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Effective implementation of the ε-constraint method for the generation of efficient solutions in Multi-objective Mathematical Programming problems

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Multi-objective programming

Mathematical Programming (single objective)	***Multi-objective Mathematical Programming***
One objective function	Several objective functions
Scalar optimization	Vector optimization
One optimal solution	Several Pareto optimal solutions (efficient, non-dominated) ✓ Equally good solutions ✓ Need more information to find the most preferred

Multi-criteria optimization has to combine two aspects:

Optimization and decision support

Classification of methods

- Hwang and Masud (1979)
- According to the stage when the DM expresses his/her preferences

- A priori methods
  - Goal programming, weighted sum e.t.c.
- Interactive (or progressive) methods
  - STEM, GDF, Z-W, NIMBUS, GUESS, LBS e.t.c.
- A posteriori (or generation methods)
  - ε-constraint, weighting method etc

Pros and Cons

	A-priori	Interactive	A-posteriori
Advantages	Easy, available software, low computational burden	The DM guides the search, iterative, the DM "learns" about the problem	The expression of preference follows the optimization phase (all available information is at hand), can produce subsets of efficient solutions
Shortcomings	Need unrealistically precise information, need extensive sensitivity analysis,	Need extensive interaction with the DM, the DM decides based on samples	Computational burden, not widely available software, need a second phase for selecting the most preferred P.O.S.

Generation methods

- Two phases
  - Generation of Pareto optimal solutions
  - Selection among them, the most preferred
- Usually the Pareto set is expressed by an adequate representation of efficient solutions
- Basic advantage:
  - The DM examines all available options before the final decision, thus reinforcing his/her confidence to the final decision

Generation methods in Mathematical Programming

- **Exact methods**
  - **Produce the entire Pareto set**
  - **Merely for special kind of problems of small size**
- **Approximate methods**
  - **Weighted sum**
    - varying the weights of objective functions in a weighted sum
  - **ε-constraint**
    - p-1 objective function as constraints
    - varying the RHS of these constraints
  - **Distance metrics**
    - Tchebycheff

ε-constraint method

MOMP

$$\max f_1(x)$$
$$\max f_2(x)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\max f_p(x)$$
$$st$$
$$x \in S$$

MP(e_2, ...e_p)

$$\max f_1(x)$$
$$st$$
$$f_2(x) \geq e_2$$
$$f_3(x) \geq e_3$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$f_p(x) \geq e_p$$
$$x \in S$$

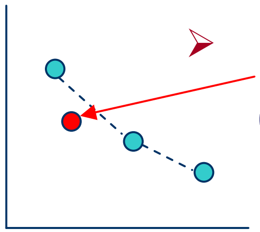
Optimal solutions of MP(e_i) are Pareto optimal for MOMP

* Unless there are alternative optima and the constraints are not binding

Weakly efficient solutions

Advantages of the ε-constraint method over the weighting method

weighting method	***ε-constraint method***
➢ produces only efficient extreme solutions	➢ produce non-extreme efficient solutions
➢ In Integer MOP problems produces only supported efficient solutions	➢ Produces also unsupported efficient solutions
➢ Proper scaling of the obj. functions is mandatory	➢ Need no scaling of the obj. functions
➢ Several combinations of weights may lead to the same efficient solution (redundancy)	➢ Each run is usually a new efficient solution → better control in the Pareto set representation



Shortcomings of the ε-constraint method

1. In the ε-constraint method we need the ranges of the objective functions in order to generate the grid points that will act as RHS (e_i, $i=2\ldots p$).
   - These ranges are difficult to obtain
   - Usually from the payoff table, or the lower bound (reservation value) is defined by the DM
2. The efficiency of the obtained optimal solution is not guaranteed in the conventional ε-constraint model

The augmented ε-constraint method

- **Three points of intervention:**
  - **The model**
    - Guarantee the generation of only Pareto optimal solutions
  - **The representation through the grid points**
    - Proper calculation of the objective function ranges
  - **The calculation procedure**
    - Early exit from the loops when infeasibilities occur

