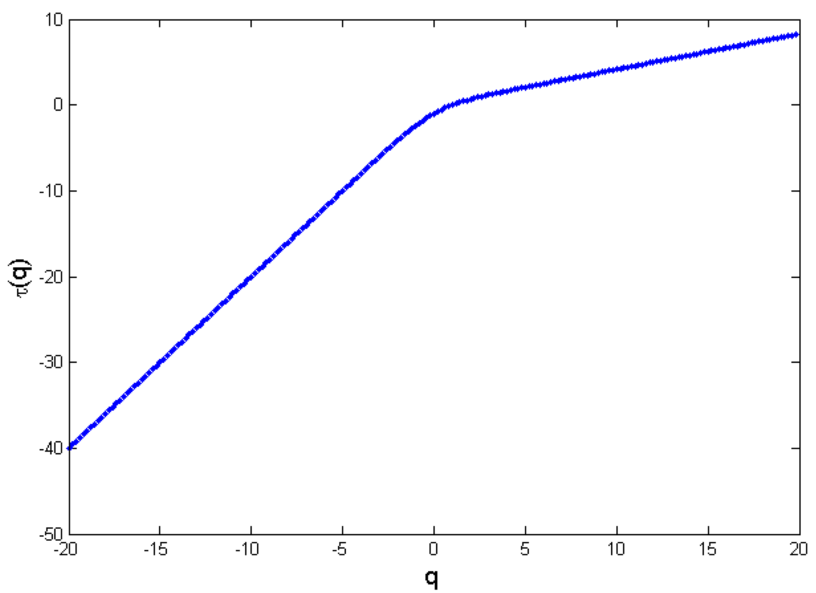
The sequence of mass exponents

q=0 $\rightarrow \mu_i^{q=0}$ = 1 and $Z_0(\varepsilon)$ is simply the number of boxes needed to cover the set

