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RESEARCH ARTICLE

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Exact algorithms for multiobjective linear optimization problems with integer variables: A state of the art survey

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Abstract

We provide a comprehensive overview of the literature of algorithmic approaches for multiobjective mixed-integer and integer linear optimization problems. More precisely, we categorize and display exact methods for multiobjective linear problems with integer variables for computing the entire set of nondominated images. Our review lists 108 articles and is intended to serve as a reference for all researchers who are familiar with basic concepts of multiobjective optimization and who have an interest in getting a thorough view on the state-of-the-art in multiobjective mixed-integer programming.

KEYWORDS

algorithms, integer, literature overview, mixed-integer, multiobjective optimization

1 | INTRODUCTION

Many real-world optimization problems possess several, often conflicting objectives that have to be optimized simultaneously. Multiobjective optimization is the discipline which is concerned with analyzing the mathematical structure of these problems and designing appropriate solution methods. Also known as multicriteria or vector optimization, this multidisciplinary field of research connects mathematics, computer science, economics, and operations research. As a consequence of its multidisciplinarity and universality, multiobjective optimization is utilized in applications across various domains. We refer the interested reader to one of several surveys of applications of multiobjective optimization to get an overview (Achilles et al., 1979; Andersson, 2000; Marler & Arora, 2004; Nehse, 1982; White, 1990).

Due to the existence of more than one objective, the notion of optimality is not unique. Typically, the notion of Pareto-optimality (Edgeworth, 1881; Pareto, 1896) is applied in multiobjective optimization. As a consequence, several efficient solutions and nondominated images exist. Here, a solution is called efficient (and its image is called nondominated), if there is no other solution whose image is at least as

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good as this image for each objective function and strictly better for at least one objective function. Finding all nondominated images is a challenging task due to the computational complexity and intractability (Ehrgott, 2005) of multiobjective optimization problems even if restricted to linear constraints and continuous variables. The existence of integer variables is often motivated by practical applications, but it also exacerbates the solution process (Figueira et al., 2017). Consequently, multiobjective (mixed-)integer and combinatorial optimization gained considerable attention in research in recent years.

Multiobjective optimization has been around since the late 50s and early 60s of the last century (see Stadler, 1979, for a historical overview that also presents the history before the seminal works of Edgeworth (1881) and Pareto (1896)). The first algorithm for a class of multiobjective optimization problems with integer variables has been presented by Pasternak and Passy (1972) in 1972. Ever since, theory on multiobjective integer optimization has been enhanced and many solution algorithms have been proposed. With this survey, we display the state-of-the-art of exact algorithms for multiobjective optimization problems with integer variables over the last five decades. We point the reader also to a variety of previous surveys with different focus and depth (Captivo, 2012; Clímaco et al., 1997; Ehrgott & Wiecek, 2005; Evans, 1984; Gal, 1986; Rasmussen, 1986; Yap, 2010; Zionts, 1979).

With this survey, we aim at supporting researchers in getting a first overview of the methods and results in the area of (mixed-)integer multiobjective optimization. However, it is not meant to be a self-contained introduction and, instead, we assume the reader to be familiar with some basic concepts of multiobjective optimization. Several books introduce multiobjective optimization and present the fundamental theory (Antunes et al., 2016; Ehrgott, 2005; Miettinen, 1998; Steuer, 1989; Wallenius et al., 2008, among others).

For an overview of the existing libraries on multiobjective instances, we refer to the vOptSolver[1] by Gandibleux and Przybylski, which includes a library, vOptLib, that also points to other existing instance libraries.[2]

As new algorithms are published with increasing frequency, a current state-of-the-art overview of existing solution concepts seems necessary. Surveying the complete literature on multiobjective mixed-integer linear optimization problems is beyond the scope of this article. Thus, we focus on exact algorithms which yield the entire set of nondominated images.[3] All publications in international journals available up to Summer 2021 have been considered. This survey has a clear focus. More precisely, specific combinatorial optimization problems, even if they are special cases of integer optimization (Ehrgott & Gandibleux, 2002, 2003; Ehrgott et al., 2016; Ulungu & Teghem, 1994, and the annotated bibliography by Ehrgott and Gandibleux (2000)), continuous variables only (Wiecek et al., 2016), nonlinear problems (Chinchuluun & Pardalos, 2007; Miettinen, 1998), approximation methods (Herzel et al., 2020; Ruzika & Wiecek, 2005), interactive methods, where a decision maker is involved during the optimization process (Alves & Clímaco, 2007; Miettinen et al., 2008), goal programming (Ignizio, 1978), metaheuristics (Jones et al., 2002), evolutionary methods (Coello et al., 2007; Zhou et al., 2011), as well as genetic algorithms (Coello, 2000) are not considered in this literature overview.

Since the number of publications exceeds 100, we decided to split the considered articles into various categories to improve the overall presentation. The division into algorithms for mixed-integer and integer linear optimization problems suggests itself since mixed-integer optimization problems possess a more complex structure with nondominated segments of different dimension and partially dominated boundaries, see Figure 1c. We then divide the algorithms for both integer and mixed-integer multiobjective optimization problems into problem-specific subgroups. Algorithms in these sub-groups share the same principle idea or approach such as branch-and-bound or a distinct scalarization method. The final categorization is shown in Figure 2. In each category, publications are discussed in a logical chronological order, that is, we go from the oldest to the most recent publication, but discuss closely related papers in one paragraph. This approach leaves some ambiguity but, nonetheless, it seemed reasonable and unavoidable given the multiplicity of ideas in several articles. In order to address these overlaps, gray lines between two categories indicate a closer connection.

The remainder of this article is structured as follows: In Section 2, we introduce multiobjective optimization problems with integer variables and provide necessary definitions and notions of multiobjective optimization. In Section 3, we present the literature on solution methods for multiobjective integer optimization problems. Afterwards, we display approaches for the mixed-integer case in Section 4. Finally, Section 5 summarizes the article.

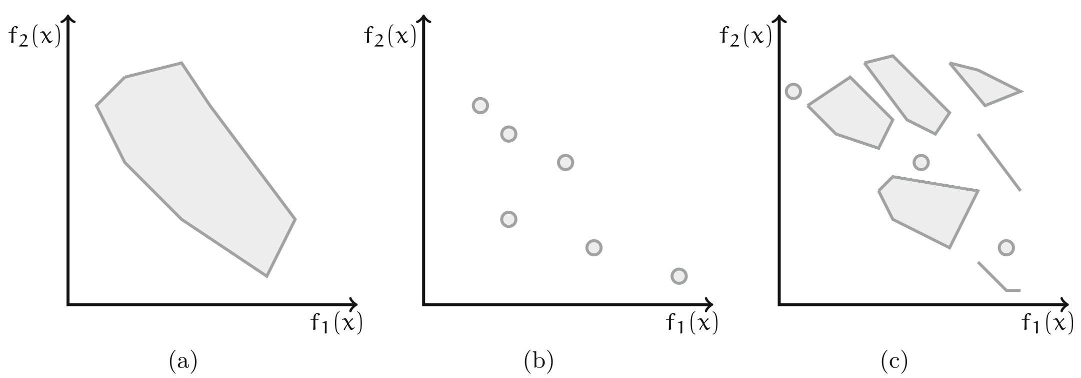

FIGURE 1 Exemplary image space of a multiobjective (a) linear, (b) integer, and (c) mixed-integer linear problem

