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Determining maximal efficient faces in multiobjective linear programming problem

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ABSTRACT

Finding an efficient or weakly efficient solution in a multiobjective linear programming (MOLP) problem is not a difficult task. The difficulty lies in finding all these solutions and representing their structures. Since there are many convenient approaches that obtain all of the (weakly) efficient extreme points and (weakly) efficient extreme rays in an MOLP, this paper develops an algorithm which effectively finds all of the (weakly) efficient maximal faces in an MOLP using all of the (weakly) efficient extreme points and extreme rays. The proposed algorithm avoids the degeneration problem, which is the major problem of the most of previous algorithms and gives an explicit structure for maximal efficient (weak efficient) faces. Consequently, it gives a convenient representation of efficient (weak efficient) set using maximal efficient (weak efficient) faces. The proposed algorithm is based on two facts. Firstly, the efficiency and weak efficiency property of a face is determined using a relative interior point of it. Secondly, the relative interior point is achieved using some affine independent points. Indeed, the affine independent property enable us to obtain an efficient relative interior point rapidly.

1. Introduction

Multiobjective linear programming (MOLP) is one of the most popular models used in multiple criteria decision making. Numerous studies and applications of multicriteria problems and MOLP have been reported in the literature in literally hundreds of books, monographs, articles and chapters in books. For an overview of these studies and applications, see, for instance [1,7,9,12,16,19] and references therein. An MOLP problem is to minimize or maximize several linear objective functions subject to a set of linear constraints. As most of the real business decision making problems involve more than one objective, the MOLP model has been widely applied in many fields and has become a useful tool for decision making [5,13,20].

An efficient solution of an MOLP problem is a solution that cannot improve some objectives without sacrificing others. A weakly efficient solution of an MOLP problem is a solution that cannot improve all the objectives simultaneously.

In a decision making process involving multiple objective programming models, to choose a most preferred efficient solution, the decision makers often analyze subsets of the efficient set. In the case when the decision maker's preference is quantifiable and given explicitly in the form of a function $f : \mathbb{R}^n \to \mathbb{R}$, then the problem for finding a most preferred efficient solution can be stated in the form "optimize $\{f(x) \mid x \in X_E\}$" where X_E is the efficient set of MOLP [19]. Since the efficient set is not convex it is a difficult task to optimize f over this set directly; thus the most convenient way for

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optimizing f over the efficient set is to decompose it into some convex sets. On the other hand, for an MOLP, it is easily seen that the set of all maximal efficient faces is a convex decomposition (its elements are convex sets) with the least number of elements of the efficient set. So, in order to optimize f over X_E it is sufficient to maximize f over all maximal efficient faces and obtain an overall optimal efficient solution. In the case when f is a linear function the above mentioned optimization problem can be solved using finite number of linear programming (LP) problems corresponding to efficient maximal faces; therefore, obtaining the maximal efficient faces is the most critical task in MOLP [19].

In the earlier time, many works were mainly conducted for identifying efficient solutions effectively, by solving multiple parametric programming or by using multiple criteria simplex method to examine the adjacent extreme points (Dantzig [6], Geoffrion [10], Yu and Zeleny [21], Philip [15]). Yu and Zeleny [21] used a global view method and the top-down search strategy, while Philip [15] used a local view method to obtain the efficient face incident to a given efficient extreme point. Isermann [12] and Ecker et al. [8] combined these two view methods. Later on, Armand and Malivert [1] and Armand [2] applied a bottom-up search strategy to develop an algorithm. However, all these previous algorithms considered the degeneration problem as their main difficulty to deal with, and the representation of the efficient solutions set was not clearly given. Benson [4] and Sayin [18] proposed to obviate the degeneration problem by employing the facial decomposition approach and the top-down search strategy first used by Yu and Zeleny [21]. Discrete representation of the efficient solutions set were given in these papers. Also, there are many approaches that just obtaining efficient extreme points and efficient extreme rays, see, for instance [4,8,11] and references therein [20]. As approach developed by Yan et al. in [20], the approach proposed in this paper avoids the degeneration problem. So, based on this point of view, the proposed algorithm is more suitable than the previous works beside for once given in [20]. Although the approach of Yan et al. avoids the degeneration problem, it has still some difficulties. As it is mentioned in Sections 3 and 6, the algorithm of Yan et al. computes all of the extreme directions of a cone and solves so many LPs in each iteration. But the approach developed in this paper using less computations than Yan et al.'s, improves it by adding the maximality criterion of efficient faces to avoid including those redundant faces.

Finding an efficient or weakly efficient solution in a multiobjective linear programming (MOLP) problem is not a difficult task. The difficulty lies in finding all these solutions and representing their structures [20]. Since there are many convenient approaches that obtain all of the (weakly) efficient extreme points and (weakly) efficient extreme rays in an MOLP [3,7,10], this paper develops an algorithm which effectively finds all of the (weakly) efficient maximal faces in an MOLP using all of the (weakly) efficient extreme points and extreme rays.

2. Preliminaries

The following definitions and results from linear algebra are needed for the next discussions [3,14].

Definition 2.1. Let $x^1, x^2, \dots, x^k \in \mathbb{R}^n$ and $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$. Then linear combination $\alpha_1 x^1 + \alpha_2 x^2 + \dots + \alpha_k x^k$ is an affine combination of $x^1, x^2, \dots, x^k$ if $\alpha_1 + \alpha_2 + \dots + \alpha_k = 1$.

Definition 2.2. Vectors $x^1, x^2, \dots, x^k \in \mathbb{R}^n$ are affine independent if $x^2 - x^1, x^3 - x^1, \dots, x^k - x^1$ are linearly independent. They are called affine dependent if they are not affine independent.

The following corollary is concluded from the previous definitions.

Corollary 2.1. *Vectors $x^1, x^2, \dots, x^k \in \mathbb{R}^n$ are affine independent if any of them cannot be represented as an affine combination of the others.*

Consider the following MOLP problem:

$$\begin{aligned} V:\quad & \min \quad Cx \\ & \text{s.t.} \quad x \in X = \left\{ x \in \mathbb{R}^n \;\middle|\; Ax \geqslant b,\ x \geqslant 0 \right\}, \end{aligned} \tag{2.1}$$

where

C is a $p \times n$ matrix, whose rows are row vectors c_i, $i = 1, 2, \dots, p$;
$x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are column vectors;
A is an $m \times n$ matrix.

Considering $\bar{A} = \binom{A}{I_{n\times n}}$ and $\bar{b} = \binom{b}{0_{n\times 1}}$, (2.1) can be rewritten as follows:

$$\begin{aligned} V:\quad & \min \quad Cx \\ & \text{s.t.} \quad \bar{A}x \geqslant \bar{b}. \end{aligned} \tag{2.2}$$

The weakly efficient solution and efficient solution are defined as follows [17].

Definition 2.3. Let $\bar{x} \in X$. Then $\bar{x}$ is called an efficient solution of (2.1) if there does not exist $x \in X$ such that

$$Cx \leqslant C\bar{x}, \qquad Cx \neq C\bar{x}.$$

The set of efficient solutions of (2.1) is denoted by X_E.

Definition 2.4. Let $\bar{x} \in X$. Then $\bar{x}$ is called a weakly efficient solution of (2.1) if there does not exist $x \in X$ such that

$$Cx < C\bar{x}.$$

The set of weakly efficient solutions of (2.1) is denoted by X_{WE}. Referring to Definitions 2.3 and 2.4, it is obvious that $X_E \subseteq X_{WE}$, and $\bar{x}$ cannot be an inner point of X if $\bar{x} \in X_E \cup X_{WE}$ [20].

Definition 2.5. A nonempty subset F of X is called a face of it if there is $I \subseteq \{1, 2, \ldots, n+m\}$ such that $F = \{x \in X \mid \bar{a}_l x = \bar{b}_l \text{ for all } l \in I\}$ [14].

Definition 2.6. A face F of X is said to be an efficient face if all of its points are efficient [17].

