The Normal Form

The description of a game can be viewed as a listing of the strategies of the players and the outcome of any set of choices of strategies, without regard to the attitudes of the players toward various outcomes. We now indicate how the final simplification of the game – the normal form – is obtained, by taking into account the preferences of the players.
The result of any set of strategies $f_1, \dots, f_k$ is a probability distribution π_f over the set R of possible outcomes. It would be particularly convenient if a given player could express his/her preference pattern in R by a bounded numerical function u defined on R, such that he or she prefers r_1 to r_2 *iff* $u(r_1) > u(r_2)$. Note that $u(r_1) = u(r_2)$ denotes indifference between r_1 to r_2. Also, the function u is such that if for any probability distgribution ξ over R we define $U(\xi)$ as the expected value of $u(r)$ computed with respect to ξ as

$$U(\xi) = \sum_{r \in R} \xi(r) u(r)$$

the player prefers $\xi_1$to ξ_2 *iff* $U(\xi_1) > U(\xi_2)$.
It is remarkable fact that, under extremely plausible hypothesis concerning the preference pattern such function u exists.

Definition (*utility function*): The function U defined for all probability distributions ξ over R, is called the player's ***utility function***.
U is unique, for a given preference pattern up to a linear transformation. We will assume that each player has such utility function.
The aim of each player in the game is to maximize his/her expected utility. If U_i is the utility function of player i, his/her aim is to make $M_i(f_1, \dots, f_k) = U_i(\pi_f)$ as large as possible where π_f is the probability distribution for fixed $f_1, \dots, f_k$ over R determined by the overall chance move.
We are in a position to give a description of the normal form of a game:

Definition (*normal form of a game*): A game consists of k spaces $F_1, \dots, F_k$ and k bounded numerical functions $M_i(f_1, \dots, f_k)$ defined on the space of all k-tuples $(f_1, \dots, f_k)$, $f_i \in F_i, i = 1, \dots, k$. The game is played as follows: Player i chooses an element f_i of F_i, the k choices being made simultaneously and independently; player i then receives the amount $M_i(f_1, \dots, f_k), i = 1, \dots, k$. The aim of Player i is to make M_i as large as possible. The statement "Player i receives the amount $M_i(f_1, \dots, f_k)$" is shorthand of saying "a situation results whose utility for Player i is $M_i(f_1, \dots, f_k)$".

Example (*two player game involving coin-toss and a number choice*):
Player I moves first and selects one of the two integers $1, 2$. The referee then tosses a coin and if the outcome is "head", he informs player II of player I's choice and not otherwise. Player II then moves and selects one of two integers $3, 4$. The fourth move is again a chance move by the referee and consists of selecting one of three integers $1, 2, 3$ with respective probabilities $0.4, 0.2, 0.4$. The numbers selected in the first, third and the fourth move are added and the amount of dollars is paid by II to I if the sum is even and by I to II if the sum is odd. Note that $|R| = 2 \times 2 \times 2 \times 3 = 24$.

Here are the two strategy spaces:
$F_1 = \{f_1, f_2\}$; $f_1 = (1)$, $f_2 = (2)$

$F^2 = \{f^1, f^2, f^3, f^4, f^5, f^6, f^7, f^8\}$; $f^1 = (3,3,3), f^2 = (3,3,4), f^3 = (3,4,3), f^4 = (3,4,4), f^5 = (4,3,3), f^6 = (4,3,4), f^7 = (4,4,3), f^8 = (4,4,4)$
Here the first position of the triple is conditioned upon coin falling *Head* and player I choosing 1, the second position in the triple is conditioned upon coin falling head and player I choosing 2, and the third position of the triple is conditioned upon coin falling *Tail*.

The set R of possible outcomes for this game where I denotes player I, 0 denotes the referee and II denotes player II is shown below:
I –> 2 – 0 –> Head – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Head – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Head – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$

I –> 2 – 0 –> Head – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
I –> 2 – 0 –> Head – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
I –> 2 – 0 –> Head – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
I –> 2 – 0 –> Tail – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 2 – 0 –> Tail – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 2 – 0 –> Tail – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$

I –> 1 – 0 –> Head – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^5), (f_1, f^6), (f_1, f^7), (f_1, f^8)$
I –> 1 – 0 –> Head – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_1, f^5), (f_1, f^6), (f_1, f^7), (f_1, f^8)$
I –> 1 – 0 –> Head – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^5), (f_1, f^6), (f_1, f^7), (f_1, f^8)$
I –> 1 – 0 –> Tail – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^2), (f_1, f^4), (f_1, f^6), (f_1, f^8)$
I –> 1 – 0 –> Tail – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_1, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 1 – 0 –> Tail – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$

I –> 1 – 0 –> Head – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^1), (f_1, f^2), (f_1, f^3), (f_1, f^4)$
I –> 1 – 0 –> Head – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_1, f^1), (f_1, f^2), (f_1, f^3), (f_1, f^4)$
I –> 1 – 0 –> Head – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^1), (f_1, f^2), (f_1, f^3), (f_1, f^4)$

I –> 1 – 0 –> Tail – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 1 – 0 –> Tail – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_1, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 1 – 0 –> Tail – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$

In the theory of games it is usual to treat first a special class of games, *the two-person zero-sum games*. The theory of these games is particularly simple and complete and we will consider only such games in our discussion.

Definition (*two-person game*): a game with $k = 2$: we have only two utility functions M_1 and M_2 and two strategy sets F_1 and F_2 for each of the two players.

Definition (*zero-sum game*): A game for which the following holds true:
$\sum_{i=1}^{k} M_i(f_1, \ldots, f_k) = 0$ for all $f_1, \ldots, f_k$
More precisely, since each M_i is unique up to a linear transformation, a game is a **zero-sum** if there is a determination of $M_1, \ldots., M_k$ for which $\sum_{i=1}^{k} M_i(f_1, \ldots, f_k) = 0$ for all $f_1, \ldots, f_k$. Thus a two-person zero-sum game is a game between two players in which their interests are diametrically opposed: one player gains at the expense of the other. Consequently, there is no motive for collusion between the players. It is precisely the fact that collusion is unprofitable that simplifies the theory.

Definition (*constant-sum game*): A **constant-sum game** i.e. one in which $\sum_{i=1}^{k} M_i(f_1, \ldots, f_k) = c$ for all $f_1, \ldots, f_k$ is zero-sum game in the sense defined above, since an alternative choice of utility functions is $M_1^* = M_1 - c$, $M_i^* = M_i$ for $i \neq 1$, and $\sum_{i=1}^{k} M_i^* = 0$. Thus the theory developed for zero sum two person games applies for constant sum two person games.

Since for two-person zero sum game we have $M_2(f_1, f_2) = -M_1(f_1, f_2)$ we need to specify only M_1. We will consider only two-person zero-sum games from now on.

Definition (*game in a normal form*): A **game in a normal form** is a triple (X, Y, M), where X, Y are arbitrary spaces and M is a bounded numerical function defined on the product space $X \times Y$ of pairs (x, y), $x \in X, y \in Y$. The points $x(y)$ are called strategies for player I (II) and the function M is called payoff. The game G is played as follows: I chooses $x \in X$, II chooses $y \in Y$, the choises being made independently and simultaneously. II then pays I the amount $M(x, y)$.