The Normal Form

The description of a game can be viewed as a listing of the strategies of the players and the outcome of any set of choices of strategies, without regard to the attitudes of the players toward various outcomes. We now indicate how the final simplification of the game – the normal form – is obtained, by taking into account the preferences of the players.
The result of any set of strategies $f_1, \dots, f_k$ is a probability distribution π_f over the set R of possible outcomes. It would be particularly convenient if a given player could express his/her preference pattern in R by a bounded numerical function u defined on R, such that he or she prefers r_1 to r_2 *iff* $u(r_1) > u(r_2)$. Note that $u(r_1) = u(r_2)$ denotes indifference between r_1 to r_2. Also, the function u is such that if for any probability distgribution ξ over R we define $U(\xi)$ as the expected value of $u(r)$ computed with respect to ξ as

$U(\xi) = \sum_{r\in R} \xi(r)u(r)$

the player prefers $\xi_1$to ξ_2 *iff* $U(\xi_1) > U(\xi_2)$.
It is remarkable fact that, under extremely plausible hypothesis concerning the preference pattern such function u exists.

Definition (*utility function*): The function U defined for all probability distributions ξ over R, is called the player's ***utility function***.
U is unique, for a given preference pattern up to a linear transformation. We will assume that each player has such utility function.
The aim of each player in the game is to maximize his/her expected utility. If U_i is the utility function of player i, his/her aim is to make $M_i(f_1, \dots, f_k) = U_i(\pi_f)$ as large as possible where π_f is the probability distribution for fixed $f_1, \dots, f_k$ over R determined by the overall chance move.
We are in a position to give a description of the normal form of a game:

Definition (*normal form of a game*): A game consists of k spaces $F_1, \dots, F_k$ and k bounded numerical functions $M_i(f_1, \dots, f_k)$ defined on the space of all k-tuples $(f_1, \dots, f_k)$, $f_i \in F_i, i = 1, \dots, k$. The game is played as follows: Player i chooses an element f_i of F_i, the k choices being made simultaneously and independently; player i then receives the amount $M_i(f_1, \dots, f_k), i = 1, \dots, k$. The aim of Player i is to make M_i as large as possible. The statement "Player i receives the amount $M_i(f_1, \dots, f_k)$" is shorthand of saying "a situation results whose utility for Player i is $M_i(f_1, \dots, f_k)$".

Example (*two player game involving coin-toss and a number choice*):
Player I moves first and selects one of the two integers $1, 2$. The referee then tosses a coin and if the outcome is "head", he informs player II of player I's choice and not otherwise. Player II then moves and selects one of two integers $3, 4$. The fourth move is again a chance move by the referee and consists of selecting one of three integers $1, 2, 3$ with respective probabilities $0.4, 0.2, 0.4$. The numbers selected in the first, third and the fourth move are added and the amount of dollars is paid by II to I if the sum is even and by I to II if the sum is odd. Note that $|R| = 2 \times 2 \times 2 \times 3 = 24$.

Here are the two strategy spaces:
$F_1 = \{f_1, f_2\}$; $f_1 = (1), f_2 = (2)$

$F^2 = \{f^1, f^2, f^3, f^4, f^5, f^6, f^7, f^8\}$; $f^1 = (3,3,3), f^2 = (3,3,4), f^3 = (3,4,3), f^4 = (3,4,4), f^5 = (4,3,3), f^6 = (4,3,4), f^7 = (4,4,3), f^8 = (4,4,4)$
Here the first position of the triple is conditioned upon coin falling *Head* and player I choosing 1, the second position in the triple is conditioned upon coin falling head and player I choosing 2, and the third position of the triple is conditioned upon coin falling *Tail*.

The set R of possible outcomes for this game where I denotes player I, 0 denotes the referee and II denotes player II is shown below:
I –> 2 – 0 –> Head – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Head – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Head – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
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I –> 2 – 0 –> Head – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
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In the theory of games it is usual to treat first a special class of games, *the two-person zero-sum games*. The theory of these games is particularly simple and complete and we will consider only such games in our discussion.

Definition (*two-person game*): a game with $k = 2$: we have only two utility functions M_1 and M_2 and two strategy sets F_1 and F_2 for each of the two players.

