

# Spectral Density of Sample Covariance Matrices of Colored Noise

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## Abstract

We study the dependence of the spectral density of the covariance matrix ensemble on the power spectrum of the underlying multivariate signal. The white noise signal leads to the celebrated Marchenko-Pastur formula. We demonstrate results for some colored noise signals.

## 1. INTRODUCTION

The covariance matrix is a fundamental object in the multivariate statistics and probability theory. A sample covariance matrix use only part of the data and is determined by the number of samples. But it has the same population size as the covariance matrix. When the population size is not large and the number of sampling points is sufficient the sample covariance matrix is a good approximate of the covariance matrix. Unfortunately, we usually investigate data with a sampling rate that is not sufficient and select the number of samples to be comparable with the population size. In this case the sample covariance matrix is no longer a good approximation to the covariance matrix.

Marchenko and Pastur [5] were discussing a limiting case when the ratio  $p$  between the population size  $m$  and the number of samples  $n$  remains constant and  $n$  grows without bounds. They studied the sample covariance matrix  $c$  defined by the formula

$$c_{i,j} = \sum_{k=1}^n x_k^i x_k^j \quad \text{or} \quad c = xx^T \quad (1)$$

where  $x_k^i$  stands for the normalized (i. e. with zero-mean) independent and identically distributed random data. The upper and lower indices denote the population and sample index respectively.

The spectral density of  $c$  depends in the limit only on the variance  $\sigma^2$  of  $x$  and on the population-to-sample ratio  $p$

$$\rho(\lambda) = \begin{cases} \frac{1}{2\pi\lambda p\sigma^2} \sqrt{(b-\lambda)(\lambda-a)} & a \leq \lambda \leq b \\ 0 & \lambda < a \vee \lambda > b \end{cases}, \quad (2)$$

where  $a = \sigma^2 (1 - \sqrt{p})^2$ ,  $b = \sigma^2 (1 + \sqrt{p})^2$ . For  $p > 1$ , there is an additional Dirac measure at  $\lambda = 0$  of mass  $1 - \frac{1}{p}$ .

The formula (2) describes the spectral density of the sample covariance matrices of a white noise signal. So the power spectrum of the signal vector  $x_k^i$  is constant. In many situations however the signal is not accessible directly. What is actually measured is its filtered image. For instance if we deal with the EEG signal we do not measure directly the cerebral signal but only its image filtered through the tissues in the skull. The natural question is of course to what degree the spectral density of the sample covariance matrix depends on such signal filtering. We show that spectral density (2) is universal in certain circumstances and that it represents a special case of the general probability distribution which depends on the power spectrum of the signal.

## 2. SIGNAL FREQUENCY ANALYSIS AND THE COVARIANCE MATRIX SPECTRAL DENSITY

The measuring device has a finite sampling rate that leads to a discrete set of the measured values. For that reason we will use a discrete Fourier transformation (DFT) for the frequency analysis. In our notation DFT is defined as

$$X_k = \frac{1}{\sqrt{n}} \sum_{j=1}^n x_j e^{-i\omega_k t_j}, \quad \omega_k = (k-1) \frac{2\pi f}{n}, \quad t_j = \frac{j-1}{f}, \quad (3)$$

where  $f$  is the sampling rate.

The Fourier transform  $X$  of a real vector  $x$  is complex and fulfills

$$X_i = \overline{X_{n-i}}, \quad 1 < i \leq n. \quad (4)$$

For real  $x$  it is therefore useful to use another transformation

$$\tilde{X}_i = \sqrt{2} \cdot \begin{cases} \operatorname{Re}(X_i) & 1 < i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ \operatorname{Im}(X_{n-i}) & \left\lfloor \frac{n}{2} \right\rfloor + 1 < i \leq n \end{cases}, \quad (5)$$

where  $[a]$  means the integer part of  $a$ . The remaining two elements are defined separately. Since  $X_1$  is real and equal to the sum of  $x_i$ , we define  $\tilde{X}_1 = X_1$ .

For even  $n$  we take  $\tilde{X}_{\frac{n}{2}+1} = X_{\frac{n}{2}+1}$  since  $X_{\frac{n}{2}+1}$  is real. For odd  $n$  we use  $\tilde{X}_{\frac{n+1}{2}} = \sqrt{2}\text{Re}\left(X_{\frac{n+1}{2}}\right)$ .