Definition 2.7. An efficient face F of X is said to be a maximal efficient face if there is no efficient face G of X such that $F \subsetneqq G$ [17].

Theorem 2.1. *Let $\bar{x} \in X$. Then $\bar{x}$ is an efficient solution of* (2.1) *if and only if there exists a $\lambda \geqslant e$ such that $\bar{x}$ is an optimal solution for the following LP problem*:

$$\begin{aligned} \min \quad & \lambda^t Cx \\ s.t. \quad & Ax \geqslant b, \\ & x \geqslant 0, \end{aligned}$$

where $e^t = (1, \ldots, 1)$ [17].

There are many efficiency criteria for examining the efficiency of a given point $\bar{x} \in X$, three of them are as follows [17].

Theorem 2.2. *Let $\bar{x} \in X$. Then $\bar{x}$ is an efficient solution of* (2.1) *if and only if the optimal value of the following problem is $e^t C\bar{x}$*:

$$\begin{aligned} \min \quad & e^t Cx \\ s.t. \quad & Cx \leqslant C\bar{x}, \\ & Ax \geqslant b, \\ & x \geqslant 0, \end{aligned}$$

in which $e^t = (1, \ldots, 1)$.

Theorem 2.3. *Let $\bar{x} \in X$. Then $\bar{x}$ is a weak efficient solution of* (2.1) *if and only if the following problem is feasible*:

$$\begin{aligned} & G^t y - C^t \lambda = 0, \\ & e^t \lambda = 1, \\ & y \geqslant 0, \quad \lambda \geqslant 0, \end{aligned}$$

in which $e^t = (1, \ldots, 1)$ and G is a matrix containing those rows of $\bar{A}$ corresponding to binding constraints in $\bar{x}$.

Theorem 2.4. *Let $\bar{x} \in X$. Then $\bar{x}$ is an efficient solution of* (2.1) *if and only if the following problem is feasible*:

$$\begin{aligned} & G^t y - C^t \lambda = C^t e, \\ & y \geqslant 0, \quad \lambda \geqslant 0, \end{aligned}$$

in which $e^t = (1, \ldots, 1)$ and G is a matrix containing those rows of $\bar{A}$ corresponding to binding constraints in $\bar{x}$.

Now, consider a well-known theorem that examines the efficiency of a given face using any relative interior point of it [17].

Theorem 2.5. *Let F be a face of X. Then F is efficient if and only if it has an efficient relative interior point.*

3. The algorithm of Yan et al. (A1)

In this section the recent worthwhile algorithm developed by Yan et al. [20] is explained. For simplicity we call it A1. Yan et al. at first determine the weak efficient solutions set of MOLP problem and give a representation of the weak efficient solutions set. To this aim, they first give an algorithm for finding all weak efficient solutions in the objective space, and present the structure of weak efficient solutions. The algorithm is given as follows [20]:

Step 1. For $i = 1, \dots, p$ solve the following linear programming problem:

$$\begin{aligned} P(e_i):\quad & c_i x \\ \text{s.t.}\quad & x \in R = \{x \in \mathbb{R}^n \mid Ax \geqslant b,\ x \geqslant 0\}. \end{aligned}$$

Let x^i, $i = 1, \dots, p$, be optimal solutions of $P(e_i)$. Denote $\bar{R}^1 = \{x^1, \dots, x^p\}$. For convenience, denote $k_1 = p$.

Step 2. Let

$$S^1 = \left\{ \begin{pmatrix} G \\ \alpha \end{pmatrix} \;\middle|\; (Cx^i)^t G \geqslant \alpha,\ G \neq 0,\ i = 1, \dots, k_1 \right\}.$$

Obtain extreme rays of S^1 and denote the rays by $\binom{\upsilon^l}{\alpha_l}$, $l = 1, \dots, r_1$, namely,

$$S^1 = \left\{ \sum_{l=1}^{r_1} \beta_l \begin{pmatrix} \upsilon^l \\ \alpha_l \end{pmatrix} \;\middle|\; \beta_l \geqslant 0,\ l = 1, \dots, r_1 \right\}.$$

Then get r_1 weights $\upsilon^1, \dots, \upsilon^{r_1}$ and denote

$$P^1 = \{F \mid (\upsilon^l)^t F \geqslant \alpha_l,\ l = 1, \dots, r_1\}.$$

Step 3. For $l = 1, \dots, r_1$, solve the weighted sum problem $P(\upsilon^l)$ associated with weight υ^l:

$$\begin{aligned} P(\upsilon^l):\quad \min\quad & (\upsilon^l)^t Cx \\ \text{s.t.}\quad & x \in R. \end{aligned}$$

Let $\bar{x}^l$ be an optimal solution of $P(\upsilon^l)$. Denote an index set $I_1 = \{l \mid C\bar{x}^l \notin P^1,\ 1 \leqslant l \leqslant r_1\}$.

Step 4. If $I_1 = \emptyset$, then denote $k = k_1$, $r = r_1$ and stop, else go to Step 5.

Step 5. Denote

$$\bar{R}^2 = \bar{R}^1 \cup \{\bar{x}^l \mid l \in I_1\}.$$

Without loss of generality, denote

$$\bar{R}^2 = \{x^1, x^2, \dots, x^p, x^{p+1}, \dots, x^{k_2}\}, \quad k_2 > p.$$

Repeat from Step 2 to Step 5 starting at $\bar{R}^2$ and k_2.

Since R has finite number of extreme points, this algorithm will finally end after finite steps. When the algorithm stops, k weak efficient solutions of (2.1) are obtained. Denote the solutions by $x^1, \dots, x^k$, and their images in the objective space are $F^1 = Cx^1$, $F^2 = Cx^2, \dots, F^k = Cx^k$. So, $Cx^1, Cx^2, \dots, Cx^k$ are weak efficient solutions [20] of

$$\begin{aligned} \min\quad & (f_1, \dots, f_p) \\ \text{s.t.}\quad & F \in F(R) = \{Cx \mid x \in R\}. \end{aligned} \tag{3.1}$$

Now

$$Q = \left\{ F \;\middle|\; F = \sum_{i=1}^{k} \lambda_i (Cx^i),\ \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geqslant 0,\ i = 1, \dots, k \right\} + \mathbb{R}^p_+$$

and

$$S = \left\{ \begin{pmatrix} G \\ \alpha \end{pmatrix} \;\middle|\; \left(Cx^i\right)^t G \geqslant \alpha,\ G \neq 0,\ i = 1, \ldots, k \right\}.$$

All extreme rays of S are denoted by $\binom{\upsilon^l}{\alpha_l}$, $l = 1, \ldots, r$, then

$$S = \left\{ \sum_{l=1}^{r} \beta_l \begin{pmatrix} \upsilon^l \\ \alpha_l \end{pmatrix} \;\middle|\; \beta_l \geqslant 0,\ l = 1, \ldots, r \right\}.$$

Let

$$P = \left\{ F \mid \left(\upsilon^l\right)^t F \geqslant \alpha_l,\ l = 1, \ldots, r \right\}.$$

It can be shown (Theorem 1 in [20]) that $P = Q$. Since the algorithm stops, we know that the optimal solution $\bar{x}^l$ of the linear programming

$$\begin{aligned} P\left(\upsilon^l\right): \quad & \min \quad \left(\upsilon^l\right)^t Cx \\ & \text{s.t.} \quad x \in R, \end{aligned}$$

must satisfy $C\bar{x}^l \in P$, $l = 1, \ldots, r$.

Theorem 3.1. *In the objective space, the weak efficient solutions set of* (3.1) *denoted by* F_{WE} *can be represented as*

$$F_{WE} = \bigcup_{l=1}^{r} \left\{ F \mid F \in F(R),\ \left(\upsilon^l\right)^t F = \alpha_l \right\}.$$

Proof. See [20]. □

Theorem 3.2. *The weak efficient solutions set of problem* (2.1) *denoted by* X_{WE} *can be represented as*

$$X_{WE} = \bigcup_{l=1}^{r} \left\{ x \mid Ax = b,\ x \geqslant 0,\ \left(\upsilon^l\right)^t Cx = \alpha_l \right\}.$$

Proof. See [20]. □

3.1. Structure of efficient solutions set

In order to obtain the efficient solutions of MOLP problem (2.1), Yan et al. [20] first consider MOLP problem (3.1) in the objective space. The following theorem gives a rule to test if $\bar{F} \in F_{WF}$ is an efficient solution of (3.1).