The model

$$\max\ (f_1(\mathbf{x}) + eps \times (s_2 + s_3 + \ldots + s_p))$$

st

$$f_2(\mathbf{x}) - s_2 = e_2$$

$$f_3(\mathbf{x}) - s_3 = e_3$$

$$\ldots$$

$$f_p(\mathbf{x}) - s_p = e_p$$

$$\mathbf{x} \in S \quad \text{and} \quad s_i \geq 0$$

Explicit incorporation of the surplus variables which also participate in the obj. function as secondary priority

where *eps* is a small number (usually between 10^{-3} and 10^{-6})

Generation of only efficient solutions

Proposition: **The above formulation (P) of the ε-constraint method produces only efficient solutions (it avoids the generation of weakly efficient solutions).**

Proof: **Assume that the problem (P) has alternative optima and one of them (depicted as x') dominates the optimal solution (depicted as x) obtained from (P).**
=> vector (z_1, z_2, …, z_p) is dominated by vector (z_1, z_2', …, z_p')
=> vector (z_1, e_2+s_2, …, e_p+s_p) is dominated by vector (z_1, e_2+s_2', …, e_p+s_p') =>

$$\left.\begin{array}{l} e_2 + s_2 \le e_2 + s_2' \\ e_3 + s_3 \le e_3 + s_3' \\ \ldots \\ e_p + s_p \le e_p + s_p' \end{array}\right\} \quad \text{with at least on strict inequality}$$

Taking the sum of these relations and based on the fact that there is at least one strict inequality we conclude that:

$$\boxed{\sum_{i=2}^{p} s_i < \sum_{i=2}^{p} s_i'}$$

But this contradicts the initial assumption that the optimal solution of (P) maximizes the sum of s_i. □

Ranges, nadir points

- ε-constraint needs reliable estimates for the range of the p-1 objective functions in order to properly adjust the grid points.
- The problem is the calculation of nadir point
- Two remedies:
  - DM assigns his/her “nadir point” (in the sense of a “reservation value”)
  - Nadir point estimation from the payoff table
    - Attention: the latter is valid only when we have Pareto optimal solutions in the payoff table
    - Lexicographic optimization

Lexicographic optimization

1. $\max f_1(x) = z_1 \mid x \in S \rightarrow z_1^*$
2. $\max f_2(x) = z_2 \mid x \in S$ and $f_1(x) = z_1^* \rightarrow z_2^{*1}$
3. $\max f_3(x) = z_3 \mid x \in S$ and $f_1(x) = z_1^*$ and $f_2(x) = z_2^{*1}$ $\rightarrow z_3^{*1,2}$

   …….

4. $\max f_p(x) = z_p \mid x \in S$ and $f_1(x) = z_1^*$and $f_2(x) = z_2^{*1}$ and… $\rightarrow z_p^{*1,2\ldots,p-1}$