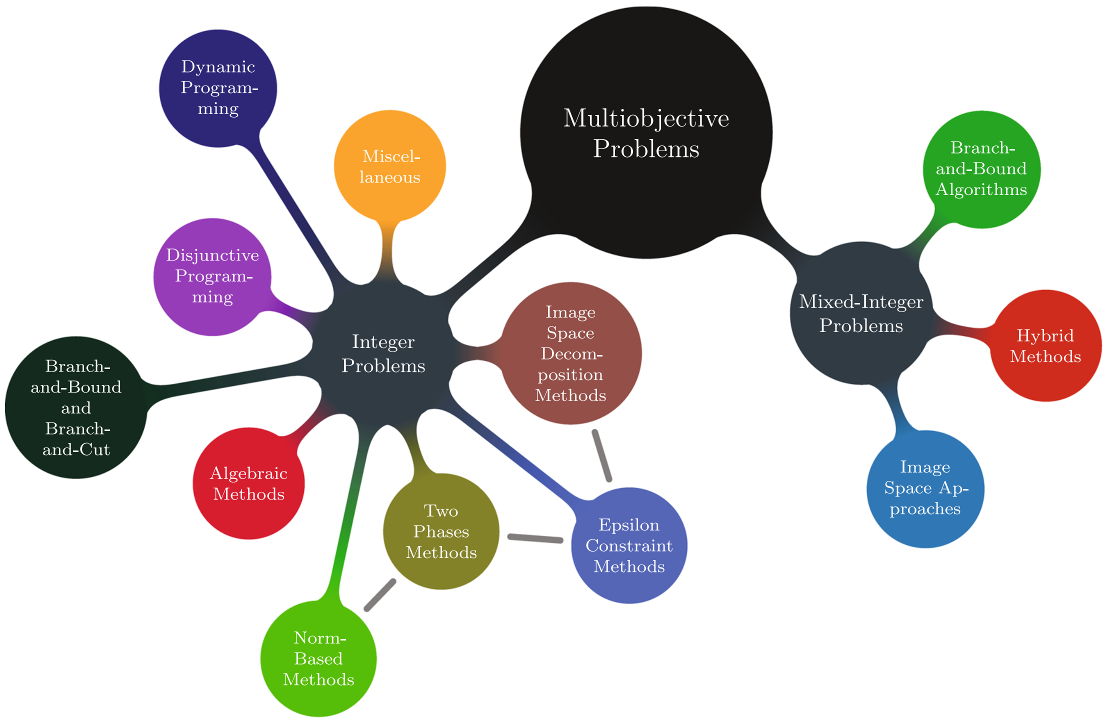

FIGURE 2 Categorization of articles presented in this survey: After algorithms are divided into problem-specific groups, they are further categorized into sub-groups regarding the principle solution approach. A thick gray line between two categories indicates an overlap between categories

2 | PRELIMINARIES

In the following, we introduce basic definitions and concepts of multiobjective optimization. For a more detailed introduction, we refer to Ehrgott (2005).

In general, a multiobjective linear problem with $p \geq 2$ objectives, $n_1 \geq 0$ discrete, and $n_2 \geq 0$ continuous variables can be concisely stated as:

$$\begin{aligned} \min \quad & f(x) = Cx \\ \text{s.t.} \quad & Ax \leq b, \\ & x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned}$$

with $n := n_1 + n_2, A \in \mathbb{R}^{m \times n}, m \in \mathbb{N}$, $C \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^m$. If this problem contains both discrete and continuous variables, we call it a multiobjective mixed-integer linear problem (MOMILP). If the MOMILP has two or three objective functions, we refer to it as a biobjective mixed-integer linear problem (BOMILP) and a triobjective mixed-integer linear problem (TOMILP), respectively. If $n_2 = 0$, we obtain a multiobjective integer linear problem (MOILP). Analogously, we use the terms biobjective and triobjective as well as the abbreviations BOILP and TOILP for two or three objective functions, respectively. On the other hand, if $n_1 = 0$, MOMILP denotes a multiobjective linear problem (MOLP). In particular, in case all integer variables of a MOMILP are fixed, we obtain a MOLP. This is also introduced as a so-called slice problem by Belotti et al. (2013) and we call this process slicing. In Figure 1, examples of the image space of various biobjective linear problems are depicted.

Let $X := \{x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} : Ax \leq b\}$ be the feasible set and $Y := \{Cx : x \in X\} \subseteq \mathbb{R}^p$ the *image set*. The Euclidean vector spaces $\mathbb{R}^n$ and $\mathbb{R}^p$ comprising the set of feasible solutions and the image set are called decision space and image space, respectively.

Since there does not exist a canonical order in $\mathbb{R}^p$ for $p \geq 2$, the following variants of the componentwise order are used. For $y^1, y^2 \in \mathbb{R}^p$ and $N = \{1,\ldots,p\}$, we define

$$\begin{aligned} & y^1 \leqq y^2 :\Leftrightarrow y^1_i \leq y^2_i \text{ for all } i \in N, \\ & y^1 \leq y^2 :\Leftrightarrow y^1 \leqq y^2 \text{ but } y^1 \neq y^2, \\ & y^1 < y^2 :\Leftrightarrow y^1_i < y^2_i \text{ for all } i \in N. \end{aligned}$$

For $p \in \mathbb{N}$, the non-negative orthant is defined as $\mathbb{R}^p_{\geq} := \{r \in \mathbb{R}^p : r \geq 0\}$ and, likewise, the sets $\mathbb{R}^p_{\geqq}$ and $\mathbb{R}^p_{>}$ are defined.

An image $y^* \in Y$ is called nondominated (weakly nondominated), if there is no other image $y \in Y$ such that $y \leq y^*$ ($y < y^*$). Analogously, $x^* \in X$ is called efficient (weakly efficient), if $f(x^*)$ is nondominated (weakly nondominated). The set of all nondominated (weakly nondominated) images is referred to as Y_N (Y_{wN}) and the set of (weakly) efficient solutions as X_E (X_{wE}). In general, for an arbitrary set $S \subset \mathbb{R}^p$ the set of nondominated images is denoted by S_N.

There are particular points of interest in a multiobjective problem: Given a permutation π of the set $\{1,\ldots,p\}$ and the associated order for the objective functions, we refer to the problem

$$\begin{aligned} &\operatorname{lex\,min}\left(f_{\pi(1)},\ldots,f_{\pi(p)}\right)(x) \qquad (\mathrm{Lex}(\pi))\\ &\text{s.t.} \quad x\in X \end{aligned}$$

as the lexicographic optimization problem with respect to permutation π. Varying the permutation π, different solutions along with the corresponding images can be found, the so-called lexicographically optimal solutions (images). Clearly, these images are nondominated. Further, we define the ideal point $y^I \in \mathbb{R}^p$ as the vector consisting of the componentwise optimal images $y_i^I := \min\{y_i : y \in Y\}$. The nadir point y^N is defined componentwise by $y_i^N := \max\{y_i : y \in Y_N\}$ for $i = 1,\ldots,p$. While for $p = 2$ the nadir point is given by the maximum value of each component of the lexicographically optimal images, no efficient algorithm for finding the nadir point for $p \geq 3$ is currently known (Ehrgott, 2005). For two nondominated images y^1 and y^2 with $y_1^1 > y_1^2$ and $y_2^1 < y_2^2$, the local nadir point is defined as $y^{LN} := (y_1^1, y_2^2)$. As we also deal with nondominated line segments in the case of a biobjective mixed-integer problem, the local nadir set of such a line segment is the segment itself.

When it comes to solving a multiobjective problem, scalarization methods that transform such a problem into a single objective optimization problem play a crucial role in finding all or a subset of the nondominated images. Here, we list the most common scalarization methods (see Ehrgott, 2006, and Eichfelder, 2008, for further methods and more detailed information).

2.1 | Weighted sum method

$\min_{x\in X} \lambda^\top f(x)$ for a weight $\lambda \in \mathbb{R}^p$.

Introduced by Zadeh (1963), an optimal solution to the weighted sum problem is a (weakly) efficient solution of the original problem if $\lambda \in \mathbb{R}^p_{>}$ ($\lambda \in \mathbb{R}^p_{\geq}$) (Geoffrion, 1968). The image of such a solution is called supported nondominated image and extreme supported nondominated image if y cannot be expressed by a convex combination of points in $Y_N \backslash \{y\}$. Examples of such images are the lexicographically optimal images. The set of extreme supported nondominated images is denoted by Y_{ESN} and we use these terms analogously for efficient solutions.

2.2 | ε-constraint method

$\min_{x\in X} f_j(x)$ s. t. $f_i(x) \leq \varepsilon_i, i \neq j$, for an $\varepsilon \in \mathbb{R}^p$.

An optimal solution to the ε-constraint problem is a weakly efficient solution of the original problem and, if the solution further denotes the unique optimal solution, it is even efficient (Haimes et al., 1971). In order to find efficient solutions, several variations have been developed such as the augmented ε-constraint method (Mavrotas, 2009), where an augmentation term $\lambda \cdot \sum_{i\neq j} f_i(x)$ is added to the objective function or the elastic and the flexible constraint method, where surplus variables (and slack variables) are added and bound violations are penalized (and bound observance rewarded) (Ehrgott, 2005; Ehrgott & Ruzika, 2008).

2.3 | Hybrid method

$\min_{x\in X} \lambda^\top f(x)$ s. t. $f_i(x) \leq \varepsilon_i, i = 1,\ldots,p$, for $\lambda, \varepsilon \in \mathbb{R}^p$.

If $\lambda > 0$, this combination of the previous two methods ensures efficiency while being capable of finding all efficient solutions (Guddat et al., 1985).

