Determination of the Efficient Set in Multiobjective Linear Programming

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Determination of the Efficient Set in Multiobjective Linear Programming[1]

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Abstract. This paper develops a method for finding the whole set of efficient points of a multiobjective linear problem. Two algorithms are presented; the first one describes the set of all efficient vertices and all efficient rays of the constraint polyhedron, while the second one generates the set of all efficient faces. The method has been tested on several examples for which numerical results are reported.

Key Words. Multiobjective linear programming, efficient points, nondominated faces, degeneracy.

1. Introduction

An important question arising in multiobjective programming consists in the determination of the whole efficient set. Indeed, the set of objective values corresponding to different efficient points is not generally totally ordered, and an appropriate answer to the decision-maker's demand is to provide him the whole set of efficient solutions. Several approaches of this problem have been proposed up to now, in the particular framework of linear programming. Although these methods are quite different, they are always based upon the complete determination of efficient vertices of the constraint polyhedron, using a modified simplex algorithm. In some papers, vertices are tested for efficiency after being obtained (Ref. 1), while other authors propose some tests to know which pivots are leading to an efficient

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vertex or ray (Refs. 2–6). These tests are sometimes completed in order to determine efficient faces adjacent to the current vertex (Refs. 4, 7, 8, 9). In this way, the efficient faces and the efficient vertices are obtained simultaneously. In Ref. 1, the determination of efficient faces is carried out in a subsequent phase when all efficient vertices are known.

As discussed in Ref. 1, the main difficulty in this kind of problem is to deal with degeneracy. For instance, in Refs. 3 and 5, the description of all efficient vertices is done by uncovering all dual-efficient bases; however, this method may lead to very high computational efforts if some vertices are degenerate. On the other hand, the characterization of an efficient face in Ref. 7 is true only if the current vertex is nondegenerate. The aim of this work is to propose two algorithms with numerical experiments overcoming these difficulties. In the next section, we recall characterizations for basic efficient solutions and then we show that, with special pivoting rules, it is possible to describe all efficient extreme points by uncovering only some efficient bases. When a degenerate vertex is encountered, we go through the subset of its corresponding bases, generating only a part of it. The third section is devoted to the determination of efficient faces. Several problems occurring from degeneracy (Refs. 4, 7) are avoided by describing an efficient face as the convex hull of its efficient vertices and its efficient rays. Using this description, we give a characterization for an efficient face and then we present an algorithm providing all maximal efficient faces. In the last section, we comment on our algorithms and report on several numerical experiments.

2. Efficiency

A linear multiobjective problem can be stated under the standard form

$$\text{(LVMP)} \quad \max\{\bar{C}X : \bar{A}X = b, X \geqq 0\},$$

where $\bar{C} = (C, 0)$ is a $k \times (n+m)$ matrix with C a $k \times n$ matrix, $\bar{A} = (A, I)$ is an $m \times (n+m)$ matrix with A an $m \times n$ matrix and I the identity matrix of $\mathbb{R}^m$, $b \in \mathbb{R}^m_+$, and $X \in \mathbb{R}^{n+m}$.

Let X be in $\mathbb{R}^p$. We use the following notations:

$X \geqq 0$, if and only if $X_j \geq 0, j = 1, \ldots, p$;

$X \geq 0$, if and only if $X \geqq 0$ and $X \neq 0$;

$X > 0$, if and only if $X_j > 0, j = 1, \ldots, p$.

We denote by

$$P = \{Y \in \mathbb{R}^n : AY \leqq b,\ Y \geqq 0\},$$

$$P_\infty = \{R \in \mathbb{R}^n : AR \leqq 0,\ R \geqq 0\},$$

$$S = \{X \in \mathbb{R}^{n+m} : \bar{A}X = b,\ X \geqq 0\},$$

$$S_\infty = \{D \in \mathbb{R}^{n+m} : \bar{A}D = 0,\ D \geqq 0\},$$

the set of feasible points, the set of recession rays, the set of feasible solutions, and the set of recession directions, respectively.

Note that the applications ϕ: P→S and ψ: $P_\infty \to S_\infty$, defined by $\phi(Y) = (Y, b - AY)$ and $\psi(R) = (R, -AR)$, are one-to-one correspondences. We will denote the set of extreme points or vertices of P by ext(P) and the set of extreme rays of P_∞ by ext(P_∞). A feasible solution X is an extreme (or basic) feasible solution if $X = \phi(Y)$ with $Y \in$ ext(P). A point $Y^* \in$ P is said to be an efficient point if there is no $Y \in$ P such that $CY \geq CY^*$ and the corresponding solution $X^* = \phi(Y^*)$ in S is called an efficient solution. The set of all efficient points is denoted by P* and the set of efficient extreme points P* $\cap$ ext(P) by ext(P*). The corresponding solutions in S are called efficient extreme (or basic) solutions. We say that $R^* \in$ ext(P_∞) is an efficient extreme ray if there exists $Y^* \in$ ext(P*) such that $\{Y^* + tR^*, t \geq 0\} \subset$ P*. The corresponding direction $D^* = \psi(R^*)$ in S_∞ is called an efficient extreme direction. A key result in all the sequel is the following characterization of efficiency which can be found in many places (see, e.g., Refs. 1 and 2).

Proposition 2.1. A feasible solution $X^* \in$ S is efficient if and only if there exists $\lambda \in \mathbb{R}^k$, $\lambda > 0$ such that X^* solves the linear scalar problem

$$(\mathrm{LP}(\lambda)) \quad \max\{{}^T\lambda \bar{C}X : X \in \mathrm{S}\}.$$

Recall that ${}^T\lambda$ stands for the transpose of the vector λ. We introduce the subset

$$\mathscr{E} = \{\mathrm{conv}(V) + \mathrm{cone}(W) : V \subset \mathrm{ext(P)}\, W \subset \mathrm{ext}(\mathrm{P}_\infty)\},$$

where 'conv' is the convex hull and 'cone' means "convex cone generated by." We deduce from Proposition 2.1 the following result.

Theorem 2.1. For $E \in \mathscr{E}$, the following three conditions are equivalent:

(i) every point of E is efficient;
(ii) there exists Y^* in the relative interior ri E such that Y^* is efficient;
(iii) there exists $\lambda > 0$ such that, for each point Y of E, $\phi(Y)$ solves LP(λ).

