An Axiomatic Approach to Image Interpolation

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Abstract

We discuss possible algorithms for interpolating data given in a set of curves and/or points in the plane. We propose a set of basic assumptions to be satisfied by the interpolation algorithms which lead to a set of models in terms of possibly degenerate elliptic partial differential equations. The Absolute Minimal Lipschitz Extension model (AMLE) is singled out and studied in more detail. We show experiments suggesting a possible application, the restoration of images with poor dynamic range.

1 Introduction

Our purpose in this paper will be to discuss possible algorithms for interpolating scalar data given on a set of points and/or curves in the plane. Our main motivation comes from the field of image processing. A number of different approaches using interpolation techniques have been proposed in the literature for 'perceptually motivated' coding applications [6, 17, 22]. The underlying image model is based on the concept of 'raw primal sketch' [18]. The image is assumed to be made mainly of areas of constant or smoothly changing intensity separated by discontinuities represented by strong edges. The coded information, also known as *sketch data*, consists of the geometric structure of the discontinuities and the amplitudes at the edge pixels. In very low bit rate applications, the decoder has to reconstruct the smooth areas in between by using the edge information. This can be posed as a scattered data interpolation problem from an arbitrary initial set (the sketch data) under certain smoothness constraints. For higher bit rates, the residual texture information has to be separately coded by means of a waveform coding technique, for instance, pyramidal or transform coding. In the following we assume that a set of curves and points is given and we want to construct a continuous function interpolating these data. Several interpolation techniques using implicitely or explicitely the solution of a partial differential equation have been used in the engineering literature [5, 6, 7]. In the spirit of [1], our approach to the problem will be based on a set of formal requirements that any interpolation operator in the plane should satisfy. Then we show that any operator which interpolates continuous data given on a set of curves can be given as the viscosity solution of a degenerate elliptic partial differential equation of a certain type. The examples include the Laplacian operator and the minimal Lipschitz extension operator [15] which is related to the work of J. Casas [6, 7]. We also discuss other interpolation schemes proposed in the literature.

Before starting with the theory, we wish to give it a flavour by very simple heuristic arguments. The main differential operators we discuss here arise inmediately from the mere consideration of which kind (linear?, nonlinear?) of mean value property an interpolant function

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$u(x)$ must have. Assume that we know $u(x)$ at all pixels except one, $x_0 \in I\!R^2$. What is the value to be chosen at $x_0 \in I\!R^2$? The possibilities are three and correspond to more and more adventurous decisions:
1) $u(x_0)$ is a mean value of the neighbouring pixels.
2) $u(x_0)$ is a median value of the neighbouring pixels.
3) $u(x_0)$ is obtained by propagation from neighbouring pixels. We shall make this definition more precise below.

Let us now assume that a function is an interpolant of itself. That is, it satisfies for all x_0, $u(x_0) = (\textit{mean value}\ (u(x)))$ on a neighbourhood, no matter what we mean by "mean value". Then, returning to the above possibilities and assuming that u is C^2, we have:
1) $u(x_0) = \frac{1}{4}(u(x_0+(h,0))+u(x_0-(h,0))+u(x_0+(0,h))+u(x_0-(0,h)))$. Taking the difference and letting $h \to 0$, it is easily seen by Taylor expansion that this implies

$$\Delta u(x_0) = \frac{\partial^2 u}{\partial x_1^2}(x_0) + \frac{\partial^2 u}{\partial x_2^2}(x_0) = 0.$$

This is the "standard" interpolation. The above calculation does not depend upon the kind of linear mean value algorithm. See [13].
2) $u(x_0) = \textit{median value}\ \{u(y),\quad y \in D(x_0,h)\}$, where $D(x_0,h)$ is a disk with radius h. In this case, it can be proved (see [13]) that by letting $h \to 0$ and some manipulations, we get

$$curv(u)(x_0) = \frac{1}{|Du|^3} D^2u(Du^\perp, Du^\perp) = \frac{u_{x_2}^2 u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_1}^2 u_{x_2x_2}}{(u_{x_1}^2 + u_{x_2}^2)^{3/2}} = 0$$

where $curv(u)(x_0)$ is the curvature of the level line passing by x_0 and $Du^\perp$ is orthogonal to Du, $|Du^\perp| = |Du|$, $Du = (u_{x_1}, u_{x_2})$ being the gradient of u and D^2u the Hessian of u, i.e. the matrix of the second derivatives of u. Here and in all what follows we shall use the notation $A(x,y) = \sum_{i,j=1}^2 a_{ij}x_iy_j$, where $A = (a_{ij})_{i,j}$ is a 2×2 matrix and $x, y \in I\!R^2$.
3) In the case of propagation (see Fig. 1), let us take as an example the remarkable Casas and Torres interpolation algorithm [7]. It is easily seen that if u is C^2 at x and if $u(x)$ is obtained by this interpolation algorithm, then we can write

$$u(x) = \frac{1}{2}\left(u(x+hDu) + u(x-hDu)\right) + o(h^2).$$

Letting $h \to 0$ and using here again a Taylor expansion, one gets easily

$$D^2u(Du, Du) = 0.$$

We shall give more details on this algorithm before the end of this introduction.

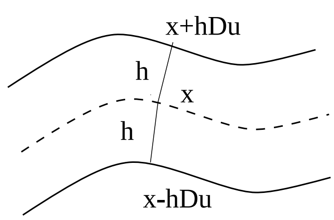

Figure 1

In conclusion, we see that three different classical or new interpolation processes suggest that the interpolant function must be solution of one of three elliptic PDE's

$$\Delta u = 0. \tag{1}$$

$$D^2u(Du^\perp, Du^\perp) = 0, \tag{2}$$

$$D^2u(Du, Du) = 0, \tag{3}$$

Notice that the first equation is nothing but the sum of the two last ones. This sum yields $|Du|^2\Delta u = 0$.

We shall not develop further the analysis of simple interpolation processes. There is no need for doing separate analyses as sketched above. Indeed, we shall show that the *axiomatic approach* not only permits to retrieve the preceding operators, but also to identify *all possible* operators, given sound assumptions on the interpolation process.

In fact, our axiomatic analysis will prove that the three above operators (1), (2), (3) essentially describe all the choices we have for an interpolation method. Now, the second one (2) will be proved not to give necessarily a solution. The first one (1) is excellent and standard, but does not permit to interpolate isolated points. It is well known that the problem $\Delta u = 0$, $u = 0$ on $\partial D(0,r)$, $u(0) = 1$ has no solution. We have the same imposibility with Equation (2). Equation (3), instead, yields a cone function $u(x) = |x| - 1$, as interpolant. Equation (3) is somewhat new as far as image interpolation is concerned. Using Aronsson's [3] and Jensen's [15] results, we shall show that we can indeed define for every Lipschitz datum defined at curves and points a Lipschitz interpolant. This method is inspired from Casas-Torres [7], but it must be made clear that the Casas-Torres algorithm does not create necessarily a continuous interpolant, in contrast with Aronsson's method [3] (see also [15]). Let us explain briefly why. In the case of an initial datum u_0 defined on a set $?_0$ of curves γ_i and points P_i, such that $u_0 = cte$ on each γ_i, the Casas-Torres definition is as follows:
a) Compute the skeleton $\tilde{?}_1$ of $I\!R^2 \setminus ?_0$.
b) For every point x in $\tilde{?}_1$: if $\tilde{?}_1$ is a simple curve at x, then there are two points in $?_0$, y and z, such that $d(x,y) = d(x,z) = d(x,?_0)$. Then set $u(x) = \frac{u(y)+u(z)}{2}$. If x is a multiple point of $\tilde{?}_1$, define similarly $u(x)$ as the mean value of all points z in $?_0$ such that $d(x,z) = d(x,?_0)$.
c) Take $?_1 = ?_0 \cup \tilde{?}_1$ and iterate.

This process defines u on a dense subset $\cup_n ?_n$ of $I\!R^2$ and an extension of u to the whole plane would be possible, was $u|_{\cup_n \Gamma_n}$ continuous. Unfortunately, this is not the case. Take (e.g.)

$$?_0 = \{(0,0),(0,2),(2,0),(2,2)\}$$

with $u(0,0) = 0$, $u(2,0) = 2$, $u(2,0) = 6$ and $u(2,2) = 4$. Then $?_1$ is made of four half lines L_i, $i = 1,2,3,4$ with end points at $(1,1)$ (see Fig. 2)

$$L_1 = \{(x,y): \quad x = 1, \quad y < 1\} \qquad u|_{L_1} = 1,$$

$$L_2 = \{(x,y): \quad y = 1, \quad x < 1\} \qquad u|_{L_2} = 3,$$

$$L_3 = \{(x,y): \quad y = 1, \quad x > 1\} \qquad u|_{L_3} = 3,$$

$$L_4 = \{(x,y): \quad x = 1, \quad y > 1\} \qquad u|_{L_4} = 5.$$

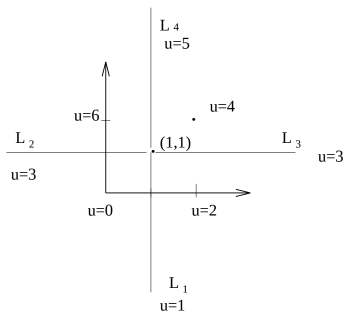

Figure 2

Clearly, $(1,1)$ is a discontinuity point for u and remains so in the iteration process, since this iteration process does not modify the acquired values of u.

We refer to the excellent and far ranging review by Powell [20] on the numerical methods for interpolation of scattered data points. From his review follows that we have essentially three classes of interpolation algorithms for scattered points.
a) The Delaunay triangulation ([11]), followed by piecewise polynomial interpolation.
b) The Shepard's global method, which permits C^∞ interpolation.
c) The radial basis function method.

It is easily seen that no one of these methods is adapted to image processing. Indeed, first of all, in image processing, we have not only points as data for interpolation, but also pieces of curves, or very dense sets of points. This makes all three methods difficult to implement and numerically unstable. Next, it is trivial to notice that none of these methods is stable. We mean that given a datum u_0 and an interpolant u, it may well be asked that if we interpolate u itself, we get back to u. This is not the case for a), b), c). Now, we will prove that it is possible to define stable interpolation methods in the preceeding sense.