2.4 | Benson's method

$\max_{x\in X, h\in \mathbb{R}^p_{\geq 0}} \sum_{i=1}^p h_i$ s. t. $f(x^0)$ - $f(x)$ - $h = 0$, for a solution $x^0 \in X$.

This method determines whether x^0 is efficient. If not, it returns a solution (x^*, h^*), where x^* is efficient (Benson, 1978).

2.5 | Weighted Tchebycheff method

$\min_{x\in X} \max_{i=1,\ldots,p} \lambda_i \left(f_i(x) - y_i^I\right)$, for a weight $\lambda \in \mathbb{R}^p_{\geq}$.

Introduced by Bowman (1976), Yu (1973), and Zeleny (1973), the optimal solution of this reference point method is weakly efficient to the original problem. Similarly to the ε-constraint method, an augmented variant (Steuer & Choo, 1983) or a lexicographic or two-stage weighted Tchebycheff method return efficient solutions. In this context, a second problem with the values of the weakly efficient solution as right-hand side bounds is solved (Sayın & Kouvelis, 2005; Steuer & Choo, 1983). The weighted Tchebycheff is also closely related to reference point methods and achievement scalarizing function methods (Wierzbicki, 1980; Wierzbicki, 1986). In these methods, changing the reference point has the same effect as changing weights in other methods.

2.6 | Pascoletti–Serafini method

$\min_{x\in X, t\in \mathbb{R}} t$ s. t. $f(x) \leqq a + t \cdot r$, for an $a \in \mathbb{R}^p$ and a direction $r \in \mathbb{R}^p_{\geq}$.

Using this method, weakly efficient solutions are returned (Pascoletti & Serafini, 1984), while additional uniqueness of the optimal solution even ensures efficiency.

In the remainder of this article, we largely omit the term optimization to shorten notation.

3 | ALGORITHMS FOR MULTIOBJECTIVE INTEGER OPTIMIZATION PROBLEMS

About 70 different algorithms and algorithmic variants exist that are able to obtain the whole nondominated set of a bi- or multiobjective integer problem, either for a distinct number of objectives, binary variables only, or for arbitrary MOILPs. In the past 48 years, numerous approaches have evolved with different origins and techniques. In the following, we present the algorithms by dividing them into nine categories, cf. Figure 2. In Figure 3, we present the publication history, while the number of papers and citations for each category are depicted in Figures 4 and 5, respectively. Remark that our assignment of individual papers to one of several possible categories has a strong impact. The assignment can be found in Table 1.

3.1 | Algebraic methods

Methods originating from algebraic theory and programming are scarce and rather new in integer optimization, as this requires a detailed knowledge in both fields, optimization and algebra. The first notable algorithm for single objective integer problems has been developed by Conti and Traverso (1991). They compute the Gröbner bases corresponding to the convex hull of the set of feasible solutions with an adaption of Buchberger's algorithm (Buchberger, 2006). A Gröbner basis is a finite generating set for an ideal in a polynomial ring over a field. In particular, such a basis can be computed for a polytope and, hence, for the convex hull of the set of feasible solutions. Using this basis, an optimal solution of the optimization problem can be computed. Improvements and more algebraic methods in integer optimization have followed (see, for example, Hoşten & Sturmfels, 1995; Sturmfels, 2003; Thomas, 1998).

Blanco and Puerto (2009) are the first to use algebraic methods in algorithms for MOILPs. They adapt methods for single objective problems that are based on the computation of Gröbner bases. As there is only a partial order for the image space of a multiobjective problem, Blanco and Puerto introduce the notion of a partial Gröbner basis. They propose a variation of Buchberger's algorithm (Buchberger, 2006) that computes such a basis using an initial set of generators of the toric ideal of the constraint matrix. Using a reduction method for the basis and adapting the algorithm of Hoşten and Sturmfels (1995), they obtain the set of nondominated images.

Independently, De Loera et al. (2009) and Blanco and Puerto (2007), Blanco and Puerto (2012) use short rational functions and Barvinok's theory (Barvinok, 1994, 1999; Barvinok & Woods, 2003) in order to obtain complexity results and algorithms for multiobjective integer problems. While De Loera, Hemmecke, and Köppe require a fixed number of decision variables and a fixed number of objectives for their results, Blanco and Puerto drop the latter requirement. The idea behind Barvinok's theory is to encode integer points in a rational polytope P in a "long" sum of monomials. This can be reformulated as a short rational function. It is shown that only polynomial time and space is needed to encode all nondominated

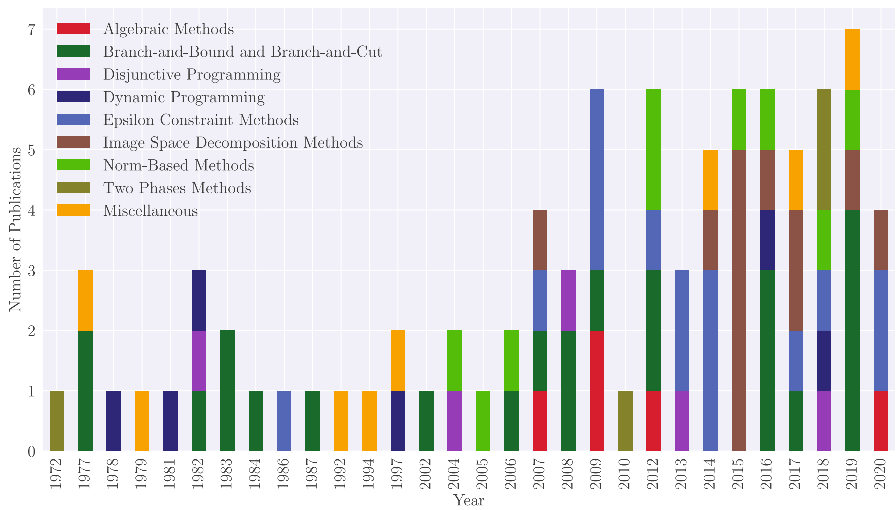

FIGURE 3 The publication history of algorithms for multiobjective integer linear problems sorted by categories

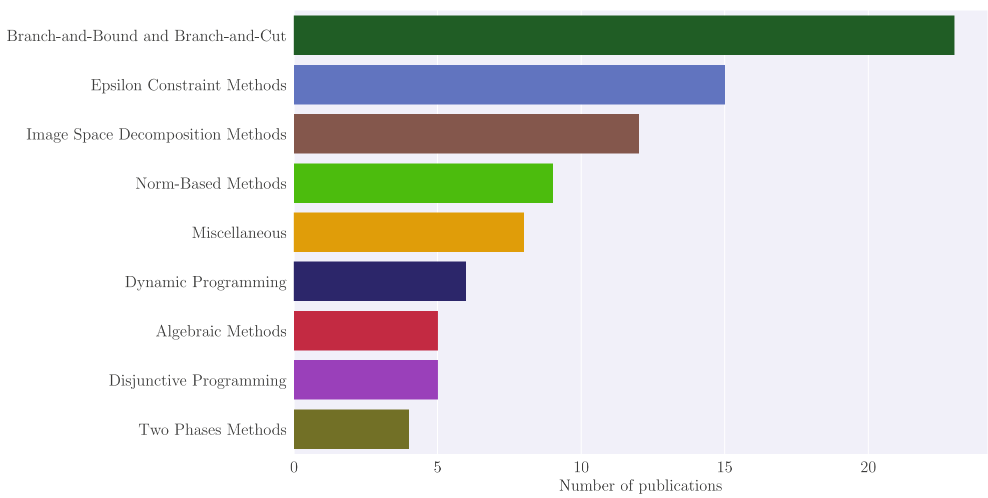

FIGURE 4 The number of papers for each category for multiobjective integer linear problems