$\tau(0)$ = -D equals the box counting dimension

Large positive q's favor the contribution of cells with larger μ_i

Large negative q's favor the contribution of cells with lower μ_i



$q \rightarrow -\infty : \tau(q) \rightarrow q\alpha_{\text{min}}$

$q = 0 : \tau(q) = -D$

$q = 1 : \tau(q) = 0$

$q \rightarrow \infty : \tau(q) \rightarrow q\alpha_{max}$

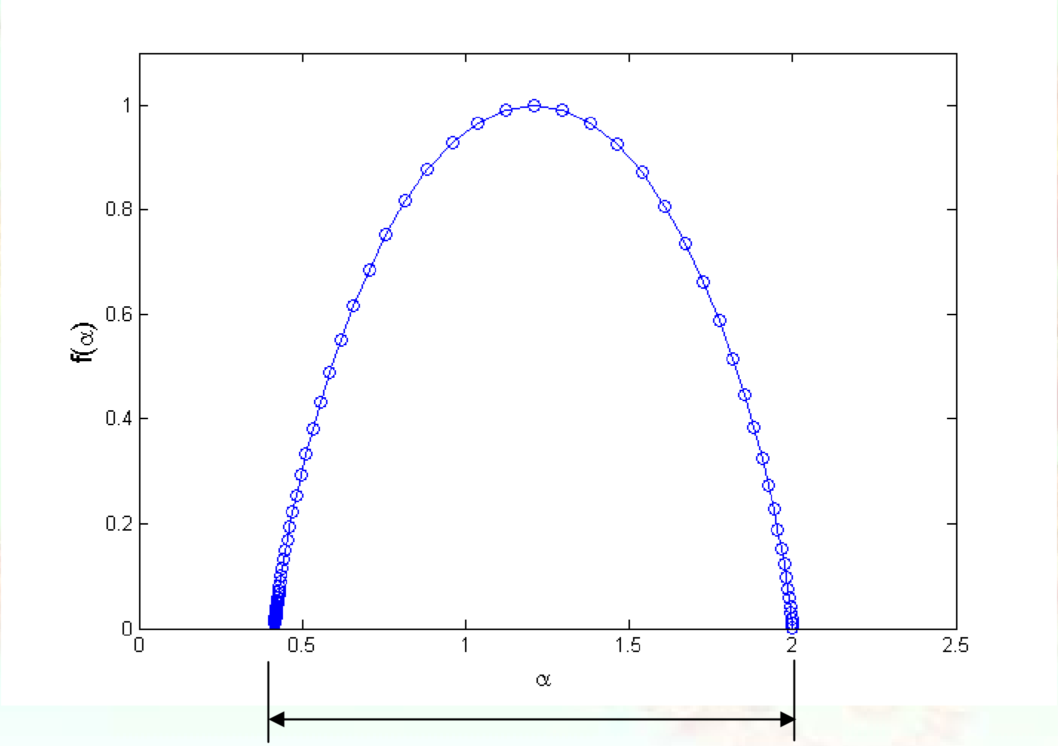
The Holder exponent

α : singularity strength, (local) Holder exponent

The span of α describes the different subsets of singularities

Monofractal: a single point

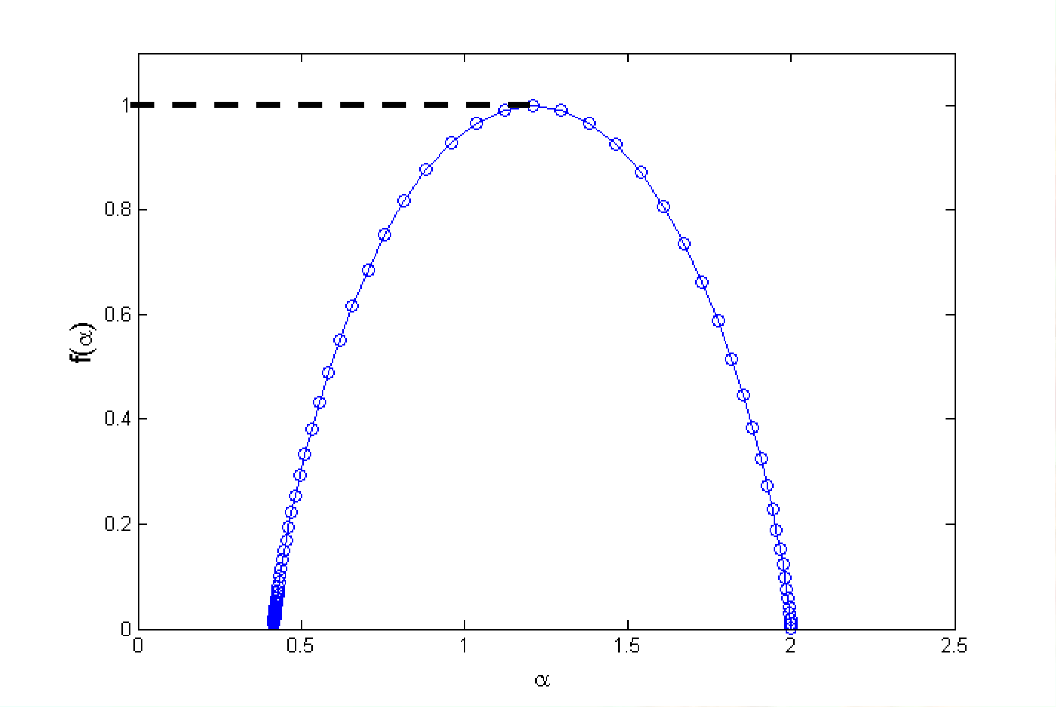
Multifractal: a distribution of points



The singularity spectrum

f(α) : the fractal dimension of the fractal set having Holder exponent α

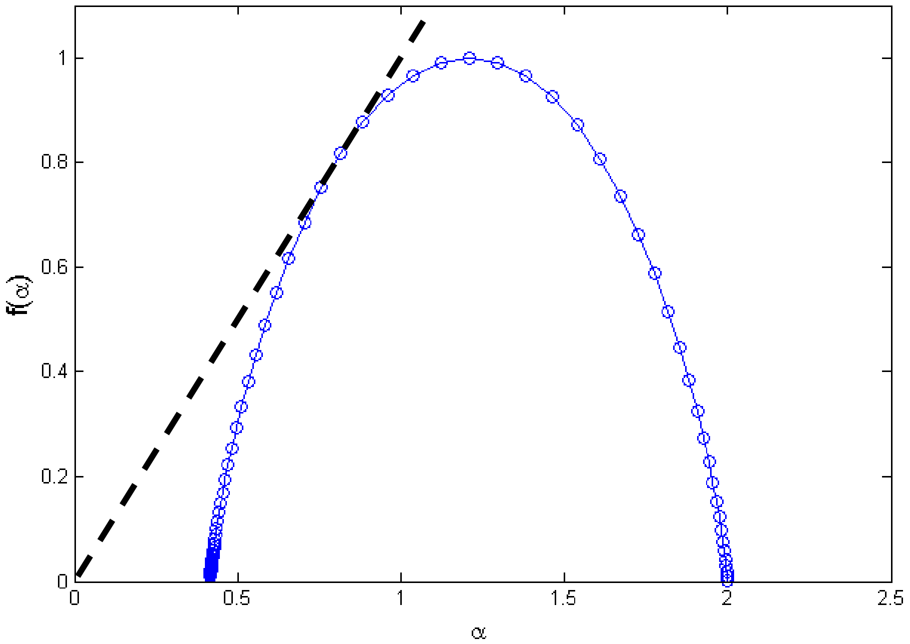
The maximum value of the subset S_α equals the fractal dimension of the support of the measure. It occurs at q = 0.



The singularity spectrum

The point intersection $f(\alpha) = \alpha$ is the fractal dimension of the measure

It also corresponds to $q = 1$

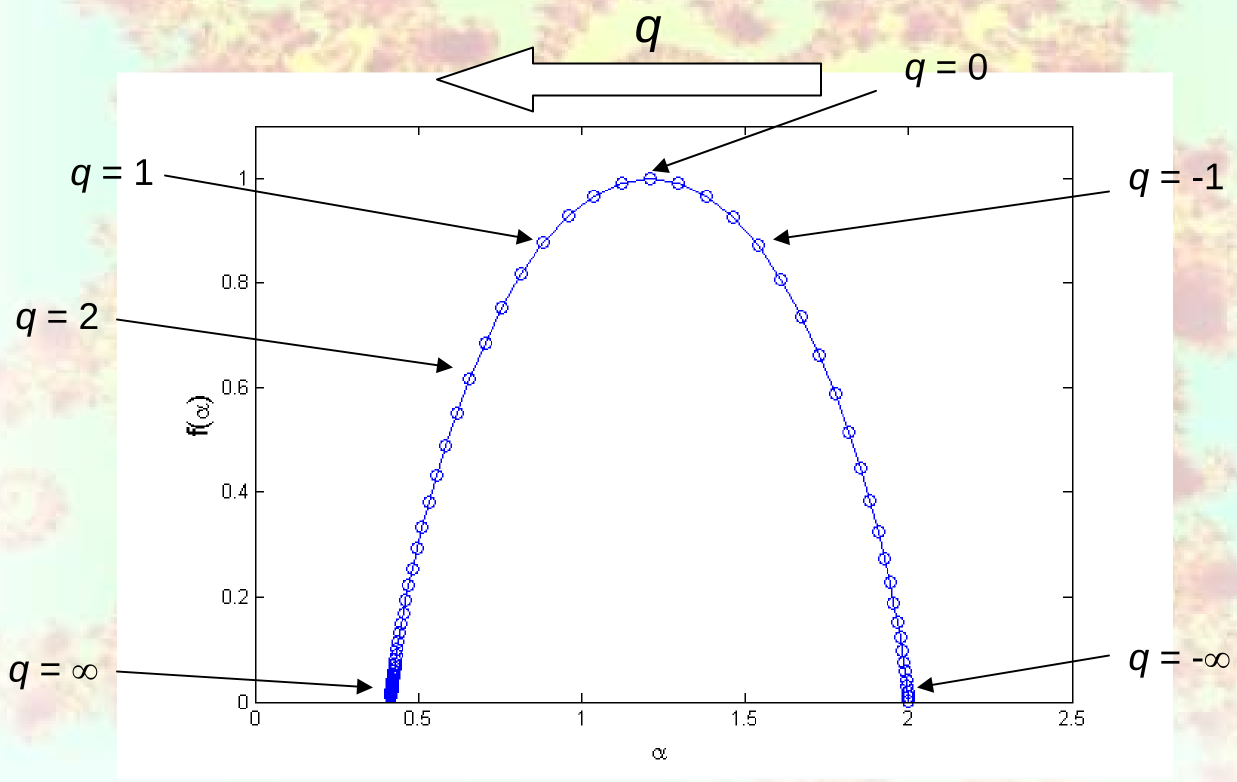


The singularity spectrum

The moments *q* are related to the singularity spectrum

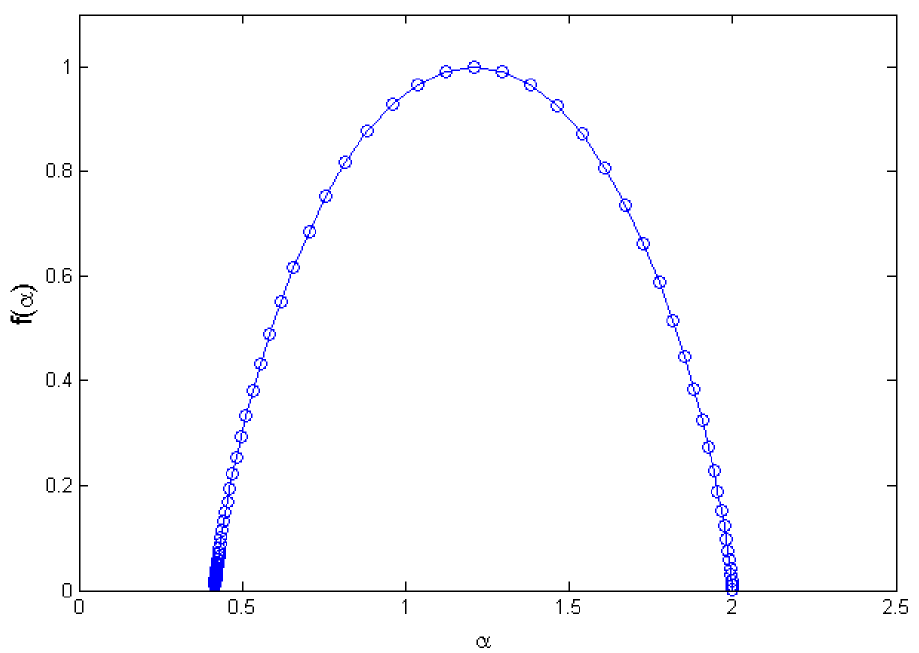
For values *q* > 1, the strongly singular measures are enhanced