Here, we meet a peculiarity of image processing with respect to classical numerical analysis. In numerical analysis, it is generally desirable that the interpolant be C^2 or more. We shall prove that no stable method can yield a C^2 interpolant if we, in addition, ask the method to interpolate data defined on both curves and points. In contrast, we shall prove that the equation $D^2u(Du, Du) = 0$ defines a stable method with smooth enough (but not C^2) interpolant. In theory, we only know that the interpolant is Lipschitz. In practice, experiments prove that its regularity is more than enough as far as visual comfort is asked.

Operator (3) is in no way new in Computer Vision. In fact, it has been proposed as edge detector by Havens and Strikwerda [14], Torre and Poggio [25] and Yuille [26]. It also appears in an early work by Prewitt [21] in the context of edge enhancement. The whole Canny edge detection theory [4] is based on it. Its use is following. As is proved below (Proposition 2), at points where $|Du|$ is maximal in the direction of the gradient one has by differentiation $D^2u(Du, Du) = 0$. Such points are defined as edge points by the above mentionned authors. Thus, it is reasonable to impose the condition $D^2u(Du, Du) = 0$ in regions where we are interpolating the image. This only means that in these regions all points are edge points, or, rather, that none of them has any advantage as a candidate to be an edge. Of course, the same considerations apply to Operator (2) and the Marr-Hildreth edge detection theory [19].

Our plan is as follows. In Sect. 2 we introduce a formal set of axioms which should be satisfied by any interpolation operator in the plane and derive the associated partial differential equation. In Sect. 3 we discuss several examples of interpolation operators relating them to the set of axioms studied in the previous section. Section 4 is devoted to a detailed study of one of the interpolation operators given in Sect. 3, the so called AMLE model, giving existence and uniqueness results for the associated PDE. Its numerical analysis is given in Sect. 5. Finally, in Sect. 6 we display some experimental results obtained by using the previous model. Although the whole theory will be developped in $\mathbb{R}^2$, there are very few alterations in order to extend it to $\mathbb{R}^n$.

2 Axiomatic analysis of interpolation operators

We begin by recalling the definition of a continuous simple Jordan curve.

Definition: A continuous function $? : [a,b] \to I\!R^2$ is a continuous simple Jordan curve if it is one-to-one in (a,b) and $?(a) = ?(b)$. By Alexandroff Theorem, such a curve surrounds a bounded simply connected domain which we denote by $D(?)$.

Let $\mathcal{C}$ be the set of continuous simple Jordan curves in $I\!R^2$. For each $? \in \mathcal{C}$, let $\mathcal{F}(?)$ be the set of continuous functions defined on $?$. We shall consider an interpolation operator as a transformation E which associates with each $? \in \mathcal{C}$ and each $\varphi \in \mathcal{F}(?)$ a unique function $E(\varphi, ?)$ defined in the region $D(?)$ inside $?$ satisfying the following axioms:

$(A1)$ Comparison principle:

$$E(\varphi, ?) \leq E(\psi, ?) \quad \text{for any } ? \in \mathcal{C} \quad \text{and any } \varphi, \psi \in \mathcal{F}(?) \quad \text{with } \varphi \leq \psi$$

$(A2)$ Stability principle:

$$E(E(\varphi, ?)\mid_{\Gamma'}, ?') = E(\varphi, ?)\mid_{D(\Gamma')}$$

for any $? \in \mathcal{C}$, any $\varphi \in \mathcal{F}(?)$ and $?' \in \mathcal{C}$ such that $D(?') \subseteq D(?)$.

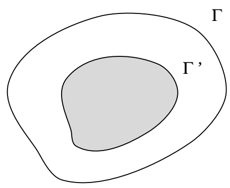

Figure 3

This principle means that no new application of the interpolation can improve a given interpolant. If this were not the case, we should iterate the interpolation operator indefinitely until a limit interpolant satisfying $(A2)$ is attained.

For the next principle, we denote by $SM(2)$ the set of symmetric two-dimensional matrices.
$(A3)$ Regularity principle: Let $A \in SM(2)$, $p \in I\!R^2 - \{0\}$, $c \in I\!R$ and

$$Q(y) = \frac{A(y-x, y-x)}{2} + < p, y-x > + c.$$

(where $< x, y >= \sum_{i=1}^{2} x_i y_i$). Let $D(x,r) = \{y \in I\!R^2 : \|y - x\| \leq r\}$ and $\partial D(x,r)$ its boundary. Then

$$\frac{E(Q\mid_{\partial D(x,r)}, \partial D(x,r))(x) - Q(x)}{r^2/2} \to F(A, p, c, x) \quad \text{as } r \to 0+ \tag{4}$$

where $F : SM(2) \times I\!R^2 - \{0\} \times I\!R \times I\!R^2 \to I\!R$ is a continuous function.
This assumption is much weaker than what it appears to be. Indeed, assume only that given A, p, c, x, we can find a C^2 function u such that $D^2u(x) = A$, $Du(x) = p$, $u(x) = c$, such that the differentiability assumption (4) holds (with u instead of Q). Then, arguing as in Theorem 1 below it is easily proven that (4) holds for all C^2 functions and in particular for Q.

Together with these basic axioms, let us consider the following axioms which express obvious independence properties of the interpolation process with respect to the observer's, standpoint and the grey level encoding scale.

$(A4)$ Translation invariance:

$$E(\tau_h\varphi, ? - h) = \tau_h E(\varphi, ?)$$

where $\tau_h\varphi(x) = \varphi(x+h)$, $h \in I\!R^2$, $\varphi \in \mathcal{F}(?)$, $? \in \mathcal{C}$. The interpolant of a translated image is the translated of the interpolant.

$(A5)$ Rotation invariance:

$$E(R\varphi, R?) = RE(\varphi, ?)$$

where $R\varphi(x) = \varphi(R^t x)$, R being an orthogonal map in $I\!R^2$, $\varphi \in \mathcal{F}(?)$, $? \in \mathcal{C}$. The interpolant of a rotated image is the rotated of the interpolant.

$(A6)$ Grey scale shift invariance:

$$E(\varphi + c, ?) = E(\varphi, ?) + c$$

for any $? \in \mathcal{C}$, any $\varphi \in \mathcal{F}(?)$, $c \in I\!R$.

$(A7)$ Linear grey scale invariance:

$$E(\lambda\varphi, ?) = \lambda E(\varphi, ?) \quad \text{for any } \lambda \in rho$$

$(A8)$ Zoom invariance:

$$E(\delta_\lambda\varphi, \lambda^{\perp 1}?) = \delta_\lambda E(\varphi, ?)$$

where $\delta_\lambda\varphi(x) = \varphi(\lambda x)$, $\lambda > 0$. The interpolant of a zoomed image is the zoomed interpolant.

Axioms (A1),(A3) and (A4) to (A8) are obvious adaptations from the axiomatic developped in [1]. The results below are also proved along the same lines.

Theorem 1 *Assume that E is an interpolation operator satisfying $(A1)$, $(A2)$, $(A3)$. Then, for any smooth function φ in $I\!R^2$ and any $x \in I\!R^2$ such that $D\varphi(x) \neq 0$ we have*

$$\frac{E(\varphi\mid_{\partial D(x,r)}, \partial D(x,r))(x) - \varphi(x)}{r^2/2} \to F(D^2\varphi(x), D\varphi(x), \varphi(x), x) \tag{5}$$

as $r \to 0+$. Moreover $F(A,p,c,x)$ is a nondecreasing function of A.

Proof. Without loss of generality we may assume that $x = 0$. Let $\varphi \in C_b^2$ be such that $D\varphi(0) \neq 0$. For each $\varepsilon \in I\!R$, let

$$Q_\varepsilon(x) = \varphi(0) + D\varphi(0)x + \frac{1}{2}D^2\varphi(0)(x,x) + \frac{\varepsilon}{2} < x, x > .$$

Then, in a neighborhood of 0 we have

$$Q_{\perp\varepsilon}(x) \leq \varphi(x) \leq Q_\varepsilon(x)$$

$\varepsilon > 0$. By the comparison principle $(A1)$ and the stability principle $(A2)$, for r small enough,

$$G(-\varepsilon, r) \leq E(\varphi\mid_{\partial D(0,r)}, \partial D(0,r))(0) \leq G(\varepsilon, r)$$

where $G(\eta, r) = E(Q_\eta\mid_{\partial D(0,r)}, \partial D(0,r))(0)$, $\eta \in I\!R$. Since $Q_{\perp\varepsilon}(0) = Q_\varepsilon(0) = \varphi(0)$,

$$G(-\varepsilon, r) - Q_{\perp\varepsilon}(0) \leq E(\varphi\mid_{\partial D(0,r)}, \partial D(0,r))(0) - \varphi(0) \leq G(\varepsilon, r) - Q_\varepsilon(0).$$

Dividing by $r^2/2$ and letting $r \to 0$ in the previous inequality, using the regularity axiom $(A3)$, we get

$$
\begin{aligned}
F(D^2\varphi(0)-\varepsilon I, D\varphi(0), \varphi(0), 0) &\le \liminf_{r\to 0} \frac{E(\varphi\mid_{\partial D(0,r)}, \partial D(0,r))(0)-\varphi(0)}{r^2/2} \\
&\le \limsup_{r\to 0} \frac{E(\varphi\mid_{\partial D(0,r)}, \partial D(0,r))(0)-\varphi(0)}{r^2/2} \\
&\le F(D^2\varphi(0)+\varepsilon I, D\varphi(0), \varphi(0), 0).
\end{aligned}
$$

Letting $\epsilon \to 0$, we get (5). To check that $F(A, p, c, x)$ is nondecreasing with respect to A, let $A_1, A_2 \in SM(2)$, $A_1 \le A_2$, $p \in I\!R^2$ $c \in I\!R$, and let

$$
Q_i(y) = \frac{A_i(y-x, y-x)}{2} + < p, y-x > + c \quad i = 1, 2.
$$

Since $Q_1(y) \le Q_2(y)$ and $Q_1(x) = Q_2(x)$, by the comparison principle $(A1)$

$$
E(Q_1\mid_{\partial D(x,r)}, \partial D(x,r)) - Q_1(x) \le E(Q_2\mid_{\partial D(x,r)}, \partial D(x,r)) - Q_2(x).
$$

Dividing by $r^2/2$ and letting $r \to 0$, we get

$$
F(A_1, p, c, x) \le F(A_2, p, c, x). \quad \square
$$

Theorem 2 *Assume that the interpolation operator E satisfies $(A1)$, $(A2)$, $(A3)$. Let $\varphi \in \mathcal{C}(?)$, $u = E(\varphi, ?)$. Then u is a viscosity solution of*

$$
\begin{aligned}
&F(D^2u, Du, u, x) = 0 \quad \text{in } D(?) \\
&u_{|\Gamma} = \varphi,
\end{aligned}
\tag{6}
$$

i. e., for any $\varphi \in C_b^\infty(D(?))$ such that $u - \varphi$ has a local maximum (minimum) at $x = x_0$ and $D\varphi(x_0) \neq 0$, then $F(D^2\varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0) \ge 0$ (resp., ≤ 0).

