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# Mathematical analysis of models for viscoelastic fluids

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## PH.D. THESIS

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**Title:** Mathematical analysis of models for viscoelastic fluids
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**Abstract:** We consider several problems in the thesis. First we summarize key ideas of fluid mechanics theory and introduce several models describing nonnewtonian behaviour of fluids. In the second chapter we prove local existence of solutions to the Oldroyd-type system achieved as a limit case with infinite relaxation and retardation times. We work with three types of boundary conditions, namely homogenous Dirichlet and periodic conditions and whole space, in 2D and 3D. We study also related system of PDE's which is equivalent to the Oldroyd-type system in 2D. In the third chapter we prove local existence of solutions to the system of PDE's describing the flow of a polymeric liquid. The polymer molecules are modeled as elastic dumbbells with spring force having the so-called FENE potential. Arising system consists of Navier–Stokes equations coupled with Fokker–Planck equation. In the fourth chapter we study asymptotic behaviour of solutions to equations desribing steady flow of a second grade fluid past an obstacle in three dimensions with prescribed nonzero velocity at infinity. Key point in the proof is using results of boundedness of convolutions with fundamental solution to the Oseen equation in weighted Lebesgue spaces. Finally, we prove global existence of solutions to another Oldroyd-type system with shear rate dependent viscosity and nonlinear stress diffusion.
**Keywords:** nonnewtonian fluids, Oldroyd model, local existence of solutions, polymeric liquid, FENE potential, second grade fluid, weighted estimates, fixed point theorem, stress diffusion, global existence of solutions

**Název práce:** Matematická analýza modelů viskoelastických tekutin
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**Abstrakt:** V této práci se věnujeme několika problémům. Nejdříve shrneme klíčové myšlenky teorie mechaniky tekutin a zavedeme několik způsobů popisu nenewtonského chování tekutin. V druhé kapitole dokážeme lokální existenci řešení systému rovnic Oldroydova typu dosaženého jako limitní případ s nekonečným relaxačním a retardačním časem. Pracujeme se třemi typy okrajových podmínek: homogenní Dirichletovou podmínkou, periodickým případem a celým prostorem. Studujeme také související systém parciálních diferenciálních rovnic, který je ekvivalentní v dimenzi 2. Ve třetí kapitole dokážeme lokální existenci řešení systému rovnic popisujících proudění polymerické tekutiny. Molekuly polymeru jsou modelovány jako elastické činky s pružnou silou mající tzv. FENE potenciál. Získaný systém sestává z Navier–Stokesových rovnic a Fokker–Planckovy rovnice. Ve čtvrté kapitole studujeme asymptotické chování řešení rovnic popisujících stacionární proudění tekutiny druhého stupně kolem překážky ve třech dimenzích s předepsanou nenulovou rychlostí v nekonečnu. Klíčovým krokem je použití výsledků o omezenosti konvolucí s fundamentálním řešením Oseenovy rovnice v Lebesgueových prostorech s vahami. Nakonec dokážeme globální existenci řešení jiného systému Oldroydova typu s viskozitou závislou na gradientu rychlosti a nelineární difuzí napětí.
**Klíčová slova:** nenewtonské tekutiny, Oldroydův model, lokální existence řešení, polymerická tekutina, FENE potenciál, tekutina druhého stupně, váhové odhady, věta o pevném bodě, difúze napětí, globální existence řešení

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# Notation

Here we summarize notation used in this thesis. Vector quantities are printed in bold small letters and tensor quantities in bold capital letters. We usually use Einstein summation convention when working with coordinates of vectors and tensors. For details see the corresponding place in text.

## Quantitites

| | |
|---|---|
| $\mathbf{A}_n$ | Rivlin-Ericksen tensor |
| $\mathbf{C}$ | right Cauchy-Green tensor |
| $\mathbf{D}$ | symmetric part of the velocity gradient |
| $\mathbf{f}$ | external body force |
| $\mathbf{F}$ | deformation gradient |
| $\mathbf{I}$ | identity tensor |
| $\mathbf{L}$ | velocity gradient |
| $\mathbf{n}$ | unit outer normal |
| $N$ | spatial dimension |
| $(\boldsymbol{\mathcal{O}}, \mathbf{e})$ | fundamental solution to the Oseen problem |
| $p, P, q, s$ | pressure |
| $\mathcal{R}$ | Reynolds number |
| $s(\mathbf{x})$ | $= \mathbf{x} - x_1$ |
| $t$ | time variable |
| $T$ | time |
| $\mathbf{T}$ | Cauchy stress tensor |
| $\mathbf{T}_E = \mathbf{T} + p\mathbf{I}$ | extra stress |
| $\mathbf{T}_e$ | elastic part of the extra stress |
| $\mathbf{u}, \mathbf{v}, \mathbf{w}$ | velocities |
| $\mathcal{W}$ | Weissenberg number |

| | |
|---|---|
| $\mathbf{x}$ | space variable |
| $\mu, \mu_0$ | viscosities |
| $\rho$ | density |
| $\Omega$ | spatial domain |
| $Q_T = (0,T) \times \Omega$ | time-space cylinder |

## Operations and function spaces

| | |
|---|---|
| $\mathbf{A}^T$ | transpose of a tensor |
| $(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}$ | gradient of a vector field |
| $(\nabla \mathbf{A})_{ijk} = \frac{\partial A_{ij}}{\partial x_k}$ | gradient of a tensor field |
| $\operatorname{div} \mathbf{v} = \frac{\partial v_i}{\partial x_i}$ | divergence of a vector field |
| $(\operatorname{div} \mathbf{A})_i = \frac{\partial A_{ij}}{\partial x_j}$ | divergence of a tensor field |
| $\mathbf{u} \cdot \mathbf{v} = u_i v_i$ | scalar product of two vector fields |
| $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$ | scalar product of two tensor fields |
| $(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j$ | tensor product of two vector fields |
| $(\mathbf{u} \otimes \mathbf{A})_{ijk} = u_i A_{jk}$ | tensor product of a vector field and a tensor field |
| $X \hookrightarrow Y$ | continuous embedding of $X$ into $Y$ |
| $X \hookrightarrow\hookrightarrow Y$ | compact embedding of $X$ into $Y$ |
| $f^n \rightharpoonup f$ | weak convergence of $f^n$ to $f$ |
| $f^n \to f$ | strong convergence of $f^n$ to $f$ |
| $L^p, L^p(\Omega)$ | Lebesgue spaces |
| $L^p(g), L^p(\Omega, g)$ | weighted Lebesgue spaces |
| $W^{k,p}, W^{k,p}(\Omega)$ | Sobolev spaces |
| $W_0^{k,p}, W_0^{k,p}(\Omega)$ | Sobolev spaces with zero trace |
| $X_{div}$ | space of vectors with zero divergence |
| $X_{per}$ | space of periodic functions |

# Introduction

Theory of nonnewtonian fluids became very popular in last couple of decades. The fluids with nonnewtonian behaviour became very important in many fields and due to increased power of computers and advances in mathematical theory these models can be nowadays treated both theoretically and numerically. A lot of different models of nonnewtonian fluids was derived and then studied. A large class among all models of nonnewtonian fluids consists of so-called rate type fluids. In such models the Cauchy stress tensor (or its elastic part) is related to other unknowns in the model through an additional partial differential equation. Mathematical analysis of such models is usually quite complicated due to the coupling of equations in the system which consists of the Navier–Stokes type equation and some kind of transport equation for a quantity related to the stress tensor. Global existence results to models which are not simplified by adding some terms into the equations are rare, one of the most significant results in this area is due to Lions and Masmoudi [27] who proved global existence of solutions to the corotational Oldroyd model. For this reason even local existence results are valuable. In this work we concentrate on several models describing flow of nonnewtonian viscoelastic fluids and prove some new results.

In Chapter 1 we present a short survey of fluid mechanics and derive models which are studied in next chapters. Chapter 2 contains local existence results for Oldroyd-type model with infinite relaxation and retardation times. We study this model with three different boundary conditions in two and three space dimensions and prove local existence of solution under less restrictive conditions than in previous works [25], [5]. Our method is based on the generalization of Banach fixed point theorem and $L^p$ estimates for the Stokes problem. We consider also related system of PDE's which is equivalent in 2D and prove similar results.

In Chapter 3 we work with a system of PDEs desribing the flow of a polymeric liquid. We use standard dumbbell model to describe the polymer molecules and assume that the spring force has so-called FENE potential. We apply the method from previous chapter also here and prove local existence of solutions. Again, we assume lower regularity of the initial conditions than in previous work [30].

Chapter 4 is devoted to a different problem. We aim to identify asymptotic structure of solution to system describing steady flow of a second grade fluid past an obstacle. We prove the existence of the wake region behind the obstacle, i.e. an area where the solution decays to the presribed velocity at infinity slower than outside this region. Such results were known for newtonian fluid and some models of viscoelastic fluid, the same problem for a second grade fluid was open. We follow the decompositon introduced in [32] and split the system into an Oseen equation and a steady transport equation. Crucial point in our proof is using the result of boundedness of convolution with the second gradient of the Oseen kernel in weighted Lebesgue spaces proved in [18].

In Chapter 5 we present a global existence result for a model with shear dependent viscosity and nonlinear stress diffusion. The final chapter is then a summary of most of the theorems used in previous chapters, mainly the results from the theory of Lebesgue and Sobolev spaces.

Parts of the results presented here were published or submitted for publishing, see [21], [22], [19].

# Chapter 1

# Fluid mechanics

## 1.1 Basic principles

We use a standard approach of continuum mechanics to derive studied models. For details see e.g. classical textbook [16]. Let us assume that the continuum occupies a region $\mathcal{B}$ in the physical space and that $\mathcal{B}$ consists of material points $p$ which are usually called particles. We assume that there is a smooth one-to-one mapping $\mathbf{p}$ (called reference configuration) of $\mathcal{B}$ onto $\Omega_0 \subset \mathbb{R}^N$

$$\mathbf{X} = \mathbf{p}(p). \tag{1.1}$$

Here $N$ corresponds to dimension of the physical space and usually $N = 2$ or 3. In this way we identify material points with points in $\Omega_0$. Further we assume that there is a smooth one-to-one mapping $\boldsymbol{\chi}$ of $\Omega_0$ onto $\Omega \subset \mathbb{R}^N$ called the deformation

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \tag{1.2}$$

so $\mathbf{x}$ corresponds to the place occupied by the particle $p$, where $\mathbf{X} = \mathbf{p}(p)$, in the deformed configuration. One can study properties of deformations by studying the deformation gradient $\mathbf{F}$

$$\mathbf{F}(\mathbf{X}) = \nabla\boldsymbol{\chi}(\mathbf{X}). \tag{1.3}$$

We have to add time variable to be able to study motions of bodies. Let us therefore consider a one-parameter family of deformations $\boldsymbol{\chi}(t, \mathbf{X})$, where the real parameter $t$ denotes the time. We assume that for all $t$ the mapping is invertible

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(t, \mathbf{x}). \tag{1.4}$$

There are two basic approaches in studying motions in the continuum mechanics, one relies on studying the motions of a certain particle through space (Lagrangean approach, connected with the material) while the other one relies on studying the motions of particles through certain point in space (Eulerian approach, connected with the space). Fluid mechanics usually uses Eulerian approach and so do we.

We have to distinguish between two types of time derivatives. Let us have a given spatial field $f_s(t, \mathbf{x})$. We can associate to it a material field $f_m(t, \mathbf{X}) = f_s(t, \boldsymbol{\chi}(t, \mathbf{X}))$. Then by the spatial time derivative, which we denote by $\frac{\partial}{\partial t}$, we understand simply the derivative of the spatial field

$$\frac{\partial}{\partial t} f_s = \frac{\partial}{\partial t} f_s(t, \mathbf{x}). \tag{1.5}$$

By the material time derivative, which we denote by $\frac{\mathrm{d}}{\mathrm{d}t}$, we understand

$$\frac{\mathrm{d}}{\mathrm{d}t} f_s = \frac{\partial}{\partial t} f_m(t, \mathbf{X}) \Big|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(t,\mathbf{x})}. \tag{1.6}$$

Now we can introduce the velocity $\mathbf{v}$ as a material derivative of $\boldsymbol{\chi}$, i.e.

$$\mathbf{v}(t, \mathbf{x}) = \frac{\partial}{\partial t} \boldsymbol{\chi}(t, \mathbf{X}) \Big|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(t,\mathbf{x})} \tag{1.7}$$

and similarly the acceleration

$$\mathbf{a}(t, \mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{v}(t, \mathbf{x}). \tag{1.8}$$

By a simple computation we can show the important relationship between the two time derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t, \mathbf{x}) = \frac{\partial}{\partial t} f(t, \mathbf{x}) + (\mathbf{v}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}}) f(t, \mathbf{x}). \tag{1.9}$$

We denote by $\mathbf{L}$ the velocity gradient $\mathbf{L}(t, \mathbf{x}) = \nabla_{\mathbf{x}} \mathbf{v}(t, \mathbf{x})$ and by $\mathbf{D}$ and $\mathbf{W}$ its symmetric and skew-symmetric part respectively, i.e.

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \qquad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T). \tag{1.10}$$

One can easily show the following formula

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \mathbf{L}\mathbf{F}, \tag{1.11}$$

which is a simple corollary of the derivation chain rule.

The equations of continuum mechanics are derived from the empirically deduced balance laws. First of them is the balance of mass which says that the mass of any part of the body $\mathcal{B}$ is conserved. Therefore we assume that there is a positive function $m$ called mass and mass density $\rho_0$ such that

$$m(\mathcal{P}) = \int_{\mathbf{p}(\mathcal{P})} \rho_0(\mathbf{X}) \mathrm{d}\mathbf{X} = \int_{\boldsymbol{\chi}(\mathbf{p}(\mathcal{P}))} \rho(t, \mathbf{x}) \mathrm{d}\mathbf{x} \tag{1.12}$$

for all $\mathcal{P}$ such that $\mathbf{p}(\mathcal{P})$ is Lebesgue measurable. The relationship between $\rho_0$ and $\rho$ is given by

$$\rho(t, \mathbf{x}) = \frac{\rho_0(\mathbf{X})}{\det \mathbf{F}(t, \mathbf{X})} \Big|_{\mathbf{X} = \boldsymbol{\chi}^{-1}(t, \mathbf{x})}. \tag{1.13}$$

The balance of mass can be now written as

$$\frac{\mathrm{d}}{\mathrm{d}t} m(\mathcal{P}) = 0, \tag{1.14}$$

which leads to the following differential form of the balance of mass

$$\frac{\partial \rho}{\partial t}(t, \mathbf{x}) + \operatorname{div}(\rho \mathbf{v})(t, \mathbf{x}) = 0. \tag{1.15}$$

In all our studied models we assume the continuum to be homogenous and incompressible, i.e. $\rho = \rho_0$ is a constant in both space and time and the differential form of the balance of mass reduces to

$$\operatorname{div} \mathbf{v}(t, \mathbf{x}) = 0. \tag{1.16}$$

Next we consider the balance of linear and angular momentum. Let us denote by $\mathcal{F}(\mathcal{P})$ total force acting on part $\mathcal{P}$ of the body $\mathcal{B}$. We distinguish two type of forces, volume forces act directly to every material point and surface forces act directly only on the surface material points. Then we have

$$\mathcal{F}(\mathcal{P}) = \int_{\mathbf{p}(\mathcal{P})} \rho_0 \mathbf{f} \mathrm{d}\mathbf{X} + \int_{\partial \mathbf{p}(\mathcal{P})} \mathbf{s} \mathrm{d}S, \tag{1.17}$$

where $\mathbf{f}$ is the density of the volume force and $\mathbf{s} = \mathbf{s}(t, \mathbf{X}, \mathbf{n})$ is the density of the surface force, called also the stress vector. Here $\mathbf{n}$ denotes the unit outer normal to $\mathbf{p}(\mathcal{P})$ at point $\mathbf{X}$. The Cauchy theorem says that the dependence of $\mathbf{s}$ on $\mathbf{n}$ is linear and therefore there exists a (Cauchy stress) tensor $\mathbf{T}(t, \mathbf{x})$ such that

$$\mathbf{s}(t, \mathbf{X}, \mathbf{n})\Big|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(t,\mathbf{x})} = \mathbf{T}(t, \mathbf{x})\mathbf{n}(t, \mathbf{x}). \tag{1.18}$$

This yields the differential form of the balance of linear momentum

$$\rho\mathbf{a}(t, \mathbf{x}) = \rho\frac{\partial \mathbf{v}}{\partial t}(t, \mathbf{x}) + \rho(\mathbf{v}(t, \mathbf{x}) \cdot \nabla_x)\mathbf{v}(t, \mathbf{x}) = \operatorname{div} \mathbf{T}(t, \mathbf{x}) + \rho\mathbf{f}(t, \mathbf{x}). \tag{1.19}$$

The balance of angular momentum then yields the symmetry of the Cauchy stress tensor

$$\mathbf{T} = \mathbf{T}^T. \tag{1.20}$$

## 1.2 Constitutive relations

In the whole thesis we consider only thermodynamicaly stable systems, so we have no more equations for internal energy, temperature or other thermodynamical quantities. The system of equations (1.16), (1.19) is underdetermined, assuming $\mathbf{f}$ and $\rho$ given we have in 3 dimensions 4 equations for 9 unknown quantities. Therefore we have to add more relations to the system which come from the properties of the material. These constitutive relations usually specify the dependence of $\mathbf{T}$ on $\mathbf{v}$ and its derivatives.

The constitutive relations can not be completely arbitrary, they have to satisfy certain physical principles, mainly the frame indifference property. Let us consider the change of observer, i.e. the following change of variables

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{q}(t), \tag{1.21}$$

where $\mathbf{Q}$ is a rotation ($\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, $\det \mathbf{Q} = 1$). Then the constitutive relation has to be frame indifferent, i.e.

$$\mathbf{T}^*(t, \mathbf{x}^*) = \mathbf{Q}(t)\mathbf{T}(t, \mathbf{x})\mathbf{Q}^T(t). \tag{1.22}$$

Moreover, the constitutive relation has to be invariant under the change of the observer (1.21), in other words the equation has to have exactly the same form when changed to a new reference frame. This physical assumption causes troubles when using time derivatives as neither spatial time derivative $\frac{\partial}{\partial t}$ nor material time derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ satisfy this assumption. Therefore we need to find proper forms of time derivation which are the objective time derivatives

$$\frac{\mathcal{D}_a \mathbf{T}}{\mathcal{D}t} = \frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W} - a(\mathbf{D}\mathbf{T} + \mathbf{T}\mathbf{D}), \tag{1.23}$$

where $a \in [-1, 1]$ is a real parameter. Special cases are achieved by choosing certain values of the parameter $a$.

- a = 1: Upper convected derivative

$$\frac{\mathcal{D}_1 \mathbf{T}}{\mathcal{D}t} = \frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T \tag{1.24}$$

- a = -1: Lower convected derivative

$$\frac{\mathcal{D}_{-1} \mathbf{T}}{\mathcal{D}t} = \frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L} \tag{1.25}$$

- a = 0: Corotational (Jaumann) derivative

$$\frac{\mathcal{D}_0 \mathbf{T}}{\mathcal{D}t} = \frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W} \tag{1.26}$$

Another restriction comes from the material symmetries. Let us consider the stress tensor $\mathbf{T}$ in the reference configuration for a moment. Up to now we did not use anywhere the fact that the considered material is a fluid. For that reason we introduce the history functional $\mathcal{G}$

$$\mathbf{T}(t, \mathbf{X}) = \mathcal{G}_{s=0}^{\infty}(\mathbf{F}(t-s, \mathbf{X}), t, \mathbf{X}), \tag{1.27}$$

where we have already made an assumption that considered material is the so-called simple material, i.e. the stress response of the material at point $\mathbf{X}$ is localized in space, while in time it remains generally unlocalized. We say that the material is a fluid if the group of symmetry of the material is the unimodular group, i.e.

$$\mathcal{G}_{s=0}^{\infty}(\mathbf{F}(t-s, \mathbf{X}), t, \mathbf{X}) = \mathcal{G}_{s=0}^{\infty}(\mathbf{F}(t-s, \mathbf{X})\mathbf{G}, t, \mathbf{X}) \tag{1.28}$$

for all tensors $\mathbf{G}$ such that $\det \mathbf{G} = \pm 1$.

## 1.3 Newtonian fluid

For details concerning this section see e.g. [24]. Assuming that the response functional $\mathcal{G}$ is localized in time we get the following form of the Cauchy's stress tensor

$$\mathbf{T} = \mathbf{T}(\mathbf{F}, \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t}). \tag{1.29}$$

Applying above mentioned principles of frame indifference and material symmetries one can derive[1]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}_E(\mathbf{D}), \tag{1.30}$$

where the scalar function $p(t, \mathbf{x})$ is called pressure. Using the representation theorem for isotropic functions we get the following form of the extra stress $\mathbf{T}_E$

$$\mathbf{T}_E = \varphi_0\mathbf{I} + \varphi_1\mathbf{D} + \varphi_2\mathbf{D}^2, \tag{1.31}$$

where $\varphi_i(I_\mathbf{D}, II_\mathbf{D}, III_\mathbf{D})$ are functions of invariants of $\mathbf{D}$.[2] Linearizing this relation we get

$$\mathbf{T}_E = \lambda(\operatorname{tr}\mathbf{D})\mathbf{I} + 2\mu\mathbf{D} = 2\mu\mathbf{D} \tag{1.32}$$

as $\operatorname{div}\mathbf{v} = 0$. The constant $\mu$ is called viscosity. Plugging in (1.16), (1.19), (1.30) and (1.32) together we derive classical incompressible Navier-Stokes equations

$$\begin{aligned} \operatorname{div}\mathbf{v} &= 0 \\ \rho\frac{\partial\mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \nabla p - \mu\Delta\mathbf{v} &= \rho\mathbf{f}. \end{aligned} \tag{1.33}$$

Let us mention that we get system of $N+1$ equations for $N+1$ unknown quantities: scalar $p$ and vector $\mathbf{v}$.

## 1.4 Nonnewtonian fluids

Real fluids often express behavior that can not be described by Navier-Stokes equations (1.33), such as shear thinning or thickening, stress relaxation, yield stress and other. Therefore one has to impose more advanced constitutive relations than (1.30) and (1.32). The aim of this section is not to describe

[1] Let us recall that we consider only homogenous and incompressible fluids and therefore the density $\rho$ is constant.

[2] We recall that $I_\mathbf{D} = \operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v}$, $II_\mathbf{D} = \frac{1}{2}((\operatorname{tr}\mathbf{D})^2 - \operatorname{tr}\mathbf{D}^2)$, $III_D = \det\mathbf{D}$.

most of possible ways how to achieve various models of nonnewtonian fluids, we rather concentrate on the models which we deal with in following chapters. Some details about constitutive equations for nonnewtonian fluids can be found for example in [45].

### 1.4.1 Oldroyd-type fluids

Similarly as in the previous section, one can show that for volume preserving fluids (which is our setup) the stress tensor can be decomposed into two parts

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}_E, \tag{1.34}$$

where the pressure part does zero work[3]. We start with the notion of *Maxwell fluid* which can be considered as generalization of the one-dimensional Maxwell model of linear viscoelasticity. The constitutive relation is then given by

$$\mathbf{T}_E + \lambda_1 \frac{\mathcal{D}_a \mathbf{T}_E}{\mathcal{D}t} = 2\mu_0 \mathbf{D}, \tag{1.35}$$

where constant $\lambda_1 > 0$ is called the stress relaxation time. One easily obtains a model of newtonian fluid letting $\lambda_1 \to 0$.

More general class of fluids are *Oldroyd fluids* with the constitutive relation

$$\mathbf{T}_E + \lambda_1 \frac{\mathcal{D}_a \mathbf{T}_E}{\mathcal{D}t} = 2\mu_0 \left( \mathbf{D} + \lambda_2 \frac{\mathcal{D}_a \mathbf{D}}{\mathcal{D}t} \right). \tag{1.36}$$

Here we add another constant $0 \leq \lambda_2 < \lambda_1$ called the retardation time. Oldroyd fluids can be considered as Maxwell fluids with additional newtonian viscosity, in fact defining

$$\mu_n = \frac{\mu_0 \lambda_2}{\lambda_1}, \qquad \mu_e = \mu_0 \left( 1 - \frac{\lambda_2}{\lambda_1} \right) \tag{1.37}$$

and setting

$$\mathbf{T}_E = 2\mu_n \mathbf{D} + \mathbf{T}_e, \tag{1.38}$$

we observe that the elastic part of the extra stress $\mathbf{T}_e$ satisfies

$$\mathbf{T}_e + \lambda_1 \frac{\mathcal{D}_a \mathbf{T}_e}{\mathcal{D}t} = 2\mu_e \mathbf{D}, \tag{1.39}$$

[3] $p\mathbf{I} : \mathbf{D} = p \operatorname{div} \mathbf{v} = 0$

i.e. the constitutive relation for a Maxwell fluid.

Maxwell [31] was probably the first to state a linear, one-dimensional theory for viscoelastic fluid based on simple superposition of viscous and elastic effects. Oldroyd [36] was first to introduce properly invariant forms of the time derivatives in constitutive equations.

In Chapter 2 we consider a model which can be obtained as a limit case of an Oldroyd fluid. Let us put $a = 1$, $\lambda_2 = \alpha\lambda_1$ with $\alpha \in (0, 1)$ being a constant and let the relaxation time $\lambda_1 \to \infty$. Then dividing (1.39) by $\lambda_1$ we end up with

$$\mathbf{T}_E = 2\mu_0\alpha\mathbf{D} + \mathbf{T}_e, \tag{1.40}$$

$$\frac{\mathcal{D}_1\mathbf{T}_e}{\mathcal{D}t} = \mathbf{0}. \tag{1.41}$$

This model can be obtained also by a completely different approach. Let us assume the extra stress in the form (1.40) and the elastic part of the extra stress given by

$$\mathbf{T}_e = \frac{\partial\mathbf{S}(\mathbf{F})}{\partial\mathbf{F}}\mathbf{F}^T, \tag{1.42}$$

where $\mathbf{S}(\mathbf{F})$ is density of the elastic energy of the fluid and $\mathbf{F}$ is the deformation gradient. Considering the simplest case of elasticity, namely $\mathbf{S}(\mathbf{F}) = |\mathbf{F}|^2 = \mathrm{tr}\,(\mathbf{F}\mathbf{F}^T)$ we recover

$$\mathbf{T}_e = \mathbf{F}\mathbf{F}^T \tag{1.43}$$

and taking into account the relation (1.11) we can show that $\mathbf{T}_e$ satisfies (1.41).

### 1.4.2 Polymeric fluids

A different large class of models of nonnewtonian fluids is achieved by considering polymeric fluids. Let us consider a mixture of a newtonian fluid with polymer molecules which are fully diluted in the solvent. The molecules are assumed not to interact with each other. This leads similarly as before to decomposition of the full stress tensor into the newtonian part and the extra stress related to the elasticity of the polymer molecules

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} + \mathbf{T}_e. \tag{1.44}$$

The difference comes now in stating constitutive relation for the elastic extra stress $\mathbf{T}_e$. As the polymer molecules are very complex objects we have to find a suitable way how to idealize and model them. This is very often done by considering dumbbell models, where the molecule is modeled by a dumbbell, which are two beads connected with the elastic spring described by the elongation vector $\mathbf{r}$. We introduce the domain of all admissible conformations $D \subset \mathbb{R}^N$. Natural assumption is that this set is convex and balanced which means that $\mathbf{r} \in D$ if and only if $-\mathbf{r} \in D$. Typical examples are $D = \mathbb{R}^N$ and $D = B_{r_0}(\mathbf{0})$. Let us suppose that the elastic spring force $\mathbf{f}_e(\mathbf{r})$ is derived from the potential $U(\mathbf{r})$

$$\mathbf{f}_e(\mathbf{r}) = \nabla_{\mathbf{r}} U(\mathbf{r}). \tag{1.45}$$

We have to keep in mind that typical length scale of the microscopic molecules is different that the macroscopic length scale of the domain occupied by the fluid. The crucial quantity describing the behaviour of the molecules is the scalar function $\psi(t, \mathbf{x}, \mathbf{r})$ which roughly speaking represents a probability of finding the centre of mass of a dumbbell at time $t$ and point $\mathbf{x}$ having elongation vector $\mathbf{r}$. More precisely $\psi$ denotes the probability density corresponding to the stochastic processes $(X(t), R(t))$, where $X(t)$ is the position of a centre of mass of a dumbbell and $R(t)$ is its conformation.

The elastic extra stress is now defined by Kramer expression

$$\mathbf{T}_e(t, \mathbf{x}) = k_B \theta \left( \int_D \mathbf{r} \otimes \nabla_{\mathbf{r}} U(\mathbf{r}) \psi(t, \mathbf{x}, \mathbf{r}) \mathrm{d}\mathbf{r} \right), \tag{1.46}$$

where $k_B$ is the Boltzmann constant and $\theta$ is the absolute temperature. It can be shown that the probability density $\psi$ satisfies the Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} + \mathbf{v} \cdot \nabla \psi = \operatorname{div}_{\mathbf{r}} [-\nabla_{\mathbf{x}} \mathbf{v} \cdot \mathbf{r} \psi + \nabla_{\mathbf{r}} \psi + \nabla_{\mathbf{r}} U \psi], \tag{1.47}$$

for details concerning derivation of this equation see [4][4]. We end up with system of equations (1.16), (1.19), (1.47) with relations (1.44), (1.46) describing the flow of polymer liquid. We will deal with this system of equations in Chapter 3.

### 1.4.3 Rivlin-Ericksen fluids

Let us introduce the Rivlin-Ericksen tensors $\mathbf{A}_n$. For this reason we first define the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$. Then the tensors $\mathbf{A}_n$ are

[4] Note that in [4] the authors derive equation (1.47) with additional term $\varepsilon \Delta_{\mathbf{x}} \psi$ on the right-hand side. Usually this term is omitted as the constant $\varepsilon$ is of several orders of magnitude smaller than all other terms in the equation.

for $n \geq 1$ defined through

$$\mathbf{A}_n = \frac{\mathrm{d}^n \mathbf{C}}{\mathrm{d}t^n}. \tag{1.48}$$

More useful is the following recurrence formula

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{L} + \mathbf{L}^T = 2\mathbf{D} \\ \mathbf{A}_n &= \frac{\partial \mathbf{A}_{n-1}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^T \mathbf{A}_{n-1}. \end{aligned} \tag{1.49}$$

The idea of models of Rivlin-Ericksen fluids is due to Rivlin and Ericksen [40], [39]. They published the theory of isotropic materials for which at time $t$ the stress depends on $\mathbf{v}$, $\mathbf{D}$ and higher time derivatives of these quantities. The general constitutive relation of a Rivlin-Ericksen fluid can be thus stated as

$$\mathbf{T}_E = \mathbf{Z}(\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n), \tag{1.50}$$

where $\mathbf{Z}$ is isotropic[5]. This is another generalization of model of newtonian fluid, indeed assuming $\mathbf{T}_E$ depends only on $\mathbf{A}_1$ and the dependence is linear we recover the constitutive relation of a newtonian fluid (1.32).

We are mainly interested in special type of Rivlin-Ericksen fluids, namely in *fluids of grade n*. Such fluids have a special form of the constitutive relation, namely

$$\mathbf{Z}(\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n) = \sum_{i=1}^{n} \mathbf{Z}_i(\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_i), \tag{1.51}$$

where each of the functions $\mathbf{Z}_i$ is isotropic and homogenous of degree $i$. This means that

$$\mathbf{Z}_i(\lambda \mathbf{A}_1, \lambda^2 \mathbf{A}_2, ..., \lambda^i \mathbf{A}_i) = \lambda^i \mathbf{Z}_i(\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_i). \tag{1.52}$$

One can show using the represantation theorem for isotropic functions that

$$\begin{aligned} \mathbf{Z}_1(\mathbf{A}_1) &= \mu \mathbf{A}_1 \\ \mathbf{Z}_2(\mathbf{A}_1, \mathbf{A}_2) &= \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 \qquad \text{etc.}, \end{aligned} \tag{1.53}$$

where $\mu, \alpha_1, \alpha_2$ are material constants. Therefore we see that fluid of grade 1 is in fact a newtonian fluid, while fluid of grade 2 or as we will call it later on *second grade fluid* has the form of the stress tensor

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2. \tag{1.54}$$

[5] A tensor function $\mathbf{Z}(\mathbf{A})$ is called isotropic if $\mathbf{Q}\mathbf{Z}(\mathbf{A})\mathbf{Q}^T = \mathbf{Z}(\mathbf{Q}\mathbf{A}\mathbf{Q}^T)$ for all orthogonal tensors $\mathbf{Q}$. Similarly for functions of multiple tensor variables.

We will study properties of solutions of three-dimensional steady flow of second grade fluid past an obstacle in Chapter 4.

### 1.4.4 Other models of nonnewtonian fluids

There is a lot of other ways how to describe nonnewtonian behavior of fluids. In Chapter 5 we consider model which involves two features which were not yet described. One of them is nonconstant viscosity. Real fluids exhibit changes of viscosity in experiments, an effect that can not be described by any of above mentioned models. In thermodynamicaly stable cases the viscosity $\mu$ is usually assumed to depend on pressure $p$ and shear rate $\mathbf{D}(\mathbf{v})$. We will study an Oldroyd-type model with shear dependent viscosity, i.e. we will assume that

$$\mathbf{T} = -p\mathbf{I} + \mu(\mathbf{D})\mathbf{D} + \mathbf{T}_e \tag{1.55}$$

and the elastic part $\mathbf{T}_e$ satisfies an Oldroyd-type differential equation. In this equation we introduce second feature which did not appear in previous models, which is the stress diffusion. The differential equation for $\mathbf{T}_e$ is then (1.39) with added diffusion term, i.e.

$$\mathbf{T}_e + \lambda_1 \frac{\mathcal{D}_a \mathbf{T}_e}{\mathcal{D}t} - \varepsilon \Delta \mathbf{T}_e = 2\mu_0 \mathbf{D}, \tag{1.56}$$

with $\varepsilon > 0$ being a constant. Final adjustment of this model consists in considering nonlinear diffusion instead of a linear one. Therefore we replace linear term $\Delta \mathbf{T}_e$ by a nonlinear one $\operatorname{div}(\gamma(\nabla \mathbf{T}_e)\nabla \mathbf{T}_e)$ and we end up with

$$\mathbf{T}_e + \lambda_1 \frac{\mathcal{D}_a \mathbf{T}_e}{\mathcal{D}t} - \varepsilon \operatorname{div}(\gamma(\nabla \mathbf{T}_e)\nabla \mathbf{T}_e) = 2\mu_0 \mathbf{D}. \tag{1.57}$$

The assumptions on nonlinear functions $\mu(\mathbf{D})$ and $\gamma(\nabla \mathbf{T}_e)$ will be specified in Chapter 5.

Let us mention here that adding a stress diffusion term is not completely artificial, models with stress diffusion are studied for example in numerics. One can get a model with stress diffusion for example by considering a polymeric fluid with additional diffusion term (see footnote in section 1.4.2), Hookean potential $U$ and domain of admissible conformations of polymers $D = \mathbb{R}^N$. Corresponding macroscopic model is then an Oldroyd model with stress diffusion. See [42] for numerical study of such model.

# Chapter 2

# Local existence for an Oldroyd-type model

## 2.1 The model

In this chapter we study an Oldroyd-type model introduced in 1.4.1 which is achieved as a limit case with infinite relaxation and retardation times. The system of equations consists of (1.16), (1.19), (1.34), (1.40) and (1.41). We have therefore system

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p - \mu\Delta\mathbf{v} &= \mathbf{f} + \operatorname{div}\mathbf{T}_e \\ \operatorname{div}\mathbf{v} &= 0 \\ \frac{\partial \mathbf{T}_e}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{T}_e - \nabla\mathbf{v}\mathbf{T}_e - \mathbf{T}_e\nabla\mathbf{v} &= \mathbf{0}, \end{aligned} \tag{2.1}$$

where $\mu = \alpha\mu_0$ is a constant viscosity and we have set the constant density $\rho = 1$. As we mentioned in 1.4.1 this model can be equivalently rewritten in the following form

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p - \mu\Delta\mathbf{v} &= \mathbf{f} + \operatorname{div}\mathbf{F}\mathbf{F}^T \\ \operatorname{div}\mathbf{v} &= 0 \\ \frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{F} &= \nabla\mathbf{v}\mathbf{F}, \end{aligned} \tag{2.2}$$

where $\mathbf{v}(t,\mathbf{x}), \mathbf{F}(t,\mathbf{x})$ and $p(t,\mathbf{x})$ are unknown quantities and $\mathbf{f}(t,\mathbf{x})$ is given. To mathematicaly complete this system we have to specify domains of interest and add initial and boundary conditions. We are searching for solutions in $(0,T)\times\Omega$ and we add to system (2.2) initial conditions

$$\left.\begin{aligned}\mathbf{v}(0,\mathbf{x}) &= \mathbf{v}_0(\mathbf{x})\\ \mathbf{F}(0,\mathbf{x}) &= \mathbf{F}_0(\mathbf{x})\end{aligned}\right\}\text{ in }\Omega. \tag{2.3}$$

We consider three types of spatial domains. We assume either $\Omega = (0,L)^N$ is a periodic box or $\Omega\subset\mathbb{R}^N$ being sufficiently smooth bounded domain or we assume the whole space case $\Omega=\mathbb{R}^N$. In the first case the boundary conditions are replaced with assumption that all functions are space periodic and we denote this problem by $P_1$. In the second case we add homogenous Dirichlet boundary condition for the velocity

$$\mathbf{v}(t,\mathbf{x}) = \mathbf{0}, \qquad t\in(0,T), \mathbf{x}\in\partial\Omega. \tag{2.4}$$

There is no need to prescribe a boundary condition on $\mathbf{F}$. We denote this problem by $D_1$ and we denote the Cauchy problem by $C_1$.

Let us make the following observation. Taking divergence of $(2.2)_3$ and taking into consideration $(2.2)_2$ we get

$$\frac{\partial \operatorname{div}\mathbf{F}}{\partial t} + (\mathbf{v}\cdot\nabla)\operatorname{div}\mathbf{F} = \mathbf{0}, \tag{2.5}$$

i.e. $\operatorname{div}\mathbf{F}$ is transported freely with the flow. Therefore assuming $\operatorname{div}\mathbf{F}_0 = \mathbf{0}$ we get $\operatorname{div}\mathbf{F} = \mathbf{0}$ for all $t\in(0,T)$. In two space dimensions we can use this divergence-free condition to rewrite our system. There exists $\boldsymbol{\phi} = (\phi_1,\phi_2)$ such that

$$\mathbf{F} = \begin{pmatrix} -\frac{\partial\phi_1}{\partial x_2} & -\frac{\partial\phi_2}{\partial x_2}\\ \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_2}{\partial x_1}\end{pmatrix}. \tag{2.6}$$

System (2.2) can be then transformed into

$$\begin{gathered}\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p = -\sum_{i=1}^{N}\Delta\phi_i\nabla\phi_i + \mathbf{f}\\ \operatorname{div}\mathbf{v} = 0\\ \frac{\partial\boldsymbol{\phi}}{\partial t} + \mathbf{v}\cdot\nabla\boldsymbol{\phi} = \mathbf{0},\end{gathered} \tag{2.7}$$

with initial conditions

$$\left.\begin{aligned}\mathbf{v}(0,\mathbf{x}) &= \mathbf{v}_0(\mathbf{x})\\ \boldsymbol{\phi}(0,\mathbf{x}) &= \boldsymbol{\phi}_0(\mathbf{x})\end{aligned}\right\}\text{ in }\Omega. \tag{2.8}$$

Systems (2.2) and (2.7) are equivalent only in the case $N = 2$. However, from the mathematical point of view it is interesting to study both systems even in higher dimensions as they share similar properties. We will study system (2.7) in same three cases as system (2.2) and we denote corresponding problems by $P_2$ (for periodic boundary conditions), $D_2$ (for the case of bounded domain with homogenous Dirichlet boundary condition) and $C_2$ (for the Cauchy problem). The system (2.7) was studied by Lin, Liu and Zhang [25]. The authors proved the existence of smooth solutions on short time intervals and global existence of smooth solutions provided the initial data are sufficiently close to the equilibrium state. The main difficulty to prove these results lies on the free transport equation of $\boldsymbol{\phi}$, which does not show any dissipative mechanism. However, in [25] quite high regularity of the initial conditions, at least $\mathbf{v}_0 \in W^{2,2}(\Omega)$ and $\boldsymbol{\phi}_0 \in W^{3,2}(\Omega)$, was required. The authors use standard technique of $L^2$ energy estimates for $\mathbf{v}, \boldsymbol{\phi}$ and its derivatives. Our approach is different, we use $L^p$ theory of the Stokes system to achieve the existence of smooth solutions on short time intervals under less regular initial conditions, namely $\nabla\boldsymbol{\phi}_0 \in W^{1,p}(\Omega)$ for $p > 2$ and $\mathbf{v}_0 \in W^{1,2}_{div}(\Omega)$.

