2

Efficiency and Nondominance

This chapter covers the fundamental concepts of efficiency and nondominance. We first present some fundamental properties of nondominated points and several existence results for nondominated points and efficient solutions in Section 2.1. Section 2.2 introduces ideal and nadir points as bounds on the set of nondominated solutions. Then we briefly review weakly and strictly efficient solutions in Section 2.3. The same section also includes a geometric characterization of the three optimality concepts, with some extensions for the case of weakly efficient solutions. Finally, in Section 2.4 we introduce several definitions of properly efficient solutions, important subsets of efficient solutions from a computational point of view and in applications, and investigate their relationships.

Most of the material in this chapter can be found in the two books Göpfert and Nehse (1990) and Sawaragi *et al.* (1985), where the results are presented in more generality. We will also refer to the original publications for the main results.

2.1 Efficient Solutions and Nondominated Points

In this chapter we consider multicriteria optimization problems of the class $\bullet/\mathrm{id}/(\mathbb{R}^p, \leq)$:

$$\begin{aligned} \min\ & (f_1(x), \ldots, f_p(x)) \\ & \text{subject to } x \in \mathcal{X}. \end{aligned} \tag{2.1}$$

The image of the feasible set $\mathcal{X}$ under the objective function mapping f is denoted as $\mathcal{Y} := f(\mathcal{X})$. Let us formally repeat the definition of efficient solutions and nondominated points. Definition 2.1 also introduces the notion of dominance.

Definition 2.1. *A feasible solution $\hat{x} \in \mathcal{X}$ is called* efficient *or* Pareto optimal, *if there is no other $x \in \mathcal{X}$ such that $f(x) \leq f(\hat{x})$. If $\hat{x}$ is efficient, $f(\hat{x})$ is called* nondominated point. *If $x^1, x^2 \in \mathcal{X}$ and $f(x^1) \leq f(x^2)$ we say x^1* dominates *x^2 and $f(x^1)$* dominates *$f(x^2)$. The set of all efficient solutions $\hat{x} \in \mathcal{X}$ is denoted $\mathcal{X}_E$ and called the* efficient set. *The set of all nondominated points $\hat{y} = f(\hat{x}) \in \mathcal{Y}$, where $\hat{x} \in \mathcal{X}_E$, is denoted $\mathcal{Y}_N$ and called the* nondominated set—.

We have to remark that these notations are not unique in literature, unfortunately. Some authors use Pareto optimal for what we call efficient and efficient for what we call nondominated (e.g. this notation was used in the first edition of this book). The term noninferior solution has also been used. We will use the terms of Definition 2.1, but whenever consulting literature, the reader should check the definitions the respective author adopts.

Several other, equivalent, definitions of efficiency are frequently used, and we shall often refer to the one which is best suited in a given context. In particular, $\hat{x}$ is efficient if

1. there is no $x \in \mathcal{X}$ such that $f_k(x) \leq f_k(\hat{x})$ for $k = 1, \ldots, p$ and $f_i(x) < f_i(\hat{x})$ for some $i \in \{1, \ldots, k\}$;
2. there is no $x \in X$ such that $f(x) - f(\hat{x}) \in -\mathbb{R}^p_{\geqq} \setminus \{0\}$;
3. $f(x) - f(\hat{x}) \in \mathbb{R}^p \setminus \left\{-\mathbb{R}^p_{\geqq} \setminus \{0\}\right\}$ for all $x \in \mathcal{X}$;
4. $f(\mathcal{X}) \cap \left(f(\hat{x}) - \mathbb{R}^p_{\geqq}\right) = \{f(\hat{x})\}$;
5. there is no $f(x) \in f(\mathcal{X}) \setminus \{f(\hat{x})\}$ with $f(x) \in f(\hat{x}) - \mathbb{R}^p_{\geqq}$;
6. $f(x) \leqq f(\hat{x})$ for some $x \in \mathcal{X}$ implies $f(x) = f(\hat{x})$.

With the exception of the last, these definitions can be illustrated graphically. Definition 2.1 and equivalent definitions 1., 4., and 5. consider $f(\hat{x})$ and check for images of feasible solutions to the left and below (in direction of $-\mathbb{R}^p_{\geqq}$) of that point. See the left part of Figure 2.1. In equivalent definitions 2. and 3., through $f(x) - f(\hat{x})$, the set $\mathcal{Y} = f(\mathcal{X})$ is translated so that the origin coincides with $f(\hat{x})$, and the intersection of the translated set $\mathcal{Y}$ with the negative orthant is checked. This intersection contains only $f(\hat{x})$ if $\hat{x}$ is efficient. See the right part of Figure 2.1.

The first questions we discuss are the existence and the properties of the efficient set $\mathcal{X}_E$ and the nondominated set $\mathcal{Y}_N$. It is convenient to consider $\mathcal{Y}_N$ first, and then use properties of f to derive results on $\mathcal{X}_E$. So let $\mathcal{Y} \subset \mathbb{R}^p$ be a set. According to our definitions, $\hat{y} \in \mathcal{Y}$ is nondominated, if there is no $y \in \mathcal{Y}$ such that $y \leq \hat{y}$.

First we show by means of an example that even for convex sets $\mathcal{X}$ and $\mathcal{Y}$ the efficient set $\mathcal{X}_E$ and the nondominated set set $\mathcal{Y}_N$ might be empty or

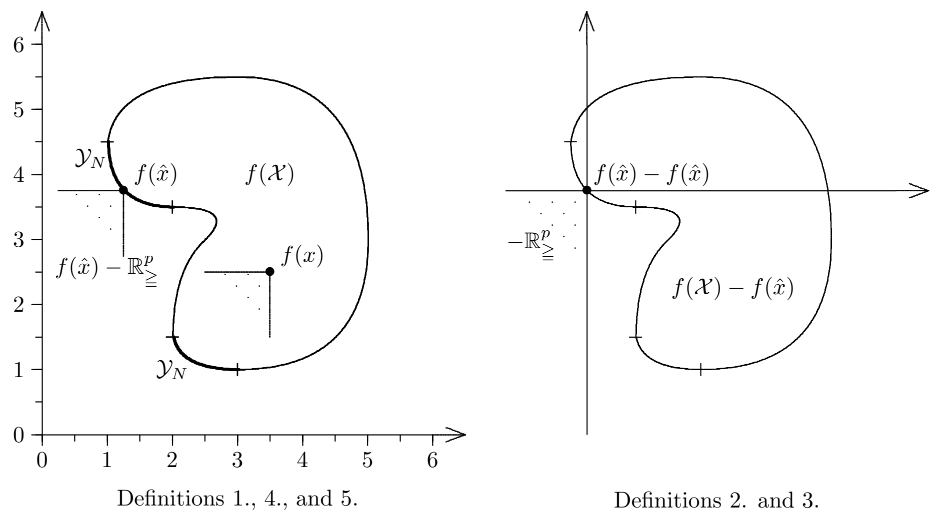


Fig. 2.1. Illustration of definitions of efficient solutions.

consist of isolated points. We will then proceed to prove some basic properties of nondominated sets, before we present several existence theorems for efficient solutions/nondominated points. Results on connectedness of $\mathcal{Y}_N$ and $\mathcal{X}_E$ will be given in Chapter 3.

Example 2.2 (Göpfert and Nehse (1990)). Consider a bicriterion optimization problem with feasible set

$$\mathcal{X} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} -1 \leq x_1 \leq 1, \\ -\sqrt{-x_1^2+1} < x_2 \leq 0 \quad \text{if} \quad -1 \leq x_1 \leq 0, \\ -\sqrt{-x_1^2+1} \leq x_2 \leq 0 \quad \text{if} \quad 0 < x_1 \leq 1 \end{array} \right\} \tag{2.2}$$

and objective function

$$f(x_1, x_2) = (x_1, x_2). \tag{2.3}$$

The feasible sets $\mathcal{X}$ in decision space and $\mathcal{Y}$ in criterion space (the latter coincides with $\mathcal{X}$ in this example) are depicted in Figure 2.2.

Clearly, there are no nondominated points, and therefore the bicriterion problem given by (2.2) and (2.3)does not have any efficient solutions: $\mathcal{Y}_N = \mathcal{X}_E = \emptyset$, even though $\mathcal{X}$ and $\mathcal{Y}$ are convex and f is continuous.

If we modify the problem slightly by letting

$$\mathcal{X} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} -1 \leq x_1 \leq 1, \\ x_2 = 0 \quad \text{if} \quad x_1 = -1, \\ -\sqrt{-x_1^2+1} < x_2 \leq 0 \quad \text{if} \quad -1 < x_1 < 0, \\ -\sqrt{-x_1^2+1} \leq x_2 \leq 0 \quad \text{if} \quad 0 \leq x_1 \leq 1 \end{array} \right\} \tag{2.4}$$

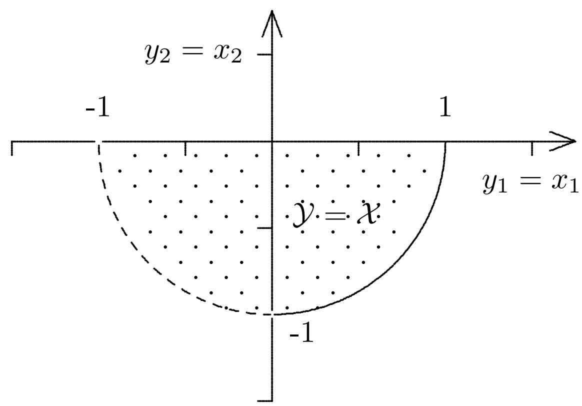


Fig. 2.2. Feasible set of the original problem in Example 2.2.

$\mathcal{Y}_N = \{(-1, 0), (0, -1)\}$ is no longer empty (Figure 2.3), but consists of only two disconnected points, which are "far apart" from one another in $\mathcal{Y}_N$.

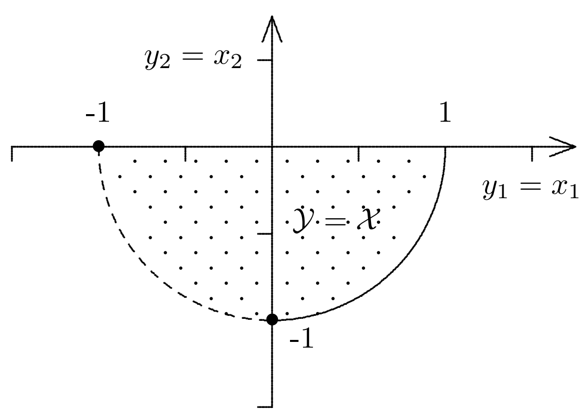


Fig. 2.3. Feasible set of the modified problem in Example 2.2.

□

Example 2.2 shows that conditions for existence of efficient solutions and nondominated points must be our first concern in the study of multicriteria optimization. In multicriteria optimization, the "trick" of Example 2.2, to use $y = f(x) = x$ is quite useful, as it allows to identify decision and criterion space and enables the study of both $\mathcal{X}_E$ and $\mathcal{Y}_N$ at the same time. We will often apply it in the examples to come.

The following properties of nondominated sets are mainly proved as tools for the proofs of theorems later in the text. However, they may well enhance an intuitive understanding of the concept of nondominance. First we show that nondominated points are located in the "lower left part" of $\mathcal{Y}$: Adding $\mathbb{R}^p_{\geqq}$ to $\mathcal{Y}$ does not change the nondominated set.

So let $\mathcal{Y} \subset \mathbb{R}^p$. Let $\mathcal{Y}_N = \{y \in \mathcal{Y} : \text{ there is no } y' \in \mathcal{Y} \text{ such that } y' \leq y\}$. In particular $\mathcal{Y}_N \subset \mathcal{Y}$.

Proposition 2.3. $\mathcal{Y}_N = \left(\mathcal{Y} + \mathbb{R}^p_{\geqq}\right)_N$.

Proof. The result is trivial if $\mathcal{Y} = \emptyset$, because $\mathcal{Y} + \mathbb{R}^p_{\geqq} = \emptyset$ and the nondominated subsets of both are empty, too.

So let $\mathcal{Y} \neq \emptyset$. First, assume $y \in (\mathcal{Y} + \mathbb{R}^p_{\geqq})_N$, but $y \notin \mathcal{Y}_N$. There are two possibilities. If $y \notin \mathcal{Y}$ there is $y' \in \mathcal{Y}$ and $0 \neq d \in \mathbb{R}^p_{\geqq}$ such that $y = y' + d$. Since $y' = y' + 0 \in \mathcal{Y} + \mathbb{R}^p_{\geqq}$ we get $y \notin (\mathcal{Y} + \mathbb{R}^p_{\geqq})_N$, a contradiction. If $y \in \mathcal{Y}$ there is $y' \in \mathcal{Y}$ such that $y' \leq y$. Let $d = y - y'$, which is in $\mathbb{R}^p_{\geqq} \setminus \{0\}$. Therefore $y = y' + d$ and $y \notin (\mathcal{Y} + \mathbb{R}^p_{\geqq})_N$, again contradicting the assumption. Hence in either case $y \in \mathcal{Y}_N$.

Second, assume $y \in \mathcal{Y}_N$ but $y \notin (\mathcal{Y} + \mathbb{R}^p_{\geqq})_N$. Then there is some $y' \in \mathcal{Y} + \mathbb{R}^p_{\geqq}$ with $y - y' = d' \in \mathbb{R}^p_{\geqq} \setminus \{0\}$. I.e. $y' = y'' + d''$ with $y'' \in \mathcal{Y}$, $d'' \in \mathbb{R}^p_{\geqq}$ and therefore $y = y' + d' = y'' + (d' + d'') = y'' + d$ with $d = d' + d'' \in \mathbb{R}^p_{\geqq} \setminus \{0\}$. This implies $y \notin \mathcal{Y}_N$, contradicting the assumption. Hence, $y \in (\mathcal{Y} + \mathbb{R}^p_{\geqq})_N$. $\square$

Proposition 2.3 is illustrated in Figure 2.4.

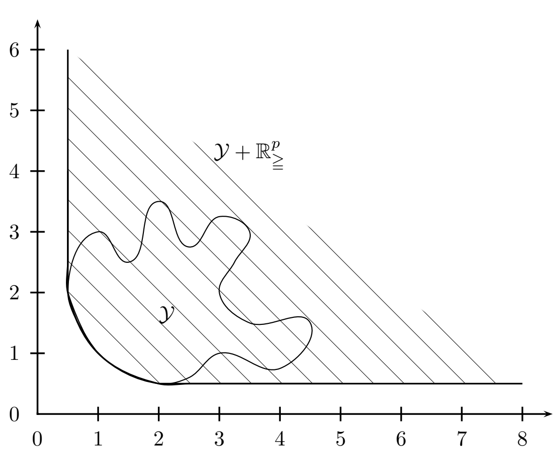


Fig. 2.4. Nondominated points of $\mathcal{Y}$ and $\mathcal{Y} + \mathbb{R}^p_{\geqq}$ are the same.

A second result, which is intuitively clear, is that efficient points must belong to the boundary of $\mathcal{Y}$.

Proposition 2.4. $\mathcal{Y}_N \subset \mathrm{bd}(\mathcal{Y})$.

Proof. Let $y \in \mathcal{Y}_N$ and suppose $y \notin \mathrm{bd}(\mathcal{Y})$. Therefore $y \in \mathrm{int}\,\mathcal{Y}$ and there exists an ε-neighbourhood $B(y,\varepsilon)$ of y (with $B(y,\varepsilon) := y + B(0,\varepsilon) \subset \mathcal{Y}$, $B(0,\varepsilon)$ is an open ball with radius ε centered at the origin). Let $d \neq 0,\ d \in \mathbb{R}^p_{\geqq}$. Then we can choose some $\alpha \in \mathbb{R},\ 0 < \alpha < \varepsilon$ such that $\alpha d \in B(0,\varepsilon)$. Now, $y - \alpha d \in \mathcal{Y}$ with $\lambda d \in \mathbb{R}^p_{\geqq} \setminus \{0\}$, i.e. $y \notin \mathcal{Y}_N$. □

From Propositions 2.3 and 2.4 we immediately get conditions for $\mathcal{Y}_N$ being empty.