Proof. Let $\varphi \in C_b^\infty(D(?))$ and suppose that $u - \varphi$ has a local maximum at $x = x_0$ and $D\varphi(x_0) \neq 0$. Then for some $r > 0$

$$
u(x) \le \varphi(x) + u(x_0) - \varphi(x_0) \quad \text{in } \partial D(x_0, r).
$$

Using the comparison principle and the grey scale shift invariance,

$$
E(u_{|\partial D(x_0,r)}, \partial D(x_0, r))(x_0) - u(x_0) \le E(\varphi + u(x_0) - \varphi(x_0), \partial D(x_0, r))(x_0) - u(x_0).
$$

By the stability principle (A2), the left hand side above is 0. We have

$$
0 \le E(\varphi + u(x_0) - \varphi(x_0), \partial D(x_0, r))(x_0) - u(x_0).
$$

Dividing by $r^2/2$ and letting $r \to 0$, we get

$$
0 \le F(D^2\varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0).
$$

Similarly, if $u - \varphi$ has a local minimum at $x = x_0$,

$$
F(D^2\varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0) \le 0.
$$

Thus, u is a viscosity solution of (6). $\square$

Corollary 1 *Assume that the interpolation operator E satisfies $(A1)$,$(A2)$, $(A3)$, $(A4)$, $(A6)$. Let $\varphi \in \mathcal{C}(?)$, $u = E(\varphi, ?)$. Then u is a viscosity solution, in the sense of the previous theorem, of*

$$
\begin{aligned}
&F(D^2u, Du) = 0 \quad \text{in } D(?) \\
&u_{|\Gamma} = \varphi.
\end{aligned}
\tag{7}
$$

Proof. It suffices to show that $F(A,p,c,x) = F(A,p)$, $A \in SM(2)$, $p \in I\!R^2$, $c \in I\!R$, $x \in I\!R^2$. Let $x_0, h \in I\!R^2$, $r > 0$ and

$$Q(x) = \frac{A(x-x_0, x-x_0)}{2} + < p, x-x_0 > +c.$$

Let $?_r = \partial D(x_0, r)$. Using $(A4)$,

$$E(\tau_h Q_{|\Gamma_r}, ?_r - h)(x_0 - h) - \tau_h Q(x_0 - h) = E(Q_{|\Gamma_r}, ?_r)(x_0) - Q(x_0).$$

According to Theorem 5, dividing by $r^2/2$ and letting $r \to 0$ we get

$$F(A,p,c,x_0 - h) = F(A,p,c,x_0) \quad \text{for any } h \in I\!R^2.$$

Hence F is independent of x and we may write $F(A,p,c,x) = F(A,p,c)$. Similarly, using that $(A6)$ it follows

$$F(A,p,c+k,x) = F(A,p,c,x) \quad \text{for all } k \in I\!R.$$

Thus, F is independent of c. Combining both informations we have (7). □

Lemma 1 *Assume that the interpolation operator E satisfies $(A1)-(A4)$ and $(A6)$. Then,*
i) if E satisfies $(A5)$ then,

$$F(R^t A R, R^t p) = F(A,p), \tag{8}$$

ii) if E satisfies $(A7)$ then,

$$F(\lambda A, \lambda p) = \lambda F(A,p), \tag{9}$$

iii) if E satisfies $(A8)$ then,

$$F(\lambda^2 A, \lambda p) = \lambda^2 F(A,p), \tag{10}$$

where $A \in SM(2)$, $p \in I\!R^2$, $\lambda > 0$ ($\lambda \in I\!R$ if we are in case ii)) and R is any orthogonal matrix in $I\!R^2$.

Proof. Given A, p, R as above, let

$$Q(x) = \frac{A(x,x)}{2} + < p, x > \quad x \in I\!R^2.$$

Let $Q_R(x) = Q(Rx)$. Then $DQ_R(0) = p$, $D^2 Q_R(0) = R^t A R$. Let $?_r = \partial D(0,r)$. By $(A6)$,

$$E(Q_{R_{|\Gamma_r}}, ?_r)(0) - Q_R(0) = E(Q_{|\Gamma_r}, ?_r)(0) - Q(0).$$

Again, dividing by $r^2/2$ and letting $r \to 0$ we get (8). Formulas (9) and (10) follow in a similar way and we shall skip the details. □

Given $p \in I\!R^2$, $p \neq 0$, let R_p be the rotation matrix such that $R_p^t p = |p| e_1$ where $e_1 = (1,0)$.

Corollary 2 *Assume that the interpolation operator E satisfies $(A1)-(A8)$. Let $\varphi \in \mathcal{C}(?)$, $u = E(\varphi, ?)$. Then u is a viscosity solution of*

$$\begin{aligned} &G(R_{\nabla u}^t D^2 u R_{\nabla u}) = 0 \quad \text{in } D(?) \\ &u_{|\Gamma} = \varphi \end{aligned} \tag{11}$$

where $G(A) = F(A, e_1)$, $A \in SM(2)$. Hence, G is a continuous and nondecreasing function of A such that $G(\lambda A) = \lambda G(A)$ for all $\lambda \in I\!R$ and any $A \in SM(2)$.

Proof. Combining (9) and (10) in Lemma 1 we have

$$F(\lambda A, p) = \lambda F(A, p), \quad F(A, \lambda p) = F(A, p)$$

for any $A \in SM(2)$, $p \in I\!R^2$, $\lambda > 0$. Hence if $p \neq 0$, we have

$$F(A, p) = F(R_p^t A R_p, R_p^t p) = F(R_p^t A R_p, |p| e_1) = F(R_p^t A R_p, e_1) = G(R_p^t A R_p)$$

Our claim follows from this and (7). Observe that $G(\lambda A) = \lambda G(A)$, $A \in SM(2)$, $\lambda > 0$. $\square$

Remark. If we assume that linear data on the boundary ? $\in \mathcal{C}$ are linearly interpolated in $D(?)$, i.e.,

$$E(\varphi_{|\Gamma}, ?) = \varphi \quad \text{in } D(?)$$

when $\varphi(x) = < p, x > +c$, $p \in I\!R^2$, $c \in I\!R$, then

$$F(\lambda I, p) = \lambda \quad \text{for any } \lambda \geq 0.$$

From now on, we shall assume that the interpolation operator E satisfies $(A1) - (A8)$. Given a matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

let us write for simplicity $G(a, b, c)$ instead of $G(A)$. Let $\nu = \frac{p}{|p|}$, $p \in I\!R^2$, $p \neq 0$. Then $R_p(x) = e_1 \otimes \nu(x) + e_2 \otimes \nu^\perp(x)$ (where $a \otimes b(x) = < a, x > b$, $a, b, x \in I\!R^2$) and we may write

$$R_p^t D^2 u R_p = \begin{pmatrix} D^2 u(\nu, \nu) & D^2 u(\nu, \nu^\perp) \\ D^2 u(\nu^\perp, \nu) & D^2 u(\nu^\perp, \nu^\perp) \end{pmatrix}$$

Thus we may write equation (11) as

$$G\left(D^2 u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right), D^2 u\left(\frac{Du}{|Du|}, \frac{Du^\perp}{|Du|}\right), D^2 u\left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}\right)\right) = 0.$$

Using the monotonicity of G, we can reduce the number of involved arguments inside G.

Proposition 1 *i) If G does not depend upon its first or its last argument, then it only depends on its last (resp. its first) argument. In other terms,*

$$\text{If} \quad G(\alpha, \beta, \gamma) = \hat{G}(\alpha, \beta), \quad \text{then} \quad G = \hat{G}(\alpha) = \alpha \hat{G}(1),$$

$$\text{If} \quad G(\alpha, \beta, \gamma) = \hat{G}(\beta, \gamma), \quad \text{then} \quad G = \hat{G}(\gamma) = \gamma \hat{G}(1).$$

$\alpha, \beta, \gamma \in I\!R$.

ii) If G is differentiable at 0 then G may be written as $G(A) = Tr(BA)$ where B is a nonnegative matrix.

Proposition 1i) is due to the fact that $A \to A(\nu, \nu^\perp)$ ($\nu \in R^2$, $|\nu| = 1$, $\nu^\perp$ being the vector obtained by rotation of $\pi/2$ of ν) is not a monotone operator with respect to A.