Theorem 3.3. *When the algorithm stops at Step* 4*, let* $\bar{F} \in F_{WF}$ *and denote index set* J *as*

$$J = \left\{ l \mid \left(\upsilon^l\right)^t F = \alpha_l,\ 1 \leqslant l \leqslant r \right\}.$$

Then $\bar{F}$ *is an efficient solution of* (3.1) *if and only if* $\sum_{l \in J} \upsilon_l > 0$.

Proof. See [20]. □

Theorem 3.3 gives a sufficient and necessary condition to test if $\bar{F} \in F_{WF}$ is an efficient solution of (3.1). The structure of efficient solutions of (3.1) may be more complicated than that of weak efficient solutions. But according to the test rule in Theorem 3.3, one can select all efficient solutions of (3.1) from F_{WF} step by step. In [20], they test if there exists an l $(1 \leqslant l \leqslant r)$, such that $\upsilon_l > 0$, then we test if there exist l_1 and l_2 $(1 \leqslant l_1 < l_2 \leqslant r)$, such that $\upsilon^{l_1} + \upsilon^{l_2} > 0$, and so forth. In each step, if an index set $B \subseteq \{1, 2, \ldots, r\}$ satisfies $\sum_{l \in B} \upsilon^l > 0$, then the sets containing B as a subset need not be tested again. After all these index sets are tested, the efficient solutions set of (3.1) is found. Like the results in Theorems 3.1 and 3.2, one can obtain F_E and X_E in the same way as that of F_{WF} and X_{WE}. Then the representation of X_E can be obtained. In particular, from Theorem 3.3, when $p = 2$, if $(P(e_1))$ and $(P(e_2))$ do not have one same optimal solution, then

$$X_E = \bigcup_{\upsilon^l > 0} \left\{ x \in X \mid \left(\upsilon^l\right)^t Cx = \alpha_l \right\}.$$

3.2. Numerical example

Here, we find the maximal efficient (weak efficient) faces of problem (3.2) using algorithm A1 [20]. In this example $X_{WE} = X_E$.

Example 3.1. Consider the following MOLP.

$$
\begin{aligned}
V:\quad \min\quad & (-x_1, -x_2, -x_3) \\
\text{s.t.}\quad & -x_1 - x_2 - 2x_3 \geqslant -10, \\
& -x_1 - x_2 - x_3 \geqslant -6, \\
& -3x_1 - 3x_2 - x_3 \geqslant -12, \\
& x_1, x_2, x_3 \geqslant 0.
\end{aligned} \tag{3.2}
$$

Solving (3.2) by A1 contains following computations:

Iteration I:

- Solving three LPs in Step 1, the set of optimal solutions is as follows:

$$\bar{R}^1 = \{(4,0,0)^t, (0,4,0)^t, (0,0,5)^t\}.$$

- Finding all of the extreme rays of S^1 which are as follows:

$$
\begin{pmatrix} 0 \\ 0 \\ 0.1667 \\ -0.8333 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.2000 \\ 0 \\ -0.8000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.1724 \\ 0.1379 \\ -0.6897 \end{pmatrix}, \begin{pmatrix} 0.0769 \\ 0 \\ 0 \\ -0.3077 \end{pmatrix},
$$
$$
\begin{pmatrix} 0.1000 \\ 0.1000 \\ 0 \\ -0.4000 \end{pmatrix}, \begin{pmatrix} 0.1020 \\ 0 \\ 0.0816 \\ -0.4082 \end{pmatrix}, \begin{pmatrix} 0.1471 \\ 0.1471 \\ 0.1176 \\ -0.5882 \end{pmatrix}.
$$

- Solving seven LPs (corresponding to weights obtained in Step 2) in Step 3, the optimal solutions are as follows:

$$
\begin{aligned}
&x^{1^*} = (0,0,5)^t, \quad x^{2^*} = (0,4,0)^t, \quad x^{3^*} = (0,3,3)^t, \quad x^{4^*} = (4,0,0)^t, \\
&x^{5^*} = (4,0,0)^t, \quad x^{6^*} = (3,0,3)^t, \quad x^{7^*} = (3,0,3)^t,
\end{aligned}
$$

in which x^{l^*} is an optimal solution of $P(v^l)$ $(1 \leqslant l \leqslant 7)$. Since $x^{3^*}, x^{5^*}, x^{7^*} \notin P^1$, $I_1 = \{3,5,7\}$ and

$$\bar{R}^2 = \{(4,0,0)^t, (0,4,0)^t, (0,0,5)^t, (3,0,3)^t, (0,3,3)^t\}.$$

Iteration II:

- Finding all of the extreme rays of S^2 (ten extreme rays) in Step 2 which are as follows:

$$
\begin{pmatrix} 0 \\ 0 \\ 0.1667 \\ -0.8333 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.1000 \\ 0.1500 \\ -0.7500 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.2000 \\ 0 \\ -0.8000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.0682 \\ 0.0227 \\ -0.2727 \end{pmatrix}, \begin{pmatrix} 0.0417 \\ 0 \\ 0.0625 \\ -0.3125 \end{pmatrix},
$$
$$
\begin{pmatrix} 0.0556 \\ 0.0556 \\ 0.0833 \\ -0.4167 \end{pmatrix}, \begin{pmatrix} 0.0556 \\ 0 \\ 0 \\ -0.2222 \end{pmatrix}, \begin{pmatrix} 0.0682 \\ 0 \\ 0.0227 \\ -0.2727 \end{pmatrix}, \begin{pmatrix} 0.0833 \\ 0.0833 \\ 0 \\ -0.3333 \end{pmatrix}, \begin{pmatrix} 0.1154 \\ 0.1154 \\ 0.0385 \\ -0.04615 \end{pmatrix}.
$$

- Solving ten LPs (corresponding to new weights obtained in Step 2) in Step 3 (two of them have been solved in iteration I), the following optimal solutions are as follows:

$$
\begin{aligned}
&x^{1^*} = (0,0,5)^t, \quad x^{2^*} = (0,2,4)^t, \quad x^{3^*} = (0,4,0)^t, \quad x^{4^*} = (0,4,0)^t, \quad x^{5^*} = (2,0,4)^t, \\
&x^{6^*} = (2,0,4)^t, \quad x^{7^*} = (4,0,0)^t, \quad x^{8^*} = (4,0,0)^t, \quad x^{9^*} = (4,0,0)^t, \quad x^{10^*} = (4,0,0)^t.
\end{aligned}
$$

Since $x^{2^*}, x^{5^*}, x^{6^*}, x^{7^*}, x^{8^*}, x^{9^*}, x^{10^*} \notin P^1$, $I_1 = \{2, 5, 6, 7, 8, 9, 10\}$ and

$$\bar{R}^3 = \{(4,0,0)^t, (0,4,0)^t, (0,0,5)^t, (3,0,3)^t, (0,3,3)^t, (0,2,4)^t, (2,0,4)^t\}.$$

Iteration III:

- Finding all of the extreme rays of S^2 (thirteen extreme rays) in Step 3 which are as follows:

$$\begin{pmatrix}0\\0\\0.1667\\-0.8333\end{pmatrix}, \begin{pmatrix}0\\0.0769\\0.1538\\-0.7692\end{pmatrix}, \begin{pmatrix}0\\0.1250\\0.1250\\-0.7500\end{pmatrix}, \begin{pmatrix}0\\0.0357\\0\\-0.1429\end{pmatrix}, \begin{pmatrix}0\\0.0455\\0.0152\\-0.1818\end{pmatrix},$$
$$\begin{pmatrix}0.0238\\0\\0.0476\\-0.2381\end{pmatrix}, \begin{pmatrix}0.0294\\0.0294\\0.0588\\-0.2941\end{pmatrix}, \begin{pmatrix}0.0417\\0\\0\\-0.1667\end{pmatrix}, \begin{pmatrix}0.0417\\0\\0.0417\\-0.2500\end{pmatrix}, \begin{pmatrix}0.0556\\0\\0.0185\\-0.2222\end{pmatrix},$$
$$\begin{pmatrix}0.0625\\0.0625\\0.0625\\-0.3750\end{pmatrix}, \begin{pmatrix}0.0625\\0.0625\\0\\-0.2500\end{pmatrix}, \begin{pmatrix}0.1000\\0.1000\\0.0333\\-0.4000\end{pmatrix}.$$