Proof. It follows from Proposition 2.1 that (iii) implies (i). Since ri E is nonempty, we have obviously (i)$\Rightarrow$(ii). To show that (ii)$\Rightarrow$(iii), consider $Y^* \in \mathrm{ri}\, E$ and the corresponding efficient solution X^*. From Proposition 2.1, there exists $\lambda > 0$ such that X^* solves LP(λ). For each $Y \in E$, we can choose $\alpha \in (0, 1)$ and $Y' \in E$ such that

$$Y^* = \alpha Y' + (1-\alpha) Y,$$

and we have, with the corresponding feasible solutions,

$${}^T\lambda \bar{C} X^* = \alpha\, {}^T\lambda \bar{C} X' + (1-\alpha)\, {}^T\lambda \bar{C} X \leq \alpha\, {}^T\lambda \bar{C} X^* + (1-\alpha)\, {}^T\lambda \bar{C} X.$$

It follows that

$${}^T\lambda \bar{C} X^* = {}^T\lambda \bar{C} X,$$

which implies that X solves LP(λ). □

Note that this result is a slight extension of Theorem 4.1.2 in Ref. 1.

We denote by $\mathscr{E}^*$ the collection of subsets of $\mathscr{E}$ satisfying the conditions of Theorem 2.1. A face of P, in the usual sense, is called an efficient face if it is entirely contained in P*. A maximal efficient face is a maximal element of the set of all efficient faces ordered by the inclusion, and we recall that the efficient set P* is the union of all maximal efficient faces (Ref. 7). Note that an efficient face belongs necessarily to $\mathscr{E}^*$, the converse implication being not necessarily true. Indeed, if we consider the problem

$$\max \begin{bmatrix} x+y \\ -x-y \end{bmatrix},$$

$$\text{s.t.} \quad 0 \leq x \leq 1, 0 \leq y \leq 1,$$

conv((0, 0), (1, 1)) belongs to $\mathscr{E}^*$, but is not a face of the constraint set. However, we have the following result.

Corollary 2.1. An element E of $\mathscr{E}^*$ is a maximal element of $\mathscr{E}^*$ if and only if it is a maximal efficient face.

Proof. Suppose that

$$E = \mathrm{conv}(Y^1, \ldots, Y^p) + \mathrm{cone}(R^1, \ldots, R^q)$$

is maximal in $\mathscr{E}^*$. It follows from the efficiency and Theorem 2.1 that there exists $\lambda > 0$ such that, every feasible solution corresponding to a point of E

solves LP(λ). Denote by α the corresponding maximum value, and consider

$$E^0 = \{Y \in \mathrm{P}: {}^T\lambda C Y = \alpha\}.$$

The subset E^0 is an efficient face which contains E. As $E^0 \in \mathscr{E}^*$, the maximality of E entails that $E = E^0$. The converse implication is obvious. □

Here and below, taking Corollary 2.1 into account, we shall concentrate on the determination of the maximal elements of $\mathscr{E}^*$. We first introduce some notations. For a matrix M, M^J [respectively M_I] denotes the submatrix of M obtained by selecting only the columns $j \in J$ [respectively rows $i \in I$] of M. We call a feasible basis, or more simply a basis, a subset B of $\{1, \ldots, n+m\}$ such that, $\bar{A}^B$ is nonsingular and $[\bar{A}^B]^{-1} b \geqq 0$.

For each basis B, we define

$$N = \{1, \ldots, n+m\} \backslash B,$$

and we call basic solution specified by B, the feasible solution $X \in \mathrm{S}$ defined by

$$X_B = [\bar{A}^B]^{-1} b, \qquad X_N = 0.$$

Recall that the basic solutions of S correspond, by the one-to-one correspondence mentioned previously, to the vertices of P. When $[\bar{A}^B]^{-1} b > 0$, we say that B is nondegenerate and, in this case, it is known that B is the single basis specifying the basic solution X. Otherwise, B is said to be degenerate, and X may be specified by several bases, all of which are degenerate bases. The number of zero components of $[\bar{A}^B]^{-1} b$ is the number of zero components of X minus m and does not depend on B. Instead of saying that B is degenerate, we shall say sometimes that X (or even the corresponding vertex in P) is degenerate. A basis B is said to be an efficient basis if it specifies an efficient basic solution (or equivalently if the corresponding vertex of P is an efficient point). An efficient basis B is called a dual-efficient basis (see Definition 5 in Ref. 4) if there exists $\lambda > 0$ such that

$$\Delta = {}^T\lambda (\bar{C}^B [\bar{A}^B]^{-1} \bar{A}^N - \bar{C}^N), \tag{1}$$

satisfies $\Delta \geqq 0$. When such a $\lambda \in \mathbb{R}^k$ is known, we say that B is a λ-dual-efficient basis or a dual-optimal basis for the scalar objective function ${}^T\lambda \bar{C}$.

If B is a dual-efficient basis, we shall denote by $\Lambda(B)$ the set of $\lambda > 0$ such that B is λ-dual efficient. Note that the condition $\Delta \geqq 0$ implies that ${}^T\lambda \bar{C}^B [\bar{A}^B]^{-1}$ is an optimal solution for the dual problem of LP(λ), namely,

$$(\mathrm{DLP}(\lambda)) \quad \min\{{}^T U b: {}^T U A \geqq {}^T \lambda C,\ U \geqq 0\}. \tag{2}$$

It is easily shown, from Proposition 2.1 and the properties of the simplex algorithm, that every efficient basic solution X can be specified by a dual-efficient basis. However, when X is a degenerate efficient extreme solution, every basis specifying X is not necessarily dual-efficient. We can only assert the following result.

Proposition 2.2. Consider a basis B and the subset D of degenerate rows,

$$D=\{i\in\{1,\ldots,m\}: ([\bar{A}^B]^{-1}b)_i=0\}.$$

Then, B is an efficient basis if and only if there exist $\lambda>0$ and $\mu\geqq 0$ such that

$$^T\lambda(\bar{C}^B[\bar{A}^B]^{-1}\bar{A}^N-\bar{C}^N)+{}^T\mu([\bar{A}^B]^{-1}\bar{A}^N)_D\geqq 0. \tag{3}$$

Furthermore, (3) implies that the solution X specified by B solves LP(λ).