For values *q* < 1, the less singular areas are emphasized



The singularity spectrum

q	$\alpha = \mathrm{d}\tau(q)/dq$	$f = q\alpha + \tau(q)$
$q \to \infty$	$\to \alpha_{max} = \lim(\ln\mu_{-}/\ln\varepsilon)$	$\to 0$
$q = 0$	α_0	$f_{max} = D$
$q = 1$	$\alpha_1 = \lim(S(\varepsilon)/\ln\varepsilon)$	$f_s = \alpha_1 = S$
$q \to -\infty$	$\to \alpha_{min} = \lim(\ln\mu_{+}/\ln\varepsilon)$	$\to 0$



Conclusions - Theory

- The concept of dimension is appropriate to characterize complex data
- A spectrum of dimensions can be necessary to fully characterize the statistics of complex data
- The partition function is equivalent to a weighted box counting
- The sequence of mass exponents describes how the partition function scales with ε
- The singularity spectrum describes the fractal dimension of the different subsets having different Holder exponents

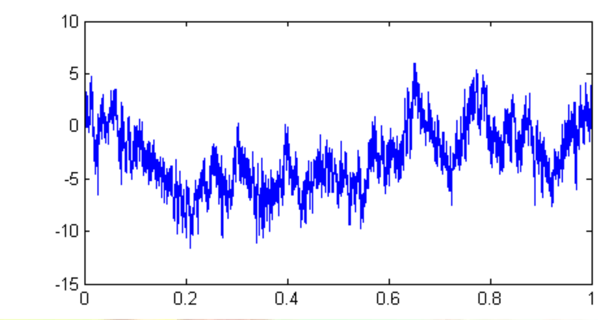
Algorithms

- The method of moments
  - Time series
  - Planar data (images)

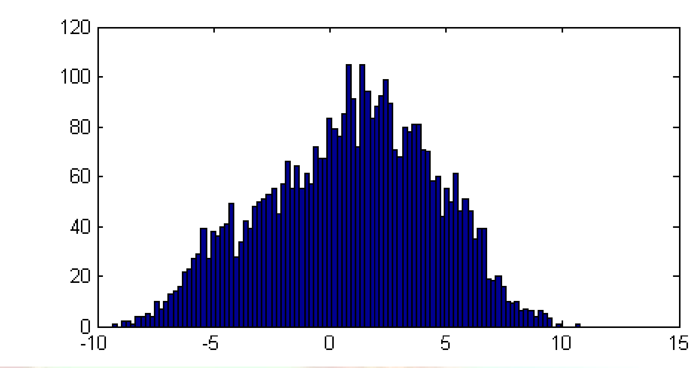
- The wavelet transform modulus maxima

The method of moments – time series

- A simple, direct approach
- Consider a time series $x_1, x_2, x_3, \ldots, x_N$



- Put the data in a histogram
  - Select bin size ε
  - The maximum x_i is denoted M and minimum x_i is denoted m
  - The bin sizes are $[m, m+\varepsilon]$, $[m+\varepsilon, m+2\varepsilon]$, …, $[m+k\varepsilon, M]$



- Count the number of x_i in every bin and denote by n_j. Ignore the empty bins.

The method of moments – time series

- Compute the partition function for q= -20…20

$$Z_{\varepsilon}^{q} = (n_0 / N)^q + \ldots + (n_k / N)^q$$

- Repeat, from building the histogram, with a smaller ε getting closer to 0, to build the partition function to different ε

- For different q, find the slope of the plot log(Z^q) vs log(ε) to determine $\tau(q)$

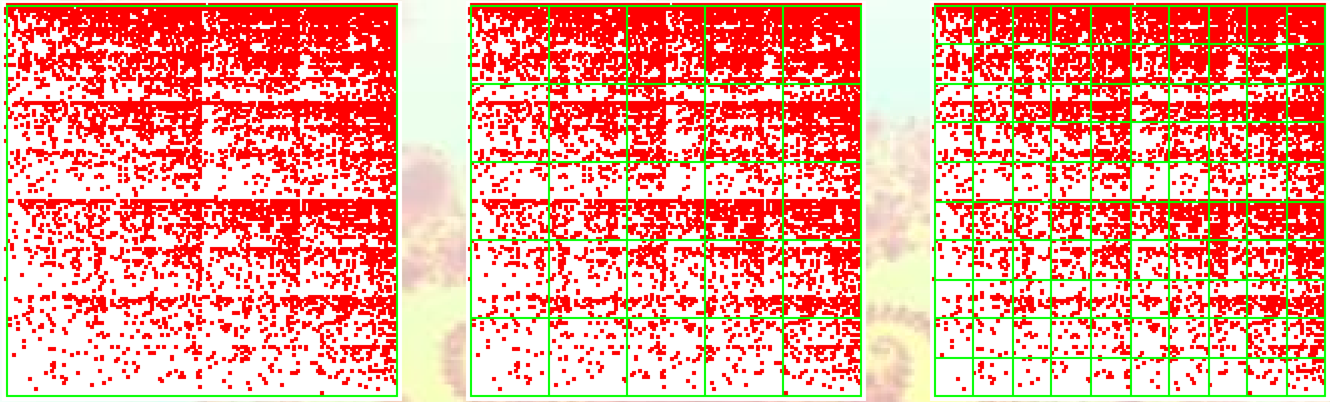
- We now have the $\tau(q)$ spectrum, we need to do its Legendre transform to get the $f(\alpha)$ spectrum

The method of moments – time series

- Start with α = 0.1, or any value close to 0
- Compute min(αq - $\tau(q)$) over all q values to get $f(\alpha)$. Ignore negative values
- Increase α until $f(\alpha)$ becomes negative again