As we mentioned earlier, in three space dimensions the situation is different. In this case the restriction $\operatorname{div} \mathbf{F} = \mathbf{0}$ does not yield a system like (2.7). Actually, one may obtain a much more complex system if one persists to find such kind of equivalent form. Nevertheless, similar results as in [25] were obtained by Chen and Zhang in [5] for system (2.2); however, they required at least the same regularity of the initial data as in [25]. Again, the authors use technique of $L^2$ energy estimates for $\mathbf{v}, \mathbf{F}$ and its derivatives. Our approach is the same as in two space dimension, we use $L^p$ theory of the Stokes system to achieve the existence of smooth solutions on short time intervals under similar initial conditions as in 2D, namely $\mathbf{F}_0 \in W^{1,p}(\Omega)$ for $p > 3$ and $\mathbf{v}_0 \in W^{1,2}_{div}(\Omega)$.

Oldroyd models and models related to them are of great interest. P.-L. Lions and N. Masmoudi [27] obtained global existence of weak solutions to the Oldroyd model with the corotational time derivative, which seems to be one of the most significant results in this area.

## 2.2 Main results

We state our main results here. Note that in comparison to the results proved in [25] or [5] we need less regular initial conditions. First, let us formulate the results for the system (2.2). For the problems $D_1$ and $P_1$ we have

**Theorem 2.2.1.** *Let $N = 2,3$, $\Omega \in C^{2,\lambda}$ with $\lambda > 0$ (for $D_1$) and let $\mathbf{f} \in L^2(0,T,L^2(\Omega)) \cap L^q(0,T,L^p(\Omega))$ for certain $p \in (N, \frac{2N}{N-2})$ and $q > 1$, $\int_\Omega \mathbf{f}\,\mathrm{d}\mathbf{x} = \mathbf{0}$ (for $P_1$). Let $\mathbf{F}_0 \in W^{1,p}(\Omega)$, $\operatorname{div}\mathbf{F}_0 = \mathbf{0}$ and $\mathbf{v}_0 \in W^{1,2}_{div}(\Omega)$. Then there exists $T^* > 0$ such that on $(0,T^*)$ there exists a strong solution to the problem (2.2) with either Dirichlet or periodic boundary conditions such that $\mathbf{v} \in L^2(0,T^*,W^{2,2}(\Omega)) \cap L^\infty(0,T^*,W^{1,2}(\Omega))$ and $\mathbf{F} \in L^\infty(0,T^*,W^{1,p}(\Omega))$.*

*Remark* 2.2.1. Let us emphasize that assumption $\mathbf{f} \in L^q(0,T,L^p)$ implies that $\mathbf{f} \in L^s(0,T,L^p)$ for all $1 \le s < q$. Thus we may work with $q$ sufficiently close to 1 in the proof. This applies also for theorems below.

Analogous result holds for (2.7)

**Theorem 2.2.2.** *Let $N = 2,3$, $\Omega \in C^{2,\lambda}$ with $\lambda > 0$ (for $D_2$) and let $\mathbf{f} \in L^2(0,T,L^2(\Omega)) \cap L^q(0,T,L^p(\Omega))$ for certain $p \in (N, \frac{2N}{N-2})$ and $q > 1$, $\int_\Omega \mathbf{f}\,\mathrm{d}\mathbf{x} = \mathbf{0}$ (for $P_2$). Let $\nabla\boldsymbol{\phi}_0 \in W^{1,p}(\Omega)$ and $\mathbf{v}_0 \in W^{1,2}_{div}(\Omega)$. Then there exists $T^* > 0$ such that on $(0,T^*)$ there exists a strong solution to the problem (2.7) with either Dirichlet or periodic boundary conditions such that $\mathbf{v} \in L^2(0,T^*,W^{2,2}(\Omega)) \cap L^\infty(0,T^*,W^{1,2}(\Omega))$ and $\nabla\boldsymbol{\phi} \in L^\infty(0,T^*,W^{1,p}(\Omega))$.*

Concerning the Cauchy problem, it is not possible to expect $\mathbf{F}_0$ (or $\nabla\boldsymbol{\phi}_0$) to be integrable with any power. However, we may consider $\mathbf{H}_0 = \mathbf{F}_0 - \mathbf{I}$ and for $\mathbf{H} = \mathbf{F} - \mathbf{I}$ we get the following problem (recall that we assume $\operatorname{div}\mathbf{F} = 0$)

$$\begin{gathered}\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p = \operatorname{div}(\mathbf{H}\mathbf{H}^T) + \operatorname{div}\mathbf{H}^T + \mathbf{f} \\ \operatorname{div}\mathbf{v} = 0 \\ \frac{\partial \mathbf{H}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{H} = \nabla\mathbf{v}\mathbf{H} + \nabla\mathbf{v},\end{gathered} \tag{2.9}$$

with initial conditions

$$\left.\begin{aligned}\mathbf{v}(0,\mathbf{x}) &= \mathbf{v}_0(\mathbf{x}) \\ \mathbf{H}(0,\mathbf{x}) &= \mathbf{H}_0(\mathbf{x})\end{aligned}\right\} \text{ in } \mathbb{R}^N. \tag{2.10}$$

Similarly, defining $\boldsymbol{\psi}(\mathbf{x}) = \boldsymbol{\phi}(\mathbf{x}) - \mathbf{x}$ we replace the problem (2.7) by

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} - \mu \Delta \mathbf{v} + \nabla p = - \sum_{i=1}^{N} \Delta \psi_i \nabla \psi_i - \Delta \boldsymbol{\psi} + \mathbf{f} \\
\operatorname{div} \mathbf{v} = 0 \\
\frac{\partial \boldsymbol{\psi}}{\partial t} + \mathbf{v} \cdot \nabla \boldsymbol{\psi} = -\mathbf{v}
\end{aligned}
\tag{2.11}
$$

with initial conditions

$$
\left. \begin{aligned} \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \\ \boldsymbol{\psi}(0, \mathbf{x}) = \boldsymbol{\psi}_0(\mathbf{x}) \end{aligned} \right\} \text{ in } \mathbb{R}^N. \tag{2.12}
$$

We have

**Theorem 2.2.3.** *Let $N = 2, 3$, $\mathbf{f} \in L^2(0, T, L^2(\mathbb{R}^N)) \cap L^q(0, T, L^p(\mathbb{R}^N))$ for certain $p \in (N, \frac{2N}{N-2})$ and $q > 1$. Let $\mathbf{H}_0 \in W^{1,p}(\mathbb{R}^N) \cap W^{1,2}(\mathbb{R}^N)$, $\operatorname{div} \mathbf{H}_0 = 0$ and $\mathbf{v}_0 \in W^{1,2}_{div}(\mathbb{R}^N)$. Then there exists $T^* > 0$ such that on $(0, T^*)$ there exists a strong solution to the problem (2.9) such that $\mathbf{v} \in L^2(0, T^*, W^{2,2}(\mathbb{R}^N)) \cap L^\infty(0, T^*, W^{1,2}(\mathbb{R}^N))$ and $\mathbf{H} \in L^\infty(0, T^*, W^{1,p}(\mathbb{R}^N))$.*

Similarly for (2.11) we have

**Theorem 2.2.4.** *Let $N = 2, 3$, $\mathbf{f} \in L^2(0, T, L^2(\mathbb{R}^N)) \cap L^q(0, T, L^p(\mathbb{R}^N))$ for certain $p \in (N, \frac{2N}{N-2})$ and $q > 1$. Let $\nabla \boldsymbol{\psi}_0 \in W^{1,p}(\mathbb{R}^N) \cap W^{1,2}(\mathbb{R}^N)$ and $\mathbf{v}_0 \in W^{1,2}_{div}(\mathbb{R}^N)$. Then there exists $T^* > 0$ such that on $(0, T^*)$ there exists a strong solution to the problem (2.11) such that $\mathbf{v} \in L^2(0, T^*, W^{2,2}(\mathbb{R}^N)) \cap L^\infty(0, T^*, W^{1,2}(\mathbb{R}^N))$ and $\nabla \boldsymbol{\psi} \in L^\infty(0, T^*, W^{1,p}(\mathbb{R}^N))$.*

## 2.3 Proofs

In this section we are going to present the proofs of the results presented in the previous section. We start with Theorem 2.2.1.

*Proof of Theorem 2.2.1.* Let us consider first $N = 2$. We take $p > 2$ and we set $X = L^\infty(0, T, W^{1,p}(\Omega))$, $Y = L^\infty(0, T, L^p(\Omega))$ and we will apply Theorem

6.3.2 on the mapping $\mathcal{T} = \mathcal{T}_1 \circ \mathcal{T}_2 : \overline{\mathbf{F}} \to \mathbf{F}$, where $\mathcal{T}_2 : \overline{\mathbf{F}} \to \mathbf{v}$ and $\mathcal{T}_1 : \mathbf{v} \to \mathbf{F}$ are the following mappings. $\mathcal{T}_2(\overline{\mathbf{F}}) = \mathbf{v}$ which is a solution of the problem

$$\left.\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p &= \operatorname{div}(\overline{\mathbf{F}}\,\overline{\mathbf{F}}^T) + \mathbf{f} \\ \operatorname{div}\mathbf{v} &= 0 \end{aligned}\right\} \text{in } (0,T)\times\Omega, \qquad \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \Omega \tag{2.13}$$

and $\mathbf{v} = 0$ on $\partial\Omega \times (0,T)$ for $D_1$, $\mathbf{v}$ space periodic for $P_1$ (then also $W^{1,p}(\Omega)$ and $L^p(\Omega)$ is replaced by $W^{1,p}_{per}(\Omega)$ and $L^p_{per}(\Omega)$).

Further $\mathcal{T}_1(\mathbf{v}) = \mathbf{F}$ which is a solution of the following problem

$$\begin{aligned} \frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{F} &= \nabla\mathbf{v}\mathbf{F} \qquad \text{in } (0,T)\times\Omega, \\ \mathbf{F}(0,\mathbf{x}) &= \mathbf{F}_0(\mathbf{x}) \qquad \text{in } \Omega\,. \end{aligned} \tag{2.14}$$

According to Theorem 6.3.2 we have to prove two things, first that $\mathcal{T}$ maps sufficiently large balls in $L^\infty(0,T,W^{1,p}(\Omega))$ into themselves and second that $\mathcal{T}$ is a contraction on $L^\infty(0,T,L^p(\Omega))$. We start by showing that $\mathcal{T}$ maps sufficiently large balls in $L^\infty(0,T,W^{1,p}(\Omega))$ into themselves. Therefore let us assume that

$$\|\overline{\mathbf{F}}\|_{L^\infty(0,T,W^{1,p}(\Omega))} \leq R \tag{2.15}$$

for some $R$ which will be specified later. Our goal is to show that

$$\|\mathcal{T}(\overline{\mathbf{F}})\|_{L^\infty(0,T,W^{1,p}(\Omega))} = \|\mathbf{F}\|_{L^\infty(0,T,W^{1,p}(\Omega))} \leq R. \tag{2.16}$$

Considering problem (2.14), i.e. the transport equation for $\mathbf{F}$, for $\mathbf{v} \in L^1(0,T,W^{2,p}(\Omega))$, $p > 2$ it is not difficult to prove the existence of the unique weak solution to (2.14). Multiplying $(2.14)_1$ by $|\mathbf{F}|^{q-2}\,\mathbf{F}$ and integrating over $\Omega$ we get[1]

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{F}\|^q_{L^q(\Omega)} = \int_\Omega \nabla\mathbf{v}\mathbf{F} : \mathbf{F}\,|\mathbf{F}|^{q-2}\,\mathrm{d}\mathbf{x}. \tag{2.17}$$

The left hand side can be rewritten in the following way

$$\|\mathbf{F}\|^{q-1}_{L^q(\Omega)}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{F}\|_{L^q(\Omega)} = \int_\Omega \nabla\mathbf{v}\mathbf{F} : \mathbf{F}\,|\mathbf{F}|^{q-2}\,\mathrm{d}\mathbf{x}. \tag{2.18}$$

[1] recall that $\operatorname{div}\mathbf{v} = 0$ and thus the second term is zero.

Using Hölder inequality on the right hand side and dividing by $\|\mathbf{F}\|_{L^q(\Omega)}^{q-1}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{F}\|_{L^q(\Omega)} \leq \|\nabla\mathbf{v}\|_{L^\infty(\Omega)}\|\mathbf{F}\|_{L^q(\Omega)} \tag{2.19}$$

and finally after integrating in time and using Gronwall inequality we end up with

$$\|\mathbf{F}\|_{L^\infty(0,T,L^q)} \leq \|\mathbf{F}_0\|_{L^q}\exp\left(\int_0^T\|\nabla\mathbf{v}\|_{L^\infty}\,\mathrm{d}t\right), \qquad 1 \leq q < \infty. \tag{2.20}$$

Moreover, passing with $q \to \infty$ we see that the same holds also for $q = \infty$.

Similarly we proceed to get the estimate on the gradient of $\mathbf{F}$. We take gradient of $(2.14)_1$:

$$\begin{aligned}\frac{\partial\nabla\mathbf{F}}{\partial t} + (\mathbf{v}\cdot\nabla)\nabla\mathbf{F} = \mathbf{B}_1(\nabla\mathbf{v},\nabla\mathbf{F}) + \mathbf{B}_2(\nabla^2\mathbf{v},\mathbf{F}) \qquad &\text{in } (0,T)\times\Omega,\\ \nabla\mathbf{F}(0,\mathbf{x}) = \nabla\mathbf{F}_0(\mathbf{x}) \qquad &\text{in } \Omega\,,\end{aligned} \tag{2.21}$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are bilinear forms[2]. We multiply it by $|\nabla\mathbf{F}|^{p-2}\nabla\mathbf{F}$ and integrate over $\Omega$. We get after using Hölder inequality and dividing by $\|\nabla\mathbf{F}\|_{L^p(\Omega)}^{p-1}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{F}\|_{L^p(\Omega)} \leq C\left(\|\nabla\mathbf{v}\|_{L^\infty(\Omega)}\|\nabla\mathbf{F}\|_{L^p(\Omega)} + \left\|\nabla^2\mathbf{v}\right\|_{L^p(\Omega)}\|\mathbf{F}\|_{L^\infty(\Omega)}\right). \tag{2.22}$$

Now integrating in time and using (2.20) and Gronwall inequality we get

$$\begin{aligned}\|\nabla\mathbf{F}\|_{L^\infty(0,T,L^p)} \leq C\left(\|\nabla\mathbf{F}_0\|_{L^p} + \|\mathbf{F}\|_{L^\infty(0,T,L^\infty)}\int_0^T\left\|\nabla^2\mathbf{v}\right\|_{L^p}\mathrm{d}t\right)\times\\ \times\exp\left(\int_0^T\|\nabla\mathbf{v}\|_{L^\infty}\,\mathrm{d}t\right).\end{aligned} \tag{2.23}$$

Note that these estimates do not depend on the studied boundary value problems ($C_1$, $D_1$, $P_1$).

Next, we need estimates of $\mathbf{v}$ in $L^q(0,T,W^{2,p})$. As $\overline{\mathbf{F}} \in L^\infty(0,T,W^{1,p})$, due to Theorem 6.2.3, there exists unique solution to problem (2.13). More-

[2] More precisely writing in components and using summation convention $[\mathbf{B}_1(A,B)]_{ijk} = A_{il}B_{ljk} - A_{li}B_{jkl}$ and $[\mathbf{B}_2(A,B)]_{ijk} = A_{ijl}B_{lk}$

over, applying Theorem 6.2.1 we have[3]

$$\left\|\nabla^2\mathbf{v}\right\|_{L^q(0,T,L^p)} \leq C\left(\left\|\mathbf{v}_0\right\|_{D_p^{1-1/q,q}} + \left\|\operatorname{div}(\overline{\mathbf{F}}\,\overline{\mathbf{F}}^T) - \mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{f}\right\|_{L^q(0,T,L^p)}\right). \tag{2.24}$$

Note that we may take $q$ sufficiently close to 1 so that

$$\left\|\mathbf{v}_0\right\|_{D_p^{1-1/q,q}} \leq C\left\|\mathbf{v}_0\right\|_{W^{1,2}}. \tag{2.25}$$

As $p > 2$,

$$\left\|\operatorname{div}(\overline{\mathbf{F}}\,\overline{\mathbf{F}}^T)\right\|_{L^q(0,T,L^p)} \leq T^{1/q}\left\|\overline{\mathbf{F}}\right\|_{L^\infty(0,T,L^\infty)}\left\|\nabla\overline{\mathbf{F}}\right\|_{L^\infty(0,T,L^p)} \leq CT^{1/q}\cdot R^2, \tag{2.26}$$

where $R$ is the diameter of the ball in $L^\infty(0,T,W^{1,p}(\Omega))$ introduced in (2.15).

Finally, taking $M > p$, using Hölder inequality in space and embedding theorems, we get

$$\left\|\mathbf{v}\cdot\nabla\mathbf{v}\right\|_{L^p} \leq \left\|\nabla\mathbf{v}\right\|_{L^M}\left\|\mathbf{v}\right\|_{L^{\frac{Mp}{M-p}}} \leq C\left\|\nabla\mathbf{v}\right\|_{W^{1,2}}^{\frac{M-2}{M}}\left\|\mathbf{v}\right\|_{W^{1,2}}^{\frac{M+2}{M}} \tag{2.27}$$

and thus using Hölder inequality in time[4]

$$\left\|\mathbf{v}\cdot\nabla\mathbf{v}\right\|_{L^q(0,T,L^p)} \leq CT^{\frac{2q+M(2-q)}{2Mq}}\left\|\nabla\mathbf{v}\right\|_{L^2(0,T,W^{1,2})}^{\frac{M-2}{M}}\left\|\mathbf{v}\right\|_{L^\infty(0,T,W^{1,2})}^{\frac{M+2}{M}}. \tag{2.28}$$

We have to apply the energy method to estimate these two terms. First, multiplying (2.13) by $\mathbf{v}$ yields after integration by parts

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}\right\|_{L^2}^2 + \mu\left\|\nabla\mathbf{v}\right\|_{L^2}^2 \leq \int_\Omega\left|\overline{\mathbf{F}}\right|^2\left|\nabla\mathbf{v}\right|\mathrm{d}\mathbf{x} + \int_\Omega\left|\mathbf{f}\cdot\mathbf{v}\right|\mathrm{d}\mathbf{x}, \tag{2.29}$$

which leads using standard operations to

$$\left\|\mathbf{v}\right\|_{L^\infty(0,T,L^2)} + \mu\left\|\nabla\mathbf{v}\right\|_{L^2(0,T,L^2)} \leq C\left(T^{1/2}R^2 + \left\|\mathbf{f}\right\|_{L^2(0,T,L^2)} + \left\|\mathbf{v}_0\right\|_{L^2}\right). \tag{2.30}$$

However this is not enough and we have to get second energy estimate to handle the term $\left\|\nabla\mathbf{v}\right\|_{L^2(0,T,W^{1,2})}$. For this reason we test the same equation

[3]See section 6.2.1 for the definition of space $D_p^{1-1/q,q}$.

[4]Note that we may take $q < 2$.

by $-P\Delta\mathbf{v}$, where $P$ is the Leray projection (thus $P\Delta\mathbf{u} = \Delta\mathbf{u}$ in the case of periodic boundary conditions):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\mathbf{v}\right\|_{L^2}^2 + \mu\left\|\nabla^2\mathbf{v}\right\|_{L^2}^2 \leq \\ \leq \int_\Omega \left|\mathrm{div}\,(\overline{\mathbf{F}}\,\overline{\mathbf{F}}^T)P\Delta\mathbf{v}\right| \mathrm{d}\mathbf{x} + \int_\Omega \left|(\mathbf{v}\cdot\nabla\mathbf{v})\cdot P\Delta\mathbf{v}\right| \mathrm{d}\mathbf{x} + \int_\Omega \left|\mathbf{f}\cdot P\Delta\mathbf{v}\right| \mathrm{d}\mathbf{x}. \tag{2.31}$$

Note that we used that $\|\nabla^2\mathbf{v}\|_{L^2} \leq \|P\Delta\mathbf{v}\|_{L^2}$ and in the case of periodic boundary conditions, the second term on the right hand side is zero.

We estimate the first term on the right hand side

$$\int_\Omega \left|\mathrm{div}\,(\overline{\mathbf{F}}\,\overline{\mathbf{F}}^T)P\Delta\mathbf{v}\right| \mathrm{d}\mathbf{x} \leq \left\|\nabla^2\mathbf{v}\right\|_{L^2}\left\|\nabla\overline{\mathbf{F}}\right\|_{L^p}\left\|\overline{\mathbf{F}}\right\|_{L^{\frac{2p}{p-2}}} \leq \\ \leq \frac{\mu}{4}\left\|\nabla^2\mathbf{v}\right\|_{L^2}^2 + C(\mu)\left\|\overline{\mathbf{F}}\right\|_{W^{1,p}}^4. \tag{2.32}$$

The second term is treated as follows

$$\int_\Omega \left|(\mathbf{v}\cdot\nabla\mathbf{v})\cdot P\Delta\mathbf{v}\right| \mathrm{d}\mathbf{x} \leq \left\|\nabla^2\mathbf{v}\right\|_{L^2}\left\|\nabla\mathbf{v}\right\|_{L^4}\left\|\mathbf{v}\right\|_{L^4} \leq \\ \leq C\left\|\nabla^2\mathbf{v}\right\|_2^{\frac{3}{2}}\left\|\nabla\mathbf{v}\right\|_{L^2}\left\|\mathbf{v}\right\|_{L^2}^{\frac{1}{2}} \leq \frac{\mu}{4}\left\|\nabla^2\mathbf{v}\right\|_{L^2}^2 + C(\mu)\left\|\nabla\mathbf{v}\right\|_{L^2}^4\left\|\mathbf{v}\right\|_{L^2}^2 \tag{2.33}$$

and the last term is simple. Puting these calculations together and using (2.31) we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\mathbf{v}\right\|_{L^2}^2 + \mu\left\|\nabla^2\mathbf{v}\right\|_{L^2}^2 \leq C(\mu)(1+TR^4)\left(\left\|\overline{\mathbf{F}}\right\|_{W^{1,p}}^4 + \left\|\mathbf{f}\right\|_{L^2}^2 + \left\|\nabla\mathbf{v}\right\|_{L^2}^4\right). \tag{2.34}$$

This leads to

$$\left\|\nabla\mathbf{v}\right\|_{L^\infty(0,T,L^2)} + \mu\left\|\nabla^2\mathbf{v}\right\|_{L^2(0,T,L^2)} \leq \\ \leq C(1+T^{\frac{1}{2}}R^2)e^{\int_0^T\|\nabla\mathbf{v}\|_{L^2}^2\mathrm{d}t}\left(T^{1/2}R^2 + \left\|\mathbf{f}\right\|_{L^2(0,T,L^2)} + \left\|\nabla\mathbf{v}_0\right\|_{L^2}\right). \tag{2.35}$$

Now we plug all these estimates together, namely (2.35) and (2.30) into (2.28) and using (2.26) and (2.24) we have

$$\begin{aligned}\|\nabla\mathbf{v}\|_{L^q(0,T,W^{1,p})} &\le C\Big(\|\mathbf{v}_0\|_{D_p^{1-1/q,q}} + T^{1/q}R^2 + \|\mathbf{f}\|_{L^q(0,T,L^p)} + \\ &+ CT^{\frac{2q+M(2-q)}{2Mq}}\Big[TR^4 + \|\mathbf{f}\|^2_{L^2(0,T,L^2)} + \|\mathbf{v}_0\|^2_{L^2} + (1+TR^4)\times \\ &\times e^{TR^4+\|\mathbf{f}\|^2_{L^2(0,T,L^2)}+\|\mathbf{v}_0\|^2_{L^2}}\Big(TR^4 + \|\mathbf{f}\|^2_{L^2(0,T,L^2)} + \|\nabla\mathbf{v}_0\|^2_{L^2}\Big)\Big]\Big). \end{aligned} \tag{2.36}$$

Therefore, we easily conclude from (2.20) and (2.23) that taking $R$ sufficiently large with respect to $\|\mathbf{F}_0\|_{W^{1,p}}$ there exists

$$T^* = T^*(q, p, R, \|\mathbf{v}_0\|_{W^{1,2}}, \|\mathbf{F}_0\|_{W^{1,p}}, \Omega, \mathbf{f}) \tag{2.37}$$

such that

$$\|\mathbf{F}\|_{L^\infty(0,T^*,W^{1,p})} \le R. \tag{2.38}$$

The first part of Theorem 6.3.2 is satisfied.

Next, we proceed with the second part. We have to verify that the mapping $\mathcal{T}$ is in fact a contraction on $L^\infty(0,T,L^p(\Omega))$. To this aim let us denote $\mathbf{v}^i = \mathcal{T}_2(\overline{\mathbf{F}}^i)$, $i = 1,2$ and $\mathbf{F}^i = \mathcal{T}_1(\mathbf{v}^i)$, thus $\mathbf{F}^i = \mathcal{T}(\overline{\mathbf{F}}^i)$. Moreover we denote $\mathbf{F}^{12} = \mathbf{F}^1 - \mathbf{F}^2$, $\mathbf{v}^{12} = \mathbf{v}^1 - \mathbf{v}^2$ and $\overline{\mathbf{F}}^{12} = \overline{\mathbf{F}}^1 - \overline{\mathbf{F}}^2$. Subtracting equations for $\mathbf{F}^1$ and $\mathbf{F}^2$ we have

$$\begin{aligned} \frac{\partial\mathbf{F}^{12}}{\partial t} + \mathbf{v}^1\cdot\nabla\mathbf{F}^{12} + \mathbf{v}^{12}\cdot\nabla\mathbf{F}^2 = \nabla\mathbf{v}^1\mathbf{F}^{12} + \nabla\mathbf{v}^{12}\mathbf{F}^2 \qquad &\text{in } (0,T)\times\Omega, \\ \mathbf{F}^{12}(0,\mathbf{x}) = \mathbf{0} \qquad \text{in } \Omega\,. & \end{aligned} \tag{2.39}$$

Similarly as in estimate (2.20) we get here

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{F}^{12}\|^p_{L^p} &\le \int_\Omega |\mathbf{v}^{12}|\,|\nabla\mathbf{F}^2|\,|\mathbf{F}^{12}|^{p-1}\,\mathrm{d}\mathbf{x} + \\ &+ \int_\Omega |\nabla\mathbf{v}^1|\,|\mathbf{F}^{12}|^p\,\mathrm{d}\mathbf{x} + \int_\Omega |\nabla\mathbf{v}^{12}|\,|\mathbf{F}^2|\,|\mathbf{F}^{12}|^{p-1}\,\mathrm{d}\mathbf{x} \le \\ &\le \|\mathbf{F}^{12}\|^{p-1}_{L^p}\big(\|\mathbf{v}^{12}\|_{L^\infty}\|\nabla\mathbf{F}^2\|_{L^p} + \\ &+ \|\nabla\mathbf{v}^1\|_{L^\infty}\|\mathbf{F}^{12}\|_{L^p} + \|\mathbf{F}^2\|_{L^\infty}\|\nabla\mathbf{v}^{12}\|_{L^p}\big), \end{aligned} \tag{2.40}$$

hence

$$\left\|\mathbf{F}^{12}\right\|_{L^\infty(0,T,L^p)} \le$$
$$\le C e^{\int_0^T \left\|\nabla \mathbf{v}^1\right\|_{L^\infty} \mathrm{d}t} \int_0^T \left(\left\|\mathbf{v}^{12}\right\|_{L^\infty} \left\|\nabla \mathbf{F}^2\right\|_{L^p} + \left\|\mathbf{F}^2\right\|_{L^\infty} \left\|\nabla \mathbf{v}^{12}\right\|_{L^p}\right) \mathrm{d}t \le$$
$$\le CR \int_0^T \left\|\mathbf{v}^{12}\right\|_{W^{1,p}} \mathrm{d}t. \quad (2.41)$$

It remains to estimate the difference $\mathbf{v}^{12}$ in $L^p(0,T,W^{1,p}(\Omega))$. We have (denoting by $p^{12}$ the difference of corresponding pressures)

$$\left.\begin{array}{c} \dfrac{\partial \mathbf{v}^{12}}{\partial t} - \mu \Delta \mathbf{v}^{12} + \nabla p^{12} = \operatorname{div}\left(\overline{\mathbf{F}}^{12} \overline{\mathbf{F}}^{1T} + \overline{\mathbf{F}}^{2} \overline{\mathbf{F}}^{12T}\right) - \\ -\mathbf{v}^1 \cdot \nabla \mathbf{v}^{12} - \mathbf{v}^{12} \cdot \nabla \mathbf{v}^2 \\ \operatorname{div} \mathbf{v}^{12} = 0 \end{array}\right\} \text{in } (0,T)\times\Omega,$$
$$\mathbf{v}^{12}(0,\mathbf{x}) = \mathbf{0} \qquad \text{in } \Omega, \quad (2.42)$$

together with corresponding boundary conditions (Dirichlet or periodic ones). Applying Theorem 6.2.2 we immediately see that

$$\left\|\nabla \mathbf{v}^{12}\right\|_{L^p(L^p)} \le C \left\|\overline{\mathbf{F}}^{12} \overline{\mathbf{F}}^{1T} + \overline{\mathbf{F}}^{2} \overline{\mathbf{F}}^{12T} - \mathbf{v}^1 \otimes \mathbf{v}^{12} - \mathbf{v}^{12} \otimes \mathbf{v}^2\right\|_{L^p(L^p)} \le$$
$$\le C \left( \sum_{i=1}^2 \left\|\mathbf{v}^i \otimes \mathbf{v}^{12}\right\|_{L^p(0,T,L^p)} + \sum_{i=1}^2 \left\|\overline{\mathbf{F}}^{i} \overline{\mathbf{F}}^{12T}\right\|_{L^p(0,T,L^p)} \right). \quad (2.43)$$

We deal with the last terms easily

$$\left\|\overline{\mathbf{F}}^{i} \overline{\mathbf{F}}^{12T}\right\|_{L^p(0,T,L^p)} \le T^{1/p} \left\|\overline{\mathbf{F}}^{i}\right\|_{L^\infty(0,T,L^\infty)} \left\|\overline{\mathbf{F}}^{12}\right\|_{L^\infty(0,T,L^p)} \le$$
$$\le CRT^{1/p} \left\|\overline{\mathbf{F}}^{12}\right\|_{L^\infty(0,T,L^p)}. \quad (2.44)$$

In the other term we have to be more careful. We can proceed in the following way (again we take $M > p$)

$$\left\|\mathbf{v}^i \mathbf{v}^{12}\right\|_{L^p} \le \left\|\mathbf{v}^i\right\|_{L^{\frac{Mp}{M-p}}} \left\|\mathbf{v}^{12}\right\|_{L^M} \le C \left\|\mathbf{v}^i\right\|_{W^{1,2}} \left\|\mathbf{v}^{12}\right\|_{L^2}^{\frac{2p+M(p-2)}{2M(p-1)}} \left\|\nabla \mathbf{v}^{12}\right\|_{L^p}^{\frac{p(M-2)}{2M(p-1)}} \quad (2.45)$$

and therefore

$$
\begin{aligned}
&\left\|\mathbf{v}^i\mathbf{v}^{12}\right\|_{L^p(0,T,L^p)} \leq \\
&\quad \leq C\left\|\mathbf{v}^i\right\|_{L^\infty(0,T,W^{1,2})}\left(\int_0^T \left\|\nabla\mathbf{v}^{12}\right\|_{L^p}^{\frac{p^2(M-2)}{2M(p-1)}}\left\|\mathbf{v}^{12}\right\|_{L^2}^{\frac{p(2p+M(p-2))}{2M(p-1)}}\,\mathrm{d}t\right)^{1/p} \leq \\
&\leq CT^{\frac{2p+M(p-2)}{2pM(p-1)}}\left\|\mathbf{v}^i\right\|_{L^\infty(0,T,W^{1,2})}\left(\left\|\nabla\mathbf{v}^{12}\right\|_{L^p(0,T,L^p)}+\left\|\mathbf{v}^{12}\right\|_{L^\infty(0,T,L^2)}\right). \qquad (2.46)
\end{aligned}
$$

Taking $T$ sufficiently small, the first term can be absorbed into the left hand-side of (2.43), while for the second term we use the energy method.

We multiply (2.42) by $\mathbf{v}^{12}$ and integrate over $\Omega$:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}^{12}\right\|_{L^2}^2+\mu\left\|\nabla\mathbf{v}^{12}\right\|_{L^2}^2 \leq \\
&\quad \leq \int_\Omega \left|\mathbf{v}^{12}\right|^2\left|\nabla\mathbf{v}^2\right|\mathrm{d}\mathbf{x}+\int_\Omega\left|\overline{\mathbf{F}}^{12}\right|\left(\left|\overline{\mathbf{F}}^1\right|+\left|\overline{\mathbf{F}}^2\right|\right)\left|\nabla\mathbf{v}^{12}\right|\mathrm{d}\mathbf{x} \leq \\
&\qquad \leq \left\|\mathbf{v}^{12}\right\|_{L^2}\left\|\nabla\mathbf{v}^{12}\right\|_{L^2}\left\|\nabla\mathbf{v}^2\right\|_{L^2}+ \\
&\qquad\quad +\left\|\nabla\mathbf{v}^{12}\right\|_{L^2}\left\|\overline{\mathbf{F}}^{12}\right\|_{L^p}\left(\left\|\overline{\mathbf{F}}^1\right\|_{W^{1,p}}+\left\|\overline{\mathbf{F}}^2\right\|_{W^{1,p}}\right) \qquad (2.47)
\end{aligned}
$$

Therefore

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}^{12}\right\|_{L^2}^2+\mu\left\|\nabla\mathbf{v}^{12}\right\|_{L^2}^2 \leq C(\mu)\left(\left\|\mathbf{v}^{12}\right\|_{L^2}^2\left\|\nabla\mathbf{v}^2\right\|_{L^2}^2+2R^2\left\|\overline{\mathbf{F}}^{12}\right\|_{L^p}^2\right), \qquad (2.48)
$$

which implies

$$
\left\|\mathbf{v}^{12}\right\|_{L^\infty(0,T,L^2)}+\mu\left\|\nabla\mathbf{v}^{12}\right\|_{L^2(0,T,L^2)} \leq CRT^{1/2}e^{\int_0^T\left\|\nabla\mathbf{v}^2\right\|_{L^2}^2\mathrm{d}t}\left\|\overline{\mathbf{F}}^{12}\right\|_{L^\infty(0,T,L^p)}. \qquad (2.49)
$$

Thus, estimates (2.43), (2.44), (2.46) and (2.49) imply

$$
\begin{aligned}
&\left\|\nabla\mathbf{v}^{12}\right\|_{L^p(0,T,L^p)} \leq \\
&\leq CRT^{1/p}\left\|\overline{\mathbf{F}}^{12}\right\|_{L^\infty(0,T,L^p)}+C(R,\mathbf{f},\mu,\mathbf{F}_0,\mathbf{v}_0)T^{\frac{2p+M(p-2)}{2pM(p-1)}+\frac{1}{2}}\left\|\overline{\mathbf{F}}^{12}\right\|_{L^\infty(0,T,L^p)}. \qquad (2.50)
\end{aligned}
$$

From (2.50) and (2.41) we get

$$
\left\|\mathbf{F}^{12}\right\|_{L^\infty(0,T,L^p)} \leq C(R,\mathbf{f},\mu,\mathbf{F}_0,\mathbf{v}_0)T^\alpha\left\|\overline{\mathbf{F}}^{12}\right\|_{L^\infty(0,T,L^p)} \qquad (2.51)
$$

for some $\alpha$ positive, which yields for $T$ sufficiently small that $\mathcal{T}$ is a contraction in $L^\infty(0,T,L^p)$. The theorem for $N=2$ is proved.

Let us now continue with the case $N=3$. The proof works basically similarly, with $p>3$. There is one main difference which reflects the fundamental difference between the regularity of solutions to the two– and three dimensional Navier–Stokes equations, see below. Except for this, there are only a few changes connected with interpolation inequalities. In Chapter 3 we use similar method to prove local existence for a different model. There we write down the detailed proof in the case $N=3$. Most of the features apply also here, therefore we only state the differences between three dimensional and two dimensional case here.

It holds for $p<6$

$$\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^p} \le \|\nabla\mathbf{v}\|_{L^p}\|\mathbf{v}\|_{L^\infty} \le C\|\nabla\mathbf{v}\|_{W^{1,2}}^{\frac{2p-3}{p}}\|\nabla\mathbf{v}\|_{L^2}^{\frac{3}{p}} \tag{2.52}$$

and therefore the estimate (2.28) is replaced by

$$\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^q(0,T,L^p)} \le CT^{\frac{2p-q(2p-3)}{2pq}}\|\nabla\mathbf{v}\|_{L^2(0,T,W^{1,2})}^{\frac{2p-3}{p}}\|\mathbf{v}\|_{L^\infty(0,T,W^{1,2})}^{\frac{3}{p}}. \tag{2.53}$$

Note that we may always take $1<q<\frac{2p}{2p-3}$ and the power at $T$ is positive.

The main difference is in estimate (2.35). While in the 2D case we were able to get the estimates immediately (note that we have global-in-time existence of strong solutions to the Navier–Stokes equations), we have now instead of (2.34)[5]

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_{L^2}^2 + \mu\left\|\nabla^2\mathbf{v}\right\|_{L^2}^2 \le C(\mu)\left(\left\|\overline{\mathbf{F}}\right\|_{W^{1,p}}^4 + \|\mathbf{f}\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^6\right), \tag{2.54}$$

from where one can only deduce that there is

$$\begin{aligned} &T^* = T^*(\|\overline{\mathbf{F}}\|_{L^\infty(0,T,W^{1,p})},\mathbf{f},\mathbf{v}_0) > 0 \text{ and}\\ &C^* = C^*(T^*,\|\overline{\mathbf{F}}\|_{L^\infty(0,T,W^{1,p})},\mathbf{f},\mathbf{v}_0), \end{aligned} \tag{2.55}$$

such that there is a strong solution to (2.13) in 3D on $(0,T^*)$ with

$$\|\nabla\mathbf{v}\|_{L^\infty(0,T^*,L^2)} + \left\|\nabla^2\mathbf{v}\right\|_{L^2(0,T^*,L^2)} \le C^*. \tag{2.56}$$

[5]See the diference between 2D and 3D cases in Corollary 6.1.11.

