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A Markov Chain Approximation to Choice Modeling

Jose Blanchet
Industrial Engineering and Operations Research, Columbia University, New York, NY. jose.blanchet@columbia.edu

Guillermo Gallego
Industrial Engineering and Operations Research, Columbia University, New York, NY. ggallego@ieor.columbia.edu

Vineet Goyal
Industrial Engineering and Operations Research, Columbia University, New York, NY. vgoyal@ieor.columbia.edu

Assortment planning is an important problem that arises in many industries such as retailing and airlines. One of the key challenges in an assortment planning problem is to identify the "right" model for the substitution behavior of customers from the data. Error in model selection can lead to highly sub-optimal decisions. In this paper, we consider a Markov chain based choice model and show that it provides a simultaneous approximation for all random utility based discrete choice models including the multinomial logit (MNL), the probit, the nested logit and mixtures of multinomial logit models. In the Markov chain model, substitution from one product to another is modeled as a state transition in the Markov chain. We show that the choice probabilities computed by the Markov chain based model are a good approximation to the true choice probabilities for any random utility based choice model under mild conditions. Moreover, they are exact if the underlying model is a generalized attraction model (GAM) of which the MNL model is a special case. We also show that the assortment optimization problem for our choice model can be solved efficiently in polynomial time. In addition to the theoretical bounds, we also conduct numerical experiments and observe that the average maximum relative error of the choice probabilities of our model with respect to the true probabilities for any offer set is less than 3% where the average is taken over different offer sets. Therefore, our model provides a tractable approach to choice modeling and assortment optimization that is robust to model selection errors. Moreover, the state transition primitive for substitution provides interesting insights to model the substitution behavior in many real-world applications.

Key words: choice modeling; assortment optimization; model selection

1. Introduction

Assortment optimization is an important problem that arises in many industries such as retailing and airlines where the decision maker needs to select an optimal subset of products to offer to maximize the expected revenue. The demand and the revenue of any product depends on the complete set of offered products since customers potentially substitute to an available product if their most preferred product is not available. Such a substitution behavior is captured by a customer choice model that can be thought of as distribution over preference lists (or permutations of products). A customer with a particular preference list purchases the most preferable product that is available (possibly the no-purchase alternative). Therefore, the choice model specifies the probability that a customer selects a particular product for every offer set. One of the key challenges of any assortment planning problem is to find the "right choice model" to describe the substitution behavior when we only observe historical sales data for a small number of assortments. The underlying customer preferences are latent and unobservable.

Many parametric choice models have been extensively studied in the literature in several areas including marketing, transportation, economics and operations management. Typically, the decision maker selects a parametric form for the choice model where the parameters are estimated from the data. The tractability of the parameter estimation and assortment optimization problems are important factors in the model selection. For these tractability reasons, the multinomial logit (MNL) model is one of the most widely used parametric choice model in practice even though the model justifications (for instance, *Independence from Irrelevant Alternatives property* (IIA) property) are not reasonable for many applications. A more complex choice model can capture a richer substitution behavior but leads to increased complexity of the assortment optimization problem and runs the risk of over-fitting the data.

1.1. Our Contributions

In this paper, we present a computationally tractable approach to choice modeling that is robust to model selection errors. Our approach is based on modeling the substitution behavior of customers by state transitions in a Markov chain. In general, the substitution behavior of any customer is captured by his preference list over the products where he selects the most preferred product that is available (possibly the no-purchase alternative). This selection process can be interpreted as sequential transitions from one product to another in the order defined by the preference list until the customer finds an available product.

Markov Chain based Choice Model. Motivated by the above interpretation, we consider a Markov chain based choice model where substitution behavior is modeled as a sequence of state transitions of a Markov chain. In particular, we consider a Markov chain where there is a state for each product including the no-purchase alternative, and model the substitution behavior as follows: a customer arrives in the state corresponding to his most preferable product. If that product is not available, he/she transitions to other product states according to the transition probabilities of the Markov chain. Therefore, the sequential transitions based on the preference list are approximated by Markovian transitions in the Markov chain based choice model.

The Markov chain based choice model is completely specified by the arrival probabilities in each state and the transition probability matrix. We show that both the arrival probabilities to each state and the transition matrix can be computed efficiently from choice probabilities for a small number of assortments ($O(n)$ where n is the number of products). Furthermore, given the arrival probabilities and the transition probabilities, we can efficiently compute the choice probabilities for all assortments for the Markovian substitution model. For any assortment $S \subseteq \mathcal{N} = \{1, \ldots, n\}$, we modify the Markov chain to make all states corresponding to products $j \in S$ as absorbing. Then the limiting distribution over all absorbing states (including the no-purchase alternative) gives us the choice probabilities of all products in S. These can be computed efficiently by solving a system of linear equations.

Approximation Bounds. A natural question that arises is to study how accurately does the Markov chain model approximate the true underlying model. We show that the Markov chain choice model provides a good approximation to any random utility discrete choice models under mild assumptions. The class of models arising from a random utility model is quite general and includes all models that can be expressed as distributions over permutations. This class includes MNL, probit, generalized attraction model (GAM), Nested logit (NL) and mixture of MNL (MMNL) models (see McFadden and Train (2000)). We present lower and upper bounds, related to the spectral properties of the Markov chain, on the ratio of the choice probability computed by the Markov chain model and the true underlying model. These bounds show that the Markov chain model provides a good approximation for all random utility based choice models under very mild assumptions. Furthermore, we show that the Markov chain model is exact if the underlying hidden model is a generalized attraction model (GAM). In other words, if the choice probabilities used to compute the Markov chain model parameters is generated from an underlying GAM, then the choice probability computed by the Markov chain model coincides with the probability given by the GAM model for all products and all assortments. We would like to note that the MNL model is a special case of GAM and therefore, the Markov chain model exactly captures the MNL model as well.

We would like to emphasize that the estimation of the Markov chain only requires the choice probabilities of certain assortments and no additional information about the underlying choice model. Therefore, the Markov chain model circumvents the challenging model selection problem for choice modeling and provides a simultaneous approximation for all random utility based discrete choice models. We would like to note that we present the choice probability approximation bounds for the case of MNL, GAM and MMNL choice models. Any other random utility based model can be approximated as closely as desired by a mixture of MNL models McFadden and Train (2000). Therefore, our approximation bounds extend to general random utility based models. However, the bounds may not be explicitly computable in general.

Assortment Optimization. We show that the assortment optimization problem can be solved optimally in polynomial time for the Markov chain choice model. In an assortment optimization problem, the goal is to find an assortment (or offer set) that maximizes the total expected revenue, i.e.,

$$\max_{S\subseteq\{1,\ldots,n\}} r(S) = \sum_{j\in S} r_j \cdot \pi(j,S),$$

where r_j is the revenue per unit of product j and $\pi(j,S)$ denotes the choice probability of product j when the offer set is S. This result is quite surprising since in the Markov chain based choice model, we can not even express $\pi(j,S)$ as a simple functional form of the model parameters. Therefore, we are not able to even formulate the assortment optimization problem as a mathematical program directly. However, we show that the assortment optimization problem is related to the optimal stopping problem and present a policy iteration algorithm to compute an optimal assortment in polynomial time for the Markov chain based choice model. Moreover, our algorithm shows that the optimal assortment is independent of the arrival rates $\lambda_i, i \in \mathcal{N}$. This provides interesting insights about the structure of the optimal assortment.

Furthermore, we show under mild conditions that if Markov chain model parameters are computed from choice probabilities generated by some underlying latent choice model, then the optimal assortment for the Markov chain model is also a good approximation for the assortment optimization problem over the underlying latent model.

Computational Study. In addition to the theoretical approximation bounds, we present a computational study to compare the choice probability estimates of the Markov chain model as compared with the choice probability of the true model. Since the mixture of MNLs model can approximate any discrete choice model arising from random utility maximization principle as closely as required (McFadden and Train (2000)), we compare the performance of the Markov chain model with respect to MMNL model. In particular, we consider several families of random instances of mixture of MNL models as well as the distribution over permeation models and compare out of sample performance of the Markov chain model with respect to the true model. The numerical experiments show that our model performs extremely well on randomly generated instances of these models and the performance is significantly better than the MNL approximation. We also study the performance of the Markov chain model in the assortment optimization problem for the MMNL model. Our results show that the Markov chain model significantly outperforms the MNL approximation for assortment optimization.

1.2. Related Work

Discrete choice models have been studied very widely in the literature in a number of areas including Transportation, Economics, Marketing and Operations. There are two broad fundamental questions in this area: i) learn the choice model or how people choose and substitute among products, and ii) develop efficient algorithms to optimize assortment or other decisions for a given choice model. The literature in Transportation, Economics and Marketing is primarily focused on the choice model learning problem while the Operations literature is primarily focused on the optimization problem over a given choice model. Since this paper considers both these fundamental problems, we give a brief but broad review of the relevant literature.

A choice model, in the most general setting, can be thought of as a distribution over permutations that arise from preferences. In the random utility model of preferences, each customer has a utility $u_j + \epsilon_j$ for product j where u_j depends on the attributes of product j and ϵ_j is a random idiosyncratic component of the utility distributed according to some unknown distribution. The preference list of the customer is given by the decreasing order of utilities of products. Therefore, the distribution of ϵ_j completely specified the distribution over permutations, and thus, the choice model. This model was introduced by Thurstone (1927) in the early 1900s. A special case of the model is obtained when ϵ_j's are i.i.d according to a normal distribution with mean 0 and variance 1. This is referred to as the *probit model*.

Another very important special case of the above model is obtained assuming ϵ_j's are i.i.d according to an extreme value distribution such as Gumbel. This model also referred to as the Plackett-Luce model and was proposed independently by Luce (1959) and Plackett (1975). It came to be known as the Multinomial logit model (or the MNL model) after McFadden (1973) referred to it as a conditional logit model. Before becoming popular in the Operations literature, the MNL model was extensively used in the areas of transportation (see McFadden (1980), Ben-Akiva and Lerman (1985)), and marketing (see Guadagni and Little (1983) and surveys by Wierenga (2008) and Chandukala et al. (2008)). In the Operations literature, the MNL model is by far the most popular model as both the estimation as well as the optimization problems are tractable for this model. The assortment optimization for the MNL model can be solved efficiently and several algorithms including greedy, local search, and linear programming based methods are known (see Talluri and Van Ryzin (2004), Gallego et al. (2004) and Farias et al. (2011)). However, the MNL model is not able to capture heterogeneity in substitution behavior and also suffers from the Independence from Irrelevant Alternatives (IIA) property (Ben-Akiva and Lerman (1985)). These drawbacks limit the applicability of the MNL model in many practical settings.

More general choice models such as the generalized attraction model (Gallego et al. (2015)), Nested logit model (Williams (1977), McFadden (1978)), and the mixture of MNL models have been studied in the literature to model a richer class of substitution behaviors. These generalizations avoid the IIA property but are still consistent with the random utility maximization principle. Natarajan et al. (2009) consider a semiparametric approach to choice modeling using limited information of the joint distribution of the random utilities such as marginals distributions or marginal moments. However, the corresponding assortment optimization over these more general discrete choice models is not necessarily tractable. For instance, Rusmevichientong et al. (2010) show that the assortment optimization is NP-hard for a mixture of MNL model even for the case of mixture of only two MNL models. Davis et al. (2011) show that the assortment optimization problem is NP-hard for the Nested logit model in general (they give an optimal algorithm for a special case of the model parameters). We refer the readers to surveys (Kök et al. (2009), Lancaster (1990), Ramdas (2003)) for a comprehensive review of the state-of-the-art in assortment optimization under general choice models.

Most of the above work is related to static assortment optimization. The dynamic assortment optimization problem we need to make both assortment and inventory decisions for demand that arrives over multiple periods has also been widely studied in the literature (see Parlar and Goyal (1984), Mahajan and van Ryzin

(2001), Kok and Fisher (2007), Chen and Bassok (2008), Gaur and Honhon (2006)). Smith and Agrawal (2000) and Netessine and Rudi (2003) consider a two-step dynamic substitution model where each customer has only two products in the preference list. If the first choice product is not available, he substitutes to the second-choice product, and leaves without purchasing if both choices are unavailable. Our Markov chain based choice model is a generalization of the two-step substitution model where the transitions continue until the customer finds an available product or ends up in the state of no-purchase alternative.

In much of the above work, the choice model is assumed to be given and the focus is on the optimization problem. In a recent paper, Rusmevichientong and Topaloglu (2012) consider a model where the choice model is uncertain and could be any one of the given MNL models. They show that an optimal robust assortment can be computed in polynomial time. However, they consider uncertainty over model parameters and not over the class of choice models. It is quite challenging to select the appropriate parametric model and model mis-specification error can be costly in terms of performance. McFadden and Train (2000) show that any choice model arising from the random utility model can be approximated as closely as required by a mixture of a finite number of MNL models. This result implies that we can focus only on the class of mixture of MNL models for the model selection problem. However, even then, the number of mixtures in the model is not known.

In this paper, we consider a different approach to choice modeling that circumvents this model selection problem. The work by Farias et al. (2012) and van Ryzin and Vulcano (2011) are most closely related to this paper. Farias et al. (2012) consider a non-parametric approach where they use the distribution over permutations with the sparsest support that is consistent with the data. Interestingly, they show that if a certain 'signature condition' is satisfied, the distribution with the sparsest support can be computed efficiently. However, the resulting assortment optimization problem can not be solved efficiently for the sparsest support distribution. van Ryzin and Vulcano (2011) consider an iterative expectation maximization algorithm to learn a non-parametric choice model where in each iteration they add a new MNL to the mixture model. However, optimization over mixture of MNLs is NP-hard (Rusmevichientong et al. (2010)). Zhang and Cooper (2005) consider a Markov chain based choice model similar to the one introduced in the paper for an airline revenue management problem and present a simulation study. However, the paper does not present any theoretical analysis for the model.

Outline. The rest of the paper is organized as follows. In Section 2, we present the Markov chain based choice model. In Section 3, we show that this model is exact if the underlying choice model is GAM. In Section 4, we present approximations bounds for the choice probability estimates computed by the Markov chain model for general choice models. In Section 5, we consider the assortment optimization problem and present an optimal algorithm for our Markov chain based choice model. In Section 6, we present results from our computation study.

2. Markov Chain Based Choice Model

In this section, we present the Markov chain based choice model. We denote the universe of n products by the set $\mathcal{N} = \{1, 2, \ldots, n\}$ and the outside or no-purchase alternative by product 0. For any $S \subseteq \mathcal{N}$, let S_+ denote the set of items including the no-purchase alternative, i.e., $S_+ = S \cup \{0\}$. And for any $j \in S_+$, let $\pi(j, S)$ denote the choice probability of item $j \in S_+$ for offer set, S.