The transformed vector  $\tilde{X}$  is real and contains the full information on the frequency properties of the original vector  $x$ . The definition of the covariance matrix (1) can be easily rewritten using the discrete Fourier transformation and the transformation (5):

$$c_{i,j} = \sum_{k=1}^n x_k^i x_k^j = \sum_{k=1}^n X_k^i \overline{X_k^j} = \sum_{k=1}^n \tilde{X}_k^i \tilde{X}_k^j \quad \text{or} \quad c = xx^T = XX^* = \tilde{X}\tilde{X}^T, \quad (6)$$

where the rows of the matrices  $X$  and  $\tilde{X}$  are the transformed rows of the matrix  $x$ .

Colored noise is a random signal with a non-flat power spectrum. We are interested in the question how the profile of the power spectrum influence the spectral density of the sample covariance matrix. In what follows we assume that the data matrix  $x$  has independent rows with identical power spectra and zero mean. Then the elements of the matrix  $\tilde{X}$  are also of mean zero - see the definition (6). Moreover the elements in the rows of the matrix  $x$  are independent. The transform  $\tilde{X}$  leads therefore also to a matrix with independent rows. Since the signal phase is random we get

$$\left\langle \text{Re}(X_k^i)^2 \right\rangle = \left\langle \text{Im}(X_k^i)^2 \right\rangle \quad (7)$$

and hence

$$\left\langle \left(\tilde{X}_k^i\right)^2 \right\rangle = \left\langle \left(\tilde{X}_{n-k}^i\right)^2 \right\rangle, \quad \left[\frac{n}{2}\right] + 1 < k \leq n, \quad 1 \leq i \leq m, \quad (8)$$

where the angle brackets denote the sample mean. To find the spectral density of the covariance matrix ensemble we use now the a theorem of Girko [4].

**Theorem 1:** Let  $A$  be a  $m \times [cm]$  random matrix with independent entries of a zero-mean that satisfy the condition

$$m\text{Var}(A_{ij}) < B, \quad (9)$$

for some bound  $B < \infty$ . Moreover, let for each  $m$  be  $v_m$  a function  $v_m: [0, 1] \times [0, c] \rightarrow \mathbb{R}$  defined by:

$$v_m(\mu, \nu) = m\text{Var}(A_{ij}), \quad \frac{i}{m} \leq \mu \leq \frac{i+1}{m}, \quad \frac{j}{m} \leq \nu \leq \frac{j+1}{m} \quad (10)$$

and suppose that  $v_m$  converges uniformly to a limiting bounded function  $v$  for  $m \rightarrow \infty$ . Then the limiting eigenvalue distribution  $\rho(\lambda)$  of the covariation matrix  $AA^T$  exists and for every  $\tau \geq 0$  satisfies:

$$\int_0^\infty \frac{\rho(\lambda) d\lambda}{1 + \tau\lambda} = \int_0^1 u(\mu, \tau) d\mu, \quad (11)$$

with  $u(\mu, \tau)$  solving the equation

$$u(\mu, \tau) = \frac{1}{1 + \tau \int_0^c \frac{v(\mu, \nu) d\nu}{1 + \tau \int_0^1 u(\xi, \tau) v(\xi, \nu) d\xi}}. \quad (12)$$

The solution of the equation (12) exists and is unique in the class of functions  $u(\mu, \tau) \geq 0$ , analytical on  $\tau$  and continuous on  $\mu \in [0, 1]$ .

Let us use this theorem taking  $A = \tilde{X}$ . We immediately see that  $c = \frac{n}{m}$ .

Since all the rows of the matrix  $\tilde{X}$  have identical power spectra, the function  $v(\mu, \nu)$  will not depend on  $\mu$ . The equation (12) shows that the function  $u(\mu, \tau)$  is also  $\mu$  independent. Inserting  $v(\nu)$  and  $u(\tau)$  into the equations (11) and (12), we get

$$\int_0^\infty \frac{\rho(\lambda) d\lambda}{1 + \tau\lambda} = u(\tau) \quad (13)$$

and

$$u(\tau) = \frac{1}{1 + \tau \int_0^c \frac{v(\nu) d\nu}{1 + \tau u(\tau) v(\nu)}}. \quad (14)$$

The spectral density is determined by the function  $v(\nu)$  (that itself is a function of the power spectrum). However, to solve the equations (13) and (14) for a general power spectrum profile is extremely difficult. So in next chapters we will try to get an exact formula for the spectral density at least in the simplest cases.