Proof: i) Assume that $G(\alpha, \beta, \gamma) = \hat{G}(\beta, \gamma)$. Let A, B two matrices. Let $a_{22} - b_{22} = \epsilon > 0$ and $a_{12} - b_{12} = \lambda$ for any $\lambda \in I\!R$. Setting $a_{11} = b_{11} + \frac{\lambda^2}{\epsilon^2}$, we see that $(A - B)((x_1, x_2), (x_1, x_2)) = \frac{\lambda^2}{\epsilon^2} x_1^2 + 2\lambda x_1 x_2 + \epsilon x_2^2 = (\frac{\lambda}{\epsilon} x_1 + \epsilon x_2)^2 \geq 0$. Thus, $A \geq B$, which implies

$$G(A) = G(b_{12} + \lambda, b_{22} + \epsilon) \geq G(b_{12}, b_{22}) = G(B)$$

for all $\epsilon > 0$ and for all $\lambda \in I\!R$. Letting $\epsilon \to 0$, we obtain

$$G(b_{12} + \lambda, b_{22}) \geq G(b_{12}, b_{22}), \quad \forall \lambda \in I\!R.$$

Thus $\hat{G}$ does not depend upon its first argument, i.e., $G = \hat{G}(\gamma)$. Moreover, since $\hat{G}$ is continuous and $\hat{G}(\lambda\gamma) = \lambda\hat{G}(\gamma)$ for all $\lambda \in I\!R$, then $\hat{G}(\gamma) = \gamma\hat{G}(1)$ for all $\gamma \in I\!R$.

ii) Let $\alpha, \beta, \gamma \in I\!R$ and $\varepsilon > 0$. Since G is differentiable at $(0,0,0)$ then

$$G(\varepsilon\alpha, \varepsilon\beta, \varepsilon\gamma) = G(0,0,0) + \varepsilon\nabla G(0,0,0) \cdot (\alpha, \beta, \gamma) + o(\varepsilon).$$

Since $G(0,0,0) = 0$ and $G(\varepsilon\alpha, \varepsilon\beta, \varepsilon\gamma) = \varepsilon G(\alpha, \beta, \gamma)$, dividing the above identity by ε and letting $\varepsilon \to 0$ we get that

$$G(\alpha, \beta, \gamma) = a\alpha + 2b\beta + c\gamma$$

where $(a, 2b, c) = \nabla G(0,0,0)$. Observe that the above expression can be written as $Tr(BA)$ where

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

Since G is an increasing function of A, then B must be a nonnegative matrix. $\square$

Thus if we assume that G is diferentiable at $(0,0,0)$ then we may write equation (11) as

$$aD^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) + 2bD^2u\left(\frac{Du}{|Du|}, \frac{Du^\perp}{|Du|}\right) + cD^2u\left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}\right) = 0. \qquad (12)$$

where $a, c \geq 0$, $ac - b^2 \geq 0$ which is the same as to say that the matrix B above is nonnegative. Let us explore which of these operators can be used to interpolate data given on a set of points and/or curves. For that we consider $D = B((0,0),1)$ the ball of center $(0,0)$ and radius 1 and look for a solution U of (12) on $D \setminus (0,0)$ such that $U(0,0) = 1$ and $U(x_1, x_2) = 0$ for $(x_1, x_2) \in \partial D$. Assume that we have existence and uniqueness of solutions of (12). Since the equation and the data are rotation invariant then we may look for a radial solution $U = f(r)$ with $r = (x_1^2 + x_2^2)^{1/2}$ of (12). If U satisfies (12) then f is a solution of

$$arf'' + cf' = 0 \quad 0 < r < 1 \qquad (13)$$

such that $f(0) = 1, f(1) = 0$. In terms of the values of a, b, c we have

i) If $a = 0$, then $b = 0$. If $c = 0$ then we have no equation. If $c > 0$ then $f' = 0$ and the only solution of (13) is $f = constant$. The boundary conditions cannot be satisfied. The are no interpolation operators in this case.

ii) Consider now $a > 0$. Since (13) is an Euler equation the solutions are of the form 1, r^z or $logr$. If $0 \leq c < a$ then $z = 1 - c/a$ and $f(r) = 1 - r^z$. Notice that ∇U is bounded if and only if $z = 1$, i.e. $c = 0$. In that case also $b = 0$ and the equation is

$$D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) = 0 \qquad (14)$$

When $0 < c < a$ the solution exists but the gradient is unbounded at $(0,0)$. If $c = a$ then the general solution of (13) is $f(r) = \alpha + \beta logr$, $\alpha, \beta \in R$ and we cannot match the boundary conditions. Similarly if $c > a$, $f(r) = \alpha + \beta r^z$, $z = 1 - c/a < 0$, $\alpha, \beta \in R$ and again we cannot match the boundary conditions.

This discussion proves that if we require to the interpolation operators described by a smooth function G to be able to interpolate data given on curves and/or points we are forced to assume model (14). As discussed above there are other possibilities with $0 < c < a$ but the gradient may become unbounded even for smooth data at the boundary which means that we are having less regularity than in model (14) which, as we shall see below, always keeps a bound on the gradient if the data have a bounded gradient.

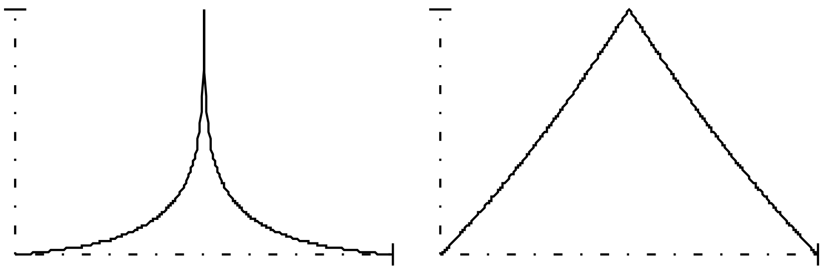

Figure 4: Sectional view of the radial solution of U for the Laplacian and model (14) respectively.

3 Examples

Example 1. Given $? \in \mathcal{C}$ and $\varphi \in \mathcal{F}(?)$ we consider $E_1(\varphi, ?)$ to be the solution of

$$\begin{aligned} &\Delta u = 0 \quad \text{in } D(?) \\ &u_{|\Gamma} = \varphi. \end{aligned} \tag{15}$$

The operator E_1 satisfies all axioms $(A1) - (A8)$ above. Just mention that the regularity axiom follows from the mean value theorem for the Laplace equation on a disk. It corresponds to the function $G(A) = -Tr(A)$, that is $G(a, b, c) = -(a + c)$. We recall that this operator does not permit to interpolate points (see introduction).

A more general situation is given by the so called p-Laplacian

$$\begin{aligned} &div(|\nabla u|^{p\perp 2} \nabla u) = 0 \quad \text{in } D(?) \\ &u_{|\Gamma} = \varphi. \end{aligned} \tag{16}$$

where $p \geq 1$. Formally, after dividing by $|\nabla u|^{p\perp 2}$, the above equation can be written as

$$(p-1) D^2 u \left(\frac{Du}{|Du|}, \frac{Du}{|Du|} \right) + D^2 u \left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|} \right) = 0. \tag{17}$$

which is contained in the family of equations (12) with $a = p - 1$, $b = 0$, $c = 1$. As it is known the value of u can be fixed at at point if and only if $p > 2$. This corresponds to the case $a > c$. From the above discussion we see that, unless $c = 0$ which corresponds to the case $p = \infty$, the gradient of u can be unbounded. The case of $p = \infty$ will be our next example.

Example 2. Our next example is more interesting and will be discussed in more details in the next section. Given a domain Ω with $\partial\Omega \in \mathcal{C}$ and $\varphi \in \mathcal{F}(\partial\Omega)$ we consider $E_2(\varphi, \partial\Omega)$ to be the viscosity solution of

$$\begin{aligned} &D^2 u \left(\frac{Du}{|Du|}, \frac{Du}{|Du|} \right) = 0 \quad \text{in } \Omega \\ &u_{|\partial\Omega} = \varphi. \end{aligned} \tag{18}$$

We consider equation (18) in the viscosity sense. Given $u \in C(\Omega)$ we say that u is a viscosity subsolution (supersolution) of (18) if for any $\psi \in C^2(\Omega)$ and any x_0 local maximum (minimum) of $u - \psi$ in Ω such that $D\psi(x_0) \neq 0$

$$D^2 \psi(x_0) \left(\frac{D\psi(x_0)}{|D\psi(x_0)|}, \frac{D\psi(x_0)}{|D\psi(x_0)|} \right) \geq 0 \quad (\leq 0)$$

A viscosity solution is a function which is a viscosity sub- and supersolution.

Equation (18) was introduced by G. Aronsson in [3] and recently it has been studied by R. Jensen [15]. Indeed, in [3] the author considered the following problem:

Given a domain Ω in $I\!R^n$, does a Lipschitz function u in Ω exist such that

$$\|Du\|_{L^\infty(\tilde{\Omega};I\!R^n)} \leq \|Dw\|_{L^\infty(\tilde{\Omega};I\!R^n)}$$

for all $\tilde{\Omega} \subseteq \Omega$ and w such that $u - w$ is Lipschitz in $\tilde{\Omega}$ and $u = w$ on $\partial\tilde{\Omega}$. If it exists, such a function will be called an absolutely minimizing Lipschitz interpolant of $w|_{\partial\Omega}$ inside Ω or $AMLE$ for short. Notice that the above definition, if it defines uniquely u, immediately implies the stability of AMLE in the sense of $(A2)$. Then it was proved in [3] that if u is an $AMLE$ and is C^2 in Ω, then u is a classical solution of

$$D^2u(Du, Du) = 0 \quad \text{in } \Omega. \tag{19}$$

Later Jensen [15] proved that if u is an $AMLE$, then u solves (19) in the viscosity sense. Moreover, the viscosity solution is unique. We shall use the viscosity solution formulation of Equation (19). Given $u \in C(\Omega)$ we say that u is a viscosity subsolution (supersolution) of (19) if for any $\psi \in C^2(\Omega)$ and any x_0 local maximum (minimum) of $u - \psi$ in Ω

$$D^2\psi(x_0)(D\psi(x_0), D\psi(x_0)) \geq 0 \quad (\leq 0).$$

A viscosity solution is a function which is a viscosity sub- and supersolution. Then Jensen proved [15] a comparison principle between sub- and supersolutions of Equation (19) together with an existence result for boundary data in the space of functions $Lip_\partial(\Omega)$ which are Lipschitz continuous with respect to the distance $d_\Omega(x,y)$. We denote by $d_\Omega(x,y)$ the geodesic distance between x and y, i.e., the minimal length of all possible paths joining x and y and contained in Ω [15]. Observe that if u is a viscosity subsolution (supersolution, solution) of (18) if and only if u is a viscosity subsolution (supersolution, solution) of (19). From this follows the corresponding comparison principle for solutions of (18).