Example 1

$\max f_1 = X_1$

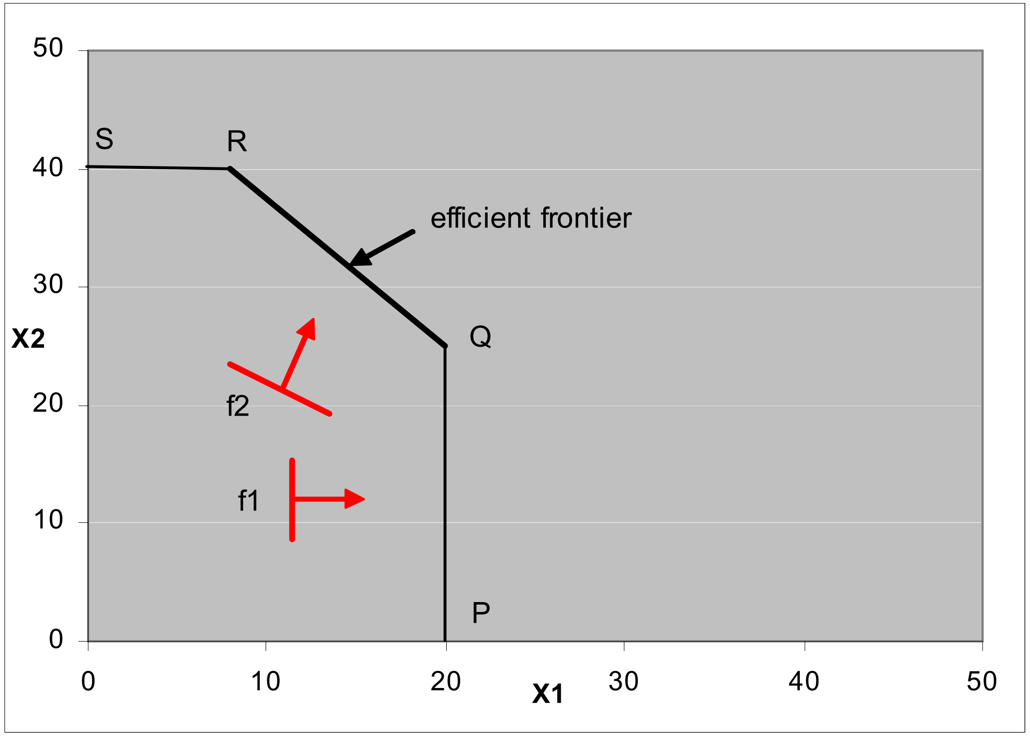
$\max f_2 = 3X_1 + 4X_2$

st

$X_1 <= 20$

$X_2 <= 40$

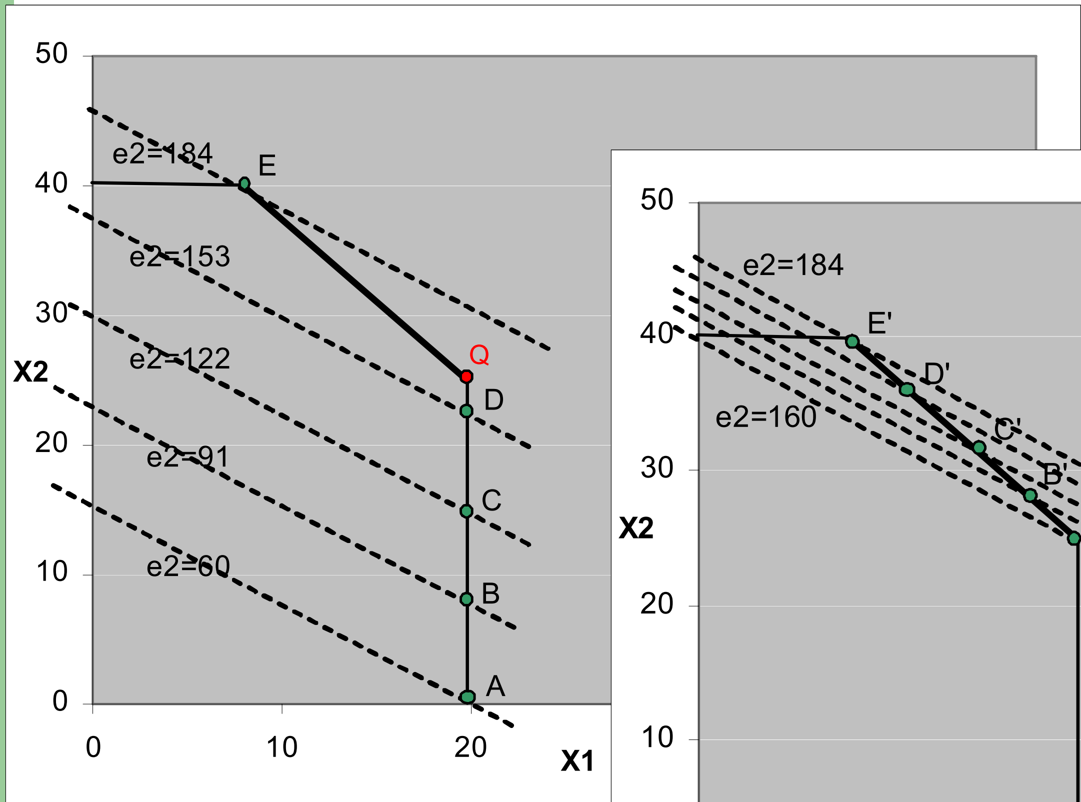