The method of moments – Planar data

- Same concept, but for 2d – or any number of dimension

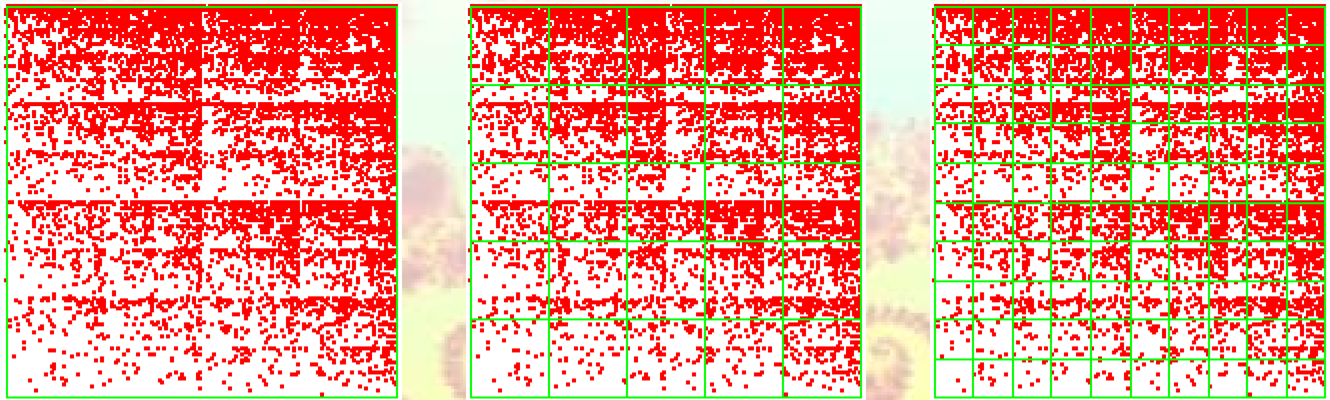


From http://classes.yale.edu/fractals/

- Subdivide the plane in smaller squares of side length ε_1, then ε_2, ε_3, and so on
- Count the number of points in each square of side length ε_1. Denote these $n(1, \varepsilon_1)$, $n(2, \varepsilon_1)$, $n(3, \varepsilon_1)$, …
- Count the number of points in each square of side length ε_2. Denote these $n(1, \varepsilon_2)$, $n(2, \varepsilon_2)$, $n(3, \varepsilon_2)$, …

The method of moments – Planar data

- Same concept, but for 2d – or any number of dimension



From http://classes.yale.edu/fractals/

- Build the partition function by computing the *q* moments for size ε_1

$$Z_{\varepsilon}^{q} = (n(1,\varepsilon_1)/N)^q + (n(2,\varepsilon_1)/N)^q + (n(3,\varepsilon_1)/N)^q + \ldots$$

- Repeat for diffent ε

$$Z_{\varepsilon}^{q} = (n(1,\varepsilon_2)/N)^q + (n(2,\varepsilon_2)/N)^q + (n(3,\varepsilon_2)/N)^q + \ldots$$

The wavelet transform modulus maxima method (WTMM)

- Wavelet transform, introduced to analyze seismic data and acoustic signals, is used in a wide diversity of fields

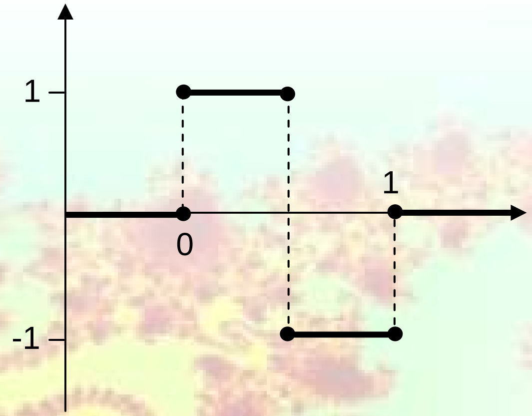
- The wavelet transform (WT) of a function s corresponds in decomposing it into elementary space-scale contribution, the wavelets

- The wavelets are constructed from a single function ψ that is dilated and translated

Different types of wavelets can be used

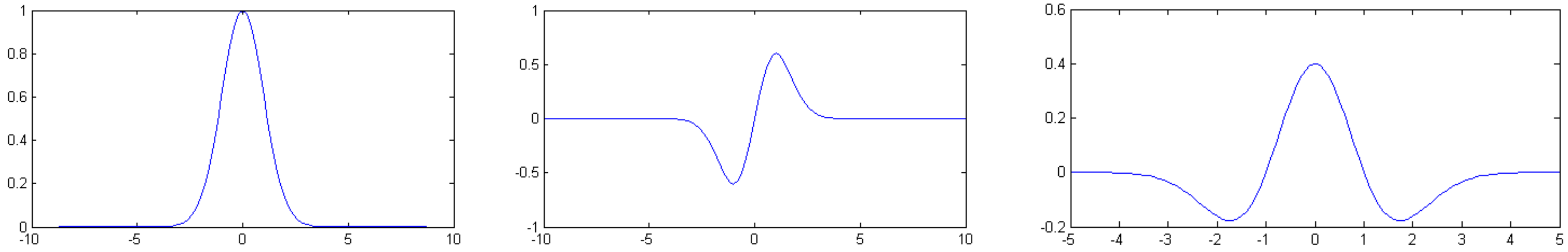
- Haar wavelet

$$f(x)=\begin{cases}1 & 0\le x<1/2\\ -1 & 1/2\le x<1\\ 0 & otherwise\end{cases}$$



- Gaussian function and its derivatives

$$\psi^{(N)}(x)=\frac{d^N}{dx^N}e^{-x^2/2}$$



The **wavelet transform** of a function *s(x)* is defined as

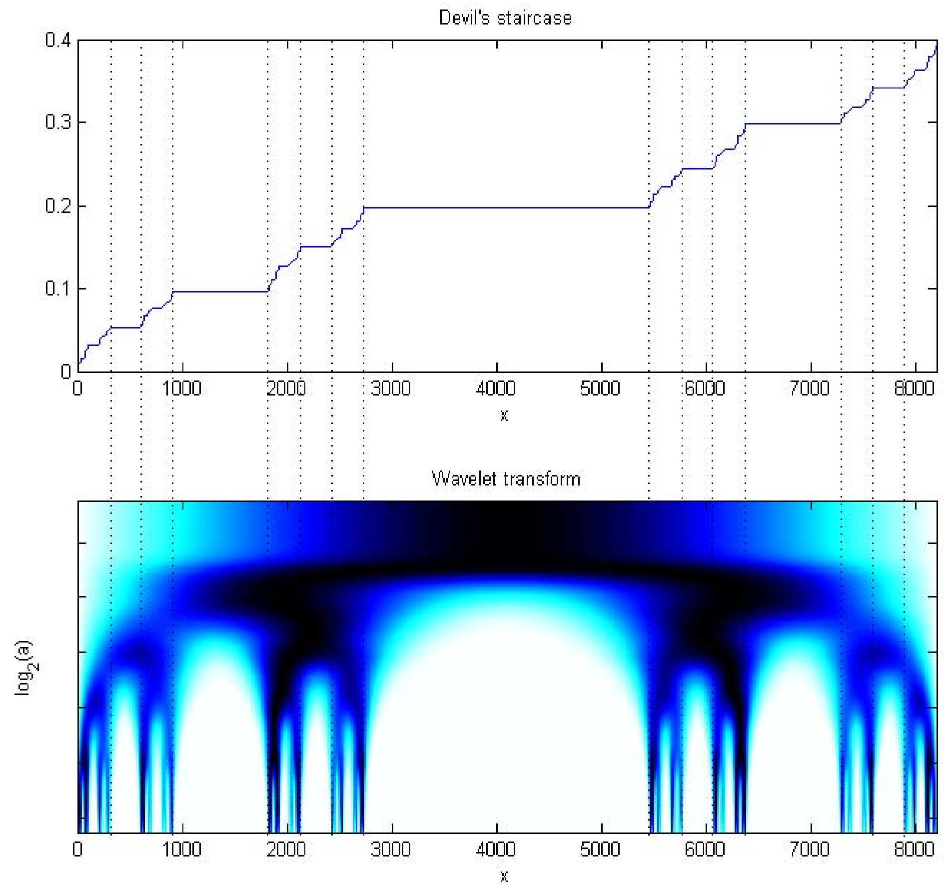
$$W_{\psi}[s](b,a) = \frac{1}{a}\int_{-\infty}^{\infty} \overline{\psi}\left(\frac{x-b}{a}\right) s(x)\,dx$$

Where $\overline{\psi}$ is the complex conjugate of ψ, *b* denotes the translation of the wavelet and *a* its dilation.