Estimate (2.46) is replaced by

$$\left\|\mathbf{v}^i\mathbf{v}^{12}\right\|_{L^p(0,T,L^p)} \leq C\left(\int_0^T \left\|\mathbf{v}^i\right\|_{L^6}^p \left\|\nabla\mathbf{v}^{12}\right\|_{L^p}^{p\frac{4p-6}{5p-6}} \left\|\mathbf{v}^{12}\right\|_{L^2}^{\frac{p^2}{5p-6}} \mathrm{d}t\right)^{1/p} \leq$$
$$\leq CT^{\frac{1}{5p-6}} \left\|\mathbf{v}^i\right\|_{L^\infty(0,T,W^{1,2})} \left(\left\|\nabla\mathbf{v}^{12}\right\|_{L^p(0,T,L^p)} + \left\|\mathbf{v}^{12}\right\|_{L^\infty(0,T,L^2)}\right). \quad (2.57)$$

Instead of (2.49) we get

$$\left\|\mathbf{v}^{12}\right\|_{L^\infty(0,T,L^2)} + \mu\left\|\nabla\mathbf{v}^{12}\right\|_{L^2(0,T,L^2)} \leq CRT^{1/2}e^{\int_0^T\left\|\nabla\mathbf{v}^2\right\|_{L^2}^4\mathrm{d}t}\left\|\overline{\mathbf{F}}^{12}\right\|_{L^\infty(0,T,L^p)} \quad (2.58)$$

and instead of (2.50) we have

$$\left\|\nabla\mathbf{v}^{12}\right\|_{L^p(0,T,L^p)} \leq CRT^{1/p}\left\|\overline{\mathbf{F}}^{12}\right\|_{L^\infty(0,T,L^p)} + CT^{\frac{5p-4}{10p-12}}\left\|\overline{\mathbf{F}}^{12}\right\|_{L^\infty(0,T,L^p)}. \quad (2.59)$$

Thus, provided $T \leq T^*$ is sufficiently small and $R$ sufficiently large, we get that the mapping $\mathcal{T}$ maps balls in $L^\infty(0,T,W^{1,p}(\Omega))$ into itself and it is a contraction in the space $L^\infty(0,T,L^p(\Omega))$. This finishes the proof of Theorem 2.2.1 for $N = 3$. □

We proceed further with the proof of Theorem 2.2.2. As systems (2.2) and (2.7) are very similar and also the statements of Theorems 2.2.1 and 2.2.2 are similar, we again only write down the differences in the proof.

*Proof of Theorem 2.2.2.* First note that

$$\frac{\partial}{\partial t}\nabla\boldsymbol{\phi} + (\mathbf{v}\cdot\nabla)\nabla\boldsymbol{\phi} + \nabla\mathbf{v}\nabla\boldsymbol{\phi} = \mathbf{0} \quad (2.60)$$

and

$$\frac{\partial}{\partial t}\nabla^2\boldsymbol{\phi} + (\mathbf{v}\cdot\nabla)\nabla^2\boldsymbol{\phi} + \nabla^2\mathbf{v}\nabla\boldsymbol{\phi} = \mathbf{0} \quad (2.61)$$

thus, in the estimates we need, $\nabla\boldsymbol{\phi} \sim \mathbf{F}$ and $\nabla^2\boldsymbol{\phi} \sim \nabla\mathbf{F}$. The proof works now more or less in the same way as in Theorem 2.2.1, with only one exception. It is impossible to write $\sum_{i=1}^{2}\nabla\phi_i\Delta\phi_i$ as a divergence of some quantity which behaves like $|\nabla\boldsymbol{\phi}|^2$. Therefore in (2.43) the second term is replaced by (again, we denote $\boldsymbol{\phi}^{12} = \boldsymbol{\phi}^1 - \boldsymbol{\phi}^2$)

$$I \equiv \left\|\nabla\boldsymbol{\phi}^i\Delta\boldsymbol{\phi}^{12}\right\|_{L^p(0,T,W^{-1,p})} + \left\|\Delta\boldsymbol{\phi}^i\nabla\boldsymbol{\phi}^{12}\right\|_{L^p(0,T,W^{-1,p})}. \quad (2.62)$$

In 2D we have

$$\begin{aligned} I &\le \left\|\nabla\boldsymbol{\phi}^i\nabla\boldsymbol{\phi}^{12}\right\|_{L^p(0,T,L^p)} + \left\|\nabla^2\boldsymbol{\phi}^i\nabla\boldsymbol{\phi}^{12}\right\|_{L^p(0,T,L^{\frac{2p}{p+2}})} \le \\ &\le CT^{1/p}\left\|\nabla\boldsymbol{\phi}^{12}\right\|_{L^\infty(0,T,L^p)}\left(\left\|\nabla\boldsymbol{\phi}^i\right\|_{L^\infty(0,T,L^\infty)} + \left\|\nabla^2\boldsymbol{\phi}^i\right\|_{L^\infty(0,T,L^2)}\right) \le \\ &\le CT^{1/p}\left\|\nabla\boldsymbol{\phi}^{12}\right\|_{L^\infty(0,T,L^p)}\left\|\nabla\boldsymbol{\phi}^i\right\|_{L^\infty(0,T,W^{1,p})} \end{aligned} \tag{2.63}$$

and in 3D

$$\begin{aligned} I &\le \left\|\nabla\boldsymbol{\phi}^i\nabla\boldsymbol{\phi}^{12}\right\|_{L^p(0,T,L^p)} + \left\|\nabla^2\boldsymbol{\phi}^i\nabla\boldsymbol{\phi}^{12}\right\|_{L^p(0,T,L^{\frac{3p}{p+3}})} \le \\ &\le CT^{1/p}\left\|\nabla\boldsymbol{\phi}^{12}\right\|_{L^\infty(0,T,L^p)}\left(\left\|\nabla\boldsymbol{\phi}^i\right\|_{L^\infty(0,T,L^\infty)} + \left\|\nabla^2\boldsymbol{\phi}^i\right\|_{L^\infty(0,T,L^3)}\right) \le \\ &\le CT^{1/p}\left\|\nabla\boldsymbol{\phi}^{12}\right\|_{L^\infty(0,T,L^p)}\left\|\nabla\boldsymbol{\phi}^i\right\|_{L^\infty(0,T,W^{1,p})}. \end{aligned} \tag{2.64}$$

The rest follows exactly as in the proof of Theorem 2.2.1. □

The whole space case is also very similar.

*Proof of Theorems 2.2.3 and 2.2.4.* Due to the presence of linear terms in the right-hand side of (2.9) and (2.11) we have to replace the spaces. We set

$$X = L^\infty(0,T,W^{1,p}(\mathbb{R}^N)\cap W^{1,2}(\mathbb{R}^N)) \tag{2.65}$$

$$Y = L^\infty(0,T,L^p(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)), \qquad p > N. \tag{2.66}$$

We additionally need estimates of $\mathbf{H}$ in $L^\infty(0,T,W^{1,2}(\mathbb{R}^N))$ and $\mathbf{H}^{12} = \mathbf{H}^1 - \mathbf{H}^2$ in $L^\infty(0,T,L^2(\mathbb{R}^N))$, similarly for $\nabla\boldsymbol{\psi}$. However, these estimates can be easily deduced from the corresponding estimates with $p > N$. These estimates are necessary as in (2.9) and in (2.11) linear terms of the type $\operatorname{div}\mathbf{H}^T$ and $\Delta\boldsymbol{\psi}$ appear additionally. We can now follow step by step the proof of Theorems 2.2.1 and 2.2.2. □

The results in this Chapter were published in [21].

# Chapter 3

# Local existence for a polymeric fluid model

## 3.1 The model

Here we study model of a polymeric liquid introduced in 1.4.2. The model consists of a Navier-Stokes equation with elastic extra stress term $\operatorname{div} \mathbf{T}_e$ on the right hand side achieved from (1.19) and (1.44), divergence free condition (1.16) and additional equation describing the evolution of $\mathbf{T}_e$. The elastic extra stress is related to the behavior of the polymers, namely to the potential of the elastic force $U(\mathbf{r})$ and the probability density $\psi(t, \mathbf{x}, \mathbf{r})$ by the Kramer expression (1.46). The key quantity here is the probability density $\psi$, its evolution is described by the Fokker-Planck equation (1.47). For simplicity we set some constants equal to 1, namely the density $\rho$ and the product of $k_B\theta$ and moreover we assume the external force $\mathbf{f} = \mathbf{0}$[1]. This yields the system

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \mu \Delta \mathbf{v} &= \operatorname{div} \mathbf{T}_e \\
\operatorname{div} \mathbf{v} &= 0 \\
\mathbf{T}_e &= \int_D \mathbf{r} \otimes \nabla_{\mathbf{r}} U(\mathbf{r}) \psi(t, \mathbf{x}, \mathbf{r}) \mathrm{d}\mathbf{r} \\
\frac{\partial \psi}{\partial t} + \mathbf{v} \cdot \nabla \psi &= \operatorname{div}_{\mathbf{r}} [-\nabla_{\mathbf{x}} \mathbf{v} \cdot \mathbf{r} \psi + \nabla_{\mathbf{r}} \psi + \nabla_{\mathbf{r}} U \psi],
\end{aligned} \tag{3.1}$$

[1]The case $\mathbf{f} \neq \mathbf{0}$ is also treatable and one could state conditions under which the results of this chapter hold, similarly as we have done in previous chapter.

where the potential of the elastic force $U(\mathbf{r})$ and the domain of admissible conformations of the polymers $D$ are given and the unknown quantities are the velocity $\mathbf{v}(t,\mathbf{x})$, pressure $p(t,\mathbf{x})$ and the probability density $\psi(t,\mathbf{x},\mathbf{r})$. We add to system (3.1) initial conditions

$$\begin{aligned} \mathbf{v}(0,\mathbf{x}) &= \mathbf{v}_0(\mathbf{x}), \\ \psi(0,\mathbf{x},\mathbf{r}) &= \psi_0(\mathbf{x},\mathbf{r}). \end{aligned} \tag{3.2}$$

We have to consider three types of boundary conditions for this problem. First, we consider boundary conditions for the velocity $\mathbf{v}$ on the boundary of the physical domain $\Omega$ occupied by the fluid. Similarly as in the previous chapter we will consider two boundary conditions on $\mathbf{v}$ – either we will assume that $\Omega = (0,L)^N$ is a periodic box and we will consider space-periodic functions or we will assume that $\Omega$ is a bounded domain with $\mathbf{v} = \mathbf{0}$ on $(0,T)\times\partial\Omega$. Next, we impose a boundary condition for $\psi$ on the boundary of the domain of possible conformations $D$. In this case in view of the equation $(3.1)_4$ it is natural to assume that

$$(-\nabla_x \mathbf{v}\cdot\mathbf{r}\psi + \nabla_r\psi + \nabla_r U\psi)\cdot\mathbf{n}_D = 0 \qquad \text{in } (0,T)\times\Omega\times\partial D, \tag{3.3}$$

where $\mathbf{n}_D$ is the unit outer normal to $D$. This condition is to be understood in the weak sense and it ensures the conservation of $\psi$. Moreover if $\int_D \psi_0(\mathbf{x},\mathbf{r})\mathrm{d}\mathbf{r} = 1$ for all $\mathbf{x}\in\Omega$ then $\int_D \psi(t,\mathbf{x},\mathbf{r})\mathrm{d}\mathbf{r} = 1$ for all $t$ and $\mathbf{x}\in\Omega$, i.e. $\psi$ really stays a probability density. Last possible type of boundary condition is to prescribe the behaviour of $\psi$ on $\partial\Omega$. However, there is no need to prescribe such condition.

We study the so-called FENE dumbbell model[2]. As the polymers are assumed to have finite extensibility, the domain $D$ is a ball with finite diameter $r_0$, $B_{r_0}(\mathbf{0})$. The FENE potential is given by

$$U(\mathbf{r}) = -k\ln\left(1 - \frac{|\mathbf{r}|^2}{r_0^2}\right), \tag{3.4}$$

with $k > 0$ being a constant, see e.g. [30]. We again make a small simplification by assuming $r_0 = 1$ and we denote $B = B_1(\mathbf{0})$. Note that there is a stationary solution to (3.1) given by $\mathbf{v} = \mathbf{0}$ and

$$\psi(\mathbf{r}) = \psi_\infty(\mathbf{r}) = \frac{e^{-U(\mathbf{r})}}{\int_B e^{-U(\mathbf{r}')}\mathrm{d}\mathbf{r}'}. \tag{3.5}$$

[2]FENE stands for Finite Extensible Nonlinear Elastic.

Especially for the FENE potential

$$\psi_\infty(\mathbf{r}) = C\left(1-|\mathbf{r}|^2\right)^k. \tag{3.6}$$

This quantity will play a crucial role in our study of this model. Now we can rewrite the system (3.1) into more convenient form

$$\left.\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p = \operatorname{div}\mathbf{T}_e \\ \operatorname{div}\mathbf{v} = 0 \end{aligned}\right\} \text{ in } (0,T)\times\Omega, \tag{3.7}$$

$$\mathbf{T}_e = 2k\int_B \frac{\mathbf{r}\otimes\mathbf{r}}{1-|\mathbf{r}|^2}\psi \mathrm{d}\mathbf{r}, \tag{3.8}$$

$$\frac{\partial \psi}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}}\psi - L(\psi) = \operatorname{div}_{\mathbf{r}}[-\nabla_{\mathbf{x}}\mathbf{v}\cdot\mathbf{r}\psi] \qquad \text{in } (0,T)\times\Omega\times B, \tag{3.9}$$

where the operator $L$ is given by

$$L(\psi) = \operatorname{div}_{\mathbf{r}}\left(\psi_\infty\nabla_{\mathbf{r}}\frac{\psi}{\psi_\infty}\right). \tag{3.10}$$

Systems coupling fluids and polymers became recently to be of great interest due to their applications in many branches of science. There are several models which describe the polymer molecule which may be very complicated. As we are interested in the macroscopic behaviour of the fluid rather than detailed structure of the molecules, we consider only a simple dumbbell model. However, even in this case we achieve an interesting system of partial differential equations consisting of a Navier–Stokes and a Fokker–Planck equation. The Navier–Stokes equation describes the behaviour on the macroscopic level while the Fokker–Planck equation desribes the evolution of conformations of the polymer molecules on the microscopic level. An approximate closure of the linear Fokker-Planck equation reduces the system to equations for the added stresses. This leads to non-newtonian systems such as the Oldroyd-type models mentioned in the previous chapter.

There are several works concerning the system on the whole micro-macro level. One of the earliest results is due to Renardy [38], where the author proved local existence with the potential $U(\mathbf{r}) = (1-|\mathbf{r}|^2)^{1-\sigma}$ for $\sigma > 1$. E, Li and Zhang [9] showed local existence in the case where $\mathbf{r}$ is taken in the whole space under some growth conditions on the potential. Jourdain, Lelièvre and Le Bris [17] proved local existence for the FENE model with $b = 2k > 6$ for a Couette flow while Zhang and Zhang [46] proved local well-posedness for the FENE model with $b = 2k > 76$. The lower bound on $b$ was due to the fact that the authors did not work with the weighted spaces $\mathcal{L}^p$ (see below) and this bound arised from the energy estimates for the Fokker-Planck equation. Masmoudi [30] showed above else the local well-posedness in energy spaces for the FENE model with $k > 0$ and also global existence for the corotational FENE model. Lions and Masmoudi [26] also dealed with the same problem. Barrett, Schwab and Süli [2] proved global existence for the FENE model which includes some $\mathbf{x}$-mollified terms in the Fokker-Planck equation. Moreover, Barrett and Süli [3] proved global existence for the FENE model with a center-of-mass diffusion and a microscopic cut-off.

## 3.2 Preliminaries

Here we summarize some useful results which we will need later. First, for $1 < p < \infty$ we define spaces

$$\mathcal{L}^p(B) \equiv \mathcal{L}^p = \left\{ \psi, \|\psi\|^p_{\mathcal{L}^p} = \int_B \psi_\infty \left| \frac{\psi}{\psi_\infty} \right|^p \mathrm{d}\mathbf{r} < \infty \right\}. \tag{3.11}$$

We have this simple lemma.

**Lemma 3.2.1.** *$\mathcal{L}^p \hookrightarrow \mathcal{L}^q$ for $p > q$.*

*Proof.* It is a straightforward consequence of Hölder inequality and properties of $\psi_\infty$. □

Moreover we introduce the following notation

$$D_p(\psi) = D_p\psi = \psi_\infty \nabla_{\mathbf{r}} \left| \frac{\psi}{\psi_\infty} \right|^{p/2}. \tag{3.12}$$

The operator $L$ defined by (3.10) can be considered as a weighted Laplace operator and in fact it shares some useful properties with the Laplacian. We summarize some of them here, for proofs see [30]. We study the operator $L$ on the space $\mathcal{L}^2$ with the domain

$$D(L) = \left\{ \psi \in \mathcal{L}^2, D_2\psi \in \mathcal{L}^2, L(\psi) \in \mathcal{L}^2, D_2\psi = 0 \text{ on } \partial B \right\}. \tag{3.13}$$

It holds

**Lemma 3.2.2.** *$L$ is self-adjoint and positive operator. Moreover it has a discrete spectrum formed by a sequence of eigenvalues $\{\lambda_n\}$ such that $\lambda_n \to \infty$ for $n \to \infty$.*

We have then the following existence theorem for the Fokker-Planck equation with given velocity field.

**Lemma 3.2.3.** *Let $\mathbf{v} \in L^\infty(0,T,W^{1,2}(\Omega)) \cap L^2(0,T,W^{2,2}(\Omega))$ be given and let $\psi_0 \in W^{1,2}(\Omega,\mathcal{L}^2)$. Then there exist a unique solution $\psi$ to equation (3.9) such that $\psi \in L^\infty(0,T,W^{1,2}(\Omega,\mathcal{L}^2))$ with $D_2\psi \in L^2(0,T,W^{1,2}(\Omega,\mathcal{L}^2))$*

*Proof.* First, let us recall that by a weak solution of the equation (3.9) we mean a function $\psi \in L^\infty(0,T,L^2(\Omega,\mathcal{L}^2))$ with $D_2\psi \in L^2(0,T,L^2(\Omega,\mathcal{L}^2))$ such that

$$
\begin{aligned}
&-\int_0^T \int_\Omega \int_B \frac{\psi}{\psi_\infty}\frac{\partial\varphi}{\partial t}\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Omega \int_B \psi_\infty \nabla_{\mathbf{r}}\frac{\psi}{\psi_\infty}\nabla_{\mathbf{r}}\frac{\varphi}{\psi_\infty}\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{x}\mathrm{d}t = \\
&= \int_\Omega \int_B \frac{\psi_0\varphi(0)}{\psi_\infty}\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{x} + \int_0^T \int_\Omega \int_B \left(\nabla_{\mathbf{x}}\mathbf{v}\cdot\mathbf{r}\psi\nabla_{\mathbf{r}}\frac{\varphi}{\psi_\infty} + \frac{\psi}{\psi_\infty}\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi\right)\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{x}\mathrm{d}t
\end{aligned}
\tag{3.14}
$$

for all functions $\varphi$ such that $\varphi, D_2\varphi \in C^\infty([0,T],C^\infty(\Omega,\mathcal{L}^2))$ with $\varphi(T) = 0$. This also gives a sense to the boundary condition (3.3).

The whole proof can be found in [30]. We will only mention the main ingrediences and ideas of the proof. We use Lemma 3.2.2 and we look at all terms in equation (3.9) only as functions of variables $t$ and $\mathbf{r}$ and prove existence using the Galerkin method. Finally, we prove the regularity in the variable $\mathbf{x}$ by rewriting the equation into Lagrangean coordinates and studying difference quotients. □

Finally, we prove the following inequality which allows us to estimate $L^p$ norms of $\mathbf{T}_e$ by norms of $\psi$.

**Lemma 3.2.4.** *For all $\varepsilon > 0$ and $p > 2$ there exists $C_\varepsilon > 0$ such that*

$$
\left(\int_B \frac{|\psi|}{1-|\mathbf{r}|}\mathrm{d}\mathbf{r}\right)^p \le \varepsilon \int_B \psi_\infty \left|\nabla_{\mathbf{r}}\left(\frac{\psi}{\psi_\infty}\right)^{p/2}\right|^2 \mathrm{d}\mathbf{r} + C_\varepsilon \int_B \frac{|\psi|^p}{\psi_\infty^{(p-1)}}\mathrm{d}\mathbf{r}. \tag{3.15}
$$

*Moreover, if $(p-1)k > 1$ we can take in (3.15) $\varepsilon = 0$.*

*Remark* 3.2.1. Inequality (3.15) can be rewritten in the following way

$$|\mathbf{T}_e|^p \leq \varepsilon \|D_p(\psi)\|_{\mathcal{L}^2}^2 + C_\varepsilon \|\psi\|_{\mathcal{L}^p}^p . \tag{3.16}$$

*Proof.* The proof is a consequence of the following 1D inequality (we denote $x = 1 - |\mathbf{r}|$)

$$\left(\int_0^1 \frac{|\psi|}{x} \mathrm{d}x\right)^p \leq \varepsilon \int_0^1 x^k \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi}{x^k}\right)^{p/2}\right|^2 \mathrm{d}x + C_\varepsilon \int_0^1 \frac{|\psi|^p}{x^{k(p-1)}} \mathrm{d}x. \tag{3.17}$$

To prove (3.17) we distinguish two cases

*Case* $k(p-1) > 1$*:* In this case we can take $\varepsilon = 0$ and the inequality (3.17) is a consequence of the Hölder inequality as $k - \frac{p}{p-1} > -1$.

*Case* $k(p-1) \leq 1$*:* Recall that $p > 2$ and therefore $k < 1$. We use similar method as in [30]. We make the following change of variables $y = x^{1-k}$, hence $\mathrm{d}y = (1-k)x^{-k}\mathrm{d}x$. We also denote $g(y) = \left(\frac{\psi}{x^k}\right)^{p/2}$. We get

$$\int_0^1 x^k \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi}{x^k}\right)^{p/2}\right|^2 \mathrm{d}x = (1-k)\int_0^1 \left(\frac{\mathrm{d}g(y)}{\mathrm{d}y}\right)^2 \mathrm{d}y, \tag{3.18}$$

$$\int_0^1 \frac{|\psi|^p}{x^{k(p-1)}} \mathrm{d}x = \frac{1}{1-k}\int_0^1 g^2(y) y^{2\alpha} \mathrm{d}y, \tag{3.19}$$

$$\int_0^1 \frac{|\psi|}{x} \mathrm{d}x = \frac{1}{1-k}\int_0^1 y^\alpha \frac{g^{2/p}(y)}{y} \mathrm{d}y, \tag{3.20}$$

where $\alpha = \frac{k}{1-k}$. Therefore it is enough to prove that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left(\int_0^1 y^\alpha \frac{g^{2/p}(y)}{y} \mathrm{d}y\right)^p \leq \varepsilon \int_0^1 \left(\frac{\mathrm{d}g(y)}{\mathrm{d}y}\right)^2 \mathrm{d}y + C_\varepsilon \int_0^1 g^2(y) y^{2\alpha} \mathrm{d}y. \tag{3.21}$$

We prove this by contradiction. Assume that for some $\varepsilon > 0$ (3.21) does not hold. Then there exists a sequence $\{g_n\}$ such that

- $\int_0^1 y^\alpha \frac{g_n^{2/p}(y)}{y} \mathrm{d}y = 1$,
- $\int_0^1 g_n^2(y) y^{2\alpha} \mathrm{d}y \to 0$,
- $\int_0^1 g_n'(y)^2 \mathrm{d}y$ is bounded.

Extracting a subsequence we deduce that there is $g$ such that $g_n \to g$ in $W^{1,2}(0,1)$. Moreover, from the second relation we see that $y^\alpha g_n \to 0$ in $L^2(0,1)$ and therefore $g = 0$. By compact embedding we have also $g_n \to 0$ in $L^\infty(0,1)$ and therefore also $g_n^{2/p} \to 0$ in $L^\infty(0,1)$. Since $y^{\alpha-1} \in L^1(0,1)$, we deduce that $\int_0^1 y^\alpha \frac{g_n^{2/p}(y)}{y} dy \to 0$ which is a contradiction. $\square$

## 3.3 Main result and proof

We formulate the main result here. Note that in comparison to the result proved in [30] we need considerably less regular initial conditions.

**Theorem 3.3.1.** *Let $p \in (N, \frac{2N}{N-2})$ $(N = 2,3)$, $\mathbf{v}_0 \in W^{1,p}(\Omega)$ and $\psi_0 \in W^{1,p}(\Omega, \mathcal{L}^p)$. Then there exists a time $T > 0$ and a solution $(\mathbf{v}, \psi)$ to system (3.7)–(3.9) such that $\mathbf{v} \in L^\infty(0,T,W^{1,2}) \cap L^2(0,T,W^{2,p})$ and $\psi \in L^\infty(0,T,W^{1,p}(\mathcal{L}^p))$ and $D_p(\psi) \in L^2(0,T,W^{1,2}(\mathcal{L}^2))$.*

*Proof.* The method of the proof is the same as in the previous chapter. We again use Theorem 6.3.2 with the following setting of spaces

$$\begin{aligned} X &= \left\{\psi, \psi \in L^\infty(0,T,W^{1,p}(\Omega,\mathcal{L}^p)), D_p\psi \in L^2(0,T,W^{1,2}(\Omega,\mathcal{L}^2))\right\}, \\ Y &= \left\{\psi, \psi \in L^\infty(0,T,L^p(\Omega,\mathcal{L}^p)), D_p\psi \in L^2(0,T,L^2(\Omega,\mathcal{L}^2))\right\}, \end{aligned} \tag{3.22}$$

and we define the mapping $\mathcal{T} = \mathcal{T}_1 \circ \mathcal{T}_2$, $\psi = \mathcal{T}(\overline{\psi})$, $\mathcal{T}_2(\overline{\psi}) = \mathbf{v}$, where $\mathbf{v}$ is the solution to the problem

$$\begin{aligned} &\left.\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p = \operatorname{div}\overline{\mathbf{T}} \\ \operatorname{div}\mathbf{v} = 0 \end{aligned}\right\} \text{in } (0,T)\times\Omega, \\ &\mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \qquad \text{in } \Omega, \end{aligned} \tag{3.23}$$

$$\overline{\mathbf{T}} = \int_B (\mathbf{r}\otimes\nabla_{\mathbf{r}}U)\overline{\psi}\,\mathrm{d}\mathbf{r}, \tag{3.24}$$

and $\mathbf{v} = 0$ on $\partial\Omega \times (0,T)$ in case of the homogeneous Dirichlet boundary conditions, while $\mathbf{v}$ space periodic for the case of periodic boundary conditions (then also all function spaces are in the space variables replaced by their periodic versions, for example $W^{1,p}(\Omega,\mathcal{L}^p)$ is replaced by $W^{1,p}_{per}(\Omega,\mathcal{L}^p)$). Further, $\mathcal{T}_1(\mathbf{v}) = \psi$, which is a solution to the problem

$$\frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}}\psi - L(\psi) = -\operatorname{div}_{\mathbf{r}}(-\nabla_{\mathbf{x}}\mathbf{v}\cdot\mathbf{r}\psi) \tag{3.25}$$

with the corresponding boundary condition (3.3).

In Chapter 2 we presented detailed proof in the case $N = 2$ and in 3D case we just mentioned the differences. Here on the contrary we present detailed proof in the 3D case and the differences in the 2D case are mentioned at the end.

a) Case $N = 3$

We need to prove that $\mathcal{T}$ maps sufficiently large balls in $X$ into themselves and that $\mathcal{T}$ is a contraction on $Y$. Let us assume that $\overline{\psi} \in X$,

$$\left\|\overline{\psi}\right\|_{L^\infty(W^{1,p}(\mathcal{L}^p))} + \left\|D_p\overline{\psi}\right\|_{L^2(W^{1,2}(\mathcal{L}^2))} < R \tag{3.26}$$

for sufficiently large $R$ which will be specified later.

We begin with the transport equation (3.25). Existence of a solution and its uniqueness for $\mathbf{v} \in L^\infty(0,T,W^{1,2}) \cap L^2(0,T,W^{2,p})$ is due to Lemma 3.2.3. Thus we concentrate ourselves on the a priori estimates. We multiply (3.25) by $\frac{\psi}{\psi_\infty}\left|\frac{\psi}{\psi_\infty}\right|^{p-2}$ and integrate $\int_B \mathrm{d}\mathbf{r}$. Using integration by parts together with Hölder and Young inequalities on the right-hand side, we get

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\psi\right\|_{\mathcal{L}^p}^p + \frac{1}{p}\mathbf{v}\cdot\nabla\left\|\psi\right\|_{\mathcal{L}^p}^p + C_1(p)\left\|D_p\psi\right\|_{\mathcal{L}^2}^2 \le C_2(p)\left|\nabla\mathbf{v}\right|^2\left\|\psi\right\|_{\mathcal{L}^p}^p. \tag{3.27}$$

Integrating $\int_\Omega \mathrm{d}\mathbf{x}$ we get[3]

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\psi\right\|_{L^p(\mathcal{L}^p)}^p + C_1(p)\left\|D_p\psi\right\|_{L^2(\mathcal{L}^2)}^2 \le C_2(p)\left\|\nabla\mathbf{v}\right\|_{L^\infty}^2\left\|\psi\right\|_{L^p(\mathcal{L}^p)}^p. \tag{3.28}$$

Next, we omit the second term on the left hand side (which is positive) and using Gronwall inequality we get after integrating $\int_0^T \mathrm{d}t$

$$\left\|\psi\right\|_{L^\infty(L^p(\mathcal{L}^p))}^p \le \left\|\psi_0\right\|_{L^p(\mathcal{L}^p)}^p \exp\left(C_3(p)\int_0^T\left\|\nabla\mathbf{v}\right\|_{L^\infty}^2\mathrm{d}t\right). \tag{3.29}$$

Finally, we return to (3.28), integrate again $\int_0^T \mathrm{d}t$ and use (3.29) to get

$$\begin{aligned}&\left\|D_p\psi\right\|_{L^2(L^2(\mathcal{L}^2))}^2 \le \\ &\le \left\|\psi_0\right\|_{L^p(\mathcal{L}^p)}^p \exp\left(C_3(p)\int_0^T\left\|\nabla\mathbf{v}\right\|_{L^\infty}^2\mathrm{d}t\right)\left(1 + C_2(p)\int_0^T\left\|\nabla\mathbf{v}\right\|_{L^\infty}^2\mathrm{d}t\right).\end{aligned} \tag{3.30}$$

[3]Second term on the left hand side is zero as $\operatorname{div}\mathbf{v} = 0$

Moreover, multiplying (3.27) by a suitable power of $\|\psi\|_{\mathcal{L}^p}$ we are able to get by a similar procedure

$$\|\psi\|_{L^\infty(L^q(\mathcal{L}^p))} \le \|\psi_0\|_{L^q(\mathcal{L}^p)} \exp\left(C_2(p)\int_0^T \|\nabla \mathbf{v}\|_{L^\infty}^2 \,\mathrm{d}t\right) \tag{3.31}$$

for any $q < \infty$ and by limiting procedure $q \to \infty$ we get the same estimate also for $q = \infty$.

Let us next consider the spatial gradient of equation (3.25), that is

$$\frac{\partial}{\partial t}\nabla_{\mathbf{x}}\psi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\nabla_{\mathbf{x}}\psi - L(\nabla_{\mathbf{x}}\psi) = -\nabla_{\mathbf{x}}\mathbf{v}\nabla_{\mathbf{x}}\psi - \operatorname{div}_{\mathbf{r}}(\nabla_{\mathbf{x}}^2\mathbf{v}\mathbf{r}\psi + \nabla_{\mathbf{x}}\mathbf{v}\mathbf{r}\nabla_{\mathbf{x}}\psi). \tag{3.32}$$

We apply the same procedure as before to get $L^p$ estimates. Namely, we multiply equation (3.32) by $\frac{\nabla_{\mathbf{x}}\psi}{\psi_\infty}\left|\frac{\nabla_{\mathbf{x}}\psi}{\psi_\infty}\right|^{p-2}$, integrate $\int_B \mathrm{d}\mathbf{r}$ and estimate the terms on the right hand side using integration by parts, Hölder and Young inequalities to get

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_{\mathbf{x}}\psi\|_{\mathcal{L}^p}^p + \frac{1}{p}\mathbf{v}\cdot\nabla\|\nabla_{\mathbf{x}}\psi\|_{\mathcal{L}^p}^p + C_4(p)\|D_p\nabla_{\mathbf{x}}\psi\|_{\mathcal{L}^2}^2 \le \\ \le |\nabla\mathbf{v}|\,\|\nabla_{\mathbf{x}}\psi\|_{\mathcal{L}^p}^p + C_5(p)\left(\left|\nabla^2\mathbf{v}\right|^2\|\nabla_{\mathbf{x}}\psi\|_{\mathcal{L}^p}^{p-2}\|\psi\|_{\mathcal{L}^p}^2 + |\nabla\mathbf{v}|^2\|\nabla_{\mathbf{x}}\psi\|_{\mathcal{L}^p}^p\right).\end{aligned} \tag{3.33}$$

Next, we integrate $\int_\Omega \mathrm{d}\mathbf{x}$, use again Hölder inequality and get

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_{\mathbf{x}}\psi\|_{L^p(\mathcal{L}^p)}^p + C_4(p)\|D_p\nabla_{\mathbf{x}}\psi\|_{L^2(\mathcal{L}^2)}^2 \le \|\nabla\mathbf{v}\|_{L^\infty}\|\nabla_{\mathbf{x}}\psi\|_{L^p(\mathcal{L}^p)}^p + \\ + C_5(p)\left(\left\|\nabla^2\mathbf{v}\right\|_{L^p}^2\|\nabla_{\mathbf{x}}\psi\|_{L^p(\mathcal{L}^p)}^{p-2}\|\psi\|_{L^\infty(\mathcal{L}^p)}^2 + \|\nabla\mathbf{v}\|_{L^\infty}^2\|\nabla_{\mathbf{x}}\psi\|_{L^p(\mathcal{L}^p)}^p\right).\end{aligned} \tag{3.34}$$

Finally, we proceed the same way as before when integrating in time. Namely, we first omit the second term on the left hand side and dividing the inequality by $\|\nabla_{\mathbf{x}}\psi\|_{L^p(\mathcal{L}^p)}^{p-2}$ and integrating in time we get

$$\begin{aligned}&\|\nabla_{\mathbf{x}}\psi\|_{L^\infty(L^p(\mathcal{L}^p))}^2 \le \\ &\le \left(\|\nabla_{\mathbf{x}}\psi_0\|_{L^p(\mathcal{L}^p)}^2 + C(p)\|\psi\|_{L^\infty(L^\infty(\mathcal{L}^p))}^2\int_0^T\left\|\nabla^2\mathbf{v}\right\|_{L^p}^2\mathrm{d}t\right)\times \\ &\times\exp\left(\int_0^T C(p)\|\nabla\mathbf{v}\|_{L^\infty} + C(p)\|\nabla\mathbf{v}\|_{L^\infty}^2\,\mathrm{d}t\right).\end{aligned} \tag{3.35}$$

Note that we can raise this inequality to suitable powers ($\frac{p}{2}$ and $\frac{p-2}{2}$) and get similar estimate for the $p$-th and $(p-2)$-th powers of the norm on the left hand side. Then we get back to (3.34), move the first term to the right hand side, integrate in time, estimate all terms and use (3.35) to end up with

$$\begin{aligned} &\|\nabla_{\mathbf{x}}\psi\|^p_{L^\infty(L^p(\mathcal{L}^p))} + C(p)\,\|D_p\nabla_{\mathbf{x}}\psi\|^2_{L^2(L^2(\mathcal{L}^2))} \leq \\ &\qquad \leq \left(\|\nabla_{\mathbf{x}}\psi_0\|^p_{L^p(\mathcal{L}^p)} + C(p)\,\|\psi\|^p_{L^\infty(L^\infty(\mathcal{L}^p))}\left\|\nabla^2\mathbf{v}\right\|^{p/2}_{L^2(L^p)}\right)\times \\ &\qquad\qquad \times\left(1 + \|\nabla\mathbf{v}\|_{L^1(L^\infty)} + \|\nabla\mathbf{v}\|_{L^2(L^\infty)}\right)\times \\ &\qquad\qquad \times \exp\left(C(p)\,\|\nabla\mathbf{v}\|_{L^1(L^\infty)} + C(p)\,\|\nabla\mathbf{v}\|_{L^2(L^\infty)}\right). \end{aligned} \tag{3.36}$$

Moreover, we can use (3.31) to estimate the $L^\infty(L^\infty(\mathcal{L}^p))$ norm of $\psi$. Together with the embedding inequality $\|\nabla\mathbf{v}\|_{L^\infty} \leq C\,\|\nabla\mathbf{v}\|_{W^{1,p}}$ (recall $p > N$) we get from (3.30) and (3.36)

$$\begin{aligned} &\|\psi\|_{L^\infty(0,T,W^{1,p}(\Omega,\mathcal{L}^p))} + \|D_p\psi\|_{L^2(0,T,W^{1,2}(\Omega,\mathcal{L}^2))} \leq \\ &\qquad\qquad \leq F(\|\psi_0\|_{W^{1,p}(\Omega,\mathcal{L}^p)}, \|\nabla\mathbf{v}\|_{L^2(0,T,W^{1,p}(\Omega))}) \end{aligned} \tag{3.37}$$

for certain function $F$ which can be specified if necessary.