$5X_1 + 4X_2 <= 200$

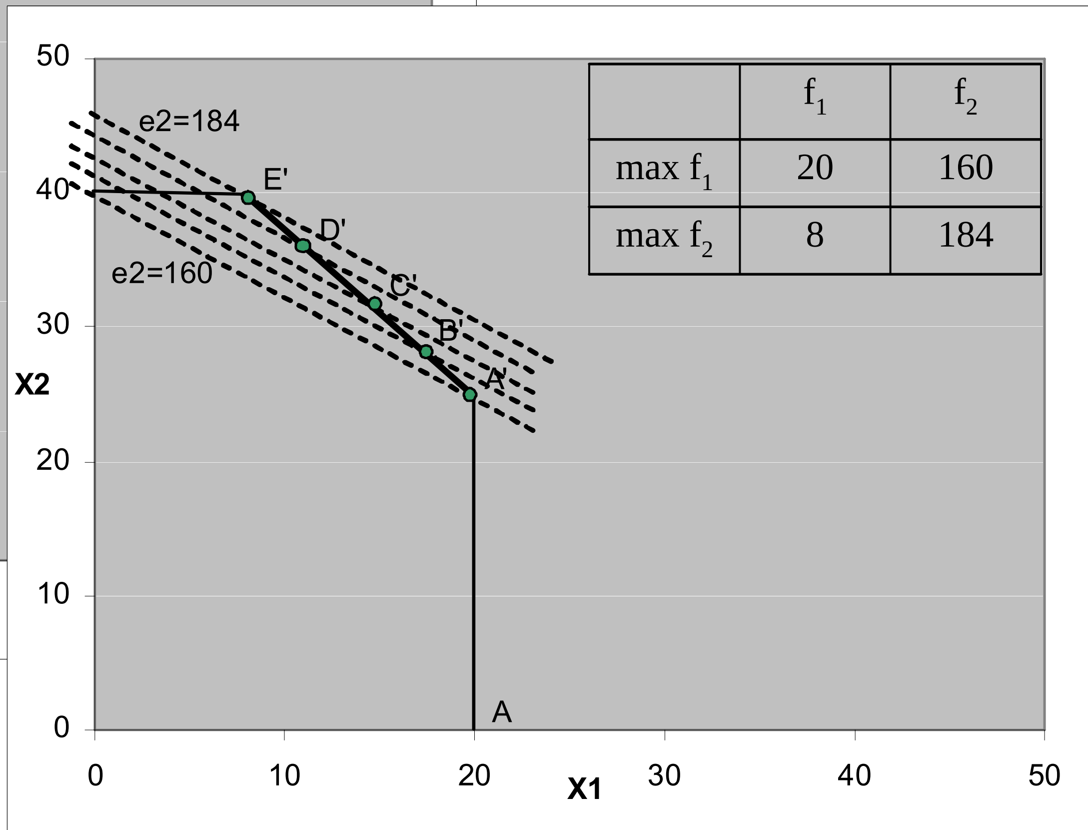


Conventional payoff table

	f_1	f_2
max f_1	20	60
max f_2	8	184

Example 1 (cont.)



