Hamilton-Jacobi-Bellman equations and optimal control

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1. Introduction

The aim of this paper is to offer a quick overview of some applications of the theory of *viscosity solutions* of *Hamilton-Jacobi-Bellman* equations connected to nonlinear optimal control problems.
The central role played by *value functions* and Hamilton-Jacobi equations in the
Calculus of Variations was recognized as early as in C. Caratheodory's work, see [1] for a survey. Similar ideas reappeared under the name of Dynamic Programming in the work of R. Bellman and his school and became a standard tool for the synthesis of feedback controls for discrete time systems. However, the lack of smoothness of value functions, even in simple problems, was recognized as a severe restriction to the range of applicability of Hamilton-Jacobi-Bellman theory (in short, HJB from now on) to continuous time processes.
The main reasons for this limitation are twofold:

(i) the very basic difficulty to give an appropriate global meaning to the HJB equation (a fully non linear partial differential equation) satisfied at all points of differentiability by the value function,

(ii) to identify value function as the unique solution of that equation; a related important issue is that of stability of value functions, specially in connection with approximation procedures required for computational purposes.

Several non classical notions of solutions have been therefore proposed to overcome these difficulties. Let us mention, in this respect, the Kruzkov theory which applies, in the case of sufficiently smooth Hamiltonians, to semiconvex functions satisfying the HJB equation almost everywhere (see [2, 3] and also [4, 5] for recent results on semiconcavity of value functions).

Only in the 80's, however, a decisive impulse to the setting of a satisfactory mathematical framework to Dynamic Programming came from the introduction by Crandall-Lions [6] of the notion of viscosity solutions of Hamilton-Jacobi equations.

The presentation here, which is mainly based on material contained in the forthcoming book [7], to which we refer for detailed proofs, will be focused on optimization problems for controlled ordinary differential equations and discrete time systems.

The content of the paper is as follows:

2 Some examples of HJB equations in optimal control
3 Some basic facts about viscosity solutions
4 Necessary and sufficient conditions
5 Approximate synthesis of feedbacks
6 Final remarks

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2. Some examples of HJB equations in optimal control

Let us consider the *control system*, whose solution will be denoted by y_x^α,

$$\dot{y}(t) = f(y(t), \alpha(t)) \quad (t > 0), \quad y(0) = x. \tag{2..1}$$

In (2..1), $f : \mathbb{R}^N \times A \longrightarrow \mathbb{R}^N$, A is a topological space and

$$\alpha \in M(A) = \{\alpha : [0, +\infty) \longrightarrow A,\, \alpha \text{ measurable}\}$$

We assume:

(A_0) A is compact, f is continuous on $\mathbb{R}^N \times A$.
(A_1) $\exists L_f \geq 0 : |f(x,a) - f(y,a)| \leq L_f |x-y|, \forall a \in A$.

Here we list a few classical examples of optimal control problems and associated HJB equations (complemented in some cases by initial or boundary conditions) for system (2..1).

2.1 The infinite horizon discounted regulator

For this problem the *value function* is

$$v(x) = Inf_{\alpha \in M(A)} \int_0^{+\infty} l(y_x^\alpha(t), \alpha(t))\, e^{-\lambda t} dt, \tag{2..2}$$

where the running cost l is a real valued function on $\mathbb{R}^N \times A$ and the discount factor λ is a positive number.
The corresponding HJB equation is

$$\lambda v(x) + Sup_{a \in A}[-f(x,a) \cdot Dv(x) - l(x,a)] = 0 \quad \text{in } \mathbb{R}^N. \tag{2..3}$$

2.2 The Mayer problem

In this case the value function is

$$v(x,t) = Inf_{\alpha \in M(A)} \quad g(y_x^\alpha(t))\,,$$

where the terminal cost is a given real valued function on $\mathbb{R}^N$.
The HJB equation takes now the form of a Cauchy problem, namely:

$$\frac{\partial v}{\partial t} + Sup_{a \in A}[-f(x,a) \cdot Dv(x)] = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, +\infty)$$
$$v(x,0) = g(x).$$