Theorem 3 *Assume that v is a subsolution and w a supersolution of* (18) *(equivalently of* (19)*. If $v \mid_{\partial\Omega}$, $w \mid_{\partial\Omega} \in Lip_\partial(\Omega)$ then*

$$\sup_{x\in\Omega}(v - w) = \sup_{x\in\partial\Omega}(v - w) \tag{20}$$

Theorem 4 *Given $g \in Lip_\partial(\Omega)$, u is the $AMLE$ of g into Ω if and only if u is the solution of* (19) *with $u \mid_{\partial\Omega}= g$.*

The following existence result for (18) follows from R. Jensen's results.

Theorem 5 *Given $g \in Lip_\partial(\Omega)$, then there exists a unique viscosity solution $u \in W^{1,\infty}(\Omega)$ of* (18) *such that $u \mid_{\partial\Omega}= g$.*

This result enables us to define the following interpolation operator. Given $\varphi \in Lip_\partial(\Omega)$, let $E_2(\varphi, \partial\Omega)$ be the viscosity solution of (18). Then

Theorem 6 *The operator E_2 defined above satisfies axioms $(A1) - (A8)$.*

Proof. According to the previous results it suffices to prove that $(A3)$ is satisfied. Let $A \in SM(2)$, $p \in I\!R^2, p \neq 0$, $c \in I\!R$. Without loss of generality we may take $c = 0$ and also $x = 0$. Let

$$Q(x) = \frac{A(x,x)}{2} + \langle p, x \rangle$$

Let $\nu = \frac{p}{|p|}$. In the canonical basis $\{\nu, \nu^\perp\}$ the matrix A can be written

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \text{so that} \quad a = D^2Q\left(\frac{DQ}{|DQ|}, \frac{DQ}{|DQ|}\right) \quad \text{and} \quad p = (|p|, 0).$$

Let us define

$$Q_\varepsilon(x) = \frac{A(x,x) - \varepsilon x_1^2}{2} + < p, x >$$

Observe that on the boundary of $D(0,r)$, $r > 0$,

$$Q_\varepsilon(x) = \frac{a}{2}r^2 - \frac{\varepsilon}{2}x_1^2 + bx_1x_2 + \frac{c-a}{2}x_2^2 + < p, x > . \tag{21}$$

We look for a supersolution ψ of (18) such that $\psi \geq Q_\varepsilon$ on $\partial D(0,r)$ for $r > 0$ small enough. We claim that

$$\psi(x) = \frac{a}{2}r^2 + \eta(x) + < p, x >$$

where

$$\eta(x) = -\frac{\varepsilon}{2}x_1^2 + bx_1x_2 + \frac{c-a}{2}x_2^2$$

is a supersolution of (18). According to (21) $\psi \geq Q_\varepsilon$ on $\partial D(0,r)$. Since

$$D^2\psi(D\psi, D\psi) = D^2\eta(p + D\eta, p + D\eta) = -\varepsilon\|p\|^2 + O(\|x\|) < 0$$

for $r > 0$ small enough, it follows that ψ is a supersolution of (18). Then, according to Theorem 3

$$E(Q_\varepsilon, \partial D(0,r)) \leq \psi \quad \text{in } D(0,r), \tag{22}$$

$$\sup_{D(0,r)} |E(Q_\varepsilon, \partial D(0,r)) - E(Q, \partial D(0,r))| \leq \sup_{D(0,r)} |Q_\varepsilon - Q| \leq \frac{\varepsilon}{2}r^2 \tag{23}$$

Then, for $r > 0$ small enough

$$\begin{aligned} E(Q, \partial D(0,r))(0) - Q(0) &\leq \frac{\varepsilon}{2}r^2 + E(Q_\varepsilon, \partial D(0,r))(0) - Q(0) \\ &\leq \frac{\varepsilon}{2}r^2 + \psi(0) \leq \frac{\varepsilon}{2}r^2 + \frac{a}{2}r^2. \end{aligned}$$

Now, dividing by $r^2/2$ and letting $r \to 0$ and $\varepsilon \to 0$ in this order we get

$$\limsup_{r\to 0} \frac{E(Q, \partial D(0,r))(0) - Q(0)}{r^2/2} \leq a.$$

Similarly we prove that

$$\liminf_{r\to 0} \frac{E(Q, \partial D(0,r))(0) - Q(0)}{r^2/2} \geq a.$$

Hence, $(A3)$ holds with

$$F(A, p) = a = A(\nu, \nu). \quad \square$$

Remark. An interesting feature of model (18) is the fact that we can interpolate data not only on boundaries which are made of Jordan curves but also on boundaries which contain isolated points. We shall discuss this in detail in the next section.

Example 3. Our next example is concerned with the curvature operator. Let Ω be a domain in $I\!R^2$ with Lipschitz boundary and φ be a Lipschitz continuous function on $\partial\Omega$. We consider the equation

$$\begin{aligned} &D^2u\left(\frac{Du^\perp}{|Du|},\frac{Du^\perp}{|Du|}\right)=0 \quad \text{in } \Omega \\ &u_{|\partial\Omega}=\varphi \end{aligned} \tag{24}$$

We consider solutions of (24) in the viscosity sense, which can be defined in the same way as solutions of (18) in Example 2. This model cannot be used as a model for interpolating data because of the following facts
a) There is no uniqueness of viscosity solutions of (24).
b) There are no viscosity solutions of (24) for general smooth curves $\partial\Omega$ and boundary data $\varphi \in \mathcal{F}(\partial\Omega)$. This is easily deduced from techniques developped in [8].

Indeed, let $\Omega = D(0,1)$, $\varphi(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2$, $\lambda_1 > \lambda_2$. Since on $\partial D(0,1)$, $x_1^2 + x_2^2 = 1$ and

$$\varphi(x_1,x_2) = \varphi(-x_1,x_2) = \varphi(x_1,-x_2),$$

the functions

$$u_1(x_1,x_2) = \varphi\left(\sqrt{1-x_2^2}, x_2\right) \qquad u_2(x_1,x_2) = \varphi\left(x_1, \sqrt{1-x_1^2}\right)$$

are two viscosity solutions of (24) in $D(0,1)$ with the same boundary data.

On the other hand, concerning existence, there are no viscosity solutions of (24) for general smooth curves $\partial\Omega$ and boundary data $\varphi \in \mathcal{F}(\partial\Omega)$. We shall not give a proof of this fact here but let us mention instead some heuristic arguments. A solution u of (24) is also a static solution of the evolution problem

$$\begin{aligned} &\frac{\partial v}{\partial t} = D^2v\left(\frac{Dv^\perp}{|Dv|},\frac{Dv^\perp}{|Dv|}\right) \quad \text{in } (0,+\infty)\times\Omega \\ &v(0,x) = u(x) \quad \text{in } \Omega \\ &v(t,x) = \varphi(x) \quad (t,x) \in (0,+\infty)\times\partial\Omega \end{aligned} \tag{25}$$

which means that all level lines of u are moving by mean curvature. This is impossible unless the level lines of u are straight lines. In general, this is not possible as can be seen in Figure 5. Figure 5 depicts a nonconvex smooth domain with boundary data φ such that φ increases when we go from A to B along the boundary in the clockwise direction and then decreases symmetrically when going from B to A.

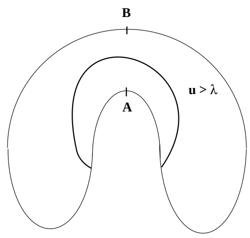

Figure 5

Example 4. Consider a set of points $\{x_i : i = 1,...,N\}$ in $I\!R^2$. Shepard [24] proposed the following formula

$$\begin{aligned} &f(x) = \frac{\sum_{i=1}^N f_i |x-x_i|^{\perp 2}}{\sum_{i=1}^N |x-x_i|^{\perp 2}} \quad x \neq x_i, \quad i=1,...,N \\ &f(x_i) = f_i \end{aligned} \tag{26}$$

to interpolate the values f_i on x_i. This formula can be extended to give the values of f on a curve. Let Ω be a domain in $I\!R^2$ whose boundary is a Lipschitz continuous simple Jordan curve and let f be a continuous function on $\partial\Omega$. If we parametrize $\partial\Omega$ by its arclength $x : [0, L] \to I\!R^2$, then the function $F : \Omega \to I\!R$ given by

$$F(x) = \frac{\int_0^L \frac{f(x(s))}{\|x \perp x(s)\|^2} ds}{\int_0^L \frac{ds}{\|x \perp x(s)\|^2}} \quad \text{if } x \notin \partial\Omega, \qquad F(x) = f(x) \quad \text{if } x \in \partial\Omega \tag{27}$$

is continuous in $\bar{\Omega}$. Moreover it is elementary to check that the operator $E_4(f, \partial\Omega) = F$ satisfies $(A1)$,$(A3) - (A8)$. Just mention that $(A3)$ follows from the fact that (27) coincides with Poisson formula for the Laplace equation when Ω is a disk. According to this, E_4 does not satisfy $(A2)$. This explains why this operator is not contained in (11). In fact, if we iterate E_4 on all disks, the iterated interpolant will converge to a solution of the heat equation, but the boundary conditions at isolated points gets lost by this process. Observe that we may use operator E_2 to interpolate data given on a finite number of points and the complexity of this algorithm is independent of the number of them, in contrast to Shepard's formula (26).

4 The minimal Lipschitz extension operator

Let us state R. Jensen's existence result for (18) in a way that makes explicit the fact that we are able to interpolate a datum which is given on a set of curves and points. Let us consider a domain Ω whose boundary $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega \cup \partial_3\Omega$ where $\partial_1\Omega$ is a finite union of rectifiable simple Jordan curves,

$$\partial_2\Omega = \cup_{i=1}^m C_i,$$

where C_i are rectifiable curves homeomorphic to a closed interval and

$$\partial_3\Omega = \{x_i : i = 1, ..., N\}$$

is a finite number of points. The boundary data to be interpolated is given by a Lipschitz function φ_1 on $\partial_1\Omega$, two Lipschitz functions $\varphi^i_{2+}, \varphi^i_{2\perp}$ on each curve C_i, which coincide on the extreme points of C_i, $i = 1, ..., m$ and a constant value u_i on each point x_i, $i = 1, ..., N$. We shall denote by C_i^+, $C_i^\perp$ the same curve C_i where we take into account the direction of the normal $\nu_i^+(x)$, $\nu_i^\perp(x) = -\nu_i^+(x)$, $x \in C_i$ as in Figure 6. When we write $u_{|_{C_i^+}} = \varphi^i_{2+}$ as in the next theorem we mean that $u(y) \to \varphi^i_{2+}(x)$ as $y \to x$ if $< y, \nu_i^+(x) >< 0$, $x \in C_i$ and similarly for $u_{|_{C_i^-}} = \varphi^i_{2\perp}$.