Definition (*zero-sum game*): A game for which the following holds true:
$\sum_{i=1}^{k} M_i(f_1, \dots, f_k) = 0$ for all $f_1, \dots, f_k$
More precisely, since each M_i is unique up to a linear transformation, a game is a **zero-sum** if there is a determination of $M_1, \dots., M_k$ for which $\sum_{i=1}^{k} M_i(f_1, \dots, f_k) = 0$ for all $f_1, \dots, f_k$. Thus a two-person zero-sum game is a game between two players in which their interests are diametrically opposed: one player gains at the expense of the other. Consequently, there is no motive for collusion between the players. It is precisely the fact that collusion is unprofitable that simplifies the theory.

Definition (*constant-sum game*): A **constant-sum game** i.e. one in which $\sum_{i=1}^{k} M_i(f_1, \dots, f_k) = c$ for all $f_1, \dots, f_k$ is zero-sum game in the sense defined above, since an alternative choice of utility functions is $M_1^* = M_1 - c$, $M_i^* = M_i$ for $i \neq 1$, and $\sum_{i=1}^{k} M_i^* = 0$. Thus the theory developed for zero sum two person games applies for constant sum two person games.

Since for two-person zero sum game we have $M_2(f_1, f_2) = -M_1(f_1, f_2)$ we need to specify only M_1. We will consider only two-person zero-sum games from now on.

Definition (*game in a normal form*): A **game in a normal form** is a triple (X, Y, M), where X, Y are arbitrary spaces and M is a bounded numerical function defined on the product space $X \times Y$ of pairs (x, y), $x \in X, y \in Y$. The points $x(y)$ are called strategies for player I (II) and the function M is called payoff. The game G is played as follows: I chooses $x \in X$, II chooses $y \in Y$, the choices being made independently and simultaneously. II then pays I the amount $M(x, y)$.

Equivalent Games

If, in a given game, one relabels the strategies of either player, the new game is essentially not different than the old. Every statement about either game can be translated into a corresponding statement about the other and we wish to consider the two games equivalent.

Another simple transformation which does not alter the essential character of the game is the deletion of duplicate strategies. In other words, if a player I has two strategies x_1, x_2 such that $M(x_1, y) = M(x_2, y)$ for all y, the deletion of x_2 from X is an inessential change in the game, even though it might, for example destroy such properties as symmetry.

Definition (*reduction of game*): Let $G_1 = (X_1, Y_1, M_1)$ and $G_2 = (X_2, Y_2, M_2)$ be two games. Then G_2 is a reduction of G_2, written $G_2\ r\ G_1$, if either:

(a) $X_2 = X_1$, and there is a function f from Y_1 onto Y_2 such that $M_1(x, y) = M_2(x, f(y))$ for all $x \in X_1$, $y \in Y_1$, or
(b) $Y_2 = Y_1$, and there is a function g from X_1 onto X_2 such that $M_1(x, y) = M_2(g(x), y)$ for all $x \in X_1$, $y \in Y_1$

If f is a $1-1$ transformation, G_2 is obtained from $G_1$by relabeling of strategies; if f is not $1-1$, G_2 is obtained from G_1 by deletion of certain duplicated strategies and relabeling.

Definition (*equivalent games*): Two games G and G' are called **equivalent**, written $G{\sim}G'$, *iff* there is a finite sequence of games $G_0, G_1, \dots, G_n$ with $G_0 = G, G_n = G'$, and for each $i = 1, \dots, n$ either $G_{i-1}\ r\ G_i$ or $G_i\ r\ G_{i-1}$.

Example (*equivalent games*): Let $G = (X, Y, M)$ be a game, where $X = (x_1, \dots, x_N)$, a set of N real numbers, $Y = (y_1, \dots, y_R)$, also a set of R real numbers; and for $x \in X, y \in Y$,

M(x,y)=
//TODO: finish this

Lower and Upper Pure Value

In a game $G = (X, Y, M)$, the consequences of strategy x_0 are described by the function $M(x_0, y)$. Using x_0, player I is certain to receive at least

$$\Lambda_G(x_0) = \inf_{y \in Y} M(x_0, y)$$

and cannot be certain of any definite larger amount. Thus, the number

$$\lambda_G^* = \sup_{x \in X} \Lambda_G(x)$$

is the upper limit to the amount I can guarantee getting: for every $\varepsilon > 0$, the player can, simply by choosing a suitable x, be certain of $\lambda_G^* - \varepsilon$, and for no $\varepsilon > 0$ is there an x which makes the player certain to receive at least $\lambda_G^* + \varepsilon$ against all y. Similarly, we define

$$\Upsilon_G(y_0) = \sup_{x \in X} M(x, y_0), \quad \upsilon_G^* = \inf_{y \in Y} \Upsilon_G(y)$$

by selecting a y suitably, player II can with certainty restrict his/her loss to $\upsilon_G^* + \epsilon$ but not to $\upsilon_G^* - \epsilon$ for any $\epsilon > 0$. For subsequent reference these statements are stated formally:

Definition (*Capital lambda of x as game infimum*): If $G = (X, Y, M)$ is a game, then, for $x_0 \in X$, $\Lambda_G(x_0) = \inf_{y \in Y} M(x_0, y)$.