Proof. We set $\Psi=[\bar{A}^B]^{-1}$ and define Δ as in (1). Suppose that B is an efficient basis. It follows from Proposition 2.1 that there exists $\lambda>0$ such that X given by $(X_B, X_N)=(\Psi b, 0)$ is an optimal solution for LP(λ). Let $H_N\in\mathbb{R}^n$ be a vector satisfying the following conditions:

$$H_N\geqq 0, \qquad (\Psi\bar{A}^N)_D H_N\leqq 0. \tag{4}$$

Choosing

$$H_B=-\Psi\bar{A}^N H_N,$$

we get that

$$H=(H_B, H_N)\in\mathbb{R}^{n+m},$$

such that $X+tH\in S$, for $t>0$ small enough. Since X is an optimal solution of LP(λ), we must have $\Delta H_N\geq 0$. Thus, we have the following implication:

$$\forall H_N\geqq 0, -(\Psi\bar{A}^N)_D H_N\geqq 0 \Rightarrow \Delta H_N\geq 0;$$

and from Farka's lemma, there exists $\mu\geqq 0$ satisfying

$$\Delta+{}^T\mu(\Psi\bar{A}^N)_D\geqq 0.$$

Conversely, suppose that (3) is satisfied, and let X be the basic solution specified by B. Consider $H\in\mathbb{R}^{n+m}$ such that $X+tH\in S$, for $t>0$ sufficiently small. We have $\bar{A}H=0$, or equivalently

$$H_B=-\Psi\bar{A}^N H_N,$$

and $H_i\geq 0$, for each i such that $X_i=0$. Then, from the definition of D, it follows that

$$(\Psi\bar{A}^N)_D H_N\leqq 0.$$

Therefore, from (3), we get $\Delta H_N \geq 0$. As from (1) and the definition of H_B,

$$\Delta H_N = -{}^T\lambda \bar{C} H,$$

we have proved that X is an optimal solution of LP(λ). It follows from Proposition 2.1 that B is an efficient basis and that the corresponding efficient solution solves LP(λ). □

The result above is a dual form of Lemma 2.4 in Ref. 2 (see also Theorem 3 in Ref. 10). Now, we develop our first algorithm in order to enumerate all efficient vertices. The general idea of our procedure is essentially the same as in Refs. 4 and 6, but we propose here a special treatment of the degenerate case. Recall that, given a feasible basis B_0 and a basic feasible solution X, we can find, using the lexicographical rule of Dantzig (Ref. 11), a finite sequence of positive pivots linking B_0 with a feasible basis B specifying X; that basis B is said to be B_0-admissible. Note that, starting from B_0 and using the lexicographical rule, we cannot necessarily obtain all the bases specifying X. Given a feasible basis B_0, we say that a basis B is B_0-dual efficient if it is dual efficient and B_0-admissible. We suppose from now that a first B_0-dual efficient basis B is known, as well as the corresponding simplex tableau $T(B)$,

$$\begin{array}{c|c|c|} & X_N & \\ \cline{2-3} X_B & A' & b' \\ \cline{2-3} & C' & f' \\ \cline{2-3} \end{array},$$

where

$$A' = [\bar{A}^B]^{-1}\bar{A}^N, \qquad C' = \bar{C}^B[\bar{A}^B]^{-1}\bar{A}^N - \bar{C}^N,$$
$$b' = [\bar{A}^B]^{-1}b, \qquad f' = \bar{C}^B[\bar{A}^B]^{-1}b.$$

Several methods exist which provide such an efficient basis B from a starting basis B_0; see, for instance, Refs. 4, 12, 13. For each $\lambda \in \Lambda(B)$, we also consider the tableau $T_\lambda(B)$,

$$\begin{array}{c|c|c|} & X_N & \\ \cline{2-3} X_B & A' & b' \\ \cline{2-3} & {}^T\lambda C' & {}^T\lambda f' \\ \cline{2-3} \end{array}.$$

A column $j \in N$ of $T(B)$ is said to be λ-degenerate if there exists $\lambda \in \Lambda(B)$ such that $[{}^T\lambda C']^j = 0$. Two dual-efficient bases (B_1, B_2), both B_0-dual-efficient, are said to be B_0-adjacent if and only if $T(B_2)$ is obtained from $T(B_1)$ by a single positive pivot on a λ-degenerate column, with the lexicographical rule induced by B_0. If such a pivot leads to an efficient extreme direction D, we say that D is B_0-adjacent to B_1. Note that, in this case, the bases B_1 and B_2 are necessarily λ-dual efficient for the same λ and that they may specify the same basic solution. A B_0-chain will be a sequence $B_1, \ldots, B_p$ of B_0-dual-efficient bases such that (B_i, B_{i+1}) are B_0-adjacent for $i = 1, \ldots, p-1$.

In order to determine the set of all efficient vertices, some graphs are often introduced in the literature. We list below some of them and make observations justifying our next results. Let us define the following graphs by their node set and edge set, respectively:

($\mathscr{G}$) $\{B: B$ is an efficient basis$\}$,
$\{(B_1, B_2): B_2$ can be obtained from B_1 by a positive pivot$\}$;

($\mathscr{G}_-$) $\{B: B$ is an efficient basis$\}$,
$\{(B_1, B_2): B_2$ can be obtained from B_1 by a positive or negative pivot$\}$;

($\mathscr{H}$) $\{B: B$ is a dual-efficient basis$\}$,
$\{(B_1, B_2): B_2$ can be obtained from B_1 by a positive pivot on a λ-degenerate column$\}$;

($\mathscr{H}_-$) $\{B: B$ is a dual-efficient basis$\}$,
$\{(B_1, B_2): B_2$ can be obtained from B_1 by a positive or negative pivot on a λ-degenerate column$\}$;

($\mathscr{L}_{B_0}$) $\{B: B$ is a B_0-dual-efficient basis$\}$,
$\{(B_1, B_2): B_1$ and B_2 are B_0-adjacent$\}$.

It is well known that $\mathscr{G}_-$ is a connected graph (Theorem 2.3 in Ref. 6). However, $\mathscr{G}$, $\mathscr{H}$, and $\mathscr{H}_-$ may fail to be connected, as shown in the following example.

Example 2.1. Consider the vector maximization problem

$$\max \begin{bmatrix} -x-y-z/4 \\ x+y+(3/2)z \end{bmatrix},$$

$$\text{s.t.}\quad x+y+z \le 3,$$
$$2x+2y+z \le 4,$$
$$x-y \le 0,$$
$$x \ge 0, \qquad y \ge 0, \qquad z \ge 0.$$

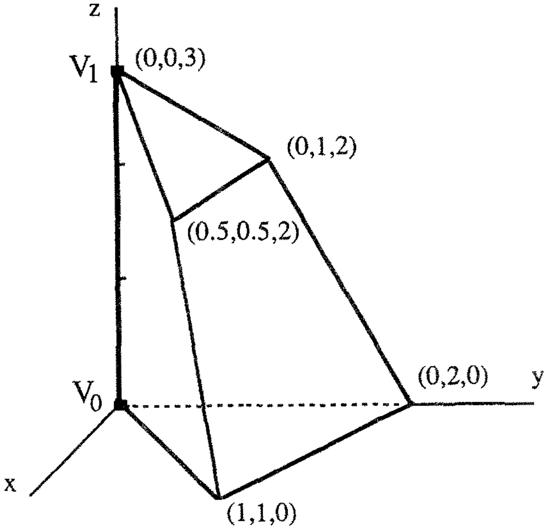

Fig. 1. Constraint polyhedron of Example 2.1. The efficient set is the edge $[V_0, V_1]$.