- Solving thirteen LPs (corresponding to new weights obtained in Step 2) in Step 3 (one of them has been solved in iteration I), the optimal solutions are as follows:

$$x^{1^*} = (0,0,5)^t,\quad x^{2^*} = (0,2,4)^t,\quad x^{3^*} = (0,3,3)^t,\quad x^{4^*} = (0,4,0)^t,\quad x^{5^*} = (0,4,0)^t,$$
$$x^{6^*} = (2,0,4)^t,\quad x^{7^*} = (2,0,4)^t,\quad x^{8^*} = (4,0,0)^t,\quad x^{9^*} = (3,0,3)^t,\quad x^{10^*} = (4,0,0)^t,$$
$$x^{11^*} = (3,0,3)^t,\quad x^{12^*} = (4,0,0)^t,\quad x^{13^*} = (4,0,0)^t.$$

Since $I_1 = \emptyset$ in Step 4, A1 terminates with the following representation for X_{WE} $(= X_E)$:

$$X_{WE} = \left\{(x_1,x_2,x_3)^t \;\middle|\; (x_1,x_2,x_3)^t \in \bigcup_{l=1}^{13} X_{WE}^l\right\}$$

where

$$X_{WE}^l = \{x \in X,\ (\upsilon^l)^t Cx = \alpha_l\},\quad l = 1,\dots,13.$$

By computing all extreme points and extreme directions of X_{WE}^l's, we have

$$\begin{aligned}
&X_{WE}^1 = \{(0,0,5)^t\};\\
&X_{WE}^2 = \{\lambda_1(0,2,4)^t + \lambda_2(0,0,5)^t \mid \lambda_1+\lambda_2 = 1,\ \lambda_1,\lambda_2 \geqslant 0\};\\
&X_{WE}^3 = \{\lambda_1(0,3,3)^t + \lambda_2(0,2,4)^t \mid \lambda_1+\lambda_2 = 1,\ \lambda_1,\lambda_2 \geqslant 0\};\\
&X_{WE}^4 = \{(0,4,0)^t\};\\
&X_{WE}^5 = \{\lambda_1(0,4,0)^t + \lambda_2(0,3,3)^t \mid \lambda_1+\lambda_2 = 1,\ \lambda_1,\lambda_2 \geqslant 0\};\\
&X_{WE}^6 = \{\lambda_1(2,0,4)^t + \lambda_2(0,0,5)^t \mid \lambda_1+\lambda_2 = 1,\ \lambda_1,\lambda_2 \geqslant 0\};\\
&X_{WE}^7 = \{\lambda_1(2,0,4)^t + \lambda_2(0,2,4)^t + \lambda_3(0,0,5)^t \mid \lambda_1+\lambda_2+\lambda_3 = 1,\ \lambda_1,\lambda_2,\lambda_3 \geqslant 0\};\\
&X_{WE}^8 = \{(4,0,0)^t\};\\
&X_{WE}^9 = \{\lambda_1(3,0,3)^t + \lambda_2(2,0,4)^t \mid \lambda_1+\lambda_2 = 1,\ \lambda_1,\lambda_2 \geqslant 0\};\\
&X_{WE}^{10} = \{\lambda_1(4,0,0)^t + \lambda_2(3,0,3)^t \mid \lambda_1+\lambda_2 = 1,\ \lambda_1,\lambda_2 \geqslant 0\};\\
&X_{WE}^{11} = \{\lambda_1(3,0,3)^t + \lambda_2(0,3,3)^t + \lambda_3(0,2,4)^t + \lambda_4(2,0,4)^t \mid \lambda_1+\lambda_2+\lambda_3+\lambda_4 = 1,\ \lambda_1,\lambda_2,\lambda_3,\lambda_4 \geqslant 0\};\\
&X_{WE}^{12} = \{\lambda_1(4,0,0)^t + \lambda_2(0,4,0)^t \mid \lambda_1+\lambda_2 = 1,\ \lambda_1,\lambda_2 \geqslant 0\};\\
&X_{WE}^{13} = \{\lambda_1(4,0,0)^t + \lambda_2(0,4,0)^t + \lambda_3(3,0,3)^t + \lambda_4(0,3,3)^t \mid \lambda_1+\lambda_2+\lambda_3+\lambda_4 = 1,\ \lambda_1,\lambda_2,\lambda_3,\lambda_4 \geqslant 0\}.
\end{aligned}$$

Since in this example $X_{WE} = X_E$, the above representation can be applied for the efficient set.

Remark 3.1. It can be seen that faces X^1_{WE}, X^2_{WE}, X^3_{WE}, X^4_{WE}, X^5_{WE}, X^6_{WE}, X^8_{WE}, X^9_{WE}, X^{10}_{WE}, X^{12}_{WE} are redundant. Indeed, they are subsets of faces X^7_{WE}, X^{11}_{WE}, X^{13}_{WE}.

4. A new procedure and its properties

In this section, an approach is introduced for determining all maximal efficient faces using efficient extreme points and efficient extreme rays. Hence, it is assumed that all of them have been found. At first, corresponding to any efficient extreme point and efficient extreme ray, an index set namely, *minimal index set* is defined. Then by using the minimal index sets, the maximal efficient faces of a given MOLP is determined. For this purpose, firstly, index all of the constraints and nonnegative restrictions binding at least in one efficient extreme point. Denote the set of these indices by I.

4.1. A procedure for finding a maximal efficient face

Let $\mu(x,d)$ denote an extreme ray emanating from extreme point x in direction d. So, $x+d$ is a relative interior point of $\mu(x,d)$. Hereafter, consider $x+d$ as the corresponding point to $\mu(x,d)$ and recall it as the indicator of $\mu(x,d)$. Taking into account all of the efficient extreme points and efficient extreme ray indicators, construct a set namely, E which contains all of the extreme points and ray indicators denoted as $E=\{x^1,\ldots,x^q\}$. Let $T:=\{1,\ldots,q\}$. Now, consider $x^i\in E$ and define I_i as a subset of I which contains the indices of all constraints binding in x^i corresponding to problem (2.2). We do this for all of the elements of E and construct a family of some subsets of I denoted as:

$$\mathcal{A}=\{I_1,\ldots,I_q\}.$$

Theorem 4.1. *If there exist x^i and x^j in E such that $I_i\cap I_j=\emptyset$, then $\frac{1}{2}(x^i+x^j)$ is efficient if and only if all of the points in X are efficient.*

Proof. Let $l\in I$. Since $I_i\cap I_j=\emptyset$, without loss of generality assume that $\bar{a}_l x^i>\bar{b}_l$. On the other hand, $\bar{a}_l x^j\geqslant\bar{b}_l$. Therefore

$$\bar{a}_l\frac{1}{2}\big(x^i+x^j\big)=\frac{1}{2}\big(\bar{a}_l x^i+\bar{a}_l x^j\big)>\bar{b}_l.$$

That is, there is no binding constraint in $\frac{1}{2}(x^i+x^j)$. i.e. it is a relative interior point of X. Since $\frac{1}{2}(x^i+x^j)$ is an efficient relative interior point of X, Theorem 2.5 shows that X is efficient. The converse of the proof is obvious. □

Remark 4.1. If all of the points in X are efficient, we say that X is totally efficient.

Remark 4.2. If there are two elements in E satisfying the conditions of Theorem 4.1, the procedure of finding maximal efficient faces terminates and the only efficient face is X.