2.3 Exit time problems

In these problems a subset $\mathcal{T}$ of $\mathbb{R}^N$ (the *target*) is given . If t_x^α denotes the first time when the trajectory y_x^α of system (2..1) hits the target, we consider the value function

$$v(x) = Inf_{\alpha \in M(A)} J(x, \alpha).$$

Here we set:

$$J(x,\alpha) = \int_0^{t_x^\alpha} l(y_x^\alpha(t), \alpha(t))\, e^{-\lambda t} dt + e^{-\lambda t_x^\alpha} g(y_x^\alpha({t_x}^\alpha)), \quad \text{if } t_x^\alpha < +\infty,$$

and

$$J(x,\alpha) = \int_0^{+\infty} l(y_x^\alpha(t), \alpha(t))\, e^{-\lambda t} dt, \quad \text{otherwise.}$$

The corresponding HJB equation is now the Dirichlet problem:

$$\lambda v(x) + Sup_{a \in A}[-f(x,a) \cdot Dv(x) - l(x,a)] = 0 \text{ in } \mathbb{R}^N \setminus \mathcal{T}\,,$$
$$v(x) = g(x) \quad \text{if} \quad x \in \partial \mathcal{T}.$$

2.4 State-constrained problems

For this kind of problems the value function is

$$v(x) = Inf_{\alpha \in M_x(A)} \int_0^{+\infty} l(y_x^\alpha(t), \alpha(t))\, e^{-\lambda t} dt.$$

Here, the feasible controls are taken in the set

$$M_x(A) = \{\alpha \in M(A) : y_x^\alpha(t) \in \overline{\Omega}, \quad \forall t \geq 0\}.$$

The set $\overline{\Omega}$ plays here the role of a constraint on the states of system (2..1).
For the present problem the HJB equation (2..3) is complemented with a quite unusual boundary condition, namely

$$\lambda v(x) + Sup_{a \in A}[-f(x,a) \cdot Dv(x) - l(x,a)] \geq 0 \quad \text{in} \quad \partial\Omega.$$

2.5 The monotone control problem

In this example the control set A is the interval $[0,1]$ and the value function
$v : \mathbb{R}^N \times [0,1] \longrightarrow \mathbb{R}$ is

$$v(x,a) = Inf_{\alpha \in M_a^m(A)} \int_0^{+\infty} l(y_x^\alpha(t), \alpha(t))\, e^{-\lambda t} dt.$$

Here we set:

$$M_a^m(A) = \{\alpha \in M(A), \quad \alpha \text{ nondecreasing}, \alpha(0) \geq a\}.$$

The HJB equation for this problem takes the form of the evolutionary variational inequality:

$$Max[\lambda v(x,a) - f(x,a) \cdot D_x v(x,a) - l(x,a); -\frac{\partial v}{\partial a}] = 0 \quad \text{in} \quad \mathbb{R}^N \times [0,1),$$

$$v(x,1) = \int_0^{+\infty} l(y_x^1(t), 1)\, e^{-\lambda t} dt \quad \text{in } \mathbb{R}^N.$$

3. Some basic facts about viscosity solutions

It is well-known that the value functions of the various optimal control problems described above satisfy the corresponding HJB equations at all points of differentiability. This fact can be proved by means of the *Dynamic Programming Principle*, a functional equation relating the value of v at the initial point x to its value at some point reached later by a trajectory of system (2..1).
Let us just indicate that, in the simplest case of the infinite horizon discounted regulator problem, the Dynamic Programming Principle is expressed by the identity

$$v(x) = Inf_{\alpha \in M(A)} \int_0^T l(y_x^\alpha(t), \alpha(t))\, e^{-\lambda t} dt + e^{-\lambda T} v((y_x^\alpha(T)), \tag{3..4}$$

which holds for all $x \in \mathbb{R}^N$ and $T > 0$.
It is well-known as well that everywhere differentiability of v cannot hold in general. Indeed, simple examples show that two different optimal trajectories for some initial point x may exist implying non differentiability of v at x.

A concept of solution which allows to understand HJB equations globally, in a weak sense, is provided by the notion of viscosity solution.

The definition is based on the notion of first order *semidifferentials* of a function $u : \Omega \longrightarrow \mathbb{R}$, Ω being an open subset of $\mathbb{R}^N$.

These are the convex sets

$$D^+u(x) = \left\{ p \in \mathbb{R}^N : \limsup_{\Omega \ni y \to x} \frac{u(y) - u(x) - p \cdot (y-x)}{|x-y|} \leq 0 \right\}$$

$$D^-u(x) = \left\{ p \in \mathbb{R}^N : \liminf_{\Omega \ni y \to x} \frac{u(y) - u(x) - p \cdot (y-x)}{|x-y|} \geq 0 \right\}.$$

Consider the partial differential equation

$$F(x, u(x), Du(x)) = 0 \quad x \in \Omega \tag{3..5}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a continuous function.