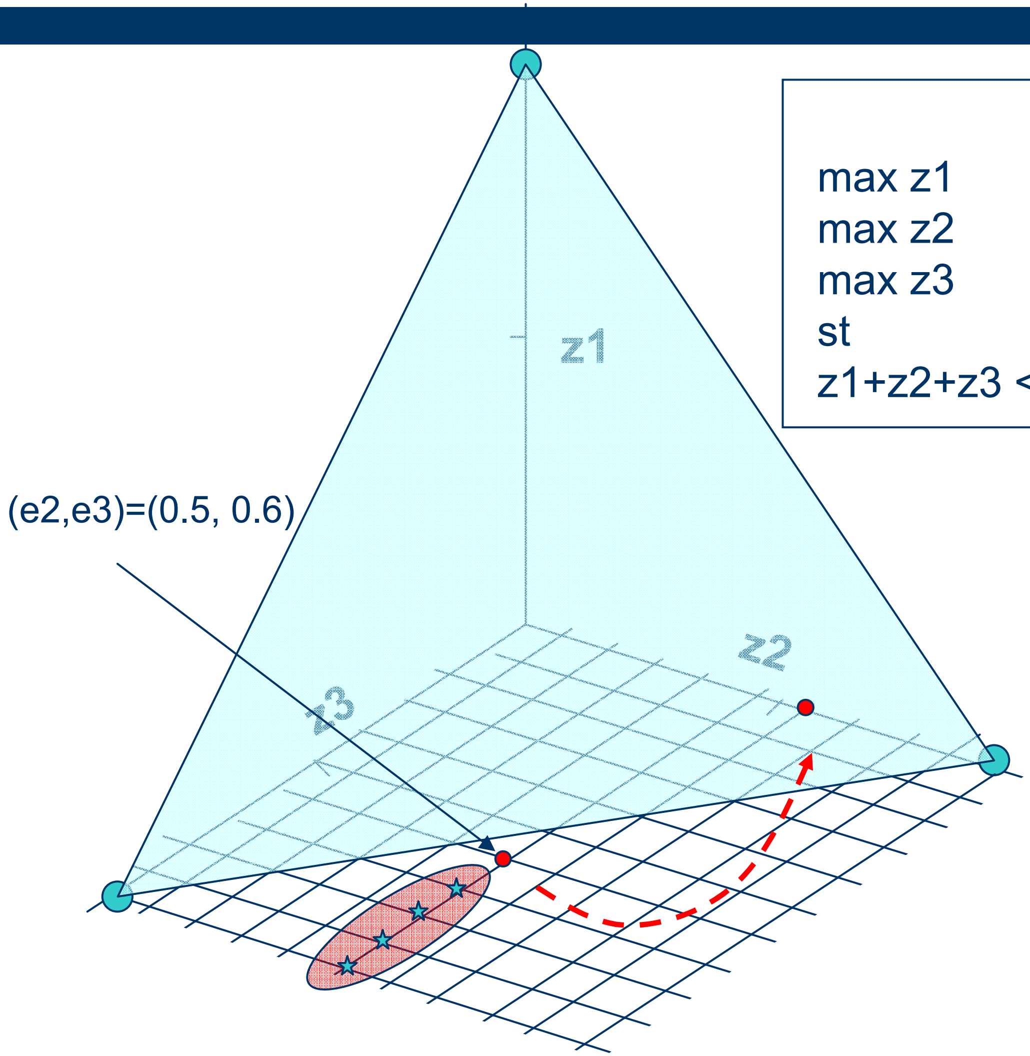


	f_1	f_2
max f_1	20	160
max f_2	8	184

Early exit from the loops

- For all the objective functions, the direction for bounding the obj. functions is from the more relaxed bound (lower bound for a maximization problem) to the more strict (individual optimum).
- When infeasibilities occur there is no need to keep on tightening the problem in the ε-constraint method
- Early exit from the respective loop
  - As the number of obj. functions increase the reduction in computation time is more apparent
  - 45% reduction in a problem with 6 obj functions, 236 variables and 96 constraints. With 5 grid point per objective function initially 2^5 runs = 3125 reduced to 1705)

Example 2



max z1 max z2 max z3 st z1+z2+z3 <= 1	***ε-constraint*** ⇒	*P(e2, e3)* max z1 st z2 >= e2 z3 >= e3 z1+z2+z3 <= 1