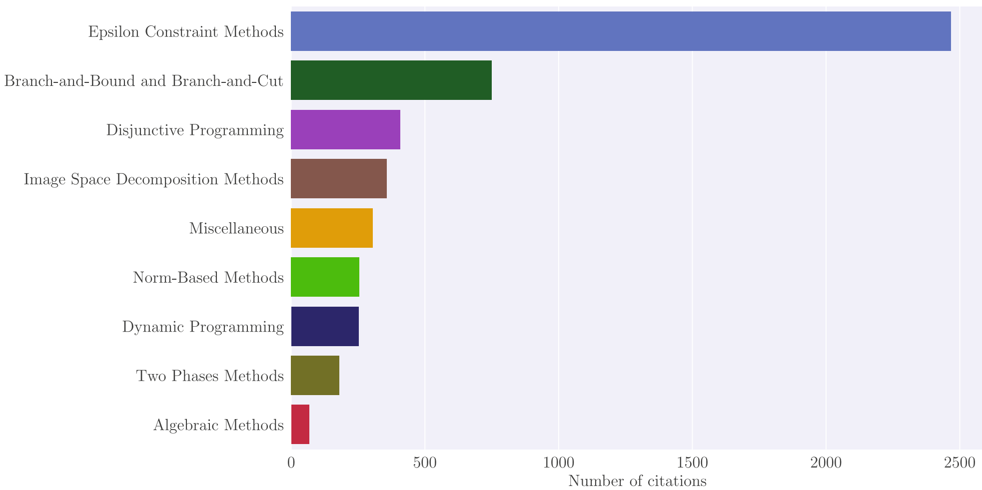

FIGURE 5 The number of citations for each category for multiobjective integer linear problems

TABLE 1 Assignment of references to the sub-groups for pure integer problems

Assignment of references to the sub-groups for pure integer problems	
Algebraic methods	Blanco and Puerto (2009, 2007), Blanco and Puerto (2012), De Loera. Hemmecke, and Köppe (2009) Hartillo-Hermoso et al. (2020)
Branch-and-Bound and Branch-and-Cut	Abbas and Chaabane (2002), Abbas et al. (2012), Bitran and Rivera (1982), Boland et al. (2016a, 2019), Chergui et al. (2009), Chergui et al. (2008), Deckro and Winkofsky (1983), Ehrgott and Gandibleux (2007), Gadegaard et al. (2019), Jozefowiez et al. (2012), Kiziltan and Yucaoğlu (1983), Parragh and Tricoire (2019), Przybylski and Gandibleux (2017), Sergienko and Perepelitsa (1987), Simopoulos (1977), Sourd and Spanjaard (2008), Sourd et al. (2006), Turgut et al. (2019), White (1984), Zionts (1977)
Disjunctive Programming	Bektaş (2018), Klein and Hannan (1982), Lokman and Köksalan (2013), Sylva and Crema (2008, 2004)
Dynamic Programming	Bergman and Cire (2016), Bergman, Bodur, et al. (2018), Klötzler (1978), Trzaskalik (1997), Villarreal and Karwan (1981), Villarreal and Karwan (1982)
Epsilon Constraint Methods	Al-Rabeeah et al. (2020), Bérubé et al. (2009), Chalmet et al. (1986), De Santis et al. (2020), Kirlik and Sayın (2014), Lokman et al. (2017), Mavrotas (2008), Mavrotas (2009), Mavrotas and Florios (2013b, 2012, 2013a), Özlen and Azizoğlu (2009), Özlen et al. (2014), Pettersson and Özlen (2017, 2019b), Sáez-Aguado and Trandafir (2018), Zhang and Reimann (2014)
Image Space Decomposition Methods	Boland, Charkhgard, and Savelsbergh (2015a, 2017, 2016b, 2014a, 2015b), Dächert and Klamroth (2015), Dächert et al. (2017), Dhaenens et al. (2010), Klamroth et al. (2015), Leitner et al. (2016), Lemesre et al. (2007), Tamby and Vanderpooten (2020)
Norm-Based Methods	Clímaco and Pascoal (2016), Dächert et al. (2012), Dumaldar (2015), Filho et al. (2019), Holzmann and Smith (2018), Jahanshahloo et al. (2004), Ralphs et al. (2006), Sayın and Kouvelis (2005), Tohidi and Razavyan (2012)
Two Phases Methods	Dai and Charkhgard (2018), Pal and Charkhgard (2018), Pasternak and Passy (1972), Przybylski et al. (2010b)
Miscellaneous	Bitran (1977, 1979), Keshavarz and Toloo (2014), Kouvelis and Carlson (1992), Neumayer and Schweigert (1994), Schweigert and Neumayer (1997)

images in a short sum of rational functions. Additionally, several algorithms are proposed. Blanco and Puerto (2007) present a digging algorithm that enumerates the set of nondominated images. This is realized by processing the lead terms and distinct coefficients applied to the Laurent expansion of the short rational functions of the polytope of feasible solutions. This algorithm has a polynomial delay, that is, the maximum computation time between two consecutive outputs and before the first and after the last output of the algorithm is bounded by a polynomial in the encoding length of the input. Blanco and Puerto (2007) and De Loera et al. (2009) propose methods where a hypercube, that embeds the short rational functions of the image set, is tested for nondominated images by an application of the digging algorithm above. Then, this hypercube is iteratively shrunk regarding its dimension, that is, addends of the short rational function that cannot return unknown nondominated images are excluded.

Hartillo-Hermoso et al. (2020) use test sets to iteratively solve ε-constraint problems for BOILPs with updated ε. For an IP in standard form, a test set is a set $\mathcal{T} \subseteq \{t \in \mathbb{Z}^n : At = 0\}$ such that for any non-optimal, feasible solution there is a $t \in \mathcal{T}$ such that x - t is feasible and improves the objective function value. If x^* is an optimal solution, x^*- t is infeasible for any $t \in \mathcal{T}$. Hence, starting with a feasible solution and substracting elements of the test set gives an optimal solution without using an IP solver. Test sets can be computed using Gröbner bases (Thomas, 1995) with respect to the lexicographic ordering, which induces a total ordering in the objective space. While showing promising running time results for unconstrained knapsack problems, the computation time of the Gröbner bases is time consuming as it highly depends on the number of variables.

3.2 | Branch-and-bound and branch-and-cut

In single objective integer programming, branch-and-bound and branch-and-cut algorithms belong to the most prominent and most basic algorithmic solution strategies. The idea of implicitly searching an enumeration tree is also applied to the multiobjective case by several authors. The basic steps of these algorithms follow the single objective approach (active node selection, branching, fathoming, etc.). Yet, they are adapted to the fact that a set of "optimal" solutions is sought. For example, instead of primal and dual bounds, so-called bound sets are utilized and maintained which comes with an extra effort. Due to their universality, branch-and-bound algorithms have been among the first algorithms for multiobjective integer problems. The application of cuts has significantly improved their performance. Anyhow, the enumeration is a time consuming task, in particular when compared to algorithms operating in the image space and considering the fact that usually we have $p < < n$.

Zionts (1977) introduces a branch-and-bound and a cutting plane algorithm. The author presents extended methods for multiobjective linear programming by using multipliers, while the decision maker is included in the algorithmic process.

Simopoulos (1977) develops a search-tree based implicit enumeration algorithm to find all efficient solutions of a multiobjective binary problem. At each node, certain variables are set to one and arcs indicate if the terminal node has set one additional variable to one. This search tree is built by an algorithm and traversed using certain rules that guarantee that successors do not have to be considered if the current solution is feasible or dominated.

For multiobjective binary problems, a classical branch-and-bound method including computation of bounds, fathoming rules, feasibility testing, branching, and dominance is suggested by Bitran and Rivera (1982). Lower bounds are generated by solving a weighted-sum problem. The authors suggest to use utility theory to equip the solution procedure with a method to find a finally preferred solution (after the nondominated set has been found). The proposed algorithm is especially designed for a class of facility location problems and computational results are provided. Kiziltan and Yucaoğlu (1983) refine this approach. Their algorithm is also based on a branch-and-bound procedure: a search tree is kept and nodes, which represent partial solutions, are explored one by one with the goal of finding/deciding feasible and efficient solutions. Lower and upper bounds are used to fathom nodes of the search tree in addition to fathoming by infeasibility and dominance. As soon as the enumeration is complete, the lower bound set constitutes the nondominated images.

Deckro and Winkofsky (1983) present an algorithm for binary MOILPs that is based on implicitly enumerating all efficient solutions. It systematically keeps track of partial solutions obtained so far which are then to be completed such that new nondominated images are found. Bounds from solutions which have been found in preceding iterations are used to limit the search space. Additionally, a dominance check is utilized to filter dominated solutions. If new solutions are found in a current iteration, that is, if an iteration was successful, then a backtracking procedure is used to return to another, yet unexplored partial solution and the procedure starts again.

Multiobjective binary maximization problems with polynomial objective functions and one integer valued polynomial equality constraint are researched by White (1984). The author first shows that a multiobjective variant of Lagrangian relaxation, where the same penalty term is added to each of the objective functions, yields the same set of efficient solutions as the original problem. Then, a basic branch-and-bound algorithm is described that is based on a particular decomposition of the objective functions. This is used to derive estimates of local ideal and nadir points, respectively, in order to exclude dominated sub-problems and, hence, iteratively fix variables.