Now, we need to estimate first and second gradient of the velocity in $L^2(L^p)$. We start with the second gradient which is more difficult. For this purpose, we use equation (3.23) and Theorem 6.2.1. We have

$$\left\|\nabla^2\mathbf{v}\right\|_{L^2(L^p)} \leq C\left(\|\mathbf{v}_0\|_{W^{1,p}} + \left\|\operatorname{div}\overline{\mathbf{T}_e}\right\|_{L^2(L^p)} + \|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^2(L^p)}\right). \tag{3.38}$$

Using Lemma 3.2.4 (for $\nabla_{\mathbf{x}}\psi$ instead of $\psi$) and Hölder inequality in time we get

$$\begin{aligned} &\left\|\operatorname{div}\overline{\mathbf{T}_e}\right\|_{L^2(L^p)} \leq \varepsilon^{1/p}\left\|D_p\nabla_{\mathbf{x}}\overline{\psi}\right\|^{2/p}_{L^{4/p}(L^2(\mathcal{L}^2))} + C_\varepsilon\left\|\nabla_{\mathbf{x}}\overline{\psi}\right\|_{L^2(L^p(\mathcal{L}^p))} \leq \\ &\qquad \leq \varepsilon^{1/p}T^{\frac{p-2}{2p}}\left\|D_p\nabla_{\mathbf{x}}\overline{\psi}\right\|^{2/p}_{L^2(L^2(\mathcal{L}^2))} + C_\varepsilon T^{1/2}\left\|\nabla_{\mathbf{x}}\overline{\psi}\right\|_{L^\infty(L^p(\mathcal{L}^p))}. \end{aligned} \tag{3.39}$$

It holds for $p > 3$ using Sobolev embedding and interpolation inequality,

$$\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^p} \leq \|\nabla\mathbf{v}\|_{L^p}\|\mathbf{v}\|_{L^\infty} \leq C\,\|\nabla\mathbf{v}\|^2_{L^p} \leq C\,\|\nabla\mathbf{v}\|^{\frac{6(p-2)}{5p-6}}_{W^{1,p}}\|\nabla\mathbf{v}\|^{\frac{4p}{5p-6}}_{L^2}, \tag{3.40}$$

and consequently for $p<6$ using Hölder and Young inequalities in time,

$$\begin{aligned}
\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^2(0,T,L^p)} &\leq CT^{\frac{6-p}{2(5p-6)}}\|\nabla\mathbf{v}\|_{L^2(0,T,W^{1,p})}^{\frac{6(p-2)}{5p-6}}\|\mathbf{v}\|_{L^\infty(0,T,W^{1,2})}^{\frac{4p}{5p-6}} \leq \\
&\leq c\left\|\nabla^2\mathbf{v}\right\|_{L^2(0,T,L^p)} + CT^{\frac{1}{2}}\|\mathbf{v}\|_{L^\infty(0,T,W^{1,2})}^{\frac{4p}{6-p}} + \\
&+ CT^{\frac{6-p}{2(5p-6)}}\|\mathbf{v}\|_{L^\infty(0,T,W^{1,2})}^{\frac{4p}{5p-6}}\|\nabla\mathbf{v}\|_{L^2(0,T,L^p)}^{\frac{6(p-2)}{5p-6}} \leq \\
&\leq c\left\|\nabla^2\mathbf{v}\right\|_{L^2(0,T,L^p)} + CT^{\frac{1}{2}}\|\mathbf{v}\|_{L^\infty(0,T,W^{1,2})}^{\frac{4p}{6-p}} + \\
+ CT^{\frac{6-p}{2(5p-6)}}&\|\mathbf{v}\|_{L^\infty(0,T,W^{1,2})}^{\frac{4p}{5p-6}}\|\nabla\mathbf{v}\|_{L^2(0,T,L^2)}^{\frac{3(6-p)(p-2)}{p(5p-6)}}\|\nabla\mathbf{v}\|_{L^2(0,T,W^{1,2})}^{\frac{9(p-2)^2}{p(5p-6)}} \quad (3.41)
\end{aligned}$$

with constant $c$ as small as we need. Together with (3.38) we get

$$\begin{aligned}
&\left\|\nabla^2\mathbf{v}\right\|_{L^2(L^p)} \leq \\
&\leq C\Bigg(\|\mathbf{v}_0\|_{W^{1,p}} + \left\|\operatorname{div}\overline{\mathbf{T}_e}\right\|_{L^2(L^p)} + c\left\|\nabla^2\mathbf{v}\right\|_{L^2(L^p)} + T^{\frac{1}{2}}\|\mathbf{v}\|_{L^\infty(0,T,W^{1,2})}^{\frac{4p}{6-p}} + \\
&\quad +T^{\frac{6-p}{2(5p-6)}}\|\mathbf{v}\|_{L^\infty(0,T,W^{1,2})}^{\frac{4p}{5p-6}}\|\nabla\mathbf{v}\|_{L^2(0,T,L^2)}^{\frac{3(6-p)(p-2)}{p(5p-6)}}\|\nabla\mathbf{v}\|_{L^2(0,T,W^{1,2})}^{\frac{9(p-2)^2}{p(5p-6)}}\Bigg). \quad (3.42)
\end{aligned}$$

We choose $c$ small enough to absorb this term to the left hand side. Similarly as in previous chapter, it remains to estimate $\mathbf{v}$ in $L^\infty(0,T,W^{1,2})\cap L^2(0,T,W^{2,2})$. We apply the energy method to estimate these two norms. First, multiplying (3.23) by $\mathbf{v}$ yields after standard operations

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_{L^2}^2 + \mu\|\nabla\mathbf{v}\|_{L^2}^2 \leq \int_\Omega\left|\overline{\mathbf{T}_e}\right||\nabla\mathbf{v}|\,\mathrm{d}\mathbf{x}. \quad (3.43)$$

We estimate the right-hand side in the following way

$$\begin{aligned}
\int_\Omega\left|\overline{\mathbf{T}_e}\right||\nabla\mathbf{v}|\,\mathrm{d}\mathbf{x} &\leq \frac{\mu}{2}\|\nabla\mathbf{v}\|_2^2 + C(\mu)\left\|\overline{\mathbf{T}_e}\right\|_p^2 \leq \\
&\leq \frac{\mu}{2}\|\nabla\mathbf{v}\|_2^2 + \varepsilon^{2/p}\left\|D_p\overline{\psi}\right\|_{L^2(\mathcal{L}^2)}^{4/p} + C(\varepsilon)\left\|\overline{\psi}\right\|_{L^p(\mathcal{L}^p)}^2, \quad (3.44)
\end{aligned}$$

which leads to

$$\|\mathbf{v}\|_{L^\infty(0,T,L^2)} + \mu\|\nabla\mathbf{v}\|_{L^2(0,T,L^2)} \leq C\left(\varepsilon^{\frac{1}{p}}T^{\frac{p-2}{2p}}R^{\frac{2}{p}} + T^{\frac{1}{2}}R + \|\mathbf{v}_0\|_2\right) \quad (3.45)$$

due to Lemmas 3.2.4 and 3.2.1 and assumption (3.26).

Finally, we test the same equation by $-P\Delta\mathbf{v}$, where $P$ is the Leray projection (thus $P\Delta\mathbf{v} = \Delta\mathbf{v}$ in the case of periodic boundary conditions):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_{L^2}^2 + \mu\left\|\nabla^2\mathbf{v}\right\|_{L^2}^2 \leq \int_\Omega \left|\operatorname{div}(\overline{\mathbf{T}_e})P\Delta\mathbf{v}\right|\mathrm{d}\mathbf{x} + \int_\Omega |(\mathbf{v}\cdot\nabla\mathbf{v})\cdot P\Delta\mathbf{v}|\,\mathrm{d}\mathbf{x}. \tag{3.46}$$

Note that we used that $\|\nabla^2\mathbf{v}\|_{L^2} \leq \|P\Delta\mathbf{v}\|_{L^2}$. Using standard tools as the Hölder inequality and the Sobolev embedding theorem together with Lemma 3.2.4, we end up with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_{L^2}^2 + \mu\left\|\nabla^2\mathbf{v}\right\|_{L^2}^2 &\leq \\ &\leq C(\mu)\left(\left\|\nabla_{\mathbf{x}}\overline{\psi}\right\|_{L^p(\mathcal{L}^p)}^2 + \varepsilon^{2/p}\left\|D_p\nabla_{\mathbf{x}}\overline{\psi}\right\|_{L^2(\mathcal{L}^2)}^{4/p} + \|\nabla\mathbf{v}\|_{L^2}^6\right),\end{aligned} \tag{3.47}$$

from where we deduce that there is

$$\begin{aligned}T^* &= T^*(\|\nabla_{\mathbf{x}}\overline{\psi}\|_{L^\infty(L^p(\mathcal{L}^p))}, \left\|D_p\nabla_{\mathbf{x}}\overline{\psi}\right\|_{L^2(L^2(\mathcal{L}^2))}, \mathbf{v}_0) = T^*(R,\mathbf{v}_0) > 0 \text{ and}\\ C^* &= C^*(T^*, \|\nabla_{\mathbf{x}}\overline{\psi}\|_{L^\infty(L^p(\mathcal{L}^p))}, \left\|D_p\nabla_{\mathbf{x}}\overline{\psi}\right\|_{L^2(L^2(\mathcal{L}^2))}, \mathbf{v}_0) = C^*(T^*, R, \mathbf{v}_0),\end{aligned} \tag{3.48}$$

such that there is a strong solution to (3.23) in 3D on $(0,T^*)$ with

$$\|\nabla\mathbf{v}\|_{L^\infty(0,T^*,L^2)} + \left\|\nabla^2\mathbf{v}\right\|_{L^2(0,T^*,L^2)} \leq C^*. \tag{3.49}$$

Plugging in together (3.38), (3.39), (3.42), (3.45) and (3.49), we get

$$\left\|\nabla^2\mathbf{v}\right\|_{L^2(L^p)} \leq G(R) \tag{3.50}$$

for certain function $G$ which depends on the choice of $T^*$ and $\varepsilon$.

We also need an estimate on the first gradient of $\mathbf{v}$. This is done in a similar way using equation (3.23) and Lemma 6.2.2. We will do this procedure later on with a similar equation and therefore we skip this part here. However, we get a similar estimate as for the second gradient:

$$\|\nabla\mathbf{v}\|_{L^2(L^p)} \leq H(R) \tag{3.51}$$

for some function $H$. Now we combine (3.37), (3.50) and (3.51) and end up with

$$\begin{aligned}\|\psi\|_{L^\infty(0,T,W^{1,2}(\Omega,\mathcal{L}^p))} + \|D_p\psi\|_{L^2(0,T,W^{1,2}(\Omega,\mathcal{L}^2))} &\leq \\ &\leq F(\|\psi_0\|_{W^{1,p}(\Omega,\mathcal{L}^p)}, G(R)+H(R)).\end{aligned} \tag{3.52}$$

Now it is enough to choose $T \leq T^*$ and $\varepsilon$ sufficiently small and $R$ sufficiently large with respect to $\|\mathbf{v}_0\|_{W^{1,2}}$ and $\|\psi_0\|_{W^{1,p}(\mathcal{L}^p)}$ and all constants in the formulas such that $F(\|\psi_0\|_{W^{1,p}(\Omega,\mathcal{L}^p)}, G(R) + H(R)) < R$ and the first part of the proof is finished.

We have proved that, taking $T$ small enough, $\mathcal{T}$ maps sufficiently large balls in $X$ into themselves. Next, we have to prove that $\mathcal{T}$ is in fact a contraction on $Y$. For that purpose, let us denote $\mathbf{v}^i = \mathcal{T}_2(\overline{\psi^i})$, $i = 1, 2$ and $\psi^i = \mathcal{T}_1(\mathbf{v}^i)$, thus $\psi^i = \mathcal{T}(\overline{\psi^i})$. Moreover, we denote $\psi^{12} = \psi^1 - \psi^2$, $\mathbf{v}^{12} = \mathbf{v}^1 - \mathbf{v}^2$ and $\overline{\psi^{12}} = \overline{\psi^1} - \overline{\psi^2}$. We have

$$\frac{\partial \psi^{12}}{\partial t} + \mathbf{v}^2 \cdot \nabla_{\mathbf{x}} \psi^{12} - L(\psi^{12}) = -\mathbf{v}^{12} \cdot \nabla_{\mathbf{x}} \psi^1 - \operatorname{div}_{\mathbf{r}} (\nabla_{\mathbf{x}} \mathbf{v}^1 \cdot \mathbf{r} \psi^{12} + \nabla_{\mathbf{x}} \mathbf{v}^{12} \cdot \mathbf{r} \psi^2), \tag{3.53}$$

with initial condition

$$\psi^{12}(0, \mathbf{x}) = 0 \qquad \text{in } \Omega. \tag{3.54}$$

We proceed in a similar way as in the first part of the proof. Namely, we multiply equation (3.53) by $\frac{\psi^{12}}{\psi_\infty} \left| \frac{\psi^{12}}{\psi_\infty} \right|^{p-2}$ and integrate $\int_B \mathrm{d}\mathbf{r}$, we apply similar procedures as before and get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left\|\psi^{12}\right\|_{\mathcal{L}^p}^p + \mathbf{v}^2 \cdot \nabla \left\|\psi^{12}\right\|_{\mathcal{L}^p}^p + C \left\|D_p \psi^{12}\right\|_{\mathcal{L}^2}^2 \leq C \left|\mathbf{v}^{12}\right| \left\|\nabla_{\mathbf{x}} \psi^1\right\|_{\mathcal{L}^p} \left\|\psi^{12}\right\|_{\mathcal{L}^p}^{p-1} + \\ + C \left|\nabla \mathbf{v}^1\right|^2 \left\|\psi^{12}\right\|_{\mathcal{L}^p}^p + C \left|\nabla \mathbf{v}^{12}\right|^2 \left\|\psi^{12}\right\|_{\mathcal{L}^p}^{p-2} \left\|\psi^2\right\|_{\mathcal{L}^p}^2 . \end{aligned} \tag{3.55}$$

Integrating $\int_\Omega \mathrm{d}\mathbf{x}$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left\|\psi^{12}\right\|_{L^p(\mathcal{L}^p)}^p + C \left\|D_p \psi^{12}\right\|_{L^2(\mathcal{L}^2)}^2 \leq \\ &\quad \leq C \left\|\mathbf{v}^{12}\right\|_\infty \left\|\nabla_{\mathbf{x}} \psi^1\right\|_{L^p(\mathcal{L}^p)} \left\|\psi^{12}\right\|_{L^p(\mathcal{L}^p)}^{p-2} \Big( \left\|\psi^1\right\|_{L^p(\mathcal{L}^p)} + \left\|\psi^2\right\|_{L^p(\mathcal{L}^p)} \Big) + \\ &\quad\quad + C \left\|\nabla \mathbf{v}^1\right\|_\infty^2 \left\|\psi^{12}\right\|_{L^p(\mathcal{L}^p)}^p + C \left\|\nabla \mathbf{v}^{12}\right\|_p^2 \left\|\psi^{12}\right\|_{L^p(\mathcal{L}^p)}^{p-2} \left\|\psi^2\right\|_{L^\infty(\mathcal{L}^p)}^2 \end{aligned} \tag{3.56}$$

and finally, when integrating in time, we again first omit the second term on the left hand side to get

$$\begin{aligned} &\left\|\psi^{12}\right\|_{L^\infty(L^p(\mathcal{L}^p))}^2 \leq \\ &\leq C \Big( \left\|\mathbf{v}^{12}\right\|_{L^1(L^\infty)} \left\|\nabla_{\mathbf{x}} \psi^1\right\|_{L^\infty(L^p(\mathcal{L}^p))} \Big( \left\|\psi^1\right\|_{L^\infty(L^p(\mathcal{L}^p))} + \left\|\psi^2\right\|_{L^\infty(L^p(\mathcal{L}^p))} \Big) + \\ &\quad\quad + \left\|\nabla \mathbf{v}^{12}\right\|_{L^2(L^p)}^2 \left\|\psi^2\right\|_{L^\infty(L^\infty(\mathcal{L}^p))}^2 \Big) \cdot \exp\Big( C \left\|\nabla \mathbf{v}^1\right\|_{L^2(L^\infty)}^2 \Big) . \end{aligned} \tag{3.57}$$

We go back to (3.56), move the first term to the right hand side, integrate in time and estimate the right hand side using (3.57) to get

$$\begin{aligned}
&\left\|\psi^{12}\right\|^p_{L^\infty(L^p(\mathcal{L}^p))} + C\left\|D_p\psi^{12}\right\|^2_{L^2(L^2(\mathcal{L}^2))} \leq \\
&\leq C\left(\left\|\mathbf{v}^{12}\right\|_{L^1(L^\infty)}\left\|\nabla_{\mathbf{x}}\psi^1\right\|_{L^\infty(L^p(\mathcal{L}^p))}\left(\left\|\psi^1\right\|_{L^\infty(L^p(\mathcal{L}^p))} + \left\|\psi^2\right\|_{L^\infty(L^p(\mathcal{L}^p))}\right) + \right.\\
&\qquad\left. + \left\|\nabla\mathbf{v}^{12}\right\|^2_{L^2(L^p)}\left\|\psi^2\right\|^2_{L^\infty(L^\infty(\mathcal{L}^p))}\right)^{\frac{p}{2}}\left(1 + \left\|\nabla\mathbf{v}^1\right\|_{L^2(L^\infty)}\right)\times \\
&\qquad\qquad \times\exp\left(C\left\|\nabla\mathbf{v}^1\right\|^2_{L^2(L^\infty)}\right). \qquad (3.58)
\end{aligned}$$

We see that we need to estimate $\nabla\mathbf{v}^{12}$ in $L^2(0,T,L^p(\Omega))$, boundedness of other terms is a consequence of the first part of the proof. We have (denoting by $p^{12}$ the difference of corresponding pressures)

$$\begin{aligned}
\frac{\partial\mathbf{v}^{12}}{\partial t} - \mu\Delta\mathbf{v}^{12} + \nabla p^{12} = \operatorname{div}(\overline{\mathbf{T}_e}^{12} - \mathbf{v}^1\otimes\mathbf{v}^{12} - \mathbf{v}^{12}\otimes\mathbf{v}^2),&\\
\operatorname{div}\mathbf{v}^{12} = 0,&\\
\mathbf{v}^{12}(0,\mathbf{x}) = \mathbf{0} \qquad \text{in } \Omega\,,&
\end{aligned} \qquad (3.59)$$

together with corresponding boundary conditions (Dirichlet or periodic ones). We apply Theorem 6.2.2 and get

$$\begin{aligned}
&\left\|\nabla\mathbf{v}^{12}\right\|_{L^2(L^p)} \leq T^{\frac{p-2}{2p}}\left\|\nabla\mathbf{v}^{12}\right\|_{L^p(L^p)} \leq \\
&\qquad\leq CT^{\frac{p-2}{2p}}\left(\left\|\overline{\mathbf{T}^{12}_e}\right\|_{L^p(L^p)} + \sum_{i=1}^2\left\|\mathbf{v}^i\mathbf{v}^{12}\right\|_{L^p(L^p)}\right). \qquad (3.60)
\end{aligned}$$

Using Lemma 3.2.4 we are able to estimate the first term

$$\left\|\overline{\mathbf{T}^{12}}\right\|_{L^p(L^p)} \leq \varepsilon^{1/p}\left\|D_p\overline{\psi^{12}}\right\|^{2/p}_{L^2(L^2(\mathcal{L}^2))} + C_\varepsilon T^{1/p}\left\|\overline{\psi^{12}}\right\|_{L^\infty(L^p(\mathcal{L}^p))}, \qquad (3.61)$$

while the second term is estimated in the following way

$$\begin{aligned}
&\sum_{i=1}^2\left\|\mathbf{v}^i\mathbf{v}^{12}\right\|_{L^p(L^p)} \leq C\left(\int_0^T\sum_{i=1}^2\left\|\mathbf{v}^i\right\|^p_{L^6}\left\|\nabla\mathbf{v}^{12}\right\|^{p\frac{4p-6}{5p-6}}_{L^p}\left\|\mathbf{v}^{12}\right\|^{\frac{p^2}{5p-6}}_{L^2}\,\mathrm{d}t\right)^{1/p} \leq \\
&\leq CT^{\frac{1}{5p-6}}\sum_{i=1}^2\left\|\mathbf{v}^i\right\|_{L^\infty(0,T,W^{1,2})}\left(\left\|\nabla\mathbf{v}^{12}\right\|_{L^p(0,T,L^p)} + \left\|\mathbf{v}^{12}\right\|_{L^\infty(0,T,L^2)}\right). \qquad (3.62)
\end{aligned}$$

Taking $T$ sufficiently small, the first term can be absorbed into the left hand-side of (3.60), while for the second term we use the energy method. We multiply (3.59) by $\mathbf{v}^{12}$ and integrate over $\Omega$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}^{12}\right\|_{L^2}^2+\mu\left\|\nabla\mathbf{v}^{12}\right\|_{L^2}^2\leq\int_\Omega\left|\mathbf{v}^{12}\right|^2\left|\nabla\mathbf{v}^2\right|\mathrm{d}\mathbf{x}+\int_\Omega\left|\overline{\mathbf{T}^{12}}\right|\left|\nabla\mathbf{v}^{12}\right|\mathrm{d}\mathbf{x}\leq$$
$$\leq\left\|\mathbf{v}^{12}\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\mathbf{v}^{12}\right\|_{L^2}^{\frac{3}{2}}\left\|\nabla\mathbf{v}^2\right\|_{L^2}+C\left\|\nabla\mathbf{v}^{12}\right\|_{L^2}\left\|\overline{\mathbf{T}^{12}}\right\|_{L^p}. \quad (3.63)$$

Therefore, using Lemma 3.2.4 similarly as in (3.44)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}^{12}\right\|_{L^2}^2+\mu\left\|\nabla\mathbf{v}^{12}\right\|_{L^2}^2\leq$$
$$\leq C(\mu)\left(\left\|\mathbf{v}^{12}\right\|_{L^2}^2\left\|\nabla\mathbf{v}^2\right\|_{L^2}^4+\varepsilon^{\frac{2}{p}}\left\|D_p\overline{\psi^{12}}\right\|_{L^2(\mathcal{L}^2)}^{\frac{4}{p}}+C_\varepsilon\left\|\overline{\psi^{12}}\right\|_{L^p(\mathcal{L}^p)}^2\right) \quad (3.64)$$

and consequently

$$\left\|\mathbf{v}^{12}\right\|_{L^\infty(0,T,L^2)}^2+\mu\left\|\nabla\mathbf{v}^{12}\right\|_{L^2(0,T,L^2)}^2\leq$$
$$\leq C\left(\varepsilon^{\frac{2}{p}}T^{\frac{p-2}{p}}\left\|D_p\overline{\psi^{12}}\right\|_{L^2(L^2(\mathcal{L}^2))}^{\frac{4}{p}}+T\left\|\overline{\psi^{12}}\right\|_{L^\infty(L^p(\mathcal{L}^p))}^2\right)\times$$
$$\times\exp\left(\int_0^T\left\|\nabla\mathbf{v}^2\right\|_{L^2}^4\mathrm{d}t\right). \quad (3.65)$$

Note again that there is a time $T^*$ such that $\int_0^{T^*}\|\nabla\mathbf{v}^2\|_{L^2}^4\,\mathrm{d}t<C$. Putting together (3.60), (3.62) and (3.65) we get

$$\left\|\nabla\mathbf{v}^{12}\right\|_{L^2(L^p)}\leq T^{\frac{p-2}{2p}}\left(\alpha\left\|D_p\overline{\psi^{12}}\right\|_{L^2(L^2(\mathcal{L}^2))}^{\frac{2}{p}}+\beta\left\|\overline{\psi^{12}}\right\|_{L^\infty(L^p(\mathcal{L}^p))}\right) \quad (3.66)$$

for $\alpha,\beta$ finite numbers (recall that $\|\psi^i\|_X<R$ and for $\mathbf{v}^i$ we have the estimates from the first part of the proof). Estimate (3.66) together with (3.58) finally implies

$$\left\|\psi^{12}\right\|_{L^\infty(L^p(\mathcal{L}^p))}+\left\|D_p\psi^{12}\right\|_{L^2(L^2(\mathcal{L}^2))}^{\frac{2}{p}}\leq$$
$$\leq CT^{\frac{p-2}{2p}}\left(\gamma\left\|D_p\overline{\psi^{12}}\right\|_{L^2(L^2(\mathcal{L}^2))}^{\frac{2}{p}}+\delta\left\|\overline{\psi^{12}}\right\|_{L^\infty(L^p(\mathcal{L}^p))}\right) \quad (3.67)$$

for $\gamma, \delta$ finite numbers. Taking $T$ sufficiently small we get that $\mathcal{T}$ is a contraction on $Y$. This completes the proof in the three dimensional case.

b) Case $N = 2$.

Similarly as in previous chapter, we mention here only the differences in the proof in 2D case, as the proof works basically similarly with $p > 2$. In fact some estimates can be done easier, see below. There are also a few changes connected with interpolation inequalities. In the first part of the proof we can use instead of (3.42) a different approach. It holds for $p < M$

$$\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^p} \le \|\nabla\mathbf{v}\|_{L^M}\|\mathbf{v}\|_{L^{\frac{Mp}{M-p}}} \le C\|\nabla\mathbf{v}\|_{W^{1,2}}^{\frac{M-2}{M}}\|\mathbf{v}\|_{W^{1,2}}^{\frac{M+2}{M}} \tag{3.68}$$

and thus

$$\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^2(0,T,L^p)} \le CT^{\frac{1}{M}}\|\nabla\mathbf{v}\|_{L^2(0,T,W^{1,2})}^{\frac{M-2}{M}}\|\mathbf{v}\|_{L^\infty(0,T,W^{1,2})}^{\frac{M+2}{M}}. \tag{3.69}$$

Then we proceed the same way as in 3D case. The main difference is in estimate (3.49). In the 2D case we are able to get the estimates immediately (note that we have global-in-time existence of strong solutions to the Navier–Stokes equations), instead of (3.47) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{v}\|_{L^2}^2 + \mu\left\|\nabla^2\mathbf{v}\right\|_{L^2}^2 \le\\ \le C(\mu)\left(\left\|\nabla_{\mathbf{x}}\overline{\psi}\right\|_{L^p(\mathcal{L}^p)}^2 + \varepsilon^{2/p}\left\|D_p\nabla_{\mathbf{x}}\overline{\psi}\right\|_{L^2(\mathcal{L}^2)}^{4/p} + \|\nabla\mathbf{v}\|_2^4\|\mathbf{v}\|_2^2\right).\end{aligned} \tag{3.70}$$

Thus

$$\begin{aligned}&\|\nabla\mathbf{v}\|_{L^\infty(0,T,L^2)}^2 + \mu\left\|\nabla^2\mathbf{v}\right\|_{L^2(0,T,L^2)}^2 \le\\ &\le C(\|\nabla\mathbf{v}_0\|_2^2 + TR^2 + \varepsilon^{\frac{2}{p}}T^{\frac{p-2}{p}}R^{\frac{4}{p}})\exp\left(C(\|\mathbf{v}_0\|_2^2 + TR^2 + \varepsilon^{\frac{2}{p}}T^{\frac{p-2}{p}}R^{\frac{4}{p}})\right)\end{aligned} \tag{3.71}$$

and we can follow the rest of the first part of the proof in 3D case.

In the second part (contraction of $\mathcal{T}$) we have the following differences.

Estimate (3.62) is replaced by

$$\sum_{i=1}^{2} \left\| \mathbf{v}^i \mathbf{v}^{12} \right\|_{L^p(L^p)} \leq$$
$$\leq C \sum_{i=1}^{2} \left\| \mathbf{v}^i \right\|_{L^\infty(W^{1,2})} \left( \int_0^T \left\| \nabla \mathbf{v}^{12} \right\|_{L^p}^{\frac{p^2}{6(p-1)}} \left\| \mathbf{v}^{12} \right\|_{L^2}^{\frac{p(5p-6)}{6(p-1)}} \mathrm{d}t \right)^{1/p} \leq$$
$$\leq C T^{\frac{5p-6}{6p-6}\frac{1}{p}} \sum_{i=1}^{2} \left\| \mathbf{v}^i \right\|_{L^\infty(W^{1,2})} \left( \left\| \nabla \mathbf{v}^{12} \right\|_{L^p(L^p)} + \left\| \mathbf{v}^{12} \right\|_{L^\infty(L^2)} \right) \quad (3.72)$$

and instead of (3.65) we get

$$\left\| \mathbf{v}^{12} \right\|_{L^\infty(0,T,L^2)}^2 + \mu \left\| \nabla \mathbf{v}^{12} \right\|_{L^2(0,T,L^2)}^2 \leq$$
$$\leq C \left( \varepsilon^{\frac{2}{p}} T^{\frac{p-2}{p}} \left\| D_p \overline{\psi^{12}} \right\|_{L^2(L^2(\mathcal{L}^2))}^{\frac{4}{p}} + T \left\| \overline{\psi^{12}} \right\|_{L^\infty(L^p(\mathcal{L}^p))}^2 \right) \exp(\left\| \nabla \mathbf{v}^2 \right\|_{L^2(L^2)}^2). \quad (3.73)$$

Thus, provided $T$ is sufficiently small and $R$ sufficiently large, we get that the mapping $\mathcal{T}$ maps balls in $X$ into itself and it is a contraction in the space $Y$. This finishes the proof of Theorem 3.3.1 for $N = 2$. □

The results of this chapter were published in [22].

# Chapter 4

# Steady flow of a second grade fluid past an obstacle

## 4.1 The model

In this chapter we study a model of a second grade fluid which was introduced in 1.4.3. Let us recall that this model consists of a linear momentum equation (1.19), divergence free condition (1.16) and a constitutive relation (1.54) with Rivlin-Ericksen tensors $\mathbf{A}_n$ defined in (1.49). Summarizing these equations we get

$$\begin{aligned}
\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} &= \operatorname{div}\mathbf{T} + \rho\mathbf{f} \\
\operatorname{div}\mathbf{v} &= 0 \\
\mathbf{T} &= -p\mathbf{I} + 2\mu\mathbf{D} + \alpha_1\mathbf{A}_2 + 4\alpha_2\mathbf{D}^2 \\
\mathbf{A}_2 &= 2\left(\frac{\partial \mathbf{D}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{D} + \mathbf{D}\mathbf{L} + \mathbf{L}^T\mathbf{D}\right).
\end{aligned} \tag{4.1}$$

Here $\alpha_1 > 0$ and $\alpha_2$ are the stress moduli. The condition of thermodynamical stability yields $\alpha_1 + \alpha_2 = 0$, see [8]. The question of signs and values of the stress moduli $\alpha_1$, $\alpha_2$ and of $\alpha_1 + \alpha_2$ in this model is however not clear. In [6] the authors show that the constraint $\alpha_1 + \alpha_2 = 0$ is not necessary for the mathematical problem being well set. In [13] the authors show that for $\alpha_1 < 0$ the rest state of flow of second grade fluid in exterior domain is instable. Our results can be easily adapted also for the case $\alpha_1 + \alpha_2 \neq 0$, however we keep this thermodynamical constraint for simplicity.

We consider a three–dimensional steady flow of a second grade fluid past an obstacle, so $\Omega$ is an exterior domain $\mathbb{R}^3 \setminus \mathcal{B}$, where $\mathcal{B}$ is a simply connected compact set and we assume that $B_{\kappa L}(\mathbf{0}) \subset \mathcal{B} \subset B_L(\mathbf{0})$ for some $\kappa > 0$ and $L > 0$. We prescribe homogenous Dirichlet boundary conditions for the velocity on $\partial\Omega$ and we prescribe a constant velocity at infinity. Plugging all equations of (4.1) together we get

$$\left.\begin{aligned} -\mu\Delta\mathbf{v} - \alpha_1(\mathbf{v}\cdot\nabla)\Delta\mathbf{v} + \nabla p = -\rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \rho\mathbf{f} + \\ +\alpha_1 \mathrm{div}\,[(\nabla\mathbf{v})^T(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)] \\ \mathrm{div}\,\mathbf{v} = 0 \end{aligned}\right\} \text{in } \Omega \tag{4.2}$$
$$\begin{aligned} \mathbf{v} &= \mathbf{0} && \text{on } \partial\Omega = \partial\mathcal{B} \\ \mathbf{v} &\to \mathbf{v}_\infty && \text{as } |\mathbf{x}| \to \infty, \end{aligned}$$

where $\mathbf{v}_\infty$ is the prescribed constant velocity at infinity. Assuming $\mathbf{v}_\infty \neq \mathbf{0}$ we can rotate the coordinate system in such a way that $\mathbf{v}_\infty = \beta\mathbf{e}_1 = (\beta, 0, 0)$ and denoting $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty$ we get from (4.2)

$$\left.\begin{aligned} -\mu\Delta\mathbf{u} - \alpha_1(\mathbf{u}\cdot\nabla)\Delta\mathbf{u} - \alpha_1\beta\Delta\frac{\partial\mathbf{u}}{\partial x_1} + \rho\beta\frac{\partial\mathbf{u}}{\partial x_1} + \nabla p = \\ -\rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \rho\mathbf{f} + \alpha_1\mathrm{div}\,[(\nabla\mathbf{u})^T(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)] \\ \mathrm{div}\,\mathbf{u} = 0 \end{aligned}\right\} \text{in } \Omega$$
$$\begin{aligned} \mathbf{u} = -\mathbf{v}_\infty &= -\beta\mathbf{e}_1 && \text{on } \partial\Omega = \partial\mathcal{B} \\ \mathbf{u} &\to \mathbf{0} && \text{as } |\mathbf{x}| \to \infty. \end{aligned} \tag{4.3}$$

Next, we rewrite the equations in dimensionless form, i.e. we introduce new velocity $\mathbf{U} = \frac{\mathbf{u}}{\beta}$ and new independent variable $\mathbf{X} = \frac{\mathbf{x}}{L}$, where $L$ is the diameter of the obstacle. We renormalize the pressure $P = \frac{p}{\rho\beta^2}$ and the external force $\mathbf{F} = \frac{\mathbf{f}L}{\beta^2}$ and introduce the Reynolds number $\mathcal{R} = \frac{\rho\beta L}{\mu}$ and the Weisenberg number $\mathcal{W} = \frac{\alpha_1\beta}{L\mu}$. However, for the sake of transparency, we keep writing small letters instead of capital letters.

After renormalization we end up with

$$
\left.\begin{aligned}
-\Delta\mathbf{u} - \mathcal{W}(\mathbf{u}\cdot\nabla)\Delta\mathbf{u} - \mathcal{W}\Delta\frac{\partial\mathbf{u}}{\partial x_1} + \mathcal{R}\frac{\partial\mathbf{u}}{\partial x_1} + \mathcal{R}\nabla p = \\
-\mathcal{R}(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathcal{R}\mathbf{f} + \mathcal{W}\mathrm{div}\,[(\nabla\mathbf{u})^T(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)] \\
\mathrm{div}\,\mathbf{u} = 0
\end{aligned}\right\}\ \text{in } \Omega
$$
$$
\begin{aligned}
\mathbf{u} &= -\mathbf{e}_1 && \text{on } \partial\Omega \\
\mathbf{u} &\to \mathbf{0} && \text{as } |\mathbf{x}| \to \infty,
\end{aligned} \tag{4.4}
$$

where the renormalized domain[1] $\Omega = \mathbb{R}^3 \setminus \mathcal{D}$ and $B_\kappa(\mathbf{0}) \subset \mathcal{D} \subset B_1(\mathbf{0})$.

Finally, we follow the decomposition procedure proposed in [32]. We introduce new pressure $q$ as a solution to

$$q + \mathcal{W}[(\mathbf{u} + \mathbf{e}_1)\cdot\nabla]q = \mathcal{R}p \tag{4.5}$$

and we denote

$$-\Delta\mathbf{u} + \mathcal{R}\frac{\partial\mathbf{u}}{\partial x_1} + \nabla q =: \mathbf{z}. \tag{4.6}$$

Then $\mathbf{z}$ satisfies

$$
\begin{aligned}
\mathbf{z} + \mathcal{W}[(\mathbf{u} + \mathbf{e}_1)\cdot\nabla]\mathbf{z} = \mathcal{R}\mathbf{f} - \mathcal{R}(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathcal{W}\mathrm{div}\,[(\nabla\mathbf{u})^T(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)] - \\
- \mathcal{W}(\nabla\mathbf{u})^T\nabla q + \mathcal{R}\mathcal{W}(\mathbf{u}\cdot\nabla)\frac{\partial\mathbf{u}}{\partial x_1} + \mathcal{R}\mathcal{W}\frac{\partial^2\mathbf{u}}{\partial x_1^2}.
\end{aligned} \tag{4.7}
$$

Note that we still have the conditions

$$
\begin{aligned}
\mathrm{div}\,\mathbf{u} &= 0 && \text{in } \Omega \\
\mathbf{u} &= -\mathbf{e}_1 && \text{on } \partial\Omega \\
\mathbf{u} &\to \mathbf{0} && \text{as } |\mathbf{x}| \to \infty.
\end{aligned} \tag{4.8}
$$

We will study the properties of the solutions by searching for the fixed points of the mapping

$$\mathcal{T} : (\mathbf{w}, s) \mapsto \mathbf{z} \mapsto (\mathbf{u}, q), \tag{4.9}$$

[1] We again use the same notation for the original domain and the renormalized domain.

where for given $(\mathbf{w}, s)$, $\mathbf{z}$ is the solution to the transport equation

$$\mathbf{z} + \mathcal{W}[(\mathbf{w}+\mathbf{e}_1)\cdot\nabla]\mathbf{z} = \mathcal{R}\mathbf{f} - \mathcal{R}(\mathbf{w}\cdot\nabla)\mathbf{w} + \mathcal{W}\mathrm{div}\,[(\nabla\mathbf{w})^T(\nabla\mathbf{w}+(\nabla\mathbf{w})^T)] - \\ - \mathcal{W}(\nabla\mathbf{w})^T\nabla s + \mathcal{R}\mathcal{W}(\mathbf{w}\cdot\nabla)\frac{\partial\mathbf{w}}{\partial x_1} + \mathcal{R}\mathcal{W}\frac{\partial^2\mathbf{w}}{\partial x_1^2} =: \mathbf{B}(\mathbf{f},\mathbf{w},s) \qquad \text{in } \Omega \tag{4.10}$$

and $(\mathbf{u}, q)$ is the solution to the Oseen problem

$$\begin{aligned} -\Delta\mathbf{u} + \mathcal{R}\frac{\partial\mathbf{u}}{\partial x_1} + \nabla q &= \mathbf{z} && \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= -\mathbf{e}_1 && \text{on } \partial\Omega \\ \mathbf{u} &\to \mathbf{0} && \text{as } |\mathbf{x}| \to \infty. \end{aligned} \tag{4.11}$$

We have decomposed the original problem into the Oseen problem (4.11) and the steady transport equation (4.10). Due to the pressence of the Oseen problem we expect the structure of solutions to correspond to the structure of the Oseen fundamental solution, especially the existence of the wake region behind the obstacle. This has been proved for some other models of fluids, namely for incompressible newtonian fluid described by Navier–Stokes equations, see [10] and [41], and for a model of a viscoelastic fluid, see [34]. Denoting

$$s(\mathbf{x}) = |\mathbf{x}| - x_1 \tag{4.12}$$

one might expect the solution $\mathbf{u}$ to satisfy

$$|\mathbf{u}(\mathbf{x})| \le C\,|\mathbf{x}|^{-1}\,(1+s(\mathbf{x}))^{-1} \tag{4.13}$$

for $|\mathbf{x}|$ sufficiently large, the same property has the Oseen fundamental solution. However, we are only able to prove

$$|\mathbf{u}(\mathbf{x})| \le C\,|\mathbf{x}|^{-1+\varepsilon}\,(1+s(\mathbf{x}))^{-1+\varepsilon} \tag{4.14}$$

for arbitrarily small $\varepsilon$. This is due to the presence of the linear term $\mathcal{R}\mathcal{W}\frac{\partial^2\mathbf{u}}{\partial x_1^2}$ on the right-hand side of the transport equation (4.10). This implies that the solution to the transport equation has the same decay as $\nabla^2\mathbf{w}$ (quadratic terms decay faster). Moreover, for the Oseen system previously available estimates for the second gradient (which were using techniques of fundamental solution) lose logarithmic factor in the weight in the $L^\infty$ norm, while $L^p$ estimates lose $\varepsilon$ in the weight, see [37]. Thus fixed point theorem argument would not work. Fortunately, due to recent results of Koch [18] we have at least $L^p$ estimates without mentioned $\varepsilon$ loss in the weight and therefore fixed point argument works.

It is worth mentioning that in [34], where the model of viscoelastic fluid is considered, authors are able to overcome these problems by introducing modified Oseen problem with the problematic term $\frac{\partial^2 \mathbf{u}}{\partial x_1}$ being included in the Oseen operator. Then all terms on the right hand side are quadratic and thus with better decay. In our problem this cannot be repeated since this linear term appears in the transport equation.

The drawback of using $L^p$ estimates is that in order to get $L^\infty$ estimate we have to use embedding theorems and thus we are able to prove

$$|\mathbf{u}(\mathbf{x})| \leq C\, |\mathbf{x}|^{-1+\varepsilon} (1 + s(\mathbf{x}))^{-1+\varepsilon} \tag{4.15}$$

for arbitrarily small $\varepsilon$. Details will be specified later on.