### 3. GENERALIZED WHITE NOISE

Consider a situation when the signals from several sources come into one given point. Every sources produce a noise in a specific frequency bands and the frequency bands are disjoint. Further, the intensity of all sources is the same. The total incoming signal has gaps in the power spectrum. The function  $v(\nu)$

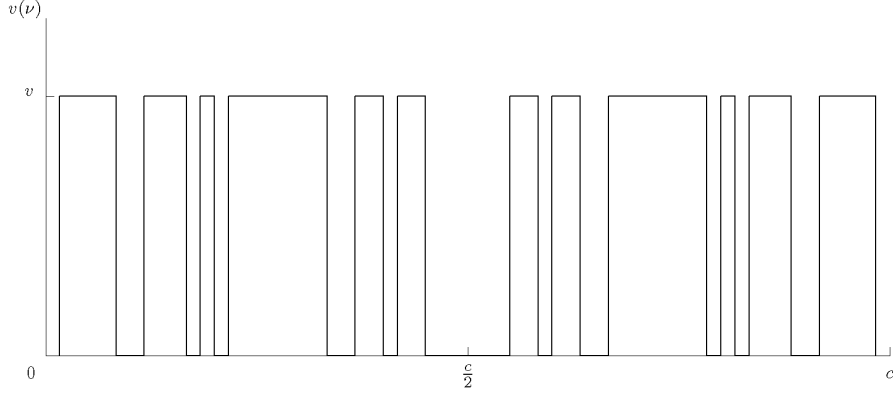


Figure 1: An example of the function  $v(\nu)$  for the generalized white noise.

(see the definition in the Theorem 1.) is a step function with steps of an equal height, see the figure (1).

In order to evaluate the spectral density we have to know the size  $d$  of the support of  $v(\nu)$  (i.e. the sum of the lengths of all intervals where  $v(\nu)$  is nonzero) and the value  $v$  of the function  $v(\nu)$  on this support (the function  $v(\nu)$  is constant on the support). The solution of the equation (14) gives

$$u(\tau) = \frac{-[1 + v\tau(d - 1)] + \sqrt{[1 + v\tau(d - 1)]^2 + 4v\tau}}{2v\tau} \quad (15)$$

and the integral equation (13) can be transformed into the form

$$\int_0^\infty \frac{\rho(\lambda)d\lambda}{\nu + \lambda} = \frac{u\left(\frac{1}{\nu}\right)}{\nu} = \tilde{u}(\nu), \quad (16)$$

The generalized Stieltjes transform

$$G(\xi) = \int_0^\infty \frac{F(\nu)d\nu}{(\nu + \xi)^q}, \quad |\arg \xi| < \pi, \quad (17)$$

has an inverse [7]

$$F(\nu) = -\frac{1}{2\pi i} \nu^q \int_{\mathcal{C}} \frac{G'(yw)dw}{(1 + w)^{1-q}}, \quad (18)$$

where  $q > 0$  and  $\mathcal{C}$  is a contour starting at the point  $w = -1$  and encircling the origin in the counterclockwise sense. For  $q = 1$  the eq. (18) can be explicitly evaluated:

$$F(\nu) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} (G(-\nu - i\varepsilon) - G(-\nu + i\varepsilon)) \quad (19)$$

for  $\nu > 0$ .

We find the spectral density  $\rho(\lambda)$  as an inverse Stieltjes transform with  $q = 1$ . The limit in (19) with  $G = \tilde{u}$  and  $F = \rho$  leads to

$$\rho(\lambda) = \begin{cases} \frac{1}{2\pi\lambda v} \sqrt{(\lambda_1 - \lambda)(\lambda - \lambda_2)} & \lambda_1 \leq \lambda \leq \lambda_2 \\ 0 & \lambda < \lambda_1 \vee \lambda > \lambda_2 \end{cases}, \quad (20)$$

where  $\lambda_{1,2} = v \left(1 \mp \sqrt{d}\right)^2$ . For  $d < 1$ , there is an additional Dirac measure at  $\lambda = 0$  of mass  $1 - d$ .