We consider a Markov chain $\mathcal{M}$ to model the substitution behavior using Markovian transitions in $\mathcal{M}$. There is a state corresponding to each product in $\mathcal{N}_+$ including a state for the no-purchase alternative 0. A customer with a random preference list is modeled to arrive in the state corresponding to the most preferable product. Therefore, for any $i \in \mathcal{N}_+$, a customer arrives in state i with probability $\lambda_i = \pi(i, \mathcal{N})$ and selects product i if it is available. Otherwise, the customer transitions to a different state $j \neq i$ (including the state corresponding to the no-purchase alternative) with probability ρ_{ij} that can be estimated from the data. After transitioning to state j, the customer behaves exactly like a customer whose most preferable product is j. He selects j if it is available or continues the transitions otherwise. Therefore, we approximate the linear substitution arising from a preference list by a Markovian transition model where transitions out of state i do not depend on the previous transitions. The model is completely specified by initial arrival probabilities λ_i for all states $i \in \mathcal{N}_+$ and the transition probabilities ρ_{ij} for all $i \in \mathcal{N}$, $j \in \mathcal{N}_+$. Note that for every state $i \in \mathcal{N}$, there is a probability of transitioning to state 0 corresponding to the no-purchase alternative in which case, the customer leaves the system. For any $j \in \mathcal{N}_+$, we use j to refer to both product j and the state corresponding to the product j in the Markov chain $\mathcal{M}$.

2.1. Computing Choice Model Parameters

The arrival probabilities, λ_i for all $i \in \mathcal{N}_+$ can be interpreted as the arrival rate of customers who prefer i when everything is offered. The transition probability ρ_{ij}, for $i \in \mathcal{N}, j \in \mathcal{N}_+$ is the probability of substituting to j from i given that product i is the most preferable but is not available. We can compute these probabilities from the choice probabilities of products for a small number of assortments.

Suppose we are given the choice probabilities for the following $(n+1)$ assortments,

$$\mathcal{S} = \{\mathcal{N}, \mathcal{N} \setminus \{i\} \mid i = 1, \ldots, n\}. \tag{2.1}$$

We compute the arrival probabilities, λ_i and transition probabilities, ρ_{ij} for all $i \in \mathcal{N}, j \in \mathcal{N}_+$ as follows.

$$\lambda_i = \pi(i, \mathcal{N}), \text{ and } \rho_{ij} = \begin{cases} 1, & \text{if } i = 0, j = 0 \\ \dfrac{\pi(j, \mathcal{N} \setminus \{i\}) - \pi(j, \mathcal{N})}{\pi(i, \mathcal{N})}, & \text{if } i \in \mathcal{N}, j \in \mathcal{N}_+, i \neq j \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Note that $\pi(i, \mathcal{N})$ is exactly the fraction of customers whose most preferable product is i. For all $i \in \mathcal{N}, j \in \mathcal{N}_+$, $i \neq j$, the numerator in the definition of ρ_{ij}, $\delta_{ij} = \pi(j, \mathcal{N} \setminus \{i\}) - \pi(j, \mathcal{N})$ is the increase in probability of selecting j when i is removed from the assortment. Therefore, we can interpret the definition of ρ_{ij} as the conditional probability that a customer substitutes from product i to product j given that product i is the most preferable product but is not offered. This is consistent with the Markov chain interpretation. For the true model, the transition probability ρ_{ij} depends on the offer set S. But here, we approximate the substitution behavior as a sequence of Markovian transitions.

In (2.2), we assume that accurate choice probability data for $(n+1)$ assortments $\mathcal{S}$ (as defined in (2.1)) is available. However, such data may not always be available in practice. Typically, we have access to only partial and noisy choice probability data for an arbitrary collection of assortments and we can not compute the Markov chain parameters using (2.2). Therefore, it is important to study robust statistical estimation procedures for directly estimating the Markov chain model parameters from noisy partial data. However, our main focus in this paper is to introduce the Markov chain based choice model, analyze its performance in modeling random utility based discrete choice models and study the corresponding assortment optimization problem. We leave the study of robust statistical estimation procedures from data for future work.

We would like to note that the Markov chain choice model is also useful in cases when the true choice model is known but the corresponding assortment optimization problem is hard (for instance, in the case of the MMNL model). Since the Markov chain model provides a good approximation of most discrete choice models and the corresponding assortment optimization problem can be solved efficiently, it can be useful to use the Markov chain model instead of the known true choice model. In this case, we can easily compute the parameters of the Markov chain model using (2.2). For instance, if the true model is a mixture of MNL model, then we compute the Markov chain parameters as in (2.5). Farias et al. (2012) estimate a distribution over permutations model with a sparse support from choice probability data under fairly general conditions, but the resulting assortment problem is hard. We can compute the Markov chain parameters from a given distribution over permutations model using (2.6).

2.2. Examples

We discuss the examples of MNL, GAM, mixture of MNL and distribution over permutations choice models and compute the parameters of the Markov chain model from the choice probabilities for $(n+1)$ assortments $\mathcal{S} = \{\mathcal{N}, \mathcal{N} \setminus \{i\} \mid i = 1, \ldots, n\}$.

Multinomial logit (MNL) model. Suppose the underlying choice model is MNL given by utility parameters $u_0, \ldots, u_n$. We can assume without loss of generality that $u_0 + \ldots + u_n = 1$. For any $S \subseteq \mathcal{N}$, $j \in S_+$,

$$\pi(j, S) = \frac{u_j}{\sum_{i \in S_+} u_i}.$$

From (2.2), the parameters λ_i and ρ_{ij} for all $i \in \mathcal{N}, j \in \mathcal{N}_+, i \neq j$, can be computed as follows.

$$\begin{aligned} \lambda_i &= \pi(i, \mathcal{N}) = u_i \\ \rho_{ij} &= \frac{\pi(j, \mathcal{N} \setminus \{i\}) - \pi(j, \mathcal{N})}{\pi(i, \mathcal{N})} = \frac{u_j}{1 - u_i}. \end{aligned} \tag{2.3}$$

Note that for all $i \neq j$, ρ_{ij} is exactly the choice probability of product j for offer set $\mathcal{N} \setminus \{i\}$. Furthermore, for any state i, the transition probability, ρ_{ij} to state j, for any $j \neq i$ is proportional to u_j. This property preserves the structure of MNL model in the Markov chain approximation. In fact, we can show that our Markov chain approximation is exact if the underlying choice model is MNL and more generally for the generalized attraction model (GAM) described below.

Generalized attraction model (GAM). The generalized attraction model, introduced in Gallego et al. (2015) is given by $(2n+1)$ parameters: $u_0, \ldots, u_n$ and $v_1, \ldots, v_n$ where for any $i = 1, \ldots, n$, v_i represents the increase in the no-purchase utility if product i is not offered in the assortment. We assume that $0 \leq v_i \leq u_i$ for all $i \in \mathcal{N}$ and $u_0 + \ldots + u_n = 1$. For any $S \subseteq \mathcal{N}$, $j \in S$, the choice probability

$$\pi(j, S) = \frac{u_j}{u_0 + \sum_{i \notin S} v_i + \sum_{i \in S} u_i)}.$$

For all $i \in \mathcal{N}$, let $\theta_i = v_i/u_i$. Note that the MNL model is a special case of GAM as we recover the MNL model by setting $v_i = 0$ for all i. The independent demand model is also a special case of GAM and can be recovered by setting $v_i = u_i$ for all i. From (2.2), the parameters λ_i and ρ_{ij} for all $i \in \mathcal{N}, j \in \mathcal{N}_+, i \neq j$, can be computed as follows.

$$\begin{aligned} \lambda_i &= \pi(i, \mathcal{N}) = u_i \\ \rho_{i0} &= \frac{u_0(1-\theta_i) + \theta_i}{1 - u_i(1-\theta_i)} \\ \rho_{ij} &= \frac{u_j(1-\theta_i)}{1 - u_i(1-\theta_i)},\ j \neq 0. \end{aligned} \tag{2.4}$$

Similar to the case of MNL model, the transition probability from any state $i \in \mathcal{N}$ to any state $j \in \mathcal{N}$ ($j \neq 0$), ρ_{ij} is proportional to u_j. Utilizing the structure of the transition probability matrix, we show that the Markov chain approximation is exact for the generalized attraction model (GAM).

Mixture of MNL model. Suppose the underlying model is a mixture of K MNL models. Let $\theta_k, k = 1, \ldots, K$ ($\theta_1 + \ldots + \theta_K = 1$) denote the probability that a random customer belongs to segment k that is given by the MNL model parameters: $u_{jk}, j = 0, \ldots, n \in \mathbb{R}_+$ such that $\sum_{j=0}^{n} u_{jk} = 1$ for all $k = 1, \ldots, K$. For any assortment $S \subseteq \mathcal{N}$, the choice probability of any $j \in S_+$ is given by

$$\pi(j, S) = \sum_{k=1}^{K} \theta_k \cdot \frac{u_{jk}}{\sum_{i \in S_+} u_{ik}} = \sum_{k=1}^{K} \theta_k \cdot \pi_k(j, S),$$

where $\pi_k(j,S)$ is the choice probability of product j for offer set S for segment k. Now, the parameters for the Markov chain can be computed as follows.

$$\begin{aligned}
\lambda_i &= \sum_{k=1}^{K} \theta_k \cdot u_{ik} = \pi(i,\mathcal{N}) \\
\rho_{ij} &= \sum_{k=1}^{K} \theta_k \cdot \frac{(\pi_k(j,\mathcal{N}\setminus\{i\}) - \pi_k(j,\mathcal{N}))}{\pi(i,\mathcal{N})} \\
&= \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i,\mathcal{N})} \cdot \frac{u_{jk}}{1-u_{ik}}.
\end{aligned} \tag{2.5}$$

To interpret the above expression for ρ_{ij}, note that for any $k=1,\ldots,K$,

$$P(\text{ customer in segment } k \mid \text{ first choice is } i) = \frac{\theta_k u_{ik}}{\pi(i,\mathcal{N})}, \text{ and } \pi_k(j,\mathcal{N}\setminus\{i\}) = \frac{u_{jk}}{1-u_{ik}}.$$

Therefore, for any $i \in \mathcal{N}, j \in \mathcal{N}_+$, we can interpret ρ_{ij} as the conditional probability that a random customer selects j in $\mathcal{N}\setminus\{i\}$ given that the first choice is product i.

Distribution over Permutations Model. Suppose we are given a distribution over a set of permutations $\sigma_1,\ldots,\sigma_K$ in S_{n+1}, i.e., for each permutation, σ_k, $k=1,\ldots,K$, we are given the probability, α_k that the customer's preference list is given by σ_k. We can compute the parameters of the Markov chain as follows.

$$\begin{aligned}
\lambda_i &= \sum_{k:\sigma_k(1)=i} \alpha_k \\
\rho_{ij} &= \sum_{k=1}^{K} \alpha_k \cdot \mathbb{1}(\sigma_k(1)=i,\, \sigma_k(2)=j),
\end{aligned} \tag{2.6}$$

where $\mathbb{1}(\cdot)$ denotes the indicator function. Note that the transition probabilities in (2.6) only depend on the distribution of first choice and second choice products for customers. If we have an explicit description of the distribution over permutations model and we have access to the complete preference lists of the customers, then an alternate Markov chain approximation for the distribution over permutation model can be computed as follows.

$$\begin{aligned}
\lambda_i &= \sum_{k:\sigma_k(1)=i} \alpha_k \\
\rho_{ij} &= \sum_{k=1}^{K} \alpha_k \cdot \mathbb{1}(\exists \ell \in [n],\, \sigma_k(\ell)=i,\, \sigma_k(\ell+1)=j).
\end{aligned} \tag{2.7}$$

In the above computation, we define ρ_{ij} as the total probability of all permutations where i and j are consecutive and i appears before j. This corresponds to the probability that a random customer transitions to state j given he is at state i in the distribution over permutation model. We would like to note that the Markov chain parameter computation in (2.7) is different from (2.6) (or (2.2)) and we do not analyze the approximation bounds for choice probabilities if the parameters are computed using (2.7). However, we compare the computational performance of both the Markov chain approximations of the distribution over permutations model and present the results in Section 6.

2.3. Computation of Choice Probabilities

Given the parameters λ_i and ρ_{ij} for all $i \in \mathcal{N}_+, j \in \mathcal{N}$, let us now describe how we compute the choice probabilities for any $S \subseteq \mathcal{N}, j \in S_+$. Our primitive for the substitution behavior is that a customer arrives in state i with probability λ_i, and continues to transition according to probabilities ρ_{ij} until he reaches a state corresponding to a product in S_+. We make the following assumption about the transition probability matrix, $\boldsymbol{\rho}(\mathcal{N},\mathcal{N})$ that denotes the transition probabilities from states $\mathcal{N}$ to $\mathcal{N}$ excluding state 0.

ASSUMPTION 2.1. *The spectral radius of the transition sub-matrix $\boldsymbol{\rho}(\mathcal{N},\mathcal{N})$ is strictly less than one.*

This assumption is satisfied for any natural choice model. In particular, in any choice model for an empty offer set, all the demand is absorbed in state 0 and there is no transient demand in non-absorbing states. This implies that the spectral radius of $\boldsymbol{\rho}(\mathcal{N},\mathcal{N})$ is strictly less than one for any natural choice model without loss of generality.

Now, for any offer set $S \subseteq \mathcal{N}$, we define a Markov chain, $\mathcal{M}(S)$ over the state space, $\mathcal{N}_+$. Since each state $i \in S_+$ is an absorbing state in $\mathcal{M}(S)$, we modify the transition probabilities, $\rho_{ij}(S)$ as follows.

$$\rho_{ij}(S) = \begin{cases} 0 & \text{if } i \in S, j \neq i \\ 1 & \text{if } i \in S, j = i \\ \rho_{ij} & \text{otherwise} \end{cases} \tag{2.8}$$

Note that there are no transitions out of any state $i \in S_+$. The choice probability, $\hat{\pi}(j,S)$, for any $j \in S_+$ can be computed as the probability of absorbing in state j in $\mathcal{M}(S)$. In particular, we have the following theorem.

THEOREM 2.1. *Suppose the parameters for the Markov chain model are given by λ_i, ρ_{ij} for all $i \in \mathcal{N}, j \in \mathcal{N}_+$. For any $S \subseteq \mathcal{N}$, let $\boldsymbol{B} = \boldsymbol{\rho}(\bar{S}, S_+)$ denote the transition probability sub-matrix from states $\bar{S} = \mathcal{N} \setminus S$ to S_+, and $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$ denote the transition sub-matrix from states in $\bar{S}$ to $\bar{S}$. Then for any $j \in S_+$,*

$$\hat{\pi}(j,S) = \lambda_j + (\boldsymbol{\lambda}(\bar{S}))^T (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{B} \boldsymbol{e}_j, \tag{2.9}$$

where $\boldsymbol{\lambda}(\bar{S})$ is the vector of arrival probabilities in $\bar{S}$ and e_j is the j^{th} unit vector.