The operation is similar to the convolution of *s(x)* and the wavelet at a given scale

Example of wavelet transform: Devil's staircase

The staircase is obtained by summing over the triadic Cantor set



The wavelet transform reveals the singular behavior of the measure

The wavelet transform can be used for **singularity detection**

- Before, we relied on boxes of length size ε to cover the measure, now we use wavelets of scale *a*
- The wavelet is a more precise box. One can even choose a different box type for different applications
- The partition function will be built from the wavelet coefficients

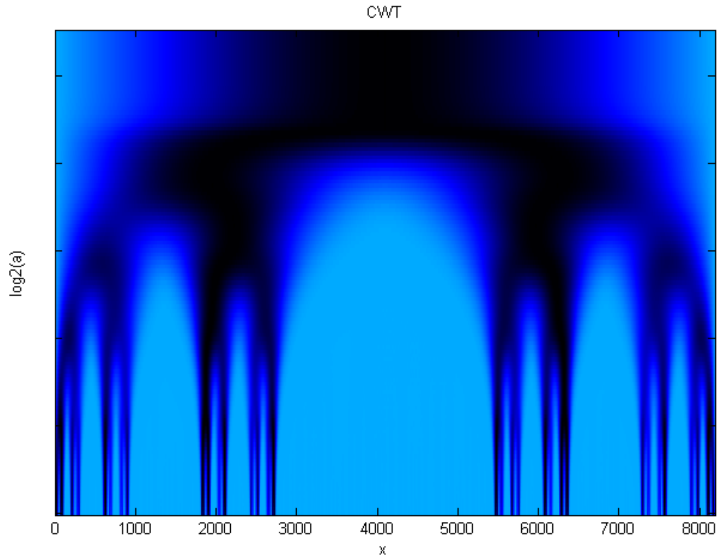
A simple method to build the partition function would be

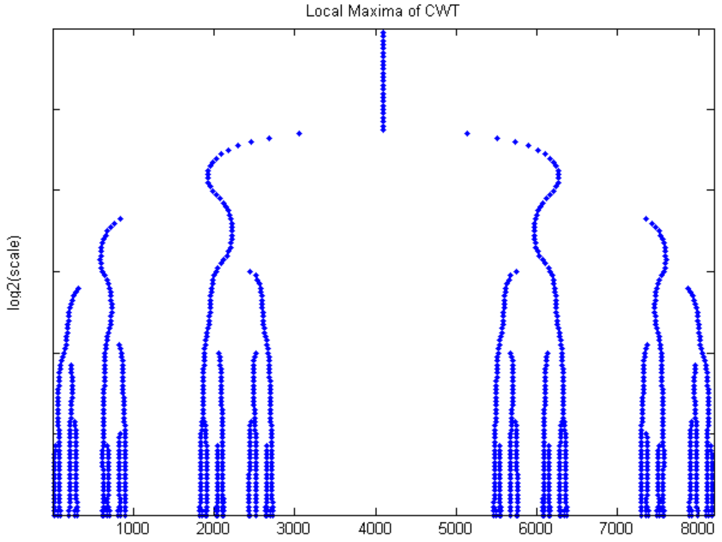
$$Z(q,a)=\int \left|W_{\psi}[s](x,a)\right|^{q}dx$$

But nothing prevents W_{ψ} from vanishing and Z would diverge for q<0.

The Wavelet Transform Modulus Maxima Method (WTMM) changes the **continuous sum over space** into a **discrete sum over the local maxima** of $|W_{\psi}[s](x,a)|$ considered as a function of x

The **skeleton of the wavelet transform** is created by all the maxima lines of the WT

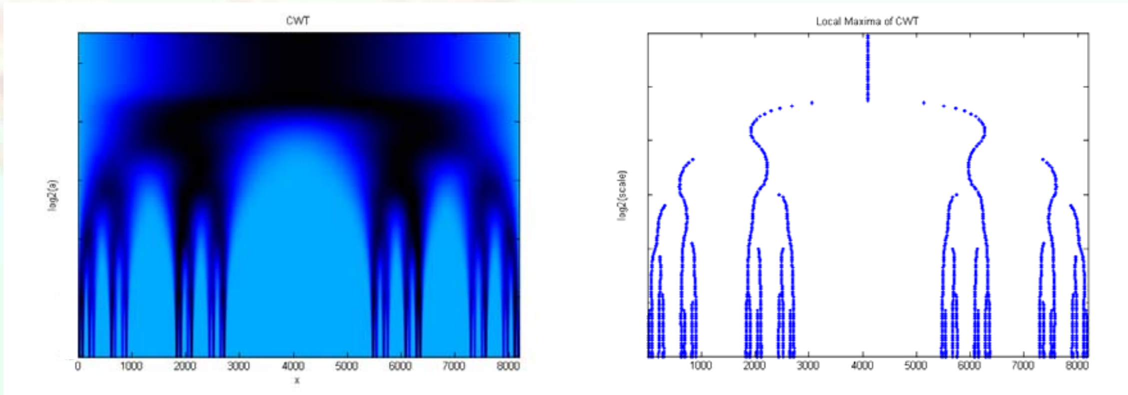




Let $L(a_0)$ be the set of all the maxima lines that exist at the scale a_0 and which contain maxima at any scale $a \leq a_0$

We expect the number of maxima lines to diverge in the limit $a \to 0$

The **TWMM skeleton** enlightens the hierarchical organization of the singularities



The partition function will be defined as

$$Z(q,a) = \sum_{l \in L(a)} \left(\sup_{(x,a') \in l} \left| W_{\psi}[s](x,a') \right|^{q} \right)$$

The skeleton indicates how to position the boxes on the measure

The sup defines a **scale-adaptative partition** preventing divergences to show up

$$Z(q,a) = \sum_{l \in L(a)} \left(\sup_{(x,a') \in l} \left| W_\psi[s](x,a') \right|^q \right)$$

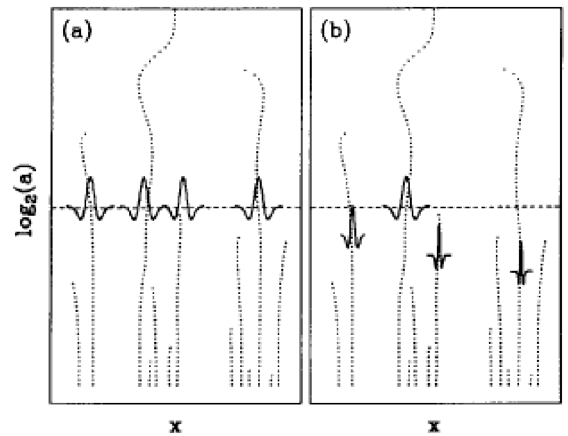


Fig. 2. Representation of the uniform and scale-adapted partitions. (a) Uniform partition: $Z(q,a)$ involves wavelets of the same size a. (b) Scale-adapted partition: $Z(q,a)$ involves wavelets of different sizes $a' \leq a$. The large scales are at the top.