Definition 1 *Let $u : \Omega \longrightarrow \mathbb{R}$. Then,*

(i) u is a viscosity subsolution of (3..5) if u is upper semicontinuous on Ω and

$$F(x, u(x), p) \leq 0 \quad \forall p \in D^+u(x), \forall x \in \Omega;$$

(ii) u is a viscosity supersolution of (3..5) if u is lower semicontinuous on Ω and

$$F(x, u(x), p) \geq 0, \quad \forall p \in D^-u(x), \forall x \in \Omega;$$

(iii) u is a viscosity solution of (3..5) if u satisfies both (i) and (ii).

Here we list some facts and remarks about this definition:
- if u is continuous on Ω then the sets $A^{\pm} = \{x \in \Omega : D^{\pm}u(x) \neq \emptyset\}$ are dense in Ω;
- if u is a viscosity solution of (3..5) then $F(x, u(x), Du(x)) = 0$ at any x where u is differentiable;
- if u is Lipschitz continuous and satisfies (3..5) in the viscosity sense then, $F(x, u(x), Du(x)) = 0$ almost everywhere;
- if $u \in C^1(\Omega)$ satisfies (3..5) at all points then u is a viscosity solution of (3..5); on the other hand, if $u \in C^1(\Omega)$ is a viscosity solution of (3..5) then u is a classical solution of (3..5);
- the equations (3..5) and $-F(x, u(x), Du(x)) = 0$ are not equivalent in the viscosity sense.
- a Lipschitz continuous function is a solution of equation (3..5) in the *extended sense* if

$$Sup_{p \in \partial u(x)} F(x, u(x), p) = 0$$

where ∂u is the Clarke's gradient of u; if u is a solution in the extended sense then u is both a viscosity subsolution of (3..5) and a supersolution of

$$-F(x, u(x), Du(x)) = 0;$$

on the other hand, a Lipschitz continuous viscosity solution u of (3..5) is a solution in the extended sense as well.

Although the notion of viscosity solution is a weak one, good *comparison*,*uniqueness* and *stability* properties hold. Sample results in these directions are the following ones where, for simplicity, we take

$$F(x, r, p) = r + H(x, p).$$

Theorem 1 *Assume that H is continuous and satisfies*

$$|H(x,p) - H(y,p)| \leq \omega(|x-y|(1+|p|))$$

and

$$|H(x,p) - H(x,q)| \leq \omega(|p-q|)$$

for all $x, y \in \Omega$,and $p, q \in \mathbb{R}^N$, where $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, nondecreasing, $\omega(0) = 0$.

If u,w are bounded continuous viscosity sub and super solutions of

$$u(x) + H(x, Du(x)) = 0 \quad in \quad \mathbb{R}^N$$

then,

$$u \equiv w \quad in \quad \mathbb{R}^N.$$

Let us just sketch the proof of the inequality $u \leq w$, the argument to prove the reverse being completely similar. Assume, by contradiction, the existence of some x_0 such that:

$$u(x_0) - w(x_0) := \delta > 0.$$

Define then

$$\Phi(x,y) = u(x) - w(y) - \frac{|x-y|^2}{2\varepsilon} - \beta((g(x) + g(y)),$$

where $\varepsilon > 0$ and

$$g(x) = \frac{1}{2} log(1 + |x|^2).$$

The parameter β is chosen as to satisfy

$$\beta \leq \frac{\delta}{4g(x_0)}, \quad \omega(2\beta) \leq \frac{\delta}{6}.$$

The above choices yield:

$$Sup_{\mathbb{R}^N \times \mathbb{R}^N}\ \Phi \geq \Phi(x_0, y_0) \geq \frac{\delta}{2}. \tag{3..6}$$

By the assumptions made on u, w it is not hard to prove
the existence of $(x_\varepsilon, y_\varepsilon)$ such that

$$\Phi(x_\varepsilon, y_\varepsilon) = Sup_{\mathbb{R}^N \times \mathbb{R}^N}\ \Phi$$

and that $(x_\varepsilon, y_\varepsilon)$ remain uniformly bounded with respect to ε. The key point is to observe that

$$p_\varepsilon := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \beta Dg(x_\varepsilon) \in D^+ u(x_\varepsilon),$$

and

$$q_\varepsilon := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \beta Dg(y_\varepsilon) \in D^- u(y_\varepsilon).$$