The polyhedron P is represented in Fig. 1, while graphs of efficient bases are shown in Fig. 2. This problem admits two degenerate efficient vertices V_0 and V_1, each of them being specified by three efficient bases, two dual-efficient bases, and only one B_0-dual efficient basis; here, B_0 is the starting basis specifying the origin.

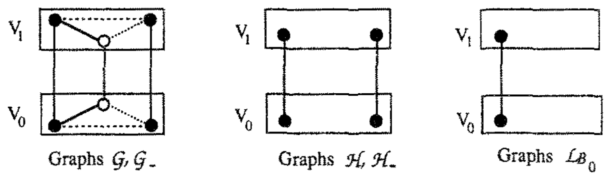

Fig. 2. Different graphs associated with Example 3.1. Black nodes represent the dual efficient bases, white nodes the efficient bases which are not dual-efficient, plain arcs the positive pivots, and dotted arcs the negative pivots.

The aim of the two following results is to prove that, for every feasible basis B_0, $\mathscr{L}_{B_0}$ is a connected graph; i.e., two B_0-dual efficient bases can be linked by a B_0-chain. Furthermore, each efficient extreme solution can be specified by a node of $\mathscr{L}_{B_0}$, and each efficient extreme direction is adjacent to a node of $\mathscr{L}_{B_0}$.

Proposition 2.3. Given two B_0-dual efficient bases B and B' specifying the same efficient extreme solution X, there exists a B_0-chain $B_1, \ldots, B_q$, with $B_1 = B$ and $B_q = B'$.

Proof. If B and B' are both λ-dual efficient, they are dual-optimal for the scalar objective ${}^T\lambda\bar{C}$, and the result follows from the simplex method with the lexicographical rule.

Suppose now that B and B' are respectively λ-dual efficient and λ'-dual efficient with $\lambda \neq \lambda'$. It is sufficient to prove the existence of a B_0-chain $B_1, \ldots, B_q$ such that $B_1 = B$ and B_q is λ'-dual efficient. If B_i is a λ_i-dual efficient basis, with

$$\lambda_i = \lambda + t_i\lambda', \qquad t_i \geq 0,$$

we set

$$c = {}^T\lambda\bar{C}, \qquad c' = {}^T\lambda'\bar{C}, \qquad c_i = c + t_i c',$$

$$\Delta_i = c_i^{B_i}[\bar{A}^{B_i}]^{-1}\bar{A} - c_i \geqq 0,$$

$$\Delta_i' = c'^{B_i}[\bar{A}^{B_i}]^{-1}\bar{A} - c'.$$

Suppose that B_i is not λ'-dual efficient. Then, Δ_i' has at least one negative component. We introduce the following nonnegative number:

$$t_{i+1} = t_i + \max\{t \geq 0 : \Delta_i + t\Delta_i' \geqq 0\}.$$

Note that it may occur that $t_{i+1} = t_i$ when there is the same component which is negative in Δ_i' and equal to zero in Δ_i. We set

$$\lambda_{i+1} = \lambda + t_{i+1}\lambda',$$

$$\Delta_{i+1} = \Delta_i + (t_{i+1} - t_i)\Delta_i',$$

$$N_{i+1} = \{j \in \{1, \ldots, n+m\} : \Delta_{i+1} = 0\},$$

and we call B_{i+1} a new basis obtained by pivoting (with the lexicographical order induced by B_0) on a column $j \in N_{i+1}$ such that $\Delta_i'^j < 0$. The basis B_{i+1} is adjacent to B_i, λ_{i+1}-dual efficient, and still specifies X. Indeed, the pivot does not change the basic solution, since $\Delta_i'^j < 0$ and X is already an optimal solution of LP(λ'). Remark also that, if $t_{i+1} > t_i$, B_{i-1} is not λ_{i+1}-dual efficient and then $B_{i-1} \neq B_i$. Actually, that follows from the equality

$${}^T\lambda_{i+1}(\bar{C}^{B_{i-1}}[\bar{A}^{B_{i-1}}]^{-1}\bar{A} - \bar{C}) = \Delta_{i-1} + (t_{i+1} - t_{i-1})\Delta_{i-1}',$$

and from the fact that one component is necessarily negative by the definition of t_i, since

$$t_{i+1} - t_{i-1} > t_i - t_{i-1}.$$

If B_{i+1} is not λ'-dual efficient, we construct in the same way B_{i+2} and so on. The number of bases being finite and from the remark above, there exists $v > 0$ such that t_i is constant for $i \geq v$. Consequently, for $i \geq v$, $\Delta_i = \Delta$ and $N_i = N$ do not depend on i. The column index j of the pivoting term between B_i

and B_{i+1} is such that $j \in N$ and $\Delta_i'^j < 0$. We know from the simplex algorithm (with the lexicographical rule) that, after a finite number of iterations, say $\delta \in \mathbb{N}$, we shall have $\Delta_{v+\delta}'^N \geqq 0$; therefore, the algorithm will stop with a λ'-dual efficient basis. □

Theorem 2.2. $\mathscr{L}_{B_0}$ is a connected graph.

Proof. Let B' and B^* be two B_0-dual efficient bases specifying respectively X' and X^*. If $X' = X^*$, it follows from Proposition 2.3 that there exists a B_0-chain linking B' and B^*. Suppose now that $X' \neq X^*$ and B' [respectively B^*] is λ'-dual efficient (respectively λ^*-dual efficient). It is easily seen that, for each λ belonging to the segment $[\lambda', \lambda^*]$, LP(λ) admits an optimal basic solution, denoted by X_λ; we choose

$$X_{\lambda'} = X' \quad \text{and} \quad X_{\lambda^*} = X^*.$$

The number of basic solutions being finite, there exists $\alpha \in \mathbb{N}$ such that

$$\{X_\lambda : \lambda \in [\lambda', \lambda^*]\} = \{X_1, \ldots, X_\alpha\}.$$

For each $i \in \{1, \ldots, \alpha\}$, we consider

$$\Lambda_i = \{\lambda \in [\lambda', \lambda^*] : X_i \text{ solves LP}(\lambda)\}.$$

By construction, Λ_i is a nonempty closed segment. These Λ_i, $i = 1, \ldots, \alpha$, form a covering of $[\lambda', \lambda^*]$, and we may suppose by permuting some indices that

$$\Lambda_{i-1} \cap \Lambda_i \neq \varnothing, \qquad i = 2, \ldots, \alpha,$$

$$X_1 = X', \qquad X_\alpha = X^*.$$

Let us choose

$$\lambda_i \in \Lambda_{i-1} \cap \Lambda_i, \quad i = 2, \ldots, \alpha, \qquad \text{and } \lambda_1 = \lambda'.$$