Notation. *Corresponding to x^{i_k}* define y^{i_k} *as follows*:

$$y^{i_1}=\frac{1}{2}\big(x^{i_0}+x^{i_1}\big),\quad y^{i_2}=\frac{1}{2}\big(y^{i_1}+x^{i_2}\big),\quad\ldots,\quad y^{i_k}=\frac{1}{2}\big(y^{i_{k-1}}+x^{i_k}\big),\quad\ldots,$$

where $i_k\in T$.

Remark 4.3. It is easily seen that in the above notation y^{i_k} is a positive convex combination of $x^{i_0},x^{i_1},\ldots,x^{i_k}$ $(k=1,2,\ldots)$.

For determining maximal efficient faces, possibly find x^i and x^j in E for which $I^i\cap I^j=\emptyset$ and $\frac{1}{2}(x^i+x^j)$ is efficient. In this case the only maximal efficient face is X and the algorithm of finding maximal efficient faces terminates. Otherwise, we consider $x^{i_0}\in E$ $(i_0\in T)$ and explain how to construct the corresponding minimal index set denoted by I_{i_0} starting from x^{i_0}. To do this, possibly select an element $I_{i_1}\in\mathcal{A}$ $(i_1\in T)$ in such a way $\emptyset\neq I_{i_0}\cap I_{i_1}\subsetneqq I_{i_0}$ and $\frac{1}{2}(x^{i_0}+x^{i_1})$ is efficient (some criteria for examining efficiency have been given in Section 2). Then possibly find $I_{i_2}\in\mathcal{A}$ $(i_2\in T)$ such that

$$\emptyset\neq(I_{i_0}\cap I_{i_1})\cap I_{i_2}\subsetneqq I_{i_0}\cap I_{i_1}$$

and $\frac{1}{2}(y^{i_1}+x^{i_2})$ is efficient. Continuing this process, find some elements of $\mathcal{A}$ namely, $I_{i_0},I_{i_1},\ldots,I_{i_{t_0}}$ so that there is no $I_j\in\mathcal{A}$ such that $j\notin\{i_1,i_2,\ldots,i_{t_0}\}$,

$$\emptyset\neq\left(\bigcap_{k=0}^{t_0}I_{i_k}\right)\cap I_j\subsetneqq\bigcap_{k=0}^{t_0}I_{i_k}$$

and $\frac{1}{2}(y^{i_k}+x^j)$ is efficient.

Now, the minimal index set corresponding to x^{i_0} denoted by $I^m_{i_0}$ is defined as follows:

$$I^m_{i_0} = \bigcap_{k=0}^{t_0} I_{i_k}.$$

Comment 4.1. *Constructing procedure of $I^m_{i_0}$ shows that it has minimality property in the sense of set inclusion; thus we have called it as minimal index set.*

Corresponding to $I^m_{i_0}$, define a face of X namely, F^{i_0} as follows:

$$F^{i_0} = \{x \in X \mid \bar{a}_l x = \bar{b}_l, \text{ for all } l \in I^m_{i_0}\}. \tag{4.1}$$

4.2. Main results

Theorem 4.2. *In the procedure of Section* 4.1, *$x^{i_0}, x^{i_1}, \ldots, x^{i_{t_0}}$ are affine independent.*

Proof. The proof, is based on induction.

For $k = 0$, the proof is obvious.

Assume $x^{i_0}, x^{i_1}, \ldots, x^{i_{j-1}}$ for $j \leqslant i_{t_0}$, are affine independent. We show that $x^{i_0}, x^{i_1}, \ldots, x^{i_j}$ are also affine independent. By contradiction assume that they are not affine independent; thus by Corollary 2.1 one of them, say $x^{i_{\bar{t}}}$ ($\bar{t} \in \{0, 1, \ldots, j\}$), can be represented in the following form:

$$x^{i_{\bar{t}}} = \sum_{t=0,\, t\neq\bar{t}}^{j} \lambda_t x^{i_t}, \qquad \sum_{t=0,\, t\neq\bar{t}}^{j} \lambda_t = 1. \tag{4.2}$$

Two cases can be considered as follows:

Case I: $\bar{t} \neq j$.

Since $x^{i_0}, x^{i_1}, \ldots, x^{i_{j-1}}$ are affine independent, $\lambda_j \neq 0$. So,

$$x^{i_j} = \frac{1}{\lambda_j} x^{i_{\bar{t}}} + \sum_{t=0,\, t\neq\bar{t}}^{j-1} \frac{-\lambda_t}{\lambda_j} x^{i_t}.$$

Let $\alpha_{\bar{t}} = \frac{1}{\lambda_j}$ and $\alpha_t = -\frac{\lambda_t}{\lambda_j}$ for $t \neq \bar{t},\ t \neq j$. Then

$$x^{i_j} = \sum_{t=0}^{j-1} \alpha_t x^{i_t}, \qquad \sum_{t=0}^{j-1} \alpha_t = 1. \tag{4.3}$$

Case II: $\bar{t} = j$.

In this case, considering $\alpha_t := \lambda_t$, (4.2) is transformed into (4.3).

Hence in each case (4.3) holds. Now, let $l \in \bigcap_{t=0}^{j-1} I_{i_t}$. Then $\bar{a}_l x^{i_t} = \bar{b}_l$, for all $t \in \{0, 1, \ldots, j-1\}$. Consequently:

$$\bar{a}_l x^{i_j} = \bar{a}_l \sum_{t=0}^{j-1} \alpha_t x^{i_t} = \sum_{t=0}^{j-1} \alpha_t \bar{b}_l = \bar{b}_l;$$

thus $l \in I_j$. That is $\bigcap_{t=0}^{j-1} I_{i_t} \subseteq I_j$. So,

$$\left(\bigcap_{t=0}^{j-1} I_{i_t}\right) \cap I_j = \bigcap_{t=0}^{j-1} I_{i_t}$$

and it is against the accepted rule in the procedure of Section 4.1 for constructing I_j which yields $(\bigcap_{t=0}^{j-1} I_{i_t}) \cap I_j \subsetneqq \bigcap_{t=0}^{j-1} I_{i_t}$; thus the contrary assumption does not hold and $x^{i_0}, x^{i_1}, \ldots, x^{i_{j-1}}, x^{i_j}$ are affine independent. $\square$

Theorem 4.3. *Denote $\bar{x}^{i_0} := y^{i_{t_0}}$. Then the index set of binding constraints in $\bar{x}^{i_0}$ is $I^m_{i_0}$ and $\bar{x}^{i_0}$ is a relative interior point of F^{i_0}.*

Proof. The proof of this theorem is divided to two parts.

Part I: Let $\bar{I}_{i_0}$ be the binding index set corresponding to $\bar{x}^{i_0}$. Assume that $l \in I_{i_0}^m$. Since $I_{i_0}^m = \bigcap_{k=0}^{t_0} I_{i_k}$,

$$\forall k \in \{0, 1, \ldots, t_0\}, \quad \bar{a}_l x^{i_k} = \bar{b}_l.$$

On the other hand, by Remark 4.3 there exist some $\lambda_k > 0$ $(k = 0, 1, \ldots, t_0)$ such that $\sum_{k=0}^{t_0} \lambda_k = 1$ and $\bar{x}^{i_0} = \sum_{k=0}^{t_0} \lambda_k x^{i_k}$. Consequently:

$$\bar{a}_l \bar{x}^{i_0} = \bar{a}_l \left(\sum_{k=0}^{t_0} \lambda_k x^{i_k} \right) = \sum_{k=0}^{t_0} \lambda_k \bar{a}_l x^{i_k} = \sum_{k=0}^{t_0} \lambda_k \bar{b}_l = \bar{b}_l;$$

that is $l \in \bar{I}_{i_0}$. Hence $I_{i_0}^m \subseteq \bar{I}_{i_0}$. Conversely, let $l \in \bar{I}_{i_0}$. By contradiction assume that there exits $k_0 \in \{0, 1, \ldots, t_0\}$ such that $\bar{a}_l x^{i_{k_0}} > \bar{b}_l$. For each $k \in \{0, 1, \ldots, t_0\}$ $\bar{a}_l x^{i_k} \geqslant \bar{b}_l$. Consequently,

$$\bar{a}_l \bar{x}^{i_0} = \bar{a}_l \left(\sum_{k=0}^{t_0} \lambda_k x^{i_k} \right) = \sum_{k=0}^{t_0} \lambda_k \bar{a}_l x^{i_k} = \sum_{k=0,\, k \neq k_0}^{t_0} \lambda_k \bar{a}_l x^{i_k} + \lambda_{k_0} \bar{a}_l x^{i_{k_0}} > \bar{b}_l;$$

thus $\bar{a}_l \bar{x}^{i_0} > \bar{b}_l$ and this contradicts assumption $l \in \bar{I}_{i_0}$ and the first part of theorem is proved.