Corollary 2.5. *If $\mathcal{Y}$ is open or if $\mathcal{Y} + \mathbb{R}^p_{\geqq}$ is open $\mathcal{Y}_N = \emptyset$.*

The next results concern the nondominated set of the Minkowski sum of two sets and of a set multiplied by a positive scalar.

Proposition 2.6. $(\mathcal{Y}_1 + \mathcal{Y}_2)_N \subset (\mathcal{Y}_1)_N + (\mathcal{Y}_2)_N$.

Proof. Let $y \in (\mathcal{Y}_1 + \mathcal{Y}_2)_N$. Then $y = y^1 + y^2$ for some $y^1 \in \mathcal{Y}_1, y^2 \in \mathcal{Y}_2$. Assuming $y^1 \notin (\mathcal{Y}_1)_N$ it follows that there must be some $y' \in \mathcal{Y}_1$ and $d \in \mathbb{R}^p_{\geq}$ such that $y^1 = y' + d$ and thus $y = y' + y^2 + d$ with $y' + y^2 \in \mathcal{Y}_1 + \mathcal{Y}_2$ whence $y \notin (\mathcal{Y}_1 + \mathcal{Y}_2)_N$, contradicting the assumption.

Analogously, $y^2 \in (\mathcal{Y}_2)_N$, i.e. $y^1 + y^2 \in (\mathcal{Y}_1)_N + (\mathcal{Y}_\in)_\mathcal{N}$. □

The inclusion $(\mathcal{Y}_1)_N + (\mathcal{Y}_2)_N \subset (\mathcal{Y}_1 + \mathcal{Y}_2)_N$ is not satisfied in general, Exercise 2.1 asks for a counterexample.

Proposition 2.7. $(\alpha\mathcal{Y})_N = \alpha(\mathcal{Y}_N)$*, for* $\alpha \in \mathbb{R},\ \alpha > 0$.

Proof. The easy proof is left to the reader, see Exercise 2.4. □

With these propositions we have some tools to facilitate working with nondominated sets. In order to prove existence results for nondominated points we have to introduce another fundamental statement, Zorn's Lemma.

Definition 2.8. *Let $(\mathcal{S}, \preceq)$ be a preordered set, i.e. $\preceq$ is reflexive and transitive. $(\mathcal{S}, \preceq)$ is* inductively ordered, *if every totally ordered subset of $(\mathcal{S}, \preceq)$ has a lower bound. A totally ordered subset of $(\mathcal{S}, \preceq)$ is also called a* chain.

Theorem 2.9 (Zorn's lemma). *Let the preordered set $(\mathcal{S}, \preceq)$ be inductively ordered. Then $\mathcal{S}$ contains a minimal element, i.e. there is $\hat{s} \in \mathcal{S}$ such that $s \preceq \hat{s}$ implies $\hat{s} \preceq s$.*

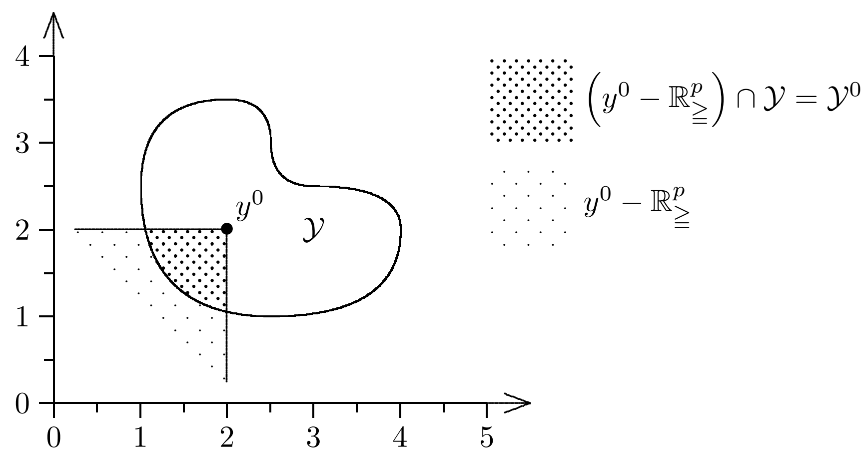


Fig. 2.5. $\mathcal{Y}^0$ is a compact section of $\mathcal{Y}$.

Theorem 2.10 (Borwein (1983)). *Let $\mathcal{Y}$ be a nonempty set and suppose there is some $y^0 \in \mathcal{Y}$ such that the section $\mathcal{Y}^0 = \{y \in \mathcal{Y} : y \leqq y^0\} = (y^0 - \mathbb{R}^p_{\geqq}) \cap \mathcal{Y}$ is compact (we say "$\mathcal{Y}$ contains a compact section"). Then $\mathcal{Y}_N$ is nonempty.*

Proof. The idea of the proof is as follows. We use the compactness of $\mathcal{Y}^0$ to show that every chain in $\mathcal{Y}^0$ has a lower bound. Thus $\mathcal{Y}^0$ is inductively ordered, and by Zorn's Lemma contains a minimal element $\hat{y}$. Showing that $\hat{y}$ is efficient in $\mathcal{Y}$ completes the proof.

Let $\mathcal{Y}^0$ be the compact section that exists by assumption and let $\mathcal{Y}^{\mathcal{I}} = \{y^i : i \in \mathcal{I}\}$, where $\mathcal{I}$ is some index set, be a chain in $\mathcal{Y}^0$. We prove that $\{y^i : i \in \mathcal{I}\}$ has a lower bound. To that end let $\mathcal{J} := \{J \subset \mathcal{I} : |J| < \infty\}$ be the set of all finite subsets of index set $\mathcal{I}$. For all $J \in \mathcal{J}$ finiteness of J and $\mathcal{Y}^{\mathcal{I}}$ being a chain in $\mathcal{Y}^0$ imply that $y^J := \inf\{y^i : i \in J\}$ exists and $y^J \in \mathcal{Y}^0$. Consider all sets $\mathcal{Y}^i := (y^i - \mathbb{R}^p_{\geqq}) \cap \mathcal{Y}^0$, where $i \in \mathcal{I}$. Obviously $\mathcal{Y}^i \subset \mathcal{Y}^0$ and $\mathcal{Y}^i$ is compact as a closed subset of the compact set $\mathcal{Y}^0$. Furthermore, if $J \in \mathcal{J}$, i.e. J is finite, $\cap_{i \in J} \mathcal{Y}^i \neq \emptyset$ because it contains y^J. Finally, by compactness of $\mathcal{Y}^0$ it follows that $\cap_{i \in \mathcal{I}} \mathcal{Y}^i \neq \emptyset$, which means there is some

$$y' \in \bigcap_{i \in \mathcal{I}} \left(y^i - \mathbb{R}^p_{\geqq}\right) \cap \mathcal{Y}^0. \tag{2.5}$$

In terms of the componentwise order this means $y' \leqq y^i$ for all $i \in \mathcal{I}$, or, in other words, $y' \in \mathcal{Y}^0$ is a lower bound of $\{y^i : i \in \mathcal{I}\}$, which is therefore inductively ordered.

We can now apply Zorn's Lemma (Theorem 2.9) to conclude that $\mathcal{Y}^0$ contains a minimal element $\hat{y}$. It remains to be shown that $\hat{y} \in \mathcal{Y}_N$. Assume the contrary. Then there would be some $y'' \in \mathcal{Y}$ with $y'' \leq \hat{y}$. For y'' we have

$$\begin{aligned} y'' \in \left(\hat{y} - \mathbb{R}^p_{\geqq}\right) \cap \mathcal{Y} &\subset \left(\left(y^0 - \mathbb{R}^p_{\geqq}\right) \cap \mathcal{Y} - \mathbb{R}^p_{\geqq}\right) \cap \mathcal{Y} \\ &\subset \left(y^0 - \mathbb{R}^p_{\geqq}\right) \cap \mathcal{Y} - \mathbb{R}^p_{\geqq} = \mathcal{Y}^0 - \mathbb{R}^p_{\geqq}. \end{aligned} \tag{2.6}$$

The first inclusion holds because $\hat{y} \in \mathcal{Y}^0$, the second is clear. Since $y'' \in \mathcal{Y}$ this implies $y'' \in \mathcal{Y}^0$, so that $y'' \leq \hat{y}$ contradicts minimality of $\hat{y}$ in $\mathcal{Y}^0$. □

Note that we have used the following fact about compact sets: If $\mathcal{Y}$ is compact and $(\mathcal{Y}^i)$, $i \in \mathcal{I}$ is a family of closed subsets of $\mathcal{Y}$ for some index set $\mathcal{I}$ such that $\cap_{k=1}^n \mathcal{Y}_{i_k} \neq \emptyset$ for all finite subsets of $\{i_1, \ldots, i_n\}$ of $\mathcal{I}$ then $\cap_{i \in \mathcal{I}} \mathcal{Y}_i \neq \emptyset$.

Another existence result does not use a compact section but a condition on $\mathcal{Y}$ which is similar to the finite subcover property of compact sets: the $\mathbb{R}^p_{\geqq}$-semicompactness condition, which considers open covers with special sets.

Definition 2.11. *A set $\mathcal{Y} \subset \mathbb{R}^p$ is called* $\mathbb{R}^p_{\geqq}$-semicompact *if every open cover of $\mathcal{Y}$ of the form $\left\{(y^i - \mathbb{R}^p_{\geqq})^c : y^i \in \mathcal{Y}, i \in \mathcal{I}\right\}$ has a finite subcover. This means that whenever $\mathcal{Y} \subset \cup_{i \in \mathcal{I}} (y^i - \mathbb{R}^p_{\geqq})^c$ there exist $m \in \mathbb{N}$ and $\{i_1, \ldots, i_m\} \subset \mathcal{I}$ such that*

$$\mathcal{Y} \subset \bigcup_{k=1}^{m} \left(y^{i_k} - \mathbb{R}^p_{\geqq}\right)^c. \tag{2.7}$$

Here $(y^i - \mathbb{R}^p_{\geqq})^c$ denotes the complement $\mathbb{R}^p \setminus (y^i - \mathbb{R}^p_{\geqq})$ of $y^i - \mathbb{R}^p_{\geqq}$. Note that these sets are always open.

Based on Zorn's Lemma again, we can prove that $\mathbb{R}^p_{\geqq}$-semicompactness guarantees existence of efficient points.

Theorem 2.12 (Corley (1980)). *If $\mathcal{Y} \neq \emptyset$ is $\mathbb{R}^p_{\geqq}$-semicompact then $\mathcal{Y}_N \neq \emptyset$.*

Proof. The main steps of the proof are the same as for Theorem 2.10. We show that $\mathcal{Y}$ is inductively ordered and apply Zorn's Lemma. First, we construct an open cover of $\mathcal{Y}$ as in Definition 2.11 and derive a contradiction when we assume that $\mathcal{Y}$ is not inductively ordered.

So assume $\mathcal{Y}$ is not inductively ordered. Then there is a totally ordered subset (a chain) of $\mathcal{Y}$, say $\mathcal{Y}' = \{y^i : i \in \mathcal{I}\}$ which has no lower bound. Therefore

$$\bigcap_{i \in \mathcal{I}} \left(\left(y^i - \mathbb{R}^p_{\geqq}\right) \cap \mathcal{Y}\right) = \emptyset. \tag{2.8}$$

As seen in the proof of Theorem 2.10, any element in this intersection would be a lower bound of $\mathcal{Y}'$. Then for each $y \in \mathcal{Y}$ there is some $y^i \in \mathcal{Y}'$ such that $y \notin y^i - \mathbb{R}^p_{\geqq}$.

Since $y^i - \mathbb{R}^p_{\geqq}$ is closed, $\{(y^i - \mathbb{R}^p_{\geqq})^c : i \in \mathcal{I}\}$ defines an open cover of $\mathcal{Y}$. Moreover, $y^i - \mathbb{R}^p_{\geqq} \subset y^{i'} - \mathbb{R}^p_{\geqq}$ if and only if $y^i \leqq y^{i'}$ and the sets of the cover are totally ordered by inclusion because $\mathcal{Y}'$ is a chain. Also, $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-semicompact and there is a finite subcover of $\{(y^i - \mathbb{R}^p_{\geqq})^c : i \in \mathcal{I}\}$.

Combining the last two observations, it follows that there is a minimal set (with respect to inclusion) in the finite subcover and hence there exists a single $y^* \in \mathcal{Y}'$ such that $\mathcal{Y} \subset (y^* - \mathbb{R}^p_{\geqq})^c$. This implies $y^* \leqq y^i$ for all $i \in \mathcal{I}$ and $y^* \notin \mathcal{Y}$, which is not possible. Therefore $\mathcal{Y}$ is inductively ordered.

Knowing that, we proceed as in the proof of Theorem 2.10 to conclude $\mathcal{Y}_N \neq \emptyset$. □

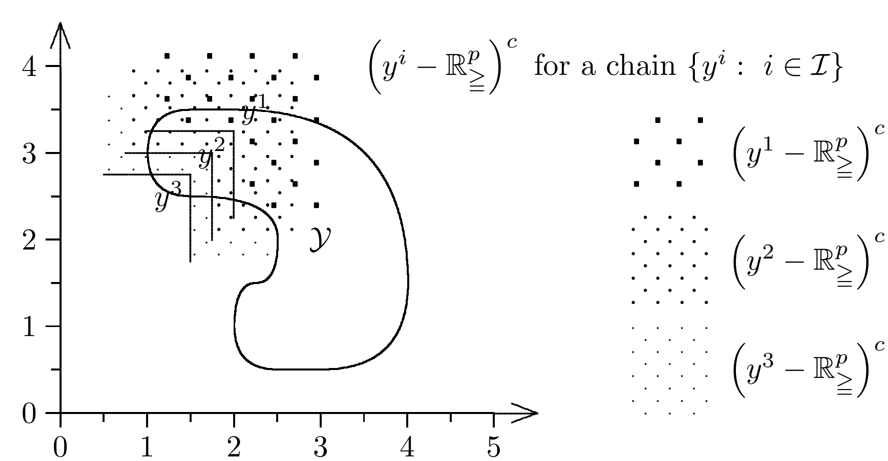


Fig. 2.6. Constructing a cover from a chain.

Although theoretically interesting, Theorem 2.12 gives a condition which is usually not easy to check: $\mathbb{R}^p_{\geqq}$-semicompactness. A weaker result is obtained if we use the stronger assumption of $\mathbb{R}^p_{\geqq}$-compactness.

Definition 2.13. *A set $\mathcal{Y} \subset \mathbb{R}^p$ is called* $\mathbb{R}^p_{\geqq}$-compact, *if for all $y \in \mathcal{Y}$ the section $(y - \mathbb{R}^p_{\geqq}) \cap \mathcal{Y}$ is compact.*

Proposition 2.14. *If $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-compact then $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-semicompact.*

Proof. Let $\{(y^i - \mathbb{R}^p_{\geqq})^c : y^i \in \mathcal{Y}, i \in \mathcal{I}\}$ be an open cover of $\mathcal{Y}$. For arbitrary $y^{i'} \in \mathcal{Y}$ take

$$\left\{ \left(y^i - \mathbb{R}^p_{\geqq} \right)^c : y^i \in \mathcal{Y}, i \in \mathcal{I}, i \neq i' \right\}. \tag{2.9}$$

(2.9) defines an open cover of $(y^{i'} - \mathbb{R}^p_{\geqq}) \cap \mathcal{Y}$, a compact set, since $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-compact. But compactness implies that the cover in (2.9) contains a finite subcover of $(y^{i'} - \mathbb{R}^p_{\geqq}) \cap \mathcal{Y}$. This finite subcover together with $(y^{i'} - \mathbb{R}^p_{\geqq})^c$ yields a finite cover of $\mathcal{Y}$, of the structure required for $\mathbb{R}^p_{\geqq}$-semicompactness. □

Corollary 2.15 (Hartley (1978)). *If $\mathcal{Y} \subset \mathbb{R}^p$ is nonempty and $\mathbb{R}^p_{\geqq}$-compact, then $\mathcal{Y}_N \neq \emptyset$.*

Proof. The result follows immediately from Theorem 2.12 and Proposition 2.14. □

So far, we focused on existence of nondominated points. Let us now consider existence of efficient solutions, i.e. conditions that guarantee $\mathcal{X}_E \neq \emptyset$, which is an important issue when practical problems are considered. We can use Theorem 2.12 and properties of f to get an existence result for $\mathcal{X}_E$. Theorem 2.19 below is a multicriteria analogon to the well known result that a lower semicontinuous function attains its minimum over a compact set.