Each X_i, $i = 1, \ldots, \alpha - 1$, appears to be optimal for both ${}^T\lambda_i \bar{C}$ and ${}^T\lambda_{i+1} \bar{C}$. Let B_i [respectively B_i'] be a λ_i-dual efficient (respectively λ_{i+1}-dual efficient) basis specifying X_i; at the first iteration, $X_1 = X'$ and $B_1 = B'$. It follows from Proposition 2.3 that there exists a B_0-chain linking B_i and B_i'. Then, X_{i+1} being also an optimal solution of LP(λ_{i+1}'), we know from the simplex theory that there exists a B_0-chain linking B_i' with a basis B_{i+1} specifying X_{i+1}, the pivoting terms being always on λ_{i+1}'-degenerate columns. Then, we apply again Proposition 2.3 to find a B_0-chain linking B_{i+1} to a basis B_{i+1}' which is λ_{i+2}-dual efficient and specifies X_{i+1}, and so on. In this way, it is possible to find a B_0-chain of dual-efficient bases such that the last one, say B'', specifies $X_\alpha = X^*$ and from Proposition 2.3, there exists a B_0-chain linking B'' to B^*. □

Remark 2.1. Suppose that D^* is an extreme efficient direction. There exists an efficient extreme solution X^* and $\lambda^* \in \mathbb{R}^k$, $\lambda^* > 0$, such that, for each $t \geq 0$, the vector $X^* + tD^*$ is an optimal solution of LP(λ^*). Denote by B^* a node of $\mathscr{L}_{B_0}$ specifying X^*. Then, the direction D^* will be obtained automatically from B^* by performing pivot operations on λ^*-degenerate columns with the lexicographical rule induced by B_0. Thus, D^* is B_0-adjacent to a node of $\mathscr{L}_{B_0}$.

Note that Theorem 2.2 implies that each connected component of graphs $\mathscr{G}$ and $\mathscr{H}$ generates all efficient extreme solutions. Indeed, such a component admits $\mathscr{L}_B$ as a subgraph where B is any dual-efficient basis belonging to the component. Furthermore, if there exists a nondegenerate efficient extreme solution, then it is specified by a single dual-efficient basis. Necessarily this basis belongs to every connected component of $\mathscr{G}$ and $\mathscr{H}$, which implies that, in this case, these graphs are connected.

3. Determination of Efficient Vertices and Efficient Extreme Rays

From Theorem 2.2, we may deduce an algorithm describing the set of all efficient vertices and all extreme rays. Suppose that we know an efficient extreme solution X_0 specified by a λ-dual efficient basis B_0. As $\mathscr{L}_{B_0}$ is a connected graph, we apply the standard depth-first search algorithm (Chapter 5 in Ref. 14) in order to enumerate all nodes of $\mathscr{L}_{B_0}$. The set of the efficient extreme solutions, the set of the efficient extreme directions, and the set of the B_0-dual efficient bases are respectively initialized to

$$\mathscr{S} = \{X_0\}, \qquad \mathscr{D} = \{\varnothing\}, \qquad \mathscr{B} = \{B_0\}.$$

Then, the method relies on a recursive subroutine denoted VERTICES(B, λ), where B is a λ-dual-efficient basis of the set $\mathscr{B}$. The process is the following.

Subroutine VERTICES(B, λ)

Step V1. Store in a list $\mathscr{C}_B$ all couples (j, λ') such that the column j of $T(B)$ is λ'-degenerate and such that the pivot, chosen on this column with the lexicographical rule induced by B_0, leads to a basis not already in $\mathscr{B}$.

Step V2. If $\mathscr{C}_B = \varnothing$, then stop; otherwise, choose (j, λ') in $\mathscr{C}_B$. Let $\mathscr{C}_B = \mathscr{C}_B - \{(j, \lambda')\}$.

Step V3. If the jth column of $T(B)$ is nonpositive, then go to Step V6; otherwise, perform the pivot operation on that column using the lexicographical rule. Denote by B' the new basis and X' the corresponding solution.

Step V4. Store B' in $\mathscr{B}$, and, if $X' \notin \mathscr{S}$, then store X' in $\mathscr{S}$.

Step V5. Call the subroutine VERTICES(B', λ'). Go to Step V2.

Step V6. An efficient extreme direction D has been discovered. Then, if D does not belong to $\mathscr{D}$, it is stored in $\mathscr{D}$. Go to Step V2.

Note that this subroutine needs another subroutine to fill in the list $\mathscr{C}_B$. Although this problem appears in many places (Refs. 1, 2, 5, 8, 15), we propose here a new method based on the knowledge of $\lambda > 0$ for which B is λ-dual-efficient.

Actually, for each column j, we have to test the existence of $\lambda' > 0$ and $\xi \geqq 0$ such that

$$-{}^TC'\lambda' + \xi = 0, \qquad \xi_j = 0,$$

where

$$C' = \bar{C}^B[\bar{A}^B]^{-1}\bar{A}^N - \bar{C}^N.$$

By homogeneity, we can replace $\lambda' > 0$ by $\lambda' \geqq \lambda$ and study whether the following problem admits the value zero or not:

$$\begin{aligned} (\mathscr{T}_j) \quad & \min \xi_j, \\ & \text{s.t.} \quad -{}^TC'\lambda' + \xi = {}^TC'\lambda, \\ & \qquad \lambda' \geqq 0, \qquad \xi \geqq 0. \end{aligned}$$

The right-hand side ${}^TC'\lambda$ is nonnegative, since B is λ-dual efficient, so that the simplex algorithm (phase II) can be directly applied. This advantage vanishes if λ is replaced by the constant vector $(1, \ldots, 1)$; see Corollary 2 in Ref. 5. Note that we need not perform the test $(\mathscr{T}_j)$ for each nonbasic j, as simple remarks show that, for some j, the value of $(\mathscr{T}_j)$ is obviously positive. For example, if $C'^j \geqq 0$, or if this column dominates another one in the sense of $\leqq$, then the value of $(\mathscr{T}_j)$ is necessarily positive (Ref. 1).