## 4.2 Preliminaries

Our goal is to show the existence of the wake region behind the obstacle, in other words we show that the decay of the solution $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty$ is slower in the direction of the flow. The crucial tool which is used to measure the rate of the decay are weighted Lebesgue spaces. Therefore we introduce several anisotropic weights which will be useful in studying the asymptotic structure of the solution. First let us remind that

$$s(\mathbf{x}) = |\mathbf{x}| - x_1. \tag{4.16}$$

We define

$$\begin{aligned}
\eta^A_B(\mathbf{x}) &= (1 + |\mathbf{x}|)^A (1 + s(\mathbf{x}))^B \\
\nu^A_B(\mathbf{x}) &= |\mathbf{x}|^A (1 + s(\mathbf{x}))^B \\
\mu^{A,\omega}_B(\mathbf{x}) &= \eta^{A-\omega}_B(\mathbf{x}) \nu^\omega_0(\mathbf{x}) \\
\eta^A_B(\mathbf{x}, \mathcal{R}) &= (1 + |\mathcal{R}\mathbf{x}|)^A (1 + s(\mathcal{R}\mathbf{x}))^B \\
\nu^A_B(\mathbf{x}, \mathcal{R}) &= |\mathbf{x}|^A (1 + s(\mathcal{R}\mathbf{x}))^B \\
\mu^{A,\omega}_B(\mathbf{x}, \mathcal{R}) &= \eta^{A-\omega}_B(\mathbf{x}, \mathcal{R}) \nu^\omega_0(\mathbf{x}, \mathcal{R}).
\end{aligned} \tag{4.17}$$

As we decomposed the original problem into an Oseen problem and a steady transport equation, we shall mention several classical results about these problems in three dimensional exterior domains. Let us start with the Oseen problem (4.11).

### 4.2.1 Oseen problem

We denote by $(\boldsymbol{\mathcal{O}}, \mathbf{e})$ the fundamental solution to the Oseen problem. It can be shown (see for example [37]) that

$$\mathbf{e}(\mathbf{x}) = \nabla \mathcal{E}(\mathbf{x}), \tag{4.18}$$

where $\mathcal{E}(\mathbf{x})$ is the fundamental solution to the Laplace equation. The tensor $\boldsymbol{\mathcal{O}}(\mathbf{x}, \mathcal{R})$ (here $\mathcal{R}$ denotes the constant standing in front of $\frac{\partial \mathbf{u}}{\partial x_1}$ in the equation) satisfies the following property

$$\boldsymbol{\mathcal{O}}(\mathbf{x}, \mathcal{R}) = \mathcal{R}\boldsymbol{\mathcal{O}}(\mathcal{R}\mathbf{x}, 1) \tag{4.19}$$

and therefore it is sufficient to study the tensor $\boldsymbol{\mathcal{O}}(\mathbf{x}, 1)$. For $|\mathbf{x}| \to \infty$ we have

$$\begin{aligned}
\boldsymbol{\mathcal{O}}(\mathbf{x}, 1) &\sim |\mathbf{x}|^{-1} (1 + s(\mathbf{x}))^{-1} \\
D^\alpha \boldsymbol{\mathcal{O}}(\mathbf{x}, 1) &\sim |\mathbf{x}|^{-1-\frac{|\alpha|}{2}} (1 + s(\mathbf{x}))^{-1-\frac{|\alpha|}{2}} \\
D^\alpha \boldsymbol{\mathcal{O}}(\mathbf{x}, 1) &\sim |\mathbf{x}|^{-1-\alpha_1-\frac{|\alpha|-\alpha_1}{2}} (1 + s(\mathbf{x}))^{-1-\frac{|\alpha|-\alpha_1}{2}},
\end{aligned} \tag{4.20}$$

i.e. the derivatives with respect to the first variable decay faster.

Next, we present some results for the general Oseen problem

$$\begin{aligned}
-\Delta \mathbf{u} + \mathcal{R}\frac{\partial \mathbf{u}}{\partial x_1} + \nabla P = \mathbf{f} = \operatorname{div} \boldsymbol{\mathcal{G}} \quad &\text{in } \Omega \\
\operatorname{div} \mathbf{u} = 0 \quad &\text{in } \Omega \\
\mathbf{u} = \mathbf{u}_* \quad &\text{on } \partial\Omega \\
\mathbf{u} \to \mathbf{0} \quad &\text{as } |\mathbf{x}| \to \infty,
\end{aligned} \tag{4.21}$$

where $\Omega$ is an exterior domain.

The proof of the following classical theorem can be found in [12] or in [37].

**Theorem 4.2.1.** *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $C^{k+2}$. Let $\mathbf{f} \in D_0^{-1,q}(\Omega) \cap W^{k,2}(\Omega)$, $\mathbf{u}_* \in W^{k+\frac{3}{2},2}(\partial\Omega)$, $q \in (\frac{3}{2}, 3)$, $k \geq 0$. Then there exists exactly one q-weak solution (i.e. weak solution such that $\mathbf{u} \in D^{1,q}(\Omega)$) to (4.21). Moreover*

$$\mathbf{u} \in L^{\frac{4q}{4-q}}(\Omega) \quad \textit{and} \quad \nabla \mathbf{u}, P \in L^q(\Omega) \cap W^{k+1,2}(\Omega) \tag{4.22}$$

*and*

$$\begin{aligned} a_2 \|\mathbf{u}\|_{\frac{4q}{4-q}} + |\mathbf{u}|_{1,q} + \|\nabla \mathbf{u}\|_{k+1,2} + \|P\|_q + \|P\|_{k+1,2} \leq \\ \leq C(|\mathbf{f}|_{-1,q} + \|\mathbf{f}\|_{k,2} + \|\mathbf{u}_*\|_{k+\frac{3}{2},2,\partial\Omega}), \end{aligned} \tag{4.23}$$

*where for $\mathcal{R} \in (0, \mathcal{R}_0]$ the constant $C = C(k, q, \Omega, \mathcal{R}_0)$ and $a_2 = \min\{1, \mathcal{R}^{\frac{1}{4}}\}$.*

We need the following integral representation of solutions to (4.21) to obtain weighted estimates. Let us denote

$$\begin{aligned} T_{ij}(\mathbf{u}, P) &= \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - P\delta_{ij} \\ T'_{ij}(\mathbf{e}) &= \frac{\partial e_i}{\partial x_j} + \frac{\partial e_j}{\partial x_i} + \mathcal{R} e_1 \delta_{ij}. \end{aligned} \tag{4.24}$$

**Theorem 4.2.2.** *Let $\Omega \in C^2$ be an exterior domain, $\boldsymbol{\mathcal{G}} \in C_0^\infty(\overline{\Omega})$ and $(\mathbf{u}, P)$ be the unique solution to (4.21). Let $T_{ij}$ be defined in (4.24) and $(\boldsymbol{\mathcal{O}}, \mathbf{e})$ be the fundamental solution to the Oseen problem. Then*

$$\begin{aligned} u_j(\mathbf{x}) = &\int_\Omega \frac{\partial}{\partial x_k} \mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}, \mathcal{R}) \mathcal{G}_{ik}(\mathbf{y}) \mathrm{d}\mathbf{y} + \\ &+ \int_{\partial\Omega} [-\mathcal{R}\mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}, \mathcal{R}) u_i(\mathbf{y}) \delta_{1k} + u_i(\mathbf{y}) T_{ik}(\boldsymbol{\mathcal{O}}_{\cdot j}, e_j)(\mathbf{x} - \mathbf{y}, \mathcal{R}) + \\ &+ \mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}, \mathcal{R}) T_{ik}(\mathbf{u}, P)(\mathbf{y}) + \mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}, \mathcal{R}) \mathcal{G}_{ik}(\mathbf{y})]\, n_k(\mathbf{y}) \mathrm{d}S \end{aligned} \tag{4.25}$$

$$\begin{aligned} D^\alpha u_j(\mathbf{x}) = &- \int_\Omega D^\alpha \frac{\partial}{\partial x_k} \mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}, \mathcal{R}) \mathcal{G}_{ik}(\mathbf{y}) \mathrm{d}\mathbf{y} + \\ &+ \int_{\partial\Omega} [-\mathcal{R} D^\alpha \mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}, \mathcal{R}) u_i(\mathbf{y}) \delta_{1k} + u_i(\mathbf{y}) D^\alpha T_{ik}(\boldsymbol{\mathcal{O}}_{\cdot j}, e_j)(\mathbf{x} - \mathbf{y}, \mathcal{R}) + \\ &+ D^\alpha \mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}, \mathcal{R}) T_{ik}(\mathbf{u}, P)(\mathbf{y}) + D^\alpha \mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}, \mathcal{R}) \mathcal{G}_{ik}(\mathbf{y})]\, n_k(\mathbf{y}) \mathrm{d}S \end{aligned} \tag{4.26}$$

*for* $|\alpha| = 1$ *and*

$$D^\alpha u_j(\mathbf{x}) = v.p. \int_\Omega D^\alpha \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y},\mathcal{R}) \frac{\partial}{\partial y_k} \mathcal{G}_{ik}(\mathbf{y}) \mathrm{d}\mathbf{y} + c_{ij\alpha_1\alpha_2} \frac{\partial \mathcal{G}_{ik}}{\partial x_k}(\mathbf{x}) + \\ + \int_{\partial\Omega} [-\mathcal{R} D^\alpha \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y},\mathcal{R}) u_i(\mathbf{y}) \delta_{1k} + u_i(\mathbf{y}) D^\alpha T_{ik}(\boldsymbol{\mathcal{O}}_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y},\mathcal{R}) + \\ + D^\alpha \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y},\mathcal{R}) T_{ik}(\mathbf{u},P)(\mathbf{y})]\, n_k(\mathbf{y}) \mathrm{d}S \quad (4.27)$$

*for* $|\alpha| = 2$.

*Remark* 4.2.1. The integral representation formulas hold for much larger classes of functions. For example it holds for a.a. $\mathbf{x} \in \Omega$ if $\mathbf{u} \in W^{2,q}_{loc}(\overline{\Omega})$ and $P \in W^{1,q}_{loc}(\overline{\Omega})$ for some $q \in (1,\infty)$ and

- (4.25) if $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$ and $\operatorname{div} \boldsymbol{\mathcal{G}} \in L^r_{loc}(\overline{\Omega})$ for $q \in (1,4)$, $r \in (1,\infty)$,
- (4.26) if $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$ and $\operatorname{div} \boldsymbol{\mathcal{G}} \in L^r_{loc}(\overline{\Omega})$ for $q, r \in (1,\infty)$,
- (4.27) if $\operatorname{div} \boldsymbol{\mathcal{G}} \in L^r_{loc}(\overline{\Omega})$ for $r \in (1,\infty)$.

For the pressure we have also integral representation formulas.

**Theorem 4.2.3.** *Let* $\Omega \in C^2$ *be an exterior domain,* $\boldsymbol{\mathcal{G}} \in C_0^\infty(\overline{\Omega})$ *and* $(\mathbf{u}, P)$ *be the unique solution to (4.21). Let* $T_{ij}$ *and* $T'_{ij}$ *be defined in (4.24). Then*

$$P(\mathbf{x}) = v.p. \int_\Omega \frac{\partial}{\partial x_k} e_i(\mathbf{x}-\mathbf{y}) \mathcal{G}_{ik}(\mathbf{y}) \mathrm{d}\mathbf{y} + c_{ik} \mathcal{G}_{ik}(\mathbf{x}) + \\ + \int_{\partial\Omega} [-\mathcal{R} e_i(\mathbf{x}-\mathbf{y}) u_i(\mathbf{y}) \delta_{1l} + u_i(\mathbf{y}) T'_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + \\ + e_i(\mathbf{x}-\mathbf{y}) T_{il}(\mathbf{u},P)(\mathbf{y}) + e_i(\mathbf{x}-\mathbf{y}) \mathcal{G}_{il}(\mathbf{y})]\, n_l(\mathbf{y}) \mathrm{d}S \quad (4.28)$$

$$D^\alpha P(\mathbf{x}) = v.p. \int_\Omega D^\alpha e_i(\mathbf{x}-\mathbf{y}) \frac{\partial}{\partial y_k} \mathcal{G}_{ik}(\mathbf{y}) \mathrm{d}\mathbf{y} + c_{ik} \frac{\partial}{\partial x_k} \mathcal{G}_{ik}(\mathbf{x}) + \\ + \int_{\partial\Omega} [-\mathcal{R} D^\alpha e_i(\mathbf{x}-\mathbf{y}) u_i(\mathbf{y}) \delta_{1l} + u_i(\mathbf{y}) D^\alpha T'_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + \\ + D^\alpha e_i(\mathbf{x}-\mathbf{y}) T_{il}(\mathbf{u},P)(\mathbf{y})]\, n_l(\mathbf{y}) \mathrm{d}S \quad (4.29)$$

*for* $|\alpha| = 1$.

*Remark* 4.2.2. The integral representation formulas for pressure hold also for much larger classes of functions. For example it holds for a.a. $\mathbf{x} \in \Omega$ if $\mathbf{u} \in W^{2,q}_{loc}(\overline{\Omega})$ and $P \in W^{1,q}_{loc}(\overline{\Omega})$ for some $q \in (1, \infty)$ and

- (4.28) if $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$ and $\operatorname{div} \boldsymbol{\mathcal{G}} \in L^r_{loc}(\overline{\Omega})$ for $q, r \in (1, \infty)$,
- (4.29) if $\operatorname{div} \boldsymbol{\mathcal{G}} \in L^r_{loc}(\overline{\Omega})$ for $r \in (1, \infty)$.

The proof of these representation formulas can be found in [12] or in [37].

### 4.2.2 Results for weighted spaces

Let $g \in L^1_{loc}(\Omega)$ be a nonnegative weight. Then $L^p(\Omega, g)$ denotes the weighted $L^p$ space with the norm

$$\|u\|_{p,(g)} = \|u\|_{p,g} = \|ug\|_p \tag{4.30}$$

for any $p \in [1, \infty]$. Similarly, $W^{k,p}(\Omega, g)$ denotes the weighted Sobolev space with the norm

$$\|u\|_{k,p,(g)} = \|u\|_{k,p,g} = \|ug\|_{k,p} \,. \tag{4.31}$$

Note that if there is no confusion we sometimes omit writing the domain and instead of $L^p(\Omega, g)$ we write simply $L^p(g)$.

We recall the classical definition of the class $A_p$.

**Definition 4.2.1.** The weight $g$ belongs to the class $A_p$ if there exists a constant $C$ such that

$$\sup_Q \left[ \left( \frac{1}{|Q|} \int_Q g^p(\mathbf{x}) \mathrm{d}\mathbf{x} \right) \left( \frac{1}{|Q|} \int_Q g^{-\frac{p}{p-1}}(\mathbf{x}) \mathrm{d}\mathbf{x} \right)^{p-1} \right] \leq C < \infty, \tag{4.32}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^3$.

We mention also an useful lemma concerning the integrability of the weigths $\eta^A_B$ defined in (4.17).

**Lemma 4.2.4.** *Let $a, b \in \mathbb{R}$. Then*

$$\int_{\mathbb{R}^3} \eta^{-a}_{-b}(\mathbf{x}) \mathrm{d}\mathbf{x} < \infty \quad \Leftrightarrow \quad a + \min\{1, b\} > 3. \tag{4.33}$$

*Proof.* The whole proof and other properties of this weight can be found in [37]. We only mention the idea of the proof here. It is based on proving that

$$\int_{\partial B_R} \eta_{-b}^{-a}(\mathbf{x})\mathrm{d}S \sim R^{2-a-\min\{1,b\}}\ (\times \ln R \text{ if } b = 1) \tag{4.34}$$

as $R \to \infty$. This can be shown by rewriting this integral in spherical coordinates, then it can be calculated explicitly. □

One of the main tools we use is the following theorem due to Koch [18]. We should mention that this estimate is an essential improvement of what has been known about weighted estimates of the solutions to the Oseen system. Without this result it was impossible to obtain weighted estimates for the model of the second grade fluid or for Maxwell and Oldroyd-type fluids, see [37].

**Theorem 4.2.5.** *Let $T$ be an integral operator with the kernel $\frac{\partial^2}{\partial x_i \partial x_j}\boldsymbol{\mathcal{O}}$ on $\mathbb{R}^3$. Then the following estimate hold:*

$$\|Tf\|_{p,(g),\mathbb{R}^3} \leq C\,\|f\|_{p,(g),\mathbb{R}^3} \tag{4.35}$$

*for $p \in (1, \infty)$ and $g = \eta_B^A(\mathbf{x})$ for $A, B$ satisfying*

$$\begin{aligned} A, B \in \left(-\frac{1}{p}, \frac{2(p-1)}{p}\right) \\ A + B > -\frac{1}{p} \\ 2A - B, 2B - A < \frac{2(p-1)+1}{p}. \end{aligned} \tag{4.36}$$

The proof of this theorem can be found in [18]. Note that in [18] the theorem is formulated only for the case $p = 2$, nevertheless the proof is given for general $p \in (1, \infty)$. The same estimate holds also for the case of an exterior domain.

**Corollary 4.2.6.** *For the same operator $T$ as in Theorem 4.2.5 we have*

$$\|Tf\|_{p,(g_1),\Omega} \leq C\mathcal{R}^{\omega}\,\|f\|_{p,(g_2),\Omega}\,, \tag{4.37}$$

*where $g_1 = \mu_B^{A,\omega}(\mathbf{x}, \mathcal{R})$, $g_2 = \mu_B^{A,2\omega}(\mathbf{x}, \mathcal{R})$, $A, B$ satisfying (4.36) and $\omega \in [0, \frac{A}{2})$.*

*Proof.* First we observe that for all $\mathbf{x} \in \Omega$ it holds

$$\mu_B^{A,\omega}(\mathbf{x}) \leq \eta_B^A(\mathbf{x}) \leq (1+\frac{1}{\kappa})^{\omega} \mu_B^{A,\omega}(\mathbf{x}), \tag{4.38}$$

where $\kappa$ was introduced earlier by condition $B_\kappa(\mathbf{0}) \subset \mathcal{D}$ and $\Omega = \mathbb{R}^3 \setminus \mathcal{D}$. In other words the weights $\eta_B^A(\mathbf{x})$ and $\mu_B^{A,\omega}(\mathbf{x})$ are equivalent in $\Omega$. The presence of the term $\mathcal{R}^\omega$ is an easy consequence of a rescaling argument, because $\nabla^2 \boldsymbol{\mathcal{O}}(\mathbf{x}, \mathcal{R}) = \mathcal{R}^3 \nabla^2 \boldsymbol{\mathcal{O}}(\mathcal{R}\mathbf{x}, 1)$. □

The proofs of the following theorems can be found for example in [20] or in [37].

**Theorem 4.2.7.** *Let $T$ be an integral operator with the kernel $|\nabla \boldsymbol{\mathcal{O}}|$, $T : f \to |\nabla \boldsymbol{\mathcal{O}}| * f$ and $p \in (1,\infty)$. Then $T$ is a well defined continuous operator:*

$$L^p(\mathbb{R}^3, \eta_B^{A+\frac{1}{2}}(\cdot, \mathcal{R})) \mapsto L^p(\mathbb{R}^3, \eta_B^A(\cdot, \mathcal{R})) \tag{4.39}$$

*for $B \in (0, \frac{3}{2} - \frac{3}{2p})$, $A + B > -\frac{1}{p}$, $A < \frac{3}{2} - \frac{2}{p}$, $A - B < \frac{1}{2} - \frac{1}{p}$. Moreover we have for $A, B$ specified above*

$$\| |\nabla \boldsymbol{\mathcal{O}}(\cdot, \mathcal{R})| * f \|_{p, \eta_B^A(\cdot, \mathcal{R}), \mathbb{R}^3} \leq C \mathcal{R}^{-1} \|f\|_{p, \eta_B^{A+\frac{1}{2}}(\cdot, \mathcal{R}), \mathbb{R}^3}. \tag{4.40}$$

**Corollary 4.2.8.** *For the same operator $T$ as in Theorem 4.2.7 one has*

$$\|Tf\|_{p,(g_1),\Omega} \leq C \mathcal{R}^{-1+\omega} \|f\|_{p,(g_2),\Omega}, \tag{4.41}$$

*where $g_1 = \mu_B^{A,\omega}(\mathbf{x}, \mathcal{R})$, $g_2 = \mu_B^{A,2\omega}(\mathbf{x}, \mathcal{R})$, $A, B$ satisfy the assertions of Theorem 4.2.7 and $\omega \in [0, \frac{A}{2})$.*

**Theorem 4.2.9.** *Let*

$$Tf(\mathbf{x}) = \frac{\partial}{\partial x_i} \int_{\mathbb{R}^3} e_j(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) \mathrm{d}\mathbf{y}, \quad i, j = 1, 2, 3, \tag{4.42}$$

*$f \in C_0^\infty(\mathbb{R}^3)$, $p \in (1, \infty)$ and let $g$ stands for one of weights $\eta_B^A$, $\nu_B^A$, $\mu_B^{A,\omega}$. Let $A, B$ be such that $g$ is an $A_p$ weight in $\mathbb{R}^3$. Then $T$ maps $C_0^\infty(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3, g)$ and*

$$\|Tf\|_{p,(g),\mathbb{R}^3} \leq C \|f\|_{p,(g),\mathbb{R}^3}. \tag{4.43}$$

*Moreover $T$ can be continuously extended onto $L^p(\mathbb{R}^3, g)$.*

**Corollary 4.2.10.** *The same holds also for the case of an exterior domain* $\Omega$.

**Theorem 4.2.11.**

- *Let* $B \in (-\frac{1}{p}, \frac{p-1}{p})$ *and* $A + B \in (-\frac{3}{p}, \frac{3(p-1)}{p})$. *Then the weight* $\eta_B^A$ *is an* $A_p$ *weight in* $\mathbb{R}^3$ *for* $p \in (1, \infty)$.
- *Let moreover* $A \in (-\frac{3}{p}, \frac{3(p-1)}{p})$ *and* $\omega \in [0, A]$. *Then the weights* $\nu_B^A$ *and* $\mu_B^{A,\omega}$ *are* $A_p$ *weights in* $\mathbb{R}^3$ *for* $p \in (1, \infty)$.

Finally we mention here an useful generalization of the Sobolev embedding theorems to weighted spaces.

**Lemma 4.2.12.** *There exist a constant* $C$ *depending only on* $\Omega, A, B, \omega$ *such that for* $p > 3$, $A, B \geq 0$, $\omega \in [0, A]$ *and* $\mathcal{R} \leq 1$

$$\|g\|_{L^\infty(\Omega, \mu_B^{A,\omega}(\cdot, \mathcal{R}))} \leq C(\|g\|_{L^p(\Omega, \mu_B^{A,\omega}(\cdot, \mathcal{R}))} + \|\nabla g\|_{L^p(\Omega, \mu_B^{A,\omega}(\cdot, \mathcal{R}))}). \tag{4.44}$$

*Proof.* It is an easy consequence of the Sobolev embedding theorem and the fact that there is a constant $C$ independent of $\mathcal{R}$ such that

$$\|g\|_{L^p(\Omega, \nabla \mu_B^{A,\omega}(\cdot, \mathcal{R}))} \leq C \|g\|_{L^p(\Omega, \mu_B^{A,\omega}(\cdot, \mathcal{R}))}. \tag{4.45}$$

□

### 4.2.3 Steady transport equation

Next we consider the steady transport equation

$$z + \mathbf{w} \cdot \nabla z = f \qquad \text{in } \Omega. \tag{4.46}$$

This equation is scalar, nevertheless all theorems below hold also for the vector case. The following theorems are proved in [33] and in [37] even for more complicated cases.

**Theorem 4.2.13.** *(i) Let* $\Omega \in C^{0,1}$ *be an exterior domain,* $\mathbf{w} \in C^{k-1}(\Omega)$, $\mathbf{w} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$, $\nabla^k \mathbf{w} \in L^3(\Omega)$, $f \in W^{k,q}$ *for* $q \in (1, 3)$, $kq > 3$. *Then there exists* $\alpha > 0$ *such that if*

$$\|\nabla \mathbf{w}\|_{C^{k-2}} + \left\|\nabla^k \mathbf{w}\right\|_3 < \alpha, \tag{4.47}$$

*then there exists unique solution* $z \in W^{k,q}(\Omega)$ *to (4.46) satisfying the estimate*

$$\|z\|_{k,q} \leq C(\alpha) \|f\|_{k,q}. \tag{4.48}$$

*(ii) Let* $\Omega \in C^{0,1}$ *be an exterior domain,* $\mathbf{w} \in C^k(\Omega)$*,* $\mathbf{w} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$*,* $f \in W^{k,q}$ *for* $kq > 3$*. Then there exists* $\alpha > 0$ *such that if*

$$\|\nabla \mathbf{w}\|_{C^{k-1}} < \alpha, \tag{4.49}$$

*then there exists unique solution* $z \in W^{k,q}(\Omega)$ *to (4.46) satisfying the estimate*

$$\|z\|_{k,q} \leq C(\alpha) \|f\|_{k,q}. \tag{4.50}$$

**Theorem 4.2.14.** *Let* $\Omega, k, q, \mathbf{w}$ *and* $f$ *satisfy the assumptions of Theorem 4.2.13 (ii). Moreover, let* $g \in C^k(\Omega)$ *be a positive weight such that*

$$W^{k,q}(\Omega, g) \subset W^{k,q}(\Omega) \tag{4.51}$$

*and let*

$$\|\mathbf{w} \cdot \nabla \ln g\|_{C^{k-1}} + |\mathbf{w} \cdot \nabla \ln g|_{k,q} \tag{4.52}$$

*be sufficiently small. Let* $f \in W^{k,q}(\Omega, g)$*. Then* $z$*, the solution to (4.46), belongs to* $W^{k,q}(\Omega, g)$ *and*

$$\|z\|_{k,q,(g)} \leq C \|f\|_{k,q,(g)}. \tag{4.53}$$

## 4.3 Main theorem and proof

Our main result is the following

**Theorem 4.3.1.** *Let* $\mathbf{f} = \operatorname{div} \mathbf{H}$*,* $\mathbf{H} \in W^{k,2}(\Omega)$*,* $k \geq 3$*. Let* $\Omega \in C^{k+1}$ *be an exterior domain in* $\mathbb{R}^3$ *and let* $\mathcal{R}_0, \mathcal{W}_0$ *be sufficiently small. Then for any* $\mathcal{R} \in (0, \mathcal{R}_0), \mathcal{W} \in (0, \mathcal{W}_0)$ *there exists a unique solution* $(\mathbf{u}, q)$ *to the problem (4.5) – (4.8) for which the following estimates hold*

$$\mathcal{R}^{\frac{1}{4}} \|\mathbf{u}\|_4 + \|\nabla \mathbf{u}\|_{k,2} + \|q\|_{k,2} \leq K. \tag{4.54}$$

*If in addition* $\mathbf{f}, \mathbf{H} \in L^p(\Omega, \mu_{1-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p}, 2\omega}(\cdot, \mathcal{R}))$ *for some* $p > 6$ *and* $\mathcal{R}$ *and* $\mathcal{W}$ *are sufficiently small, the previously obtained solution* $(\mathbf{u}, q)$ *has the following* *properties*

$$\begin{aligned}
\mathbf{u} &\in L^p(\Omega, \mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot, \mathcal{R}))\\
\nabla\mathbf{u}, \nabla^2\mathbf{u} &\in L^p(\Omega, \mu_{1-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot, \mathcal{R}))\\
q, \nabla q &\in L^p(\Omega, \mu_{\frac{1}{2}-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot, \mathcal{R})).
\end{aligned} \tag{4.55}$$

*This yields in particular*

$$\mathbf{u} \in L^\infty(\Omega, \mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot, \mathcal{R})). \tag{4.56}$$

*Remark* 4.3.1. The weights $\mu_B^{A,\omega}$ are defined in (4.17). As the power $p$ can be chosen arbitrarily large, we get almost the same asymptotic structure as for the fundamental solution $\boldsymbol{\mathcal{O}}$ of the Oseen system.

*Remark* 4.3.2. Note that for $\mathbf{f} \in L^\infty(\Omega, \eta_1^{\frac{3}{2}}(\mathbf{x}))$ it holds

$$\|\mathbf{f}\|_{L^p(\Omega, \mu_{1-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},2\omega}(\cdot,\mathcal{R}))} \le C\mathcal{R}^{-2\omega-\frac{3}{p}} \|\mathbf{f}\|_{L^\infty(\Omega, \eta_1^{\frac{3}{2}}(\cdot))}. \tag{4.57}$$

### 4.3.1 Existence of solution

The proof of existence of solutions in Sobolev spaces is based on the method described for example in [37] and [34]. The solution is obtained as a limit of successive approximations

$$(\mathbf{u}_{n+1}, q_{n+1}) = \mathcal{T}(\mathbf{u}_n, q_n), \qquad n \ge 0, \tag{4.58}$$

where the mapping $\mathcal{T}$ was introduced in (4.9).

**Theorem 4.3.2.** *Let $\Omega \in C^{k+1}$ be an exterior domain in $\mathbb{R}^3$ and let $\mathbf{f} = \operatorname{div}\mathbf{H}$, $\mathbf{H} \in W^{k,2}(\Omega)$, $k \ge 3$. Let $\mathcal{R}_0$, $\mathcal{W}_0$ be sufficiently small. Then for any $\mathcal{R} \in (0, \mathcal{R}_0)$, $\mathcal{W} \in (0, \mathcal{W}_0)$ there exists $(\mathbf{u}, q)$ a solution to the system (4.5)-(4.8) such that $\mathbf{u} \in L^4(\Omega)$ and $\nabla\mathbf{u}, q \in W^{k,2}(\Omega)$.*

*Proof.* We use Theorem 6.3.2 for the following choice of spaces: $X = V_k$, $Y = V_{k-1}$, where

$$V_k = \left\{(\mathbf{u}, q) : \mathbf{u} \in L^4(\Omega), \nabla\mathbf{u}, q \in W^{k,2}(\Omega)\right\} \tag{4.59}$$

with the norm

$$\|(\mathbf{u},q)\|_{V_k} = \mathcal{R}^{\frac{1}{4}} \|\mathbf{u}\|_4 + \|\nabla\mathbf{u}\|_{k,2} + \|q\|_{k,2}\,. \tag{4.60}$$

First, we have to prove that $\mathcal{T}$ maps sufficiently large balls into themselves in $V_k$. Let us assume that $\|(\mathbf{w},s)\|_{V_k} \le R$. Using Theorem 4.2.1 we get that

$$\|(\mathbf{u},q)\|_{V_k} \le C(1 + |\mathbf{z}|_{-1,2} + \|\mathbf{z}\|_{k-1,2}). \tag{4.61}$$

Now we use Theorem 4.2.13 to get (recall that we denote the right hand side of the transport equation (4.10) by $\mathbf{B}(\mathbf{f},\mathbf{w},s)$)

$$\|\mathbf{z}\|_{k-1,2} \le C\,\|\mathbf{B}(\mathbf{f},\mathbf{w},s)\|_{k-1,2}\,. \tag{4.62}$$

One of the key observations is that the right hand side of the transport equation can be written in the divergence form as

$$\begin{aligned}\mathbf{B}(\mathbf{f},\mathbf{w},s) = \operatorname{div}\Big[&\mathcal{R}\mathbf{H} - \mathcal{R}\mathbf{w}\otimes\mathbf{w} + \mathcal{W}(\nabla\mathbf{w})^T(\nabla\mathbf{w} + (\nabla\mathbf{w})^T) - \\ -\mathcal{W}(\nabla\mathbf{w})^T s &+ \mathcal{R}\mathcal{W}\frac{\partial\mathbf{w}}{\partial x_1}\otimes\mathbf{w} + \mathcal{R}\mathcal{W}\frac{\partial\mathbf{w}}{\partial x_1}\otimes\mathbf{e}_1\Big] =: \operatorname{div}\mathbf{C}(\mathbf{H},\mathbf{w},s)\end{aligned} \tag{4.63}$$

assuming $\mathbf{f} = \operatorname{div}\mathbf{H}$. Therefore

$$\|\mathbf{B}(\mathbf{f},\mathbf{w},s)\|_{k-1,2} \le \|\nabla\mathbf{C}(\mathbf{H},\mathbf{w},s)\|_{k-1,2}\,. \tag{4.64}$$

Moreover, one can rewrite the transport equation (4.7) in the following way due to the fact that $\operatorname{div}\mathbf{w} = 0$

$$\mathbf{z} = \operatorname{div}\,(-\mathcal{W}\mathbf{z}\otimes(\mathbf{w}+\mathbf{e}_1) + \mathbf{C}(\mathbf{H},\mathbf{w},s)) = \operatorname{div}\mathbf{Z}, \tag{4.65}$$

thus $\mathbf{z}$ is in fact a divergence of some quantity which we denote by $\mathbf{Z}$. This enables us to estimate also the seminorm

$$|\mathbf{z}|_{-1,2} \le C\,\|\mathbf{Z}\|_2 \le C(\|\mathbf{C}(\mathbf{H},\mathbf{w},s)\|_2 + (1 + \|\mathbf{w}\|_\infty)\,\|\mathbf{z}\|_2). \tag{4.66}$$

Plugging this together with (4.61), (4.62) and (4.64) we end up with

$$\|(\mathbf{u},q)\|_{V_k} \le C(1 + \|\mathbf{w}\|_\infty)\,\|\mathbf{C}(\mathbf{H},\mathbf{w},s)\|_{k,2}\,. \tag{4.67}$$

We estimate $\|\mathbf{w}\|_\infty$ using Lemma 6.1.15 and all quadratic terms of $\mathbf{C}(\mathbf{H},\mathbf{w},s)$ are easily estimated using Lemma 6.1.16 while the linear terms are trivial. Thus we end up with

$$\|(\mathbf{u},q)\|_{V_k} \le C(1+(R+R^2)(\mathcal{R}^{\frac{1}{2}}+\mathcal{W}\mathcal{R}^{\frac{3}{4}}+\mathcal{W})) \le R \tag{4.68}$$

taking $R$ sufficiently large and $\mathcal{R}$ and $\mathcal{W}$ sufficiently small.

Next, we have to verify that $\mathcal{T}$ is a contraction on $V_{k-1}$. We take couples $(\mathbf{w}^1,s^1),(\mathbf{w}^2,s^2)\in V_k$ and denote corresponding images by $(\mathbf{u}^1,q^1)$ and $(\mathbf{u}^2,q^2)$, with $\mathbf{z}^1$, $\mathbf{z}^2$ being corresponding solutions to the transport equation. We denote similarly as in previous chapters $\xi^{12}=\xi^1-\xi^2$ for any quantity $\xi$. We have to prove that

$$\left\|(\mathbf{u}^{12},q^{12})\right\|_{V_{k-1}} \le \rho \left\|(\mathbf{w}^{12},s^{12})\right\|_{V_{k-1}} \tag{4.69}$$

for some $\rho<1$. For that reason we work with the differences of equations for $\mathbf{u}^1$ and $\mathbf{u}^2$ and for $\mathbf{z}^1$ and $\mathbf{z}^2$. The Oseen equation is linear and therefore the obtained equation for $\mathbf{u}^{12}$ is again an Oseen equation

$$\begin{aligned} -\Delta\mathbf{u}^{12}+\mathcal{R}\frac{\partial\mathbf{u}^{12}}{\partial x_1}+\nabla q^{12} &= \mathbf{z}^{12} \qquad \text{in } \Omega \\ \operatorname{div}\mathbf{u}^{12} &= 0 \qquad \text{in } \Omega \\ \mathbf{u}^{12} &= \mathbf{0} \qquad \text{on } \partial\Omega \\ \mathbf{u}^{12} &\to \mathbf{0} \qquad \text{as } |\mathbf{x}|\to\infty. \end{aligned} \tag{4.70}$$

The equation for $\mathbf{z}^{12}$ is again a steady transport equation but the situation is more complicated as there appear quadratic terms. We have

$$\begin{aligned} &\mathbf{z}^{12}+\mathcal{W}[(\mathbf{w}^1+\mathbf{e}_1)\cdot\nabla]\mathbf{z}^{12} = \\ &\qquad = \mathcal{W}\mathbf{w}^{12}\cdot\nabla\mathbf{z}^2+\operatorname{div}\left(\mathbf{C}(\mathbf{H},\mathbf{w}^1,s^1)-\mathbf{C}(\mathbf{H},\mathbf{w}^2,s^2)\right), \end{aligned} \tag{4.71}$$

where the differences of according quadratic terms can be rewritten similarly as we did with the convective term. Now the procedure is similar as in the previous step where we have proved that $\mathcal{T}$ maps balls into themselves. The estimates work basically in the same way and the smallness of the contraction constant is achieved again by taking $\mathcal{R}$ and $\mathcal{W}$ sufficiently small.

Let us emphasize that as a part of the proof we obtained the following estimates

$$\begin{aligned} \mathcal{R}^{\frac{1}{4}}\|\mathbf{u}_n\|_4+\|\nabla\mathbf{u}_n\|_{k,2}+\|q_n\|_{k,2} &\le K \\ \|\mathbf{C}(\mathbf{H},\mathbf{u}_n,q_n)\|_{k,2} &\le K \end{aligned} \tag{4.72}$$

for all $n\ge 0$, for some constant $K>0$ depending only on $\mathcal{R}_0,\mathcal{W}_0$ and $\mathbf{H}$. $\square$

### 4.3.2 Weighted estimates

In this section we study weighted estimates which are crucial to obtain asymptotic behaviour of the solution. As we have used Theorem 6.3.2 to prove existence of solutions in Sobolev spaces, it is now sufficient to prove that $\mathcal{T}$ maps sufficiently large balls in proper weighted spaces into themselves. Then choosing $(\mathbf{u}_1, q_1)$ from this ball the solution, as the limit of the sequence $(\mathbf{u}_n, q_n)$, belongs to the same ball. We estimate the sequence in the following space

$$V = \Big\{(\mathbf{u}, q) : \mathbf{u} \in L^p(\Omega, \mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot, \mathcal{R})), \nabla\mathbf{u}, \nabla^2\mathbf{u} \in L^p(\Omega, \mu_{1-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot, \mathcal{R})),$$
$$q, \nabla q \in L^p(\Omega, \mu_{\frac{1}{2}-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot, \mathcal{R}))\Big\} \tag{4.73}$$

with the norm

$$\|(\mathbf{u}, q)\|_V = \|\mathbf{u}\|_{p,\mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot,\mathcal{R}),\Omega} + \left\|\nabla\mathbf{u}, \nabla^2\mathbf{u}\right\|_{p,\mu_{1-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot,\mathcal{R}),\Omega} +$$
$$+ \|q, \nabla q\|_{p,\mu_{\frac{1}{2}-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot,\mathcal{R}),\Omega}, \tag{4.74}$$

where $p$ is sufficiently large and $\omega < \frac{1}{2} - \frac{3}{2p}$.

Let us mention that using Lemma 4.2.12 we get $\mathbf{u} \in L^\infty(\Omega, \mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot, \mathcal{R}))$ with $p$ arbitrarily large and therefore almost the same asymptotic structure as $\boldsymbol{\mathcal{O}}$.

Let us assume

$$\|(\mathbf{w}, s)\|_V \le C_0, \tag{4.75}$$

where $C_0$ is sufficiently large constant which will be determined later. It is important to mention that this constant is determined by the estimates (4.72) and is independent of $\mathcal{R}$ and $\mathcal{W}$.

Our aim is to prove that also

$$\|\mathcal{T}(\mathbf{w}, s)\|_V = \|(\mathbf{u}, q)\|_V \le C_0. \tag{4.76}$$

We recall that we also assume that $(\mathbf{w}, s)$ satisfy (4.72).