Sergienko and Perepelitsa (1987) consider the complexity of a sequence of biobjective combinatorial problems on graphs with the goal of computing a minimal complete set of efficient solutions. A polynomially solvable, yet non-trivial biobjective integer problem on graphs is presented. This problem is a combination of weighted-sum and bottleneck objectives. First, lexicographic solutions are found and then a threshold-type algorithm is presented.

Ehrgott and Gandibleux (2007) discuss general concepts of lower and upper bound sets on the nondominated set for general MOILPs (see also Ehrgott & Gandibleux, 2001, for a prior work on the biobjective case). Moreover, the article suggests a method to compute lower and upper bound sets for biobjective problems based on the iterative solution of weighted-sum scalarizations. In the case of lower bound sets, problem relaxations can be additionally used to further reduce the computational burden. Special attention is given to particular combinatorial problems. Here, the quality of the resulting bound sets is numerically analyzed and compared using different quality indicators at a large number of test instances.

A general multiobjective branch-and-bound framework based on lower and upper bound sets is presented by Sourd and Spanjaard (2008) (see also Sourd et al., 2006, for a prior version). In this work, the upper bound set is based on supported nondominated images and/or efficient solutions that are computed using a heuristic algorithm. Nodes of the enumeration tree are fathomed if the lower bound set of the corresponding sub-problem and the (global) upper bound set can be separated by an appropriate hypersurface. The method is exemplified on a biobjective spanning tree problem. Moreover, a problem specific pre-processing strategy is applied. Numerical tests on different types of instances with up to 500 nodes show the importance of high quality bound sets for the efficiency of the branch-and-bound algorithm.

Abbas et al. (2012), Chergui et al. (2008), and Chergui et al. (2009) first observe some polyhedral properties and then introduce a cutting plane algorithm. It first finds a minimizer of one of the individual objective functions and then iteratively introduces Dantzig cuts and Gomory cuts to enumerate systematically all efficient solutions. The dual simplex method is then applied to restore feasibility. A cutting plane method is proposed by Abbas et al. (2012). Two types of nodes are distinguished in the search tree: In the first kind of nodes, non-integer solutions are found and some branching is needed. In the second, integer solutions are found and cuts are introduced to remove dominated solutions or fathom the nodes themselves. A (dual) simplex-type algorithm is used to generate these cuts. If a solution is non-integer, a branching step is performed yielding two child nodes in the search tree. A comparison with Sylva and Crema (2004) and Özlen and Azizoğlu (2009) and a computational study are provided.

The single-objective branch-and-cut concept is extended by Jozefowiez et al. (2012) using lower and upper bound sets according to Ehrgott and Gandibleux (2007) to prune subproblems. The method is particularly tailored to problems, where all but one objective have only a limited number of feasible outcome values (as, for example, min – max objective functions). Upper bound sets are derived from already known feasible solutions (initialized, for example, by heuristics). Lower bound sets are defined by sets of outcome vectors that are obtained. Parallel branching is used and cutting planes are generated based on fractional variables in the LP relaxations. Cutting planes are in some cases also obtained from partial dominance between lower and upper bound sets. This is exemplified at biobjective problems. The method is compared to a biobjective image space method that successively solves ε-constraint scalarizations.

Przybylski and Gandibleux (2017) review the general single and multiobjective branch-and-bound concept, and the main ingredients of multiobjective branch-and-bound algorithms. This includes a discussion of lower and upper bound sets, dominance tests and pruning techniques, global and local branch and bound strategies as well as pre-processing techniques. The article thus contributes to the multiobjective branch-and-bound literature. First and foremost, however, it contains an extensive and thorough review of the literature on multiobjective branch-and-bound up to the year 2017. The respective contributions are described in detail and categorized according to the identified characteristics of multiobjective branch-and-bound methods.

A generic branch-and-bound algorithm, which does not use problem specific structures or information, is presented by Parragh and Tricoire (2019). It efficiently utilizes the fact that for biobjective problems, there exists a total ordering among the nondominated outcome vectors. As opposed to generic image space methods, the suggested approach can take advantage of problem specific speed-up techniques. This includes, for example, column generation for efficient bound computations, and integrality of objective functions for bound improvements. Infeasible areas of the search region are pruned using a sophisticated analysis of lower bound sets (obtained from linear programming relaxations) and upper bound sets (given by incumbent solutions). The bound set filtering procedure generates independent subproblems using a variant of Pareto branching/image space branching (see Gadegaard et al., 2019; Stidsen et al., 2014, below and in Section 4.1, respectively).

Gadegaard et al. (2019) suggest a biobjective LP-based branch-and-cut algorithm. Lower and upper bound sets are defined according to Ehrgott and Gandibleux (2007). Various strategies for lower bound computation, dominance testing and pruning, cut generation, and branching strategies are proposed and compared in extensive numerical tests. This also includes a strengthened version of Pareto branching/image space branching (Parragh & Tricoire, 2019; Stidsen et al., 2014). This branching strategy is also discussed by Forget et al. (2020), where a general branch-and-bound framework and a generalization of Pareto branching/image space branching to more than two objectives is presented for multiobjective combinatorial problems.

Boland et al. (2019) (see also Boland et al., 2016a, for an earlier version) study conditions on sets of objective functions that guarantee that the ideal point is feasible for all possible feasible constraint sets. Sets of objective functions satisfying this condition are called universally co-ideal. To determine whether a set of objective functions is universally co-ideal is NP-hard, but can be detected in pseudo-polynomial time. Moreover, all universally co-ideal objective functions can be replaced by a single unified objective function without changing the efficient set. The concept can be used for problem simplification, for the computation of bounds on the size of the nondominated set, and for cut generation using, for example, restrictions to variable subsets and to subsets of the efficient set.

Turgut et al. (2019) have combined a branch-and-bound algorithm with domination and infeasibility fathoming and a pre-processing step with parallelization strategies. They report an extensive computational study and show the potential of parallelization for MOILP algorithms in general and branch-and-bound algorithms.

3.3 | Disjunctive programming

Disjunctive programming, also called disjunctive constraints or disjunctive inequalities, is a general concept in optimization meant for feasible sets consisting of different parts that are logically linked by "or"-statements. In the context of integer programming, these constraints can be reformulated with the help of a big-M-approach and artificial binary variables: they are used to activate or deactivate parts of the feasible set. Thus, the binary variables indicate in which part of the feasible set the optimal solution lies. For multiobjective optimization problems, this concept is useful to describe the search region as the union of rectangular sets. It is then possible to consider all these parts of the search region simultaneously, that is, to search for a new nondominated image by solving one single IP. However, with every new nondominated image, the number of constraints and artificial binary variables increases. Thus, this approach is computationally demanding, in general.

Klein and Hannan (1982) use disjunctive constraints to solve multiobjective integer minimization problems. They choose any of the given p objective functions as objective of the disjunctive constraint problem. All others appear in the constraints. With each new nondominated image found, they add a set of $p - 1$ constraints to the problem. In their numerical study, problems with two to five objectives are presented, however, having only a rather limited set of nondominated images.

Sylva and Crema (2004) use a reformulation of the disjunctive constraints by a big-M-approach together with artificial binary variables. Moreover, they change the objective function to a weighted sum, which results in obtaining nondominated instead of only weakly nondominated images. In every iteration, p binary variables as well as $p + 1$ constraints are added, which increases the computational effort. Indeed, Sylva and Crema (2004) only generate complete representations for the biobjective case. For three objectives, they restrict the numerical study to the generation of incomplete representations.

Sylva and Crema (2008) extend Sylva and Crema (2004). They still use binary variables to indicate active constraints. However, in contrast to the previous work, the binary variables are extracted from the disjunctive optimization problem and "optimized" separately. The computational time of the resulting method has been decreased compared to Sylva and Crema (2004).

Lokman and Köksalan (2013) present two algorithms for integer multiobjective optimization problems in maximization format. The first one directly builds upon the method by Sylva and Crema (2004) by augmenting the objective by all other objectives scaled by a small constant. Thereby, one constraint and one binary variable can be omitted in each iteration. However, the algorithm still suffers from the quickly growing number of constraints and binary variables. For the second algorithm, the authors observe that for each feasible image, at most one constraint from the disjunctive formulation is sufficient for each of the $p - 1$ objectives. In each iteration, a set of $n + 1$ ε-constraint problems with augmentation terms are solved. Here, the right-hand side values depend on components of previously determined nondominated images. Hence, this approach also fits into the ε-constraint category, Section 3.5.