Proof The transition probability matrix for Markov chain $\mathcal{M}(S)$ where states in S_+ are absorbing is given (after permuting the rows and columns appropriately) by

$$\mathcal{P}(S) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix}. \tag{2.10}$$

Here the first $|S|+1$ rows and columns of $\mathcal{P}(S)$ correspond to states in S_+ and the remaining rows and columns correspond to states in $\bar{S}$. Let $p = |S|$. Then, $\boldsymbol{I}$ is a $(p+1) \times (p+1)$ identity matrix, $\boldsymbol{B} \in \mathbb{R}_+^{(n-p)\times(p+1)}$ and $\boldsymbol{C} \in \mathbb{R}_+^{(n-p)\times(n-p)}$. For any $j \in S_+$, the choice probability estimate $\hat{\pi}(j,S)$ can be computed as follows.

$$\hat{\pi}(j,S) = \lim_{q\to\infty} \boldsymbol{\lambda}^T (\mathcal{P}(S))^q \boldsymbol{e}_j = \lim_{q\to\infty} \boldsymbol{\lambda}^T \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \left(\sum_{j=0}^{q} \boldsymbol{C}^j\right) \boldsymbol{B} & \boldsymbol{C}^q \end{bmatrix} \boldsymbol{e}_j = \boldsymbol{\lambda}^T \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \boldsymbol{e}_j. \tag{2.11}$$

Note that $\boldsymbol{C}$ is a principal sub-matrix of $\boldsymbol{\rho}(\mathcal{N},\mathcal{N})$ that has spectral radius is strictly less than one (Assumption 2.1). Therefore, the spectral radius of $\boldsymbol{C}$ is also strictly less than one and

$$\sum_{j=0}^{\infty} \boldsymbol{C}^j = (\boldsymbol{I}-\boldsymbol{C})^{-1}, \; \lim_{q\to\infty} \boldsymbol{C}^q = \boldsymbol{0}.$$

Therefore,

$$\hat{\pi}(j,S) = \lambda_j + (\boldsymbol{\lambda}(\bar{S}))^T (\boldsymbol{I}-\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{e}_j,$$

where $\boldsymbol{e}_j$ is the j^{th} unit vector in $\mathbb{R}^{p+1}$. Note that $\boldsymbol{Y} = (\boldsymbol{I}-\boldsymbol{C})^{-1}\boldsymbol{B}$ is a $|\bar{S}|\times|S|$ matrix where for any $i\in\bar{S}$, $j\in S_+$, Y_{ij} denotes the resulting probability of absorbing in state j conditional on arrival in state i. In other words, Y_{ij} is the probability of substituting to product j given that product i is the most preferable product but is not available in offer set S. □

Therefore, the parameters λ_i and ρ_{ij} for $i\in\mathcal{N}_+, j\in\mathcal{N}$, give a compact representation of choice probabilities for all offer sets and we can efficiently compute the choice probability for any j,S. However, unlike several commonly used parametric choice models such as MNL, GAM, Nested logit and mixture of MNL, we do not have a simple functional form for the choice probabilities as a function of the parameters.

3. Generalized Attraction Model (GAM) and Markov Chain Model

We show that the Markov chain model is exact if the underlying choice model is GAM. In other words, if the parameters λ_i, ρ_{ij} for all $i\in\mathcal{N}$, $j\in\mathcal{N}_+$ are computed using choice probability data from GAM, then for any assortment $S\subseteq\mathcal{N}$, the choice probability for any $j\in S_+$ computed using the Markov chain model is exactly equal to the choice probability given by the underlying GAM. We formalize this in the theorem below.

THEOREM 3.1. *Suppose the underlying model is GAM for the given choice probabilities,* $\pi(j,S)$ *for all* $S\in\{\mathcal{N},\mathcal{N}\setminus\{i\}, i=1,\ldots,n\}$. *Then for all* $S\subseteq\mathcal{N}, j\in S_+$,

$$\hat{\pi}(j,S) = \pi(j,S),$$

where $\hat{\pi}(j,S)$ *is the choice probability computed by the Markov chain model* (2.9) *and* $\pi(j,S)$ *is true choice probability given by the underlying GAM.*

Proof Suppose the parameters for the underlying GAM are $u_0,\ldots,u_n>0$, $v_1,\ldots,v_n$ such that $u_0+\ldots+u_n=1$ and $0\le v_i\le u_i$ for all $i\in\mathcal{N}$. Let $\theta_i = v_i/u_i$ for all $i\in\mathcal{N}$. The exactness of GAM depends on the structure of transition probabilities ρ_{ij} from state $i\in\mathcal{N}$ to $j\neq i$. From (2.4), we know that for any $i\in\mathcal{N}$,

$$\begin{aligned}
\lambda_i &= u_i\\
\rho_{i0} &= \frac{u_0(1-\theta_i)+\theta_i}{1-u_i(1-\theta_i)}\\
\rho_{ij} &= \frac{u_j(1-\theta_i)}{1-u_i(1-\theta_i)}, \; j\neq 0, i.
\end{aligned}$$

Consider an alternate Markov chain $\hat{\mathcal{M}}$ with the transition matrix $\hat{\rho}$ where for any $i \in \mathcal{N}$,

$$\begin{aligned}\hat{\rho}_{i0} &= u_0(1-\theta_i)+\theta_i \\ \hat{\rho}_{ij} &= u_j(1-\theta_i),\ \forall j \neq 0.\end{aligned} \tag{3.1}$$

Note that $\hat{\rho}_{ii} = u_i(1-\theta_i)$ for all $i \in \mathcal{N}$ (unlike the original Markov chain where $\rho_{ii}=0$). In the new Markov chain $\hat{\mathcal{M}}$, for any $i,j \in \mathcal{N}$, $i \neq j$, the probability of visiting state j starting from state i before visiting any other state $k \neq i,j$ is given by

$$\sum_{q=0}^{\infty} (\hat{\rho}_{ii})^q \hat{\rho}_{ij} = \sum_{q=0}^{\infty} (u_i(1-\theta_i))^q u_j(1-\theta_i) = \frac{u_j(1-\theta_i)}{1-u_i(1-\theta_i)} = \rho_{ij}.$$

Moreover, the probability of reaching state 0 starting from state i before visiting any other state $k \neq i, 0$ is

$$\sum_{q=0}^{\infty} (\hat{\rho}_{ii})^q \hat{\rho}_{i0} = \sum_{q=0}^{\infty} (u_i(1-\theta_i))^q (u_0(1-\theta_i)+\theta_i) = \frac{u_0(1-\theta_i)+\theta_i}{1-u_i(1-\theta_i)} = \rho_{i0}.$$

This proves that the two Markov chains, $\mathcal{M}$ and $\hat{\mathcal{M}}$ are equivalent, i.e., the limiting distributions of the two Markov chains for any subset S of absorbing states are the same. Note that the transition probability matrix of $\hat{\mathcal{M}}$ over the states $\mathcal{N}$ has rank 1.

Now, consider any offer set $S \subseteq \mathcal{N}$. For ease of notation, let $S = \{1,\ldots,p\}$ for some $p \leq n$. Now, $\hat{\pi}(j,S)$ is the absorption probability of state j in Markov chain $\mathcal{M}(S)$ where the states corresponding to S_+ are absorbing states. Since the Markov chains, $\mathcal{M}$ and $\hat{\mathcal{M}}$ are equivalent with respect to limiting distributions, we can compute the absorption probability of state j in Markov chain $\hat{\mathcal{M}}(S)$ where states in S_+ are absorbing. Let $\bar{S} = \mathcal{N} \setminus S$ and let $\boldsymbol{B} = \hat{\boldsymbol{\rho}}(\bar{S}, S)$, $\boldsymbol{C} = \hat{\boldsymbol{\rho}}(\bar{S}, \bar{S})$. From Theorem 2.1, for any $j \in S$, the choice probability given by the Markov chain model,

$$\hat{\pi}(j,S) = \pi(j,\mathcal{N}) + \boldsymbol{\lambda}(\bar{S})^T(\boldsymbol{I}-\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{e}_j = \pi(j,\mathcal{N}) + \sum_{q=0}^{\infty} \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{C}^q \boldsymbol{B}\boldsymbol{e}_j.$$

Let

$$u(\bar{S}) = \sum_{i \in \bar{S}} u_i,\ v(\bar{S}) = \sum_{i \in \bar{S}} v_i.$$

Note that $\boldsymbol{\lambda}(\bar{S})$ is a left eigenvector of $\boldsymbol{C}$ with eigenvalue $(u(\bar{S})-v(\bar{S}))$ as

$$\boldsymbol{\lambda}(\bar{S})^T \boldsymbol{C} = (u(\bar{S})-v(\bar{S}))\boldsymbol{\lambda}(\bar{S})^T.$$

Therefore,

$$\begin{aligned}\hat{\pi}(j,S) &= \pi(j,\mathcal{N}) + \sum_{q=0}^{\infty} \left(u(\bar{S})-v(\bar{S})\right)^q \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{B}\boldsymbol{e}_j \\ &= u_j + \frac{1}{1-u(\bar{S})+v(\bar{S})} \cdot \sum_{i \in \bar{S}} u_i \hat{\rho}_{ij} \\ &= u_j + \frac{1}{1-u(\bar{S})+v(\bar{S})} \cdot u_j(u(\bar{S})-v(\bar{S})) \\ &= \frac{u_j}{u_0 + \sum_{i \in S} u_i + \sum_{i \in \bar{S}} v_i},\end{aligned}$$

which is equal to $\pi(j,S)$ for GAM. □

Since the MNL model is a special case of GAM with $v_i = 0$ for all $i \in \mathcal{N}$, we have the following corollary.

COROLLARY 3.1. *Suppose the underlying model is MNL for the given choice probabilities, $\pi(j,S)$ for all $S \in \{\mathcal{N}, \mathcal{N} \setminus \{i\}, i = 1,\dots,n\}$. Then for all $S \subseteq \mathcal{N}, j \in S_+$,*

$$\hat{\pi}(j,S) = \pi(j,S),$$

where $\hat{\pi}(j,S)$ is the choice probability computed by the Markov chain model (2.9) *and $\pi(j,S)$ is true choice probability given by the underlying MNL model.*

The proof of Theorem 3.1 shows that GAM can be represented exactly by a Markov chain where the transition sub-matrix over states $\mathcal{N}$ is a rank one matrix. In particular, we can represent GAM using the transition matrix, $\hat{\boldsymbol{\rho}}$ as defined in (3.1) where $\hat{\boldsymbol{\rho}}(\mathcal{N},\mathcal{N}) = \boldsymbol{\beta}\boldsymbol{u}^T$ is rank one (with $\beta_i = (1-\theta_i)$ for all $i \in \mathcal{N}$). However, we would like to note that not every rank one transition sub-matrix over $\mathcal{N}$ represents a GAM. In the case of GAM, $\boldsymbol{\lambda} = \boldsymbol{u}$ and $\boldsymbol{\rho}(\mathcal{N},\mathcal{N}) = \boldsymbol{\beta}\boldsymbol{u}^T$ (where $\boldsymbol{\beta}$ is a non-negative vector with $\beta_i = (1-\theta_i)$). For general $\boldsymbol{\lambda}$ and rank one transition matrix $\boldsymbol{\rho}(\mathcal{N},\mathcal{N})$, the Markov chain model does not necessarily represent a GAM.

4. General Choice Models

In this section, we discuss the performance of the Markov chain based model for general choice models. From the previous section, we know that the model is exact if the underlying choice model is MNL (or more generally GAM). But this is not the case for general discrete choice models. In fact, we can construct pathological instances where the two probabilities are significantly different (see Section 4.2). However, under a fairly general assumption, we show that the Markov chain model is a good approximation for all random utility based discrete choice model. Furthermore, we also show that we can compute a good approximation to the optimal assortment problem for the true model using the Markov chain model.

4.1. Bounds for Markov chain approximation for general choice models

To prove such an approximation result, we need to show that for every discrete choice model, the Markov chain model computes good choice probability estimates if the choice probability data arises from that choice model. The following theorem from McFadden and Train (2000) allows us to restrict to proving our result for the mixture of multinomial logit (MMNL) model.

THEOREM 4.1 **(McFadden and Train (2000))**. *Consider any random utility based discrete choice model where utility of a product j for customer i depends on the product features, customer characteristics and random idiosyncratic components that depend on both. Each customer selects the product with the highest random utility. Let $\pi(j,S)$ denote the choice probability of product j for offer set S based on the random* *utility maximization. Then for any $\epsilon > 0$, there exists a mixture of multinomial logit model (MMNL) such that*

$$|\pi(j,S) - \pi_{\mathsf{MMNL}}(j,S)| \leq \epsilon,\ \forall j \in S,\ \forall S \subseteq \mathcal{N},$$

where $\pi_{\mathsf{MMNL}}(j,S)$ is the choice probability given by the MMNL model for product j and offer set S.

The above theorem is an existential result that shows that for any random utility model and error bound $\epsilon > 0$, there exists a MMNL model that approximates all choice probabilities within an additive error of ϵ. For more details, see Theorem 1 in McFadden and Train (2000). Using the above theorem, it suffices to prove that the Markov chain model is a good approximation for the mixture of multinomial logit (MMNL) model with an arbitrary number of segments. Consider a MMNL model given by a mixture of K multinomial logit models. Using the same notation as before, let θ_k, $k = 1, \ldots, K$ denote the probability that a random customer belongs to the MNL model k (or segment k) and $u_{jk}, j = 0, \ldots, n$ denote the utility parameters for segment k such that $\sum_{j=0}^{n} u_{jk} = 1$, and $u_{0k} > 0$. Also, for any $k = 1, \ldots, K$ and any $S \subseteq \mathcal{N}_+$, let

$$u_k(S) = \sum_{j \in S} u_{jk}.$$

Let $\bar{S} = \mathcal{N} \setminus S$. The choice probability $\pi(j,S)$ for any offer set S and $j \in S_+$ for the mixture of MNLs model can be expressed as follows.

$$\begin{aligned} \pi(j,S) &= \sum_{k=1}^{K} \theta_k \cdot \frac{u_{jk}}{1 - \sum_{i \in \bar{S}} u_{ik}} \\ &= \sum_{k=1}^{K} \theta_k u_{jk} \cdot \left(1 + \sum_{q=1}^{\infty} \left(u_k(\bar{S})\right)^q\right), \end{aligned} \tag{4.1}$$

where the last equality follows from the fact that $u_k(\bar{S}) < 1$ since $u_{0k} > 0$ for all $k = 1, \ldots, K$. For any $S \subseteq \mathcal{N}$, let

$$\alpha(S) = \max_{k=1}^{K} \sum_{i \in S} \pi_k(i, \mathcal{N}) = \max_{k=1}^{K} \sum_{i \in S} u_{ik}, \tag{4.2}$$

where $\pi_k(i,\mathcal{N})$ is the probability that a random customer from segment k selects $i \in S$ when the offer set is $\mathcal{N}$. Therefore, $\alpha(S)$ is the maximum probability that the most preferable product for a random customer from any segment $k = 1, \ldots, K$ belongs to S. Also, for any $S \subseteq \mathcal{N}$, let

$$\tau(S) = \min\{\kappa \mid \boldsymbol{v} \leq \boldsymbol{\lambda}(S) \leq \kappa \cdot \boldsymbol{v},\ \boldsymbol{v} \text{ is a top left eigenvector of } \boldsymbol{\rho}(S,S)\}. \tag{4.3}$$

Note that $\tau(S)$ intuitively is a measure of geometric distance between $\boldsymbol{\lambda}(S)$ and the direction of the top left eigenvector of $\boldsymbol{\rho}(S,S)$, where $\tau(S)$ is small if v and $\boldsymbol{\lambda}(S)$ are nearly parallel and large otherwise. For any $S \subseteq \mathcal{N}$, $\boldsymbol{\rho}(S,S)$ is the transition sub-matrix from states S to S. Note that $\boldsymbol{\rho}(S,S)$ is irreducible since all off-diagonal entries are greater than 0. Therefore, from the Perron-Frobenius theorem (see Theorem 1.4.4

in Bapat and Raghavan (1997)), the top eigenvalue is real and positive, and the corresponding eigenvector, $\boldsymbol{v} > \mathbf{0}$ which implies that $\tau(S)$ is finite. We prove lower and upper bounds on the relative error between the Markov chain choice probability, $\hat{\pi}(j,S)$, and $\pi(j,S)$ that depend on $\alpha(\bar{S})$ and $\tau(\bar{S})$. In particular, we prove the following theorem.