4. Determination of All Efficient Points

It follows from Theorem 2.1 that the set of all efficient points can be written as the union of maximal subsets of $\mathscr{E}^*$, also called maximal efficient faces. The aim of the following is to construct iteratively those maximal elements. More precisely, suppose that

$$E = \operatorname{conv}(V^1, \dots, V^p) + \operatorname{cone}(R^1, \dots, R^q)$$

is an element of $\mathscr{E}^*$ with $X^1, \dots, X^p$ and $D^1, \dots, D^q$ the corresponding efficient basic solutions and efficient extreme directions. Let X^s be a new basic solution specified by a λ^s-dual efficient basis B^s and corresponding to a new vertex V^s. Consider the associated tableau $T(B^s)$,

$$\begin{array}{c|c|c|}
 & X_{N^s} & \\
\cline{2-3}
X_{B^s} & A^s & b^s \\
\cline{2-3}
 & C^s & f^s \\
\cline{2-3}
\end{array}.$$

Theorem 4.1. The set

$$E^s = \operatorname{conv}(V^1, \dots, V^p, V^s) + \operatorname{cone}(R^1, \dots, R^q)$$

belongs to $\mathscr{E}^*$ if and only if $\gamma = 0$ is a solution of the problem

$$(\mathscr{H}(E, V^s)) \quad \min \gamma,$$

$$\text{s.t.} \quad -{}^T C^s \lambda - {}^T[A^s]_D \mu + w = {}^T C^s \lambda^s, \tag{5}$$

$$-{}^T X_{N^s} {}^T C^s \lambda + \gamma = {}^T X_{N^s} {}^T C^s \lambda^s, \tag{6}$$

$$\lambda \geqq 0, \qquad \mu \geqq 0, \qquad w \geqq 0, \qquad \gamma \geqq 0,$$

where X is any point in $\operatorname{ri}[\operatorname{conv}(X^1, \dots, X^p) + \operatorname{cone}(D^1, \dots, D^q)]$

Proof. From Proposition 2.2, (5) means that X^s solves $\mathrm{LP}(\lambda + \lambda^s)$. As

$$\bar{C}X = \bar{C}X^s - C^s X_{N^s}, \qquad \text{for every } X \in S,$$

the equality (6) with $\gamma = 0$ is equivalent to

$${}^T(\lambda + \lambda^s)\bar{C}(X - X^s) = 0.$$

Thus, $\mathscr{H}(E, V^s)$ admits the value 0 if and only if there exists $\lambda + \lambda^s > 0$ such that X^s and X solve $\mathrm{LP}(\lambda + \lambda^s)$. It follows from Theorem 2.1 that this latter condition means that E^s is efficient. □

Remark 4.1. The right-hand sides of (5) and (6) are nonnegative, so that the simplex algorithm (phase II) can be applied to solve $(\mathscr{H}(E, V^s))$.

Now, instead of a new efficient vertex, we consider a new efficient extreme ray R^s corresponding to the extreme direction D^s, and we suppose that this direction has been obtained from $T(B^p)$, where B^p is a dual efficient basis specifying X^p. The subset E is the same as above and, as previously, the different parts of the tableau $T(B^p)$ are named: A^p, C^p, b^p, f^p.

Theorem 4.2. The set

$$E^s = \operatorname{conv}(V^1, \ldots, V^p) + \operatorname{cone}(R^1, \ldots, R^q, R^s)$$

belongs to $\mathscr{E}^*$ if and only if the value of $(\mathscr{H}(E, V^p))$ is zero, where X is a relative interior point of $\operatorname{conv}(X^1, \ldots, X^p) + \operatorname{cone}(D^1, \ldots, D^q, D^s)$.

Proof. The proof is similar to the one of Theorem 4.1 and is omitted. □

Theorems 4.1 and 4.2 may be used to test whether E^s, obtained by addition of a new vertex or a new extreme ray, is efficient or not. Suppose that all efficient extreme points $V^1, \ldots, V^p$ and all efficient extreme rays $R^{p+1}, \ldots, R^{p+q}$ are known. Furthermore, for each $i = 1, \ldots, p$, suppose that we have a λ^i-dual efficient basis B^i specifying V^i and the corresponding tableau $T(B^i)$. For convenience, we denote an element

$$E = \operatorname{conv}(V^{i_1}, \ldots, V^{i_k}) + \operatorname{cone}(R^{i_{k+1}}, \ldots, R^{i_s})$$

by the ordered set of its indices, i.e.,

$$E = \{i_1, \ldots, i_s\}, \qquad \text{where } 1 \le i_1 < \cdots < i_s \le p+q;$$

and we introduce, for such an element E, the subset of $\mathscr{E}$,

$$\mathscr{E}_E = \{E \cup \{i_{s+1}, \ldots, i_r\} : i_s < i_{s+1} < \cdots < i_r \le p+q\}.$$

Given an efficient subset E of $\mathscr{E}^*$, the subroutine EFFICIENCY(E) described below provides all maximal elements of $\mathscr{E}_E \cap \mathscr{E}^*$. If we restrict E to a single efficient extreme point V^i, we find in this way all maximal elements of $\mathscr{E}^*$ containing V^i. It follows from Corollary 2.1 that these elements are also the maximal efficient faces containing V^i. In order to obtain all maximal efficient faces, we call the subroutine EFFICIENCY(V^i), $i = 1, \ldots, p$, this task being performed by the subroutine FACE. Let E be an ordered subset of $\{1, \ldots, p+q\}$ and denote by Ter E the largest index in E.

Subroutine EFFICIENCY(E)

Step E1. Initialize a boolean variable M to "true."

Step E2. For $j = \text{Ter}(E)+1$ to $p+q$, let $G = E \cup \{j\}$ be a new subset of $\mathscr{E}_E$. If G is efficient, then assign the value "false" to M and call the subroutine EFFICIENCY(G).

At the end of Step E2, if M is false, then E is not a maximal element of $\mathscr{E}_E \cap \mathscr{E}^*$, and if M is true, then for each $j \in \{\text{Ter}(E)+1, \ldots, p+q\}$, $E \cup \{j\}$ is not efficient.

Step E3. If M is true and if there does not exist $F \in \mathscr{F}$ such that $E \subset F$, then $\mathscr{F} = \mathscr{F} \cup \{E\}$.

Subroutine FACE

Step E1. Initialize $\mathscr{F}$ to $\varnothing$.

Step E2. For $i = 1$ to p, set $E = \{i\}$, and call EFFICIENCY(E).

The procedure EFFICIENCY(E) can be improved in different ways in order to reduce the number of convex hulls to be checked. For instance, the Step E2 can be avoided if $E \cup \{\text{Ter}(E)+1, \ldots, p+q\}$ has already been obtained as an efficient face. Moreover, in the case where the entire set P is not efficient, every efficient set

$$E = \text{conv}(V^1, \ldots, V^p) + \text{cone}(R^1, \ldots, R^p)$$

is included into an efficient face distinct from P. Thus, all efficient extreme solutions and all efficient extreme directions corresponding respectively to $V^1, \ldots, V^p$ and $R^1, \ldots, R^p$ must have a common zero component. Remark also that the tests $(\mathscr{H}(E, V^s))$ and $(\mathscr{H}(E, V^p))$ are simpler when the first vertex of E is nondegenerate. Thus, efficient vertices will be rearranged, in order to place the nondegenerate ones in the first positions. We conclude this section with the following example.

