The Normal Form

The description of a game can be viewed as a listing of the strategies of the players and the outcome of any set of choices of strategies, without regard to the attitudes of the players toward various outcomes. We now indicate how the final simplification of the game – the normal form – is obtained, by taking into account the preferences of the players.

The result of any set of strategies $f_1, \dots, f_k$ is a probability distribution π_f over the set R of possible outcomes. It would be particularly convenient if a given player could express his/her preference pattern in R by a bounded numerical function u defined on R, such that he or she prefers r_1 to r_2 *iff* $u(r_1) > u(r_2)$. Note that $u(r_1) = u(r_2)$ denotes indifference between r_1 to r_2. Also, the function u is such that if for any probability distgribution ξ over R we define $U(\xi)$ as the expected value of $u(r)$ computed with respect to ξ as

$$U(\xi) = \sum_{r \in R} \xi(r) u(r)$$

the player prefers $\xi_1$to ξ_2 *iff* $U(\xi_1) > U(\xi_2)$.

It is remarkable fact that, under extremely plausible hypothesis concerning the preference pattern such function u exists.

Definition (*utility function*): The function U defined for all probability distributions ξ over R, is called the player's ***utility function***.

U is unique, for a given preference pattern up to a linear transformation. We will assume that each player has such utility function.

The aim of each player in the game is to maximize his/her expected utility. If U_i is the utility function of player i, his/her aim is to make $M_i(f_1, \dots, f_k) = U_i(\pi_f)$ as large as possible where π_f is the probability distribution for fixed $f_1, \dots, f_k$ over R determined by the overall chance move.

We are in a position to give a description of the normal form of a game:

Definition (*normal form of a game*): A game consists of k spaces $F_1, \dots, F_k$ and k bounded numerical functions $M_i(f_1, \dots, f_k)$ defined on the space of all k-tuples $(f_1, \dots, f_k)$, $f_i \in F_i, i = 1, \dots, k$. The game is played as follows: Player i chooses an element f_i of F_i, the k choices being made simultaneously and independently; player i then receives the amount $M_i(f_1, \dots, f_k), i = 1, \dots, k$.

TODO: finish it

Example (*two player game involving coin-toss and a number choice*):

Player I moves first and selects one of the two integers $1, 2$. The referee then tosses a coin and if the outcome is "head", he informs player II of player I's choice and not otherwise. Player II then moves and selects one of two integers $3, 4$. The fourth move is again a chance move by the referee and consists of selecting one of three integers $1, 2, 3$ with respective probabilities $0.4, 0.2, 0.4$. The numbers selected in the first, third and the fourth move are added and the amount of dollars is paid by II to I if the sum is even and by I to II if the sum is odd. Note that $|R| = 2 \times 2 \times 2 \times 3 = 24$.

Here is the set R of possible outcomes for this game where I denotes player I, 0 denotes the referee and II denotes player II:

I –> 2 – 0 –> Head – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Head – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Head – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$

I –> 2 – 0 –> Head – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
I –> 2 – 0 –> Head – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
I –> 2 – 0 –> Head – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
I –> 2 – 0 –> Tail – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 2 – 0 –> Tail – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 2 – 0 –> Tail – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$

I –> 1 – 0 –> Head – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^5), (f_2, f^6), (f_2, f^7), (f_2, f^8)$
I –> 1 – 0 –> Head – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^5), (f_2, f^6), (f_2, f^7), (f_2, f^8)$
I –> 1 – 0 –> Head – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^5), (f_2, f^6), (f_2, f^7), (f_2, f^8)$
I –> 1 – 0 –> Tail – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 1 – 0 –> Tail – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 1 – 0 –> Tail – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$

I –> 1 – 0 –> Head – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$
I –> 1 – 0 –> Head – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$
I –> 1 – 0 –> Head – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$
I –> 1 – 0 –> Tail – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$
I –> 1 – 0 –> Tail – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$
I –> 1 – 0 –> Tail – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$

Here are the two strategy spaces:
$F_1 = \{f_1, f_2\}$; $f_1 = (1), f_2 = (2)$

$F^2 = \{f^1, f^2, f^3, f^4, f^5, f^6, f^7, f^8\}$; $f^1 = (3,3,3), f^2 = (3,3,4), f^3 = (3,4,3), f^4 = (3,4,4), f^5 = (4,3,3), f^6 = (4,3,4), f^7 = (4,4,3), f^8 = (4,4,4)$