THEOREM 4.2. *For any $S \subseteq \mathcal{N}$, $j \in S_+$, let $\hat{\pi}(j,S)$ be the choice probability computed by the Markov chain model, and $\pi(j,S)$ be the true choice probability given by the mixture of MNL model. Then,*

$$\left(1-\alpha(\bar{S})^2\right) \cdot \pi(j,S) \leq \hat{\pi}(j,S) \leq \left(1+\frac{\tau(\bar{S})\cdot\alpha(\bar{S})}{1-\alpha(\bar{S})}\right)\cdot \pi(j,S).$$

If the offer set S is sufficiently large, then $\alpha(\bar{S})$ would typically be small and we get a sharp lower bound for $\hat{\pi}(j,S)$ with respect to the true probability $\pi(j,S)$. For instance, if $\alpha(\bar{S}) = 1/4$, then for any $j \in S_+$, the bounds in Theorem 4.2 imply

$$\hat{\pi}(j,S) \geq 0.94 \cdot \pi(j,S).$$

If in addition, $\tau(\bar{S})$ is also small, we also get sharp upper bounds. We can also interpret the bounds as follows: if the customers are able to substitute to alternatives that are close to their original choice (which happens if the offer set is sufficiently large), the Markov chain model provides a good approximation for the choice probability of the true underlying model. However, the bounds get worse as $\alpha(\bar{S})$ increases or the size of offer set S decreases. It is reasonable to expect this degradation for the following reason. We compute the transition probability parameters from choice probabilities of different products for offer sets $\mathcal{N}$ and $\mathcal{N} \setminus \{i\}$ for $i \in \mathcal{N}$ and make a Markovian assumption for the substitution behavior. For smaller offer sets (say of size $o(n)$), the number of state transitions for a random demand before reaching an absorbing state is large with high probability. Therefore, the error from the Markovian assumption gets worse.

The approximation bounds in Theorem 4.2 hold for the MMNL choice model. We can extend the bounds for general random utility based models using Theorem 4.1. However, we would like to note that Theorem 4.1 is an existential result and therefore, the approximation bounds may not be explicitly computable for general models.

To prove the above theorem, we first compute upper and lower bounds on the choice probability $\hat{\pi}(j,S)$ for any $S \subseteq \mathcal{N}$, $j \in S_+$ in the following two lemmas.

LEMMA 4.1. *For any $S \subseteq \mathcal{N}$, $j \in S_+$, let $\hat{\pi}(j,S)$ denote the choice probability of product j for offer set S computed by the Markov chain model. Then*

$$\hat{\pi}(j,S) \geq \sum_{k=1}^{K} \theta_k u_{jk} \left(1+\sum_{i\in\bar{S}}\left(\sum_{q=1}^{\infty}(u_{ik})^q\right)\right).$$

The above lower bound on the choice probability $\hat{\pi}(j,S)$ for $j \in S_+$ is computed by considering only a one-step substitution to another product if the first choice product is not available. According to our Markov chain model, a customer with product i as the first choice product transitions to another state if product i is not available. The transitions continue according to the transition matrix $\mathcal{P}(S)$ until the customer ends up in an absorbing state. Therefore, by considering only a single transition in the Markov chain $\mathcal{M}(S)$, we obtain a lower bound on $\hat{\pi}(j,S)$ for any $j \in S_+$. We present the proof of Lemma 4.1 in Appendix A.

In the following lemma, we prove an upper bound on $\hat{\pi}(j,S)$ for any $S \subseteq \mathcal{N}$, $j \in S_+$. The bound depends on the spectral radius of the transition sub-matrix $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S},\bar{S})$ of transition probabilities from $\bar{S}$ to $\bar{S}$.

LEMMA 4.2. *For any $S \subseteq \mathcal{N}$, $j \in S_+$, let $\hat{\pi}(j,S)$ be the choice probability of product $j \in S_+$ for offer set S computed by the Markov chain model. Let $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S},\bar{S})$ denote the sub-matrix of transition probabilities from states $\bar{S} = \mathcal{N} \setminus S$ to $\bar{S}$, and let γ be the maximum eigenvalue of $\boldsymbol{C}$. Then*

$$\hat{\pi}_j(S) \le \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{\tau(\bar{S})}{1-\gamma} \cdot \left(\sum_{i \in \bar{S}} \sum_{q=1}^{\infty} (u_{ik})^q \right)\right),$$

where $\tau(\cdot)$ is defined in (4.3) *constant that depends on the eigenvector corresponding to the maximum eigenvalue of $\boldsymbol{C}$.*

In the following lemma, we show the spectral radius of the transition sub-matrix $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S},\bar{S})$ is related to the parameter $\alpha(\bar{S})$ defined in Theorem 4.2.

LEMMA 4.3. *Consider any $S \subseteq \mathcal{N}$ and let $\alpha = \alpha(\bar{S})$. Let $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S},\bar{S})$ be the probability transition sub-matrix of $\mathcal{P}(S)$ from states $\bar{S}$ to $\bar{S}$. Then the maximum eigenvalue of $\boldsymbol{C}$, γ is at most α.*

We present the proofs of Lemmas 4.2 and 4.3 in Appendix A. Now, we are ready to prove the main theorem.

Proof of Theorem 4.2 Let $\alpha = \alpha(\bar{S})$ and $\tau = \tau(\bar{S})$. From (4.1), we know that

$$\pi(j,S) = \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{q=1}^{\infty} \left(u_k(\bar{S})\right)^q\right),$$

and from Lemma 4.1, we have that

$$\begin{aligned}\hat{\pi}(j,S) &\ge \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{i \in \bar{S}} \left(\sum_{q=1}^{\infty} (u_{ik})^q\right)\right)\\ &\ge \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{i \in \bar{S}} u_{ik}\right),\end{aligned}$$

where the second inequality follows as $u_{ik} \ge 0$ for all $i \in \mathcal{N}_+, k = 1, \ldots, K$. Therefore,

$$\pi(j,S) - \hat{\pi}(j,S) \le \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{q=1}^{\infty} \left(u_k(\bar{S})\right)^q\right) - \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{i \in \bar{S}} u_{ik}\right)$$

$$
\begin{aligned}
&= \sum_{k=1}^{K} \theta_k u_{jk} \cdot \left(u_k(\bar{S})\right)^2 \cdot \left(\sum_{q=0}^{\infty} u_k(\bar{S})^q\right) \\
&\leq \alpha^2 \cdot \left(\sum_{k=1}^{K} \theta_k u_{jk} \cdot \left(\sum_{q=0}^{\infty} u_k(\bar{S})^q\right)\right) \qquad (4.4) \\
&= \alpha^2 \pi(j,S), \qquad (4.5)
\end{aligned}
$$

where (4.4) follows from the fact that $u_k(\bar{S}) \leq \alpha$ for all $k = 1, \ldots, K$ and (4.5) follows from (4.1).

Let γ denote the maximum eigenvalue of $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$, the transition sub-matrix from $\bar{S}$ to $\bar{S}$. From Lemma 4.3, we know that $\gamma \leq \alpha < 1$. Therefore, from Lemma 4.2, we have that

$$
\begin{aligned}
\hat{\pi}_j(S) &\leq \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{\tau}{1-\alpha} \cdot \left(\sum_{i \in \bar{S}} \sum_{q=1}^{\infty} (u_{ik})^q\right)\right) \\
&\leq \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{\tau}{1-\alpha} \cdot \sum_{q=1}^{\infty} \left(u_k(\bar{S})\right)^q\right), \qquad (4.6)
\end{aligned}
$$

where (4.6) follows as $u_{ik} \geq 0$ for all $i \in \mathcal{N}_+, k = 1, \ldots, K$. Therefore,

$$
\begin{aligned}
\pi(j,S) - \hat{\pi}(j,S) &\geq \sum_{k=1}^{K} \theta_k u_{jk} \left(1 - \frac{\tau}{1-\alpha}\right) \cdot \left(\sum_{q=1}^{\infty} \left(u_k(\bar{S})\right)^q\right) \\
&> \frac{-\alpha\tau}{1-\alpha} \cdot \sum_{k=1}^{K} \theta_k u_{jk} \cdot \left(\sum_{q=0}^{\infty} \left(u_k(\bar{S})\right)^q\right) \\
&= \frac{-\alpha\tau}{1-\alpha} \cdot \pi(j,S), \qquad (4.7)
\end{aligned}
$$

where the first inequality follows from (4.6) and the second inequality follows as $u_k(\bar{S}) \leq \alpha$ for all $k = 1, \ldots, K$. Therefore, from (4.5) and (4.7) we have

$$
\left(1 - (\alpha(\bar{S}))^2\right) \cdot \pi(j,S) \leq \hat{\pi}(j,S) \leq \left(1 + \frac{\tau(\bar{S}) \cdot \alpha(\bar{S})}{1 - \alpha(\bar{S})}\right) \cdot \pi(j,S).
$$

$\square$

We would like to emphasize that the lower and upper bounds in Theorem 4.2 can be quite conservative in practice. For instance, in computing the lower bound on $\hat{\pi}(j,S)$ in Lemma 4.1, we only considers a one-step substitution in the Markov chain to approximate the absorption probability of state $j \in S_+$ in Markov chain $\mathcal{M}(S)$. To further investigate this gap between theory and practice, we do a computational study to compare the performance of the Markov chain model with the true underlying model. In the computational results, we observe that the performance of our model is significantly better than the theoretical bounds. The results are presented in Section 6.

4.2. A Tight Example

While the bounds in Theorem 4.2 can be quite conservative for many instances, we show that these are tight. In particular, we present a family of instances of the mixture of MNL choice model where the ratio of the Markov chain choice probability and the true choice probability for some $S \subseteq \mathcal{N}$, $j \in S_+$ is almost equal to the bound in Theorem 4.2.

THEOREM 4.3. *For any $\epsilon > 0$, there is a MMNL choice model over $\mathcal{N}$, $S \subseteq \mathcal{N}, j \in S$ such that the approximation bound in Theorem 4.2 is tight up to a factor of $O(n^{\epsilon})$.*

Proof Consider the following MMNL choice model that is a mixture of two MNL models each with probability $1/2$, i.e., $\theta_1 = \theta_2 = 1/2$. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_+^{n+1}$ denote the utility parameters of the two MNL models. Let $n_0 = (n - n^{1-\epsilon})$. For all $j = 0, \ldots, n$,

$$u_j = \begin{Bmatrix} n^{2-\epsilon}, & \text{if } j = 0 \\ 1, & \text{if } j = 1 \text{ or } j > n_0 \\ n - j + 1, & \text{otherwise} \end{Bmatrix}, \text{ and } v_j = \begin{Bmatrix} 1, & \text{if } j = 0 \text{ or } j \geq n_0 \\ j, & \text{otherwise.} \end{Bmatrix} \tag{4.8}$$

Let $S = \{1\}, j = 1$. The true choice probability $\pi(1, S) > \frac{1}{4}$. Let

$$s_1 = \sum_{j=0}^{n} u_j = \Theta(n^2), \; s_2 = \sum_{j=0}^{n} v_j = \Theta(n^2).$$

For all $j = 2, \ldots, n$, $\rho_{j1} = O(1/n^2)$, and

$$\lambda_j = \begin{Bmatrix} \Theta\left(\dfrac{1}{n}\right), \text{ if } 2 \leq j \leq n_0 \\ \Theta\left(\dfrac{1}{n^2}\right), \text{ otherwise} \end{Bmatrix}, \text{ and } \rho_{j0} = \begin{Bmatrix} \Omega\left(\dfrac{1}{n^{2\epsilon}}\right) & \text{if } j \leq (n - n^{1-\epsilon}) \\ \Omega\left(\dfrac{1}{n^{1+\epsilon}}\right) & \text{otherwise} \end{Bmatrix}$$

Let c, c_1, c_2 be some constants. Therefore, we can bound $\hat{\pi}(1, S)$ as follows.

$$\begin{aligned} \hat{\pi}(1,S) &\leq \pi(1,\mathcal{N}) + \left(\sum_{i=2}^{n_0} \lambda_i\right) \cdot \left(\sum_{q=0}^{\infty} \left(1 - \frac{c_1}{n^{2\epsilon}}\right)^q\right) \cdot \frac{c}{n^2} + \sum_{i=n_0}^{n} \lambda_i \cdot \left(\sum_{q=0}^{\infty} \left(1 - \frac{c_2}{n^{1+\epsilon}}\right)^q\right) \cdot \frac{c}{n^2} \\ &\leq O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n^{2-2\epsilon}}\right) + O\left(\frac{1}{n^2}\right) \\ &= O\left(\frac{1}{n^{2-2\epsilon}}\right) \cdot \pi(1,S), \end{aligned} \tag{4.9}$$

where the first inequality follows as $\rho_{j1} = O(1/n^2)$ for all $j = 2, \ldots, n$, $\rho_{j0} = \Omega(1/n^{2\epsilon})$ for all $j = 2, \ldots, n_0$ and $\rho_{j0} = \Omega(1/n^{1+\epsilon})$ for $j \geq n_0$. Inequality (4.9) follows as $\pi(1, \mathcal{N}) = O(1/n^2)$, and

$$\sum_{j=2}^{n_0} \lambda_j \leq 1, \quad \sum_{j=n_0+1}^{n} \lambda_j \leq O\left(\frac{1}{n^{1+\epsilon}}\right).$$

Also, $(1 - \alpha(\bar{S})^2) = \Theta\left(\frac{1}{n^2}\right)$. From the lower bound in Theorem 4.2, we have

$$\hat{\pi}(1,S) \geq (1 - \alpha(\bar{S})^2) \cdot \pi(1,S) = \Theta\left(\frac{1}{n^2}\right) \cdot \pi(1,S),$$

which implies that the lower bound is tight up to a factor of $n^{2\epsilon}$. □

In the MMNL instance in Theorem 4.3, the parameters for the MNL segments are designed such that both the arrival probability for state 1 (corresponding to product 1) and the transition probability, ρ_{j1} from any state $j = 2, \ldots, n$ are $O(1/n^2)$. This is because the transition probabilities, ρ_{j1} are computed from assortments $\mathcal{N}$ and $\mathcal{N} \setminus \{j\}$ and product 1 has a small probability of selection in both these assortments. Therefore, even though for small offer set $S = \{1\}$, the probability of selecting 1 in MNL segment 2 is $1/2$, the absorption probability in state 1 remains small for any set of absorbing states.