Part II: To prove this part, it is sufficient to show that $\forall l \in I \setminus I_{i_0}^m$, $\bar{a}_l \bar{x}^{i_0} > \bar{b}_l$.

For this purpose, let $l \in I \setminus I_{i_0}^m$. By contradiction assume that

$$\forall k \in \{0, 1, \ldots, t_0\}, \quad \bar{a}_l x^{i_k} = \bar{b}_l. \tag{4.4}$$

Consequently,

$$\bar{a}_l \bar{x}^{i_0} = \bar{a}_l \left(\sum_{k=0}^{t_0} \lambda_k x^{i_k} \right) = \sum_{k=0}^{t_0} \lambda_k \bar{a}_l x^{i_k} = \sum_{k=0}^{t_0} \lambda_k \bar{b}_l = \bar{b}_l;$$

thus $l \in I_{i_0}^m$. So, contrary assumption is false; i.e. there exists a $k' \in \{0, 1, \ldots, t_0\}$ for which $\bar{a}_l x^{i_{k'}} > \bar{b}_l$. Consequently, $\bar{a}_l \bar{x}^{i_0} > \bar{b}_l$. Since $l \in I \setminus I_{i_0}^m$ has been chosen arbitrarily, $\bar{x}^{i_0}$ is a relative interior point of F^{i_0}. □

Corollary 4.1. *Face F^{i_0} is an efficient face.*

Proof. The constructing procedure for determining x^{i_k}'s in Section 4.1 guarantees the efficiency of x^{i_k}'s and so $\bar{x}^{i_0}$. Now, Theorems 4.3 and 2.5 show the efficiency of F^{i_0}. □

Theorem 4.4. *F^{i_0} is a maximal efficient face of X.*

Proof. By Corollary 4.1, F^{i_0} is efficient. So, it is sufficient to show its maximality property. Let I_F be the index set of constraints binding in all of points in F. By contradiction assume that there exists another efficient face namely, F for which $F^{i_0} \subsetneqq F$. Then $I_F \subsetneqq I_{i_0}^m$. Now, let $l_0 \in I_{i_0}^m \setminus I_F$. Since any face is constructed by convex combination of its extreme points and nonnegative combination of its extreme rays, thus there exists an extreme point or a ray indicator point in F namely, $\bar{x}_t$ for which

$$\bar{a}_{l_0} \bar{x}_t > \bar{b}_{l_0}. \tag{4.5}$$

Therefore there exists $\bar{x}_t \in F \cap E$ for which (4.5) holds. Since $l_0 \in I_{i_0}^m \setminus I_F$,

$$I_{i_0}^m \cap I_F \subsetneqq I_{i_0}^m. \tag{4.6}$$

On the other side, since x^{i_k}'s $(k \in \{0, 1, \ldots, t_{t_0}\})$ and $\bar{x}_t$ are in F which is an efficient face thus $\frac{1}{2}(y^{i_k} + \bar{x}_t) = \frac{1}{2}(\bar{x}_{i_0} + \bar{x}_t)$ is an efficient point of F. This fact and (4.6) contradict the minimality property of $I_{i_0}^m$ which mentioned in Comment 4.1 Thus F^{i_0} is a maximal efficient face. □

Lemma 4.1. *Let F be a face with* $\dim(F) = k$, *then F does not contain more than $k + 1$ affine independent points* [3].

The following theorem is a critical result which shows that the proposed approach performs very effectively for determining maximal efficient faces such as F^{i_0}.

Theorem 4.5. *In the procedure of Section* 4.1, $t_0 \leqslant \dim(F^{i_0})$.

Proof. By Lemma 4.1 and Theorem 4.2 the proof is obvious. □

Corollary 4.2. *F^{i_0} can be determined easily using at most* $\dim(F^{i_0})+1$ *numbers of points in E.*

Remark 4.4. Corollary 4.2 shows that in the case that $F^{i_0}\cap E$ contains a lot of points, in the worst case F^{i_0} can be determined easily using $\dim(F^{i_0})+1$ elements in E.

So far, only one maximal efficient face have been determined using the procedure of Section 4.1. Now, we are in a position to determine other maximal efficient faces. For this purpose choose $x^j \in E$ in such a way that $x^j \notin F^{i_0}$ $(I_{i_0}^m \nsubseteq I_j)$. Starting from x^j, in a similar manner as applied for x^{i_0}, a minimal index set namely, I_j^m and a maximal efficient face namely, F^j can be constructed corresponding to x^j. Now, all of the results which said about F^{i_0} also holds for F^j.

We continue this process and find efficient maximal faces such as $F^{i_0}, F^{i_1}, \ldots, F^{i_s}$ such that $E \subseteq \bigcup_{k=0}^{s} F^{i_k}$.

Corollary 4.3. *All of the efficient maximal faces of X are $F^{i_0}, F^{i_1}, \ldots, F^{i_s}$.*

Proof. Assume on the contrary that there exists another maximal efficient face namely, F. Since $F \notin \{F^{i_0}, F^{i_1}, \ldots, F^{i_s}\}$, there exists $\bar{x} \in E \cap F$ such that $\bar{x} \notin \bigcup_{k=0}^{s} F^{i_k}$ this contradicts assumption $E \subseteq \bigcup_{k=0}^{s} F^{i_k}$. Thus all of the efficient maximal faces of X are $F^{i_0}, F^{i_1}, \ldots, F^{i_s}$. □

Remark 4.5. The proposed approach can also be applied for determining maximal weakly efficient faces if we have all of the weakly efficient extreme points and weakly efficient extreme rays. In this case instead of efficiency criteria, weakly efficiency criteria (Theorem 2.3) must be used to examine the weakly efficiency of related points.

5. Introduction of the new algorithm

The proposed algorithm uses the results from Section 4 to generate the set of all maximal efficient faces. The proposed algorithm, we call it A2 for simplicity, can be stated as follows.

5.1. New algorithm (A2)

Let all of the efficient extreme points and the indicators of efficient extreme rays be $x^1, x^2, \ldots, x^q$ and I_j denote the index set of all binding constraints in x^j $(j \in T)$.

Initialization step. $E := \{x^1, x^2, \ldots, x^q\}$. Compute $I_1, \ldots, I_q$. $T := \{1, 2, \ldots, q\}$,

$$J := \{1\}, \qquad k := 1, \qquad k' := 1, \qquad I_1^m := I_1, \qquad y^1 := x^1, \qquad r := 2.$$

Possibly find x^i and x^j in E such that $I^i \cap I^j = \emptyset$. If $\frac{1}{2}(x^i + x^j)$ is efficient then stop.
In this case the only maximal efficient face is X. Otherwise go to Step 1.

Step 1.

1.1: If $r \neq k'$, $\emptyset \neq I_k^m \cap I_r \subsetneqq I_k^m$ and $\frac{1}{2}(y^k + x^r)$ is efficient then

$$y^k := \frac{1}{2}\left(y^k + x^r\right), \qquad I_k^m := I_k^m \cap I_r, \qquad J := J \cup \{r\} \quad \text{and go to 1.2.}$$

1.2: If $r < q$, $r := r+1$ and go to 1.1;
else $F^k := \{x \in X \mid \bar{a}_l x = \bar{b}_l$ for all $l \in I_k^m\}$ is a maximal efficient face.