Definition 2.16. *A function $f : \mathbb{R}^n \to \mathbb{R}^p$ is* $\mathbb{R}^p_{\geqq}$-semicontinuous *if*

$$f^{-1}\left(y - \mathbb{R}^p_{\geqq} \right) = \left\{ x \in \mathbb{R}^n : y - f(x) \in \mathbb{R}^p_{\geqq} \right\} \tag{2.10}$$

is closed for all $y \in \mathbb{R}^p$, i.e. the preimage of the translated negative orthant is always closed.

Lemma 2.17 below establishes $\mathbb{R}^p_{\geqq}$-semicontinuity as a proper generalization of lower semicontinuity of scalar valued functions.

Lemma 2.17. *A function $f : \mathbb{R}^n \to \mathbb{R}^p$ is $\mathbb{R}^p_{\geqq}$-semicontinuous if and only if the component functions $f_k : \mathbb{R}^n \to \mathbb{R}$ are lower semicontinuous for all $k = 1, \ldots, p$.*

The proof is left to the reader.

Proposition 2.18. *Let $\mathcal{X} \subset \mathbb{R}^n$ be nonempty and compact, $f : \mathbb{R}^n \to \mathbb{R}^p$ be $\mathbb{R}^p_{\geqq}$-semicontinuous. Then $\mathcal{Y} = f(\mathcal{X})$ is $\mathbb{R}^p_{\geqq}$-semicompact.*

Proof. Let $\{(y^i - \mathbb{R}^p_{\geqq})^c : y^i \in \mathcal{Y}, i \in \mathcal{I}\}$ be an open cover of $\mathcal{Y}$. By $\mathbb{R}^p_{\geqq}$-semicontinuity of f, $\{f^{-1}((y^i - \mathbb{R}^p_{\geqq})^c) : y^i \in \mathcal{Y}, i \in \mathcal{I}\}$ is an open cover of $\mathcal{X}$. Because $\mathcal{X}$ is compact there is a finite subcover in this open cover. The image of this subcover is a finite subcover of $\mathcal{Y}$ whence $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$ semicompact. □

Theorem 2.19. *Let $\mathcal{X} \subset \mathbb{R}^n$ be a nonempty and compact set. Let f be $\mathbb{R}^p_{\geqq}$-semicontinuous. Then $\mathcal{X}_E \neq \emptyset$.*

Proof. The result follows directly from Theorem 2.12 and Proposition 2.18. □

Given a set $\mathcal{Y} \subset \mathbb{R}^p$ with nonempty nondominated set $\mathcal{Y}_N \neq \emptyset$, it is clear that for any $y \in \mathcal{Y} \setminus \mathcal{Y}_N$ there is some $\hat{y} \in \mathcal{Y}$ such that $\hat{y} \leq y$. But is it always guaranteed that a nondominated $\hat{y}$ dominating y exists? It turns out that under existence conditions for nondominated points this is true.

Definition 2.20. *The nondominated set $\mathcal{Y}_N$ is said to be* externally stable, *if for each $y \in \mathcal{Y} \setminus \mathcal{Y}_N$ there is $\hat{y} \in \mathcal{Y}_N$ such that $y \in \hat{y} + \mathbb{R}^p_{\geqq}$.*

Theorem 2.21. *Let $\mathcal{Y} \subset \mathbb{R}^p_{\geqq}$ be nonempty and $\mathbb{R}^p_{\geqq}$-compact. Then $\mathcal{Y}_N$ is externally stable, i.e.*

$$\mathcal{Y} \subset \mathcal{Y}_N + \mathbb{R}^p_{\geqq}.$$

Proof. Let $y \in \mathcal{Y}$. Define

$$\mathcal{Y}' := \left(y - \mathbb{R}^p_{\geqq}\right) \cap \mathcal{Y},$$

i.e. all points in $\mathcal{Y}$ dominating y. We need to show that $\mathcal{Y}' \cap \mathcal{Y}_N \neq \emptyset$. To do so it is enough to show that $\mathcal{Y}'_N \neq \emptyset$ and that $\mathcal{Y}'_N \subset \mathcal{Y}_N$.

$\mathcal{Y}'$ is $\mathbb{R}^p_{\geqq}$-compact since $\mathcal{Y}$ is (see Definition 2.13). Therefore $\mathcal{Y}'_N \neq \emptyset$ according to Corollary 2.15.

Assume that y' is not in $\mathcal{Y}_N$, but $y' \in \mathcal{Y}'$ (otherwise y' is certainly not contained in $\mathcal{Y}'_N$). Thus $y' \in \mathcal{Y}$ and there is some $y'' \in \mathcal{Y}$ such that $y'' \leq y'$. Therefore $y'' \leq y' \leq y$ and $y'' \in \mathcal{Y}'$. This implies $y' \notin \mathcal{Y}'_N$. □

2.2 Bounds on the Nondominated Set

In this section, we define the ideal and nadir points as lower and upper bounds on nondominated points. These points give an indication of the range of the values which nondominated points can attain. They are often used as reference points in compromise programming (see Section 4.5) or in interactive methods the aim of which is to find a most preferred solution for a decision maker.

We assume that $\mathcal{X}_E$ and $\mathcal{Y}_N$ are nonempty, and want to find real numbers $\underline{y}_k, \overline{y}_k,\ k = 1, \ldots, p$ with $\underline{y}_k \leq y_i \leq \overline{y}_i$ for all $y \in \mathcal{Y}_N$, as shown in Figure 2.7.

An obvious possibility is to choose

$$\underline{y}_k := \min_{y \in \mathcal{Y}} y_i, \tag{2.11}$$

$$\overline{y}_k := \max_{y \in \mathcal{Y}} y_i. \tag{2.12}$$

While the lower bound (2.11) is tight (there is always an efficient point $y \in \mathcal{Y}_N$ with $y_k = \underline{y}_k$), the upper bound (2.12) tends to be far away from actual nondominated points. For this reason, the upper bound is defined as the maximum over nondominated points only.

Definition 2.22. *1. The point $y^I = (y_1^I, \ldots, y_p^I)$ given by*

$$y_k^I := \min_{x \in \mathcal{X}} f_k(x) = \min_{y \in \mathcal{Y}} y_k \tag{2.13}$$

is called the ideal point *of the multicriteria optimization problem* $\min_{x \in \mathcal{X}} (f_1(x), \ldots, f_p(x))$.

2. The point $y^N = (y_1^N, \ldots, y_p^N)$ given by

$$y_k^N := \max_{x \in \mathcal{X}_E} f_k(x) = \max_{y \in \mathcal{Y}_N} y_k \tag{2.14}$$

is called the nadir point *of the multicriteria optimization problem.*

The ideal and nadir points for a nonconvex problem are shown in Figure 2.7.

Obviously, we have $y_k^I \leq y_k$ and $y_k \leq y_k^N$ for any $y \in \mathcal{Y}_N$. Furthermore y^I and y^N are tight lower and upper bounds on the efficient set. Since the ideal point is found by solving p single objective optimization problems its computation can be considered easy (from a multicriteria point of view). On the other hand, computation of y^N involves optimization over the efficient set, a very difficult problem. No efficient method to determine y^N for a general MOP is known.

Due to the difficulty of computing y^N, heuristics are often used. A basic estimation of the nadir point uses pay-off tables. We describe the approach now.

First, we solve p single objective problems $\min_{x \in \mathcal{X}} f_k(x)$. Let the optimal solutions be $x^k,\ k = 1, \ldots, p$, i.e. $f_k(x^k) = \min_{x \in X} f_k(x)$. Using these optimal solutions compute the pay-off table shown in Table 2.1.

Finally, from the pay-off table, clearly $y_k^I = f_k(x^k), k = 1, \ldots, p$. We define

$$\tilde{y}_i^N := \max_{k=1,\ldots,p} f_i(x^k), \tag{2.15}$$

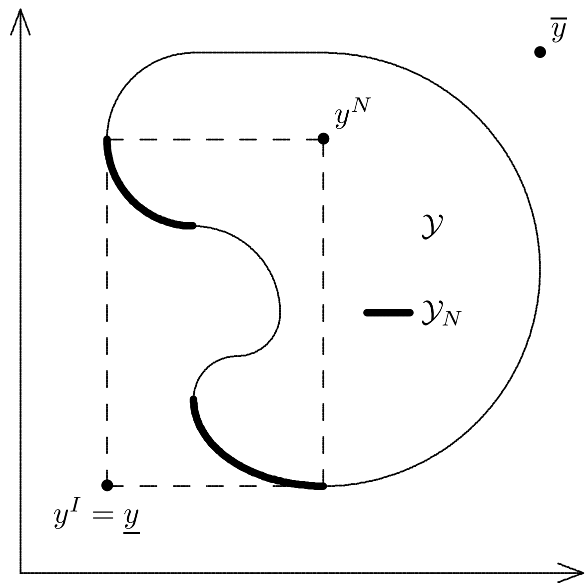


Fig. 2.7. Efficient set, ideal, and nadir point.

Table 2.1. Pay-off table and ideal point.

	x^1	x^2	$\cdots$	x^{p-1}	x^p
f_1	y_1^I	$f_1(x^2)$	$\cdots$	$f_1(x^{p-1})$	$f_1(x^p)$
f_2	$f_2(x^1)$	$\ddots$	$\cdots$	$\cdots$	$f_2(x^p)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
f_{p-1}	$f_{p-1}(x^1)$	$\cdots$	$\cdots$	$\ddots$	$f_{p-1}(x^p)$
f_p	$f_p(x^1)$	$f_p(x^2)$	$\cdots$	$f_p(x^{p-1})$	y_p^I

the largest element in row i, as an estimate for y_i^N.

Although appealing at first glance, the problem with pay-off tables is that $\tilde{y}^N$ may over- or under-estimate y^N, when more than two objectives are present, and when there are multiple optimal solutions of the single objective problems $\min_{x \in \mathcal{X}} f_k(x)$. The example below illustrates the phenomenon.

Example 2.23 (Korhonen et al. *(1997)).* Consider the multicriteria linear programming problem

$$\begin{array}{lrrrrrrrl}
\min & & -11\ x_2 & -11\ x_3 & -12\ x_4 & -9\ x_5 & -9\ x_6 & +9\ x_7 & \\
\min & -11\ x_1 & & -11\ x_3 & -9\ x_4 & -12\ x_5 & -9\ x_6 & +9\ x_7 & \\
\min & -11\ x_1 & -11\ x_2 & & -9\ x_4 & -9\ x_5 & -12\ x_6 & -12\ x_7 & \\
\text{subject to} & x_1 & +\ x_2 & +\ x_3 & +\ x_4 & +\ x_5 & +\ x_6 & +\ x_7 & = 1 \\
 & & & & & & & x & \geqq 0.
\end{array}$$

The image of the feasible set $\mathcal{Y} = f(\mathcal{X})$ is illustrated in Figure 2.8.

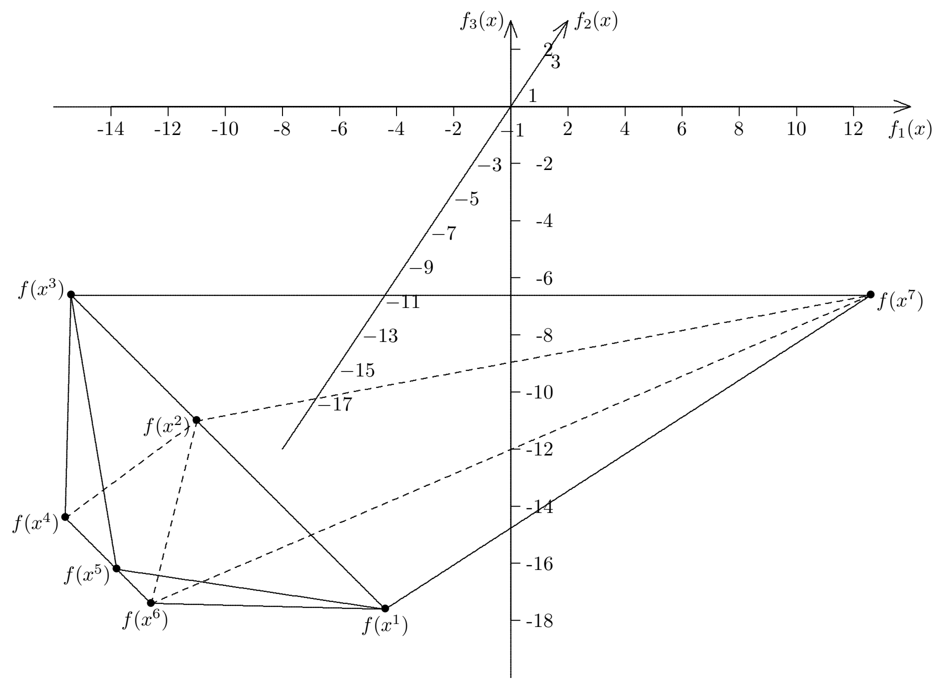


Fig. 2.8. Feasible set in Example 2.23.

To check the pay-off table approach, we proceed as follows. Solving the single objective problems, we get the solutions shown in Table 2.2, where e^i denotes the i-th unit vector.

The pay-off table is shown in Table 2.3, with two different choices of the optimal solution of the third problem, namely $x = e^6$ and $x = e^7$.

We shall now show that the nadir point cannot be obtained from the pay-off table. By solving appropriate weighted sum problems with positive weights, it can be seen that $x^i = e^i, i \in \{1, \ldots, 6\}$ are (properly) efficient (cf. Chapter 3) The feasible solution $x^7 = e^7$ is obviously weakly efficient, as a minimizer of one objective, but not efficient since x^6 dominates x^7.

For $x = e^1 \in \mathcal{X}_E$ we have $f(x) = (0, -11, -11)$. For $x = e^2 \in \mathcal{X}_E$ we have $f(x) = (-11, 0, -11)$ and for $x = e^3 \in \mathcal{X}_E$ we have $f(x) = (-11, -11, 0)$.

Table 2.2. Single objectives and minimizers in Example 2.23.

Problem	All optimal solutions
$\min_{x\in\mathcal{X}} f_1(x)$	$x_4 = 1,\ x_i = 0,\ i \neq 4$, i.e. $x = e^4$
$\min_{x\in\mathcal{X}} f_2(x)$	$x_5 = 1,\ x_i = 0,\ i \neq 5$, i.e. $x = e^5$
$\min_{x\in\mathcal{X}} f_3(x)$	$x_6 = \alpha,\ x_7 = 1-\alpha,\ x_i = 0,\ i \neq 6,7$,
	where $\alpha \in [0,1]$, i.e. $x = \alpha e^6 + (1-\alpha)e^7$

Table 2.3. Pay-off table in Example 2.23.

	e^4	e^5	e^6	e^7
f_1	-12	-9	-9	9
f_2	-9	-12	-9	9
f_3	-9	-9	-12	-12

Therefore $y_i^N \geq 0,\ i = 1,2,3$. But because no efficient solution can have positive objective values in this example, the Nadir point is $y^N = (0,0,0)$.

For the values in the pay-off table, we observe that

- with $x = e^7$ we overestimate y_1^N (arbitrarily far: replace +9 by $M > 0$ arbitrarily large), whereas
- with $x = e^6$ we underestimate y_1^N severely (arbitrarily far, if we modify the cost coefficients appropriately).

□

The reason for overestimation in Example 2.23 is, that $\overline{x}^3$ is only weakly efficient. If we choose efficient solutions to determine x^i, overestimation is of course impossible. The presence of weakly efficient solutions is caused by the multiple optimal solutions of $\min_{x\in\mathcal{X}} f_3(x)$. In general, it is difficult to be sure that the single objective optimizers are efficient.

The only case where y^N can be determined is for $p = 2$. Here the worst value for y_2 is attained when y_1 is minimal and vice versa, and by a two step optimization process, we can eliminate weakly efficient choices in the pay-off table.