Example 4.1. Consider the vector maximization problem

$$\max \begin{bmatrix} x - 100y \\ x + 100y \\ z \end{bmatrix},$$

$$\begin{aligned}
\text{s.t.}\quad & x\cos(j\pi/6) + y\sin(j\pi/6) + z \le 1 + 2\sin(j\pi/6), && j = 0, \ldots, 3, \\
& x\cos(j\pi/6) - y\sin(j\pi/6) + z \le 1 - \sin(j\pi/6), && j = 1, \ldots, 3, \\
& x \ge 0, \quad y \ge 0, \quad z \ge 0.
\end{aligned}$$

It is easily seen that this problem has five efficient faces and eight efficient vertices. The graph of B_0-dual efficient bases is represented in Fig. 3.

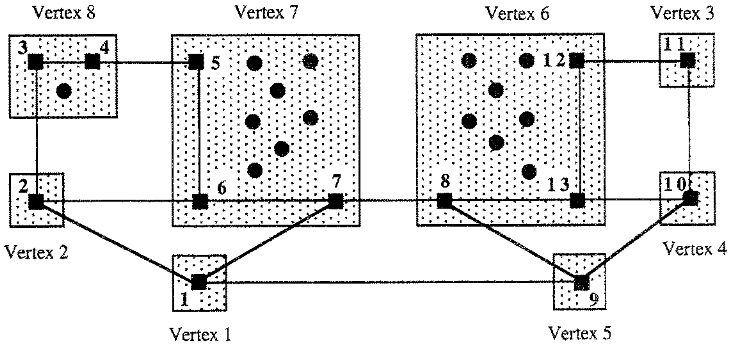

Fig. 3. Graph L_{B_0} associated to Example 4.1. Each efficient vertex is represented by a shaded rectangle which contains all dual-efficient bases specifying this vertex. Black squares describe B_0-dual efficient bases, while black discs represent the other dual-efficient bases which are not B_0-admissible; here, B_0 is the basis specifying the vertex 1. An arc between two bases means that they are B_0-adjacent. The B_0-dual efficient bases are numbered to indicate the order in which they are obtained.

When all efficient vertices are obtained, they are reordered as indicated in Fig. 3. Using the notations "ef" for efficient, "ne" for nonefficient, and "ps" for previously seen, the description of maximal efficient faces proceeds as follows.

New face(s) containing the vertex 1: $\{1, 2\}$ ef, $\{1, 2, 3\}$ ne, $\{1, 2, 4\}$ ne, $\{1, 2, 5\}$ ne, $\{1, 2, 6\}$ ne, $\{1, 2, 7\}$ ef, $\{1, 2, 7, 8\}$ ne, storage of $\{1, 2, 7\}$, $\{1, 2, 8\}$ ne, $\{1, 3\}$ ne, $\{1, 4\}$ ne, $\{1, 5\}$ ef, $\{1, 5, 6\}$ ef, $\{1, 5, 6, 7\}$ ef, $\{1, 5, 6, 7, 8\}$ ne, storage of $\{1, 5, 6, 7\}$, $\{1, 5, 7\}$ ps, $\{1, 5, 7, 8\}$ ne, $\{1, 5, 8\}$ ne, $\{1, 6\}$ ps, $\{1, 6, 7\}$ ps, $\{1, 6, 7, 8\}$ ne, $\{1, 6, 8\}$ ne, $\{1, 7\}$ ps, $\{1, 7, 8\}$ ne, $\{1, 8\}$ ne.

New face(s) containing the vertex 2: $\{2, 3\}$ ne, $\{2, 4\}$ ne, $\{2, 5\}$ ne, $\{2, 6\}$ ne, $\{2, 7\}$ ps, $\{2, 7, 8\}$ ef, storage of $\{2, 7, 8\}$, $\{2, 8\}$ ps.

New face(s) containing the vertex 3: $\{3, 4\}$ ef, $\{3, 4, 5\}$ ne, $\{3, 4, 6\}$ ef, $\{3, 4, 6, 7\}$ ne, $\{3, 4, 6, 8\}$ ne, storage of $\{3, 4, 6\}$, $\{3, 4, 7\}$ ne, $\{3, 4, 8\}$ ne, $\{3, 5\}$ ne, $\{3, 6\}$ ps, $\{3, 6, 7\}$ ne, $\{3, 6, 8\}$ ne, $\{3, 7\}$ ne, $\{3, 8\}$ ne.

New face(s) containing the vertex 4: $\{4, 5\}$ ef, $\{4, 5, 6\}$ ef, $\{4, 5, 6, 7\}$ ne, $\{4, 5, 6, 8\}$ ne, storage of $\{4, 5, 6\}$, $\{4, 5, 7\}$ ne, $\{4, 5, 8\}$ ne, $\{4, 6\}$ ps, $\{4, 6, 7\}$ ne, $\{4, 6, 8\}$ ne, $\{4, 7\}$ ne, $\{4, 8\}$ ne.

New face(s) containing the vertex 5: $\{5, 6\}$ ps, $\{5, 6, 7\}$ ps, $\{5, 6, 7, 8\}$ ne, $\{5, 6, 8\}$ ne, $\{5, 7\}$ ps, $\{5, 7, 8\}$ ne, $\{5, 8\}$ ne.

New face(s) containing the vertex 6: $\{6, 7\}$ ps, $\{6, 7, 8\}$ ne, $\{6, 8\}$ ne.

New face containing the vertex 7: $\{7, 8\}$ ps.

5. Numerical Results

The following examples have been computed in NOS/VE Pascal language on a CYBER 990, and the computing times for the subroutines VERTICES and FACE are respectively denoted by CPU1 and CPU2 (in seconds).

We begin with the nondegenerate case. Consider, for any integer k, the problem:

$$(\text{Tub(k)}) \quad \max \begin{bmatrix} -x/2+y \\ x-y/2 \end{bmatrix},$$

$$\text{s.t.} \quad x\cos(j\pi/2(k-2)) + y\sin(j\pi/2(k-2)) \leq 1,$$

$$j=0, \ldots, k-2,$$

$$z \leq 1,$$

$$x \geq 0, \qquad y \geq 0, \qquad z \geq 0.$$

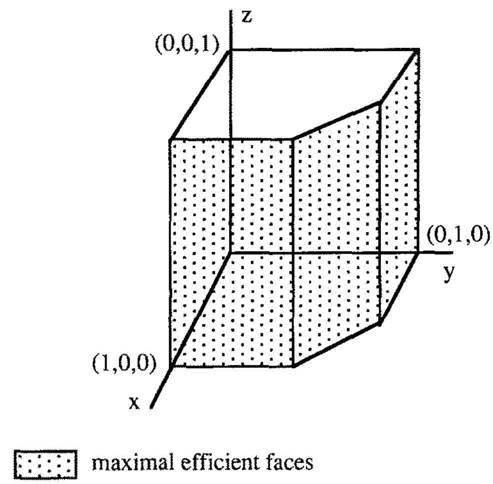

Fig. 4. Constraint polyhedron of the problem Tub (4).