Step 2. If $k = q$ then stop;

else
find $k' \in T \setminus J$ such that $x^{k'} \notin \bigcup_{i=1}^{k} F^i$ (i.e. $\forall i \in \{1, \ldots, k\}$ $I_i^m \nsubseteq I_{k'}$), and
$k := k+1$, $y^k := x^{k'}$, $J := J \cup \{k'\}$, $I_{k+1}^m := I_{k'}$, $r := 1$ and go to 1.1.
If there does not exist such k', stop.

Proposition 5.1. *Algorithm A2 terminates after finite number of iterations.*

Proof. For a fixed k', Step 1 terminates after at most q iterations. It is due to whether the conditions of Step 1.1 are satisfied or not, we go to Step 1.2 in which r increases by one and this procedure continues until $r = q$. Then the next k' which is selected in Step 2 is different from the previous ones because in this step $k' \in T \setminus J$ in which J contains all of previous k's. Since T has at most q elements, algorithm A2 terminates after finite number of iterations. $\square$

5.2. Potential advantages

The proposed algorithm has a number of potential advantages both in practice and computationally. Here, we indicate only some key potential advantages, with special attention to those not shared by other vector optimization approaches. They can be summarized as follows:

(a) In initialization step, the total efficiency of X is examined (determined) using only one LP problem and if X is totally efficient, algorithm terminates without any more computations.
(b) It may strongly happen, particularly, in higher dimensional spaces, that there are considerably many numbers of x^r's for which in Step 1.1, $I_k^m \cap I_r = I_k^m$ or $I_k^m \cap I_r = \emptyset$. As be seen in Step 1.1, in these cases this algorithm goes to Step 1.2 without any more computations.
(c) As seen in Step 1.1, the algorithm determines maximal efficient faces in an explicit way. Therefore, it is easy for decision maker to select a most preferred efficient solution.

5.3. Numerical example

Here, for a better comparison between algorithms A1 and A2, the MOLP (3.2) of Example 3.1 is rewritten as follows:

Example 5.1. Consider the following MOLP.

$$\begin{aligned} V\colon\quad \min\quad & (-x_1, -x_2, -x_3) \\ \text{s.t.}\quad & 1 \leftarrow -x_1 - x_2 - 2x_3 \geqslant -10, \\ & 2 \leftarrow -x_1 - x_2 - x_3 \geqslant -6, \\ & 3 \leftarrow -3x_1 - 3x_2 - x_3 \geqslant -12, \\ & 4 \leftarrow x_1 \geqslant 0, \\ & 5 \leftarrow x_2 \geqslant 0, \\ & 6 \leftarrow x_3 \geqslant 0. \end{aligned} \tag{5.1}$$

Solving (5.1) by A2 contains following computations:

Initialization step.

$$\begin{aligned} & E := \{(0,0,5)^t,\ (0,2,4)^t,\ (2,0,4)^t,\ (0,3,3)^t,\ (3,0,3)^t,\ (0,4,0)^t,\ (4,0,0)^t\}, \\ & I_1 = \{1,4,5\}, \quad I_2 = \{1,2,4\}, \quad I_3 = \{1,2,5\}, \quad I_4 = \{2,3,4\}, \quad I_5 = \{2,3,5\}, \\ & I_6 = \{3,4,6\}, \quad I_7 = \{3,5,6\}, \\ & T := \{1,2,3,4,5,6,7\}, \quad J := \{1\}, \quad k := 1, \quad k' := 1, \quad I_1^m := \{1,4,5\}, \quad y^1 := (0,0,5)^t, \quad r := 2. \end{aligned}$$

Iteration I:

Since $I_2 \cap I_7 = \emptyset$ and $\frac{1}{2}(x^2 + x^7)$ is not weak efficient; thus X is not totally weak efficient.

Step 1.

1.1: $r \neq k', \quad \emptyset \neq I_1^m \cap I_2 = \{1,4\} \subsetneqq I_1^m$.

So, do a weak efficiency criterion for $\frac{1}{2}(y^1 + x^2) = \frac{1}{2}((0,0,5)^t + (0,2,4)^t) = (0, 1, \frac{9}{2})^t$.
Since $(0, 1, \frac{9}{2})^t$ is weak efficient, $y^1 := (0, 1, \frac{9}{2})^t$, $I_1^m := I_1^m \cap I_2 = \{1,4\}$, $J := \{1,2\}$.

1.2: $2 < 7, \quad r := 3$. **1.1:** $r \neq k', \quad \emptyset \neq I_1^m \cap I_3 = \{1\} \subsetneqq I_1^m$.

So, do a weak efficiency criterion for $\frac{1}{2}(y^1 + x^3) = \frac{1}{2}((0, 1, \frac{9}{2})^t + (2,0,4)^t) = (1, \frac{1}{2}, \frac{17}{4})^t$.

Since $(1, \frac{1}{2}, \frac{17}{4})^t$ is weak efficient, $y^1 := (1, \frac{1}{2}, \frac{17}{4})^t$, $I_1^m := I_1^m \cap I_3 = \{1\}$, $J := \{1, 2, 3\}$.

Since I_1^m does not have any proper nonempty subset, Step 1 terminates and the first maximal weak efficient face namely, F^1 is determined as follows:

$$F^1 = \{x \in X \mid x_1 + x_2 + 2x_3 = 10\} = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + 2x_3 = 10, \\ x_1 + x_2 + x_3 \leqslant 6, \\ 3x_1 + 3x_2 + x_3 \leqslant 12, \\ x_1, x_2, x_3 \geqslant 0 \end{array} \right\}.$$

Step 2. $1 = k \neq q = 7$, $T \setminus J = \{4, 5, 6, 7\}$,

$$k' := 4, \qquad k := 2, \qquad y^2 := x^4, \qquad J := \{1, 2, 3, 4\}, \qquad I_2^m := I_4 = \{2, 3, 4\}, \qquad r := 1.$$

Iteration II:

Step 1.

1.1: $r \neq k'$, $\emptyset \neq I_2^m \cap I_1 = \{4\} \subsetneq I_2^m$.

So, do a weak efficiency criterion for $\frac{1}{2}(y^2 + x^1) = \frac{1}{2}((0, 3, 3)^t + (0, 0, 5)^t) = (0, \frac{3}{2}, 4)^t$.

Since $(0, \frac{3}{2}, 4)^t$ is not weak efficient, go to Step 1.2.

1.2: $1 < 7$, $r := 2$. **1.1:** $r \neq k'$, $\emptyset \neq I_2^m \cap I_2 = \{2, 4\} \subsetneq I_2^m$.

So, do a weak efficiency criterion for $\frac{1}{2}(y^2 + x^2) = (0, \frac{5}{2}, \frac{7}{2})^t$.

Since $(0, \frac{5}{2}, \frac{7}{2})^t$ is weak efficient,

$$y^2 := \left(0, \frac{5}{2}, \frac{7}{2}\right)^t, \qquad I_2^m := I_2^m \cap I_2 = \{2, 4\}, \qquad J := \{1, 2, 3, 4\}.$$

1.2: $2 < 7$, $r := 3$. **1.1:** $r \neq k'$, $\emptyset \neq I_2^m \cap I_3 = \{2\} \subsetneq I_2^m$.

So, do a weak efficiency criterion for $\frac{1}{2}(y^2 + x^3) = (1, \frac{5}{4}, \frac{15}{4})^t$.

Since $(1, \frac{5}{4}, \frac{15}{4})^t$ is weak efficient,

$$y^2 := \left(1, \frac{5}{4}, \frac{15}{4}\right)^t, \qquad I_2^m := I_2^m \cap I_2 = \{2\}, \qquad J := \{1, 2, 3, 4\}.$$

Step 1 terminates with

$$F^2 = \{x \in X \mid x_1 + x_2 + x_3 = 6\} = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + 2x_3 \leqslant 10, \\ x_1 + x_2 + x_3 = 6, \\ 3x_1 + 3x_2 + x_3 \leqslant 12, \\ x_1, x_2, x_3 \geqslant 0 \end{array} \right\}.$$

Step 2. $2 = k \neq q = 7$, $T \setminus J = \{5, 6, 7\}$, $I_2^m \subseteq I_5$. So,

$$k' := 6, \qquad k := 3, \qquad y^3 := x^6, \qquad J = \{1, 2, 3, 4, 6\}, \qquad I_3^m := I_6 = \{3, 4, 6\}, \qquad r := 1.$$

Iteration III:

Step 1.