We would also like to note that the MNL approximation also performs badly for the above MMNL instance with offer set $S = \{1\}$. The parameters for the MNL approximation are given by $v_0 = n^{2-\epsilon}/2$ and $v_1 = 1$. Therefore, the probability of selecting 1 given by the MNL approximation is $O(1/n^{2-\epsilon})$. Therefore, the performance of the MNL approximation is similar to the Markov chain model for this instance. However, it is important to note that the family of instances in Theorem 4.3 are pathological cases where the parameters are carefully chosen to show that the bound is tight. The choice model is a mixture of two MNLs where the MNL parameters of one class are increasing and the second class are decreasing for almost all products. Such utility parameters would not usually arise in practical settings. If we change the example slightly, we can observe that the performance of the Markov chain model is significantly better and the bounds in Theorem 4.2 are conservative. In Section 6, we present extensive computational results that compare the performance of the Markov chain model and the MNL approximation in modeling choice probabilities for random MMNL instances similar to the tight example in Theorem 4.3.

5. Assortment Optimization for Markov Chain Model

In this section, we consider the problem of finding the optimal revenue assortment for the Markov chain based choice model. For any $j \in \mathcal{N}$, let r_j denote the revenue of product j. The goal in an assortment optimization problem is to select an offer set $S \subseteq \mathcal{N}$ such that the total expected revenue is maximized, i.e.,

$$\max_{S \subseteq \mathcal{N}} \sum_{j \in S} r_j \cdot \hat{\pi}(j, S), \tag{5.1}$$

where for all $j \in S$, $\hat{\pi}(j, S)$ is the choice probability of product j given by the Markov chain model (2.9).

We present a polynomial time algorithm for the assortment optimization problem (5.1) for the Markov chain based choice model. The result is quite surprising as we can not even express the choice probability $\hat{\pi}(j, S)$ for any $S \subseteq \mathcal{N}$, $j \in S_+$ using a simple functional form of the model parameters. The computation of $\hat{\pi}(j, S)$ in (2.9) requires a matrix inversion where the coefficients of the matrix depend on the assortment decisions.

For all $i \in \mathcal{N}$ and $S \subseteq \mathcal{N}$, let $g_i(S)$ denote the expected revenue from a customer that arrives in state i (i.e. the most preferable product is i) when the offer set is S. If product $i \in S$, then the customer selects product i and $g_i(S) = r_i$. Otherwise, the customer substitutes according to the transitions in the Markov chain and

$$g_i(S) = \sum_{j \in \mathcal{N}} P_{ij} g_j(S),$$

and the total expected revenue for offer set S is given by $\sum_{j \in \mathcal{N}} \lambda_j \cdot g_j(S)$. Therefore, we can reformulate the assortment optimization (5.1) as follows.

$$\max_{S \subseteq \mathcal{N}} \sum_{j \in \mathcal{N}} \lambda_j \cdot g_j(S), \tag{5.2}$$

where $g_j(S)$ denotes the expected revenue from a customer with most preferable product is j when the offer set is S. This optimization problem is equivalent to selecting an optimal set of stopping (or absorbing) states in the Markov chain, $\mathcal{M}$, and is related to the classical optimal stopping time problem (see Chow et al. (1971)) where the goal is to decide on a stopping rule that maximizes the expected reward.

Motivated by the reformulation (5.2), we consider the following approach. For all $i \in \mathcal{N}$, let g_i be the maximum expected revenue that can be obtained from a customer whose first choice is product i, where the maximization is taken over all offer sets, i.e.,

$$g_i = \max_{S_i \subset \mathcal{N}} g_i(S_i). \tag{5.3}$$

Note that we allow for the maximizing offer set depend on the first choice product i. However, later we show that the expected revenues starting from all states are maximized for the same offer set. Given g_i for all $i \in \mathcal{N}$, we can easily compute the optimal assortment. In particular, we have the following theorem.

THEOREM 5.1. *Let g_i for all $i \in \mathcal{N}$ be as defined in* (5.3). *Let*

$$S = \{i \in \mathcal{N} \mid g_i = r_i\}. \tag{5.4}$$

Then S is optimal for the assortment optimization problem (5.2).

We begin by presenting a polynomial time algorithm to compute g_i for all $i \in \mathcal{N}$. We first present an iterative algorithm that computes $\boldsymbol{g}$ in a polynomial number of iterations under the assumption that for all $i \in \mathcal{N}$, ρ_{i0} is polynomially bounded away from zero, i.e., $\rho_{i0} = \Omega(1/n^c)$ for some constant c. The iterative algorithm provides useful insights towards reformulating the problem of computing $\boldsymbol{g}$ as a fixed point computation problem. We show that the fixed point computation can be formulated as an LP where we can relax the assumption the assumption that ρ_{i0} is polynomially bounded away from zero for all $i \in \mathcal{N}$.

Iterative Algorithm to compute $\boldsymbol{g}$. We can compute g_i using the following iterative procedure under the assumption that ρ_{i0} is polynomially bounded away from 0 for all $i \in \mathcal{N}$. For $t \in Z_+, i \in \mathcal{N}$, let g_i^t denote the maximum expected revenue starting from state i in at most t state transitions where we can stop at any state.

Algorithm 1 Iterative Algorithm to compute $\boldsymbol{g}$

Initialize: $g_j^0 := r_j$ for all $j \in \mathcal{N}$, $\Delta := 1$, $t := 0$
while $(\Delta > 0)$ **do**
  $t := t+1$
  for $i = 1 \to n$ **do**
    $g_i^t := \max\left(r_i,\ \sum_{j \neq i} \rho_{ij} \cdot g_j^{t-1}\right)$
  end for
  $\Delta := ||\boldsymbol{g}^t - \boldsymbol{g}^{t-1}||_\infty$
end while
Return: $\boldsymbol{g}$

Stopping at any state j corresponds to selecting product j resulting in revenue r_j. Therefore, for all $i \in \mathcal{N}$, $g_i^0 = r_i$ since the only possibility is to stop at state i when no state transitions are allowed. Algorithm 1 describes the iterative procedure to compute g_j for all $j \in \mathcal{N}$, by computing g_j^t for $t \geq 1$ until they converge.

We show that Algorithm 1 converges in a polynomial number of iterations and correctly computes g_j for all $j \in \mathcal{N}$. Let

$$\delta = \min_j\ \rho_{j0},\ r_{\max} = \max_{j \in \mathcal{N}}\ r_j,\ r_{\min} = \min_{j \in \mathcal{N}}\ r_j. \tag{5.5}$$

LEMMA 5.1. *Suppose δ as defined in (5.5) is polynomially bounded away from 0, i.e., $\delta = \Omega(1/n^c)$ for some constant c. Then for all $j \in \mathcal{N}$, g_j computed by Algorithm 1 is the maximum possible expected revenue from a customer arriving in state j . Furthermore, Algorithm 1 converges in polynomial number of iterations.*

Proof We first prove that for all $t \in \mathbb{Z}_+$, for all $j \in \mathcal{N}$, g_j^t computed in Algorithm 1 is the maximum expected revenue starting from state j after at most t state transitions, and g_j^t is increasing in t. We prove this by induction.

Base Case ($t = 0$**,** $t = 1$). Clearly, $g_j^0 = r_j$ for all $j \in \mathcal{N}$ when no state transitions are allowed. For $t = 1$, we can either stop in state j or stop after exactly one transition. Therefore, for all $j \in \mathcal{N}$,

$$g_j^1 = \max\left(r_j, \sum_{i \in \mathcal{N}} \rho_{ji} \cdot r_i\right) = \max\left(r_j, \sum_{i \in \mathcal{N}} \rho_{ji} \cdot g_i^0\right) \geq r_j = g_j^0.$$

Induction Step ($t = T$). For any $j \in \mathcal{N}$, the maximum expected revenue in at most T transitions starting from state j is either obtained by zero state transition (i.e. stopping at state j) or at least one transition. In the former case, the revenue is r_j. For the latter, we transition to state i with probability ρ_{ji}. From state i, we can make at most $T-1$ transitions and by induction hypothesis, g_i^{T-1} is the maximum expected revenue that can be obtained starting from state i in at most $T-1$ transitions. Therefore, g_j^T for all $j \in \mathcal{N}$ computed in Algorithm 1 as

$$g_j^T = \max\left(r_j, \sum_{i\in\mathcal{N}} \rho_{ji} \cdot g_i^{T-1}\right),$$

is the maximum expected revenue in at most T state transitions starting from state j. Also, by induction hypothesis, $g_i^{T-1} \geq g_i^{T-2}$ for all $i \in \mathcal{N}$. Therefore, for all $j \in \mathcal{N}$.

$$g_j^T = \max\left(r_j, \sum_{i\in\mathcal{N}} \rho_{ji} \cdot g_i^{T-1}\right) \geq \max\left(r_j, \sum_{i\in\mathcal{N}} \rho_{ji} \cdot g_i^{T-2}\right) = g_j^{T-1}.$$

Therefore, for all $j \in \mathcal{N}$, g_j^t is an increasing sequence upper bounded by $r_{\max} = \max_{i\in\mathcal{N}} r_i$ and it converges to g_j. This implies that Algorithm 1 correctly computes g_j for all $j \in \mathcal{N}$.

To prove that the algorithm converges in a polynomial number of iterations, observe that in each transition, there is at least a probability of δ to transition to state 0 which corresponds to zero revenue. After t transitions starting from any state j, the maximum possible expected revenue is $(1-\delta)^t \cdot r_{\max}$. Consider T such that $(1-\delta)^T \cdot r_{\mathsf{max}} \leq r_{\min}$. Then the algorithm converges in at most T iterations since starting from any state j, it is better to stop at state j itself and get revenue r_j, instead of transitioning for more than T steps. This implies that Algorithm 1 converges in at most $(1/\delta) \cdot \log(r_{\max}/r_{\min})$ iterations. Since δ is polynomially bounded away from zero, the algorithm converges in polynomial number of iterations. $\square$

LP formulation to compute $\boldsymbol{g}$. Let $\boldsymbol{P} = \boldsymbol{\rho}(\mathcal{N},\mathcal{N})$ denote the probability transition matrix. Algorithm 1 to compute $\boldsymbol{g}$ motivates the following alternate definition of g_j for all $j \in \mathcal{N}$.

$$g_j = \max(r_j, \boldsymbol{e}_j^T \boldsymbol{P}\boldsymbol{g}),\ \forall j \in \mathcal{N}, \tag{5.6}$$

where $\boldsymbol{e}_j$ denotes the j^{th} unit vector. Therefore, computing $\boldsymbol{g}$ can be formulated as a problem of computing the fixed point of the above equations. Here, we present an LP formulation to compute $\boldsymbol{g}$. The LP formulation only assumes that the substitution matrix, $\boldsymbol{\rho}(\mathcal{N},\mathcal{N})$ has spectral radius strictly less than one (Assumption 2.1). As discussed earlier, this assumption is without loss of generality for any natural choice model. We do not require any assumption on δ (5.5). Consider the following LP.

$$\min\{\boldsymbol{e}^T\boldsymbol{g} \mid \boldsymbol{g} \geq \boldsymbol{r},\ \boldsymbol{g} \geq \boldsymbol{P}\boldsymbol{g}\}. \tag{5.7}$$

We would like to note that the above LP is related to Markov decision processes (see for instance Puterman (2014)). We can interpret the assortment optimization problem for the Markov chain model as a MDP where for each state i, the possible actions are to stop and get revenue r_i or continue with the transitions. In the following lemma, we show that (5.7) correctly computes g_j for all $j \in \mathcal{N}$.

LEMMA 5.2. *Suppose $\boldsymbol{P}$ has spectral radius strictly less than one. Then the LP* (5.7) *correctly computes g_j for all $j \in \mathcal{N}$ as defined in* (5.3).

We present the proof of Lemma 5.2 and Theorem 5.1 in Appendix B.

Since the Markov chain model is exact for the generalized attraction model (Theorem 3.1), the optimal assortment for the Markov chain model also corresponds to an optimal assortment if the underlying model is GAM. Therefore, we have the following corollary.

COROLLARY 5.1. *Suppose the parameters of the Markov chain based choice model, λ_j, ρ_{ij} for all $i \in \mathcal{N}, j \in \mathcal{N}_+$ are computed from choice probabilities arising from an underlying GAM (using* (2.2)*). Then by solving LP* (5.7)*, we can compute an optimal assortment for the underlying generalized attraction choice model.*

5.1. Assortment Optimization for General Choice Models

In this section, we show that if the Markov chain model parameters are computed from choice probabilities for different assortments arising from a general random utility choice model, then the optimal assortment for the resulting Markov chain model is a good approximation for the underlying choice model as well. Since a general random utility based choice model can be approximated as closely as required by a mixture of MNL model as McFadden and Train (2000), we can assume wlog. that the underlying choice model is a mixture of MNL model. In Theorem 4.2, we show that a mixture of MNL model can be well approximated by a Markov chain model. Therefore, an optimal assortment for the corresponding Markov chain model can be a good approximation for the underlying mixture of MNL model. In particular, we prove the following theorem.

THEOREM 5.2. *Suppose the parameters of the Markov chain model are computed from choice probabilities, $\pi(\cdot,\cdot)$ arising from an underlying mixture of MNL model. Let S^* be an optimal assortment for the mixture of MNL model and let S be an optimal assortment for the Markov chain model as defined in* (5.4)*. Then*

$$\sum_{j\in S}\pi(j,S)\cdot r_j \geq \frac{(1-\alpha(\bar{S}^*)^2)(1-\alpha(\bar{S}))}{1-\alpha(\bar{S})+\alpha(\bar{S})\cdot\tau(\bar{S})}\cdot\left(\sum_{j\in S^*}\pi(j,S^*)\cdot r_j\right),$$

where $\alpha(\cdot)$ is as defined in (4.2) *and $\tau(\cdot)$ is defined in* (4.3)*.*

Proof Suppose the underlying mixture model is a mixture of K MNL segments with parameters u_{jk} for $j = 0,\ldots,n$, $k = 1,\ldots,K$ such that $u_{0k}+\ldots+u_{nk} = 1$ for all $k = 1,\ldots,K$. Let the Markov chain parameters λ_j, ρ_{ij} for all $i = 1,\ldots,n$ and $j = 0,\ldots,n$ be computed as in (2.2) and let $\hat{\pi}(\cdot,\cdot)$ denote the choice probabilities computed from the Markov chain model (2.9). Therefore, we know that the assortment S computed in (5.4) maximizes $\sum_{j\in S}\hat{\pi}(j,S)r_j$. For brevity, let

$$\alpha_1 = \left(1+\frac{\alpha(\bar{S})\cdot\tau(\bar{S})}{1-\alpha(\bar{S})}\right),\ \alpha_2 = (1-\alpha(\bar{S}^*)^2).$$

From Theorem 4.2, we know that

$$\hat{\pi}(j,S) \leq \alpha_1 \cdot \pi(j,S),\ \forall j \in S, \text{ and } \hat{\pi}(j,S^*) \geq \alpha_2 \cdot \pi(j,S^*),\ \forall j \in S^*. \tag{5.8}$$

Then

$$\sum_{j\in S} \pi(j,S)\cdot r_j \geq \frac{1}{\alpha_1} \cdot \sum_{j\in S} \hat{\pi}(j,S)\cdot r_j \geq \frac{1}{\alpha_1} \cdot \sum_{j\in S^*} \hat{\pi}(j,S^*)\cdot r_j \geq \frac{\alpha_2}{\alpha_1} \cdot \sum_{j\in S^*} \pi(j,S^*)\cdot r_j,$$

where the first inequality follows from (5.8). The second inequality follows as S is an optimal assortment for the Markov chain choice model, $\hat{\pi}$ and the last inequality follows from (5.8). $\square$

The performance bound of the assortment, S computed from the Markov chain choice model depends on $\alpha(\bar{S}^*)$ and $\alpha(\bar{S})$ where S^* is the optimal assortment for the true model. We would like to note that if either of $\alpha(\bar{S}^*)$ or $\alpha(\bar{S})$ are close to one (that could be the case if size of either S or S^* is $o(n)$), then the approximation bound is not good. The approximation bound worsens in this case because the Markov chain model does not provide a good approximation for choice probabilities when offer sets have size $o(n)$. As we discuss earlier, the transition probabilities in our model are computed from offer sets $\mathcal{N}, \mathcal{N} \setminus \{i\}$ and they may not approximate the substitution behavior well for significantly smaller offer sets. One possible approach could be to re-compute the transition probabilities when the size of offer set is small. For instance, we could consider a family of offer sets $\{S, S \setminus \{i\} \mid i \in S\}$ where the cardinality of S is small. It is an interesting open question to study the performance of such a heuristic.