The above formula is exactly equal to the Marchenko-Pastur result (2). The existence of the Dirac measure is consequence of singularity of covariance matrices for  $d < 1$ .

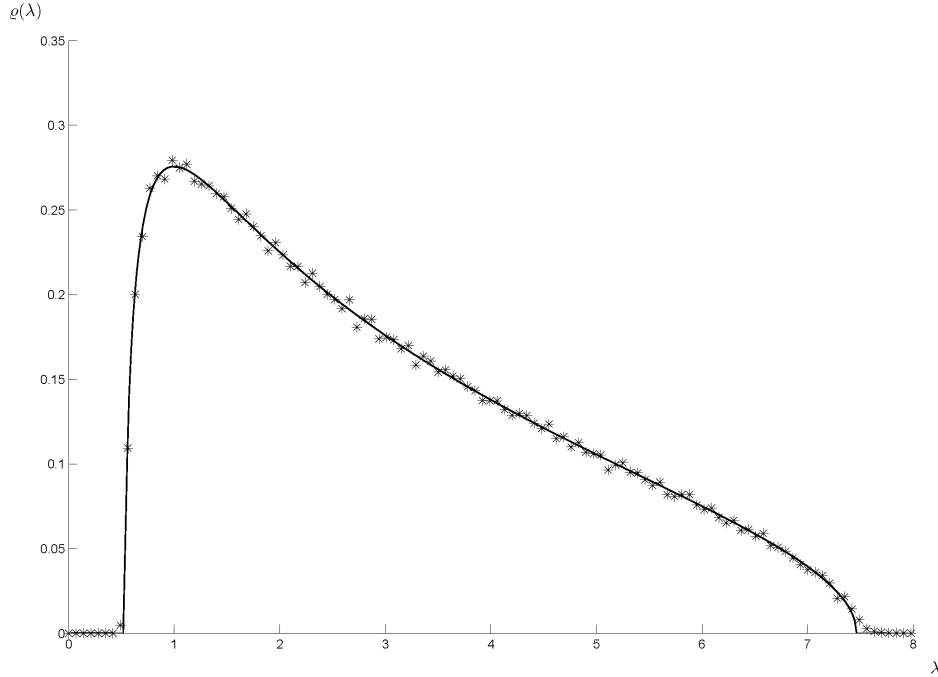


Figure 2: Spectral density. Numerical results (stars) are compared with the theoretical results for  $\rho(\lambda)$  with the parameters  $v = 1$  and  $d = 3$ .

The interesting point is that while the Marchenko-Pastur result was derived

for a white noise signal (i.e. the power spectrum was constant over the whole frequency range) we get the same result also when the power spectrum has a finite support and contains a finite number of gaps. Moreover - the exact position of the gaps is irrelevant and the result depends on the total support size only.

#### 4. COLORED NOISE

Let us now pass to the case when the highs of the power spectrum segments are unequal. This is a quite general case since in fact any power spectrum profile can be approximated by a step function.

Inserting the function  $v(\nu)$  into the integral (14) and using the definition (16) gives

$$\tau \tilde{u}(\tau) + \sum_{i=1}^K d_i - 1 = \sum_{i=1}^K \frac{d_i}{1 + v_i \tilde{u}(\tau)}, \quad (21)$$

where  $K$  is number of the nonzero segments in the function  $v(\nu)$ . In what follows the symbols  $v$  and  $d$  denote vectors (in contrast to their previous meaning as constants) with elements  $v_i$  and  $d_i$  denoting the highs and lengths of segments respectively.

To solve the equation (21) means to find the roots of a polynomial of degree  $(K + 1)$ . This cannot be done explicitly. However - in similarity to the previous case with the bars of equal height - the solution of the equation (21) does not depend on the exact location of power spectrum bars and is positive on the positive real axis.