1.1: $r \neq k'$, $\emptyset \neq I_3^m \cap I_1 = \{4\} \subsetneq I_3^m$.

So, do a weak efficiency criterion for $\frac{1}{2}(y^3 + x^1) = \frac{1}{2}((0, 4, 0)^t + (0, 0, 5)^t) = (0, 2, \frac{5}{2})^t$.

Since $(0, 2, \frac{5}{2})^t$ is not weak efficient, go to Step 1.2.

1.2: $1 < 7$, $r := 2$. **1.1:** $r \neq k'$, $\emptyset \neq I_3^m \cap I_2 = \{2\} \subsetneq I_2^m$.

So, do a weak efficiency criterion for $\frac{1}{2}(y^3 + x^2) = (0, 3, 2)^t$.

Since $(0, 3, 2)^t$ is not weak efficient, go to Step 1.2.

1.2: $2 < 7$, $r := 3$. **1.1:** $r \neq k'$, $\emptyset = I_3^m \cap I_3$.

1.2: $3 < 7$, $r := 4$. **1.1:** $r \neq k'$, $\emptyset \neq I_3^m \cap I_4 = \{3, 4\} \subsetneqq I_3^m$.

So, do a weak efficiency criterion for $\frac{1}{2}(y^3 + x^4) = (0, \frac{7}{2}, \frac{3}{2})^t$.
Since $(0, \frac{7}{2}, \frac{3}{2})^t$ is weak efficient,

$$y^3 := \left(0, \frac{7}{2}, \frac{3}{2}\right)^t, \qquad I_3^m := I_3^m \cap I_4 = \{3, 4\}, \qquad J = \{1, 2, 3, 4, 6\}.$$

1.2: $4 < 7$, $r := 5$. **1.1:** $r \neq k'$, $\emptyset \neq I_3^m \cap I_5 = \{3\} \subsetneqq I_3^m$.

So, do a weak efficiency criterion for $\frac{1}{2}(y^3 + x^5) = (\frac{3}{2}, \frac{7}{4}, \frac{9}{4})^t$.
Since $(\frac{3}{2}, \frac{7}{4}, \frac{9}{4})^t$ is weak efficient,

$$y^3 := \left(\frac{3}{2}, \frac{7}{4}, \frac{9}{4}\right)^t, \qquad I_3^m := I_3^m \cap I_5 = \{3\}, \qquad J := \{1, 2, 3, 4, 5, 6\}.$$

Step 1 terminates with

$$F^3 = \{x \in X \mid 3x_1 + 3x_2 + x_3 = 12\} = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + 2x_3 \leqslant 10, \\ x_1 + x_2 + x_3 \leqslant 6, \\ 3x_1 + 3x_2 + x_3 = 12, \\ x_1, x_2, x_3 \geqslant 0 \end{array} \right\}.$$

Step 2. The terminate condition of A2 holds.

Remark 5.1. It can be easily seen that faces F^1, F^2 and F^3 are respectively equivalent to faces X_{WE}^7, X_{WE}^{11} and X_{WE}^{13} in Example 3.1. Indeed, A2 does not generate any redundant face.

If this approach is applied for MOLPs in higher dimensional spaces it is expected that (in view point of item (b) of Section 5.2) the number of LP problems that should be solved are considerable less than the number of points in E. This is due to the fact that in this case the number of extreme points or ray indicators on a given face F are usually considerable more than $\dim(F) + 1$.

6. Comparison of A1 and A2

In this section A1 and A2 are compared based on three points of view:

1. A1 starts with minimizing each objective function over the feasible set in order to obtain some weak efficient solutions. But it is possible that there are some weak efficient solution but all of the objective functions be unbounded over the feasible set [17]. Hence, A1 is not able to determine any weak efficient solution. In this case, A1 cannot go to the next step.
2. In A1, Step 2, using weak efficient solutions obtained in previous steps and iterations, a cone namely, S, is constructed and all of its extreme directions is computed. Corresponding to each of these extreme directions, an LP problem is solved in Step 3. But computing all of extreme directions of S is very time consuming and it is a difficult task in view point of computation. Since, often, there exist so many extreme directions, there exist so many LPs that must be solved in Step 3.
   The above mentioned large amount of computations iterated as new optimal solutions are generated in Step 3, and this task is repeated in various iterations of A1. Thus, each iteration of A1 have considerable computations and the algorithm has more considerably computations when the number of iterations increases.
   In contrast, A2 at first, need computing all of extreme efficient (weak efficient) points and extreme efficient (weak efficient) rays one time for ever. Then possibly in initialization step one LP is solved. The most important computations or iterative computations in A2 contains possibly (if the conditions of Step 1.1 are satisfied) solving one LP (efficiency or weak efficiency test) in each iteration of Step 1.1. As seen in given numerical examples (5.1), these conditions are not satisfied in some iterations. Indeed, there are effective conditions because they are satisfied when the next point be affine independent with respect to previous points (whose indices are in I_k^m). For example, in numerical example solved by A1 and A2, they computationally performed as follows: A1 contains solving 30 LPs, finding 30 extreme directions and finding 15 alternative optimal solutions corresponding to $P(v^l)(s)$ in order to obtaining X_{WE}^ls. But A2 contains finding 7 efficient extreme points and solving 9 LPs (the weak efficiency criteria).
   One can seen that A2 is more efficient computationally than the recent algorithm A1.

3. A1 represent the efficient (weak efficient) set as a union of some efficient (weak efficient) faces. But these faces are not necessarily maximal efficient (weak efficient). Thus based on the good property of maximal efficient (weak efficient) faces mentioned in Section 1, the representation given in A2 is better than one given in A1. This fact is illustrative in the given numerical example as mentioned in Remark 3.1
   However, as proved in Theorem 4.4, A2 gives a representation of efficient (weak efficient) set as a union of maximal efficient (weak efficient) faces. This shows that A1 is more suitable than A2.

7. Conclusion

In this paper an algorithm is proposed for finding maximal efficient faces of a multiobjective linear programming problem (MOLP). It is well known that the set of all maximal efficient faces of an MOLP is a convex decomposition (its elements are convex sets) with the least number of elements of the efficient set. Therefore the maximal efficient faces define (represent) the structure of efficient set in the best way. So, obtaining the maximal efficient faces is the most critical task.

Since there are many convenient approaches that obtain all of the (weakly) efficient extreme points and (weakly) efficient extreme rays in an MOLP, this paper develops an algorithm which effectively finds all of the (weakly) maximal efficient faces in an explicit way using (weakly) efficient extreme points and extreme rays.

The proposed algorithm avoids the degeneration problem, which is the major problem of the most of previous algorithms and gives an explicit structure for maximal efficient (weak efficient) faces. Consequently, it gives a convenient representation of efficient (weak efficient) set using maximal efficient (weak efficient) faces. The proposed algorithm is based on the two facts. Firstly, the efficiency and weak efficiency property of a face is determined using a relative interior point of it. Secondly, the relative interior point is achieved using some affine independent points. The affine independent property enable us to obtain an efficient relative interior point rapidly.

It is needed in the proposed algorithm to solve some LPs within the process. However, If this approach is applied for MOLPs in higher dimensional spaces, it is expected that the number of LP problems that should be solved are considerably less than the number of points in E (the set of extreme points and ray indicators). This is due to facts that in this case the number of extreme points and ray indicators on a given face F are usually considerably more than $\dim(F) + 1$. Furthermore, use of affine independent property reduces the number of LPs that should be solved to at most $\dim(F) + 1$.

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