6. Computational Results

In this section, we present a computational study on the performance of the Markov chain choice model in modeling random utility based discrete choice models. In Theorem 4.2, we present theoretical bounds on the relative error between the choice probabilities computed by the Markov chain model and the true choice probability. While in Theorem 4.3, we present a family of instances where the bound is tight but these can be conservative in general. In the computational study, we compare the performance of the Markov chain model with respect to the mixture of MNL (MMNL) model as well as distribution over permutations model. We also study the performance of the Markov chain model in assortment optimization.

6.1. Comparison for Choice Probabilities

In this section, we compare the performance of the Markov chain choice model in modeling choice probabilities with respect to several family of choice models including MMNL model and the distribution over permutations model. For each instance of the choice model, we compute the Markov chain parameters, $\lambda_i, i \in \mathcal{N}$ and transition probabilities, $\rho_{ij}, i \in \mathcal{N}, j \in \mathcal{N}_+$ using the choice probabilities for only assortments $\mathcal{S} = \{\mathcal{N}, \mathcal{N} \setminus \{i\} \mid; i = 1, \ldots, n\}$ using (2.2). For the MMNL choice, we also consider the MNL approximation as a benchmark. We then compare the choice probability compute by the Markov chain model with the true choice probability for out of sample offer sets, $S \subseteq \mathcal{N}$ where $S \notin \mathcal{S}$.

Experimental Setup. To compare the out of sample performance of the Markov chain model, we generate L random offer sets of size between $n/3$ and $2n/3$ and compare the choice probability computed by the Markov chain model with the true choice probability. We use $n \geq 10$, $K = \lceil \log n \rceil$ and $L = 100$. For all assortments $S_1, \ldots, S_L$, we compute the maximum relative errors of the choice probability of the Markov chain model with respect to the true choice probability. For any $S \subseteq \mathcal{N}, j \in S_+$, let $\pi^{\mathsf{MC}}(j, S)$, $\pi^{\mathsf{MNL}}(j, S)$, and $\pi(j, S)$ denote the choice probability of the Markov chain model, approximate MNL model, and the true MMNL model respectively. Then for all $\ell = 1, \ldots, L$,

$$\mathsf{errMC}(\ell) = 100 \cdot \max_{j \in S_\ell} \frac{|\pi^{\mathsf{MC}}(j, S_\ell) - \pi(j, S_\ell)|}{\pi(j, S_\ell)},$$

$$\mathsf{errMNL}(\ell) = 100 \cdot \max_{j \in S_\ell} \frac{|\pi^{\mathsf{MNL}}(j, S_\ell) - \pi(j, S_\ell)|}{\pi(j, S_\ell)},$$

and we compute the average maximum relative error as

$$\mathsf{avg\text{-}errMC} = \frac{1}{L} \cdot \sum_{\ell=1}^{L} \mathsf{errMC}(\ell), \quad \mathsf{avg\text{-}errMNL} = \frac{1}{L} \cdot \sum_{\ell=1}^{L} \mathsf{errMNL}(\ell).$$

We also report the maximum relative error over all offer sets and all products where

$$\max\text{-}\mathsf{errMC} = \max_{\ell=1}^{L} \mathsf{errMC}(\ell), \quad \max\text{-}\mathsf{errMNL} = \max_{\ell=1}^{L} \mathsf{errMNL}(\ell).$$

We describe the details of the experiments below.

6.1.1. Random MMNL instances. We generate the random instances of the MMNL model as follows. Let n denote the number of products and K denote the number of customer segments in the MMNL model. For each $k = 1, \ldots, K$, the MNL parameters of segment k, $u_{0k}, \ldots, u_{nk}$ are i.i.d samples of the uniform distribution in $[0, 1]$. For the probability distribution over different MNL segments, we consider both uniform as well as random distribution. For the uniform distribution, the probability of segment k, $\theta_k = 1/K$ for all $k = 1, \ldots, K$, and for the random distribution, θ_k for all $k = 1, \ldots, K$ is chosen independently from a uniform distribution in $[0, 1]$ and normalized appropriately. For each random instance, we use the choice probabilities for assortments $\mathcal{S} = \{\mathcal{N}, \mathcal{N} \setminus \{i\} \mid ; i = 1, \ldots, n\}$ to compute the Markov chain choice model parameters as described in (2.2). Note that in computing the parameters for the Markov chain model, we do not use any other knowledge of the underlying MMNL model parameters except the choice probabilities of assortments $\mathcal{S}$. We consider the following MNL approximation of the MMNL model with parameters $v_j = \sum_{k=1}^{K} \theta_k u_{jk}$ for all $j = 0, \ldots, n$.

We present the computational results for the maximum relative error for the case of uniform mixture and random mixture of MNL in Tables 1 and 2 respectively. We observe a similar performance for the Markov chain and MNL approximation for both the uniform mixture and the random mixture of MNL models. The

n	K	errMNL(%)		errMC(%)	
		avg	max	avg	max
10	3	12.53	31.75	3.34	17.80
20	3	11.07	21.79	3.04	13.12
30	4	7.62	19.82	2.42	10.86
40	4	5.94	12.71	2.42	6.84
60	5	4.33	9.96	1.54	3.83
80	5	8.27	19.66	2.10	8.72
100	5	6.07	11.52	1.74	5.76
150	6	3.61	8.28	1.49	3.71
200	6	3.90	9.39	1.30	4.07
500	7	2.21	4.02	0.77	1.39
1000	7	1.60	3.21	0.58	1.16

Table 1 Relative Error of Markov chain and MNL models with respect to uniform mixture of random MNLs.

n	K	errMNL(%)		errMC(%)	
		avg	max	avg	max
10	3	4.90	11.63	1.20	5.05
20	3	9.32	29.92	2.90	12.79
30	4	10.39	20.6	2.87	7.13
40	4	6.45	25.27	2.75	14.51
60	5	7.05	13.20	1.92	4.92
80	5	4.34	9.41	1.7	4.17
100	5	3.40	9.21	1.76	4.78
150	6	5.1	9.24	1.36	3.66
200	6	4.5	8.75	1.24	2.6
500	7	1.98	4.14	0.79	1.53
1000	7	2.13	4.95	0.69	1.84

Table 2 Relative Error of Markov chain and MNL models with respect to random mixture of random MNLs.

average relative error of the choice probability computed by the Markov chain model with respect to the true MMNL choice probability is less than 3.5%, and the maximum relative error is less than 18% for all values of n, K for both families of the MMNL models. Moreover, the Markov chain model performs significantly better than the MNL approximation for the MMNL model on both metrics: average relative error and the maximum relative error. In particular, for all values of n and K in our computational experiments, the average relative error for the MNL approximation is more than twice the average relative error for the Markov chain model. We observe a similar comparison for the maximum relative error as well. We would also like to note that the average size of the offer sets, $S_1, \ldots, S_L$ is approximately $n/2$. Therefore, $|\bar{S}_\ell|$ and $\alpha(\bar{S}_\ell)$ is large on average and the approximation bounds in Theorem 4.2 are quite conservative as compared to our computational results.

6.1.2. Alternate Family of MMNL Instances. We consider another random family of instances of the mixture of MNLs model to compare the performance of the Markov chain choice model. Motivated by the bad example in Theorem 4.3 that shows that the tightness of approximation bounds for the Markov chain model, we consider the following family of MMNL instances. As before, let K denote the number of customer segments each occurring with probability $1/K$. For the first two segments, the parameters are given by

$$u_{j1} = j+1,\ u_{j2} = n+1-j,\ \forall j = 0, \ldots, n,$$

i.e. the MNL parameters belong to $\{1, \ldots, n+1\}$ and are in increasing order for segment 1 and in decreasing for segment 2. For segments $k = 3, \ldots, K$, the $u_{jk}, j = 0, \ldots, n$ are a random permutation of $\{1, \ldots, n+1\}$.

n	K	errMNL(%)		errMC(%)	
		avg	max	avg	max
10	3	7.1	21.57	3.11	13.5
20	3	7.18	19.61	3.26	12.69
30	4	5.01	14.96	2.19	6.9
40	4	4.67	16.35	2.17	8.39
60	5	3.97	12.85	1.89	7.49
80	5	3.98	14.04	1.86	7.25
100	5	3.75	9.94	1.76	5.31
150	6	2.95	7.7	1.4	4.26
200	6	2.39	6.25	1.13	2.97
500	7	1.62	3.16	0.78	1.59
1000	7	1.38	2.88	0.66	1.40

Table 3 Relative Error of Markov chain and MNL models with respect to uniform mixture of random permutation MNLs.

n	K	errMNL(%)		errMC(%)	
		avg	max	avg	max
10	3	7.05	27.57	3.11	21.5
20	3	6.85	26.3	2.92	12.28
30	4	5.33	21.88	2.59	16.753
40	4	5.64	17.17	2.68	8.75
60	5	5.82	15.79	2.46	8.69
80	5	4.12	13.37	1.92	6.73
100	5	4.17	12.61	1.97	6.79
150	6	2.9	6.28	1.38	3.16
200	6	2.66	5.83	1.27	3.19
500	7	1.79	4.08	0.86	1.94
1000	7	1.29	2.84	0.62	1.33

Table 4 Relative Error of the Markov chain and the MNL models with respect to random mixture of random permutation MNLs.

This construction is similar to the structure of the bad example in Theorem 4.3 that is a mixture of two MNL models with increasing parameters for one model and decreasing for another for almost all the products.

As before, we consider both a uniform mixture of MNL segments where each segment occurs with probability $1/K$ as well as a random mixture of MNLs. We use $n \geq 10$, $K = \lceil \log n \rceil$, and generate $L = 100$ random offer sets of size between $n/3$ and $2n/3$ as in the earlier experiments. We present our results in Tables 3 and 4. The computational results are similar to the other family of random MMNL instances for both the uniform as well as the random mixture of MNLs. The average relative error less than 3.2% and the maximum relative error less than 17% for the Markov chain model for all values of n, K in our experiments. Furthermore, as in the other case, the relative error of the Markov chain model is significantly lower than that of the MNL approximation (average relative error is less than half and the maximum relative error is also significantly lower).

We would like to note that even though the structure of the MMNL instances is similar to the bad example in Theorem 4.3, our computational performance is quite good. This is because we use offer sets with large cardinality (size between $n/3$ and $2n/3$) for our computational experiments. On the other hand, in the bad example of Theorem 4.3, we show that the performance bounds are tight for offer set of cardinality 1. Therefore, it is not surprising to observe small relative errors in the performance in our experiments. As the theoretical performance bounds in Theorem 4.2 indicate, the performance of the Markov chain model worsens as we decrease the size of the offer sets.

6.1.3. Distribution over Permutations Model. We also consider the distribution over permutations choice model to compare the performance of our Markov chain model. In particular, we consider a random distribution over a small number of randomly generated permutations over $\{0,1,\ldots,n\}$ (where 0 denotes the no-purchase alternative). A permutation represents a preference list of a customer who substitutes products in the order given by the list. We truncate each permutation at 0 as no-purchase is always an available option and the order of products appearing after 0 in the permutation does not affect the substitution behavior.

The distribution over permutations model is specified by K permutations $\{\sigma_1,\ldots,\sigma_K\}$ where permutation σ_k occurs with probability α_k for any $k=1,\ldots,K$. From a given distribution over permutations model, we compute the parameters λ_i and ρ_{ij} for all $i=1,\ldots,n$, $j=0,\ldots,n$ as given by (2.6). Let $\sigma_k(j)$ be the j^{th} product in permutation σ_k. Then

$$\lambda_i = \sum_{k:\sigma_k(1)=i} \alpha_k,\ \forall i=1,\ldots,n$$
$$\rho_{ij} = \sum_{k=1}^{K} \alpha_k \cdot \mathbb{1}(\sigma_k(1)=i,\sigma_k(2)=j),\ \forall i=1,\ldots,n, j=0,\ldots,n,$$

where $\mathbb{1}(\cdot)$ denotes the standard indicator function that is 1 if the argument is true and 0 otherwise. Therefore, for a given distribution over permutations model with a small number of permutations, we can easily compute the Markov chain parameters as described above. We also use the following alternate formulation as decribed in (2.7) to compute the Markov chain parameters and compare the computational performance of both the Markov chain approximations.

$$\lambda_i = \sum_{k:\sigma_k(1)=i} \alpha_k$$
$$\rho_{ij} = \sum_{k=1}^{K} \alpha_k \cdot \mathbb{1}(\exists \ell \in [n],\ \sigma_k(\ell)=i,\ \sigma_k(\ell+1)=j).$$

For our computational experiments, we use $n \geq 10$ and $K = \lceil \log n \rceil$. For each value of n, K, we generate an instance of the distribution over permutations model and compute the Markov chain parameters as described above. We generate $L=100$ random offer sets $S_1,\ldots,S_L$ of size between $n/3$ and $2n/3$ and compare the choice probability for each product in S_ℓ, $\ell=1,\ldots,L$ computed by the Markov chain model with respect to the probability given by the true model. For each $\ell=1,\ldots,L$, we compute the maximum relative error across all products in S_ℓ and then take the average across L different offer sets.