By definition of viscosity sub and super solution, then

$$u(x_\varepsilon) + H(x_\varepsilon, p_\varepsilon) \leq 0 \leq w(y_\varepsilon) + H(y_\varepsilon, q_\varepsilon).$$

By the assumptions on H this implies:

$$u(x_\varepsilon) - w(y_\varepsilon) \leq \omega(|p_\varepsilon - q_\varepsilon|) + \omega(|x_\varepsilon - y_\varepsilon|(1 + |p_\varepsilon|)).$$

Therefore, by the choice of β and the fact that $|Dg| \leq 1$,

$$\Phi(x_\varepsilon, y_\varepsilon) \leq \frac{\delta}{6} + \omega(\varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2 + \frac{\delta}{4g(x_0)}|x_\varepsilon - y_\varepsilon|). \tag{3..7}$$

Observe now that the inequality

$$\Phi(x_\varepsilon, x_\varepsilon) + \Phi(y_\varepsilon, y_\varepsilon) \leq 2\Phi(x_\varepsilon, y_\varepsilon)$$

yields, since u and w are bounded, the following estimate

$$|x_\varepsilon - y_\varepsilon| \leq (\varepsilon C)^{\frac{1}{2}} \quad \text{for some } C > 0$$

and, consequently,

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

At this point it is easy to realize that inequality (3..7) is contradictory with (3..6).
This concludes the proof that $u \leq w$.

As for stability we have:

Theorem 2 *Assume that* H_n *is continuous on* $\Omega \times \mathbb{R}^N$ *for each* $n = 1, 2, ...$ *and that*

$$u_n(x) + H_n(x, Du_n(x)) = 0 \quad \textit{in } \Omega, \quad \textit{in the viscosity sense.}$$

Assume also that, as $n \to +\infty$,

$$u_n \to u, \quad H_n \to H \quad \textit{locally uniformly in} \quad \Omega \times \mathbb{R}^N.$$

Then,

$$u(x) + H(x, Du(x)) = 0 \quad \textit{in } \Omega, \quad \textit{in the viscosity sense.}$$

The uniform convergence of u_n to u guarantees that for any $x \in \Omega$ and $p \in D^+u(x)$ there exist $x_n \in \Omega$ and $p_n \in D^+u_n(x)$ such that

$$x_n \to x, \quad p_n \to p.$$

From this fact it follows easily that u is a subsolution of the limit equation.
A completely similar argument, with D^+ replaced by D^-, shows that u is a supersolution as well.

The theory outlined up to now does not depend on the *convexity* of the map

$$p \longrightarrow H(x, p).$$

When this property holds true (this fact is typical of Hamiltonians occurring in optimal control problems), then some special results are valid. Let us just mention the non obvious fact that in this case a function u is a viscosity solution of equation (3..5) if and only if

$$F(x, u(x), Du(x) = 0 \quad \forall p \in D^-u(x)$$

(see ([14, 15]).

4. Necessary and sufficient conditions

The value functions in the examples 2.1, 2.2, 2.5 of Section 2 are continuous under the assumptions (A_0), (A_1) plus some uniform continuity conditions on the costs l, g. Continuity of v in problems 2.3 and 2.4 is guaranteed under an additional restriction involving the behaviour of the dynamics f on the boundary of Ω or of $\mathcal{T}$. For problem 2.4 this condition is

$$Inf_{a \in A}\, f(x, a) \cdot n(x) < 0 \quad \forall x \in \partial\Omega$$

where $n(x)$ denotes the outward normal to $\partial\Omega$ at x, see ([16, 17]).

The link between the optimal control problem and the HJB equation is provided by the Dynamic Programming Principle. In all the examples presented the value function turns out to be the viscosity solution of the corresponding HJB equation.
Let us be more specific on this point with reference to Example 2.1 ; similar results hold, however, for the various examples shown in Section 2.
For the infinite horizon discounted regulator problem the Dynamic Programming Principle is expressed by the identity

$$v(x) = Inf_{\alpha \in M(A)} \int_0^T l(y_x^\alpha(t), \alpha(t))\, e^{-\lambda t} dt + e^{-\lambda T} v((y_x^\alpha(T)), \tag{4..8}$$

which holds for all $x \in \mathbb{R}^N$ and $T > 0$.