Algorithm 2.1 (Nadir point for $p = 2$.)
Input: *Feasible set $\mathcal{X}$ and objective function f of an MOP.*

Solve the single objective problems $\min_{x \in \mathcal{X}} f_1(x)$ *and* $\min_{x \in \mathcal{X}} f_2(x)$*. Denote the optimal objective values by* y_1^I, y_2^I.

Solve $\min_{x \in \mathcal{X}} f_2(x)$ *with the additional constraint* $f_1(x) \leq y_1^I$.

Solve $\min_{x \in \mathcal{X}} f_1(x)$ *with the additional constraint* $f_2(x) \leq y_2^I$.

Denote the optimal objective values by y_2^N, y_1^N*, respectively.*

Output: $y^N = (y_1^N, y_2^N)$ *is the nadir point,* $y^I = (y_1^I, y_2^I)$ *is the ideal point.*

It is easy to see from the definition of y^N that the procedure indeed finds y^N. The optimal solutions of the constrained problems in the second step are efficient. Unfortunately, this approach cannot be generalized to more than two objectives, because if $p > 2$ we do not know, which objectives to fix in the second step. Indeed, the reader can check, that in Example 8.5 of Section 8.1, the Nadir point is $y^N = (11, 9, 11, 8)$, where y_2^N and y_4^N are determined by efficient solutions, which are not optimal for any of the single objectives.

2.3 Weakly and Strictly Efficient Solutions

Nondominated points are defined by the componentwise order on $\mathbb{R}^p$. When we use the the weak and strict componentwise order instead, we obtain definitions of strictly and weakly nondominated points, respectively. In this section, we prove an existence result for weakly nondominated points and weakly efficient solutions. We then give a geometric characterization of all three types of efficiency and some further results on the structure of weakly efficient solutions of convex multicriteria optimization problems.

Definition 2.24. *A feasible solution $\hat{x} \in \mathcal{X}$ is called* weakly efficient *(weakly Pareto optimal) if there is no $x \in \mathcal{X}$ such that $f(x) < f(\hat{x})$, i.e. $f_k(x) < f_k(\hat{x})$ for all $k = 1, \ldots, p$. The point $\hat{y} = f(\hat{x})$ is then called* weakly nondominated.

A feasible solution $\hat{x} \in \mathcal{X}$ is called strictly efficient *(strictly Pareto optimal) if there is no $x \in \mathcal{X}$, $x \neq \hat{x}$ such that $f(x) \leqq f(\hat{x})$. The weakly (strictly) efficient and nondominated sets are denoted $\mathcal{X}_{wE}(\mathcal{X}_{sE})$ and $\mathcal{Y}_{wE}$, respectively.*

Some authors say that a weakly nondominated point is a nondominated point with respect to $\operatorname{int} \mathbb{R}^p_{\geqq} = \mathbb{R}^p_{>}$, a notation that is quite convenient in the context of cone-efficiency and cone-nondominance. Because in this text we focus on the case of the nonnegative orthant we shall distinguish between efficiency/nondominance and their weak counterparts.

From the definitions it is obvious that

$$\mathcal{Y}_N \subset \mathcal{Y}_{wN} \tag{2.16}$$

and

$$\mathcal{X}_{sE} \subset \mathcal{X}_E \subset \mathcal{X}_{wE}. \tag{2.17}$$

As in the case of efficiency, weak efficiency has several equivalent definitions. We mention only two. A feasible solution $\hat{x} \in \mathcal{X}$ is weakly efficient if and only if

1. there is no $x \in \mathcal{X}$ such that $f(\hat{x}) - f(x) \in \operatorname{int} \mathbb{R}^p_{\geqq} = \mathbb{R}^p_{>}$
2. $(f(\hat{x}) - \mathbb{R}^p_{>}) \cap \mathcal{Y} = \emptyset$.

It is also of interest that there is no such concept as strict nondominance for sets $\mathcal{Y} \subset \mathbb{R}^p$. By definition, strict efficiency prohibits solutions x^1, x^2 with $f(x^1) = f(x^2)$, i.e. strict efficiency is the multicriteria analogon of unique optimal solutions in scalar optimization:

$$\hat{x} \in \mathcal{X}_{sE} \Longleftrightarrow \hat{x} \in \mathcal{X}_E \text{ and } |\{x : f(x) = f(\hat{x})\}| = 1. \tag{2.18}$$

It is obvious that all existence results for $\mathcal{Y}_N$ imply existence of $\mathcal{Y}_{wN}$ as well. However, we shall see that $\mathcal{Y}_{wN}$ can be nonempty, even if $\mathcal{Y}_N$ is empty. Therefore, independent conditions for $\mathcal{Y}_{wN}$ to be nonempty are interesting. We give a rather weak one here, another one is in Exercise 2.6. Note that the proof does not require Zorn's Lemma.

Theorem 2.25. *Let $\mathcal{Y} \subset \mathbb{R}^p$ be nonempty and compact. Then $\mathcal{Y}_{wN} \neq \emptyset$.*

Proof. Suppose $\mathcal{Y}_{wN} = \emptyset$. Then for all $y \in \mathcal{Y}$ there is some $y' \in \mathcal{Y}$ such that $y \in y' + \mathbb{R}^p_{>}$. Taking the union over all $y \in \mathcal{Y}$ we obtain

$$\mathcal{Y} \subset \bigcup_{y'} (y' + \mathbb{R}^p_{>}). \tag{2.19}$$

Because $\mathbb{R}^p_{>}$ is open, (2.19) defines an open cover of $\mathcal{Y}$. By compactness of $\mathcal{Y}$ there exists a finite subcover, i.e.

$$\mathcal{Y} \subset \bigcup_{i=1}^{k} (y^i + \mathbb{R}^p_{>}). \tag{2.20}$$

Choosing y^i on the left hand side, this yields that for all $i = 1, \ldots, k$ there is some $1 \leq j \leq k$ with $y^i \in y^j + \mathbb{R}^p_{>}$. In other words, for all i there is some j such that $y^j < y^i$. By transitivity of the strict componentwise order $<$ and because there are only finitely many y^i there exist i^*, m, and a chain of inequalities s.t. $y^{i^*} < y^{i_1} < \ldots < y^{i_m} < y^{i^*}$, which is impossible. $\square$

The essential difference as compared to the proofs of Theorems 2.10 and 2.12 is that in those theorems we deal with sets $y - \mathbb{R}^p_{\geqq}$ which are closed. Here, we have sets $y + \mathbb{R}^p_{>}$ which are open. Note that $y \notin y + \mathbb{R}^p_{>}$.

Theorem 2.25 and continuity of f can now be used to prove existence of weakly efficient solutions.

Corollary 2.26. *Let $\mathcal{X} \subset \mathbb{R}^n$ be nonempty and compact. Assume that $f : \mathbb{R}^n \to \mathbb{R}^p$ is continuous. Then $\mathcal{X}_{wE} \neq \emptyset$.*

Proof. The result follows from Theorem 2.19 and $\mathcal{X}_E \subset \mathcal{X}_{wE}$ or from Theorem 2.25 and the fact that $f(\mathcal{X})$ is compact for compact $\mathcal{X}$ and continuous f. □

As indicated earlier, the inclusion $\mathcal{Y}_N \subset \mathcal{Y}_{wN}$ is in general strict. The following example shows that $\mathcal{Y}_{wN}$ can be nonempty, even if $\mathcal{Y}_N$ is empty, and also, of course, if $\mathcal{Y}$ is not compact. It also illustrates that $\mathcal{Y}_{wN} \setminus \mathcal{Y}_N$ might be a rather large set.

Example 2.27. Consider the set

$$\mathcal{Y} = \left\{(y_1, y_2) \in \mathbb{R}^2 : \ 0 < y_1 < 1, \ 0 \leq y_2 \leq 1\right\}. \tag{2.21}$$

Then $\mathcal{Y}_N = \emptyset$ but $\mathcal{Y}_{wN} = (0,1) \times \{0\} = \{y \in \mathcal{Y} : \ 0 < y_1 < y_2, y_2 = 0\}$ (Figure 2.9).

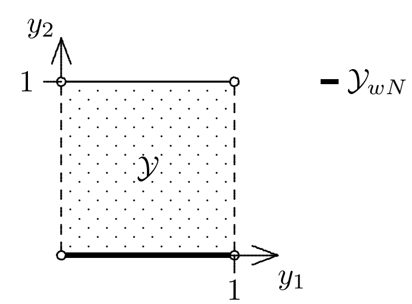


Fig. 2.9. $\mathcal{Y}_N$ is empty, $\mathcal{Y}_{wN}$ is not.

Let us now look at the closed square, i.e.

$$\mathcal{Y} = \{(y_1, y_2) \in \mathbb{R}^2 : \ 0 \leq y_i \leq 1\}. \tag{2.22}$$

We have $\mathcal{Y}_N = \{0\}$ and $\mathcal{Y}_{wN} = \{(y_1, y_2) \in \mathcal{Y} : \ y_1 = 0 \text{ or } y_2 = 0\}$. (Figure 2.10)

□

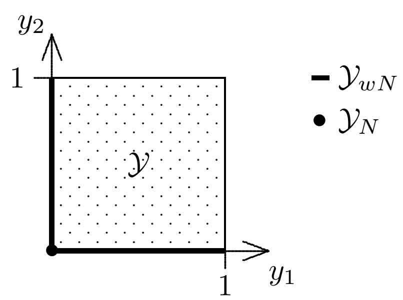


Fig. 2.10. Nondominated and weakly nondominated points.

$\mathcal{X}_E, \mathcal{X}_{sE}$ and $\mathcal{X}_{wE}$ can be characterized geometrically. To derive this characterization, we introduce *level sets* and *level curves* of functions.

Definition 2.28. *Let $\mathcal{X} \subset \mathbb{R}^n$, $f : \mathcal{X} \to \mathbb{R}$, and $\hat{x} \in \mathcal{X}$.*

$$\mathcal{L}_{\leq}(f(\hat{x})) = \{x \in \mathcal{X} : \ f(x) \leq f(\hat{x})\} \tag{2.23}$$

is called the level set *of f at $\hat{x}$.*

$$\mathcal{L}_{=}(f(\hat{x})) = \{x \in \mathcal{X} : \ f(x) = f(\hat{x})\} \tag{2.24}$$

is called the level curve *of f at $\hat{x}$.*

$$\begin{aligned} \mathcal{L}_{<}(f(\hat{x})) &= \mathcal{L}_{\leq}(f(\hat{x})) \setminus \mathcal{L}_{=}(f(\hat{x})) \\ &= \{x \in \mathcal{X} : \ f(x) < f(\hat{x})\} \end{aligned} \tag{2.25}$$

is called the strict level set *of f at $\hat{x}$.*

Obviously $\mathcal{L}_{=}(f(\hat{x})) \subset \mathcal{L}_{\leq}(f(\hat{x}))$ and $x \in \mathcal{L}_{=}(f(\hat{x}))$.

Example 2.29. We use an example with $\mathcal{X} = \mathbb{R}^2$ for illustration purposes. Let $f(x_1, x_2) = x_1^2 + x_2^2$. Let $\hat{x} = (3, 4)$. Hence

$$\mathcal{L}_{\leq}(f(\hat{x})) = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 \leq 25\right\}, \tag{2.26}$$

$$\mathcal{L}_{=}(f(\hat{x})) = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 = 25\right\}. \tag{2.27}$$

The level set and level curve are illustrated in Figure 2.11, as disk and circle in the x_1-x_2-plane, respectively.

□

For a multicriteria optimization problem we consider the level sets and level curves of all objectives $f_1, \ldots, f_p$ at $\hat{x}$. The following observation shows how level sets can be used to decide efficiency of $\hat{x}$.

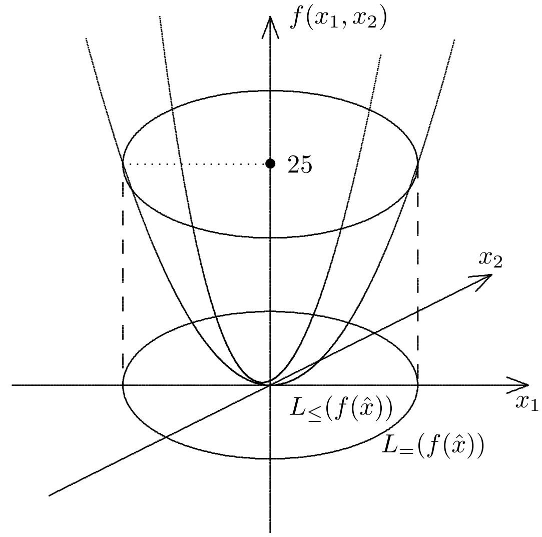


Fig. 2.11. Level set and level curve in Example 2.29.

Let us consider a bicriterion problem, and assume that we have determined $\mathcal{L}_{\leq}(f_1(\hat{x}))$ and $\mathcal{L}_{\leq}(f_2(\hat{x}))$ for feasible solution $\hat{x}$, as shown in Figure 2.12. We shall assume that the level curves are the boundaries of the level sets and the strict level sets are the interiors of the level sets.

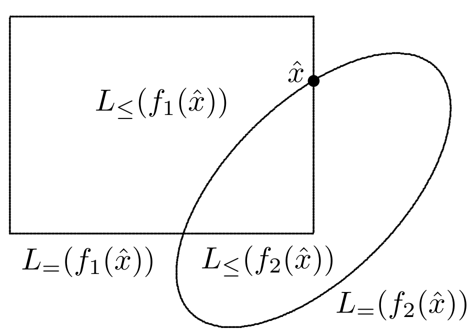


Fig. 2.12. Level sets and efficiency.

Can $\hat{x}$ be efficient? The answer is no: It is possible to move into the interior of the intersection of both level sets and thus find feasible solutions, which are better with respect to both f_1 and f_2. In fact, $\hat{x}$ is not even weakly efficient. Thus, $\hat{x}$ can only be (weakly) efficient if the intersection of strict level sets is empty or level sets intersect in level curves, respectively. We can now formulate the characterization of (strict, weak) efficiency using level sets.

Theorem 2.30 (Ehrgott *et al.* (1997)). *Let $\hat{x} \in \mathcal{X}$ be a feasible solution and define $\hat{y}_k := f_k(\hat{x})$, $k = 1, \ldots, p$. Then*

1. $\hat{x}$ is strictly efficient *if and only if*

$$\bigcap_{k=1}^{p} \mathcal{L}_{\leq}(\hat{y}_k) = \{\hat{x}\}. \tag{2.28}$$

2. $\hat{x}$ is efficient *if and only if*

$$\bigcap_{k=1}^{p} \mathcal{L}_{\leq}(\hat{y}_k) = \bigcap_{k=1}^{p} \mathcal{L}_{=}(\hat{y}_k). \tag{2.29}$$

3. $\hat{x}$ is weakly efficient *if and only if*

$$\bigcap_{k=1}^{p} \mathcal{L}_{<}(\hat{y}_k) = \emptyset. \tag{2.30}$$

Proof. 1. $\hat{x}$ is strictly efficient
$\Longleftrightarrow$ there is no $x \in \mathcal{X}, x \neq \hat{x}$ such that $f(x) \leqq f(\hat{x})$
$\Longleftrightarrow$ there is no $x \in \mathcal{X}, x \neq \hat{x}$ such that $f_k(x) \leq f_k(\hat{x})$ for all $k = 1, \ldots, p$
$\Longleftrightarrow$ there is no $x \in \mathcal{X}, x \neq \hat{x}$ such that $x \in \cap_{k=1}^{p} \mathcal{L}_{\leq}(\hat{y}_k)$
$\Longleftrightarrow \cap_{k=1}^{p} \mathcal{L}_{\leq}(\hat{y}_k) = \{\hat{x}\}$

2. $\hat{x}$ is efficient
$\Longleftrightarrow$ there is no $x \in \mathcal{X}$ such that both $f_k(x) \leq f_k(\hat{x})$ for all $k = 1, \ldots, p$ and $f_j(x) < f_j(\hat{x})$ for some j
$\Longleftrightarrow$ there is no $x \in \mathcal{X}$ such that both $x \in \cap_{k=1}^{p} \mathcal{L}_{\leq}(\hat{y}_q)$ and $x \in \mathcal{L}_{<}(\hat{y}_j)$ for some j
$\Longleftrightarrow \cap_{k=1}^{p} \mathcal{L}_{\leq}(\hat{y}_k) = \cap_{k=1}^{p} \mathcal{L}_{=}(y_k)$

3. $\hat{x}$ is weakly efficient
$\Longleftrightarrow$ there is no $x \in \mathcal{X}$ such that $f_k(x) < f_k(\hat{x})$ for all $k = 1, \ldots, p$
$\Longleftrightarrow$ there is no $x \in \mathcal{X}$ such that $x \in \cap_{k=1}^{p} \mathcal{L}_{<}(\hat{y}_q)$
$\Longleftrightarrow \cap_{k=1}^{p} \mathcal{L}_{<}(\hat{y}_k) = \emptyset$. □

Clearly, Theorem 2.30 is most useful when the level sets are available graphically, i.e. when $n \leq 3$. We illustrate the use of the geometric characterization by means of an example with two variables. Exercises 2.8 – 2.11 show how the(strictly, weakly) efficient solutions can be described explicitly for problems with one variable.