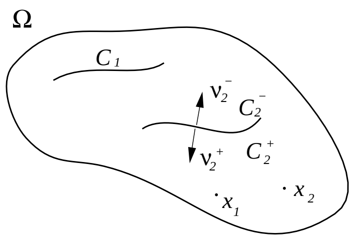

Figure 6

Theorem 7 *Given Ω, φ_1, $\varphi^i_{2+}, \varphi^i_{2\perp}$, u_j, $i = 1, ..., m$, $j = 1, ..., N$, as above then there exists a unique viscosity solution $u \in W^{1,\infty}(\bar{\Omega})$ of*

$$\begin{aligned} & D^2 u \left(\frac{Du}{|Du|}, \frac{Du}{|Du|} \right) = 0 \quad \text{in } \Omega \\ & u_{|\partial_1 \Omega} = \varphi_1 \\ & u_{|_{C_i^+}} = \varphi^i_{2+} \\ & u_{|_{C_i^-}} = \varphi^i_{2\perp} \qquad i = 1, ..., m \\ & u(x_i) = u_i, \qquad i = 1, ..., N \end{aligned} \tag{28}$$

The proof of Theorem 7 can be reduced to an application of Theorem 4. For that, let, for $r > 0$ small enough, $\Omega_r = \Omega - \cup_{i=1}^m (C_i + D(0,r)) - \cup_{i=1}^N D(x_i, r)$. Then $\partial\Omega_r$ consists of a finite union of rectifiable simple Jordan curves. First we need to extend our boundary data to $\partial\Omega_r = \partial_1\Omega \cup (\cup_{i=1}^m \partial(C_i + D(0,r))) \cup \left(\cup_{i=1}^N \partial D(x_i, r)\right)$. We keep the same boundary data on $\partial_1\Omega$, let $\varphi_r = \varphi_1$ on $\partial_1\Omega$. We define $\varphi_r = u_i$ on $\partial D(x_i, r)$, $i = 1, ..., N$. To define φ_r on $\partial(C_i + D(0,r))$ we need to parametrize this boundary as in Figure 7. The boundary of $C_i + D(0,r)$ can be described in four pieces (see Figure 6)

$$\partial(C_i + D(0,r)) = C_i^+(r) \cup C_i^\perp(r) \cup S(e_i^+) \cup S(e_i^\perp)$$

where $C_i^+(r)$ can be parametrized by $\sigma_i^+(x,r) = x + r\nu_i^+(x)$, $C_i^\perp(r)$ by $\sigma_i^\perp(x,r) = x + r\nu_i^\perp(x)$, $x \in C_i$ and $S(e_i^+)$, $S(e_i^\perp)$ are the semicircles centered at the extreme points e_i^+, $e_i^\perp$ of C_i. Then we define φ_r on $\partial(C_i + D(0,r))$ by

$$\varphi_r(x + r\nu_i^+(x)) = \varphi^i_{2+}(x), \quad \varphi_r(x + r\nu_i^\perp(x)) = \varphi^i_{2\perp}(x), \quad x \in C_i$$

$$\varphi_r \mid_{S(e_i^+)} = \varphi^i_{2+}(e_i^+) = \varphi^i_{2\perp}(e_i^+)$$

$$\varphi_r \mid_{S(e_i^-)} = \varphi^i_{2+}(e_i^\perp) = \varphi^i_{2\perp}(e_i^\perp)$$

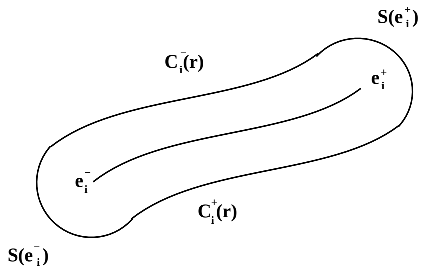

Figure 7

Then using Jensen's existence result Theorem 5 there exists a unique viscosity solution $u_r \in W^{1,\infty}(\bar{\Omega}_r)$ of

$$\begin{aligned} & D^2 u \left(\frac{Du}{|Du|}, \frac{Du}{|Du|} \right) = 0 \quad \text{in } \Omega_r \\ & u \mid_{\partial\Omega_r} = \varphi_r \end{aligned} \tag{29}$$

which satisfies

$$\|Du_r\|_\infty \leq \|\varphi_r\|_{Lip_\partial(\Omega_r)} \tag{30}$$

Proof of Theorem 7. For each $r > 0$, let $u_r \in W^{1,\infty}(\bar{\Omega}_r)$ be the viscosity solution of (29) satisfying (30). Since $\|\varphi_r\|_{Lip_\partial(\Omega_r)}$ is bounded independently of $r > 0$, then there exists a subsequence of u_r converging to a function $u \in W^{1,\infty}(\bar{\Omega})$. By the stability result of viscosity solutions (see [9]) u is a viscosity solution of (28). Uniqueness follows from Theorem 3. □

For numerical reasons it is interesting to study the asymptotic behavior of the evolution problem corresponding to Eq. (18). Then, under certain smoothness assumptions on the boundary data we prove that the solution of the evolution problem converges to the solution of Eq. (18). Let us consider the evolution equation

$$\begin{array}{ll} \dfrac{\partial u}{\partial t} = D^2 u\left(\dfrac{Du}{|Du|}, \dfrac{Du}{|Du|}\right) & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x) & x \in \Omega \\ u(t,x) = \varphi(x) & (t,x) \in (0,+\infty)\times\partial\Omega \end{array} \tag{31}$$

where we suppose that $u_0(x) = \varphi(x)$ for all $x \in \Omega$. We say that $u \in C([0,+\infty)\times\Omega)$ is a viscosity subsolution of (31) if $u(0,x) = u_0(x)$, $u(t,x) = \varphi(x)$ for all $(t,x) \in (0,+\infty)\times\partial\Omega$ and for any $\psi \in C^2((0,+\infty)\times\Omega)$ and any (t_0,x_0) local maximum of $u-\psi$ in $(0,+\infty)\times\Omega$

$$\psi_t(t_0,x_0) \le D^2\psi(t_0,x_0)\left(\frac{D\psi(t_0,x_0)}{|D\psi(t_0,x_0)|}, \frac{D\psi(t_0,x_0)}{|D\psi(t_0,x_0)|}\right) \tag{32}$$

if $D\psi(t_0,x_0) \neq 0$ and

$$\psi_t(t_0,x_0) \le sup_{|v|\le 1} D^2\psi(t_0,x_0)(v,v)$$

if $D\psi(t_0,x_0) = 0$. Similarly we define a viscosity supersolution. A viscosity solution is a viscosity sub- and supersolution.

Theorem 8 *Let Ω be a generalized domain in $I\!R^2$ as described above. Suppose that $\partial_1\Omega$, C_i^+, $C_i^\perp$, $i = 1,...,m$, have bounded curvature, and the initial condition $u_0(x)$ and the boundary data $\varphi(x)$ have bounded second derivatives . Then there exists a unique continuous viscosity solution $u(t,x)$ of* (31) *such that $u(t)$ is Lipschitz for all $t > 0$ with uniformly bounded Lipschitz norm. Moreover $u(t,.) \to u_\infty$ where u_∞ is the unique viscosity solution of* (28).

Proof. The uniqueness result follows as in the uniqueness proof given in [2] ([8]). The existence follows by considering the smooth approximation

$$\frac{\partial u}{\partial t} = D^2 u\left(\frac{Du}{(\epsilon+|Du|^2)^{1/2}}, \frac{Du}{(\epsilon+|Du|^2)^{1/2}}\right) + \epsilon\Delta u \quad \text{in } \Omega_r \tag{33}$$

with the same initial and boundary conditions than in (31), $\epsilon, r > 0$, $r > 0$ small enough. The Lipschitz estimate of the solution of (33) follows also as in [2] ([8]) after proving the Lipschitz estimate on the boundary which is proved using standard techniques for boundary gradient estimates ([12], Chapter 14). Since these estimates are independent of ϵ, r we may let $\epsilon \to 0+$ and $r \to 0+$ in this order to get a viscosity solution of Equation (31).