As an illustration we give the formula for the case with two steps:

$$K = 2, \quad v = \left(1, \frac{1}{2}\right), \quad d = \left(\frac{1}{2}, \frac{1}{2}\right). \quad (22)$$

The spectral density is than

$$\rho(\lambda) = \sqrt{\frac{\left(-\sqrt[3]{-27\lambda - 8\lambda^3 + 36\lambda^2 + 27 + 3\sqrt{3}\sqrt{\lambda}\sqrt{54 - 16\lambda^3 + 72\lambda^2 - 81\lambda} + 2\lambda - 3}\right)^2}{8\pi^2\lambda\sqrt[3]{-27\lambda - 8\lambda^3 + 36\lambda^2 + 27 + 3\sqrt{3}\sqrt{\lambda}\sqrt{54 - 16\lambda^3 + 72\lambda^2 - 81\lambda}}}}. \quad (23)$$

The spectral density of the covariance matrices will in this case again not depend on the exact location of power spectrum steps. Also the order of the steps is not important. Moreover segments of the same height can be linked into one segment with the width equal to the sum of the widths of the two particular segments. In this sense the spectral density does not depend on the reshuffling of the power spectrum.

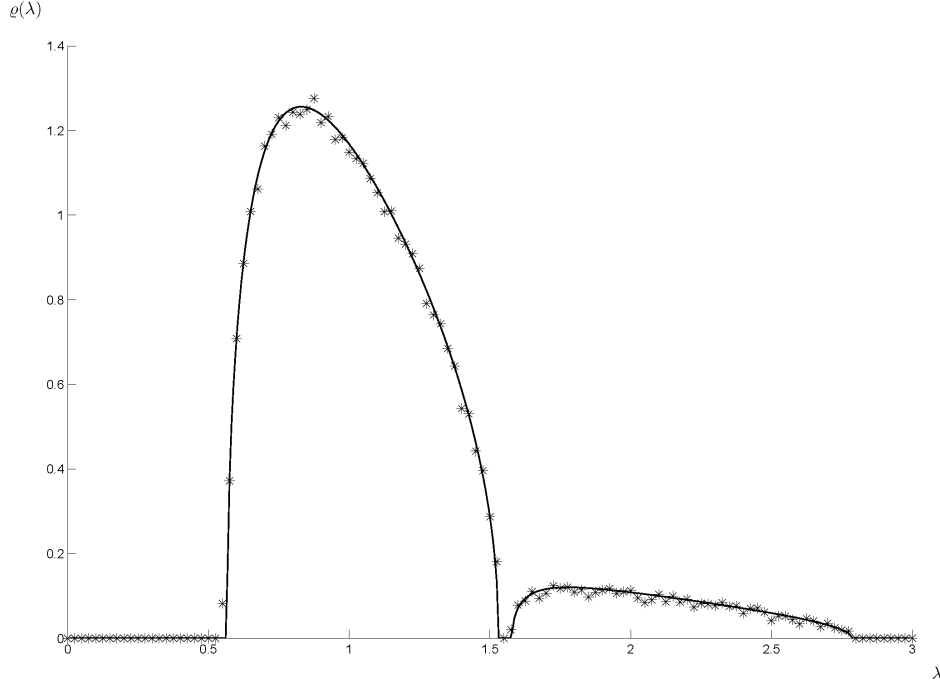


Figure 3: Spectral density. The numerical results (stars) are compared with the theoretical prediction of  $\rho(\lambda)$  for  $K = 2$ ,  $v = \left(1, \frac{1}{15}\right)$ ,  $d = \left(\frac{1}{10}, 15\right)$ .

## 5. SUMMARY

The spectral density of the covariance matrix is used in many fields of physics and economy (see [1], [2], [3]). To analyze the system the power spectrum of the signal has to be taken into account. An example is the spectral analysis of the EEG signal [8],[6].

The power spectrum directly influence the signal correlation properties. For instance the particular matrix elements of the covariance matrix depend on it. Nevertheless the spectral density of the covariance matrix ensemble remains nearly invariant.

In the presented paper we discuss the spectral density of the covariance matrix and its dependence on the power spectrum profile of the underlying signal. The results show that the spectral density is invariant under the reshuffling of its power spectrum coefficients and hence independent on the exact spectral profile of the signal.



## 6. ACKNOWLEDGEMENT

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