We have:

Theorem 3 *Assume* (A_0),(A_1)*; assume also* l *continuous and bounded on* $\mathbb{R}^N \times A$ *and*

$$(A_2) \quad \exists L_l \geq 0 : |l(x,a) - l(y,a)| \leq L_l |x-y|, \quad \forall a \in A.$$

Then, the value function v *of the infinite horizon discounted regulator problem is a bounded, continuous viscosity solution of (2..3).*
Moreover, v *is the unique viscosity solution of (2..3) in the class of bounded, continuous functions on* $\mathbb{R}^N$*.*

For the proof, note that the second statement follows from the first and the uniqueness Theorem 1.
Let just indicate how to prove that v is a viscosity solution of (2..3). Observe that (4..8) with $\alpha(t) \equiv a \in A$ yields

$$\frac{v(y_x^a(T) - v(x)}{T} \geq \frac{1}{T} \int_0^T l(y_x^a(t), a)\, e^{-\lambda t} dt + (e^{-\lambda T} - 1) v(y_x^a(T)) \tag{4..9}$$

for all $T > 0$.
On the other hand, $p \in D^+ v(x)$ implies

$$\frac{v(y_x^a(T) - v(x)}{T} \leq \frac{1}{T} p \cdot (y_x^a(T) - x) + \frac{o(T)}{T}, \quad \text{as}\, T \to 0^+.$$

Hence, taking (2..1) into account,

$$\frac{v(y_x^a(T) - v(x)}{T} \leq p \cdot \frac{1}{T} \int_0^T f(y_x^a(t), a)\, dt + \frac{o(T)}{T}, \quad \text{as}\, T \to 0^+.$$

for all $p \in D^+ v(x)$.
This and (4..9) imply, by continuity of f, l and v that

$$p \cdot f(x,a) \geq -l(x,a) + \lambda v(x)$$

for all $p \in D^+ v(x)$ and $a \in A$.
This shows that v is a viscosity subsolution of (2..3); a similar, but slightly more difficult, argument shows that v is a supersolution as well.

From the above result, which characterizes the value function of the infinite horizon problem in terms of the Hamilton-Jacobi-Bellman equation (2..3), one can deduce necessary and sufficient conditions of optimality. Under the assumptions of Theorem 3 we get the following weak formulation of the Pontryagin Maximum Principle.

Theorem 4 *Assume* $\alpha^\star = \alpha^\star(x)$ *is an optimal control, corresponding to the initial position* x*, for the infinite horizon discounted regulator, i.e.*

$$v(x) = \int_0^{+\infty} l(y_x^{\alpha^\star}(t), \alpha^\star(t))\, e^{-\lambda t} dt.$$

Then the following hold for all $p \in D^+ v(y_x^{\alpha^\star}(t)) \bigcup D^- v(y_x^{\alpha^\star}(t))$*:*

$$p \cdot \dot{y}_x^\star(t) + l(y_x^{\alpha^\star}(t), \alpha^\star(t)) = \lambda v(y_x^{\alpha^\star}(t)) \quad a.e.\, t > 0$$

$$p \cdot f(y_x^{\alpha^\star}(t), \alpha^\star(t)) + l(y_x^{\alpha^\star}(t), \alpha^\star(t)) = Min_{a \in A}[p \cdot f(y_x^{\alpha^\star}(t), a) + l(y_x^{\alpha^\star}(t), a)]$$

As for sufficient conditions we have:

Theorem 5 *Under the assumptions of Theorem 3 and*

$$D^{+}v(x)\bigcup D^{-}v(x)\neq\emptyset,$$

if $\alpha^\star$ *and* x *satisfy*

$$p\cdot f(y_x^{\alpha^\star}(t),\alpha^\star(t))+l(y_x^{\alpha^\star}(t),\alpha^\star(t))=\lambda v(y_x^{\alpha^\star}(t))\quad a.e.\, t>0$$

for all $p\in D^{+}v(y_x^{\alpha^\star}(t))\bigcup D^{-}v(y_x^{\alpha^\star}(t))$, *then* $\alpha^\star$ *is optimal for the initial position* x.