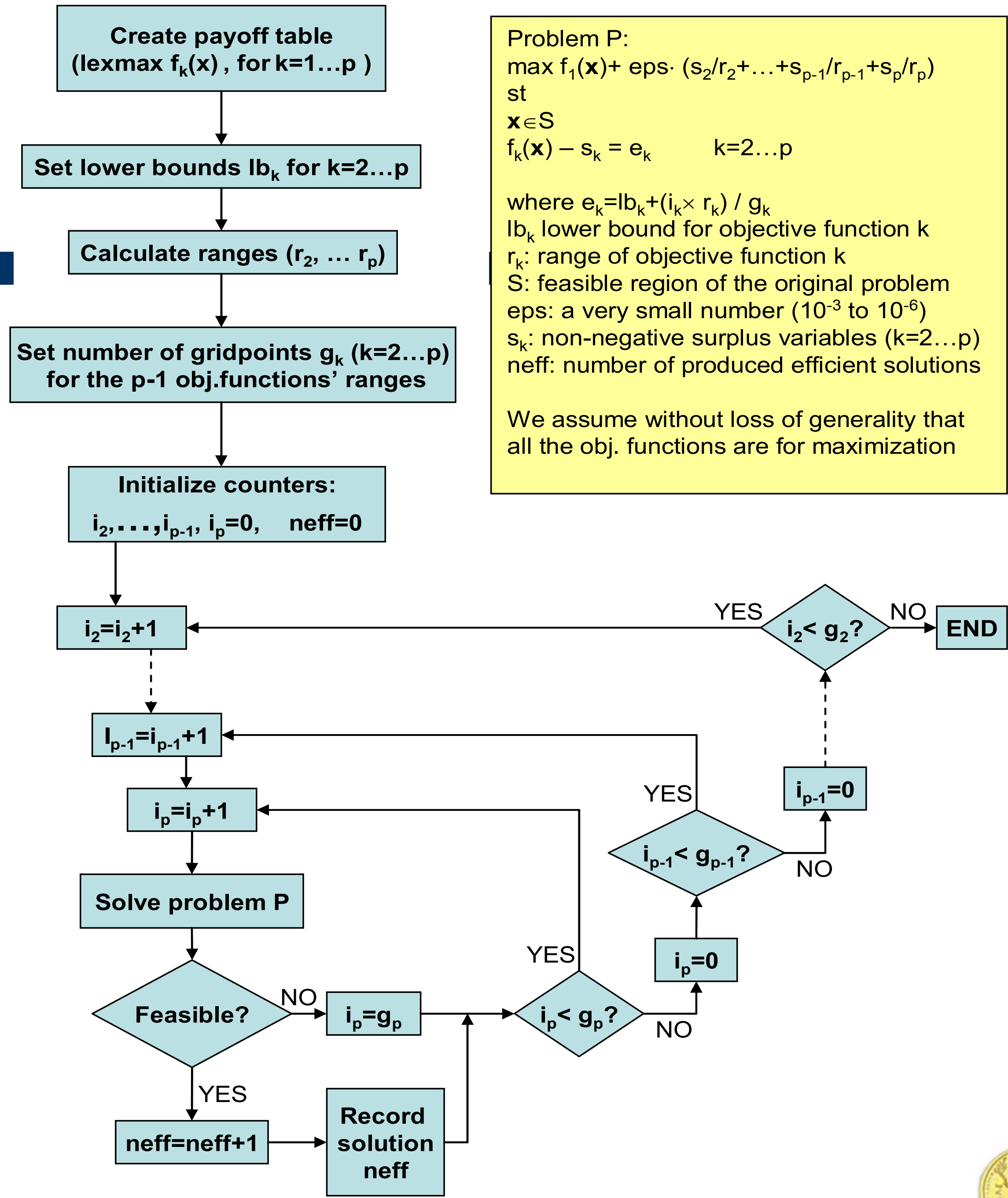
P(e2=0.5, e3=0.6) = INF

Bypass four grid points
Go to P(e2=0.6, e3=0.0)