Example 2.31. Consider three points in the Euclidean plane, $x^1 = (1,1), x^2 = (1,4)$, and $x^3 = (4,4)$. The l_2^2-location problem is to find a point $x = (x_1, x_2) \in \mathbb{R}^2$ such that the sum of weighted squared distances from x to the three points $x^i, i = 1,2,3$ is minimal. We consider a bicriterion l_2^2-location problem, i.e. two weights for each of the points x^i are given through two weight vectors $w^1 = (1,1,1)$ and $w^2 = (2,1,4)$.

The two objectives measuring weighted distances are given by

$$f_k(x) = \sum_{i=1}^{3} w_i^k ((x_1^i - x_1)^2 + (x_2^i - x_2)^2). \tag{2.31}$$

Evaluating these functions we obtain

$$\begin{aligned} f_1(x) &= 2(1-x_1)^2 + (4-x_1)^2 + (1-x_2)^2 + 2(4-x_2)^2 \\ &= (x_1^2 - 4x_1 + x_2^2 - 6x_2) + 51 \\ f_2(x) &= 3(1-x_1)^2 + 4(4-x_1)^2 + 2(1-x_2)^2 + 5(4-x_2)^2 \\ &= 7\left(x_1^2 - \frac{38}{7}x_1 + x_2^2 - \frac{44}{7}x_2\right) + 149. \end{aligned}$$

We want to know if $x = (2,2)$ is efficient. So we check the level sets and level curves of f_1 and f_2 at $(2,2)$. The objective values are $f_1(2,2) = 15$ and $f_2(2,2) = 41$.

The level set $\mathcal{L}_=(f_1(2,2)) = \{x \in \mathbb{R}^2 : f_1(x) = 15\}$ is given by

$$\begin{aligned} f_1(x) = 15 &\iff 3(x_1^2 - 4x_1 + x_2^2 - 6x_2) + 51 = 15 \\ &\iff (x_1 - 2)^2 + (x_2 - 3)^2 = 1, \end{aligned}$$

i.e. $\mathcal{L}_=(f_1(2,2)) = \{x \in \mathbb{R}^2 : (x_1-2)^2 + (x_2-3)^2 = 1\}$, a circle with center $(2,3)$ and radius 1. Analogously, for f_2 we have

$$\begin{aligned} f_2(x) = 41 &\iff 7\left(x_1^2 - \frac{38}{7}x_1 + x_2^2 - \frac{44}{7}x_2\right) + 149 = 41 \\ &\iff \left(x_1 - \frac{19}{7}\right)^2 + \left(x_2 - \frac{22}{7}\right)^2 = \frac{89}{49}, \end{aligned}$$

and $\mathcal{L}_=(f_2(2,2)) = \left\{x \in \mathbb{R}^2 : (x_1 - 19/7)^2 + (x_2 - 22/7)^2 = 89/49\right\}$, a circle around $(19/7, 22/7)$ with radius $\sqrt{89}/7$.

In Figure 2.13 we see that $\cap_{i=1}^2 \mathcal{L}_\leq(f_i(2,2)) \neq \cap_{i=1}^2 \mathcal{L}_=(f_i(2,2))$ because the intersection of the discs has nonempty interior. Therefore, from Theorem 2.30 $x = (2,2)$ is not efficient. Note that in this case the level sets are simply the whole discs, the level curves are the circles and the strict level sets are the interiors of the discs.

Let us now check $x = (2, 3)$. We have $f_1(2, 3) = 12$ and $f_2(2, 3) = 32$. Repeating the computations from above, we obtain

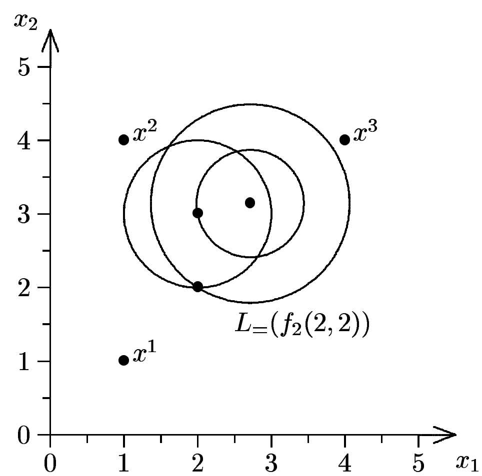
$$f_1(x) = 12 \Longleftrightarrow (x_1 - 2)^2 + (x_2 - 3)^2 = 0,$$

whence $\mathcal{L}_=(f_1(2, 3)) = \{x \in \mathbb{R}^2 : (x_1 - 2)^2 + (x_2 - 3)^2 = 0\} = \{(2, 3)\}$. For f_2

$$f_2(x) = 32 \Longleftrightarrow \left(x_1 - \frac{19}{7}\right)^2 + \left(x_2 - \frac{22}{7}\right)^2 = \frac{26}{49}$$

and $\mathcal{L}_=(f_2(2, 3)) = \left\{x \in \mathbb{R}^2 : (x_1 - 19/7)^2 + (x_2 - 22/7)^2 = 26/49\right\}$, a circle around $(19/7, 22/7)$ with radius $\sqrt{26}/7$.

We have to check if $\mathcal{L}_=(f_1(2, 3)) \cap \mathcal{L}_=(f_2(2, 3))$ is the same as $\mathcal{L}_\leq(f_1(2, 3)) \cap \mathcal{L}_\leq(f_2(2, 3))$. But for $x = (2, 3)$ $\mathcal{L}_=(f_1(2, 3)) = \{(2, 3)\}$, i.e. the level set consists of only one point, which is on the boundary of $\mathcal{L}_\leq(f_2(2, 3))$. Thus $(2, 3)$ is efficient. In fact, it is even strictly efficient.



$\mathcal{L}_=(f_1(2, 2))$ is the circle around $(2, 3)$ with radius 1.

$\mathcal{L}_=(f_1(2, 3))$ is $\{(2, 3)\}$.

$\mathcal{L}_=(f_2(2, 2))$ is the circle around $(19/7, 22/7)$ with radius $\sqrt{89}/7$.

$\mathcal{L}_=(f_2(2, 3))$ is the circle around $(19/7, 22/7)$ with radius $\sqrt{26}/7$.

Fig. 2.13. Location problem of Example 2.31.

□

Theorem 2.30 shows that sometimes not all the criteria are needed to see if a feasible solution $\hat{x}$ is weakly or strictly efficient: Once the intersection of some level sets contains only $\hat{x}$, or the intersection of some strict level sets is empty, it will remain so when intersected with more (strict) level sets. This observation leads us to investigating the question of how many objectives are actually needed to determine if a feasible solution $\hat{x}$ is (strictly, weakly) efficient or not.

Let $\mathcal{P} \subset \{1, \ldots, p\}$ and denote by $f^{\mathcal{P}} := (f_j : j \in \mathcal{P})$ the objective function vector that only contains $f_j, j \in \mathcal{P}$.

Corollary 2.32. *Let $\mathcal{P} \subset \{1, \ldots, p\}$ be nonempty and let $\hat{x} \in \mathcal{X}$. Then the following statements hold.*

1. *If $\hat{x}$ is a weakly efficient solution of $(\mathcal{X}, f^{\mathcal{P}}, \mathbb{R}^{|\mathcal{P}|})/\mathrm{id}/(\mathbb{R}^{|\mathcal{P}|}, <)$ it is also a weakly efficient solution of $(\mathcal{X}, f, \mathbb{R}^p)/\mathrm{id}/(\mathbb{R}^p, <)$.*
2. *If $\hat{x}$ is a strictly efficient solution of $(\mathcal{X}, f^{\mathcal{P}}, \mathbb{R}^{|\mathcal{P}|})/\mathrm{id}/(\mathbb{R}^{|\mathcal{P}|}, \leqq)$ it is also a strictly efficient solution of $(\mathcal{X}, f, \mathbb{R}^p)/\mathrm{id}/(\mathbb{R}^p, \leqq)$.*

Corollary 2.32 says that weak or strict efficiency of some solution $\hat{x}$ for a problem with a subset of the p objectives implies weak (strict) efficiency for the problem with all objectives. Let us now investigate whether it is possible to find *all* weakly (strictly) efficient solutions by solving only problems with less than p objectives. For weakly efficient solutions this is possible for convex functions..

For the rest of this section we suppose that $\mathcal{X} \subset \mathbb{R}^n$ is a convex set and that $f_k : \mathbb{R}^n \to \mathbb{R}$ are convex functions. This implies that all level sets are convex. Theorem 2.30 is then about intersections of convex sets. A fundamental theorem on such intersections is known in convex analysis: Helly's Theorem.

Theorem 2.33 (Helly (1923)). *Let $p > n$ and let $C_1, \ldots, C_p \subset \mathbb{R}^n$ be convex sets. Then*

$$\bigcap_{i=1}^{p} C_i \neq \emptyset$$

if and only if for all collections of $n+1$ sets $C_{i_1}, \ldots, C_{i_{n+1}}$

$$\bigcap_{j=1}^{n+1} C_{i_j} \neq \emptyset.$$

Equivalently stated, we can say that

$$\bigcap_{i=1}^{p} C_i = \emptyset$$

if and only if there is a subset of $n+1$ sets C_i, $\{C_{i_1}, \ldots, C_{i_{n+1}}\}$ such that

$$\bigcap_{j=1}^{n+1} C_{i_j} = \emptyset.$$

In the multicriteria optimization context we will choose strict level sets as C_i. Combining Theorem 2.30, Corollary 2.32 and Helly's Theorem we immediately obtain the following "reduction result" for weakly efficient solutions of convex multicriteria optimization problems.

Proposition 2.34. *Consider the multicriteria optimization problem* $(\mathcal{X}, f, \mathbb{R}^p)/\mathrm{id}/(\mathbb{R}^p, <)$, *where* $\mathcal{X} \subset \mathbb{R}^n$ *is convex,* $f_k : \mathbb{R}^n \to \mathbb{R}, k = 1 \ldots, p$ *are convex and* $p > n$. *Then* $\hat{x} \in \mathcal{X}$ *is weakly efficient if and only if there is a subset* $\mathcal{P} \subset \{1, \ldots, p\},\ 0 < |\mathcal{P}| \leq n + 1$ *such that* $\hat{x}$ *is a weakly efficient solution of* $(\mathcal{X}, f^{\mathcal{P}}, \mathbb{R}^{|\mathcal{P}|})/\mathrm{id}/(\mathbb{R}^{|\mathcal{P}|}, <)$.

We shall adopt the notation $\mathcal{X}_{wE}(f)$, $\mathcal{X}_{wE}(f^{\mathcal{P}})$, and $\mathcal{X}_E(f)$, $\mathcal{X}_E(f^{\mathcal{P}})$ here to refer to the (weakly) efficient sets of the problems with f and $f^{\mathcal{P}}$, to avoid confusion. Proposition 2.34 is called a "reduction result", because it shows that the p-criteria problem $(\mathcal{X}, f, \mathbb{R}^p)/\mathrm{id}/(\mathbb{R}^p, <)$ can be solved by solving problems with at most $n + 1$ criteria $(\mathcal{X}, f^{\mathcal{P}}, \mathbb{R}^{|\mathcal{P}|})/\mathrm{id}/(\mathbb{R}^{|\mathcal{P}|}, <)$ at a time. Indeed, we observe that the structure of $\mathcal{X}_{wE}(f)$ is described by

$$\mathcal{X}_{wE}(f) = \bigcup_{\substack{\mathcal{P} \subset \{1,\ldots,p\} \\ |\mathcal{P}| \leq n+1}} \mathcal{X}_{wE}\left(f^{\mathcal{P}}\right). \tag{2.32}$$

Investing some more effort, it is even possible to describe $\mathcal{X}_{wE}(f)$ in terms of efficient solutions of subproblems with at most $n+1$ objectives. The following results show that on the right hand side of (2.32), $\mathcal{X}_{wE}$ can be replaced by $\mathcal{X}_E$.

Proposition 2.35 (Malivert and Boissard (1994)). *When the objective functions* f_k *are convex functions and the feasible set* $\mathcal{X}$ *is convex we have*

$$\mathcal{X}_{wE}(f) = \bigcup_{\substack{\mathcal{P} \subset \{1,\ldots,p\} \\ \mathcal{P} \neq \emptyset}} \mathcal{X}_E(f^{\mathcal{P}}). \tag{2.33}$$

Proof. We prove both set inclusions by showing the contrapositive.

"$\supset$" Choose $x \in \mathcal{X}$ with $x \notin \mathcal{X}_{wE}(f)$. Consequently, there is some $x' \in \mathcal{X}$ with $f_k(x') < f_k(x)$ for all $k = 1, \ldots, p$, which implies that x cannot be in $\mathcal{X}_E(f^{\mathcal{P}})$ for any choice of $\mathcal{P} \subset \{1, \ldots, p\}$.

"$\subset$" We prove, by induction, that for each $l = 1, \ldots, p$ there is a subset $\mathcal{P}_l$ of $\{1, \ldots, p\}$ of cardinality $p - l$ and a feasible solution x^l such that

$f_i(x^l) \leq f_i(x)$ whenever $i \in \mathcal{P}_l$ and $f_i(x^l) < f_i(x)$ otherwise. For $p = l$ this implies that x is not weakly efficient.

Choose $x \in \mathcal{X}$ with $x \notin \cup_{\mathcal{P} \subset \{1,\ldots,p\}} \mathcal{X}_E(f^{\mathcal{P}})$. In particular, $x \notin \mathcal{X}_E(f)$. Thus letting $\mathcal{P} = \{1, \ldots, p\}$, there is some $i_1 \in \mathcal{P}$ and some $x^1 \in \mathcal{X}$ such that $f_{i_1}(x^1) < f_{i_1}(x)$ and $f_i(x^1) \leq f_i(x)$, $i \neq i_1$. We now define $\mathcal{P}_1 := \mathcal{P} \setminus \{i_1\}$.