From Arneodo et al. Physica A 213 (1995) 232-275

And again, *Z(q,a)~a*$^{\tau(q)}$, so the familiar exponents can be determined

Results for the Devil's staircase

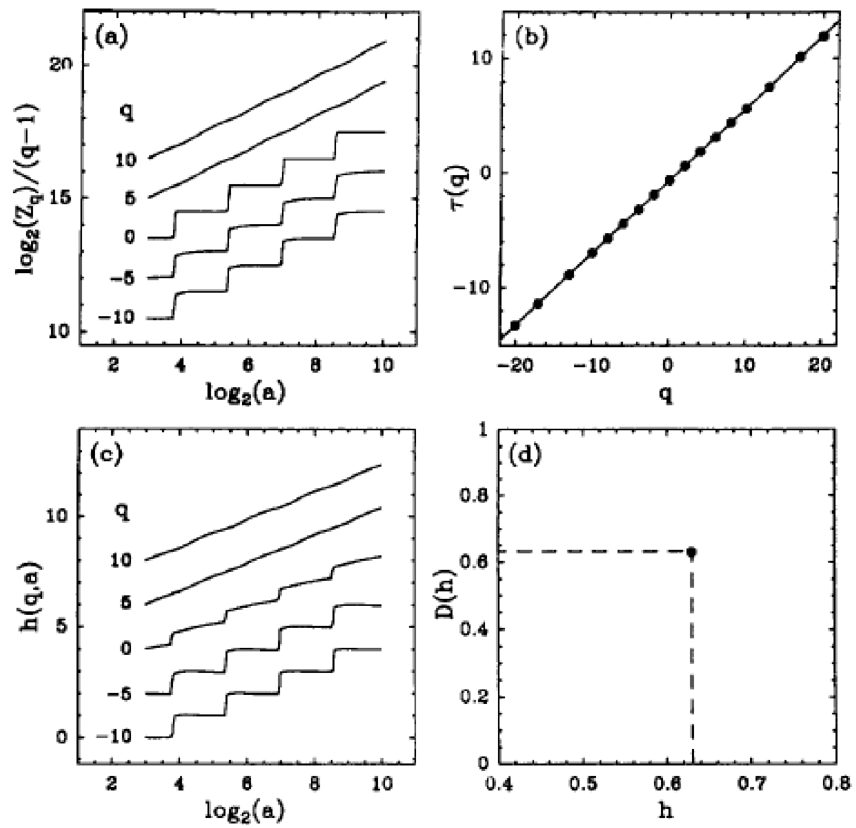


Fig. 3. Determination of the multifractal spectra of the devil's staircase associated to the uniform triadic Cantor set using the WTMM method. (a) $\log_2 Z(q,a)/(q-1)$ versus $\log_2 a$. (b) $\tau(q)$ versus q; the solid line corresponds to the theoretical curve $\tau(q) = (q-1)\ln 2/\ln 3$. (c) Determination of the exponents $h(q)$; $h(q,a)$ is plotted versus $\log_2 a$ according to Eq. (29). (d) $D(h)$ versus h. The analyzing wavelet is $\psi^{(1)}$.

From Arneodo et al. Physica A 213 (1995) 232-275

The **order of the wavelet** matters

- It can be shown that

$$W_\psi[s](x_0, a) \sim a^{\alpha(x_0)}$$

in the limit $a \to 0^+$, provided the number of vanishing moments of the wavelet $n_\psi > \alpha(x_0)$

- If $n_\psi < \alpha(x_0)$, a power law behavior still exist but with a scaling exponent n_ψ

$$W_\psi[s](x_0, a) \sim a^{n_\psi}$$

- Around x_0, the faster the wavelet transform decreases when the scale a goes to zero, the more regular s is around that point

The WTMM reveals **phase transitions** in the multifractal spectra

- Lets assume $f(x) = s(x) + r(x)$
  - $s(x)$ is a multifractal singular function with $\alpha_{\max} < \infty$
  - $r(x)$ is regular function (C^{∞})

- $n_{\psi} > \alpha(x_0)$ is impossible because of $r(x)$

- The set of maxima lines will be composed of two disjoint sets from $s(x)$ (slightly perturbated) and $r(x)$

- The partition function splits into two parts

$$Z_f(q,a) = Z_s(q,a) + Z_r(q,a) \sim a^{\tau_s(q)} + a^{qn_{\psi}}$$

The WTMM reveals **phase transitions** in the multifractal spectra

$$Z_f(q,a) = Z_s(q,a) + Z_r(q,a) \sim a^{\tau_s(q)} + a^{qn_\psi}$$

- There will be a critical value q_{crit}<0 for which there is a phase transition (a discontinuity):

$$\tau(q) = \begin{cases} \tau_s(q) & \text{for } q > q_{crit} \\ qn_\psi & \text{for } q < q_{crit} \end{cases}$$

- This discontinuity in the spectrum expresses the breakdown of the self-similarity of the singular signal *s(x)* by the perturbation of *r(x)*

Checking whether $\tau(q)$ is sensitive to the order n_{ψ} of ψ is a very good **test for the presence of highly regular parts in the signal**

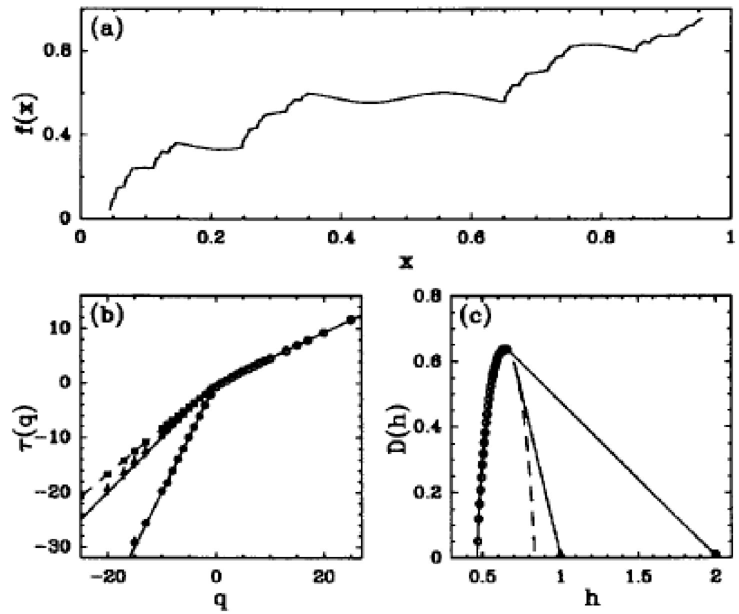


Fig. 7. WTMM analysis of a signal which is not singular on some intervals. (a) Graph of the signal $f(x) = s(x) + r(x)$, with $r(x) = R\sin(8\pi x)$ and $s(x)$ is a multifractal devil's staircase (see text). (b) $\tau(q)$ vs q as obtained with $\psi^{(1)}$ ((o) and (▲)), $\psi^{(2)}$ ((o) and (•)) and $\psi^{(4)}$ ((o) and (■)); the solid lines correspond to the theoretical predictions (Eq. (33)); the dashed line is the part $q < q_{crit}$ of $\tau_s(q)$. (c) $D(h)$ vs h from the Legendre transform of $\tau(q)$; the symbols are the same as in (b).