Let $K > 2\|u_0\| + 1$. To prove the assertion on the asymptotic behavior of $u(t,x)$ let us consider for each $T > 0$ an increasing smooth function $f(t,T)$ defined in $[0,+\infty[$ such that $f(t,T) = -K$ for all $t \in [0,1]$ and all $T > 0$ and such that $f(t,T) = 0$ for $t \ge T$. Moreover we may assume that $\|f_t(.,T)\|_\infty \to 0$ as $T \to +\infty$. Let

$$u_T(x) = sup_{t\ge 0}\{u(t,x) - \frac{t}{T^2} + f(t,T)\}.$$

By the Lipschitz estimate on $u(t,x)$, u_T is Lipschitz. Then, modulo a subsequence, we may assume that $u_T \to w$ for some Lipschitz function w. We claim that w is a viscosity solution of (28). Indeed, let ψ be any smooth function in Ω and x_0 be such that $w-\psi$ has a strict local maximum at x_0. Then, for $T \geq 1$ large enough, $u_T - \psi$ has a local maximum at x_T where $x_T \to x_0$ as $T \to +\infty$. Thus

$$u(t,x) - \frac{t}{T^2} + f(t,T) - \psi(x) \leq u_T(x) - \psi(x) \leq u_T(x_T) - \psi(x_T)$$

for all $(t,x) \in (0,+\infty) \times \Omega$. Now we observe that the supremum at the definition of u_T is attained at some $t_T \in (1,+\infty)$. Indeed, clearly the supremum is attained at some $t_T < +\infty$. On the other hand, if $t_T \leq 1$,

$$u(t_T,x) - \frac{t_T}{T^2} + f(t_T,T) \leq \|u_0\| - K$$

while we have

$$u(T,x) - \frac{T}{T^2} + f(T,T) \geq -\|u_0\| - \frac{1}{T}.$$

This contradicts our choice of K. Hence $t_T > 1$. We have

$$u(t,x) - \frac{t}{T^2} + f(t,T) - \psi(x) \leq u(t_T,x_T) - \frac{t_T}{T^2} + f(t_T,T) - \psi(x_T)$$

for all $(t,x) \in (0,+\infty) \times \Omega$. Then

$$\frac{1}{T^2} - f_t(t_T,T) \leq F^*(D^2\psi(x_T), D\psi(x_T)), \tag{34}$$

where

$$F^*(D^2\psi(x_T), D\psi(x_T)) = D^2\psi(x_T)\left(\frac{D\psi(x_T)}{|D\psi(x_T)|}, \frac{D\psi(x_T)}{|D\psi(x_T)|}\right)$$

if $D\psi(x_T) \neq 0$ and

$$F^*(D^2\psi(x_T), D\psi(x_T)) = sup_{v:|v|=1} D^2\psi(x_T)(v,v)$$

if $D\psi(x_T) = 0$. Since $\|f_t(.,T)\|_\infty \to 0$ as $T \to \infty$, letting $T \to \infty$ in (34) we get that

$$0 \leq F^*(D^2\psi(x_0), D\psi(x_0)).$$

We have shown that for some sequence $T_n \to +\infty$ u_{T_n} converges uniformly to a subsolution w of (28) satisfying the boundary data given by φ. Moreover, it is easy to check that

$$\limsup_{n\to\infty} u(T_n,x) \leq w(x) \tag{35}$$

for any $x \in \Omega$. Now, define

$$v_{T_n}(x) = \sup_{t\geq 0}\{-u(t,x) - \frac{t}{T_n^2} + f(t,T_n)\}.$$

As above we get that there exists a subsequence T_{n_i} of T_n such that $v_{T_{n_i}}$ converges uniformly to a subsolution $\overline{w}$ of (28) satisfying the boundary data given by $-\varphi$. We also get that

$$\limsup_{i\to\infty} -u(T_{n_i},x) \leq \overline{w}(x). \tag{36}$$

Now, let $w_0(x) = -\overline{w}(x)$. Then w_0 is a viscosity supersolution of (28) satisfying the boundary data given by φ. Using this and (35), (36), it follows that

$$w_0(x) \leq \liminf_{i\to\infty} u(T_{n_i},x) \leq \limsup_{i\to\infty} u(T_{n_i},x) \leq \limsup_{n\to\infty} u(T_n,x) \leq w(x) \leq w_0(x).$$

Thus,

$$\lim_{i\to\infty} u(T_{n_i},x) = w(x) = w_0(x).$$

We conclude that $\lim_{i\to\infty} u(T_{n_i},x)$ is a viscosity solution of (28) with boundary data given by φ. By uniqueness of viscosity solutions of (28), we conclude that $u(t,.) \to u_\infty$ as $t \to \infty$ where u_∞ is the unique viscosity solution of (28). $\square$

4.1 Geometric interpretation of Equation (19)

Proposition 2 *Let u be C^2 and satisfy $D^2u(Du, Du) = 0$. We define a gradient line as a curve $x(t)$, $t \in (a,b)$ such that*

$$Du(x(t)) \neq 0 \quad \text{and} \quad x'(t) = \frac{Du}{|Du|}(x(t)).$$

Then, there is a constant C such that for every $t \in (a,b)$

$$|Du|(x(t)) = C.$$

Proof: Let $\phi(t) = |Du|^2(x(t))$. We differentiate ϕ with respect to t and we obtain

$$\phi'(t) = D^2u(Du, x'(t)) = D^2u\left(Du, \frac{Du}{|Du|}\right) = 0.$$

Thus, there exists a constant C such that $\phi(t) = C$ on (a,b). □

Corollary 3 *Viscosity solutions of (18) are not necessarily C^2.*

For example, let u be defined on the boundary of the square $[0,1]^2$ by $u(0,1) = 1 = u(1,0)$, $u(0,0) = 0 = u(1,1)$ and u is affine on each side of the square. Let u_0 the AMLE of u. By symmetry of the datum and uniqueness, $u_0\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$. Thus, there is some point y on the segment L joining $(0,1)$ and $(1,0)$ such that $Du_0(y) \neq 0$, then there is a neighborhood of y such that $Du_0 \neq 0$. By symmetry again, $Du_0(y)$ is parallel to L. Thus, a segment of L containing y is a gradient line. By Proposition 2, on this segment we have $Du_0(z) = Du_0(y)$. So we conclude that the maximal segment is the whole line L. This is a contradiction with $u(0,1) = u(1,0)$. □

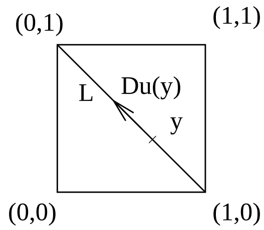

Figure 8

5 Numerical analysis of the AMLE model

We shall use the AMLE model studied above as the basic equation to interpolate data given on a set of curves and/or a set of points which may be irregularly sampled. Thus, we discretize the equation

$$D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) = 0. \tag{37}$$

It is easy to see that there is a relation between iterative methods for the solution of elliptic problems and time stepping finite difference methods for the solution of the corresponding parabolic problems. Because of that and thanks to Theorem 8, we study the equation

$$\frac{\partial u}{\partial t} = D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right), \qquad u(0,x) = u_0(x) \tag{38}$$

with corresponding initial and boundary data. Using an implicit Euler scheme we transform this evolution problem into a sequence of nonlinear elliptic problems. Thus, we may write the following implicit difference scheme in the image grid

$$u_{i,j}^{(n+1)} = u_{i,j}^{(n)} + \Delta t D^2 u_{i,j}^{(n+1)} \left(\frac{Du_{i,j}^{(n+1)}}{|Du_{i,j}^{(n+1)}|}, \frac{Du_{i,j}^{(n+1)}}{|Du_{i,j}^{(n+1)}|} \right) \tag{39}$$

$i, j = 1, \ldots, N$. To solve the above nonlinear system we use a nonlinear over-relaxation method (NLOR). Writing the system as a set of $k = N^2$ algebraic equations, one for each unknown $u_{i,j}^{(n+1)}$ $(i, j = 1, ..., N)$,

$$f_p(x_1, x_2, \ldots, x_k) = 0, \quad p = 1, 2, \ldots, k, \tag{40}$$

the basic idea of NLOR is to introduce a relaxation factor ω and iteratively compute

$$x_i^{(n+1)} = x_i^{(n)} - \omega \frac{f_i(x_1^{(n+1)}, \ldots, x_{i\perp 1}^{(n+1)}, x_i^{(n)}, \ldots, x_k^{(n)})}{f_{ii}(x_1^{(n+1)}, \ldots, x_{i\perp 1}^{(n+1)}, x_i^{(n)}, \ldots, x_k^{(n)})}, \qquad i = 1, 2, \ldots, k \tag{41}$$

where $f_{ii} = \frac{\partial f_i}{\partial x_i}$. The convergence criterion can be shown to be the same as the over-relaxation method for linear systems, replacing the matrix by the Jacobian of the equations $f_p = 0$, and stability is guaranteed for values of the relaxation parameter $0 < \omega < 2$.

6 Experimental Results

We display some experiments using the numerical scheme described in the previous section. Figures 9 and 10 show some experiments with synthetic images. Figure 9a) displays the original image, a single white point inside a rectangle. We impose $u = 0$ on the boundary of the rectangle. Figure 9b) shows the result of the interpolation algorithm with Dirichlet boundary conditions. As one would expect, the result is a pyramid whose levels sets are displayed in Fig. 9c). Figure 10a) displays a synthetic image where we combined open curves, closed curves and points. Figure 10b) shows the interpolant and Figure 10c) shows the level lines of the interpolant.

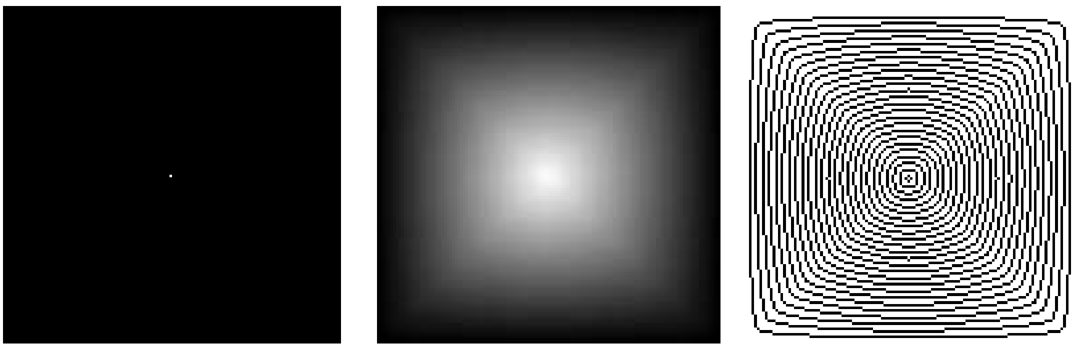

Figure 9: Left 9a): original image. Middle 9b): interpolant. Right 9c): level lines of the interpolant

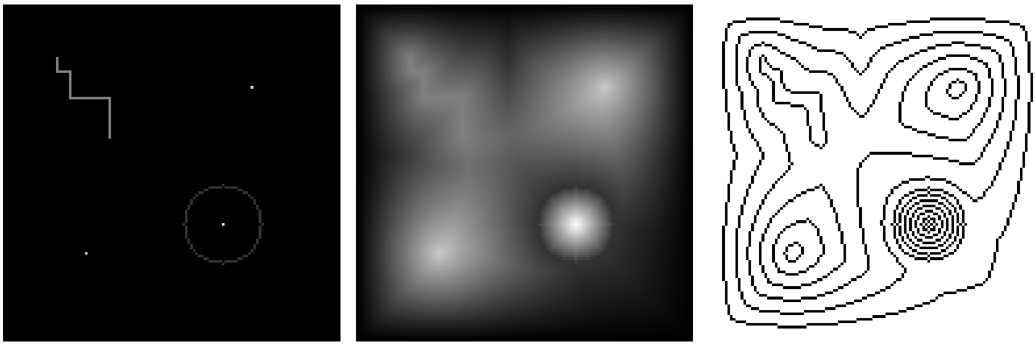

Figure 10: Left 10a): original image. Middle 10b): interpolant with $u = 0$ on the boundary. Right 10c) level lines of the interpolant