Note that condition $D^{+}v(x)\bigcup D^{-}v(x)\neq\emptyset$ is rather restrictive; it is fulfilled, for example, when v is *semiconcave*, i.e.

$$v(x+z)-2v(x)+v(x-z)\leq C|z|^2$$

for some C and all x,z in $\mathbb{R}^N$.

5. Approximate synthesis

Consider again the infinite horizon problem 2.1, the associated HJB equation

$$\lambda v(x)+Sup_{a\in A}[-f(x,a)\cdot Dv(x)-l(x,a)]=0\quad \text{in } \mathbb{R}^N$$

and set for $h>0$:

$$u_h(x)+Sup_{a\in A}[-(1-\lambda h)u_h(x+hf(x,a))-hl(x,a)]=0\quad\text{in}\quad \mathbb{R}^N. \tag{5..10}$$

Under the assumptions of Theorem 3, the Contraction Mapping Principle applies to show that for each $h\in(0,\frac{1}{\lambda})$ there exists of a unique bounded, continuous solution u_h to the above functional equation.

The functions u_h can be interpreted as value functions of a *discrete time* version of the infinite horizon problem. Let us define at this purpose

$$M_h(A)=\{\alpha\in M(A):\alpha(t)\equiv\text{costant}\quad\forall t\in[kh,(k+1)h)\}$$

and, for $\alpha\in M_h(A)$, the discrete time control system

$$y_h(x,k+1)=y_h(x,k)+hf(y_h(x,k),a(kh))\quad y_h(x,0)=x, \tag{5..11}$$

where $k=0,1,...,$.
Define then, a feedback law $a_h^\star:\mathbb{R}^N\longrightarrow A$ by selecting

$$a_h^\star(x)\in argmax_{a\in A}[-(1-\lambda h)u_h(x+hf(x,a))-hl(x,a)]$$

where u_h is the solution of equation (5..10). Consider now the control $\alpha_h^\star\in M_h(A)$ given by

$$\alpha_h^\star(t)=a_h^\star(y_h^\star([t/h]))$$

where $y_h^\star$ is obtained from (5..11).
It not hard to check then that the solution u_h of (5..10) is given by

$$u_h(x)=h\sum_{k=0}^{+\infty}(1-\lambda h)^k l(y_h^\star(x,k),\alpha^\star(kh))$$

and also that

$$u_h(x)=Inf_{\alpha\in M_h(A)}\,h\sum_{k=0}^{+\infty}(1-\lambda h)^k l(y_h(x,k),\alpha(kh)).$$

The next result states that u_h converges to the value function v of the infinite horizon problem as the time step $h\to 0^+$.

Theorem 6 *Under the assumptions of Theorem 3 we have:*

$$Sup_K|u_h(x) - v(x)| \longrightarrow 0 \quad as \quad h \to 0^+$$

for all $K \subset\subset \mathbb{R}^N$*. Under the further conditions* $\lambda > 2L_f$*,* f *smooth and* l *semiconcave, the estimate*

$$Sup_{\mathbb{R}^N}|u_h(x) - v(x)| \leq Ch \quad as \quad h \to 0^+$$

hold for some constant C*.*

As a consequence of this convergence result it follows that any optimal pair $(\alpha_h^\star, y_h^\star)$ for the above described discrete time problem converges weakly to an optimal relaxed pair $(\mu^\star, y^\star)$ for the original problem (2..2), see [11, 12].Theorem 6 is also the starting point for a numerical approach to the computation of value functions and optimal feedbacks. We refer for example to [18, 19, 21, 22, 23].

6. Final remarks

In this paper we restricted our attention to the role played by viscosity solutions in optimal control problems for systems governed by ordinary differential equations. Only a few examples have been shortly described but many more can be approached in a similar way, see [7] for impulse and switching control problems,the minimum time problem and H_∞ control.
Some important topics we did not touch as well are discontinuous viscosity solutions and their applications to control and game problems with discontinuos value functions (e.g. the classical Zermelo navigation problem). Discontinuous viscosity solutions and the closely related *weak limits technique* are relevant also in the analysis of some asymptotic problems occurring, for example, in connections with ergodic systems (see [24]), large deviations (see [25]) or control of singularly perturbed systems (see [26]).

Let us mention, finally, that the viscosity solutions approach is flexible enough to be applicable to control problems for stochastic and distributed parameters systems as well as to differential games (we refer at this purpose to [8, 9, 10, 13]).

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