From Arneodo et al. Physica A 213 (1995) 232-275

When the data shows clear **trends**, removing them will significantly increase the accuracy of the measure

- Removing trends makes the data more singular
- But filtering removes long-range correlations in the signal, which has an impact on the determination of the singularity spectrum
- Requires judgment

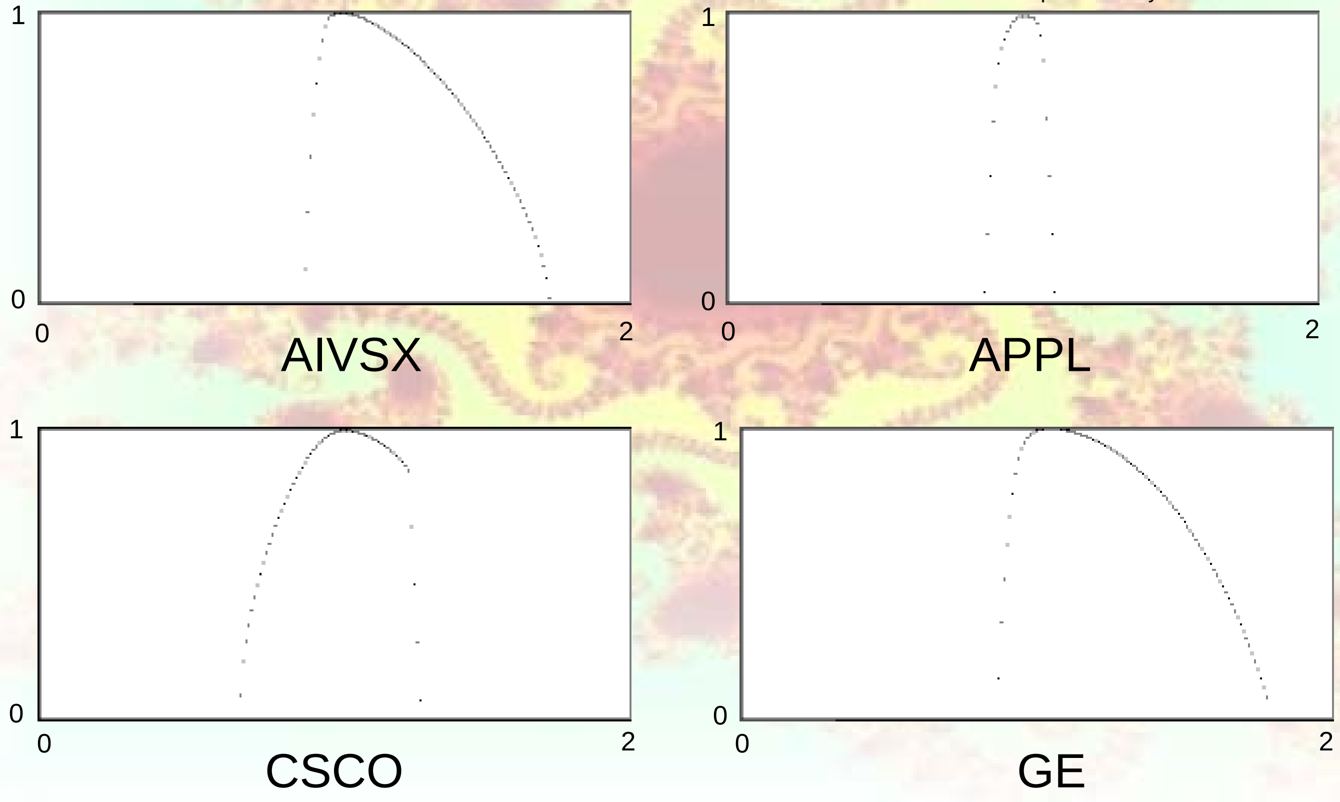
Conclusion - Algorithms

- Wavelet transform reveals the scaling structure of the singularities
- The method is scale-adaptative
- The method reveals non-singular behavior
- The formalism offers a clear link to thermodynical concepts such as entropy, free energy, …

Applications

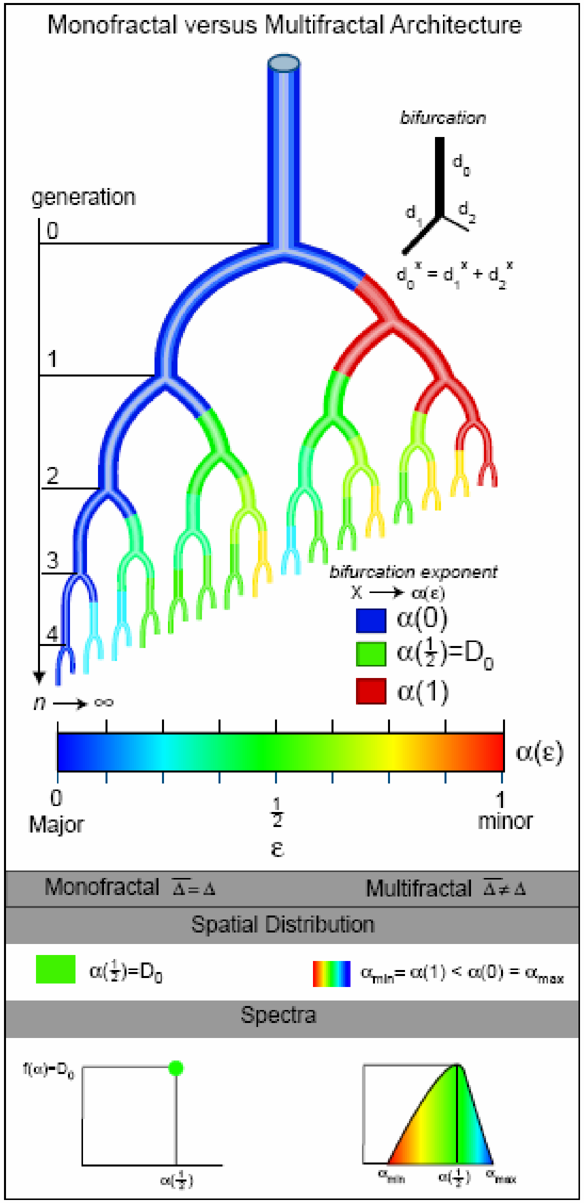
- A wide variety of system exhibit multifractal properties
- Stock market data

http://classes.yale.edu/fractals/



The branching of lungs

- The structure of the lungs as a fractal geometry



Bennet et al. Whitepaper of the University of California, 2001 - neonatology.ucdavis.edu