↓

early exit from the e3-loop

Flowchart



Problem P:
max $f_1(\mathbf{x})$+ eps$\cdot$ ($s_2/r_2+\dots+s_{p-1}/r_{p-1}+s_p/r_p$)
st
$\mathbf{x}\in S$
$f_k(\mathbf{x}) - s_k = e_k$ k=2…p

where $e_k=lb_k+(i_k\times r_k) / g_k$
lb_k lower bound for objective function k
r_k: range of objective function k
S: feasible region of the original problem
eps: a very small number (10^{-3} to 10^{-6})
s_k: non-negative surplus variables (k=2…p)
neff: number of produced efficient solutions

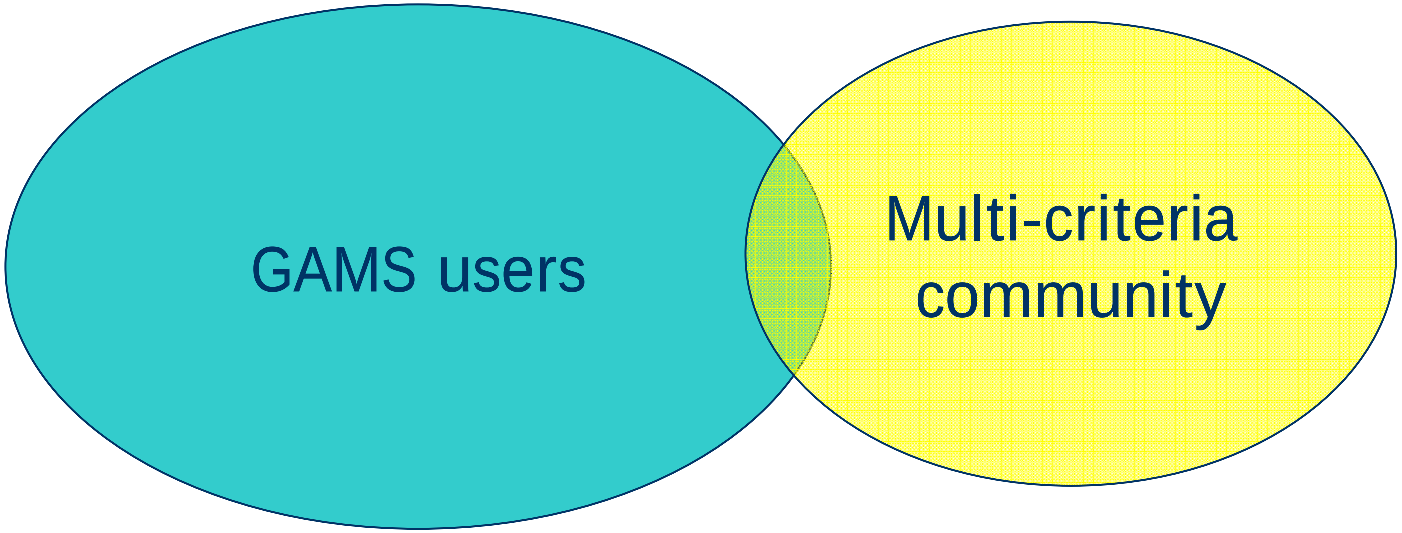
We assume without loss of generality that all the obj. functions are for maximization

Implementation in GAMS

General Algebraic Modeling System (www.gams.com)

Modeling Language

Applications in economy, energy, agriculture, engineering, management etc



GAMS structure

- **Sets**
  - Act as parameters' variables' and equations' indices
- **Parameters**
  - in vector or tabular form
- **Variables**
  - continuous, discrete etc
- **Equations**
  - model's constraints, objective function
- **Control instructions**
  - (loops, if…then…else, etc)

Model building in GAMS for implementing AUGMECON

- **GAMS performs single objective optimization**
- **Code customized (specific parameters, variables, constraints, control commands) to implement AUGMECON**
- **Constructs the set of differently parameterized single-objective problems and applies multiple runs automatically**

User input and automatic issues

- **The user specifies:**
  - **Number and direction of the obj. functions**
  - **Lower bounds in the obj. functions (if any…)**
  - **Number of grid points per obj. function**
- **Automatic calculation of:**
  - **Payoff table (through lexicographic optimization)**
  - **Pareto optimal solutions through the iterative process**