The *L*-shape approach of Boland et al. (2016b) can be seen as a hybrid between disjunctive programming and image space decomposition. It restricts the use of disjunctive constraints to only one nondominated image, which results in an *L*-shape in the triobjective case. It is discussed in more detail in Section 3.6.

Bektaş (2018) studies the disjunctions formulated by Klein and Hannan (1982) more closely and observes that only certain conjunctions of disjunctions need to be considered. Others that lead to dominated sets of inequalities can be excluded via a filtering step in the proposed algorithm. Note that the idea resembles the redundancy elimination approach of Klamroth et al. (2015) in Section 3.6.

3.4 | Dynamic programming

Dynamic programming has been formalized first by Bellman (1966) and, since then, it has been enhanced and specialized for different types of problems. However, its roots can be traced back to ancient Greece Carraway & Morin, 1988). It is widely applicable (Bellman & Dreyfus, 2015), the application to knapsack problems and Dijkstra's algorithm for shortest path are prime examples. In general, a system of stages and states is applied. At each stage, the state variables are set and the current value is determined by the value of previous stages via recursive equations until the final stage is computed. By Bellman's principle of optimality (Bellman & Dreyfus, 2015), the final stage returns an optimal solution. We refer to Lew and Mauch (2006) and Wolsey (1998) for a more detailed introduction and to Li and Haimes (1989) for an overview in multiobjective optimization. The universality of dynamic programming and its significant performance for particular problem structures comes with the cost of keeping track of numerous solutions at once that exponentially increases with the number of objective functions.

Klötzler (1978) is the first to introduce the principle of dynamic programming to the multiobjective case. The classical single objective dynamic programming approach is adapted and suitable changes are made regarding efficiency. Instead of separability and monotonicity, a more sophisticated property has to be satisfied. Trzaskalik (1997) extends the notions and properties of multiobjective dynamic programming and provides a brief survey of algorithms.

Villarreal and Karwan (1981) introduce dynamic programming recursive equations for multiple objectives. They consider a multi-dimensional knapsack-like formulation to solve multiobjective integer problems. Negative objective function coefficients can be considered and not all solutions have to be computed in each stage. Further, they propose a hybrid approach with lower bounds based on heuristic solutions to weighted sum scalarizations and a variant of local nadir points as upper bounds. Villarreal and Karwan (1982) consider a similar approach. Their algorithm is based on lifting the separability and monotonicity properties to multiple objective functions and making changes regarding efficiency. Further, they consider the incorporation of bounds.

Bergman, Bodur, et al. (2018), Bergman and Cire (2016) use a method similar to dynamic programming that involves two approaches of transforming a multiobjective binary or integer problem into a network model. First, they propose the usage of (binary) decision diagrams (Akers, 1978) that have also been used for other optimization problems (Bergman et al., 2016; Bergman, Cire, et al., 2018). A binary decision diagram is a layered-acyclic digraph, where each feasible solution resembles a path in the graph and arcs indicate whether a variable is set to one or zero. The second approach uses a multiobjective recursive formulation that is closely related to the state-transition graph of dynamic programming (Villarreal & Karwan, 1982). Based on this formulation, they construct a network model and reinterpret the elements of the formulation as nodes and arcs. Variants of multiobjective shortest path algorithms tailored to this network are applied to solve this model. Hence, nondominated images are interpreted as efficient paths for a multiobjective shortest path algorithm.

3.5 | Epsilon constraint methods

The ε-constraint scalarization, see Section 2, is one of the most prominent scalarization techniques in multiobjective optimization. Although introduced by Haimes et al. (1971) to find a single weakly efficient solution, this technique can iteratively be used to find the entire nondominated set of multiobjective integer optimization problems. Typically, ε-constraint methods can easily be implemented in a rudimentary fashion while providing various and significant possibilities for improvement of the performance via constraint selection and bound shifting. In general, the application of ε-constraints produces single objective problems that, from a complexity theoretical point of view, are often hard to solve compared to the single objective pendant to the multiobjective problem. Thus, the running time is decelerated. The latter point can be overcome by the usage of fast single objective IP solvers.

Chalmet et al. (1986) formulate a procedure to compute the entire set of nondominated images of a biobjective integer programming problem, which is later extended to the general multiobjective case. During the execution of their algorithm, a set Q of candidate adjacent pairs of nondominated images is maintained. This set is initialized with the two lexicographically optimal images. In each iteration, two adjacent nondominated images are removed from Q and are used to solve a weighted sum problem with constraints on both objectives that are based on the removed images. In a similar fashion, De Santis et al. (2020) introduce the frontier partitioning algorithm to solve biobjective integer optimization problems. Their procedure relies on the solution of $|Y_N|+2$ single objective integer programs. Additionally, the authors show that their algorithm can also be applied to biobjective nonlinear integer programs.

Bérubé et al. (2009) deal with the classical ε-constraint method for biobjective combinatorial optimization problems with integer objective values, where one objective is transformed into a constraint. They show how to generate the entire nondominated set by iteratively solving ε-constraint problems. Improvements to reduce the bound on the constrained objective by more than a value of one as well as a speed up heuristic are presented. Computational results for the traveling salesman problem with profits show the effectiveness of their approach.

"AUGMECON" (Mavrotas, 2008; Mavrotas, 2009) denotes an augmented ε-constraint method. In contrast to the classical ε-constraint method, the objective function constraints are transformed to equality constraints by introducing slack or surplus variables. These variables are added as a weighted penalty term to the objective function. Using this formulation, the computation of weakly nondominated images can be avoided. "AUGMECON2" (Mavrotas & Florios, 2012, 2013b, 2013a) is an improvement of AUGMECON. Here the information obtained from the slack or surplus variables in each iteration is exploited by introducing a so-called bypass coefficient. This bypass avoids redundant iterations yielding the same nondominated images as obtained in previous iterations. Further, the weights are modified such that the objectives are considered in a kind of lexicographica manner. In "SAUGMECON" (Zhang and Reimann (2014)), which is a variant of AUGMECON, uses the ε-constraint and adds a weighted sum of the constrained objectives to the objective function. To improve the efficiency for solving multiobjective integer programming problems, two acceleration mechanisms are proposed: an acceleration algorithm with early exit and an acceleration algorithm with bouncing steps.

Özlen and Azizoğlu (2009) define for the biobjective case a constrained weighted single objective integer programming problem (CWSOIP) that minimizes the first objective f_1 plus a weighted version of the second objective f_2 with f_2 being constrained by an upper bound. It is shown how to choose the weight and bound to compute the set of all nondominated images by iteratively solving CWSOIP and reducing the upper bound on f_2. Again, for the triobjective case, they define the constrained weighted biobjective integer programming problem (CWBOIP) with the first objective minimizing the original first objective function f_1 plus a weighted version of the third objective f_3, whereas the second objective minimizes f_2 plus a weighted version of f_3. Similar to CWSOIP, f_3 is constrained by an upper bound. By iteratively solving CWSOIP, this upper bound can be computed with the same objectives as in CWBOIP minimizing over all feasible solutions with an appropriate chosen weight for f_3. Their method iteratively solves CWBOIP and their recursive algorithm is then generalized to the multiobjective case. Özlen et al. (2014) improve the recursive algorithm presented by Özlen and Azizoğlu (2009) using information of images to relaxations of previously solved subproblems. Al-Rabeeah et al. (2020) develop an improvement of the method presented by Özlen et al. (2014), which reduces CPU times and the number of integer programs that have to be solved.

Pettersson and Özlen (2017) examine how parallelization can be applied to biobjective integer algorithms in order to improve running times. In particular, they consider the algorithm by Özlen et al. (2014). They propose two methods to parallelize the algorithm: splitting the range of the second objective function into intervals either statically or dynamically. The former just splits the range into equal intervals and applies the algorithm. The latter one starts with the lexicographic optimal images and solves lexicographic problems with an additional constraint. Information about the current constraints are exchanged until the two threads meet in the middle. This doubles the speed of the algorithm and also outperforms a CPLEX parallelization. Pettersson and Özlen (2019b) generalize the methods to an arbitrary number of objectives and the exchange of information is discussed in more detail. The static splitting is just performed on the last objective function. The dynamic splitting uses an auxiliary problem selecting k of p objective functions. The remaining objective functions are considered in a lexicographic manner in the optimization process and bounds on these objectives are applied. These problems are solved by parallel threads and both bounds and solutions are shared. Two policies for the selection of objective functions are developed and tested against existing algorithms of Dächert and Klamroth (2015), Dhaenens, Lemerse, and Talbi (2010), Özlen et al. (2014), and a CPLEX parallelization of Özlen et al. (2014). For three and four objectives they show that their parallelization performs better than the other algorithms.