Now for $l \geq 1$ suppose we found $\mathcal{P}_l = \{1, \ldots, p\} \setminus \{i_1, \ldots, i_l\}$ and $x^l \in \mathcal{X}$ such that $f_i(x^l) < f_i(x)$ for all $i \in \{i_1, \ldots, i_l\}$ and $f_i(x^l) \leq f_i(x)$ for all $i \in \mathcal{P}_l$. Since $x \notin \mathcal{X}_E(f^{\mathcal{P}_l})$ by assumption, there is some $i_{l+1} \in \mathcal{P}_l$ and $\tilde{x}^{l+1} \in \mathcal{X}$ such that $f_{i_{l+1}}(\tilde{x}^{l+1}) < f_{i_{l+1}}(x)$ and $f_i(\tilde{x}^{l+1}) \leq f_i(x)$ for all $i \in P_l$. However, $\tilde{x}^{l+1}$ itself does not suffice to prove the condition for objectives f_i, $i \in \{i_1, \ldots, i_l\}$. We exploit convexity here. Let $x^{l+1} = \alpha x^l + (1-\alpha)\tilde{x}^{l+1}$, where $\lambda \in (0, 1)$. Then

$$f_i(x^{l+1}) < f_i(x) \text{ for all } i \in \{i_1, \ldots, i_l\}, \tag{2.34}$$

whenever $(1 - \alpha)$ is sufficiently small, due to the continuity of f_i and applying $f_i(x^l) < f_i(x)$ from the induction hypothesis. Furthermore,

$$\begin{aligned} f_{i_{l+1}}(x^{l+1}) &\leq \alpha f_{i_{l+1}}(x^l) + (1-\alpha) f_{i_{l+1}}(\tilde{x}^{l+1}) \\ &< \alpha f_{i_{l+1}}(x) + (1-\alpha) f_{i_{l+1}}(x) \\ &= f_{i_{l+1}}(x) \end{aligned} \tag{2.35}$$

by applying convexity for the first inequality and the induction hypothesis as well as the choice of $\tilde{x}^{l+1}$ for the second. Finally,

$$f_i(x^{l+1}) \leq f_i(x) \text{ for all } i \in P_{l+1} = \{1, \ldots, p\} \setminus \{i_1, \ldots, i_{l+1}\} \tag{2.36}$$

follows from convexity and the choice of $\tilde{x}^{l+1}$.

After p applications of this construction we have found x^p such that $f_i(x^p) < f_i(x)$ for $i = 1, \ldots, p$, i.e. $x \notin \mathcal{X}_{wE}(f)$. □

The preliminary result of Proposition 2.35 can now be combined with Helly's Theorem to obtain the structure result for weakly efficient solutions of convex multicriteria problems.

Theorem 2.36 (Malivert and Boissard (1994)). *Assume that $\mathcal{X}$ is a nonempty convex set and that the objective functions $f_k, k = 1, \ldots, p$ are convex functions. Then*

$$\mathcal{X}_{wE}(f) = \bigcup_{\substack{\mathcal{P} \subset \{1,\ldots,p\} \\ 1 \leq |\mathcal{P}| \leq n+1}} \mathcal{X}_E(f^{\mathcal{P}}). \tag{2.37}$$

Proof. Of course we need only consider the case $p > n+1$ and we only have to prove "$\subset$", because the other inclusion is an immediate consequence of Proposition 2.35 and the fact that $\mathcal{X}_E(f^{\mathcal{P}}) \subset \mathcal{X}_{wE}(f^{\mathcal{P}})$.

So, again, choose $x \in \mathcal{X}$, where $x \notin \cup_{1 \leq |\mathcal{P}| \leq n+1} \mathcal{X}_E(f^{\mathcal{P}})$ and let $\mathcal{J} \subset \{1, \ldots, p\}$, $\mathcal{J} \neq \emptyset$, $|\mathcal{J}| \leq n+1$ be any nonempty subset of at most $n+1$ indices. By the assumption on x we know that $x \notin \cup_{\mathcal{I} \subset \mathcal{J}} \mathcal{X}_E(f^{\mathcal{I}})$. Then by Proposition 2.35 $x \notin \mathcal{X}_{wE}(f^{\mathcal{J}})$ and there is some $x^{\mathcal{J}} \in \mathcal{X}$ such that

$$f_j(x^{\mathcal{J}}) < f_j(x) \text{ for all } j \in \mathcal{J}. \tag{2.38}$$

For all indices $i \in \{1, \ldots, p\}$ we define

$$C_i = \operatorname{conv}\left\{x^{\mathcal{J}} : \ \mathcal{J} \subset \{1, \ldots, p\},\ \mathcal{J} \neq \emptyset,\ |\mathcal{J}| \leq n+1,\ i \in \mathcal{J}\right\}. \tag{2.39}$$

By (2.38) it follows that $f_i(x^{\mathcal{J}}) < f_i(x)$ for each $\mathcal{J} \subset \{1, \ldots, p\}$, $1 \leq |\mathcal{J}| \leq n+1$ and each $i \in \mathcal{J}$. Furthermore by convexity

$$f_i(x') < f_i(x) \text{ for all } x' \in C_i. \tag{2.40}$$

When we look at some $\mathcal{J}$, fixed for the moment, we know that $\cap_{i \in \mathcal{J}} C_i \supset \{x^{\mathcal{J}}\}$, i.e. $\cap_{i \in \mathcal{J}} C_i \neq \emptyset$. Therefore we can apply Helly's Theorem to conclude that there is at least one $\hat{x} \in \cap_{i=1}^{p} C_i$ and (2.40) tells us $f_i(\hat{x}) < f_i(x)$, thus $x \notin \mathcal{X}_{wE}(f)$. $\square$

With a reduction result like (2.32) and a structure result like Theorem 2.36 for weakly efficient solutions, we may ask if similar results are possible for (strictly) efficient solutions. We give a counterexample to see that

$$\mathcal{X}_{sE}(f) = \bigcup_{\substack{\mathcal{P} \subset \{1, \ldots, p\} \\ |\mathcal{P}| \leq n+1}} \mathcal{X}_{sE}\left(f^{\mathcal{P}}\right)$$

does not hold for strictly efficient solutions.

Example 2.37 (Ehrgott and Nickel (2002)). Consider the MOP

$$\begin{aligned} \min \quad & (x_1, \ldots, x_n, -x_1, \ldots - x_n) \\ \text{subject to } & x \in [-1, 1]^n. \end{aligned}$$

$\hat{x} = 0$ is a strictly efficient solution. Consider subsets $\mathcal{P} \subset \{1, \ldots, p\}$ with $|\mathcal{P}| < p = 2n$. If $\mathcal{P}$ is such that $|\mathcal{P}| = 2k$ and $i \in \mathcal{P} \Leftrightarrow 2i \in \mathcal{P}$ then $\hat{x} \in \mathcal{X}_E(f^{\mathcal{P}})$. However, for such subsets $\hat{x}$ is not strictly efficient, because all vectors $e^j, j \notin \mathcal{P}$ have the same objective function values for objectives in $\mathcal{P}$ (e^j denotes the i-th unit vector in $\mathbb{R}^n$).

In any other case there is some $i \leq n$ such that either $i \in \mathcal{P}, 2i \notin \mathcal{P}$ or $2i \in \mathcal{P}, i \notin \mathcal{P}$. Thus either $-e^i$ or e^i dominate $\hat{x}$. $\square$

This shows that strict efficiency of $\hat{x}$ can only be confirmed using all p objectives. Thus for a problem with n variables, a lower bound on the maximal number of criteria needed to decide strict efficiency is $2n$.

2.4 Proper Efficiency and Proper Nondominance

According to Definition 2.1, an efficient solution does not allow improvement of one objective function while retaining the same values on the others. Improvement of some criterion can only be obtained at the expense of the deterioration of at least one other criterion. These trade-offs among criteria can be measured by computing the increase in objective f_i, say, per unit decrease in objective f_j. In some situations such trade-offs can be unbounded. We give an example below and introduce Geoffrion's definition of efficient solutions with bounded trade-offs, so called *properly efficient solutions*. Then some further definitions of proper efficiency by Borwein, Benson, and Kuhn and Tucker are presented. The results proved thereafter give an overview about the relationships between the various types of proper efficiency.

Example 2.38. Let the feasible set in decision and objective space be given by

$$\mathcal{X} = \{(x_1, x_2) \in \mathbb{R}^2 : \ (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1, \ 0 \leq x_1, x_2 \leq 1\},$$

and $\mathcal{Y} = \mathcal{X}$ as shown in Figure 2.14.

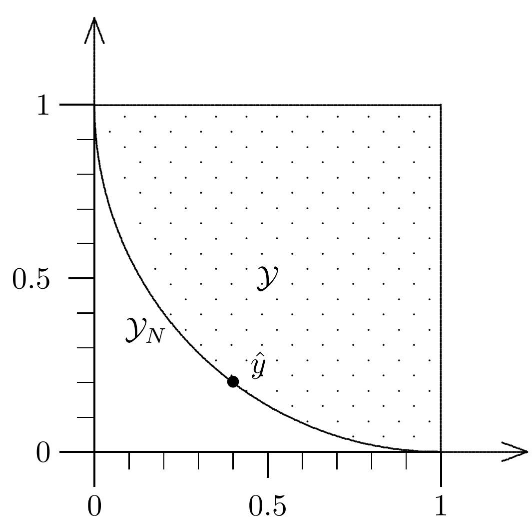


Fig. 2.14. Properly nondominated point $\hat{y}$.

Clearly, $\mathcal{Y}_N = \{(y_1, y_2) \in \mathcal{Y} : (y_1 - 1)^2 + (y_2 - 1)^2 = 1\}$. We observe that the closer $\hat{y}$ is moved towards $(1, 0)$, the larger an increase of y^1 is necessary to achieve a unit decrease in y_2. In the limit, an infinite increase of y_1 is needed to obtain a unit decrease in y_2. □

Definition 2.39 (Geoffrion (1968)). *A feasible solution $\hat{x} \in \mathcal{X}$ is called* properly efficient*, if it is efficient and if there is a real number $M > 0$ such that for all i and $x \in \mathcal{X}$ satisfying $f_i(x) < f_i(\hat{x})$ there exists an index j such that $f_j(\hat{x}) < f_j(x)$ such that*

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M. \tag{2.41}$$

The corresponding point $\hat{y} = f(\hat{x})$ is called properly nondominated.

According to Definition 2.39 properly efficient solutions therefore are those efficient solutions that have bounded trade-offs between the objectives.

Example 2.40. In Example 2.38 consider the solution $\hat{x} = (1, 0)$. We show that $\hat{x}$ is not properly efficient. To do so, we have to prove that for all $M > 0$ there is an index $i \in \{1, 2\}$ and some $x \in \mathcal{X}$ with $f_i(x) < f_i(\hat{x})$ such that

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} > M$$

for all $j \in \{1, 2\}$ with $f_j(x) > f_j(\hat{x})$.

Let $i = 1$ and choose x^ε with $x_1^\varepsilon = 1 - \varepsilon,\ 0 < \varepsilon < 1$ and $x_2^\varepsilon = 1 - \sqrt{1 - \varepsilon^2}$, i.e. x^ε is efficient because $(x_1^\varepsilon - 1)^2 + (x_2^\varepsilon - 1)^2 = 1$. Since $x^\varepsilon \in \mathcal{X}$, $x_1^\varepsilon < \hat{x}_1$ and $x_2^\varepsilon > \hat{x}_2$ we have $i = 1, j = 2$. Thus

$$\frac{f_i(\hat{x}) - f_i(x^\varepsilon)}{f_j(x^\varepsilon) - f_j(\hat{x})} = \frac{1 - (1 - \varepsilon)}{1 - \sqrt{1 - \varepsilon^2}} = \frac{\varepsilon}{1 - \sqrt{1 - \varepsilon^2}} \xrightarrow{\varepsilon \to 0} \infty. \tag{2.42}$$

□

The main results about properly efficient solutions show that they can be obtained by minimizing a weighted sum of the objective functions where all weights are positive. For convex problems optimality for the weighted sum scalarization is a necessary and sufficient condition for proper efficiency. We will prove these results in Section 3.2.

In the previous section we have given conditions for the existence of nondominated points/efficient solutions. These imply, of course, existence of weakly nondominated points/weakly efficient solutions. They do not guarantee existence of properly nondominated points. This can be seen from the following example.

Example 2.41. Let $\mathcal{Y} = \{y \in \mathbb{R}^2 : y_1 < 0, y_2 = 1/y_1\}$. Then $\mathcal{Y}_N = \mathcal{Y}$, but $\mathcal{Y}_{pN} = \textit{empty}$. To see this, take any $\hat{y} \in \mathcal{Y}_N$ and a sequence y^k with $y_2^k > \hat{y}_2$ and $y_1^k \to -\infty$ or $y_1^k > \hat{y}_1$ and $y_2^k \to -\infty$. □

As mentioned in the introduction of this section, Geoffrion is not the only one to introduce properly efficient solutions. Before we can present the definitions of Borwein and Benson, we have to introduce two cones related to sets $\mathcal{Y} \subset \mathbb{R}^p$.

Definition 2.42. *Let $\mathcal{Y} \subset \mathbb{R}^p$ and $y \in \mathcal{Y}$.*

1. *The* tangent cone *of $\mathcal{Y}$ at $y \in \mathcal{Y}$ is*

$$T_{\mathcal{Y}}(y) := \left\{d \in \mathbb{R}^p : \ \exists \{t_k\} \subset \mathbb{R}, \{y^k\} \subset \mathcal{Y} \ s.t.\ y^k \to y, t_k(y^k - y) \to d\right\}. \tag{2.43}$$

2. *The* conical hull *of $\mathcal{Y}$ is*

$$\operatorname{cone}(\mathcal{Y}) = \{\alpha y : \ \alpha \geq 0, \ y \in \mathcal{Y}\} = \bigcup_{\alpha \geq 0} \alpha \mathcal{Y}. \tag{2.44}$$

Note that the conditions $y^k \to y$ and $t_k(y^k - y) \to d$ in the definition of the tangent cone imply that $t_k \to \infty$. One could equivalently require $y^k \to y$ and $(1/(t_k))(y^k - y) \to d$, whence $t_k \to 0$. Both definitions can be found in the literature. Examples of the conical hull of a set $\mathcal{Y}$ and the tangent cone of $\mathcal{Y}$ at a point y are shown in Figure 2.15. The tangent cone is translated from the origin to the point y to illustrate where its name comes from: It is the cone of all directions tangential to $\mathcal{Y}$ in y.

Proposition 2.43 on properties of tangent cones and conical hulls will be helpful later.

Proposition 2.43. *1. The tangent cone $T_{\mathcal{Y}}(y)$ is a closed cone.*
2. If $\mathcal{Y}$ is convex then $T_{\mathcal{Y}}(y) = \operatorname{cl}(\operatorname{cone}(\mathcal{Y} - y))$, which is a closed convex cone.

Proof. 1. Note first that $0 \in T_{\mathcal{Y}}(y)$ (take $y^k = y$ for all k) and $T_{\mathcal{Y}}(y)$ is indeed a cone: For $\alpha > 0$, $d \in T_{\mathcal{Y}}(y)$ we have $\alpha d \in T_{\mathcal{Y}}(y)$. To see this, just take αt_k instead of t_k when constructing the sequence t_k.
To see that it is closed take a sequence $\{d^l\} \subset T_{\mathcal{Y}}(y)$, $y \in Y$, with $d^l \to d$, for some $d \in \mathbb{R}^p$. Since $d_l \in T_{\mathcal{Y}}(y)$, for all l there are sequences $\{y^{l,k}\}$, $\{t_{l,k}\}$ as in the Definition 2.42. From the convergence we get that for fixed l there is some k_l s.t.

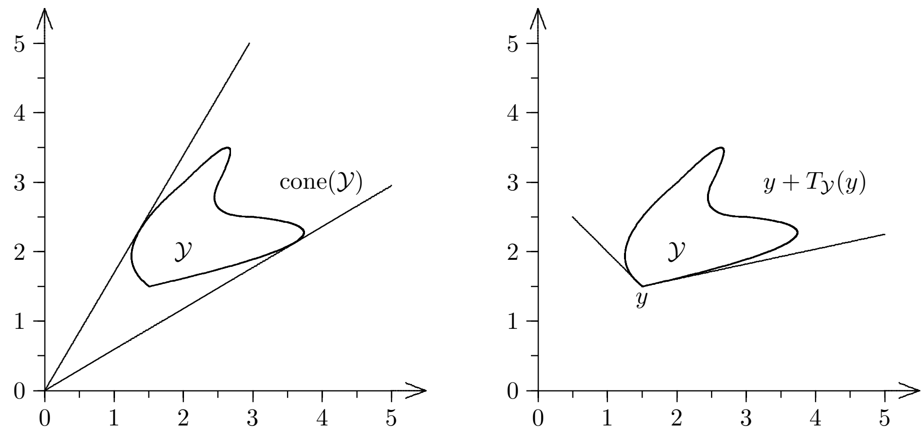


Fig. 2.15. Conical hull and tangent cone.

$$\left\|t_{l,k_l}(y^{l,k_l} - y) - d^l\right\| \le \frac{1}{l} \tag{2.45}$$

for all $k \ge k_l$. We fix the k_l and observe that because of (2.45) if $l \to \infty$ the sequence $t_{l,k_l}(y^{l,k_l} - y) \to d$, i.e. $d \in T_{\mathcal{Y}}(y)$.