Code sample

Original model

```
LIGNSUP.. sum(I,LIGN(I)) =l= ligncap ;
OILSUP.. sum(I,OIL(I)) =l= oilcap ;
NGSUP.. sum(I,NG(I)) =l= ngcap ;
RENSUP.. sum(I,REN(I)) =l= rencap ;
EQDEMAND(I).. LIGN(I) + OIL(I) + NG(I) + REN(I) =g= demand(I) ;

OBJF(K).. lcoef(K)*sum(I,LIGN(I))+ocoef(K)*sum(I,OIL(I))+ngcoef(K)*sum(I,NG(I))+
          rcoef(K)*sum(I,REN(I))=e= z(K) ;
CON_OBJ(K)..   Z(K) - dir(K)*SL(K) =E= RHS(K) ;
AUGM_OBJ..   dir('1')*Z('1')+10**(-3)*SUM(K$KM1(K),SL(K)/(MAXOBJ(K)-MINOBJ(K)))=E=
A_OBJVAL  ;
```

Code added for AUGMECON

Example

Simplified version of the power generation problem

Cover the power demand divided in base, middle and peak load by a number of different technologies (lignite, oil, natural gas, renewable) based on three criteria: cost, CO_2 emissions and energy independency

MIN 30 LIGN + 75 OIL + 60 NG + 90 RES
MIN 1.44 LIGN + 0.72 OIL + 0.45 NG
MIN OIL + NG
ST
LIGN – LIGN1 – LIGN2 = 0
OIL – OIL2 – OIL3 = 0
NG – NG1 – NG2 – NG3 = 0
RES – RES1 – RES3 = 0
LIGN <= 31000
OIL <= 15000
NG <= 22000
RES <= 10000
LIGN1 + NG1 + RES1 >= 38400
LIGN2 + OIL2 + NG2 >= 19200
OIL3 + NG3 + RES3 >= 6400

Output of the GAMS model

5 grid points per objective function

PAYOFF TABLE

3075000	62460	33000
3855000	45180	37000
3225000	55260	23000

NON DOMINATED POINTS FROM E-CONSTRAINT METHOD

#	z1	z2	z3
1	3075000	62460	33000
2	3075000	62460	33000
3	3120000	60300	30000
4	3172500	57780	26500
5	3225000	55260	23000
6	3165000	58140	27000
7	3165000	58140	27000
8	3165000	58140	27000
9	3172500	57780	26500
10	3225000	55260	23000
11	3315000	53820	25000
12	3315000	53820	25000
13	3315000	53820	25000
14	3315000	53820	25000
15	****	infeasible	****
16	3585000	49500	31000
17	3585000	49500	31000
18	****	infeasible	****
19	3855000	45180	37000
20	****	infeasible	****

9 P.o.s.

10 grid points per objective function and filtering

PAYOFF TABLE

3075000	62460	33000
3855000	45180	37000
3225000	55260	23000

NON DOMINATED POINTS FROM E-CONSTRAINT METHOD

#	z1	z2	z3
1	3075000	62460	33000
2	3078000	62316	32800
3	3099000	61308	31400
4	3120000	60300	30000
5	3141000	59292	28600
6	3162000	58284	27200
7	3183000	57276	25800
8	3204000	56268	24400
9	3225000	55260	23000
10	3111000	60732	30600
11	3147000	59004	28200
12	3219000	55548	23400
13	3315000	53820	25000
14	****	infeasible	****
15	3423000	52092	27400
16	3531000	50364	29800
17	3639000	48636	32200
18	3747000	46908	34600
19	3855000	45180	37000

18 P.o.s.

Concluding remarks

- Generation methods have significant advantages but remain unexploited mainly due to scarcity of available tools
- A new version of the ε-constraint method was proposed that overcomes its major drawbacks
  - Guarantees Pareto optimality of derived solutions
  - Proper representation of the efficient set
  - Computational efficiency
- GAMS modeling in order to familiarize the GAMS users with the multi-objective issues

Merci beaucoup…