Throughout the rest of this chapter we will use the following notation to simplify things

$$X^\omega = L^p(\Omega, \mu_{1-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot, \mathcal{R})) \tag{4.77}$$

We start with estimating both $\mathbf{B}(\mathbf{f},\mathbf{w},s)$ and $\mathbf{C}(\mathbf{H},\mathbf{w},s)$ in $X^{2\omega}$. Due to presence of the Reynolds and Weissenberg numbers in front of each term on the right hand side it is sufficient to show the presence of all terms in $X^{2\omega}$, smallness of these terms is achieved by assuming $\mathcal{R},\mathcal{W}$ sufficiently small. We will proceed term by term and denote the terms on the right hand side of (4.10) by $B_1,...,B_6$ and the corresponding terms of $\mathbf{C}$ by $C_1,...,C_6$. First we use assumption

$$\mathbf{f},\mathbf{H}\in X^{2\omega}, \tag{4.78}$$

which allows us to estimate $B_1, C_1$ in $X^{2\omega}$. We estimate $B_2$ in the following way using Hölder inequality

$$\|\mathbf{w}\nabla\mathbf{w}\|^p_{X^{2\omega}} \le \|\mathbf{w}\|^p_{L^\infty(\Omega,\mu^{1-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \|\nabla\mathbf{w}\|^p_{X^\omega} \left\|\eta^{3-p}_{2-p}(\cdot,\mathcal{R})\right\|_{L^\infty(\Omega)}. \tag{4.79}$$

The last term is finite for $p>3$ and therefore using Lemma 4.2.12

$$\|B_2\|_{X^{2\omega}} \le C\mathcal{R}C_0^2. \tag{4.80}$$

We proceed in the similar way also in the divergence form case. Here

$$\|\mathbf{w}\otimes\mathbf{w}\|^p_{X^{2\omega}} \le \|\mathbf{w}\|^p_{L^\infty(\Omega,\mu^{1-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \|\mathbf{w}\|^p_{L^p(\Omega,\mu^{1-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \left\|\eta^{3-\frac{p}{2}}_{2-p}(\cdot,\mathcal{R})\right\|_{L^\infty(\Omega)} \tag{4.81}$$

and therefore for $p>6$ we get

$$\|C_2\|_{X^{2\omega}} \le C\mathcal{R}C_0^2. \tag{4.82}$$

Similar procedure works also for terms $B_3$, $B_4$, $B_5$ and $C_3$, $C_4$, $C_5$, we only show the estimates for $B_3$ and $B_4$ which are most restritive:

$$\begin{aligned}&\left\|\nabla^2\mathbf{w}\nabla\mathbf{w}\right\|^p_{X^{2\omega}} \le\\ &\le \|\nabla\mathbf{w}\|^p_{L^\infty(\Omega,\mu^{\frac{3}{2}-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \left\|\nabla^2\mathbf{w}\right\|^p_{L^p(\Omega,\mu^{\frac{3}{2}-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \left\|\eta^{3-\frac{3p}{2}}_{2-p}(\cdot,\mathcal{R})\right\|_{L^\infty(\Omega)}\end{aligned} \tag{4.83}$$

$$\begin{aligned}&\|\nabla\mathbf{w}\nabla s\|^p_{X^{2\omega}} \le\\ &\quad\le \|\nabla\mathbf{w}\|^p_{L^\infty(\Omega,\mu^{\frac{3}{2}-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \|\nabla s\|^p_{L^p(\Omega,\mu^{\frac{3}{2}-\frac{3}{p},\omega}_{\frac{1}{2}-\frac{2}{p}}(\cdot,\mathcal{R}))} \left\|\eta^{3-\frac{3p}{2}}_{2-\frac{p}{2}}(\cdot,\mathcal{R})\right\|_{L^\infty(\Omega)}\end{aligned} \tag{4.84}$$

Linear terms $B_6$, $C_6$ are trivial. We get

$$\begin{aligned} \|B_3, B_4, C_3, C_4\|_{X^{2\omega}} &\leq C\mathcal{W}C_0^2, \\ \|B_5, C_5\|_{X^{2\omega}} &\leq C\mathcal{R}\mathcal{W}C_0^2, \\ \|B_6, C_6\|_{X^{2\omega}} &\leq \mathcal{R}^{1-\omega}\mathcal{W}C_0 \end{aligned} \tag{4.85}$$

and putting all calculations together we end up with

$$\begin{aligned} \|\mathbf{B}(\mathbf{f}, \mathbf{w}, s)\|_{X^{2\omega}} &\leq C((\mathcal{R}+\mathcal{W})C_0^2 + \mathcal{R}^{1-\omega}\mathcal{W}C_0), \\ \|\mathbf{C}(\mathbf{H}, \mathbf{w}, s)\|_{X^{2\omega}} &\leq C((\mathcal{R}+\mathcal{W})C_0^2 + \mathcal{R}^{1-\omega}\mathcal{W}C_0), \end{aligned} \tag{4.86}$$

for $p > 6$. Now we can use Theorem 4.2.14 on the equation (4.10) to get

$$\|\mathbf{z}\|_{X^{2\omega}} \leq C\,\|\mathbf{B}(\mathbf{f}, \mathbf{w}, s)\|_{X^{2\omega}} \leq C((\mathcal{R}+\mathcal{W})C_0^2 + \mathcal{R}^{1-\omega}\mathcal{W}C_0). \tag{4.87}$$

Moreover, as we have done before, we can write the equation (4.10) in the following form

$$\mathbf{z} = \operatorname{div}\,[\mathbf{C}(\mathbf{H}, \mathbf{w}, s) - \mathcal{W}\mathbf{z} \otimes (\mathbf{w} + \mathbf{e}_1)] \tag{4.88}$$

since $\operatorname{div}(\mathbf{w} + \mathbf{e}_1) = 0$. Hence $\mathbf{z} = \operatorname{div}\mathbf{Z}$ for some tensor $\mathbf{Z}$ and

$$\begin{aligned} \|\mathbf{Z}\|_{X^{2\omega}} \leq \|\mathbf{C}(\mathbf{H}, \mathbf{w}, s)\|_{X^{2\omega}} + \mathcal{W}\,\|\mathbf{z}\|_{X^{2\omega}}\,\|(\mathbf{w} + \mathbf{e}_1)\|_\infty \leq \\ \leq C((\mathcal{R}+\mathcal{W})C_0^2 + \mathcal{R}^{1-\omega}\mathcal{W}C_0). \end{aligned} \tag{4.89}$$

Now we can proceed with the Oseen equation (4.11) with the right hand side $\mathbf{z} = \operatorname{div}\mathbf{Z}$. We use the integral representation formulas (4.25) - (4.27) and (4.28) - (4.29) to estimate $(\mathbf{u}, q)$ in $V$. We can split $\mathbf{u}$ into $\mathbf{u} = \mathbf{u}^V + \mathbf{u}^S$, where $\mathbf{u}^V$ denotes the volume integral and $\mathbf{u}^S$ denotes the surface integrals. Similarly we split $\nabla\mathbf{u}$, $\nabla^2\mathbf{u}$, $q$ and $\nabla q$.

We start with the estimates of the volume parts. For the estimate of $\mathbf{u}^V \in L^p(\Omega, \mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot, \mathcal{R}))$ we use Theorem 4.2.7 and its Corollary and get

$$\begin{aligned} \left\|\mathbf{u}^V\right\|_{L^p(\mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \leq C\mathcal{R}^{-1+\omega}\,\|\mathbf{Z}\|_{X^{2\omega}} \leq \\ \leq C((\mathcal{R}^\omega + \mathcal{R}^{-1+\omega}\mathcal{W})C_0^2 + \mathcal{W}C_0), \end{aligned} \tag{4.90}$$

which can be made sufficiently small by choosing $\mathcal{R}, \mathcal{W}$ small.

For the estimates of $(\nabla\mathbf{u})^V$ and $(\nabla^2\mathbf{u})^V$ we use Theorem 4.2.5 and its Corollary and get

$$\left\|(\nabla\mathbf{u})^V\right\|_{L^p(\mu_{1-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \le C\mathcal{R}^\omega \left\|\mathbf{Z}\right\|_{X^{2\omega}} \le$$
$$\le C((\mathcal{R}^{1+\omega}+\mathcal{R}^\omega\mathcal{W})C_0^2+\mathcal{R}\mathcal{W}C_0) \quad (4.91)$$

$$\left\|(\nabla^2\mathbf{u})^V\right\|_{L^p(\mu_{1-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \le C\mathcal{R}^\omega \left\|\mathbf{z}\right\|_{X^{2\omega}} \le$$
$$\le C((\mathcal{R}^{1+\omega}+\mathcal{R}^\omega\mathcal{W})C_0^2+\mathcal{R}\mathcal{W}C_0). \quad (4.92)$$

Again, terms on the right hand sides can be made sufficiently small by choosing $\mathcal{R},\mathcal{W}$ small.

For the estimates of $q^V$ and $(\nabla q)^V$ we use Theorem 4.2.9 and its Corollary and we obtain

$$\left\|q^V\right\|_{L^p(\mu_{\frac{1}{2}-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \le C\left\|\mathbf{Z}\right\|_{L^p(\mu_{\frac{1}{2}-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \le$$
$$\le C\left\|\mathbf{Z}\right\|_{X^{2\omega}} \le C((\mathcal{R}+\mathcal{W})C_0^2+\mathcal{R}^{1-\omega}\mathcal{W}C_0) \quad (4.93)$$

$$\left\|(\nabla q)^V\right\|_{L^p(\mu_{\frac{1}{2}-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \le C\left\|\mathbf{z}\right\|_{L^p(\mu_{\frac{1}{2}-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \le$$
$$\le C\left\|\mathbf{z}\right\|_{X^{2\omega}} \le C((\mathcal{R}+\mathcal{W})C_0^2+\mathcal{R}^{1-\omega}\mathcal{W}C_0). \quad (4.94)$$

The terms on the right hand sides can be made small same way as before.

Next, we proceed with the surface integrals. Here we distinguish three cases

$$\begin{aligned}\Omega_1 &= \{\mathbf{x}\in\Omega, |\mathbf{x}|\le 1\}\\ \Omega_2 &= \left\{\mathbf{x}\in\Omega, 1\le|\mathbf{x}|\le\frac{1}{\mathcal{R}}\right\}\\ \Omega_3 &= \left\{\mathbf{x}\in\Omega, |\mathbf{x}|\ge\frac{1}{\mathcal{R}}\right\}.\end{aligned} \quad (4.95)$$

In the case $\Omega_1$ all our weights $\sim 1$ and we do not use the integral representation formulas. We rather use the following estimate

$$\|\mathbf{u}\|_{L^p(\Omega_1)} \le C(1+\|\nabla\mathbf{u}\|_{W^{1,2}(\Omega_1)}) \le C(1+\|\nabla\mathbf{u}\|_{W^{1,2}(\Omega)}) \le C(1+K), \quad (4.96)$$

which is due to Friedrichs inequality 6.1.8 and (4.72). Arising term $C(1+K)$ can be made small in comparison with $C_0$ by choosing $C_0$ large enough. Together with (4.90) we get

$$\left\|\mathbf{u}^S\right\|_{L^p(\Omega_1,\mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \leq C(1+K) + C((\mathcal{R}^\omega + \mathcal{R}^{-1+\omega}\mathcal{W})C_0^2 + \mathcal{W}C_0). \quad (4.97)$$

We use analogous procedure also for $\nabla\mathbf{u}, \nabla^2\mathbf{u}$ and get

$$\left\|\nabla\mathbf{u}, \nabla^2\mathbf{u}\right\|_{L^p(\Omega_1)} \leq C(1 + \|\nabla\mathbf{u}\|_{W^{3,2}(\Omega_1)}) \leq \\ \leq C(1 + \|\nabla\mathbf{u}\|_{W^{3,2}(\Omega)}) \leq C(1+K) \quad (4.98)$$

and therefore

$$\left\|(\nabla\mathbf{u})^S, (\nabla^2\mathbf{u})^S\right\|_{L^p(\Omega_1,\mu_{1-\frac{2}{p}}^{\frac{3}{2}-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \leq \\ \leq C(1+K) + C((\mathcal{R}^{1+\omega} + \mathcal{R}^\omega\mathcal{W})C_0^2 + \mathcal{R}\mathcal{W}C_0). \quad (4.99)$$

Analogously for the pressure.

Next, we proceed with the case $\Omega_2$. We start with $\mathbf{u}^S$ and denote four terms in the surface integral (4.25) by $\mathbf{u}^{S,1}$, ..., $\mathbf{u}^{S,4}$. For $\mathbf{u}^{S,1}$ we have

$$\left|\mathbf{u}^{S,1}(\mathbf{x})\right|^p |\mathbf{x}|^{p\omega} (1+\mathcal{R}|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \leq \\ \leq \mathcal{R}^p |\mathbf{x}|^{p\omega} (1+\mathcal{R}|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \left|\int_{\partial\Omega} \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y},\mathcal{R})\mathrm{d}\mathbf{y}\right|^p \leq \\ \leq C\mathcal{R}^p |\mathbf{x}|^{p\omega} (1+\mathcal{R}|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \left|\mathcal{O}_{ij}(\mathbf{x},\mathcal{R}) + \nabla\mathcal{O}_{ij}(\frac{\mathbf{x}}{2},\mathcal{R})\right|^p \leq \\ \leq C\mathcal{R}^p(1+\mathcal{R}|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \left(\frac{1}{|\mathbf{x}|^{p-p\omega}} + \frac{1}{|\mathbf{x}|^{2p-p\omega}}\right), \quad (4.100)$$

where the crucial estimate is the following

$$\left|\nabla^k \boldsymbol{\mathcal{O}}(\mathbf{x},\mathcal{R})\right| \leq C\frac{\mathcal{R}^{\frac{k}{2}}}{|\mathbf{x}|^{1+\frac{k}{2}}} \quad (4.101)$$

for $k \geq 0$. We use this estimate throughout the rest of the procedure in the case $\Omega_2$. Therefore

$$\left\|\mathbf{u}^{S,1}\right\|^p_{L^p(\Omega_2,\mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot,\mathcal{R}))} \leq C\mathcal{R}^p \int_{\Omega_2} \left(\frac{1}{|\mathbf{x}|^{p-p\omega}} + \frac{1}{|\mathbf{x}|^{2p-p\omega}}\right)\mathrm{d}\mathbf{x}. \quad (4.102)$$

Arising functions are integrable and $\int_{\Omega_2} |\mathbf{x}|^{p\omega-p} \,\mathrm{d}\mathbf{x} \leq \frac{4\pi}{p-p\omega-3} \leq 4\pi$ for $p > 6$ and $\omega < \frac{1}{2} - \frac{3}{2p}$, i.e. integrals of such functions over $\Omega_2$ are bounded independently of $\mathcal{R}$ by universal constant $4\pi$. Therefore this term can be estimated choosing $C_0$ large and at this point we do not require $\mathcal{R}$ to be small, even if we have $\mathcal{R}^p$ at our disposal. This fact will play a role in estimating $\mathbf{u}^{S,2}$, $\mathbf{u}^{S,3}$ and $\mathbf{u}^{S,4}$, where there is no power of $\mathcal{R}$ available.

Next for $\mathbf{u}^{S,2}$ we proceed similarly

$$\begin{aligned}
&\left|\mathbf{u}^{S,2}(\mathbf{x})\right|^p |\mathbf{x}|^{p\omega} (1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \leq \\
&\qquad \leq C\,|\mathbf{x}|^{p\omega}(1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \times \\
&\qquad \times \left|\nabla\mathcal{O}_{ij}(\mathbf{x},\mathcal{R}) + \nabla^2\mathcal{O}_{ij}(\frac{\mathbf{x}}{2},\mathcal{R}) + e_i(\mathbf{x}) + \nabla e_i(\frac{\mathbf{x}}{2})\right|^p \leq \\
&\qquad \leq C(1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2}\left(\frac{1}{|\mathbf{x}|^{2p-p\omega}} + \frac{1}{|\mathbf{x}|^{3p-p\omega}}\right)
\end{aligned} \tag{4.103}$$

and therefore

$$\left\|\mathbf{u}^{S,2}\right\|^p_{L^p(\Omega_2,\mu^{1-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \leq C\int_{\Omega_2}\left(\frac{1}{|\mathbf{x}|^{2p-p\omega}} + \frac{1}{|\mathbf{x}|^{3p-p\omega}}\right)\mathrm{d}\mathbf{x} \tag{4.104}$$

and we are in similar situation as in the case $\mathbf{u}^{S,1}$.

We treat $\mathbf{u}^{S,3}$ and $\mathbf{u}^{S,4}$ together

$$\begin{aligned}
&\left|\mathbf{u}^{S,3} + \mathbf{u}^{S,4}(\mathbf{x})\right|^p |\mathbf{x}|^{p\omega}(1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \leq \\
&\leq C\,|\mathbf{x}|^{p\omega}(1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2}\left|\mathcal{O}_{ij}(\mathbf{x},\mathcal{R}) + \nabla\mathcal{O}_{ij}(\frac{\mathbf{x}}{2},\mathcal{R})\right|^p \times \\
&\qquad \times\left(\|\nabla\mathbf{u}\|_{W^{1,2}(\Omega)} + \|q\|_{W^{1,2}(\Omega)} + \|\mathbf{Z}\|_{W^{1,2}(\Omega)}\right) \leq \\
&\qquad \leq CK(1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2}\left(\frac{1}{|\mathbf{x}|^{p-p\omega}} + \frac{1}{|\mathbf{x}|^{2p-p\omega}}\right)
\end{aligned} \tag{4.105}$$

and therefore

$$\left\|\mathbf{u}^{S,3} + \mathbf{u}^{S,4}\right\|^p_{L^p(\Omega_2,\mu^{1-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \leq CK\int_{\Omega_2}\left(\frac{1}{|\mathbf{x}|^{p-p\omega}} + \frac{1}{|\mathbf{x}|^{2p-p\omega}}\right)\mathrm{d}\mathbf{x}. \tag{4.106}$$

Here we have used also (4.72).

For higher gradients of $\mathbf{u}$ and pressure and its gradient we use similar procedure, in this case higher gradients are even easier to estimate.

We finish with the case $\Omega_3$. Here the situation is a little different. We have

$$\begin{aligned}
&\left|\mathbf{u}^{S,1}(\mathbf{x})\right|^p |\mathbf{x}|^{p\omega} (1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \leq \\
&\leq \mathcal{R}^p |\mathbf{x}|^{p\omega} (1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \left|\int_{\partial\Omega} \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y},\mathcal{R})\mathrm{d}\mathbf{y}\right|^p \leq \\
&\leq C\mathcal{R}^p |\mathbf{x}|^{p\omega} (1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2} \left|\mathcal{O}_{ij}(\mathbf{x},\mathcal{R}) + \nabla\mathcal{O}_{ij}(\frac{\mathbf{x}}{2},\mathcal{R})\right|^p \leq \\
&\leq C\mathcal{R}^p |\mathbf{x}|^{p\omega} (1+\mathcal{R}\,|\mathbf{x}|)^{p-3-p\omega}(1+s(\mathcal{R}\mathbf{x}))^{p-2}\times \\
&\times \left(\frac{\mathcal{R}^p}{|\mathcal{R}\mathbf{x}|^p (1+s(\mathcal{R}\mathbf{x}))^p} + \frac{\mathcal{R}^{2p}}{|\mathcal{R}\mathbf{x}|^{\frac{3p}{2}} (1+s(\mathcal{R}\mathbf{x}))^{\frac{3p}{2}}}\right). \qquad (4.107)
\end{aligned}$$

Here we have used that

$$\left|\nabla^k \boldsymbol{\mathcal{O}}(\mathbf{x},\mathcal{R})\right| \leq C\frac{\mathcal{R}^{\frac{k}{2}}}{|\mathbf{x}|^{1+\frac{k}{2}} (1+s(\mathcal{R}\mathbf{x}))^{1+\frac{k}{2}}} \qquad (4.108)$$

for $k \geq 0$. Therefore

$$\begin{aligned}
&\left\|\mathbf{u}^{S,1}\right\|^p_{L^p(\Omega_3,\mu^{1-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \leq C\mathcal{R}^{2p-p\omega}\int_{\Omega_3}\left(\frac{1}{(1+|\mathcal{R}\mathbf{x}|)^3}\frac{1}{(1+s(\mathcal{R}\mathbf{x}))^2}\right)\mathrm{d}\mathbf{x} + \\
&+ C\mathcal{R}^{3p-p\omega}\int_{\Omega_3}\left(\frac{1}{(1+|\mathcal{R}\mathbf{x}|)^{3+\frac{p}{2}}}\frac{1}{(1+s(\mathcal{R}\mathbf{x}))^{2+\frac{p}{2}}}\right)\mathrm{d}\mathbf{x} \leq \\
&\leq C\mathcal{R}^{2p-p\omega-3}\int_{\mathbb{R}^3}\eta^{-3}_{-2}(\mathbf{y})\mathrm{d}\mathbf{y} + C\mathcal{R}^{3p-p\omega-3}\int_{\mathbb{R}^3}\eta^{-3-\frac{p}{2}}_{-2-\frac{p}{2}}(\mathbf{y})\mathrm{d}\mathbf{y}. \qquad (4.109)
\end{aligned}$$

Arising integrals are finite due to Lemma 4.2.4.

For $\mathbf{u}^{S,2}$ we obtain in the similar way

$$\begin{aligned}
&\left\|\mathbf{u}^{S,2}\right\|^p_{L^p(\Omega_3,\mu^{1-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \leq \\
&\leq C\mathcal{R}^{2p-p\omega}\int_{\Omega_3}\left(\frac{1}{(1+|\mathcal{R}\mathbf{x}|)^{3+\frac{p}{2}}}\frac{1}{(1+s(\mathcal{R}\mathbf{x}))^{2+\frac{p}{2}}}\right)\mathrm{d}\mathbf{x}+ \\
&+C\mathcal{R}^{2p-p\omega}\int_{\Omega_3}\left(\frac{1}{(1+|\mathcal{R}\mathbf{x}|)^{3+p}}\frac{1}{(1+s(\mathcal{R}\mathbf{x}))^{2-p}}\right)\mathrm{d}\mathbf{x}+ \\
&+C\mathcal{R}^{3p-p\omega}\int_{\Omega_3}\left(\frac{1}{(1+|\mathcal{R}\mathbf{x}|)^{3+p}}\frac{1}{(1+s(\mathcal{R}\mathbf{x}))^{2+p}}\right)\mathrm{d}\mathbf{x}+ \\
&+C\mathcal{R}^{3p-p\omega}\int_{\Omega_3}\left(\frac{1}{(1+|\mathcal{R}\mathbf{x}|)^{3+2p}}\frac{1}{(1+s(\mathcal{R}\mathbf{x}))^{2-p}}\right)\mathrm{d}\mathbf{x}. \qquad (4.110)
\end{aligned}$$

Treating $\mathbf{u}^{S,3}$ and $\mathbf{u}^{S,4}$ together we get

$$\begin{aligned}
&\left\|\mathbf{u}^{S,3}+\mathbf{u}^{S,4}\right\|^p_{L^p(\Omega_3,\mu^{1-\frac{3}{p},\omega}_{1-\frac{2}{p}}(\cdot,\mathcal{R}))} \leq \\
&\leq C\mathcal{R}^{p-p\omega}\int_{\Omega_3}\left(\frac{1}{(1+|\mathcal{R}\mathbf{x}|)^{3}}\frac{1}{(1+s(\mathcal{R}\mathbf{x}))^{2}}\right)\mathrm{d}\mathbf{x}+ \\
&+C\mathcal{R}^{2p-p\omega}\int_{\Omega_3}\left(\frac{1}{(1+|\mathcal{R}\mathbf{x}|)^{3+\frac{p}{2}}}\frac{1}{(1+s(\mathcal{R}\mathbf{x}))^{2+\frac{p}{2}}}\right)\mathrm{d}\mathbf{x} \leq \\
&\leq C\mathcal{R}^{p-p\omega-3}\int_{\mathbb{R}^3}\eta^{-3}_{-2}(\mathbf{y})\mathrm{d}\mathbf{y}+C\mathcal{R}^{2p-p\omega-3}\int_{\mathbb{R}^3}\eta^{-3-\frac{p}{2}}_{-2-\frac{p}{2}}(\mathbf{y})\mathrm{d}\mathbf{y}. \qquad (4.111)
\end{aligned}$$

Again, we proceed similarly with the estimates of gradients of $\mathbf{u}$ and pressure and its gradient. Putting all calculations together, choosing first $C_0$ sufficiently large and then $\mathcal{R},\mathcal{W}$ sufficiently small we finally end up with

$$\|(\mathbf{u},q)\|_V < C_0 \qquad (4.112)$$

and the proof of Theorem 4.3.1 is finished.

The results of this chapter are submitted for publication in [19].

# Chapter 5

# Global existence for a stress diffusion model

## 5.1 The model

The model studied in this chapter was introduced in 1.4.4. It combines several features which were not used in previous chapters, namely non-constant shear rate dependent viscosity and nonlinear stress diffusion. The system of equations consists of the linear momentum equation (1.19), divergence free condition (1.16) and a constitutive relations (1.55) and (1.57). For simplicity we set $\rho = 1$, $\lambda_1 = 1$ and we assume zero external force $\mathbf{f} = \mathbf{0}$. Also, for the sake of simplicity of notation, we will use in this chapter $\mathbf{T}$ instead of $\mathbf{T}_e$ for the elastic part of the stress tensor. Thus we get the following system

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \operatorname{div}\,[\mu(\mathbf{D})\mathbf{D}] &= \operatorname{div} \mathbf{T} \\
\operatorname{div} \mathbf{v} &= 0 \\
\frac{\partial \mathbf{T}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{T} - \varepsilon \operatorname{div}\,[\gamma(\nabla \mathbf{T})\nabla \mathbf{T}] + \varepsilon\, |\mathbf{T}|^{q-2}\, \mathbf{T} + \mathbf{T} &= \\
= 2\mu_0 \mathbf{D} + \mathbf{W}\mathbf{T} - \mathbf{T}\mathbf{W} + a(\mathbf{T}\mathbf{D} + \mathbf{D}\mathbf{T}),
\end{aligned} \tag{5.1}$$

where $\mathbf{D} = \mathbf{D}(\mathbf{v})$ is the symmetric part of the velocity gradient and $\mathbf{W} = \mathbf{W}(\mathbf{v})$ is the skew-symmetric part, functions $\mu$ and $\gamma$ are given and $\varepsilon, \mu_0 > 0$, $a \in [-1, 1]$ are constants. We will use the following notation in the rest of this chapter

$$\mathbf{B}(\mathbf{v},\mathbf{T}) = \mathbf{WT} - \mathbf{TW} + a(\mathbf{TD} + \mathbf{DT}). \tag{5.2}$$

We have added term $\varepsilon\,|\mathbf{T}|^{q-2}\,\mathbf{T}$ to the equation for $\mathbf{T}$. We discuss the presence of this term in the final remark of this chapter.

We add initial conditions to system (5.1)

$$\left.\begin{aligned} \mathbf{v}(0,\mathbf{x}) &= \mathbf{v}_0(\mathbf{x}) \\ \mathbf{T}(0,\mathbf{x}) &= \mathbf{T}_0(\mathbf{x}) \end{aligned}\right\} \quad \text{in } \Omega \tag{5.3}$$

and consider two types of boundary conditions. Either we assume the periodic case, i.e. $\Omega = (0,L)^3$ and all functions are space periodic, or we assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain and prescribe the so–called no–stick boundary condition for the velocity. We also have to prescribe a boundary condition for the elastic stress tensor $\mathbf{T}$ due to the presence of stress diffusion. Here we prescribe Neumann boundary condition as the homogenous Dirichlet boundary condition for $\mathbf{T}$ is not physical. We have then[1]

$$\left.\begin{aligned} \mathbf{v}\cdot\mathbf{n} &= 0 \\ [\mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})\mathbf{n}]_\tau &= \mathbf{0} \\ \gamma(\nabla\mathbf{T})\frac{\partial\mathbf{T}}{\partial\mathbf{n}} &= \mathbf{0} \end{aligned}\right\} \quad \text{on } \partial\Omega. \tag{5.4}$$

*Remark* 5.1.1. Instead of the no–stick boundary condition we can also work with partial slip boundary condition, where instead of $(5.4)_2$ we prescribe

$$[\mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})\mathbf{n}]_\tau = -\alpha\,[\mathbf{v}]_\tau \tag{5.5}$$

for $\alpha \in (0,\infty)$. This brings only minor changes in proof of the main theorem of this chapter and moreover we do not need to assume domain $\Omega$ not to be axisymmetric as we control also the tangent part of the velocity.

We assume that the function $\mu(\mathbf{D})$ is positive, continuous and such that the tensor $\mu(\mathbf{D})\mathbf{D}$ satisfies the conditions of $(p > 1)$

- $p$-coercivity, i.e. there is $c_1 > 0$ and $\varphi_1(t,\mathbf{x}) \in L^1((0,T)\times\Omega)$ such that

$$\mu(\mathbf{D})\,|\mathbf{D}|^2 \geq c_1\,|\mathbf{D}|^p - \varphi_1 \tag{5.6}$$

  for all $\mathbf{D} \in \mathbb{R}^{3\times3}_{sym}$

[1] The subscript $[\cdot]_\tau$ denotes the tangential component of a vector $[\cdot]$

- $(p-1)$-growth, i.e. there is $c_2 > 0$ and $\varphi_2(t,\mathbf{x}) \in L^{\frac{p}{p-1}}((0,T)\times\Omega)$ such that
$$\mu(\mathbf{D})\,|\mathbf{D}| \leq c_2\,|\mathbf{D}|^{p-1} + \varphi_2 \tag{5.7}$$
for all $\mathbf{D} \in \mathbb{R}^{3\times 3}_{sym}$

- strict monotonicity, i.e.
$$(\mu(\mathbf{D}_1)\mathbf{D}_1 - \mu(\mathbf{D}_2)\mathbf{D}_2) : (\mathbf{D}_1 - \mathbf{D}_2) > 0 \tag{5.8}$$
for all $\mathbf{D}_1, \mathbf{D}_2 \in \mathbb{R}^{3\times 3}_{sym}$, $\mathbf{D}_1 \neq \mathbf{D}_2$.

Typical example of such function is a simple $(p-2)$-th power $\mu(\mathbf{D}) = \beta\,|\mathbf{D}|^{p-2}$, which makes of the corresponing term in the equation $(5.1)_1$ a $p$-laplacian, or some generalization like
$$\mu(\mathbf{D}) = \beta_1\left(\beta_2 + |\mathbf{D}|^2\right)^{\frac{p-2}{2}} \qquad \beta_1, \beta_2 > 0. \tag{5.9}$$

We impose similar conditions on the continuous function $\gamma(\nabla\mathbf{T})$, for $q > 1$ we assume

- $q$-coercivity, i.e. there is $c_3 > 0$ and $\varphi_3(t,\mathbf{x}) \in L^1((0,T)\times\Omega)$ such that
$$\gamma(\nabla\mathbf{T})\,|\nabla\mathbf{T}|^2 \geq c_3\,|\nabla\mathbf{T}|^q - \varphi_3 \tag{5.10}$$
for all $\mathbf{T} \in \mathbb{R}^{3\times 3}_{sym}$

- $(q-1)$-growth, i.e. there is $c_4 > 0$ and $\varphi_4(t,\mathbf{x}) \in L^{\frac{q}{q-1}}((0,T)\times\Omega)$ such that
$$\gamma(\nabla\mathbf{T})\,|\nabla\mathbf{T}| \leq c_4\,|\nabla\mathbf{T}|^{q-1} + \varphi_4 \tag{5.11}$$
for all $\mathbf{T} \in \mathbb{R}^{3\times 3}_{sym}$

- monotonicity, i.e.
$$(\gamma(\nabla\mathbf{T}_1)\nabla\mathbf{T}_1 - \gamma(\nabla\mathbf{T}_2)\nabla\mathbf{T}_2) : (\nabla\mathbf{T}_1 - \nabla\mathbf{T}_2) \geq 0 \tag{5.12}$$
for all $\mathbf{T}_1, \mathbf{T}_2 \in \mathbb{R}^{3\times 3}_{sym}$.

It is sufficient for us to assume monotonicity of $\gamma(\nabla\mathbf{T})\nabla\mathbf{T}$, we do not need strict monotonicity here. Note that this $q$, which characterizes the coercivity and growth of the nonlinear function $\gamma$, and $q$ in the power of term $\varepsilon\,|\mathbf{T}|^{q-2}\,\mathbf{T}$ in the equation $(5.1)_3$ are the same.

## 5.2 Main theorem

Here we state the main theorem of this chapter. First, we introduce some function spaces used in this chapter. We denote in the case of periodic boundary conditions (recall $\Omega = (0, L)^3$)

$$\begin{aligned} V &= \left\{ \mathbf{v}, \mathbf{v} \in C^\infty(\overline{\Omega}), \operatorname{div} \mathbf{v} = 0, \int_\Omega \mathbf{v} \mathrm{d}\mathbf{x} = \mathbf{0}, \mathbf{v} \text{ periodic} \right\} \\ L^2_{BC,div} &= \overline{V}^{\|\cdot\|_{L^2}} \\ W^{1,p}_{BC,div} &= \overline{V}^{\|\cdot\|_{W^{1,p}}} \\ H &= \left\{ \mathbf{T}, \mathbf{T} \in C^\infty(\overline{\Omega}), \mathbf{T} \text{ symmetric, periodic} \right\} \\ L^2_{BC} &= \overline{H}^{\|\cdot\|_{L^2}} \\ W^{1,p}_{BC} &= \overline{H}^{\|\cdot\|_{W^{1,p}}}, \end{aligned} \tag{5.13}$$

while in the case of boundary conditions (5.4) we denote

$$\begin{aligned} V &= \left\{ \mathbf{v}, \mathbf{v} \in C^\infty(\overline{\Omega}), \operatorname{div} \mathbf{v} = 0 \right\} \\ L^2_{BC,div} &= \overline{V}^{\|\cdot\|_{L^2}} \\ W^{1,p}_{BC,div} &= \left\{ \mathbf{v}, \mathbf{v} \in W^{1,p}(\Omega), \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\} \\ H &= \left\{ \mathbf{T}, \mathbf{T} \in C^\infty(\overline{\Omega}), \mathbf{T} \text{ symmetric} \right\} \\ L^2_{BC} &= \overline{H}^{\|\cdot\|_{L^2}} \\ W^{1,p}_{BC} &= \overline{H}^{\|\cdot\|_{W^{1,p}}}. \end{aligned} \tag{5.14}$$

Moreover we denote

$$\begin{aligned} (\mathbf{a}, \mathbf{b}) &= \int_\Omega a_i b_i \mathrm{d}\mathbf{x} \quad \text{for } \mathbf{a}, \mathbf{b} \text{ vectors} \\ (\mathbf{A}, \mathbf{B}) &= \int_\Omega A_{ij} B_{ij} \mathrm{d}\mathbf{x} \quad \text{for } \mathbf{A}, \mathbf{B} \text{ second order tensors} \end{aligned} \tag{5.15}$$

and similarly for higher order tensors.

Next, we introduce the definition of weak solution to the system (5.1) with initial conditions (5.3) and boundary conditions — either periodic ones or (5.4).

**Definition 5.2.1.** Let $\mathbf{v}_0 \in L^2_{BC,div}(\Omega)$, $\mathbf{T}_0 \in L^2_{BC}(\Omega)$. We say that a couple $(\mathbf{v}, \mathbf{T})$ is a weak solution if

$$\begin{aligned} \mathbf{v} &\in L^\infty(0,T,L^2_{BC,div}(\Omega)) \cap L^p(0,T,W^{1,p}_{BC,div}(\Omega)) \\ \mathbf{T} &\in L^\infty(0,T,L^2_{BC}(\Omega)) \cap L^q(0,T,W^{1,q}_{BC}(\Omega)) \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} &\left\langle \frac{\partial \mathbf{v}}{\partial t}(t), \boldsymbol{\phi} \right\rangle_p + (\mathbf{v}(t) \cdot \nabla \mathbf{v}(t), \boldsymbol{\phi}) + \\ &\qquad + (\mu(\mathbf{D}(\mathbf{v}(t)))\mathbf{D}(\mathbf{v}(t)), \mathbf{D}(\boldsymbol{\phi})) = -(\mathbf{T}(t), \mathbf{D}(\boldsymbol{\phi})) \end{aligned} \tag{5.17}$$

$$\begin{aligned} &\left\langle \frac{\partial \mathbf{T}}{\partial t}(t), \boldsymbol{\Phi} \right\rangle_q + (\mathbf{v}(t) \cdot \nabla \mathbf{T}(t), \boldsymbol{\Phi}) + \varepsilon\,(\gamma(\nabla \mathbf{T}(t))\nabla \mathbf{T}(t), \nabla \boldsymbol{\Phi}) + \\ &\qquad + \varepsilon \left( |\mathbf{T}(t)|^{q-2}\,\mathbf{T}(t), \boldsymbol{\Phi} \right) + (\mathbf{T}(t), \boldsymbol{\Phi}) = \\ &\qquad = 2\mu_0\,(\mathbf{D}(\mathbf{v}(t)), \boldsymbol{\Phi}) + (\mathbf{B}(\mathbf{v}(t), \mathbf{T}(t)), \boldsymbol{\Phi}) \end{aligned} \tag{5.18}$$

for almost all $t \in (0,T)$ and

$$\begin{aligned} \forall \boldsymbol{\phi} &\in W^{1,p}_{BC,div}(\Omega) \cap L^\infty(\Omega) \\ \forall \boldsymbol{\Phi} &\in W^{1,q}(\Omega). \end{aligned} \tag{5.19}$$

Here $\langle \cdot,\cdot \rangle_p$ denotes the duality between $W^{1,p}_{BC,div}(\Omega)$ and its dual and $\langle \cdot,\cdot \rangle_q$ denotes the duality between $W^{1,q}(\Omega)$ and its dual.

**Theorem 5.2.1.** *Let $\Omega \subset \mathbb{R}^3$ be either a periodic box $(0,L)^3$ or a bounded domain with a Lipschitz boundary which is not axisymmetric. Let $\mathbf{v}_0 \in L^2_{BC,div}(\Omega)$, $\mathbf{T}_0 \in L^2_{BC}(\Omega)$ and $T > 0$. Let $p > \frac{8}{5}$ and*

$$q > \max \left\{ 3, \frac{11p - 6 + \sqrt{25p^2 - 12p + 36}}{8p - 10} \right\}. \tag{5.20}$$

*Then there exists a weak solution $(\mathbf{v}, \mathbf{T})$ to problem (5.1) with initial conditions (5.3) and boundary conditions either periodic or (5.4).*

*Remark* 5.2.1. We point out the bounds on $q$ for certain key values of $p$. For $p$ near $\frac{8}{5}$, $q > 7.4$, for $p = 2$, $q > 4.5$ and for $p \geq 3$ applies the bound $q > 3$.

*Remark* 5.2.2. The assumption that $\Omega$ is not axisymmetric is due to the fact that we need to use the Korn inequality 6.1.12 for $\mathbf{v}$ such that $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ in the proof. In the case of partial slip boundary conditions this assumption can be omitted.

## 5.3 Proof of the main theorem

### 5.3.1 A priori estimates

The proof is based on the Galerkin method. Key steps are derivation of the a priori estimates and the limiting procedure. Before we start with the proof we mention some observations. It holds

$$\begin{aligned}(\mathbf{WT}-\mathbf{TW}):\mathbf{T}=0\\ \mathbf{DT}:\mathbf{T}=\mathbf{TD}:\mathbf{T}\\ \mathbf{T}:\nabla\mathbf{v}=\mathbf{T}:\mathbf{D}(\mathbf{v})\end{aligned} \tag{5.21}$$

for all $\mathbf{T},\mathbf{D}$ symmetric tensors and $\mathbf{W}$ skew-symmetric tensors.