Sáez-Aguado and Trandafir (2018) review the ε-constraint method and its variants and discuss their corresponding complexity with respect to the number of integer programs to be solved. These methods require solving at least $|Y_N|+3$ integer programs. Therefore, they propose generalizations of the methods designed by Neumayer and Schweigert (1994) and Özlen and Azizoğlu (2009), where only $|Y_N|+1$ integer programs have to be solved.

Kirlik and Sayın (2014) formulate a two-stage ε-constraint formulation for general multiobjective discrete optimization problems with p objectives that avoids the computation of weakly nondominated images. Rectangles in the $(p-1)$-dimensional constraint space are searched for nondominated images. These rectangles are partitioned into smaller disjoint rectangles during the execution of their algorithm depending on their associated volume measure.

A web-based solution platform to generate all nondominated images, a local subset or a representative nondominated set with desired quality level is developed by Lokman et al. (2017).

3.6 | Image space decomposition methods

In this section, we deal with criterion search methods that aim at decomposing the search space to facilitate the search for nondominated images. Therefore, geometrical properties are extensively exploited to discard several regions of the image space which do not need to be considered. If combined with ε-constraint scalarizations, these methods can also be interpreted as a special variant of ε-constraint methods, cf. Section 3.5. Due to the increased efficiency of single objective IP solvers, researchers have lately focused on image space decomposition methods obtaining algorithms with a remarkable performance in computational studies. However, since they solely operate in the image space, the problem structure containing valuable information is largely discarded in the design of algorithms.

Lemesre et al. (2007) propose a method to compute the entire set of nondominated images of a biobjective problem in three stages. In the first stage, the two extreme nondominated images are calculated, while in the second stage the search space (defined by the extreme nondominated images) is equally split with respect to one objective. In each of these splits, a nondominated image is computed. Supported as well as unsupported nondominated images can be found using this procedure. In the third stage, rectangles defined by two adjacent nondominated images are explored. Dhaenens et al. (2010) propose a generalization of this method for any number of objectives.

Boland et al. (2015a, 2015b) present the so-called "Balanced Box Method" for BOILPs, which is an extension of the box algorithm by Hamacher et al. (2007). Initially, the method divides the rectangle defined by the images of the two lexicographically optimal solutions horizontally into a lower and an upper rectangle. Then, the lower rectangle is searched for a nondominated image by solving a lexicographic optimization problem. In case a nondominated image is found, the portion of the upper triangle that is dominated gets discarded. Afterwards, the upper rectangle is searched for a nondominated image by solving again a lexicographic optimization problem with respect to the other objective. The same procedure is repeated for all newly created rectangles. Another search method for biobjective binary programs to find all nondominated images (later called "Adaptive Search in Objective Space" (ASOS) by Leitner et al., 2016) is discussed. Again, rectangles are used to search for nondominated images. Based on the current image pool of (not yet dominated) images, their method either uses the ε-constraint method (similar to Bazgan et al., 2017) or binary search in image space to explore a rectangle. Leitner et al. (2016) introduce an exact method for biobjective binary programs based on ASOS. It explores rectangles in the image space by combining the ε-constraint method and the binary search in image space (BSOS).

The "L-Shape Method" for finding all nondominated images of a triobjective integer program is presented by Boland et al. (2016b). The L-Shape Search Method uses rectangles and L-shapes in the projected space defined by the first two objectives to search the image space for nondominated images. Rectangles are explored depending on their area by searching for a nondominated image with its projection lying in the rectangle. Either a nondominated image with its projection in the rectangle is found or it is determined that no nondominated image is contained in the rectangle. In the former case, the nondominated image induces an L-shape contained in the rectangle, which is then further explored. In the latter case, the rectangle gets discarded from consideration. The exploration of a rectangle is basically done by repeatedly solving two consecutive integer programs.

Dächert and Klamroth (2015) observe that for more than two objectives redundancies occur when decomposing the search region based on a nondominated image in a standard way. They develop a split criterion based on a neighbourhood relation between local upper bound sets to avoid the generation of redundant boxes and to keep the number of subproblems to be solved low. They present a method for finding the entire set of nondominated images of a triobjective optimization problem for which the number of subproblems that have to be solved is linear with respect to the number of nondominated images. Their algorithm is independent of the chosen scalarization technique. However, they show how to reduce the linear bound further by using the ε-constraint method. Concise representations of the search region of general multiobjective optimization problems based on so-called local upper bounds (also referred to as local nadir points) are suggested and thoroughly analyzed by Klamroth et al. (2015). Assuming that the number of objectives is fixed, it is shown that the number of search zones grows only polynomially with the number of nondominated images. The complexity of the update operation is numerically tested on randomly generated instances with up to six objectives. Dächert et al. (2017) combine the results of Klamroth et al. (2015) and Dächert and Klamroth (2015). A neighbourhood concept is established to update the local upper bound sets more efficiently. Although the established concept does not improve the asymptotic (worst-case) running time, computational results are provided showing a significant improvement in running time, especially for instances with a high number of nondominated images. Lacour et al. (2017) utilize the results to compute the so-called hyper-volume indicator of a set S. Tamby and Vanderpooten (2020) expand the work of Klamroth et al. (2015) by exploiting particular properties of the ε-constraint scalarization in combination with the structure of the search region to further reduce the number and the intricacy of the subproblems that need to be solved when iteratively exploring search regions. Extensive numerical tests validate the efficiency of their approach.

The "Quadrant Shrinking Method" (QSM) (Boland et al., 2014a, 2017) is another method that computes the entire set of nondominated images of a triobjective integer optimization problem.

QSM is a criterion search algorithm working in a projected 2-dimensional image space defined by the first two objectives. Thereby, the authors use a two-stage scalarization technique similar to the one by Kirlik and Sayın (2014). This technique is used to explore quadrants defined by upper bounds in the projected space for so far unknown nondominated images.

Doğan et al. (2021) propose a box method based on Pascoletti-Serafini scalarizations to generate previously unknown efficient solutions for biobjective problems. Several variants of the box method are proposed mainly differing in the way the boxes and the direction vectors of the scalarization method are defined.

3.7 | Norm-based methods

One of the classic scalarization approaches is to determine some reference point and then to minimize the distance to this point. These distances can be measured by ℓ-norms. Of special interest are the cases of $\ell = 1{,}2$ and $\ell = \infty$. The latter case means that we minimize the maximum distance componentwise. The resulting optimization problem is also called Tchebycheff problem, or weighted Tchebycheff problem when equipped with weights, see Section 2. Norm-based methods also mainly perform in the image space. The properties of the weighted Tchebycheff scalarization are closely related to the ε-constraint method. Thus, advantages and disadvantages are similar to those discussed in Sections 3.5 and 3.6.

Sayın and Kouvelis (2005) use a two-stage weighted Tchebycheff method similar to the one by Eswaran et al. (1989) to solve a biobjective discrete optimization problem. They present two variants, one using the ideal point and the other using the origin as a fixed reference point. The weights are computed based on the fixed reference point and the local nadir point of two adjacent nondominated images.

Ralphs et al. (2006) use both a weighted and an augmented weighted Tchebycheff method to solve biobjective integer problems. They compute the weights based on the local ideal and local nadir point between two adjacent nondominated images. In each iteration, either a new nondominated image is computed or the considered region can be discarded. The augmented method requires the selection of an auxiliary parameter in the objective function that must be chosen with care, since, if the parameter is too large, some nondominated images can not be generated. However, if too small, numerical issues might appear.

Dächert et al. (2012) consider the problem of how to choose the augmentation parameter in the augmented weighted Tchebycheff problem in the biobjective case. The idea is to avoid the problems previously discussed by appropriately choosing this parameter. On the one hand, it is chosen small enough so that no nondominated image is missed, but on the other hand, as large as possible to avoid numerical issues.

Clímaco and Pascoal (2016) present a two phases approach with a weighted sum method in the first phase and a Tchebycheff method with varying reference points in the second phase. The numerical comparison between their approach and a "one-phase"-method only solving Tchebycheff problems reveals that the two phases method is slightly better. They also present a variant for computing approximations as well as an interactive variant.

Holzmann and Smith (2018) use a modified augmented weighted Tchebycheff norm which has already been proposed in a general form by Kaliszewski (2000). They consider discrete multiobjective optimization problems with any number of objectives. A numerical study includes multiobjective cardinality constrained knapsack and assignment problem instances for three to six objectives.