It is easily seen that this problem has $k-1$ efficient faces and $2k$ efficient vertices (Fig. 4). The computing times for different k are displayed in Table 1.

Table 1. CPU times (sec).

Problem	CPU1	CPU2
Tub (20)	0.2	2.5
Tub (30)	0.4	7.4
Tub (40)	0.7	16.5
Tub (50)	1.0	30.8

Now, we consider the case where one vertex is degenerate. For any integer k, we consider the problem:

$$(\text{Pyr(k)}) \quad \max \begin{bmatrix} x-y/2 \\ -x/2+y \\ -x-y+z/2 \end{bmatrix},$$

$$\text{s.t. } x\cos(j\pi/2(k-1))+y\sin(j\pi/2(k-1))+z\leq 1,$$

$$j=0,\ldots,k-1,$$

$$x\geq 0, \qquad y\geq 0, \qquad z\geq 0.$$

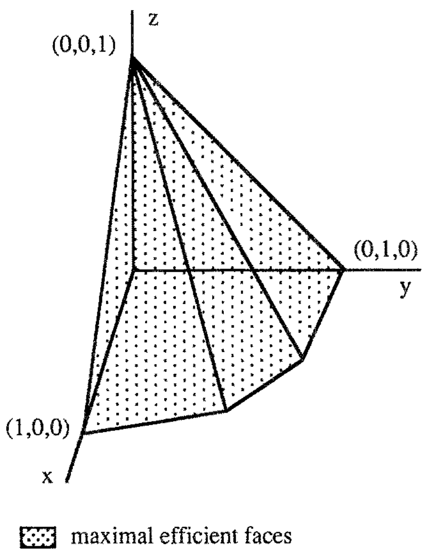

Fig. 5. Constraint polyhedron of the problem Pyr (3).

It is easily seen that this problem has k efficient faces and $k+2$ efficient vertices (Fig. 5). The computing times for several k are displayed in Table 2.

Table 2. CPU times (sec).

	CPU1	CPU2
Pyr (20)	0.4	0.3
Pyr (30)	1.1	0.7
Pyr (40)	2.4	1.3
Pyr (50)	4.3	2.3

To deal with the case where two adjacent vertices are degenerate, consider for any odd integer k the following problem:

$$\text{(Tent(k))} \quad \max \begin{bmatrix} x-100y \\ x+100y \\ z \end{bmatrix},$$

$$\begin{aligned} \text{s.t.}\ & x\cos(j\pi/(k-1))+y\sin(j\pi/(k-1))+z\leq 100 \\ & [1+2\sin(j\pi/(k-1))], \qquad j=0,\ldots,(k-1)/2, \\ & x\cos(j\pi/(k-1))-y\sin(j\pi/(k-1))+z\leq 100 \\ & [1-\sin(j\pi/(k-1))], \qquad j=1,\ldots,(k-1)/2, \\ & x\geq 0, \qquad y\geq 0, \qquad z\geq 0. \end{aligned}$$

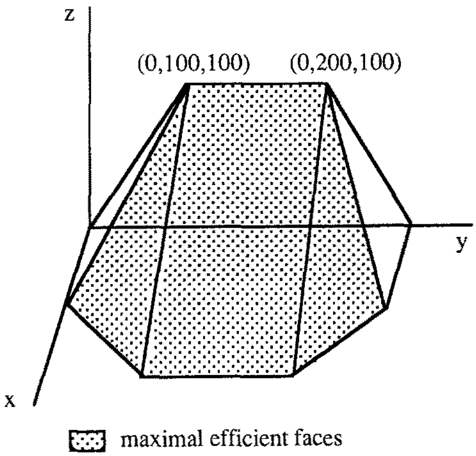

Fig. 6. Constraint polyhedron of the problem Tent (5).

This problem has $k-2$ efficient faces and $k+1$ efficient vertices (Fig. 6). Computing times for several k are displayed in Table 3:

In the literature about the determination of the entire efficient set (see, e.g., Refs. 1, 3, 4, 7, 8, 15), there is no numerical experiment with which to

Table 3. CPU times (sec).

	CPU1	CPU2
Tent (21)	0.4	0.3
Tent (31)	1.0	0.8
Tent (41)	1.6	1.5
Tent (51)	3.8	2.4

compare our method. In Refs. 2 and 3, Evans and Steuer give computing times for the determination of efficient vertices, but it is impossible to draw any comparisons, as their problems are generated randomly.

The following example, introduced by Yu and Zeleny in Ref. 15, and also considered in Ref. 16, has five objectives and eight constraints. The problem is stated as follows:

$$\max \begin{bmatrix} 3 & -7 & 4 & 1 & 0 & -1 & -1 & 8 \\ 2 & 5 & 1 & -1 & 6 & 8 & 3 & -2 \\ 5 & -2 & 5 & 0 & 6 & 7 & 2 & 6 \\ 0 & 4 & -1 & -1 & -3 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{bmatrix},$$

$$\text{s.t.} \begin{bmatrix} 1 & 3 & -4 & 1 & -1 & 1 & 2 & 4 \\ 5 & 2 & 4 & -1 & 3 & 7 & 2 & 7 \\ 0 & 4 & -1 & -1 & -3 & 0 & 0 & 1 \\ -3 & -4 & 8 & 2 & 3 & -4 & 5 & -1 \\ 12 & 8 & -1 & 4 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 8 & -12 & -3 & 4 & -1 & 0 & 0 & 0 \\ 15 & -6 & 13 & 1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{bmatrix} \leq \begin{bmatrix} 40 \\ 84 \\ 18 \\ 100 \\ 40 \\ -12 \\ 30 \\ 100 \end{bmatrix},$$

$$x_1, \ldots, x_8 \geq 0.$$

We have obtained 29 efficient extreme solutions in 0.3 seconds and 18 maximal efficient faces in 5.8 seconds.

6. Conclusions

It has been shown that the entire set of efficient solutions of a multiobjective linear problem can be determined in two steps. At first, we find out the set of efficient extreme points using the connectedness of the graph $\mathscr{L}_{B_0}$ of B_0-dual efficient bases, and then we obtain all the maximal efficient faces by a combinatorial process. As conjectured by Gal in Ref. 17 and Kruse in Ref. 18, we have observed on several examples in the degenerate case, that the number of nodes of $\mathscr{L}_{B_0}$, which depends on B_0, is generally much smaller than the number of dual-efficient bases. The possibility of determining simultaneously the efficient faces and the efficient extreme solutions in order to reduce the combinatorial part of our algorithm will be discussed in a future paper.

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