Let us take $\{\mathbf{w}^i\}_{i=1}^{\infty}\subset L^{\infty}(\Omega)$ orthogonal basis of $W^{1,p}_{BC,div}$ and $\{\mathbf{S}^i\}_{i=1}^{\infty}$ orthogonal basis of $W^{1,q}_{BC}$. We search for the approximative solutions

$$\begin{aligned}\mathbf{v}^n(t,x)&=\sum_{i=1}^{n}c_i^n(t)\mathbf{w}^i(x)\\ \mathbf{T}^n(t,x)&=\sum_{i=1}^{n}d_i^n(t)\mathbf{S}^i(x),\end{aligned} \tag{5.22}$$

which satisfy the weak formulation of the problem

$$\begin{aligned}\left\langle\frac{\partial\mathbf{v}^n}{\partial t}(t),\mathbf{w}^i\right\rangle_p+\left(\mathbf{v}^n(t)\cdot\nabla\mathbf{v}^n(t),\mathbf{w}^i\right)+\\ +\left(\mu(\mathbf{D}(\mathbf{v}^n(t)))\mathbf{D}(\mathbf{v}^n(t)),\mathbf{D}(\mathbf{w}^i)\right)=-\left(\mathbf{T}^n(t),\mathbf{D}(\mathbf{w}^i)\right)\end{aligned} \tag{5.23}$$

$$\begin{aligned}\left\langle\frac{\partial\mathbf{T}^n}{\partial t}(t),\mathbf{S}^i\right\rangle_q+\left(\mathbf{v}^n(t)\cdot\nabla\mathbf{T}^n(t),\mathbf{S}^i\right)+\varepsilon\left(\gamma(\nabla\mathbf{T}^n(t))\nabla\mathbf{T}^n(t),\nabla\mathbf{S}^i\right)+\\ +\varepsilon\left(|\mathbf{T}^n|^{q-2}\,\mathbf{T}^n,\mathbf{S}^i\right)+(\mathbf{T}^n(t),\mathbf{S}^i)=2\mu_0\left(\mathbf{D}(\mathbf{v}^n(t)),\mathbf{S}^i\right)+\\ +\left(\mathbf{B}(\mathbf{v}^n(t),\mathbf{T}^n(t)),\mathbf{S}^i\right)\end{aligned} \tag{5.24}$$

for all test functions $\mathbf{w}^i$, $\mathbf{S}^i$, $i=1,...,n$. For fixed $n$ this yields a system of ordinary differential equations for functions $c_i^n(t)$, $d_i^n(t)$. We can use Carathéodory theory of ordinary differential equations on this system to ensure that there exists a solution on $[0,T^*)$. Moreover if $T^*<T$ then

$\max \left|c_j^n\right|, \left|d_j^n\right| \to \infty$ as $t \to T^*-$. The a priori estimates which we derive later ensure that this does not happen and therefore $T^* = T$.

Let us take $\mathbf{v}^n$ as the test function in the equation (5.23) and $\mathbf{T}^n$ as the test function in the equation (5.24). This means multiplying corresponding ODE's by $c_i^n(t)$ and $d_i^n(t)$ and summing over $i$. We multiply the first equation by $2\mu_0$ and sum both equations and get

$$\begin{aligned} &2\mu_0 \left\langle \frac{\partial \mathbf{v}^n}{\partial t}(t), \mathbf{v}^n(t) \right\rangle_p + \left\langle \frac{\partial \mathbf{T}^n}{\partial t}(t), \mathbf{T}^n(t) \right\rangle_q + \\ &+ 2\mu_0 \left(\mu(\mathbf{D}(\mathbf{v}^n(t)))\mathbf{D}(\mathbf{v}^n(t)), \mathbf{D}(\mathbf{v}^n(t))\right) + \varepsilon \left(\gamma(\nabla \mathbf{T}^n(t))\nabla \mathbf{T}^n(t), \nabla \mathbf{T}^n(t)\right) + \\ &+ \varepsilon \left(|\mathbf{T}^n(t)|^{q-2}\, \mathbf{T}^n(t), \mathbf{T}^n(t)\right) + \left(\mathbf{T}^n(t), \mathbf{T}^n(t)\right) = \left(\mathbf{B}(\mathbf{v}^n(t), \mathbf{T}^n(t)), \mathbf{T}^n(t)\right). \end{aligned} \tag{5.25}$$

Due to the observation (5.21) the right hand side of this equation can be rewritten simply as

$$\left(\mathbf{B}(\mathbf{v}^n, \mathbf{T}^n), \mathbf{T}^n\right) = 2a \left(\mathbf{D}(\mathbf{v}^n)\mathbf{T}^n, \mathbf{T}^n\right) \tag{5.26}$$

and integrating by parts

$$2a \left(\mathbf{D}(\mathbf{v}^n)\mathbf{T}^n, \mathbf{T}^n\right) = -2a \left(\mathbf{v}^n \otimes \operatorname{div} \mathbf{T}^n, \mathbf{T}^n\right) - 2a \left(\nabla \mathbf{T}^n, \mathbf{v}^n \otimes \mathbf{T}^n\right). \tag{5.27}$$

Focusing on the left hand side of (5.25) we can use coercivity assumptions on functions $\mu$ and $\gamma$ together with Korn inequality to get

$$\begin{aligned} &\mu_0 \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{v}^n\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{T}^n\|_{L^2}^2 + 2\mu_0 \|\nabla \mathbf{v}^n\|_{L^p}^p + \varepsilon \|\nabla \mathbf{T}^n\|_{L^q}^q + \varepsilon \|\mathbf{T}^n\|_{L^q}^q + \\ &\qquad + \|\mathbf{T}^n\|_{L^2}^2 \le C + 2a \left|\left(\mathbf{v}^n \otimes \operatorname{div} \mathbf{T}^n, \mathbf{T}^n\right) + \left(\nabla \mathbf{T}^n, \mathbf{v}^n \otimes \mathbf{T}^n\right)\right|. \end{aligned} \tag{5.28}$$

We observe that we have to be able to deal with terms consisting of products of components of $\mathbf{v}^n$, $\mathbf{T}^n$ and first order derivatives of $\mathbf{T}^n$. Using Hölder inequality we get

$$\int_\Omega v_i^n T_{jk}^n \frac{\partial T_{lr}^n}{\partial x_s} \mathrm{d}\mathbf{x} \le \|\mathbf{v}^n\|_{L^M} \|\nabla \mathbf{T}^n\|_{L^q} \|\mathbf{T}^n\|_{L^\alpha}, \tag{5.29}$$

where $M = \frac{3p}{3-p}$ for $p < 3$ while $M$ is arbitrarily large for $p \ge 3$. In both cases $\alpha = \frac{Mq}{Mq-M-q}$. Next we use Sobolev embedding and the fact that $\Omega$ is a bounded domain to get

$$\|\mathbf{v}^n\|_{L^M} \|\nabla \mathbf{T}^n\|_{L^q} \|\mathbf{T}^n\|_{L^\alpha} \le C \|\nabla \mathbf{v}^n\|_{L^p} \|\nabla \mathbf{T}^n\|_{L^q} \|\mathbf{T}^n\|_{L^\beta} \tag{5.30}$$

for $\beta > \alpha$ arbitrary. Interpolating $L^\beta$ between $L^2$ and $W^{1,q}$ yields

$$C \|\nabla \mathbf{v}^n\|_{L^p} \|\nabla \mathbf{T}^n\|_{L^q} \|\mathbf{T}^n\|_{L^\beta} \leq C \|\nabla \mathbf{v}^n\|_{L^p} \|\mathbf{T}^n\|_{W^{1,q}}^{2-\lambda} \|\mathbf{T}^n\|_{L^2}^{\lambda} \tag{5.31}$$

with $\frac{1}{\beta} = \frac{\lambda}{2} + (1-\lambda)\left(\frac{1}{q} - \frac{1}{3}\right)$. Finally we use Young inequality to end up with

$$C \|\nabla \mathbf{v}^n\|_{L^p} \|\nabla \mathbf{T}^n\|_{L^q}^{2-\lambda} \|\mathbf{T}^n\|_{L^2}^{\lambda} \leq \mu_0 \|\nabla \mathbf{v}^n\|_{L^p}^p + \frac{\varepsilon}{2} \|\mathbf{T}^n\|_{W^{1,q}}^q + C \|\mathbf{T}^n\|_{L^2}^2 \tag{5.32}$$

with $\frac{1}{p} + \frac{2-\lambda}{q} + \frac{\lambda}{2} = 1$. First two terms can be absorbed into the left hand side of (5.28)

$$\mu_0 \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{v}^n\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{T}^n\|_{L^2}^2 + \mu_0 \|\nabla \mathbf{v}^n\|_{L^p}^p + \frac{\varepsilon}{2} \|\mathbf{T}^n\|_{W^{1,q}}^q + \|\mathbf{T}^n\|_{L^2}^2 \leq \\ \leq C + C \|\mathbf{T}^n\|_{L^2}^2 \tag{5.33}$$

and the remaining term will be handled using Gronwall inequality. First integrating $\int_0^t \mathrm{d}t$ we see that $c_i^n(t)$, $d_i^n(t)$ are bounded in time and therefore $T^* = T$ and then taking $t = T$ we end up with

$$\mu_0 \|\mathbf{v}^n\|_{L^\infty(0,T,L^2)}^2 + \frac{1}{2} \|\mathbf{T}^n\|_{L^\infty(0,T,L^2)}^2 + \mu_0 \|\nabla \mathbf{v}^n\|_{L^p(0,T,L^p)}^p + \\ + \frac{\varepsilon}{2} \|\mathbf{T}^n\|_{L^q(0,T,W^{1,q})}^q + \|\mathbf{T}^n\|_{L^2(0,T,L^2)}^2 \leq C(1 + \|\mathbf{v}_0\|_{L^2} + \|\mathbf{T}_0\|_{L^2}) =: K_0. \tag{5.34}$$

Note that in particular denoting $s = \frac{5p}{8}$, a priori estimate (5.34) yields that

$$\|\mathbf{v}^n \cdot \nabla \mathbf{v}^n\|_{L^s(\Omega)}^s \leq \|\nabla \mathbf{v}^n\|_{L^p(\Omega)}^p \|\mathbf{v}^n\|_{L^2(\Omega)}^{\frac{p}{4}}, \tag{5.35}$$

where we have used Hölder and interpolation inequalities, and therefore

$$\|\mathbf{v}^n \cdot \nabla \mathbf{v}^n\|_{L^s(0,T,L^s(\Omega))}^s \leq K_0^{1+\frac{p}{4}}. \tag{5.36}$$

We have two conditions which have to be satisfied, one coming from the interpolation inequality (5.31) and one coming from Young inequality (5.32)

$$\begin{aligned} \frac{1}{\beta} &= \frac{\lambda}{2} + (1-\lambda)\left(\frac{1}{q} - \frac{1}{3}\right) \\ 1 &= \frac{1}{p} + \frac{2-\lambda}{q} + \frac{\lambda}{2}. \end{aligned} \tag{5.37}$$

We can express $\lambda$ in terms of $\beta$ and $q$ from the first equation

$$\lambda = \frac{2q - 6 + 6\frac{q}{\beta}}{5q - 6}, \tag{5.38}$$

as $\lambda$ has to be positive we need $q > \frac{3}{1-\frac{3}{\beta}}$ with $\beta$ arbitrarily large, so $q > 3$. Consequently, we express $p$ in terms of $\beta$ and $q$

$$p = \frac{q(5q - 6)}{4q^2 - 11q + 6 - 3\frac{q^2}{\beta} + 6\frac{q}{\beta}}. \tag{5.39}$$

It is easy to observe that inf of this expression for $\beta \in (1, \infty)$, $q \in (3, \infty)$ is $\frac{5}{4}$. For $p > \frac{5}{4}$ we can calculate the lower bound on $q$. We get a quadratic equation

$$q^2(4p - 5 - 3\frac{p}{\beta}) - q(11p - 6 - 6\frac{p}{\beta}) + 6p = 0 \tag{5.40}$$

with one root $q < 1$ (therefore not interesting) and the other giving the condition

$$q = \frac{11p - 6 - 6\frac{p}{\beta} + \sqrt{25p^2 - 12p + 36 + \frac{1}{\beta}(-60p^2 + 72p) + 36\frac{p^2}{\beta^2}}}{8p - 10 - 6\frac{p}{\beta}}. \tag{5.41}$$

For fixed $p$ we get the lowest $q$ by taking $\beta$ near infinity and thus we get the condition from Theorem 5.2.1

$$q > \frac{11p - 6 + \sqrt{25p^2 - 12p + 36}}{8p - 10}. \tag{5.42}$$

## 5.3.2 Limiting procedure

The final step of the proof is the limiting procedure. From the a priori estimates we see that[2]

$$\begin{aligned}
\mathbf{v}^n &\rightharpoonup^* \mathbf{v} \quad \text{in } L^\infty(0, T, L^2(\Omega)) \\
\mathbf{v}^n &\rightharpoonup \mathbf{v} \quad \text{in } L^p(0, T, W^{1,p}_{BC,div}(\Omega)) \\
\mathbf{T}^n &\rightharpoonup^* \mathbf{T} \quad \text{in } L^\infty(0, T, L^2(\Omega)) \\
\mathbf{T}^n &\rightharpoonup \mathbf{T} \quad \text{in } L^q(0, T, W^{1,q}_{BC}(\Omega)).
\end{aligned} \tag{5.43}$$

[2] up to a subsequence $n_k \subset \mathbb{N}$ which we denote again $n$ for the sake of simplicity

Moreover we can derive a priori estimates for time derivatives $\frac{\partial \mathbf{v}^n}{\partial t}$ and $\frac{\partial \mathbf{T}^n}{\partial t}$. Let us denote

$$X = L^p(0,T,W^{1,p}_{BC,div}) \cap L^{s'}(Q_T) \tag{5.44}$$

with $s' = \frac{5p}{5p-8}$ being a dual exponent to $s = \frac{5p}{8}$. We derive apriori estimate for $\frac{\partial \mathbf{v}^n}{\partial t}$ in $X^*$.

$$\begin{aligned}
&\left\| \frac{\partial \mathbf{v}^n}{\partial t} \right\|_{X^*} \leq \\
&\leq \sup \int_0^T \int_\Omega |\mathbf{v}^n \cdot \nabla \mathbf{v}^n| \, |\boldsymbol{\psi}| + (|\mu(\mathbf{D}(\mathbf{v}^n))\mathbf{D}(\mathbf{v}^n)| + |\mathbf{T}^n|) \, |\nabla \boldsymbol{\psi}| \, \mathrm{d}\mathbf{x}\mathrm{d}t \leq \\
&\qquad\leq \sup \|\mathbf{v}^n \cdot \nabla \mathbf{v}^n\|_{L^s(Q_T)} \|\boldsymbol{\psi}\|_{L^{s'}(Q_T)} + \\
&\qquad + \left( C + \|\nabla \mathbf{v}^n\|_{L^p(Q_T)} + \|\mathbf{T}^n\|_{L^{\frac{p}{p-1}}(Q_T)} \right) \|\nabla \boldsymbol{\psi}\|_{L^p(Q_T)} \leq C
\end{aligned} \tag{5.45}$$

where the sup is taken over all $\boldsymbol{\psi} \in X$ such that the corresponding norm $\|\boldsymbol{\psi}\|_X = 1$ and we have used that $q > 3 > \frac{p}{p-1}$. Similarly we show the boundedness of $\frac{\partial \mathbf{T}^n}{\partial t}$ in some space $L^r(0,T,(W^{1,s}_{BC})^*)$.

We can add to (5.43) also

$$\begin{aligned}
\frac{\partial \mathbf{v}^n}{\partial t} &\rightharpoonup \frac{\partial \mathbf{v}}{\partial t} \quad \text{in } X^* \\
\frac{\partial \mathbf{T}^n}{\partial t} &\rightharpoonup \frac{\partial \mathbf{T}}{\partial t} \quad \text{in } L^r(0,T,(W^{1,s}_{BC}(\Omega))^*)
\end{aligned} \tag{5.46}$$

and due to Aubin-Lions theorem

$$\begin{aligned}
\mathbf{v}^n &\to \mathbf{v} \quad \text{in } L^t(0,T,L^{\bar{p}}(\Omega)) \\
\mathbf{T}^n &\to \mathbf{T} \quad \text{in } L^t(0,T,L^{\bar{q}}(\Omega))
\end{aligned} \tag{5.47}$$

for all $t < \infty$ and $\bar{p} \in [1, \frac{3p}{3-p})$, $\bar{q} \in [1, \infty]$[3].

The key step in the limiting procedure is to prove that $\mathbf{D}(\mathbf{v}^n) \to \mathbf{D}(\mathbf{v})$ almost everywhere, which enables us to pass to the limit in the nonlinear terms. To show this, we follow the procedure introduced in [11].

Let us denote $K$ the constant which maximizes all the constants appearing in the apriori estimates we derived above and set

$$\begin{aligned}
g^n &= (|\nabla \mathbf{v}^n| + |\nabla \mathbf{v}|)^p + \\
&\quad + (1 + \mu(\mathbf{D}(\mathbf{v}^n)) \, |\mathbf{D}(\mathbf{v}^n)| + \mu(\mathbf{D}(\mathbf{v})) \, |\mathbf{D}(\mathbf{v})|)(|\mathbf{D}(\mathbf{v}^n)| + |\mathbf{D}(\mathbf{v})|).
\end{aligned} \tag{5.48}$$

[3] $\bar{p} \in [1, \infty]$ for $p \geq 3$ and recall that it is always $q > 3$.

Using the apriori estimate (5.34) it is easy to see that $g^n$ is bounded in $L^1(Q_T)$ by the constant $K$. It holds

**Lemma 5.3.1.** *There is a subsequence* $\{\mathbf{v}^k\} \subset \{\mathbf{v}^n\}$ *such that for every* $\delta \in (0,1)$ *there exists* $L \leq \frac{\delta}{K}$ *independent of* $k$ *and a set*

$$E^k = \left\{(t,\mathbf{x}) \in Q_T, L^2 \leq \left|\mathbf{v}^k(t,\mathbf{x}) - \mathbf{v}(t,\mathbf{x})\right| \leq L\right\} \tag{5.49}$$

*such that*

$$\left\|g^k\right\|_{L^1(E^k)} \leq \delta. \tag{5.50}$$

We present the proof from [11] for the sake of completness.

*Proof.* Let $\delta \in (0,1)$ be given and set $L_0 = \frac{\delta}{K}$. Take $N \in \mathbb{N}$ large enough such that $N\delta > K$. Define iteratively $L_i = L_{i-1}^2$ for $i = 1, ..., N$ and set

$$E_i^n = \left\{(t,\mathbf{x}) \in Q_T, L_i^2 \leq |\mathbf{v}^n(t,\mathbf{x}) - \mathbf{v}(t,\mathbf{x})| \leq L_i\right\}. \tag{5.51}$$

For fixed $n$ the sets $E_i^n$ are disjoint and due to the boundedness of $g^n$ it holds

$$\sum_{i=1}^{N} \|g^n\|_{L^1(E_i^n)} \leq K. \tag{5.52}$$

Then there certainly is $i_0(n)$ such that

$$\|g^n\|_{L^1(E_{i_0}^n)} \leq \delta. \tag{5.53}$$

For each $n$ is $i_0 \in \{1, ..., N\}$ and thus there is necessarily a subsequence $\{\mathbf{v}^k\} \subset \{\mathbf{v}^n\}$ such that $i_0(k)$ is the same and therefore independent of $k$. We denote corresponding $L_{i_0}$ by $L$ and $E_{i_0}^k$ by $E^k$ and the proof is complete. $\square$

Next we introduce a sequence

$$\mathbf{\Psi}^k = (\mathbf{v}^k - \mathbf{v})\left(1 - \min\left(1, \frac{\left|\mathbf{v}^k - \mathbf{v}\right|}{L}\right)\right). \tag{5.54}$$

It is easy to observe that $\mathbf{\Psi}^k = \mathbf{0}$ on $Q_L^k = \left\{(t,\mathbf{x}) \in Q_T, \left|\mathbf{v}^k - \mathbf{v}\right| \geq L\right\}$ and $\left|\mathbf{\Psi}^k\right| \leq L$ otherwise. So $\mathbf{\Psi}^k \in L^\infty(Q_T) \cap L^p(0,T,W_{BC}^{1,p})$ and using Lebesgue dominated convergence theorem one can show that

$$\left\|\mathbf{\Psi}^k\right\|_{L^r(Q_T)} \to 0 \quad \forall r \in [1,\infty) \quad \text{as } k \to \infty. \tag{5.55}$$

Unfortunately it does not hold that $\operatorname{div}\mathbf{\Psi}^k = 0$, on the other hand we are able to show that for given $\delta \in (0,1)$ it holds

$$\left\|\operatorname{div}\mathbf{\Psi}^k\right\|_{L^p(Q_T)} \leq 2\delta. \tag{5.56}$$

Again we proceed the same way as in [11].

$$\begin{aligned}\left\|\operatorname{div}\mathbf{\Psi}^k\right\|_{L^p(Q_T)} \leq & \left\|\nabla(\mathbf{v}^k - \mathbf{v})\frac{1}{L}\left|\mathbf{v}^k - \mathbf{v}\right|\right\|_{L^p(Q_T\setminus Q_L^k)} = \\ = & \left\|\nabla(\mathbf{v}^k - \mathbf{v})\frac{1}{L}\left|\mathbf{v}^k - \mathbf{v}\right|\right\|_{L^p((Q_T\setminus Q_L^k)\setminus E^k)} + \\ & + \left\|\nabla(\mathbf{v}^k - \mathbf{v})\frac{1}{L}\left|\mathbf{v}^k - \mathbf{v}\right|\right\|_{L^p(E^k)} \leq LK + \delta \leq 2\delta. \end{aligned} \tag{5.57}$$

We would like to use a test function with zero divergence. Therefore we consider an auxillary problem

$$\begin{aligned} -\Delta h^k &= -\operatorname{div}\mathbf{\Psi}^k \quad \text{in } \Omega, \\ \frac{\partial h^k}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega \qquad \text{or } h^k \quad \text{periodic}, \\ \int_\Omega h^k \mathrm{d}\mathbf{x} &= 0, \\ \left\|h^k\right\|_{W^{2,p}(Q_T)} &\leq C\left\|\operatorname{div}\mathbf{\Psi}^k\right\|_{L^p(Q_T)}, \\ \left\|h^k\right\|_{W^{1,s'}(Q_T)} &\leq C\left\|\mathbf{\Psi}^k\right\|_{L^{s'}(Q_T)}, \end{aligned} \tag{5.58}$$

where $s' = \frac{5p}{5p-8}$ is the dual exponent to $s = \frac{5p}{8}$. The existence of such $h^k$ is standard. We set

$$\boldsymbol{\phi}^k = \mathbf{\Psi}^k - \nabla h^k, \tag{5.59}$$

which will be our test function. Such defined $\boldsymbol{\phi}^k$ is bounded in the space $L^p(0,T,W^{1,p}_{BC,div})$ and moreover

$$\left\|\boldsymbol{\phi}^k\right\|_{L^{s'}(Q_T)} \leq \delta \tag{5.60}$$

for sufficiently large $k$.

Our aim is to show that the following condition holds: $\forall\theta \in (0,1)\forall\varepsilon_0 \in (0,1)\exists k_0 \in \mathbb{N}\forall k \geq k_0$

$$0 < \int_{Q_T} \left((\mu(\mathbf{D}(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k) - \mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})) : \mathbf{D}(\mathbf{v}^k - \mathbf{v})\right)^\theta \mathrm{d}\mathbf{x}\mathrm{d}t < \varepsilon_0. \tag{5.61}$$

This enables us to conclude that

$$\mathbf{D}(\mathbf{v}^k) \to \mathbf{D}(\mathbf{v}) \quad \text{almost everywhere in } Q_T. \tag{5.62}$$

Indeed, due to (5.61) we have

$$\limsup_{k\to\infty} \int_{Q_T} \left( (\mu(\mathbf{D}(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k) - \mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})) : \mathbf{D}(\mathbf{v}^k - \mathbf{v}) \right)^\theta \mathrm{d}\mathbf{x}\mathrm{d}t = 0 \tag{5.63}$$

and thus, up to a subsequence

$$(\mu(\mathbf{D}(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k) - \mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})) : \mathbf{D}(\mathbf{v}^k - \mathbf{v}) \to 0 \quad \text{a. e. in } Q_T. \tag{5.64}$$

Finally, we use the procedure described for example in [7] to show the a.e. convergence of $\mathbf{D}(\mathbf{v}^k)$. Let us emphasize that this procedure strongly relies on the strict monotonicity of $\mu(\mathbf{D})\mathbf{D}$. We argue by contradiction. Let us assume that $(t, \mathbf{x})$ is such that $\mathbf{D}^k := \mathbf{D}(\mathbf{v}^k(t, \mathbf{x}))$ does not converge to $\mathbf{D} := \mathbf{D}(\mathbf{v}(t, \mathbf{x}))$ and that the set of such points $(t, \mathbf{x})$ has positive measure. Then there exists $\delta_0$ and a subsequence (still denoted by $(\mathbf{D}^k)$) such that $\left|\mathbf{D}^k - \mathbf{D}\right| \geq \delta_0$ for every $k$. Let us denote $t_k = \frac{\delta_0}{\left|\mathbf{D}^k - \mathbf{D}\right|}$ and moreover

$$\mathbf{Z}^k = t_k \mathbf{D}^k + (1 - t_k)\mathbf{D}. \tag{5.65}$$

Then $\left|\mathbf{Z}^k - \mathbf{D}\right| = t_k \left|\mathbf{D}^k - \mathbf{D}\right| = \delta_0$, thus the sequence $(\mathbf{Z}^k)$ is bounded and therefore there is a subsequence, still denoted $(\mathbf{Z}^k)$, such that $\mathbf{Z}^k \to \mathbf{Z}$ with $|\mathbf{Z} - \mathbf{D}| = \delta_0$. Let us for simplicity denote

$$\beta(\xi) := \mu(\xi)\xi. \tag{5.66}$$

Due to the monotonicity assumption (5.8) we have $(\beta(\mathbf{Z}^k) - \beta(\mathbf{D})) : (\mathbf{Z}^k - \mathbf{D}) \geq 0$ and as $t_k \leq 1$ also

$$(\beta(\mathbf{Z}^k) - \beta(\mathbf{D})) : (\mathbf{D}^k - \mathbf{D}) \geq (\beta(\mathbf{Z}^k) - \beta(\mathbf{D})) : (\mathbf{Z}^k - \mathbf{D}) \geq 0. \tag{5.67}$$

Similarly we have also

$$(\beta(\mathbf{D}^k) - \beta(\mathbf{Z}^k)) : (\mathbf{D}^k - \mathbf{D}) \geq (\beta(\mathbf{D}^k) - \beta(\mathbf{Z}^k)) : (\mathbf{D}^k - \mathbf{Z}^k) \geq 0. \tag{5.68}$$

Combining (5.67) and (5.68) we get

$$\begin{aligned}
&(\beta(\mathbf{D}^k) - \beta(\mathbf{D})) : (\mathbf{D}^k - \mathbf{D}) = \\
&\qquad = (\beta(\mathbf{D}^k) - \beta(\mathbf{Z}^k)) : (\mathbf{D}^k - \mathbf{D}) + (\beta(\mathbf{Z}^k) - \beta(\mathbf{D})) : (\mathbf{D}^k - \mathbf{D}) \geq \\
&\quad \geq (\beta(\mathbf{Z}^k) - \beta(\mathbf{D})) : (\mathbf{D}^k - \mathbf{D}) \geq (\beta(\mathbf{Z}^k) - \beta(\mathbf{D})) : (\mathbf{Z}^k - \mathbf{D}) \geq 0.
\end{aligned} \tag{5.69}$$

The first term on the left hand side converges to zero due to assumption (5.64) and therefore also

$$(\beta(\mathbf{Z}^k) - \beta(\mathbf{D})) : (\mathbf{Z}^k - \mathbf{D}) \to 0. \tag{5.70}$$

On the other hand $\mathbf{Z}^k \to \mathbf{S}$ and $\beta$ is continuous, therefore

$$(\beta(\mathbf{Z}^k) - \beta(\mathbf{D})) : (\mathbf{Z}^k - \mathbf{D}) \to (\beta(\mathbf{Z}) - \beta(\mathbf{D})) : (\mathbf{Z} - \mathbf{D}) = 0 \tag{5.71}$$

and this contradicts the strict monotonicity (5.8), which is assumed to hold almost everywhere, since $|\mathbf{Z} - \mathbf{D}| = \delta_0$.

We have proved that condition (5.61) implies the a.e. convergence of $\mathbf{D}(\mathbf{v}^k)$ to $\mathbf{D}(\mathbf{v})$. It remains to prove (5.61). We use the weak formulation of the equation for $\mathbf{v}^k$ with the test function $\boldsymbol{\phi}^k$. Note that $\boldsymbol{\phi}^k = \mathbf{0}$ on $Q_L^k$. We denote by $\chi$ the characteristic function of $Q_T \setminus Q_L^k$. We obtain the following identity

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial(\mathbf{v}^k - \mathbf{v})}{\partial t}, \boldsymbol{\Psi}^k \right\rangle_p \mathrm{d}t + \\
&\quad + \int_{Q_T} (\mu(\mathbf{D}(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k) - \mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})) : \mathbf{D}(\mathbf{v}^k - \mathbf{v})\chi \mathrm{d}\mathbf{x}\mathrm{d}t = \\
&\qquad = - \int_0^T \left\langle \frac{\partial(\mathbf{v}^k - \mathbf{v})}{\partial t}, \nabla h^k \right\rangle_p \mathrm{d}t - \int_0^T \left\langle \frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\phi}^k \right\rangle_p \mathrm{d}t + \\
&\qquad + \int_{Q_T} (\mu(\mathbf{D}(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k) - \mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})) : \mathbf{D}(\nabla h^k) \mathrm{d}\mathbf{x}\mathrm{d}t + \\
&\quad + \int_{Q_T} (\mu(\mathbf{D}(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k) - \mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})) : \mathbf{D}(\mathbf{v}^k - \mathbf{v}) \frac{|\mathbf{v}^k - \mathbf{v}|}{L} \chi \mathrm{d}\mathbf{x}\mathrm{d}t - \\
&\qquad\qquad - \int_{Q_T} \mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v}) : \mathbf{D}(\boldsymbol{\phi}^k) \mathrm{d}\mathbf{x}\mathrm{d}t - \\
&\qquad\qquad - \int_{Q_T} (\mathbf{v}^k \cdot \nabla \mathbf{v}^k) \cdot \boldsymbol{\phi}^k \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{Q_T} \mathbf{T}^k : \mathbf{D}(\boldsymbol{\phi}^k) \mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned} \tag{5.72}$$

The first term on the left hand side is equal to $H(T) - H(0)$, where $H$ is appropriate nonnegative primitive function. Moreover $H(0) = 0$ and thus the first term is nonnegative. We denote $I_1, ..., I_7$ the terms on the right hand side and we show smallness of all of these terms. First, $I_1 = 0$ which can be seen using integration by parts. The second term $I_2$ tends to zero due to the fact that $\boldsymbol{\phi}^k \rightharpoonup \mathbf{0}$ weakly in $X$[4].

The third term is estimated using Hölder inequality

$$|I_3| \leq \left\|g^k\right\|_{L^1(Q_T)} \left\|h^k\right\|_{W^{2,p}(Q_T)} \leq CK\delta. \tag{5.73}$$

We estimate the fourth term similarly as we estimated div $\boldsymbol{\Psi}^k$ using the fact that the domain of integration is $Q_T \setminus Q_L^k$

$$|I_4| \leq \left\|g^k\right\|_{L^1(Q_T)} \left\|\mathbf{D}(\mathbf{v}^k - \mathbf{v})\frac{\left|\mathbf{v}^k - \mathbf{v}\right|}{L}\right\|_{L^p(Q_T \setminus Q_L^k)} \leq CK\delta. \tag{5.74}$$

The fifth term tends to zero as $\boldsymbol{\phi}^k \rightharpoonup \mathbf{0}$ weakly in $L^p(0,T,W^{1,p}_{BC,div})$. The sixth term is estimated easily by Hölder inequality

$$|I_6| \leq \left\|\mathbf{v}^k \cdot \nabla \mathbf{v}^k\right\|_{L^s(Q_T)} \left\|\boldsymbol{\phi}^k\right\|_{L^{s'}(Q_T)} \leq CK\delta. \tag{5.75}$$

Finally, we deal with the last term similarly as with $I_5$ and we use strong convergence of $\mathbf{T}^k$ (5.47).

This way we have obtained

$$\int_{Q_T} \left((\mu(\mathbf{D}(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k) - \mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})) : \mathbf{D}(\mathbf{v}^k - \mathbf{v})\chi\right) \mathrm{d}\mathbf{x}\mathrm{d}t \leq CK\delta. \tag{5.76}$$

We use this information in the following way. Let us denote

$$f^k = (\mu(\mathbf{D}(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k) - \mu(\mathbf{D}(\mathbf{v}))\mathbf{D}(\mathbf{v})) : \mathbf{D}(\mathbf{v}^k - \mathbf{v}). \tag{5.77}$$

Then

$$\begin{aligned}
\int_{Q_T} (f^k)^\theta \mathrm{d}\mathbf{x}\mathrm{d}t &= \int_{Q_T \setminus Q_L^k} (f^k)^\theta \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{Q_L^k} (f^k)^\theta \mathrm{d}\mathbf{x}\mathrm{d}t \leq \\
&\leq \left|Q_T \setminus Q_L^k\right|^{1-\theta} \left(\int_{Q_T \setminus Q_L^k} (f^k)\mathrm{d}\mathbf{x}\mathrm{d}t\right)^\theta + \\
&\quad + \left|Q_L^k\right|^{1-\theta} \left(\int_{Q_L^k} (f^k)\mathrm{d}\mathbf{x}\mathrm{d}t\right)^\theta \leq \\
&\leq |Q_T|^{1-\theta} (CK\delta)^\theta + \left|Q_L^k\right|^{1-\theta} K^\theta
\end{aligned} \tag{5.78}$$

[4]See (5.44) for definition of space $X$

and the second term is small due to the strong convergence of $\mathbf{v}^k$ and definition of the set $Q_L^k$. This proves the condition (5.61) and consequently the almost everywhere convergence of $\mathbf{D}(\mathbf{v}^k)$.

Now we can pass to the limit in the Navier–Stokes equation. There is no problem in passing to the limit in the linear terms. In the convective term we may proceed similarly as in proving existence of solution to the classical Navier–Stokes equations. Observing that

$$|(\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{w})| = |(\mathbf{v} \cdot \nabla \mathbf{w}, \mathbf{v})|\,, \tag{5.79}$$

we see that it is enough to use the strong convergence $(5.47)_1$. Finally we pass to the limit in the viscous term. Using a priori estimates together with Vitali theorem we get

$$\mathbf{D}(\mathbf{v}^k) \to \mathbf{D}(\mathbf{v}) \quad \text{ in } L^p(Q_T). \tag{5.80}$$

Due to the continuity of $\mu$ we have also

$$\mu(\mathbf{D}(\mathbf{v}^k)) \to \mu(\mathbf{D}(\mathbf{v})) \quad \text{ almost everywhere in } Q_T \tag{5.81}$$

and this enables us to pass to the limit in the term $\mu(\mathbf{D}(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k)$.

We use the so called Minty trick to pass to the limit in the equation $(5.1)_3$. Let us take $\mathbf{\Phi}^k \to \mathbf{\Phi}$ strongly in $L^q(0,T,W^{1,q}_{BC})$ such that $\mathbf{\Phi}^k \in C^\infty(0,T,H_k)$ where $H_k$ denotes the finite dimensional subspace of $W^{1,q}_{BC}$ with base $\mathbf{S}^1, ..., \mathbf{S}^k$. Moreover, let $k \geq l$ and let $0 \leq \psi \in C_0^\infty(0,T)$. Due to the monotonicity of $\gamma(\nabla \mathbf{T})\nabla \mathbf{T}$ we have

$$\begin{aligned} 0 \leq \int_0^T & \left[ \int_\Omega \varepsilon(\gamma(\nabla \mathbf{T}^k)\nabla \mathbf{T}^k - \gamma(\nabla \mathbf{\Phi}^l)\nabla \mathbf{\Phi}^l) : (\nabla \mathbf{T}^k - \nabla \mathbf{\Phi}^l) \mathrm{d}\mathbf{x} \right] \psi \mathrm{d}t = \\ = \int_0^T & \left[ \int_\Omega -\varepsilon\gamma(\nabla \mathbf{\Phi}^l)\nabla \mathbf{\Phi}^l : (\nabla \mathbf{T}^k - \nabla \mathbf{\Phi}^l) - \left\langle \frac{\partial \mathbf{T}^k}{\partial t}, \mathbf{T}^k - \mathbf{\Phi}^l \right\rangle_q + \right. \\ & + \left( -\mathbf{v}^k \cdot \nabla \mathbf{T}^k - \mathbf{T}^k - \varepsilon \left|\mathbf{T}^k\right|^{q-2} \mathbf{T}^k \right) : (\mathbf{T}^k - \mathbf{\Phi}^l) + \\ & \left. + \left(2\mu_0 \mathbf{D}(\mathbf{v}^k) + \mathbf{B}(\mathbf{v}^k, \mathbf{T}^k)\right) : (\mathbf{T}^k - \mathbf{\Phi}^l) \mathrm{d}\mathbf{x} \right] \psi \mathrm{d}t, \end{aligned} \tag{5.82}$$

where we have used the equation for $\mathbf{T}^k$. Next, we pass to the limit with $k \to \infty$. The linear terms are easy while in the nonlinear terms we use the strong convergence (5.47) of $\mathbf{T}^k$ and $\mathbf{v}^k$. Thus we get

$$0 \leq \int_0^T \Bigg[ \int_\Omega -\varepsilon\gamma(\nabla\mathbf{\Phi}^l)\nabla\mathbf{\Phi}^l : (\nabla\mathbf{T} - \nabla\mathbf{\Phi}^l) - \left\langle \frac{\partial \mathbf{T}}{\partial t}, \mathbf{T} - \mathbf{\Phi}^l \right\rangle_q + \\ + \left(-\mathbf{v}\cdot\nabla\mathbf{T} - \mathbf{T} - \varepsilon\left|\mathbf{T}\right|^{q-2}\mathbf{T}\right) : (\mathbf{T} - \mathbf{\Phi}^l) + \\ + (2\mu_0\mathbf{D}(\mathbf{v}) + \mathbf{B}(\mathbf{v},\mathbf{T})) : (\mathbf{T} - \mathbf{\Phi}^l)\mathrm{d}\mathbf{x} \Bigg] \psi \mathrm{d}t \quad (5.83)$$

and we proceed with passing to the limit with $l \to \infty$

$$0 \leq \int_0^T \Bigg[ \int_\Omega -\varepsilon\gamma(\nabla\mathbf{\Phi})\nabla\mathbf{\Phi} : (\nabla\mathbf{T} - \nabla\mathbf{\Phi}) - \left\langle \frac{\partial \mathbf{T}}{\partial t}, \mathbf{T} - \mathbf{\Phi} \right\rangle_q + \\ + \left(-\mathbf{v}\cdot\nabla\mathbf{T} - \mathbf{T} - \varepsilon\left|\mathbf{T}\right|^{q-2}\mathbf{T}\right) : (\mathbf{T} - \mathbf{\Phi}) + \\ + (2\mu_0\mathbf{D}(\mathbf{v}) + \mathbf{B}(\mathbf{v},\mathbf{T})) : (\mathbf{T} - \mathbf{\Phi})\mathrm{d}\mathbf{x} \Bigg] \psi \mathrm{d}t. \quad (5.84)$$

Now we set $\mathbf{\Phi}(t,\mathbf{x}) = \mathbf{T}(t,\mathbf{x}) - \delta\mathbf{S}(\mathbf{x})$

$$0 \leq \delta \int_0^T \Bigg[ \int_\Omega -\varepsilon\gamma(\nabla(\mathbf{T} - \delta\mathbf{S}))\nabla(\mathbf{T} - \delta\mathbf{S}) : \nabla\mathbf{S} - \left\langle \frac{\partial \mathbf{T}}{\partial t}, \mathbf{S} \right\rangle_q + \\ + \left(-\mathbf{v}\cdot\nabla\mathbf{T} - \mathbf{T} - \varepsilon\left|\mathbf{T}\right|^{q-2}\mathbf{T} + 2\mu_0\mathbf{D}(\mathbf{v}) + \mathbf{B}(\mathbf{v},\mathbf{T})\right) : \mathbf{S}\mathrm{d}\mathbf{x} \Bigg] \psi \mathrm{d}t. \quad (5.85)$$

Dividing this inequality by $\delta$ and passing with $\delta \to 0$ we end up with[5]

$$0 = \int_0^T \Bigg[ \int_\Omega -\varepsilon\gamma(\nabla\mathbf{T})\nabla\mathbf{T} : \nabla\mathbf{S} - \left\langle \frac{\partial \mathbf{T}}{\partial t}, \mathbf{S} \right\rangle_q + \\ + \left(-\mathbf{v}\cdot\nabla\mathbf{T} - \mathbf{T} - \varepsilon\left|\mathbf{T}\right|^{q-2}\mathbf{T} + 2\mu_0\mathbf{D}(\mathbf{v}) + \mathbf{B}(\mathbf{v},\mathbf{T})\right) : \mathbf{S}\mathrm{d}\mathbf{x} \Bigg] \psi \mathrm{d}t \quad (5.86)$$

which holds for all $0 \leq \psi \in C_0^\infty(0,T)$ and thus we recover the weak formulation of the equation $(5.1)_3$ and the proof is finished.