Definition (*Capital upsilon of y as game supremum*): If $G = (X, Y, M)$ is a game, then, for $y_0 \in Y$, $\Upsilon_G(y_0) = \sup_{x \in X} M(x, y_0)$

Definition (*Lower pure value as game supremum of infimum i.e.* ***best against the best***): If $G = (X, Y, M)$ is a game, then the lower pure value of G is the number $\lambda_G^* = \sup_{x \in X} \Lambda_G(x) = \sup_{x \in X} \inf_{y \in Y} M(x, y)$

Definition (*Upper pure value as game infimum of supremum i.e.* ***worst against worst***): If $G = (X, Y, M)$ is a game, then the lower pure value of G is the number $\upsilon_G^* = \inf_{y \in Y} \Upsilon_G(x) = \inf_{y \in Y} \sup_{x \in X} M(x, y)$

Theorem (*Inequality between pure lower value and pure upper value*): If $G = (X, Y, M)$ is a game, then, for $x_0 \in X$ and $y_0 \in Y$,
$\Lambda_G(x_0) \le \Upsilon_G(y_0)$ and $\lambda_G^* \le \upsilon_G^*$
Proof: $\Lambda_G(x_0) \le M(x_0, y_0) \le \Upsilon_G(y_0)$. Thus $\lambda_G^* \le \Upsilon_G(y_0)$ for all $y_0 \in Y$ and $\lambda_G^* \le \upsilon_G^*$.

Consider now any game G. No method of play for I can guarantee the player more than υ_G^* since II can restrict his loss to υ_G^* and no method of play for II can with certainty reduce his loss below λ_G^* since I can guarantee this amount. Thus, if $\lambda_G^* = \upsilon_G^* = \upsilon$ no method of play can guarantee either player any improvement over υ and we have seen that each player can attain υ (more precisely, approximate υ as closely as the player wishes).

Perfect Information Games

Among the games that do have a pure value are the *perfect information games* of which chess, checkers and tic-tac-toe are examples.
Essentially, a game of perfect information is one that can be described in terms of successive moves in such a way that, at each personal move, the mover knows the choices and the outcomes of all preceding personal and chance moves. Perfect information game is a game in which every information set is a unit set. It is intuitively clear that this condition is equivalent to the requirement that every branch of the tree of the game also be a tree of some game. The latter condition leads to an inductive definition for games in normal form. In this definition the order of a perfect information game intuitively corresponds to the maximum number of moves in that game.

Definition (*perfect information game*): A game $G = (X, Y, M)$ is a perfect information game of order 0 *iff* $M(x, y)$ is constant. A Game $G = (X, Y, M)$ is a perfect information game of order $n + 1$ *iff* there is a set A and a class $\mathcal{G}_A$ of games $G_a = (X_a, Y_a, M_a)$ for $a \in A$, such that each G_a is a perfect information game of order n, and such that either:

$Case\ 1)$ X consists of all pairs $x = (a, z)$ with $a \in A$, $z \in X_a$, Y consists of all functions y defined on A with $y(a) \in Y_a$ for all a, and

$M\big((a, z), y\big) = M_a\big(z, y(a)\big)$ *or*

$Case\ 2)$ Y consists of all pairs $y = (a, z)$ with $a \in A$, $z \in Y_a$, X consists of all functions x defined on A with $x(a) \in X_a$ for all a, and

$M\big(x, (a, z)\big) = M_a(x(a), z)$ *or*

$Case\ 3)$ X, Y consist of all functions x, y defined on A with $x(a) \in X_a$, $y(a) \in Y_a$ for all a, and

$M(x, y) = \sum_{a \in A} p(a) M_a\big(x(a), y(a)\big)$

where $p(a) \geq 0$, $\sum_{a \in A} p(a) = 1$
A game G is called perfect information game if G is perfect information game of order n for some n.

Our inductive description corresponds to the fact that the result of the first move in perfect information game with n moves is another perfect information game with $n - 1$ moves.
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