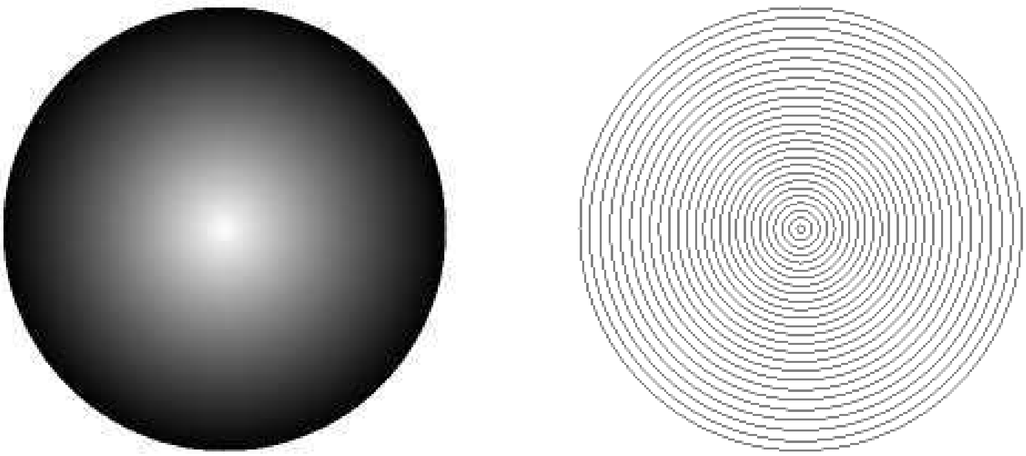

Figure 11: Left 11a): The result of the interpolation of function such that $u(0,0) = 1$ and $u(x_1, x_2) = 0$ if $(x_1, x_2) \in \partial D((0,0), R)$ with Eq. (37). Right 11b): The level lines of the above f

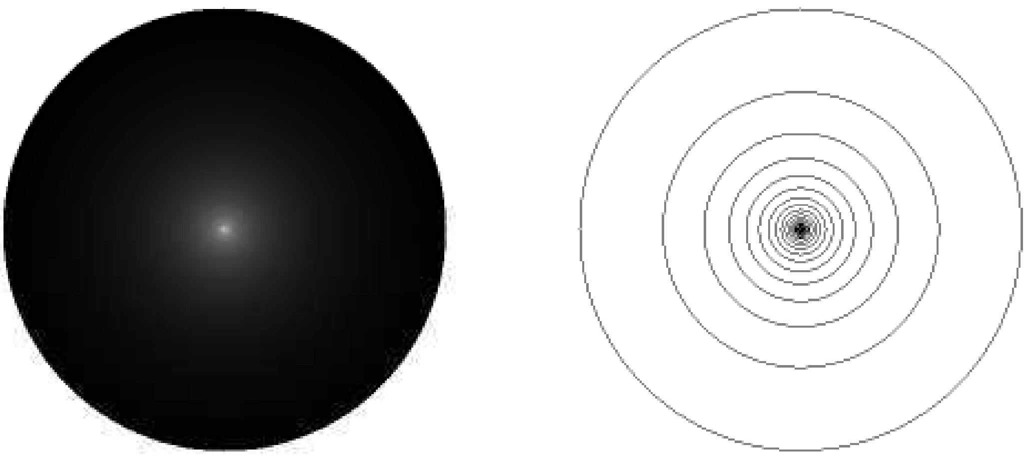

Figure 12: The same as the Figure 11 using the Laplacian.

Figures 13, 14 and 15 show how we can interpolate an image from the quantized level curves, obtaining a better result than the corresponding quantized image. Figures a) display the original images u which take integer values between 0 and 255. Then we quantize them by giving the grey levels between $r\delta \leq u < (r+1)\delta$ the value $r\delta$, $r = 0, ..., M$, $M = [255/\delta]$. Figures c) and e) display the result of this operation on Figures a) for values $\delta = 20$ and $\delta = 30$. Figures b) displays the boundaries of the level sets $[u \geq r\delta]$ at the corresponding grey level $r\delta$ (here, we have displayed the level sets for $\delta = 30$). We define the boundary values on the pixels belonging to the boundaries of the level sets B and the neighbouring pixels belonging to the boundary of the complement B'. For each pixel (i,j) we define $m(i,j) = inf\{r : u(i,j) \geq r\delta\}$, $M(i,j) = sup\{r : u(i,j) \geq r\delta\}$. Then we set $u(i,j) = m(i,j)\delta$ if $(i,j) \in B'$, $u(i,j) = M(i,j)\delta$ if $(i,j) \in B$, and we solve Eq. (39) with these boundary data. The results are displayed in Figures d) and f).

In practice, the interpolation must keep smooth the regular regions of the image. So if we quantize the image at levels multiple of 30 (e.g.), the jump across the level line after quantization is either 0 or $30, 60$, etc. The behaviour of the algorithm is following: if the jump $M(i,j) - m(i,j)$ is just 0, it is likely that the region around is smoothly perceptual, so our interpolation maintains it by giving a Lipschitz interpolation. If the jump across the level line is larger (e.g.) than 20, 30, etc., our decision is to maintain the jump because we consider that there must be an edge here. Since a jump larger than 20 is perceptible as edge, we maintain the existing edge by this choice, without significant attenuation or enhancement.

Figure 13: Left top 13a): original image. Right top 13b): level lines for $\delta = 30$. Left middle 13c): quantized image for $\delta = 20$. Right middle 13d): the interpolant for $\delta = 20$. Left bottom 13e): quantized image for $\delta = 30$. Right bottom 13f): the interpolant for $\delta = 30$.

Figure 14: The same as in Figure 13.

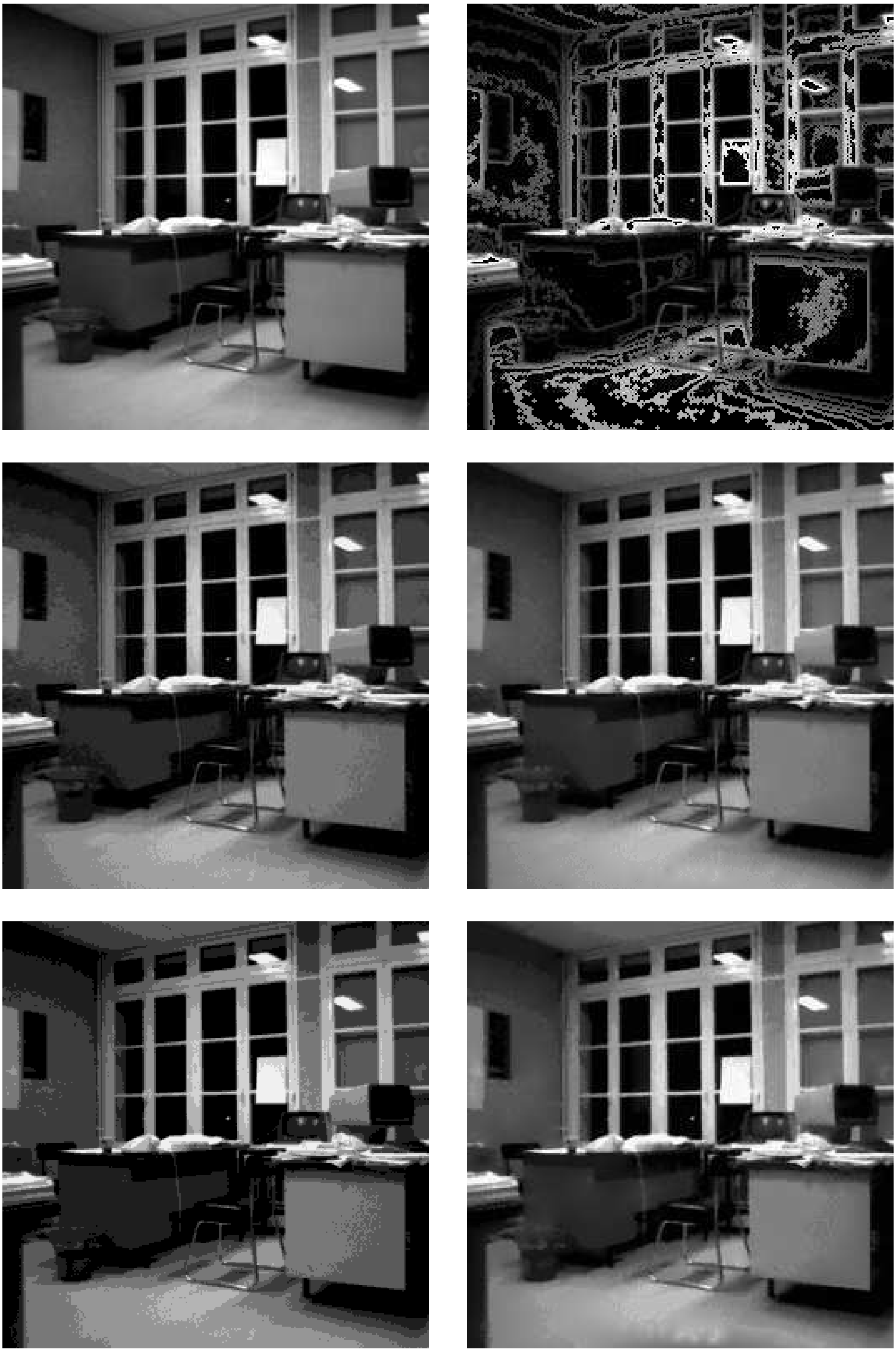

Figure 15: The same as in Figure 13.

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