*Remark* 5.3.1. The presence of the stress diffusion term $\varepsilon\gamma(\nabla\mathbf{T})\nabla\mathbf{T}$ is crucial for our proof. On the other hand we have to add the term $\varepsilon\left|\mathbf{T}\right|^{q-2}\mathbf{T}$ merely due to technical reasons. We use this term only in the formula (5.31), where

[5]Note that $\delta$ can be positive or negative and therefore we get equality.

we interpolate $\mathbf{T}$ between $L^2$ and $W^{1,q}$. At this point we need full $W^{1,q}$ norm and unfortunately the $L^q$ norm of $\nabla\mathbf{T}$ is not equivalent to the full norm. This is due to the fact that we neither prescribe homogenous Dirichlet condition for $\mathbf{T}$ and we are nor able to control $\int_\Omega \mathbf{T}\mathrm{d}\mathbf{x}$ in the case of periodic boundary condition.

Omitting term $\varepsilon\,|\mathbf{T}|^{q-2}\,\mathbf{T}$ generally leads to worse conditions for $p$ and $q$. In fact we do not need this term in the case $\beta = 2$ in step (5.30) which leads $\lambda = 1$ in (5.31). Consequently one gets conditions $p > 2$ and $q \geq \frac{2p}{p-2}$.

Instead of $\varepsilon\,|\mathbf{T}|^{q-2}\,\mathbf{T}$ we can add different monotone nonlinearity with $r$–coercivity and $(r-1)$–growth with $r \geq q$. This brings only minor changes in the proof.

We do not need to add this lower order term in the corotational case $a = 0$, where there is no term on the right hand side of (5.28). In this case the problem is simplified and one can get better results for $p$ and $q$. See also [27] for the global existence for the corotational Oldroyd model without stress diffusion.

# Chapter 6

# Appendix

In this chapter we summarize most of the mathematical tools used in previous chapters.

## 6.1 Function spaces

### 6.1.1 Spaces of continuously differentiable and Hölder continuous functions

For a domain $\Omega \subset \mathbb{R}^N$ we use standard notation for spaces of continuous functions $C(\overline{\Omega})$ as well as spaces of continuously differentiable functions up to order $k \in \mathbb{N}$ $C^k(\overline{\Omega})$[1]. All derivatives are understood in classical sense, spaces are endowed with standard norms. Then these spaces are Banach spaces.

For a multiindex $\alpha$ such that $|\alpha| \leq k$, $\mu \in (0,1]$ and $u \in C^k(\overline{\Omega})$ let us take

$$H_{\alpha,\mu}(u) = \sup_{\substack{\mathbf{x} \neq \mathbf{y} \\ \mathbf{x},\mathbf{y} \in \Omega}} \frac{|D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\mu}. \tag{6.1}$$

Then we define spaces

$$C^{k,\mu}(\overline{\Omega}) = \left\{ u \in C^k(\Omega); \sum_{|\alpha|=k} H_{\alpha,\mu}(u) < \infty \right\} \tag{6.2}$$

[1] Here $\overline{\Omega}$ is a closure of an open set $\Omega$ in $\mathbb{R}^N$.

endowed with the norm

$$\|u\|_{C^{k,\mu}(\overline{\Omega})} = \sum_{|\alpha|=k} H_{\alpha,\mu}(u) + \|u\|_{C^k(\overline{\Omega})} \,. \tag{6.3}$$

Space $C^{k,\mu}(\overline{\Omega})$ is a Banach space.

*Remark* 6.1.1. Functions belonging to $C^{0,\mu}$ for $\mu \in (0,1)$ are usually called Hölder continuous functions while $C^{0,1}$ functions are often called Lipschitz continuous. We refer e.g. to [23] for more details about Hölder and Lipschitz continuous functions.

### 6.1.2 Lebesgue and Sobolev spaces

Throughout the whole thesis we use standard notation for Lebesgue spaces $L^p(\Omega)$ for $1 \le p \le \infty$ endowed with the norm $\|\cdot\|_{L^p(\Omega)}$. However, we may skip writing the domain $\Omega$ if no confusion arise and denote the norm as $\|\cdot\|_{L^p}$ or simply $\|\cdot\|_p$. Similarly we denote Sobolev spaces $W^{k,p}(\Omega)$ for $k \in \mathbb{N}$, $1 \le p \le \infty$ endowed with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ which we may denote also by $\|\cdot\|_{W^{k,p}}$ or $\|\cdot\|_{k,p}$.

Also throughout the whole thesis we do not distinguish between scalar valued spaces (for example $L^p(\Omega)$), vector valued spaces ($L^p(\Omega)^N$) and tensor valued spaces ($L^p(\Omega)^{N\times N}$). In this way writing $\mathbf{D} \in L^p(\Omega)$ means that all components $D_{ij}$ of $\mathbf{D}$ belong to $L^p(\Omega)$.

The following three classical inequalities are basic tools which are used in this thesis. We use the convention that $\frac{1}{\infty} = 0$ in cases where parameters are allowed to be equal $\infty$.

**Theorem 6.1.1** (Young). *Let $p, p' \in (1,\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$ and let $\varepsilon > 0$. Then there exists a constant $C = C(p,\varepsilon)$ such that for any $A, B > 0$ it holds*

$$AB \le \varepsilon A^p + C B^{p'}.$$

**Theorem 6.1.2** (Hölder). *Let $p, p' \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Let $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$. Then $fg \in L^1(\Omega)$ and it holds*

$$\|fg\|_1 \le \|f\|_p \, \|g\|_{p'}.$$

**Theorem 6.1.3** (Gronwall). *Let $T > 0$, $K, L \in \mathbb{R}^+$ and $g(t) \in L^1((0,T))$ be a nonnegative function. Let $f(t)$ be a nonnegative function such that*

$$f(t) \le K + L\int_0^t f(s)g(s)ds$$

*for* $0 \le t \le T$. *Then*

$$f(t) \le K \exp\left(L\int_0^t g(s)ds\right)$$

*for* $0 \le t \le T$.

We recall several embedding theorems of Sobolev spaces.

**Theorem 6.1.4** (Sobolev embedding I)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with lipschitz boundary,* $1 \le kp < N$ *and* $q \le \frac{Np}{N-kp}$*. Then there is a constant* $C = C(\Omega, N, k, p, q)$ *such that*

$$\|u\|_q \le C\,\|u\|_{k,p},$$

*for all* $u \in W^{k,p}(\Omega)$*. This means that* $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$*.*

*Remark* 6.1.2. Theorem 6.1.4 holds also for exterior domains with the restriction $q \ge p$.

**Theorem 6.1.5** (Sobolev embedding II)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with lipschitz boundary,* $kp = N$ *and* $q < \infty$*. Then there is a constant* $C = C(\Omega, N, k, p, q)$ *such that*

$$\|u\|_q \le C\,\|u\|_{k,p},$$

*for all* $u \in W^{k,p}(\Omega)$*. This means that* $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$*.*

*Remark* 6.1.3. Theorem 6.1.5 holds also for exterior domains with the restriction $q \ge p$. It can be shown that $W^{k,p}(\Omega)$ is not embedded into $L^\infty(\Omega)$ in the case $kp = N$, except for the case $k = N$ and $p = 1$.

**Theorem 6.1.6** (Sobolev embedding III)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a domain with lipschitz boundary,* $kp > N$ *and let*

$$\mu \begin{cases} = k - \frac{N}{p} & \text{if } k - \frac{N}{p} < 1, \\ < 1 & \text{if } k - \frac{N}{p} = 1, \\ = 1 & \text{if } k - \frac{N}{p} > 1. \end{cases} \tag{6.4}$$

*Let* $u \in W^{k,p}(\Omega)$*. Then there is a constant* $C = C(\Omega, N, k, p)$ *independent of* $u$ *and a representative* $\bar{u} = u$ *a.e. in* $\Omega$ *such that*

$$\|\bar{u}\|_{C^{0,\mu}(\overline{\Omega})} \le C \|u\|_{k,p}.$$

*This means that* $W^{k,p}(\Omega) \hookrightarrow C^{0,\mu}(\overline{\Omega})$.

In some cases one can show that embeddings are compact.

**Theorem 6.1.7** (Sobolev embedding IV). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with lipschitz boundary. Then*

- *for $kp < N$, $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [1, \frac{Np}{N-kp})$*
- *for $kp = N$, $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty)$*
- *for $kp > N$, $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$.*

At some points we need also Friedrichs inequality.

**Theorem 6.1.8** (Friedrichs). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with lipschitz boundary, $\Gamma \subset \partial\Omega$ be a part of boundary with positive Lebesgue $N-1$ dimensional measure. Then there exists $C = C(\Omega, N, p, q)$ such that*

$$\|u\|_{L^q(\Omega)} \le C \left( \|\nabla u\|_{L^p(\Omega)} + \int_\Gamma |u| \,\mathrm{d}S \right), \tag{6.5}$$

*for all $u \in W^{1,p}(\Omega)$ and $q \in \left[1, \frac{Np}{N-p}\right]$ if $p < N$ and $q \in [1, \infty)$ if $p \ge N$.*

We also recall two important interpolation inequalities.

**Theorem 6.1.9** (Interpolation I). *Let $\Omega$ be any domain and let $u \in L^p(\Omega) \cap L^q(\Omega)$, $1 \le p < q \le \infty$. Then $u \in L^r(\Omega)$ for all $r \in [p, q]$ and*

$$\|u\|_r \le \|u\|_p^\lambda \|u\|_q^{1-\lambda}, \tag{6.6}$$

*where* $\frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$

**Theorem 6.1.10** (Interpolation II). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with lipschitz boundary. Let $u \in L^q(\Omega)$ and $u \in W^{1,s}(\Omega)$. Then there is a constant $C = C(\Omega, r, s, q, \lambda)$ such that*

$$\|u\|_r \le C \|u\|_{1,s}^\lambda \|u\|_q^{1-\lambda}, \tag{6.7}$$

*where*

- *if* $s \in [1, N)$ *and* $q < \frac{Ns}{N-s}$ *then* $r \in [q, \frac{Ns}{N-s})$ *and* $\frac{1}{r} = \frac{\lambda(N-s)}{Ns} + \frac{1-\lambda}{q}$, $\lambda \in [0, 1)$
- *if* $s \in [1, N)$ *and* $q \geq \frac{Ns}{N-s}$ *then* $r \in [\frac{Ns}{N-s}, q]$ *and* $\frac{1}{r} = \frac{\lambda(N-s)}{Ns} + \frac{1-\lambda}{q}$, $\lambda \in [0, 1]$
- *if* $s \in [N, \infty)$ *then* $r \in [q, \infty)$ *and* $\frac{1}{r} = \frac{\lambda(N-s)}{Ns} + \frac{1-\lambda}{q}$, $\lambda \in [0, 1)$.

Let us write down a useful special case of this theorem

**Corollary 6.1.11.** *Let* $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, *be a bounded domain with lipschitz boundary. Let* $u \in W^{1,2}(\Omega)$. *Then there is a constant* $C = C(\Omega)$ *such that*

$$\begin{aligned} \|u\|_4 &\leq C \|u\|_{1,2}^{\frac{1}{2}} \|u\|_2^{\frac{1}{2}} \qquad \text{for } N = 2 \\ \|u\|_4 &\leq C \|u\|_{1,2}^{\frac{3}{4}} \|u\|_2^{\frac{1}{4}} \qquad \text{for } N = 3. \end{aligned} \tag{6.8}$$

Korn inequality is used several times in the thesis.

**Lemma 6.1.12** (Korn)**.** *a) Let* $\Omega$ *be a bounded domain with lipschitz boundary and let* $p > 1$. *Then there is a constant* $C$ *such that*

$$\|\nabla \mathbf{v}\|_p \leq C \|\mathbf{D}(\mathbf{v})\|_p \quad \forall \mathbf{v} \in W_0^{1,p}(\Omega), \tag{6.9}$$

*where* $\mathbf{D}(\mathbf{v})$ *denotes the symmetric part of the gradient. The same holds also in the case of periodic functions* $W_{per}^{1,p}(\Omega)$.

*b) Let* $\Omega$ *be a bounded domain with lipschitz boundary which is not axisymmetric and let* $p > 1$. *Then there is a constant* $C$ *such that*

$$\|\mathbf{v}\|_{1,p} \leq C \|\mathbf{D}(\mathbf{v})\|_p \quad \forall \mathbf{v} \in W^{1,p}(\Omega) \text{ such that } \mathbf{v} \cdot \mathbf{n} = 0. \tag{6.10}$$

While the first part of this Lemma is classical, the proof of part b) is based on the procedure mentioned in [35].

In the last chapter we need also Aubin-Lions theorem and Vitali convergence theorem.

**Theorem 6.1.13** (Aubin-Lions)**.** *Let* $X_0, X_1, X$ *be a triple of Banach spaces such that* $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$. *Let moreover* $X_0, X_1$ *be reflexive and let* $\alpha_0, \alpha_1 \in (1, \infty)$. *Then for* $0 < T < \infty$ *the space* $W$ *defined as*

$$W = \left\{ v \in L^{\alpha_0}(0, T, X_0), \frac{\partial v}{\partial t} \in L^{\alpha_1}(0, T, X_1) \right\} \tag{6.11}$$

*is compactly embedded into* $L^{\alpha_0}(0, T, X)$.

**Theorem 6.1.14** (Vitali)**.** *Let $\Omega$ be a bounded domain. Let $p \geq 1$ and let $\{f_n\} \subset L^p(\Omega)$. Then $f_n \to f$ strongly in $L^p(\Omega)$ if and only if*

- *$f_n \to f$ in measure*
- *$|f_n|^p$ is uniformly integrable, i.e. $\forall \varepsilon > 0 \exists t > 0 \forall n \in \mathbb{N}$*

$$\int_{\{|f_n|>t\}} |f_n(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} < \varepsilon. \tag{6.12}$$

### 6.1.3 Homogenous Sobolev spaces

In the case when $\Omega$ is an exterior domain, one may meet the situation when classical Sobolev spaces are not applicable. Therefore we introduce the homogenous Sobolev spaces

$$\begin{aligned} D^{m,q}(\Omega) &= \left\{u \in L^1_{loc}(\Omega), D^\alpha u \in L^q(\Omega), \forall\, |\alpha| = m\right\} \\ D^{m,q}_0(\Omega) &= \overline{C^\infty_0(\Omega)}^{|\cdot|_{m,q,\Omega}}, \end{aligned} \tag{6.13}$$

where

$$|u|_{m,q,\Omega} = \left( \sum_{|\alpha|=m} \|D^\alpha u\|^q_{L^q(\Omega)} \right)^{\frac{1}{q}}. \tag{6.14}$$

If $\Omega$ is a bounded domain with lipschitz boundary, the spaces $D^{m,q}(\Omega)$ and $W^{m,q}(\Omega)$ coincide. It is obvious that the seminorm do not distinguish functions which differ by a polynomial of degree $m-1$ a.e. in $\Omega$.

Let us denote by $(D^{1,2}_0(\Omega))^*$ the dual space to $D^{1,2}_0(\Omega)$ and consider the functional

$$\langle \mathcal{G}, u \rangle = \int_\Omega f u \mathrm{d}\mathbf{x} \tag{6.15}$$

for $f \in C_0(\Omega)$. Set

$$|\mathcal{G}|_{-1,2} = \sup_{|u|_{1,2}=1} |\langle \mathcal{G}, u \rangle| \tag{6.16}$$

and denote by $D^{-1,2}_0(\Omega)$ the completition of $C_0(\Omega)$ in this norm. One can show that $(D^{1,2}_0(\Omega))^*$ and $(D^{-1,2}_0(\Omega))$ are isometrically and topologically isomorphic.

An important case is when $\mathcal{G}$ is represented by a divergence of some function. Then using integration by parts

$$|\mathcal{G}|_{-1,2} = \sup_{|u|_{1,2}=1} \left| \int_\Omega (\operatorname{div} \mathbf{Z}) u \mathrm{d}\mathbf{x} \right| = \sup_{|u|_{1,2}=1} \left| \int_\Omega \mathbf{Z} \cdot \nabla u \mathrm{d}\mathbf{x} \right| \leq \|\mathbf{Z}\|_{L^2} . \quad (6.17)$$

We mention the following estimate which is proved in [29].

**Lemma 6.1.15.** *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with lipschitz boundary. Let $u \in L^4(\Omega) \cap D^{2,2}(\Omega)$. Then*

$$\|u\|_\infty \leq C \|u\|_4^{\frac{2}{5}} |u|_{2,2}^{\frac{3}{5}} + C(\varepsilon) \|u\|_4^{\frac{2}{5}+\varepsilon} |u|_{2,2}^{\frac{3}{5}-\varepsilon} \quad (6.18)$$

*for $\varepsilon \in (0, \frac{3}{5}]$.*

Combining this Lemma with the embedding theorems if necessary we get also

**Lemma 6.1.16.** *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with lipschitz boundary and $k \geq 2$. Then we have*

$$\begin{aligned}
&\|uv\|_{k,2} \leq \|u\|_{k,2} \|v\|_{k,2} \qquad \text{for } u, v \in W^{k,2}(\Omega) \\
&\|uv\|_{1,2} \leq \|u\|_{1,2} \|v\|_{2,2} \qquad \text{for } u \in W^{1,2}(\Omega), v \in W^{2,2}(\Omega) \\
&\|uv\|_{k,2} \leq (\|u\|_4 + \|\nabla u\|_{k-1,2}) \|v\|_{k,2} \\
&\qquad\qquad \text{for } u \in L^4(\Omega), \nabla u \in W^{k-1,2}(\Omega), v \in W^{k,2}(\Omega) \\
&\|uv\|_{1,2} \leq (\|u\|_4 + \|\nabla u\|_2) \|v\|_{2,2} \\
&\qquad\qquad \text{for } u \in L^4(\Omega), \nabla u \in L^2(\Omega), v \in W^{2,2}(\Omega) \\
&\|uv\|_{k,2} \leq (\|u\|_4 + \|\nabla u\|_{k-1,2})(\|v\|_4 + \|\nabla v\|_{k-1,2}) \\
&\qquad\qquad \text{for } u, v \in L^4(\Omega), \nabla u, \nabla v \in W^{k-1,2}(\Omega) \\
&\|uv\|_{1,2} \leq (\|u\|_4 + \|\nabla u\|_2)(\|v\|_4 + \|\nabla v\|_{1,2}) \\
&\qquad\qquad \text{for } u, v \in L^4(\Omega), \nabla u \in L^2(\Omega), \nabla v \in W^{1,2}(\Omega).
\end{aligned} \quad (6.19)$$

### 6.1.4 Real interpolation

We need to define spaces obtained by real interpolation. Let $X$ and $Y$ be Banach spaces. We define

$$X + Y = \{u; u = u_1 + u_2, u_1 \in X, u_2 \in Y\} \quad (6.20)$$

endowed with norm

$$\|u\|_{X+Y} = \inf\left\{\|u_1\|_X + \|u_2\|_Y\,; u = u_1 + u_2, u_1 \in X, u_2 \in Y\right\}. \tag{6.21}$$

**Definition 6.1.1.** For any $u \in X + Y$ and $t \geq 0$ let

$$K(t,u) = \min_{u=u_1+u_2} \|u_1\|_X + t\,\|u_2\|_Y\,. \tag{6.22}$$

Let $\theta \in [0,1]$, $q \in (1,\infty)$ and let

$$\|u\|_{\theta,q;K} = \left(\int_0^\infty t^{\theta-1}\,(K(t,u))^q\,dt\right)^{\frac{1}{q}}. \tag{6.23}$$

Then the interpolation space $(X,Y)_{\theta,q}$ is defined as

$$(X,Y)_{\theta,q} = \left\{u; \|u\|_{\theta,q;K} < \infty\right\} \tag{6.24}$$

and $\|u\|_{\theta,q;K}$ is a norm on $(X,Y)_{\theta,q}$.

For details about interpolation spaces see e.g. [43].

## 6.2 Stokes and Navier-Stokes equations

### 6.2.1 Stokes problem

Let us consider the following problem

$$\left.\begin{array}{r}\dfrac{\partial \mathbf{v}}{\partial t} - \mu\Delta\mathbf{v} + \nabla p = \mathbf{f}\\ \operatorname{div}\mathbf{v} = 0\end{array}\right\}\text{in } (0,T)\times\Omega, \qquad \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \Omega\,, \tag{6.25}$$

with either $\Omega = \mathbb{R}^N$ (we denote this problem $C_S$), $\Omega = (0,L)^N$ and periodic boundary conditions (we denote this problem $P_S$) or $\Omega$ smooth with homogeneous Dirichlet boundary conditions (we denote it $D_S$).

We introduce the following space obtained by a real interpolation

$$D_q^{1-\frac{1}{s},s}(\Omega) = \left(L^q_{div}(\Omega), W^{2,q}_{div}(\Omega)\right)_{1-\frac{1}{s},s}\,, \tag{6.26}$$

where

$$L^q_{div}(\Omega) = \overline{\{\mathbf{u} \in C_0^\infty(\Omega), \operatorname{div} \mathbf{u} = 0\}}^{\|\cdot\|_{L^q}} \qquad \text{for } D_S \text{ and } C_S, \tag{6.27}$$

or

$$L^q_{div}(\Omega) = \overline{\left\{\mathbf{u} \in C^\infty_{per}(\overline{\Omega}), \operatorname{div} \mathbf{u} = 0, \int_\Omega \mathbf{u}\, \mathrm{d}\mathbf{x} = 0\right\}}^{\|\cdot\|_{L^q}} \qquad \text{for } P_S, \tag{6.28}$$

and

$$W^{2,q}_{div}(\Omega) = \left\{\mathbf{u} \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega), \operatorname{div} \mathbf{u} = 0\right\} \qquad \text{for } D_S \text{ and } C_S, \tag{6.29}$$

or

$$W^{2,q}_{div}(\Omega) = \left\{\mathbf{u} \in W^{2,q}_{per}(\Omega), \operatorname{div} \mathbf{u} = 0, \int_\Omega \mathbf{u}\, \mathrm{d}\mathbf{x} = 0\right\} \qquad \text{for } P_S. \tag{6.30}$$

We have

**Theorem 6.2.1** (Giga, Sohr)**.** *Let* $\mathbf{v}_0 \in D_p^{1-1/q,q}(\Omega)$*,* $\Omega \in C^{2,\lambda}$ *for* $\lambda > 0$ *(for* $D_S$*),* $\mathbf{f} \in L^q(0,T,L^p(\Omega))$*,* $1 < p,q < \infty$*. Then there exists unique solution to* $C_S$*,* $P_S$ *or* $D_S$ *and the solution satisfies*

$$\begin{aligned} \left\|\nabla^2 \mathbf{v}\right\|_{L^q(0,T,L^p)} + \|\nabla p\|_{L^q(0,T,L^p)} + \left\|\frac{\partial \mathbf{v}}{\partial t}\right\|_{L^q(0,T,L^p)} \le \\ \le C\left(\|\mathbf{f}\|_{L^q(0,T,L^p)} + \|\mathbf{v}_0\|_{D_p^{1-1/q,q}}\right). \end{aligned} \tag{6.31}$$

*The constant* $C$ *does not depend on* $T$*.*

*Proof.* See [15]. □

Next we assume $\mathbf{f} = \operatorname{div} \mathbf{F}$ in (6.25) with $\mathbf{v}_0 = 0$. The nonzero initial condition may also be treated, however, it is more complicated and we do not need it. We have

**Theorem 6.2.2** (Giga, Giga, Sohr)**.** *Let* $\Omega \in C^{2,\lambda}$*,* $\lambda > 0$ *for the problem* $D_S$*,* $1 < p,q < \infty$ *and* $\mathbf{F} \in L^q(0,T,L^p(\Omega))$*. Then there exists unique weak solution to (6.25) (either* $C_S$ *or* $D_S$ *or* $P_S$*) with* $\mathbf{f} = \operatorname{div} \mathbf{F}$ *and* $\mathbf{v}_0 = 0$ *and it satisfies*

$$\|\nabla \mathbf{v}\|_{L^q(0,T,L^p)} \le C\, \|\mathbf{F}\|_{L^q(0,T,L^p)}\,. \tag{6.32}$$

*The constant* $C$ *does not depend on* $T$*.*

*Proof.* See [14]. □

### 6.2.2 Navier-Stokes problem

Next we consider the Navier-Stokes problem

$$\left.\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p = \mathbf{f} \\ \operatorname{div}\mathbf{v} = 0 \end{aligned}\right\} \text{in } (0,T)\times\Omega, \qquad \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \Omega\,, \tag{6.33}$$

with the same boundary conditions as for the Stokes problem (we denote the corresponding problems by $D_{NS}$, $P_{NS}$ or $C_{NS}$ as above). We also denote

$$W^{1,2}_{div}(\Omega) = \left\{\mathbf{u}\in W^{1,2}_0(\Omega), \operatorname{div}\mathbf{u} = 0\right\} \qquad \text{for } D_{NS} \text{ and } C_{NS}, \tag{6.34}$$

$$W^{1,2}_{div}(\Omega) = \left\{\mathbf{u}\in W^{1,2}_{per}(\Omega), \operatorname{div}\mathbf{u} = 0, \int_\Omega \mathbf{u}\,d\mathbf{x} = \mathbf{0}\right\} \qquad \text{for } P_{NS}. \tag{6.35}$$

Concerning the existence, we have the following classical results (for proofs see e.g. [44])

**Theorem 6.2.3** (2D)**.** *Let $\Omega\in C^2$ (for $D_{NS}$), $\mathbf{f}\in L^2((0,T)\times\Omega)$, $\int_\Omega \mathbf{f}\,d\mathbf{x} = \mathbf{0}$ (for $P_{NS}$) $\mathbf{v}_0 \in W^{1,2}_{div}(\Omega)$. Then there exists unique global-in-time strong solution to the Navier-Stokes system (6.33) such that*

$$\mathbf{v}\in L^\infty(0,T,W^{1,2}_{div}(\Omega))\cap L^2(0,T,W^{2,2}(\Omega)) \tag{6.36}$$

*and*

$$\frac{\partial\mathbf{v}}{\partial t}, \nabla p \in L^2((0,T)\times\Omega). \tag{6.37}$$

**Theorem 6.2.4** (3D)**.** *Let $\Omega$ be an open set, $\mathbf{f}\in L^2((0,T)\times\Omega)$, $\int_\Omega \mathbf{f}\,d\mathbf{x} = \mathbf{0}$ (for $P_{NS}$), $\mathbf{v}_0\in W^{1,2}_{div}(\Omega)$. Then there exists a global-in-time weak solution to the Navier-Stokes system (6.33) such that*

$$\mathbf{v}\in L^\infty(0,T,L^2(\Omega))\cap L^2(0,T,W^{1,2}_{div}(\Omega)) \tag{6.38}$$

*and*

$$\frac{\partial\mathbf{v}}{\partial t} \in L^{4/3}((0,T),(W^{1,2}_{div}(\Omega))^*). \tag{6.39}$$

*Moreover, if $\Omega\in C^2$ (for $D_{NS}$), there exists $T^*>0$ such that in $(0,T^*)$ this solution is the strong one, unique in the class of all weak solutions satisfying the energy inequality, i.e.*

$$\mathbf{v}\in L^\infty(0,T^*,W^{1,2}_{div}(\Omega))\cap L^2(0,T^*,W^{2,2}(\Omega)) \tag{6.40}$$

*and*

$$\frac{\partial\mathbf{v}}{\partial t}, \nabla p \in L^2((0,T^*)\times\Omega). \tag{6.41}$$

## 6.3 Fixed point theorems

We recall here a classical version of Banach fixed point theorem.

**Theorem 6.3.1** (Banach). *Let $X$ be a complete metric space with the metric $d$. Let $\mathcal{T} : X \to X$ be a contraction mapping on $X$, this means that there is $\rho \in [0, 1)$ such that*

$$d(\mathcal{T}u, \mathcal{T}v) \leq \rho d(u, v) \quad \forall u, v \in X. \tag{6.42}$$

*Then the mapping $\mathcal{T}$ admits one and only one fixed point $u^* \in \mathbf{X}$, i.e. $\mathcal{T}(u^*) = u^*$.*

*Proof.* This theorem was first proved in [1]. □

Most of our proofs are based on the following generalization of this classical theorem

**Theorem 6.3.2** (Banach generalized). *Let $X$ be a reflexive Banach space or let $X$ have a separable pre-dual. Let $H$ be a convex, closed and bounded subset of $X$ and let $X \hookrightarrow Y$, where $Y$ is a Banach space. Let $\mathcal{T} : X \to X$ maps $H$ into $H$ and let*

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_Y \leq \rho \|u - v\|_Y \quad \forall u, v \in H, \quad \rho < 1. \tag{6.43}$$

*Then there exists unique fixed point of $\mathcal{T}$ in $H$.*

*Proof.* Let $u_0 \in H$ be an arbitrary element. Define $u_n = \mathcal{T}u_{n-1}$ $\forall n \in \mathbb{N}$. Operator $\mathcal{T}$ is contraction in $Y$, hence there exists $u^* \in Y$ such that $u_n \to u^*$ in $Y$. As $u_n$ is bounded in $X$, there exists $\overline{u} \in X$ such that $u_n \rightharpoonup \overline{u}$ (or $u_n \rightharpoonup^* \overline{u}$) in $X$. Evidently, $u^* = \overline{u}$ due to the continuous embedding and also $u^* = \overline{u} \in H$ due to the fact that $H$ is weakly (weakly*) closed. The fact that $u$ is a fixed point of $\mathcal{T}$ as well as its uniqueness follows exactly as in the proof of the classical Banach fixed point theorem 6.3.1. □

# Bibliography


[1] Banach, S.: *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales.* Fund. Math., **3** (1922), 133-181.

[2] Barrett, J. W., Schwab, Ch., Süli, E.: *Existence of global weak solutions for some polymeric flow models*, Math. Models Methods Appl. Sci., **15** (2005), 939–984.

[3] Barrett, J. W., Süli, E.: *Existence of global weak solutions to dumbbell models for dilute polymers with microscopic cut-off*, Math. Models Methods Appl. Sci., **18** (2008), 935–971.

[4] Barrett, J.W., Süli, E.: *Existence of global weak solutions to some regularized kinetic models for dilute polymers*, Multiscale Model. Simul., **6** (2007), 506–546.

[5] Chen Y., Zhang P.: *The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions*, Comm. Part. Diff. Eq., **31** (2006), 1793–1810.

[6] Coscia, V., Galdi, G. P.: *Existence, Uniqueness and Stability of Regular Steady Motions of a Second-Grade Fluid*, Int. J. Non-Linear Mechanics, **29** No. 4 (1994), 493–506.

[7] DalMaso, G., Murat, F.: *Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems*, Nonlinear Anal. TMA, **31** (3–4) (1998), 405–412.

[8] Dunn, J. E., Fosdick, R. L.: *Thermodynamics, Stability and Boundedness of Fluids of Complexity 2 and Fluids of Second Grade*, Arch. Rational Mech. Anal., **56** (1974), 191–252.

[9] E, W., Li, T., Zhang, P.: *Well-posedness for the dumbbell model of polymeric fluids*, Comm. Math. Phys., **248** (2004), 409–427.

[10] Finn, R.: *Estimates at infinity for stationary solution of the Navier-Stokes equations*, Bult. Math. de la Soc. Sci. Math. de la R. P. R., Tome 3, **51** 4 (1959), 387–418.

[11] Frehse, J., Málek, J., Steinhauer, M.: *On Existence Results for Fluids with Shear Dependent Viscosity - Unsteady Flows*, Partial Differential Equations, Proceedings of the ICM'98 satellite conference, Praha, 1998, Chapman & Hall/CRC, Res. Notes Math., Boca Raton, FL, **406** (2000), 121–129.

[12] Galdi, G. P.: **An Introduction to the Mathematical Theory of the Navier-Stokes Equations I**, Springer Tracts in Natural Philosophy, Vol. **38**, Springer, New York, (1994).

[13] Galdi, G. P., Padula, M., Rajagopal, K. R.: *On the conditional stability of the rest state of a fluid of second grade in unbounded domains*, Arch. Rational Mech. Anal., **109** No. 2 (1990), 173–182.

[14] Giga, M., Giga, Y., Sohr, H.: *$L^p$ estimates for the Stokes system*, in: **Functional analysis and related topics, 1991 (Kyoto)**, Lecture Notes in Math., 1540, Springer, Berlin, (1993), 55–67.

[15] Giga, Y., Sohr, H.: *Abstract $L^p$ Estimates for the Cauchy Problem with Applications to the Navier–Stokes Equations in Exterior Domains*, J. Funct. Anal., **102** no. 1 (1991), 72–94.

[16] Gurtin, M.E.: **An Introduction to Continuum Mechanics**, Academic Press, (1981).

[17] Jourdain, B., Lelièvre, T., Le Bris, C.: *Existence of solution for a micro-macro model of polymeric liquid: The FENE model*, J. Func. Anal., **209** (2004), 162–193.

[18] Koch, H.: *Partial differential equations and singular integrals*, Dispersive nonlinear problems in mathematical physics, Quad. Mat., Dept. Math., Seconda Univ. Napoli, Caserta, **15** (2004), 59–122.

[19] Konieczny, P., Kreml, O.: *On the 3D steady flow of a second grade fluid past an obstacle*, J. Math. Fluid Mech., submitted.

[20] Kračmar, S., Novotný, A., Pokorný, M.: *Estimates of three-dimensional Oseen kernels in weighted $L^p$ spaces*, in: da Veiga, B., Sequeira, A., Videman, J. (Eds.): **Applied Functional Analysis**, (1999), 281–316.

[21] Kreml, O., Pokorný, M.: *On the local strong solutions for a system describing the flow of a viscoelastic fluid*, Banach Center Publications, **86** (2009), 195–206.

[22] Kreml, O., Pokorný, M.: *On the local strong solutions for the FENE dumbbell model*, to appear in DCDS-S (Vol. 3, No. 2).

[23] Kufner, A., Fučík, S., John, O.: **Function Spaces**, Academia, Praha, (1977).

[24] Leigh, D.C.: **Nonlinear Continuum Mechanics**, McGraw-Hill, New York, (1968).

[25] Lin F., Liu C., Zhang P.: *On hydrodynamics of viscoelastic fluids*, Comm. Pure Appl. Math., **58** (2005), 1437–1471.

[26] Lions, P.-L., Masmoudi, N.: *Global existence of weak solutions to micro-macro models*, C. R. Math. Acad. Sci. Paris, **345** (2007), 15–20.

[27] Lions P.-L., Masmoudi N.: *Global solutions for some Oldroyd models of non-Newtonian flows*, Chinese Ann. Math. Ser. B, **21** no. 2 (2000), 131–146.

[28] Liu C., Walkington, N. J.: *An Eulerian description of fluids containing visco-elastic particles*, Arch. Ration. Mech. Anal., **159** no. 3 (2001), 229–252.

[29] Maremonti, P.: *Some interpolation inequalities Involving Stokes Operator and First Order Derivatives*, Annali di Mathematica Pura ed Applicata, **175** no. 1 (1998), 59–91.

[30] Masmoudi, N.: *Well posedness for the FENE dumbbell model of polymeric flows*, Comm. Pure Appl. Math., **61** (2008), 1685–1714.

[31] Maxwell, J.C.: *On the Dynamical Theory of Gases*, Phil. Trans. Roy. Soc. London A157, (1867), 49–88.

[32] Mogilevskii, I. Sh., Solonnikov V. A.: *Problem on a Stationary Flow of a Second-Grade Fluid in Hölder Classes of Functions*, Zapisky Nauc. Sem. LOMI, **243** (1997), 154–165.

[33] Novotný, A.: *About the steady transport equation*, Proceedings of the Fifth Winter School at Paseky, Pitman Research Notes in Mathematics, (1998).

[34] Novotný, A., Pokorný, M.: *Three-dimensional Steady Flow of Viscoelastic Fluid past an Obstacle*, J. Math. Fluid Mech., **2** (2000), 294–314.

[35] Novotný, A., Straškraba I.: **Introduction to the Mathematical Theory of Compressible Flow**, in "Oxford Lecture Series in Mathematics and Its Applications", Oxford University Press, **27** (2004).

[36] Oldroyd, J.G.: *On the Formulation of Rheological Equations of State*, Proc. Roy. Soc. London A200, (1950), 523–541.

[37] Pokorný, M.: **Asymptotic behaviour of solutions to certain PDE's describing the flow of fluids in unbounded domains**, Ph.D. thesis, Charles University in Prague and University of Toulon and Var, Toulon-La Garde, (1999).

[38] Renardy, M.: *An existence theorem for model equations resulting from kinetic theories of polymer solutions*, SIAM J. Math. Anal., **22** (1991), 313–327.

[39] Rivlin, R.S.: *Further remarks on the stress-deformation relations for isotropic materials*, J. Rational Mech. Anal., **4** (1955), 681–702.

[40] Rivlin, R.S., Ericksen, J.L.: *Stress-deformation relations for isotropic materials*, J. Rational Mech. Anal., **4** (1955), 323–425.

[41] Smith, D. R.: *Estimates at Infinity for Stationary Solutions of the N. S. Equations in Two Dimensions*, Arch. Rat. Mech. Anal., **20** (1965), 341–372.

[42] Sureshkumar, R., Beris, A.N.: *Effect of artificial stress diffusivity on the stability of numerical calculations and the flow dynamics of time-dependent viscoelastic flows*, J. Non-Newtonian Fluid Mech., **60** (1995), 53–80.

[43] Tartar, L.: **An Introduction to Sobolev Spaces and Interpolation**, Springer, Berlin, (2007).

[44] Temam, R.: **Navier-Stokes equations. Theory and numerical analysis**, Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, (2001).

[45] Truesdell, C., Noll, W.: **The Nonlinear Field Theories of Mechanics**, Springer, Berlin, (1992).

[46] Zhang, H., Zhang, P.: *Local existence for the FENE-dumbbell model of polymeric liquids*, Arch. Ration. Mech. Anal., **181** (2006), 373–400.