Table 5 describes the average maximum relative error in our computational experiments for both the Markov chain approximations. The results show that the Markov chain model provides a good approximation for the choice probabilities in the distribution over permutations model. Although the approximation is slightly worse as compared to the case of random MMNL models. For instance, for $n=10$, $K=3$, the average relative error is roughly 22%. For $n = 20$, $K = 3$, the relative error is around 14%. The relative error in the Markov chain approximation for random MMNL models for the corresponding values of n, K are significantly smaller. However, it is interesting to note that the Markov chain approximation gets better for larger values of n and K. The maximum relative error between the Markov chain approximation and the true choice probability over all offer sets and products is large. For all values of (n, K), the maximum relative error is more than 50% and for several instances, it is more than 100%. Therefore, in the worst case, the performance of the Markov chain model in approximating the distribution over permutations choice model is not as good as its performance in the case of MMNL model. However, the average performance is quite good as indicated by the results. We also observe that the performance of the Markov chain model where the parameters are computed using the alternate equations (2.7) is quite similar to the Markov chain model computed using (2.6).

	n	K	avg-errMC($\%$) using (2.6)	avg-errMC($\%$) using (2.7)
1.	10	3	22.3	21.2
2.	20	3	14.8	13.3
3.	30	4	10.7	10.2
4.	40	4	9.51	8.24
5.	60	5	4.23	3.97
6.	80	5	3.36	3.51
7.	100	5	3.11	4.16
8.	150	6	2.83	2.15
9.	200	6	1.81	1.65
10.	500	7	0.58	0.66
11.	1000	7	0.25	0.33

Table 5 Relative Error in choice probabilities of the Markov chain model with respect to distribution over permutations model using approximations (2.6) and (2.7).

6.2. Comparison for Assortment Optimization

In this section, we compare the performance of the Markov chain model in assortment optimization for the MMNL model. Rusmevichientong et al. (2010) show that the assortment optimization is NP-hard for a mixture of MNL model even for the case of mixture of only two MNL models. However, we can use the LP (5.7) to efficiently compute an approximation for the assortment optimization problem for the MMNL model. Since our computational experiments show that the Markov chain model provides a good approximation for choice probabilities in the MMNL model, it is plausible to expect that the Markov chain model provides a good approximation for the assortment optimization problem.

Example (Optimal assortment for MMNL model). We first present an instance where the Markov chain model computes an optimal assortment for the MMNL model when the optimal assortment is not a revenue ordered set, i.e., set of top few revenue products. Consider the following MMNL instance which is mixture of two MNL segments over 3 products with parameters

$$\boldsymbol{u}_1 = (u_{0,1}, u_{1,1}, u_{2,1}, u_{3,1}) = (1, 10, 1, 1),\ \boldsymbol{u}_2 = (u_{0,2}, u_{1,2}, u_{2,2}, u_{3,2}) = (10, 1, 1, 100).$$

Also, let the revenues $r_1 = 8, r_2 = 4$ and $r_3 = 3$ and the mixture probabilities, $\theta_1 = \theta_2 = 0.5$. By exhaustive enumeration, we can verify that the assortment $S^* = \{1, 3\}$ is optimal for the MMNL model. Note that this is not a revenue ordered assortment as it skips product 2 with $r_2 > r_3$.

For the above MMNL instance, the optimal assortment, S_{MC} for the corresponding Markov chain model is also $\{1, 3\}$ (same as the optimal assortment S^*). On the other hand, if we consider a MNL approximation for the MMNL model with parameters

$$\boldsymbol{v} = (v_0, v_1, v_2, v_3) = (5.5, 5.5, 1, 50.5),$$

the optimal assortment is the revenue ordered set, $S_{\mathsf{MNL}} = \{1, 2\}$ which is sub-optimal.

The above example illustrates that the Markov chain model is more general than the MNL model and can possibly provide significantly better approximation for the assortment optimization problem. We conduct extensive numerical experiments to study the performance of the Markov chain model.

Experimental Setup. To compare the performance of the Markov chain model in assortment optimization, we generate random instances of the MMNL model with mixture of K MNLs. For any $k = 1, \ldots, K$, let θ_k denote the probability of segment k and $u_{1k}, \ldots, u_{nk}$ denote the MNL parameters of segment k. Assume wlog. $u_{0k} = 1$ for all k. Also, let $r_1, \ldots, r_n$ denote the revenues of the n products. For each instance of the MMNL choice model, we compute the Markov chain parameters as before using (2.2) and compute the optimal assortment S_{MC} for the Markov chain model using the LP formulation (5.7) and (5.4). We compute the expected revenue for S_{MC} as

$$r(S_{\mathsf{MC}}) = \sum_{k=1}^{K} \theta_k \cdot \sum_{j \in S_{\mathsf{MC}}} \frac{r_j u_{jk}}{1 + \sum_{i \in S_{\mathsf{MC}}} u_{ik}}.$$

We consider the MNL approximation as a benchmark. In particular, we consider the MNL approximation as before for the MMNL model with parameters

$$v_j = \sum_{k=1}^{K} \theta_k u_{jk},\ \forall j = 0, \ldots, n.$$

Suppose $r_1 \geq r_2 \geq \ldots \geq r_n$. We compute the optimal assortment, S_{MNL} for the corresponding MNL model by searching over all the nested sets $\{1,\ldots,j\}$ for $j=1,\ldots,n$ (see Talluri and Van Ryzin (2004) and Gallego et al. (2004)). We compute the expected revenue for S_{MNL} as

$$r(S_{\mathsf{MNL}}) = \sum_{k=1}^{K} \theta_k \cdot \sum_{j \in S_{\mathsf{MNL}}} \frac{r_j u_{jk}}{1 + \sum_{i \in S_{\mathsf{MNL}}} u_{ik}}.$$

In addition, we also compute the optimal assortment for the MMNL model by solving a Mixed Integer Program. In particular, let x_j be a binary variable that denotes whether we select product j in the assortment or not. Also, let p_{jk} denote the conditional probability that a customer selects product j given that he belongs to MNL segment k. Then, we can formulate the optimization problem as follows.

$$\begin{aligned}
\mathsf{OPT} = \max \; & \sum_{k=1}^{K} \sum_{j=1}^{n} \theta_k r_j p_{jk} \\
& \sum_{j=0}^{n} p_{jk} = 1, \; \forall k = 1,\ldots,K \\
& p_{jk} \leq u_{jk} p_{0k}, \; \forall j = 1,\ldots,n, \; k = 1,\ldots,K \\
& p_{jk} \leq x_j, \; \forall j = 1,\ldots,n, \; k = 1,\ldots,K \\
& p_{jk} + u_{jk}(1 - x_j) \geq u_{jk} p_{0k}, \; \forall j = 1,\ldots,n, \; k = 1,\ldots,K \\
& p_{jk} \geq 0, \; \forall j = 0,\ldots,n, \forall k = 1,\ldots,K \\
& x_j \in \{0,1\}, \; \forall j = 1,\ldots,n.
\end{aligned} \tag{6.1}$$

It is easy to observe that for any feasible solution of the above MIP, for all $j=1,\ldots,n$ and $k=1,\ldots,K$, $p_{jk}=0$ or $u_{jk}p_{0k}$ depending on whether $x_j=0$ or 1. Therefore, p_{jk} represents the correct conditional probability and (6.1) accurately models the assortment optimization for the MMNL model (recall that we assume wlog. $u_{0k}=1$ for all $k=1,\ldots,K$)

MMNL Instances. We consider a slightly different family of MMNL instances than the random MMNL instances we use in the comparison for choice probability estimates. For the random MMNL instances, while there is a significant difference in performance of the Markov chain model and the MNL approximation for random offer sets, the assortments computed by both approximations have similar performance that are quite close to the optimal MMNL assortment for most instances. Therefore, we consider the following variant of the random MMNL instances. For any number of products n, we use $K=n$. The MMNL parameters, $\boldsymbol{u}_k$ for segment k, revenue r_j, $j=1,\ldots,n$ and segment probabilities α_k, $k=1,\ldots,K=n$ are generated as follows.

$$\begin{aligned}
u_{0k} &= u_{kk} = 1 \\
u_{jk} &= 0,\ \forall j > k \\
u_{jk} &= \begin{cases} 2^{n/2} & \text{with prob. } \frac{2}{n} \\ 0 & \text{otherwise.} \end{cases} \quad \forall j < k \\
r_j &= 2^j,\ \forall j = 1, \ldots, n \\
\theta_k &= \frac{\tau}{2^k},\ \forall k = 1, \ldots, K,
\end{aligned} \tag{6.2}$$

where τ is an appropriate normalizing factor. Let us first present a comparison of choice probabilities between the Markov chain and the MNL approximation for these set of instances. As before, we generate $L = 100$ random offer sets of size between $n/3$ and $2n/3$ and compute the average relative error for both the Markov chain as well as the MNL approximations. Table 6 presents the comparison of the choice probabilities for $n \in [10, 100]$. The performance of the Markov chain model for the MMNL instances (6.2) is slightly worse as compared to its performance for the random MMNL instances. However,it significantly outperforms the MNL approximation. For example, the average relative error in the Markov chain approximation is always less than 23% and less than 15% for most instances. Whereas, the relative error in the MNL approximation is more than 50% in most instances and for several instances even more than 200%. It is interesting to note that unlike the other families of choice models considered in our experiments, the performance of the Markov chain approximation (as well as the MNL approximation) worsens as n increases for the MMNL model above.

We now present our results for the performance of the two models in assortment optimization. Let OPT denote the expected revenue of the optimal assortment for the MMNL model. We compute the relative gaps for the expected revenue of assortments computed by the Markov chain model and the MNL approximation as follows.

$$\mathsf{gapMC} = 100 \cdot \frac{\mathsf{OPT} - r(S_{\mathsf{MC}})}{\mathsf{OPT}}$$

$$\mathsf{gapMNL} = 100 \cdot \frac{\mathsf{OPT} - r(S_{\mathsf{MNL}})}{\mathsf{OPT}},$$

where for any $S \subseteq [n]$, $r(S)$ denote the expected revenue of offer set S in the MMNL model. We use $n \in [10, 100]$ for our computational experiments. Since we need to solve a MIP to compute OPT, the instances with larger values of n can not be solved in a reasonable time. We use Gurobi to solve the MIP (6.1). For each value of n, we generate $L = 25$ instances of the MMNL assortment optimization problem and we report the average relative gaps as well as the maximum relative gaps over all instances for both the Markov chain as well as the MNL approximation. The results are summarized in Table 7.

Our results show that the Markov chain model performs significantly better than the MNL approximation on both the average relative gap as well as the maximum relative gap metrics. In fact, for many instances ($n \leq 40$), the expected revenue of the assortment computed using the Markov chain model is within 2%

n	avg-errMNL(%)	avg-errMC(%)
10	49.5	2.6
20	55.4	5.6
30	85.2	5.8
40	104.7	9.6
50	131.2	11.9
60	170.9	11.7
70	165.1	16.9
80	259.3	13.9
90	282.6	14.1
100	334.2	22.8

Table 6 Relative Error in choice probabilities of the Markov chain model and the MNL model with respect to MMNL model (6.2).

n	gapMNL(%)		gapMC(%)	
	avg	max	avg	max
10	0.5	0.7	0.00018	0.002
20	7.1	21.9	0.0003	0.005
30	26.86	35.23	0.01	0.12
40	35.34	45.4	0.41	2.2
50	42.68	48.36	0.75	5.13
60	45.25	52.38	3.38	9.45
70	51.43	55.25	8.47	16.2
80	55.1	58.5	9.6	24.8
90	56.62	63.1	11.34	22.1
100	59.4	65.64	15.62	26.15

Table 7 Relative Gap in Expected Revenue from OPT for Markov chain and the MNL models for MMNL model (6.2)

of the optimal. The MNL approximation performs significantly worse. For most instances, the expected revenue of the MNL assortment is more than 25% away from optimal and for $n \geq 70$, the average performance is more than 50% away from optimal. The performance of both models worsens as n increases. However, these results show that the Markov chain model provides a good approximation for the assortment optimization problem for the MMNL model.

7. Concluding Remarks

In this paper, we consider a Markov chain based choice model to address the problem of selecting the "right" choice model in assortment optimization. In this model, substitutions from one product to another are modeled as Markovian transitions between the states corresponding to the two products. We give a procedure to compute the parameters for the Markov chain choice model that uses choice probabilities of a certain family of assortments and does not require any additional knowledge of the underlying choice model. We also show that if the choice probabilities arise from an underlying GAM (or MNL as a special case), the Markov chain model is exact. Furthermore, under mild assumptions, the Markov chain model is a good approximation for general random utility based choice models and we give approximation bounds and a family of instances that show that the bound is tight (although the bounds may not be explicitly computable for general models). We also consider the assortment optimization problem for the Markov chain choice model and present a policy iteration and LP based algorithms that compute the optimal assortment in polynomial time.

In addition to the theoretical bounds, we also present computational results to compare the performance of the Markov chain model. Our results show that for random instances of the MMNL model, the Markov chain model performs extremely well. The empirical performance is significantly better than the theoretical bounds and the Markov chain model performs significantly better than the MNL approximation for the MMNL model. We also study the performance of the Markov chain model in assortment optimization for the MMNL model and observe that it outperforms the MNL approximation. The assortment computed by the Markov chain model has expected revenue quite close to the optimal (within 5% of optimal) for many instances. The theoretical and computational results presented in this paper make the Markov chain model a promising practical data-driven approach to modeling choice as well as assortment optimization. In this paper, we present estimation procedures for the Markov chain model assuming we have noiseless and complete data for certain collection of assortments. An important future step would be to study statistical estimation methods to compute Markov chain model parameters from partial noisy data that is typical in most practical applications.

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Appendix A: Proof of Lemmas 4.1, 4.2 and 4.3

Proof of Lemma 4.1 Let $\boldsymbol{\lambda}(S)$ denote the vector of choice probabilities for $j \in S_+$ and $\boldsymbol{\lambda}(\bar{S})$ denote the choice probabilities for $j \notin S$. Therefore,

$$\boldsymbol{\lambda}^T = [\boldsymbol{\lambda}(S)^T \, \boldsymbol{\lambda}(\bar{S})^T].$$

From Theorem 2.1, we know that

$$\begin{aligned}
\hat{\pi}(j,S) &= \lambda_j + \left(\boldsymbol{\lambda}(\bar{S})\right)^T (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{B}\boldsymbol{e}_j \\
&= \lambda_j + \boldsymbol{\lambda}(\bar{S})^T \left(\sum_{t=0}^{\infty} \boldsymbol{C}^t\right) \boldsymbol{B}\boldsymbol{e}_j \\
&\geq \lambda_j + \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{B}\boldsymbol{e}_j \\
&= \lambda_j + \sum_{i \in \bar{S}} \lambda_i \rho_{ij},
\end{aligned}$$

where the second last inequality follows as $C_{ij} \geq 0$ for all $i, j \in \bar{S}$ and $\boldsymbol{\lambda} \geq \mathbf{0}$. From (2.5), for any $i \in \bar{S}$, we have that

$$\rho_{ij} = \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i,\mathcal{N})} \cdot \frac{u_{jk}}{1 - u_{ik}}.$$

Therefore,

$$\hat{\pi}(j,S) \geq \lambda_j + \sum_{i \in \bar{S}} \lambda_i \cdot \left(\sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i,\mathcal{N})} \cdot \frac{u_{jk}}{1 - u_{ik}}\right)$$

$$\begin{aligned}
&= \pi(j,\mathcal{N}) + \sum_{i\in\bar{S}} \pi(i,\mathcal{N}) \left(\sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i,\mathcal{N})} \cdot \frac{u_{jk}}{1-u_{ik}} \right) \\
&= \sum_{k=1}^{K} \theta_k u_{jk} + \sum_{i\in\bar{S}} \sum_{k=1}^{K} \theta_k \cdot u_{ik} \cdot \frac{u_{jk}}{1-u_{ik}} \qquad \text{(A.1)} \\
&= \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{i\in\bar{S}} \frac{u_{ik}}{1-u_{ik}} \right) \\
&= \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{i\in\bar{S}} \left(\sum_{q=1}^{\infty} (u_{ik})^q \right) \right),
\end{aligned}$$

where the second equality follows as $\lambda_j = \pi(j,\mathcal{N})$ for all $j \in \mathcal{N}_+$ and (A.1) follows from the definition of $\pi(j,\mathcal{N})$ for the mixture of MNL model. The last equality follows as $u_{ik} < 1$ for all $i \in \mathcal{N}_+, k = 1,\ldots,K$. $\square$