2. Let $\mathcal{Y}$ be convex, $y \in \mathcal{Y}$. By definition of closure and conical hull, it is obvious that $\mathrm{cl}(\mathrm{cone}(\mathcal{Y} - y))$ is a closed convex cone.
To see that $T_{\mathcal{Y}}(y) \subset \mathrm{cl}(\mathrm{cone}(\mathcal{Y} - y))$ let $d \in T_{\mathcal{Y}}(y)$. Then there are sequences $\{t_k\}, \{y^k\}$ with $t_k(y^k - y) \to d$. Since $t_k(y^k - y) \in \alpha(\mathcal{Y} - y)$ for some $\alpha > 0$closedness implies $d \in \mathrm{cl}(\mathrm{cone}(\mathcal{Y} - y))$.
For $\mathrm{cl}(\mathrm{cone}(\mathcal{Y} - y)) \subset T_{\mathcal{Y}}(y)$ we know that $T_{\mathcal{Y}}(y)$ is closed and only show $\mathrm{cone}(\mathcal{Y} - y) \subset T_{\mathcal{Y}}(y)$. Let $d \in \mathrm{cone}(\mathcal{Y} - y)$, i.e. $d = \alpha(y' - y)$ with $\alpha \ge 0$, $y' \in \mathcal{Y}$. Now define $y^k := (1-1/k)y + (1/k)y' \in \mathcal{Y}$ and $t_k = \alpha k \ge 0$. Hence

$$t_k(y^k - y) = \alpha k \left(\left(\frac{k-1}{k} y + \frac{1}{k} y' \right) - y \right) = \alpha((k-1)y + y' - ky) = \alpha(y' - y).$$

So $y^k \to y$ and $t_k(y^k - y) \to d$ implying $d \in T_{\mathcal{Y}}(y)$. □

Definition 2.44. *1. (Borwein (1977)) A solution $\hat{x} \in \mathcal{X}$ is called* properly efficient *(in Borwein's sense) if*

$$T_{\mathcal{Y}+\mathbb{R}^p_{\geqq}}(f(\hat{x})) \cap \left(-\mathbb{R}^p_{\geqq}\right) = \{0\}. \tag{2.46}$$

2. (Benson (1979)) A solution $\hat{x} \in \mathcal{X}$ is called properly efficient *if*

$$\mathrm{cl}\left(\mathrm{cone}\left(\mathcal{Y} + \mathbb{R}^p_{\geqq} - f(\hat{x})\right)\right) \cap \left(-\mathbb{R}^p_{\geqq}\right) = \{0\}. \tag{2.47}$$

As we observed in Proposition 2.43 it is immediate from the definitions of conical hulls and tangent cones that

$$T_{\mathcal{Y}+\mathbb{R}^p_{\geqq}}(f(\hat{x})) \subset \mathrm{cl}\left(\mathrm{cone}\left(\mathcal{Y}+\mathbb{R}^p_{\geqq} - f(\hat{x})\right)\right) \tag{2.48}$$

so that Benson's definition is stronger than Borwein's.

Theorem 2.45. *1. If $\hat{x}$ is properly efficient in Benson's sense, it is also properly efficient in Borwein's sense.*
2. If $\mathcal{X}$ is convex and $f_k : \mathbb{R}^n \to \mathbb{R}$ are convex then both definitions coincide.

Example 2.46. Consider $\mathcal{X} = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ and, as usual, $f_1(x) = x_1,\ f_2(x) = x_2$. Then $(-1, 0)$ and $(0, -1)$ are efficient, but not properly efficient in the sense of Borwein (and thus not in the sense of Benson).

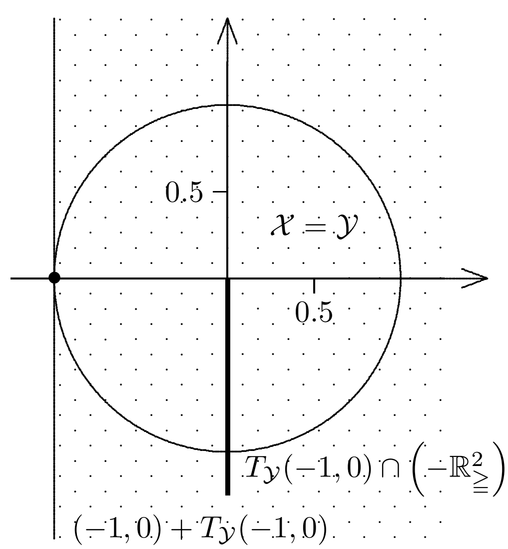


Fig. 2.16. Benson's proper efficiency.

The tangent cone translated to the point $y = (-1, 0)$ contains all directions in which $\mathcal{Y}$ extends from y, including the limits, i.e. the tangents. The tangent to the circle at $(-1, 0)$ is a vertical line, and therefore

$$T_{\mathcal{Y}}(-1, 0) = \{(y_1, y_2) \in \mathbb{R}^2 : \ y_1 \geq 0\}. \tag{2.49}$$

The intersection with the nonpositive orthant is therefore not $\{0\}$:

$$T_{\mathcal{Y}}(-1, 0) \cap (-\mathbb{R}^p_{\geqq}) = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 = 0,\ y_2 \leq 0\}, \tag{2.50}$$

indicated by the bold line in Figure 2.16. A similar interpretation applies to $(0, -1)$. □

That convexity is indeed needed for Borwein's definition to imply Benson's can be seen in Exercise 2.13. Definition 2.44 does not require $\hat{x}$ to be efficient, as does Definition 2.39. It is therefore legitimate to ask whether properly efficient solutions in Benson's or Borwein's sense are always efficient.

Proposition 2.47. *If $\hat{x}$ is properly efficient in Borwein's sense, then $\hat{x}$ is efficient.*

Proof. The proof is left to the reader as Exercise 2.12. □

Benson's and Borwein's definitions of proper efficiency are not restricted to the componentwise order. In fact, in these definitions $\mathbb{R}^p_{\geqq}$ can be replaced by an arbitrary closed convex cone $\mathcal{C}$. They are therefore applicable in the more general context of orders defined by cones. Geoffrion's definition on the other hand explicitly uses the componentwise order. Our next result shows that in the case of $\mathcal{C} = \mathbb{R}^p_{\geqq}$ the definitions of Geoffrion and Benson actually coincide, so that Benson's proper efficiency is a proper generalization of Geoffrion's.

Theorem 2.48 (Benson (1979)). *Feasible solution $\hat{x} \in \mathcal{X}$ is properly efficient in Geoffrion's sense (Definition 2.39) if and only if it is properly efficient in Benson's sense.*

Proof. "$\Longrightarrow$" Suppose $\hat{x}$ is efficient, but not properly efficient in Benson's sense. Then we know that a nonzero $d \in \mathrm{cl}(\mathrm{cone}(\mathcal{Y} + \mathbb{R}^p_{\geqq} - f(\hat{x}))) \cap (-\mathbb{R}^p_{\geqq})$ exists. Without loss of generality we may assume that $d_1 < -1$, $d_i \leq 0$, $i = 2, \ldots, p$ (otherwise we can reorder the components of f and rescale d). Consequently there are sequences $\{t_k\} \subset \mathbb{R}_>$, $\{x^k\} \subset \mathcal{X}$, $\{r^k\} \subset \mathbb{R}^p_{\geqq}$ such that $t_k(f(x^k) + r^k - f(\hat{x})) \to d$.

Choosing subsequences if necessary, we can assume that $\mathcal{Q} := \{i \in \{1, \ldots, p\} : f_i(x^k) > f_i(\hat{x})\}$ is the same for all k and nonempty (since $\hat{x}$ is efficient). Now let $M > 0$. From convergence we get existence of k_0 such that for all $k \geq k_0$

$$f_1(x^k) - f_1(\hat{x}) < -\frac{1}{2 \cdot t_k} \tag{2.51}$$

$$\text{and } f_i(x^k) - f_i(\hat{x}) \leq \frac{1}{2 \cdot M t_k} \quad i = 2, \ldots, p \tag{2.52}$$

because $t_k \to \infty$. In particular, for $i \in \mathcal{Q}$, we have

$$0 < f_i(x^k) - f_i(\hat{x}) \leq \frac{1}{2 \cdot M t_k} \quad \forall\ k \geq k_0 \tag{2.53}$$

and therefore, from (2.51) and (2.53)

$$\frac{f_1(\hat{x}) - f_1(x^k)}{f_i(x^k) - f_i(\hat{x})} > \frac{\frac{1}{2 \cdot t_k}}{\frac{1}{2 \cdot M t_k}} = M. \tag{2.54}$$

Because M was chosen arbitrarily chosen, $\hat{x}$ is not properly efficient in Geoffrion's sense.

"$\Longleftarrow$" Suppose $\hat{x}$ is efficient, but not properly efficient in Geoffrion's sense. Let $M_k > 0$ be an unbounded sequence of positive real numbers. Without loss of generality we assume that for all M_k there is an $x^k \in \mathcal{X}$ such that $f_1(x^k) < f_1(\hat{x})$ and

$$\frac{f_1(\hat{x}) - f_1(x^k)}{f_j(x^k) - f_j(\hat{x})} > M_k \quad \forall\, j \in \{2, \ldots, p\} \text{ with } f_j(x^k) > f_j(\hat{x}). \tag{2.55}$$

Again, choosing a subsequence if necessary, we can assume $\mathcal{Q} = \{i \in \{1, \ldots, p\} : \ f_i(x^k) > f_i(\hat{x})\}$ is constant for all k and nonempty. We construct appropriate sequences $\{t_k\}$, $\{r^k\}$ such that the limit of $t_k(f(x^k) + r^k - f(\hat{x}))$ converges to $d \in \mathrm{cl}(\mathrm{cone}(f(\mathcal{X}) + \mathbb{R}^p_{\geqq} - f(\hat{x}))) \cap (-\mathbb{R}^p_{\geqq})$.

Define $t_k := (f_1(\hat{x}) - f_1(x^k))^{-1}$ which means $t_k > 0$ for all k. Define $r^k \in \mathbb{R}^p_{\geqq}$ through

$$r_i^k := \begin{cases} 0 & i = 1,\ i \in \mathcal{Q} \\ f_i(\hat{x}) - f_i(x^k) & \text{otherwise.} \end{cases} \tag{2.56}$$

With these sequences we compute

$$t_k(f_i(x^k) + r_i^k - f_i(\hat{x})) \begin{cases} = -1 & i = 1 \\ = 0 & i \neq 1,\ i \notin \mathcal{Q} \\ \in (0, M_k^{-1}) & i \in \mathcal{Q}. \end{cases} \tag{2.57}$$

This sequence converges due to the choice of $M_k \to \infty$ to some $d \in \mathbb{R}^p$, where $d_i = \lim\limits_{k \to \infty} t_k(f_i(x^k) + r_i^k - f_i(\hat{x}))$ for $i = 1, \ldots, p$. Thus, from (2.57) $d_1 = -1$, $d_i = 0$, $i \neq 1$, $i \notin \mathcal{Q}$, $d_i = 0$, $i \in \mathcal{Q}$. Because $d = (-1, 0, \ldots, 0) \in \mathrm{cl}(\mathrm{cone}(f(\mathcal{X}) + \mathbb{R}^p_{\geqq} - f(\hat{x}))) \cap (-\mathbb{R}^p_{\geqq})$, $\hat{x}$ is not properly efficient in Benson's sense. □

In multicriteria optimization, especially in applications, we will often encounter problems, where $\mathcal{X}$ is given implicitly by constraints, i.e.

$$\mathcal{X} = \{x \in \mathbb{R}^n : \ (g_1(x), \ldots, g_m(x)) \leq 0\}. \tag{2.58}$$

For such constrained multicriteria optimization problems yet another definition of proper efficiency can be given. Let us assume that the objective functions $f_i,\ i = 1, \ldots, p$ as well as the constraint functions $g_j,\ j = 1, \ldots, m$ are continuously differentiable. We consider the multiobjective programme

$$\begin{aligned} \min f(x) \\ \text{subject to } g(x) \leqq 0, \end{aligned} \tag{2.59}$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^p$.

Definition 2.49 (Kuhn and Tucker (1951)). *A feasible solution $\hat{x} \in \mathcal{X}$ is called* properly efficient *(in Kuhn and Tucker's sense) if it is efficient and if there is no $d \in \mathbb{R}^n$ satisfying*

$$\langle \nabla f_k(\hat{x}), d \rangle \leq 0 \quad \forall\ k = 1, \ldots, p \tag{2.60}$$

$$\langle \nabla f_i(\hat{x}), d \rangle < 0 \quad \textit{for some } i \in \{1, \ldots, p\} \tag{2.61}$$

$$\langle \nabla g_j(\hat{x}), d \rangle \leq 0 \quad \forall\ j \in \mathcal{J}(\hat{x}) = \{j = 1, \ldots, m : \ g_j(\hat{x}) = 0\} \tag{2.62}$$

The set $\mathcal{J}(\hat{x})$ is called the set of active indices. As for Geoffrion's definition, efficiency according to the componentwise order is implicitly assumed here, and the definition is not applicable to orders derived from closed convex cones. Intuitively, the existence of a vector d satisfying (2.60) – (2.62) means that moving from $\hat{x}$ in direction d no objective function increases (2.60), one strictly decreases (2.61), and the feasible set is not left (2.62). Thus d is a feasible direction of descent. Note that a slight movement in *all* directions is always possible without violating any inactive constraint.

We prove equivalence of Kuhn and Tucker's and Geoffrion's definitions under some constraint qualification. This constraint qualification of Definition 2.50 below means that the feasible set $\mathcal{X}$ has a local description as a differentiable curve (at a feasible solution $\hat{x}$): Every feasible direction d can be written as the gradient of a feasible curve starting at $\hat{x}$.

Definition 2.50. *A differentiable MOP (2.59) satisfies the KT constraint qualification at $\hat{x} \in X$ if for any $d \in \mathbb{R}^n$ with $\langle \nabla g_j(\hat{x}), h \rangle \leq 0$ for all $j \in \mathcal{J}(\hat{x})$ there is a real number $\bar{t} > 0$, a function $\theta : [0, \bar{t}] \to \mathbb{R}^n$, and $\alpha > 0$ such that $\theta(0) = \hat{x}, g(\theta(t)) \leq 0$ for all $t \in [0, \bar{t}]$ and $\theta'(0) = \alpha d$.*

Theorem 2.51 (Geoffrion (1968)). *If a differentiable MOP satisfies the KT constraint qualification at $\hat{x}$ and $\hat{x}$ is properly efficient in Geoffrion's sense, then it is properly efficient in Kuhn and Tucker's sense.*

Proof. Suppose $\hat{x}$ is efficient, but not properly efficient according to Definition 2.49. Then there is some $d \in \mathbb{R}^n$ such that (without loss of generality, after renumbering the objectives)

$$\langle \nabla f_1(\hat{x}), d \rangle < 0 \tag{2.63}$$

$$\langle \nabla f_k(\hat{x}), d \rangle \leq 0 \quad \forall\ k = 2, \ldots, p \tag{2.64}$$

$$\langle \nabla g_j(\hat{x}), d \rangle \leq 0 \quad \forall\ j \in \mathcal{J}(\hat{x}). \tag{2.65}$$

Using the function θ from the constraint qualification we take a sequence $t_k \to 0$, and if necessary a subsequence such that

$$\mathcal{Q} = \{i : \ f_i(\theta(t_k)) > f_i(\hat{x})\} \tag{2.66}$$

is the same for all k. Since for $i \in \mathcal{Q}$ by the Taylor expansion of f_i at $\theta(t_k)$)

$$f_i(\theta(t_k)) - f_i(\hat{x}) = t_k \langle \nabla f_i(\hat{x}), \alpha d \rangle + o(t_k) > 0 \tag{2.67}$$

and $\langle \nabla f_i(\hat{x}), d \rangle \leq 0$ it must be that

$$\langle \nabla f_i(\hat{x}), \alpha d \rangle = 0 \quad \forall\ i \in \mathcal{Q}. \tag{2.68}$$

But since $\langle \nabla f_1(\hat{x}), d \rangle < 0$ the latter implies

$$\Longrightarrow \quad \frac{f_1(\hat{x}) - f_1(\theta(t_k))}{f_i(\theta(t_k)) - f_i(\hat{x})} = \frac{-\langle \nabla f_1(\hat{x}), \alpha d \rangle + \frac{o(t_k)}{t_k}}{\langle \nabla f_i(\hat{x}), \alpha d \rangle + \frac{o(t_k)}{t_k}} \to \infty \tag{2.69}$$

whenever $i \in \mathcal{Q}$. Hence $\hat{x}$ is not properly efficient according to Geoffrion's definition. □

The converse of Theorem 2.51 holds without the constraint qualification. It turns out that this result is an immediate consequence of Theorem 3.25 (necessary conditions for Kuhn and Tucker's proper efficiency) and Theorem 3.27 (sufficient conditions for Geoffrion's proper efficiency). These results are proved in Section 3.3. In Section 3.3 we shall also see that without constraint qualification, Geoffrion's proper efficiency does not necessarily imply proper efficiency in Kuhn and Tucker's sense.