Web data

- Internet data has been studied extensively

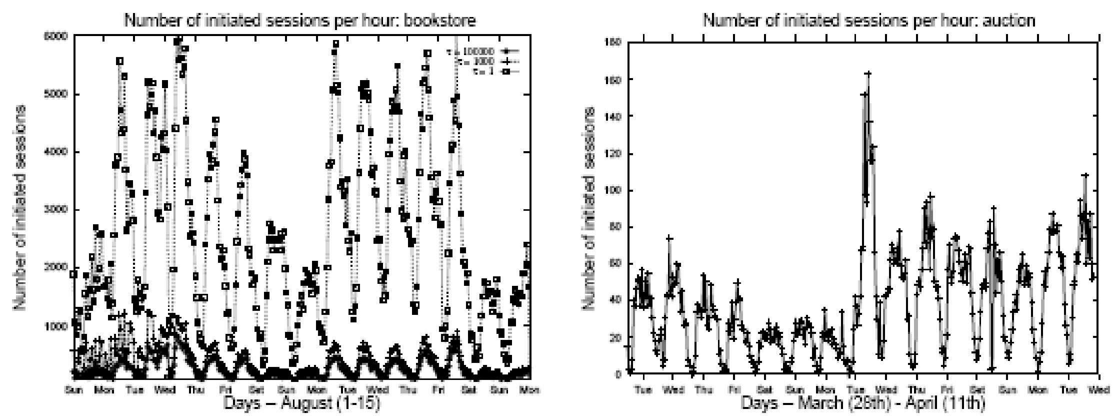


Fig. 17. Number of initiated sessions per hour for the bookstore and the auction site.

Menasce et al. A hierarchical and multiscale approach to analyze E-business workloads, Performance Evaluation 2003, 33-57.

Internet traffic

- Internet traffic
- Traffic in general: roads, flows, …

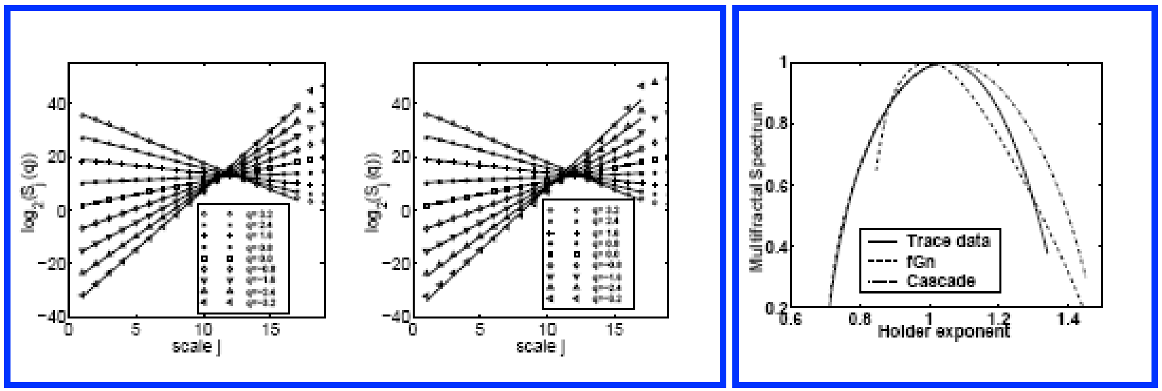


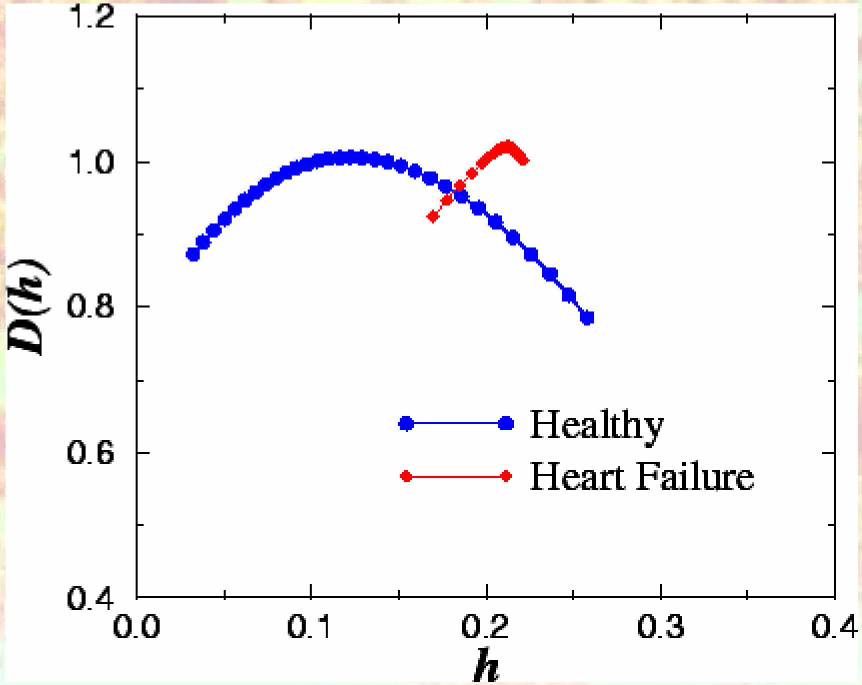
Fig. 9. Left: Superimposed log-log plots, at several values of q, of the partition sum against scale for a time series of bytes per bin of TCP traffic (taken from the LBL-TCP3 trace [24]) (on left), and a matched binomial cascade (right).

Fig. 10. Right: Multifractal spectrum of local Hölder exponents estimated via the Legendre transform.

Abry et al. *IEEE Signal Processing Magazine* 19, 28--46

Physiological data

- Neuron signals
- Collection of neurons
- Heart



From http://www.physionet.org/tutorials/multifractal/humanheart.htm

Projects at PNNL

- Solid-state studies
- Internet activity
- Protein sequence data
- EEG data
- Energy markets, energy grid behavior
- Light detection and ranging (LIDAR)
  - Effect of turbulence on laser propagation

Conclusion

- Many systems have fractal geometry in nature
- Multifractals provide information on
  - Local scaling
  - Dimension of sets having a given scaling
  - Statistical representation of data, especially well suited for distributions with long tails
- Several techniques exist to calculate the singularity spectrum

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- Paul D. Whitney, PNNL

References

- Feder, *Fractals*, Plenum Press (1988).
- Halsey et al. “Fractal measures and their singularities: the characterization of strange sets,” Phys. Rev. A 33, 1141-1151 (1986).
- Muzy et al. “Multifractal formalism for fractal signals: the structure-function approach versus the wavelet-transform modulus-maxima method,” *Phys. Rev. E* 47, 875-884 (1993).
- Arneodo et al. “The thermodynamics of fractals revisited with wavelets,” *Physica A* 213, 232-275 (1995).
- Kantelhardt et al. “Multifractal detrended fluctuation analysis of nonstationary time series,” *Physica A* 316, 87-114 (2002).
- Gleick, *Chaos*, The Viking Press (1987).

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