Proof of Lemma 4.2 From Theorem 2.1, we know that

$$\hat{\pi}(j,S) = \lambda_j + \left(\boldsymbol{\lambda}(\bar{S})\right)^T (\boldsymbol{I}-\boldsymbol{C})^{-1} \boldsymbol{B}\boldsymbol{e}_j = \lambda_j + \sum_{q=0}^{\infty} \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{C}^q \boldsymbol{B}\boldsymbol{e}_j.$$

Note that for all $i \in \mathcal{N}$, there exist $k = 1,\ldots,K$ such that u_{ik} is strictly positive. Therefore, all off-diagonal entries in both matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are strictly positive. Since all row sums of $\boldsymbol{C}$ are strictly less than one, the maximum eigenvalue $\gamma < 1$ and there is an eigenvector $\boldsymbol{v} > \boldsymbol{0}$ corresponding to the maximum eigenvalue of $\boldsymbol{C}$ such that

$$\boldsymbol{v} \le \boldsymbol{\lambda}(\bar{S}) \le \tau(\bar{S})\boldsymbol{v}.$$

Let $\tau = \tau(\bar{S})$. Therefore, for any $q \in \mathbb{Z}_+$,

$$\boldsymbol{\lambda}(\bar{S})^T \boldsymbol{C}^q \boldsymbol{B}\boldsymbol{e}_j \le \tau\gamma^q \cdot \boldsymbol{v}^T \boldsymbol{B}\boldsymbol{e}_j \le \tau\gamma^q \cdot \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{B}\boldsymbol{e}_j. \qquad \text{(A.2)}$$

Now,

$$\begin{aligned}
\hat{\pi}(j,S) &= \lambda_j + \sum_{q=0}^{\infty} \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{C}^q \boldsymbol{B}\boldsymbol{e}_j \\
&\le \lambda_j + \sum_{q=0}^{\infty} \tau \cdot \gamma^q \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{B}\boldsymbol{e}_j \qquad \text{(A.3)} \\
&= \lambda_j + \frac{\tau}{1-\gamma} \cdot \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{B}\boldsymbol{e}_j \\
&= \pi(j,\mathcal{N}) + \frac{\tau}{1-\gamma} \cdot \sum_{i\in\bar{S}} \pi(i,\mathcal{N}) \rho_{ij} \qquad \text{(A.4)} \\
&= \sum_{k=1}^{K} \theta_k u_{jk} + \frac{\tau}{1-\gamma} \cdot \sum_{i\in\bar{S}} \pi(i,\mathcal{N}) \left(\sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i,\mathcal{N})} \cdot \frac{u_{jk}}{1-u_{ik}} \right) \qquad \text{(A.5)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{\tau}{1-\gamma} \cdot \sum_{i \in \bar{S}} \frac{u_{ik}}{1-u_{ik}} \right) \\
&= \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{\tau}{1-\gamma} \cdot \left(\sum_{i \in \bar{S}} \sum_{q=1}^{\infty} (u_{ik})^q \right) \right),
\end{aligned}
$$

where (A.3) follows from (A.2). Equations(A.4) and (A.5) follows from substituting the values of the parameters λ_j and ρ_{ij} for all $i \in \bar{S}$ as computed in (2.5). $\square$

Proof of Lemma 4.3 Note that $\alpha < 1$ since $u_{0k} > 0$ for all $k \in [K]$. For all $j \in \mathcal{N}$, there exists $k \in [K]$ such that $u_{jk} > 0$. Therefore, the transition probability $\rho_{ij} > 0$ for all $i, j \in \mathcal{N}$ as computed in (2.5) which implies that $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$ is irreducible (all entries in $\boldsymbol{C}^2$, the square of $\boldsymbol{C}$ are strictly positive). From Perron-Frobenius theorem (see Theorem 1.4.4 in Bapat and Raghavan (1997)), the maximum eigenvalue, γ is real and positive and the corresponding eigenvector $\boldsymbol{v} > 0$. We show that the maximum eigenvalue of $\boldsymbol{C}$ is bounded by the maximum row sum of $\boldsymbol{C}$. Consider wlog. that $v_1 = 1$ and $0 < v_j \leq 1$ for all $j > 1$. Therefore,

$$
\gamma = \boldsymbol{e}_1^T \boldsymbol{C} \boldsymbol{v} \leq \sum_{j \in \bar{S}} C_{1j} \leq \max_{i \in \bar{S}} \sum_{j \in \bar{S}} C_{ij}.
$$

Now,

$$
\begin{aligned}
\max_{i \in \bar{S}} \sum_{j \in \bar{S}} C_{ij} &= \max_{i \in \bar{S}} \sum_{j \in \bar{S}} \rho_{ij} \\
&= \max_{i \in \bar{S}} \sum_{j \in \bar{S}, j \neq i} \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} \cdot \frac{u_{jk}}{1-u_{ik}} \\
&= \max_{i \in \bar{S}} \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} \cdot \frac{\sum_{j \in \bar{S}, j \neq i} u_{jk}}{1-u_{ik}} \\
&\leq \max_{i \in \bar{S}} \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} \cdot \frac{\alpha - u_{ik}}{1-u_{ik}} &\text{(A.6)} \\
&= \alpha \max_{i \in \bar{S}} \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} \cdot \frac{1-u_{ik}/\alpha}{1-u_{ik}} \\
&< \alpha \max_{i \in \bar{S}} \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} &\text{(A.7)} \\
&= \alpha
\end{aligned}
$$

where (A.6) follows as $u_k(\bar{S}) \leq \alpha < 1$ for all $k = 1, \ldots, K$. Inequality (A.7) follows as $\alpha < 1$ and for any $i \in \bar{S}$, $k = 1, \ldots, K$, $(1 - u_{ik}/\alpha) < (1 - u_{ik})$. The last equality follows as $\sum_{k=1}^{K} \theta_k u_{ik} = \pi(i, \mathcal{N})$. Therefore, $\gamma \leq \alpha$. $\square$

Appendix B: Proofs of Lemma 5.2 and Theorem 5.1

Proof of Lemma 5.2 Based on the proof of Lemma 5.1, we know that $\boldsymbol{g}$ (as defined in (5.3)) is a fixed point of the set of equations (5.6). We first show that (5.6) has a unique fixed point if $\boldsymbol{P}$ has spectral radius strictly less than one. In particular, we show that the following function

$$f : \boldsymbol{g} \rightarrow \{(\max(r_j, \boldsymbol{e}_j^T \boldsymbol{P} \boldsymbol{g})) \mid j = 1, \ldots, n\},$$

is a contraction, i.e., there is a distance metric, d such that for any $\boldsymbol{x}, \boldsymbol{y}$

$$d(f(\boldsymbol{x}), f(\boldsymbol{y})) < d(\boldsymbol{x}, \boldsymbol{y}).$$

Consider the following directed graph $G = (V, A)$ induced by the transition matrix P with $V = \{1, \ldots, n\}$ and $A = \{(i,j) \mid i, j \in \mathcal{N}, P_{ij} > 0\}$. Let $V_1, \ldots, V_L \subseteq V$ be the strongly connected components of G (a component $U \subseteq V$ is *strongly connected* if for all $i, j \in U$ there is a directed path from i to j). Let $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_L$ denote the transition sub-matrices corresponding to each of the strongly connected components. Note that $\boldsymbol{P}_\ell$ for all $\ell = 1, \ldots, L$ is irreducible and let $\boldsymbol{u}_\ell$ be the left eigenvector corresponding to the highest eigenvalue, λ_ℓ of $\boldsymbol{P}_\ell$. From Perron-Frobenius theorem, $\boldsymbol{u}_\ell$ is real and strictly positive for all $\ell = 1, \ldots, L$. Note that for all $\ell = 1, \ldots, L$, $\boldsymbol{u}_\ell$ (with zeroes in remaining components) is also an eigenvector of $\boldsymbol{P}$ with eigenvalue λ_ℓ. Let

$$\boldsymbol{u} = \sum_{\ell=1}^{L} \boldsymbol{u}_\ell, \ \lambda = \max_{\ell=1}^{L} \lambda_\ell.$$

Here we consider each $\boldsymbol{u}_\ell$ in n dimension with $n - |V_\ell|$ components as zeroes. Since $\boldsymbol{P}$ has spectral radius strictly less than one, $0 < \lambda < 1$. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, consider the following distance norm,

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{n} u_j |x_j - y_j|.$$

Now, consider any $\boldsymbol{x}, \boldsymbol{y}$ and consider $d(f(\boldsymbol{x}), f(\boldsymbol{y}))$.

$$\begin{aligned}
d(f(\boldsymbol{x}), f(\boldsymbol{y})) &= \sum_{j=1}^{n} u_j \left| \max(r_j, \boldsymbol{e}_j^T \boldsymbol{P}\boldsymbol{x}) - \max(r_j, \boldsymbol{e}_j^T \boldsymbol{P}\boldsymbol{y}) \right| \\
&\leq \sum_{j=1}^{n} u_j \left| \boldsymbol{e}_j^T \boldsymbol{P}\boldsymbol{x} - \boldsymbol{e}_j^T \boldsymbol{P}\boldsymbol{y} \right| \qquad \text{(B.1)} \\
&= \sum_{j=1}^{n} u_j \left| \sum_{k=1}^{n} P_{jk}(x_k - y_k) \right| \\
&\leq \sum_{j=1}^{n} \sum_{k=1}^{n} u_j P_{jk} \left| x_k - y_k \right| \\
&= \sum_{k=1}^{n} |x_k - y_k| \left(\sum_{\ell=1}^{L} \boldsymbol{u}_\ell^T \boldsymbol{P} \boldsymbol{e}_k \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} |x_k - y_k| \left(\sum_{\ell=1}^{L} \lambda_\ell \boldsymbol{u}_\ell^T \boldsymbol{e}_k \right) \\
&\leq \sum_{k=1}^{n} |x_k - y_k| \, \lambda \left(\sum_{\ell=1}^{L} \boldsymbol{u}_\ell \right)^T \boldsymbol{e}_k \qquad \text{(B.2)} \\
&= \sum_{k=1}^{n} |x_k - y_k| \, \lambda u_k \\
&= \lambda d(\boldsymbol{x}, \boldsymbol{y}) < d(\boldsymbol{x}, \boldsymbol{y}),
\end{aligned}
$$

where (B.1) follows as

$$|\max(a,b) - \max(a,c)| \leq |b - c|,$$

for any $a, b, c \in \mathbb{R}$. Inequality (B.2) follows as $\lambda_\ell \leq \lambda$ for all $\ell = 1, \ldots, L$ and $\boldsymbol{u} = \boldsymbol{u}_1 + \ldots + \boldsymbol{u}_L$, and the last inequality follows as $0 < \lambda < 1$. Therefore, f is a contraction map. From Banach's fixed point theorem (see Theorem 9.23 in Rudin (1964)), f has a unique fixed point.

We now show that the optimal solution for the LP (5.7), say $\boldsymbol{g}^*$ is the unique fixed point of (5.6). For any $j \in \mathcal{N}$, we claim that

$$g_j^* = r_j, \text{ or } g_j^* = \boldsymbol{e}_j^T \boldsymbol{P} \boldsymbol{g}^*.$$

Suppose this is not the case. Then we can decrease g_j^* to obtain an alternate feasible solution for (5.7) with a strictly smaller objective value; a contradiction. Therefore, for all $j \in \mathcal{N}$,

$$g_j^* = \max(r_j, \boldsymbol{e}_j^T \boldsymbol{P} \boldsymbol{g}^*),$$

which implies that $\boldsymbol{g}^*$ is the required fixed point. □

Proof of Theorem 5.1 Let $\boldsymbol{g}$ be the optimal solution of LP (5.7) and $S = \{j \in \mathcal{N} \mid g_j = r_j\}$. Note that $\boldsymbol{g}$ is the fixed point of (5.6). From Lemma 5.2, we know that for all $j \in \mathcal{N}$, g_j is the maximum expected revenue starting from state j. Recall that for any $Q \subseteq \mathcal{N}$, $g_j(Q)$ denotes the expected revenue starting from state j in Markov chain $\mathcal{M}(Q)$ where all states in Q are absorbing states.

We first show that for each starting state $j \in \mathcal{N}$, the maximum expected revenue is achieved for the offer set (or absorbing states) S, i.e. $g_j(S) = g_j$ for all $j \in \mathcal{N}$. Clearly, for any $j \in S$, $g_j = r_j = g_j(S)$. For all $j \notin S$, $g_j > r_j$. Therefore,

$$g_j = \sum_{i \in \mathcal{N}} \rho_{ji} \cdot g_i = \sum_{i \in S} \rho_{ji} \cdot r_i + \sum_{i \notin S} \rho_{ji} g_i, \ \forall j \notin S. \qquad \text{(B.3)}$$

Let $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$ be the probability transition sub-matrix from states in $\bar{S}$ to $\bar{S}$, and $\boldsymbol{B} = \boldsymbol{\rho}(\bar{S}, S)$. Then, the above equation can be formulated as

$$(\boldsymbol{I} - \boldsymbol{C}) \boldsymbol{g}_{\bar{S}} = \boldsymbol{B} \boldsymbol{r}_S, \qquad \text{(B.4)}$$

where $\boldsymbol{g}_{\bar{S}}$ is the restriction of $\boldsymbol{g}$ to $\bar{S}$ and similarly, $\boldsymbol{r}_S$ is the restriction of $\boldsymbol{r}$ on S. Note that for all $j \notin S$, $g_j(S)$ also satisfy (B.3) and consequently (B.4). Since $(\boldsymbol{I} - \boldsymbol{C})$ is an M-matrix, (B.4) has a unique non-negative solution which implies $g_j = g_j(S)$ for all $j \in \mathcal{N}$.

Now, consider an optimal assortment, S^* for (5.2). Therefore,

$$\mathsf{OPT} = r(S^*) = \sum_{j \in \mathcal{N}} \lambda_j \cdot g_j(S^*),$$

where $g_j(S^*)$ is the expected revenue starting from state j and stopping on the set of states in S^*. Clearly, for all $j \in \mathcal{N}$, $g_j(S^*) \leq g_j$. Therefore,

$$r(S) = \sum_{j \in \mathcal{N}} \lambda_j \cdot g_j(S) = \sum_{j \in \mathcal{N}} \lambda_j \cdot g_j \geq \sum_{j \in \mathcal{N}} \lambda_j \cdot g_j(S^*) = \mathsf{OPT},$$

where the second equality follows from the fact that $g_j(S) = g_j$ for all $j \in \mathcal{N}$, and the third inequality follows as $g_j \geq g_j(S^*)$ for all j. $\square$