Theorem 2.52. *Let $f_k, g_j : \mathbb{R}^n \to \mathbb{R}$ be convex, continuously differentiable functions and suppose $\hat{x}$ is properly efficient in Kuhn and Tucker's sense. Then $\hat{x}$ is properly efficient in Geoffrion's sense.*

Let us conclude the section by a summary of the definitions of proper efficiency and their relationships. Figure 2.17 illustrates these (see also Sawaragi *et al.* (1985)). The arrows indicate implications. Corresponding results and the conditions under which the implications hold are mentioned alongside the arrows. On the right of the picture, the orders and problem types for which the respective definition is applicable are given.

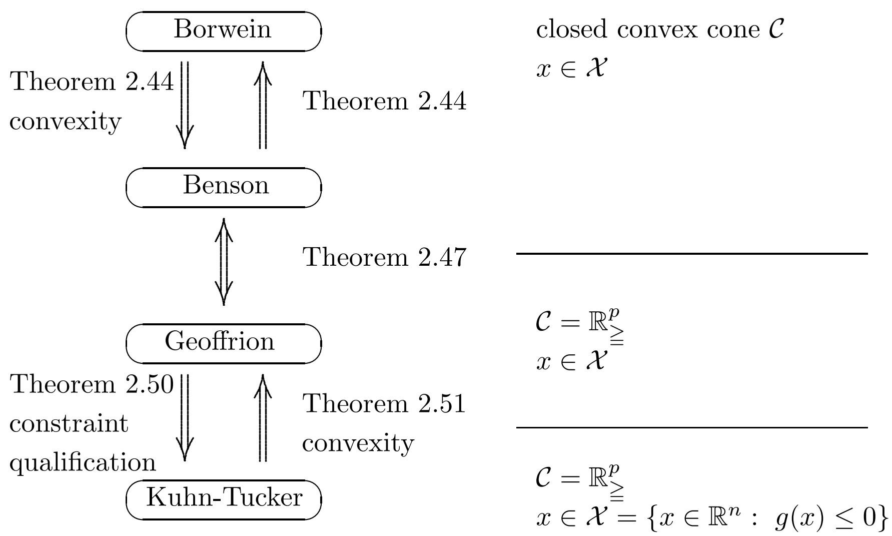


Fig. 2.17. Relationships among definitions of proper efficiency for the case $\mathcal{C} = \mathbb{R}^p_{\geqq}$.

In order to derive further results on proper efficiency and important properties of (weakly) efficient sets we have to investigate weighted sum scalarizations in greater detail, i.e. the relationships between those types of solutions and optimal solutions of single objective optimization problems

$$\min_{x \in \mathcal{X}} \sum_{k=1}^{p} \lambda_k f_k(x),$$

where $\lambda \in \mathbb{R}^p_{\geq}$ is a vector of nonnegative weights of the objective functions. This is the topic of Chapter 3.

2.5 Notes

As we have pointed out after the definition of efficient solutions and nondominated points (Definition 2.1) notation for efficient solutions and nondominated points is not unique in the literature. Table 2.4 below gives an overview of some of the notations used. Another term for efficient point is admissible point (Arrow *et al.*, 1953), but this is rarely used today. Although some authors distinguish between the case that the decision space is $\mathbb{R}^n$ or a more general vector space (Jahn) or the order is defined by $\mathbb{R}^p_{\geqq}$ or a more general cone (Miettinen), most of these definitions use the same terms in decision and

Table 2.4. Terminology for efficiency and nondominance.

Author	Decision space	Objective space
Sawaragi *et al.* (1985)	efficient	efficient
	solution	element
Chankong and Haimes (1983)	noninferior	noninferior
	solution	solution
Yu (1985)	Pareto optimal	Pareto optimal
	point	outcome
	N-point	N-point
Miettinen (1999)	Pareto optimal	Pareto optimal
	decision vector	criterion vector
	efficient	efficient
	decision vector	criterion vector
Deb (2001)	Pareto optimal	Pareto optimal
	solution	solution
Jahn (2004)	Edgeworth-Pareto	minimal
	optimal point	element
	minimal	minimal
	solution	element
Göpfert and Nehse (1990)	Pareto optimal	efficient
	solution	element
Steuer (1985)	efficient	nondominated
	point	criterion vector

criterion space, which might cause confusion and does not help distinguish two very different things.

The condition of $\mathbb{R}^p_{\geqq}$-compactness in Corollary 2.15 can be replaced by $\mathbb{R}^p_{\geqq}$-closedness and $\mathbb{R}^p_{\geqq}$-boundedness, which are generalizations of closedness and boundedness, see Exercises 2.4 and 2.5. For closed convex sets $\mathcal{Y}$ it can be shown that the conditions of Theorem 2.10, Corollary 2.15 and $\mathbb{R}^p_{\geqq}$-closedness and $\mathbb{R}^p_{\geqq}$-boundedness coincide, see for example Sawaragi *et al.* (1985, page 56). Other existence results are known, which often use a more general setting than we adopt in this text. We refer, e.g.xs to Göpfert and Nehse (1990); Sawaragi *et al.* (1985); Hazen and Morin (1983) or Henig (1986). A review of existence results for nondominated and efficient sets is provided by Sonntag and Zălinescu (2000).

We remark that all existence results presented in this Chapter are still valid, if $\mathbb{R}^p_{\geqq}$ is replaced by a convex, pointed, nontrivial, closed cone $\mathcal{C}$ with the proofs unchanged. Furthermore, the closedness assumption for $\mathcal{C}$ is not required if $(y - \mathrm{cl}\,\mathcal{C})$ is used instead of $(y - \mathcal{C})$ everywhere. In Exercises 2.2 and 2.7 nondominance with respect to a cone is formally defined, and the reader is asked to check some of the results about efficient sets in this more general context.

Similarly, Theorem 2.21 is valid for any nonempty, closed, convex cone $\mathcal{C}$. In fact, $\mathcal{C}$-compactness can be replaced by $\mathcal{C}$-closedness and $\mathcal{C}$-boundedness, see Sawaragi *et al.* (1985) for more details. External stability of $\mathcal{Y}_N$ has been shown for closed convex $\mathcal{Y}$ by Luc (1989). More results can be found in Hirschberger (2002). A counterpart to the external stability is internal stability of a set. A set $\mathcal{Y}$ is called internally stable with respect to $\mathcal{C}$, if $y - y' \notin \mathcal{C}$ for all $y, y' \in \mathcal{Y}$. Obviously, nondominated sets are always internally stable.

The computation of the nadir point is difficult, because it amounts to solving an optimization problem over the efficient set of a multicriteria optimization problem, see Yamamoto (2002) for a survey on that topic. Nevertheless, interactive methods often assume that the ideal and nadir point are known (see Miettinen (1999)[Part II, Chapter 5] for an overview on interactive methods). A discussion of heuristics and exact methods to compute nadir points and implications for interactive methods can be found in Ehrgott and Tenfelde-Podehl (2003).

For reduction results on the number of criteria to determine (strict, weak) efficiency of a feasible solution $\hat{x}$, we remark that the case of efficiency is much more difficult than either strict or weak efficiency. Ehrgott and Nickel (2002) show that a reduction result is true for strictly quasi-convex problems with $n = 2$ variables. For the general case of $n > 2$ neither a proof nor a counterexample is known.

In addition to the definitions of proper efficiency mentioned here, the following authors define properly efficient solutions Klinger (1967), Wierzbicki (1980) and Henig (1982). Borwein and Zhuang (1991, 1993) define super efficient solutions. Henig (1982) gives two definitions that generalize the definitions of Borwein and Benson (Definition 2.44) but that coincide with these in the case $\mathcal{C} = \mathbb{R}^p_{\geqq}$ discussed in this book.

Exercises

2.1. Give a counterexample to the converse inclusion in Proposition 2.6.

2.2. Given a cone $\mathcal{C} \subset \mathbb{R}^p$ and the induced order $\leq_{\mathcal{C}}$, $\hat{y} \in \mathcal{Y}$ is said to be $\mathcal{C}$-nondominated if there is no $y \in \mathcal{Y}, y \neq \hat{y}$ such that $y \in \hat{y} - \mathcal{C}$. The set of $\mathcal{C}$-nondominated points is denoted $\mathcal{Y}_{\mathcal{C}N}$. Let $\mathcal{C}_1, \mathcal{C}_2$ be two cones in $\mathbb{R}^p$ and assume $\mathcal{C}_1 \subset \mathcal{C}_2$. Prove that if $\hat{y}$ is $\mathcal{C}_2$-nondominated it is also $\mathcal{C}_1$-nondominated. Illustrate this "larger cone – fewer nondominated points" result graphically.

2.3. Prove that $(\alpha\mathcal{Y})_N = \alpha(\mathcal{Y}_E)$ where $\mathcal{Y} \subset \mathbb{R}^p$ is a nonempty set and α is a positive real number.

2.4. Let $\mathcal{Y} \subset \mathbb{R}^p$ be a convex set. The recession cone (or asymptotic cone) $\mathcal{Y}_\infty$ of $\mathcal{Y}$, is defined as

$$\mathcal{Y}_\infty := \{d \in \mathbb{R}^p : \exists\ y \text{ s.t. } y + \alpha d \in \mathcal{Y} \quad \forall\ \alpha > 0\},$$

i.e. the set of directions in which $\mathcal{Y}$ extends infinitely.

1. Show that $\mathcal{Y}$ is bounded if and only if $\mathcal{Y}_\infty = \{0\}$.
2. Let $\mathcal{Y} = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \geq y_1^2\}$. Determine $\mathcal{Y}_\infty$.

2.5. A set $\mathcal{Y} \subset \mathbb{R}^p$ is called $\mathbb{R}^p_{\geqq}$-closed, if $\mathcal{Y} + \mathbb{R}^p_+$ is closed and $\mathbb{R}^p_{\geqq}$-bounded, if $\mathcal{Y}_\infty \cap (-\mathbb{R}^p_{\geqq}) = \{0\}$. Give examples of sets $\mathcal{Y} \subset \mathbb{R}^2$ that are

1. $\mathbb{R}^2_{\geqq}$-compact, $\mathbb{R}^2_{\geqq}$-bounded, but not $\mathbb{R}^2_{\geqq}$-closed,
2. $\mathbb{R}^2_{\geqq}$-bounded, $\mathbb{R}^2_{\geqq}$-closed, but not $\mathbb{R}^2_{\geqq}$-compact.

2.6. Prove the following existence result for weakly nondominated points. Let $\emptyset \neq \mathcal{Y} \subset \mathbb{R}^p$ be $\mathbb{R}^p_{\geqq}$-compact. Show that $\mathcal{Y}_{wN} \neq \emptyset$. Do not use Corollary 2.15 nor the fact that $\mathcal{Y}_N \subset \mathcal{Y}_{wN}$.

2.7. Recall the definition of $\mathcal{C}$-nondominance from Exercise 2.2: $\hat{y} \in \mathcal{Y}$ is $\mathcal{C}$-nondominated if there is no $y \in \mathcal{Y}$ such that $\hat{y} \in y + \mathcal{C}$. Verify that Proposition 2.3 is still true if $\mathcal{C}$ is a pointed, convex cone. Give examples that the inclusion $\mathcal{Y}_{\mathcal{C}N} \subset (\mathcal{Y} + \mathcal{C})_{\mathcal{C}N}$ is not true when $\mathcal{C}$ is not pointed and when $\mathcal{C}$ is not convex.

2.8. Let $[a, b] \subset \mathbb{R}$ be a compact interval. Suppose that all $f_k : \mathbb{R} \to \mathbb{R}$ are convex, $k = 1, \ldots, p$. Let

$$x_k^m = \min \left\{ x \in [a, b] : f_k(x) = \min_{x \in [a,b]} f_k(x) \right\}$$

and

$$x_k^M = \max\left\{x \in [a,b] : f_k(x) = \min_{x\in[a,b]} f_k(x)\right\}.$$

Using Theorem 2.30 show that

$$\mathcal{X}_E = \left[\min_{k=1,\dots,p} x_k^M, \quad \max_{k=1,\dots,p} x_k^m\right] \cup \left[\max_{k=1,\dots,p} x_k^m, \quad \min_{k=1,\dots,p} x_k^M\right]$$
$$\mathcal{X}_{wE} = \left[\min_{k=1,\dots,p} x_k^m, \quad \max_{k=1,\dots,p} x_k^M\right].$$

2.9. Use the result of Exercise 2.8 to give an example of a multicriteria optimization problem with $\mathcal{X} \subset \mathbb{R}$ where $\mathcal{X}_{sE} \subset \mathcal{X}_E \subset \mathcal{X}_{wE}$, with strict inclusions. Use two or three objective functions.

2.10 (Hirschberger (2002)). Let $\mathcal{Y} = \{y \in \mathbb{R}^2 : y_1 < 0, y_2 = 1/y_1\}$. Show that $\mathcal{Y}_N = \mathcal{Y}$ but $\mathcal{Y}_{pN} = \emptyset$.

2.11. Let $\mathcal{X} = \{x \in \mathbb{R} : x \geq 0\}$ and $f_1(x) = e^x$,

$$f_2(x) = \begin{cases} \frac{1}{x+1} & 0 \leq x \leq 5 \\ (x-5)^2 + \frac{1}{6} & x \geq 5. \end{cases}$$

Using the result of Exercise 2.8, determine $\mathcal{X}_E$. Which of these solutions are strictly efficient? Can you prove a sufficient condition on f for $x \in \mathbb{R}$ to be a strictly efficient solution of $\min_{x\in\mathcal{X}\subset\mathbb{R}} f(x)$? Derive a conjecture from the example and try to prove it.

2.12. Show that if $\hat{x}$ is properly efficient in the sense of Borwein, then $\hat{x}$ is efficient.

2.13 (Benson (1979)). Consider the following example:

$$\mathcal{X} = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 + x_2 \geq 0\right\} \cup \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 1\right\}$$
$$\cup \left\{(x_1,x_2) \in \mathbb{R}^2 : x_2 \geq 1\right\}$$

with $f_1(x) = x_1, f_2(x) = x_2$. Show that $x = 0$ is properly efficient in the sense of Borwein, but not in the sense of Benson.

2.14. Consider an MOP $\min_{x\in\mathcal{X}} f(x)$ with p objectives. Add a new objective f_{p+1}. Is the efficient set of the new problem bigger or smaller than that of the original problem or does it remain unchanged?

2.15. The following definition of an ideal point was given by Balbás *et al.* (1998). Let $\min_{x\in\mathcal{X}}(f_1(x),\dots,f_p(x))$ be a multicriteria optimization problem. A point $y \in \mathbb{R}^p$ is called an ideal point if there exists a closed, convex, pointed cone $\mathcal{C} \subseteq \mathbb{R}^p$ such that $\mathcal{Y} \subset y + \mathcal{C}$. If in addition $y \in \mathcal{Y}$, y is a proper ideal point.