Inpainting Fundamentals

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# Introduction

Definition of inpainting:

Filling image information on a blank domain or several domains , based on the information available outside of these inpainting domains. On such domains , the original image has been compromised.

But the main question is how to model mathematically the process of inpainting those regions denoted with in the figure above.

inpainting domain

*is given*

Assumption of Locality of the domain

Assumption of Smoothness of the Image Function

To develop a rigorous mathematical framework for inpaintings, we make a simple assumption in which the accuracy of the inpainting can be studied. This is the assumption that the target image function is smooth, that is – the inpainting domain is contained in the interior of a smooth 2D object.

Let be a smooth image function defined on a 2D domain (usually rectangular domain). We denote by the domain to be inpainted and its diameter. We denote the restriction of on by . Then inpainting is the task to find a function defined on such that is a good approximation to .

**Definition** *Linear Inpainting procedure*

An inpainting procedure is linear if for any given smooth image

//TODO: finish the paragraph with the Linear inpainting procedure

## Patch-based methods for Image Inpainting

### Efros and Leung’s Algorthm

The image gap is filled in recursively, inward from the gap boundary: each “empty” pixel at the boundary is filled with the value of the pixel which lies outside of the image gap. Thus, is a pixel with valid information. The valid pixel Q is selected in such way that the neighborhood of which is a square patch centered at is most similar to the neighborhood of . Formally, this is expressed as an optimization problem:

, , , (I.1)

where represents the sum of squared differences among the patches and

(I.2)

Here the indices span the extent of the patches . For example, if is an patch then .

Once P is filled-in, the algorithm marches on to the next pixel at the boundary of the gap, never going back to .

Let be a given image represented as a vector in . Let the matrices , of sizes and represent the dictionaries adapted to geometry and texture, respectively. If

//TODO: finish the section on the Introduction of Inpainting Problems and Definitions

# Variational Problems and Optimal Approximations by Piecewise Smooth Functions for Image Inpaintings

We will look at three functionals which measure the degree of match between an image and a segmentation. We have a general functional which depends on two parameters and and two limiting cases and which depend only one parameter ( and accordingly) and correspond to the limits of as the parameter tends to 0 and respectively.

(II.1)

The first functional is given with

, (II.2)

where stands for the total length of the arcs making up . The smaller is, the better segments .

The second functional is simply a restriction of to piecewise constant functions i.e. . Thus (II.2) becomes:

(II.3)

where . It is obvious that (II.3) is minimized in terms of the variables by setting

(II.4)

so we are minimizing

(II.5)

We will show that if is fixed and , the which minimizes tends to a piecewise constant limit, so one can prove that

(II.6)

**Note**: can be viewed as a modification of the usual Plateau problem functional, minimizing , by an external force (or pressure) term that keeps the regions (soap bubbles in the Plateau problem) from collapsing.

Whereas the two-dimensional Plateau problem has only rather uninteresting extrema with made up of straight-line segments the addition of the pressure term makes the infimum more interesting.

The third functional depends only on and is given by

(II.7)

where is a constant, is an arc length along and is a unit normal to . With , may be rewritten as an integral along of a *generalized Finsler metric* which represents a function such that . For details on Finsler metrics see [21].

So, using generalized Finsler metric the integral (II.7) can be rewritten as:

where (II.8)

**Note 2**: Intuitively, minimizing can be viewed as a generalized geodesic problem. It asks for paths such that (i) the length of is as short as possible while (ii) while normal to , has the largest possible derivative.

It can be shown that is equal to with a special choice of .

Here are details on this surprising statement.

We consider only smooth parts of and take outside an infinitesimal neighborhood of . Near , set

where , are curvilinear coordinates defined by the tangent and the normal at every point of .

//TODO: finish the section on Variational Problems for Image Inpaintings

# Appendix

## Fourier Transform and the Nyquist-Shannon Sampling Theorem

### Fourier Transform

**Definition**: The *Fourier transform pair* is given with

(A.1)

(A.2)

The synthesis formula (A.1) represents as a superposition of infinitesimally small complex sinusoids of the form

with ranging over an interval of length and with determining the relative amount of each complex sinusoidal component.

### Sampling in Frequency Domain

Notation

– continuous signal

– discrete sampled signal

– periodic impulse train

//TODO: finish the Appendix section on Fourier transform and Shannon-Nyquist theorem

## Harmonic Functions

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## Green’s Identities

Divergence theorem

Let is a subset of which is compact and has a piecewise smooth boundary .

If is a continuously differentiable vector field defined on a neighborhood of , then:

(D.1)

The closed measurable set is oriented by outward pointing normal at almost each point on the boundary . Here is a shorthand for .

The left-hand side of (D.1) represents the total of sources in the volume and the right-hand side represents the total flow across the boundary .

Figure: A region bounded by the surface with the surface normal .

Informal Derivation of the Divergence Theorem

The divergence theorem follows from the fact that if a volume V is partitioned into separate parts, the flux out of the original volume is equal to the sum of the flux out of each component volume. This is true despite the fact that the new sub-volumes have surfaces that were not part of the original volume’s surface, because these surfaces are just partitions between two of the sub-volumes and the flux through them just passes from one volume to the other and so cancels out when the flux out of the sub-volumes is summed.

Figure: A volume divided into two sub-volumes. At right the two sub-volumes are separated to show the flux out of the different surfaces

A close-up of a cartoon

Description automatically generated

Figure: the volume can be divided into any number of sub-volumes and the flux out of V is equal to the sum of the flux out of each sub-volume, because the flux through the green surfaces cancels out the sum. In (b) the volumes are separated showing that each green partition is part of boundary of two adjacent volumes.

A diagram of a green arrow

Description automatically generated with medium confidence

Figure: As the volume is subdivided into smaller parts, the ratio of the flux out of each volume to the volume approaches .

//TODO: finish the Appendix section on Green’s Identities

## Eikonal Equation

The eikonal equation (from , image) is a non-linear first order PDE encountered in the problems of wave propagation.

The classical eikonal equation in geometric optics is given as:

(E.1)

where is in an open subset of , is a positive function. The function and we are looking for solutions . More generally, an eikonal equation is an equation of the form

(E.2)

where is a function of two variables. The function is given and is the solution. If , then (E.2) becomes (E.1).

Eikonal equations arise with the application of the WKB method to electro-magnetic wave propagation, providing a link between physical wave propagation / optics and geometric ray optics. A fast computational algorithm to obtain approximate solution to the eikonal equation is the *fast-marching method*.

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## Poisson Equation

//TODO: finish the Appendix section on Poisson Equation

## Navier-Stokes Equations

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