Filho et al. (2019) base their method on the non-inferior set estimation (Cohon, 2003) and on the works by Ralphs et al. (2006) and Solanki (1991). They find the whole nondominated set for biobjective problems by iteratively solving Tchebycheff problems with equal weights while changing both the reference point and the bounding box with respect to previously found images.

Jahanshahloo et al. (2004) propose a method that iteratively uses the 1-norm to obtain the set of all efficient solutions for problems with binary decision variables. Initialized with the set of unique minimizers of the individual objective functions, they solve the single objective problem $\min_{x \in X} \sum_{i=1}^{p} C_i x$ to obtain additional efficient solutions. Then, they add constraints that are motivated by the observation that for any new solution x and any already found solution $\bar{x}$ there exists an index i such that $C_i x < C_i \bar{x}$. All efficient solutions are returned, provided that the algorithm that solves the single objective problems is capable of returning all optimal solutions. This algorithm is extended to integer problems by Tohidi and Razavyan (2012). Dumaldar (2015) identified an error in the proofs of Jahanshahloo et al. (2004) and corrected this issue. An error in the algorithm is also mentioned, however, we can only confirm an error in the execution of the algorithm in the illustrative example but not in the algorithm itself. Nevertheless, Dumaldar adapted the method and implemented additional constraints to the original algorithm.

3.8 | Two phases methods

Generally speaking, a two phases method utilizes two algorithmic approaches to obtain the whole set of nondominated images. The first algorithm finds the rather "easily obtainable" images and the search region is narrowed. In phase two, a more sophisticated algorithm is applied to also find the remaining (usually unsupported) nondominated images. Initially, this has been applied for biobjective problems with a dichotomic search based on the weighted sum method in the first phase and the ε-constraint method in the second (for example, see Ulungu & Teghem, 1995, for combinatorial problems). Pasternak and Passy (1972) improve this procedure by replacing the ε-constraint method by a hybrid scalarization. The first algorithm provides a tighter search region that is a union of smaller regions for the second one, which may lead to improved running times and opens up the second phase to parallelization. However, the number of supported nondominated images usually tends to be small (Visée et al., 1998), so the advantage may be limited.

Przybylski et al. (2010b) generalize the two phases method to solve MOIPs with $p > 2$ objectives. First, all supported nondominated images are computed using the algorithm by Przybylski et al. (2010a) along with the supporting hyperplanes of the facets of the nondominated set and the nadir point. Next, the authors propose a procedure of computing a set of images D that play the same role as local nadir points in the biobjective context. These points together with the hyperplanes are used to specifically explore the search regions. Afterwards, the set D is updated. The algorithm terminates if no search region is left.

The algorithm by Clímaco and Pascoal (2016) is also a two phases method and is discussed in Section 3.7.

Dai and Charkhgard (2018) propose a two phases method that utilizes the balanced box method (Boland et al., 2015a) for BOILPs, see Section 3.6, in the first phase and the ε-constraint method in the second. The balanced box method finds several nondominated images and from there splits the search region into small rectangles. Then, they switch to the ε-constraint method to check whether a rectangle is empty, or to search the remaining rectangles otherwise. A switching technique based on user-defined parameters is proposed and their method only needs $\lceil 2.5 \cdot Y_N \rceil$ calls to a single objective solver.

The work of Pal and Charkhgard (2018) does not fit completely in this category as they include not two but three phases in their biobjective algorithm. Nevertheless, the "Multi-Stage Exact Algorithm" (MSEA) consists of a combination of several exact and approximate algorithms known from the literature and, hence, the main contribution is an integrated framework of all these algorithms and their respective variations in MSEA. The first phase approximates the nondominated set to provide a warm start to the solution process in phase three. The second phase is initialized by computing the lexicographically optimal images and $K - 1$ additional nondominated images. Then, K boxes defined by two consecutive nondominated images are generated. In the third phase, these boxes are explored in parallel on K different processors. In each iteration, a box is searched for at most two nondominated images and new boxes are created and ordered. The approximation and the box exploration methods can be chosen by the user or it can be even switched during the algorithm.

3.9 | Miscellaneous

This section presents algorithms and methods that either do not fall into any of the previous categories or are so specific that they have to be discussed individually.

For multiobjective binary problems, Bitran (1977) considers an auxiliary problem with binary variables only and without constraints. Hence, every efficient solution of this problem that is feasible to the original problem is efficient for the latter. A set V that characterizes the reverse polar cone to the cone defined by the rows of the objective function matrix C is introduced. Further, a point to set map M defined on V has as image of $v \in V$ the set of solutions that is dominated in the direction of v. By enumerating the efficient solutions of the auxiliary problem and computing both V and $M(v)$ for $v \in V$, the proposed algorithm is able to compute the efficient solutions of the original problem. However, the algorithm's applicability is limited to small problems. Bitran (1979) improves this algorithm by a backward scheme and an additional map N that maps V onto the solutions that dominate solutions in direction of a $v \in V$. Hence, in contrast to the original method, it works in the opposite direction. However, it is still time consuming. The work is extended to interval linear MOPs, where elements of the matrix C are not given as single values but as an interval.

Neumayer and Schweigert (1994) present three algorithms for finding all efficient solutions of a biobjective integer linear problem with positive values only. The first algorithm, which is generalized to arbitrary objectives by Schweigert and Neumayer (1997), uses a hybrid scalarization between the weighted sum and the ε-constraint method, where the parameter for one of the additional constraints is adjusted in each step while the second algorithm is basically a dichotomic search approach using the Tchebycheff norm. The third algorithm replaces the two objectives by a quadratic function of the form $g(x) = a \cdot C_1x \cdot C_2x + b \cdot C_2x + c \cdot C_1x$ with parameters a, b and c based on two initially known nondominated images that correspond to the lexicographically optimal solutions. Then, the quadratic problem is maximized and either a new solution is obtained or one of the previous solutions is optimal. In the former case, new parameters are computed similar to the weight vector for the dichotomic search and the procedure is repeated until no new solution is found.

While single objective integer linear programming problems with totally unimodular constraint matrices can be solved by solving the corresponding LP-relaxation, this is, in general, not true for multiobjective IPs, even in the case of two objectives, due to the existence of unsupported nondominated images. Kouvelis and Carlson (1992) identify a class of biobjective integer linear programming problems with totally unimodular constraint matrices and objective functions operating on disjoint variable sets, referred to as variable partitioned unimodular programs, where no unsupported nondominated images exist. Hence, in this case, biobjective linear programming relaxations yield the complete nondominated set. This result is extended to so-called variable partitioned concave separable unimodular biobjective integer programs with concave objective functions.

Data envelopment analysis (DEA) (Cooper et al., 1999) is a tool to measure the relative efficiency of solutions in operations research and decision theory. In particular, so-called decision making units are implemented to compute, based on linear programming, a solution with the best performance which is measured by the ratio between outputs and inputs. Hence, this method is often used to maximize the output while minimizing the input. Kesharvarz and Toloo (2014) consider a special biobjective problem, where one objective function is minimized while the other one is maximized using non-negative integer valued objective functions. Translated to the DEA methodology, a connection between DEA models and the biobjective problem is established: an optimal solution to the model by Deprins et al. (2006) is efficient to this biobjective problem and vice versa. The same holds true for optimal solutions of the model by Banker et al. (1984) and supported efficient solutions. An algorithm is proposed that uses these models with iteratively changing parameters to obtain all efficient solutions of a biobjective problem.

4 | ALGORITHMS FOR MULTIOBJECTIVE MIXED-INTEGER OPTIMIZATION PROBLEMS

In contrast to MOILPs, multiobjective mixed-integer optimization problems additionally have continuous variables that have to be considered when solving these problems. This higher degree of difficulty is the reason why the first algorithm for these problems has been proposed as late as 1998 by Mavrotas and Diakoulaki (1998). Since then, about 15 algorithms and algorithmic variants have been developed that can roughly be divided into three categories: branch-and-bound algorithms, algorithms that work in the image space, and hybrid methods that combine the previous two approaches. In Figure 6, we present the publication history, while the number of papers and citations for each category are depicted in Figures 7 and 8, respectively. Remark that our assignment of individual papers to one of several possible categories has a strong impact here. The assignment can be found in Table 2.

4.1 | Branch-and-bound algorithms

Branch-and-bound algorithms for MOMILPs are very similar to the ones designed for MOILPs, see Section 3.2. Note that the branch-and-bound tree only considers the integer valued variables. However, the existence of continuous variables obviously complicates the solution process. This involves not only adjustments to the bound sets and fathoming rules but also the computation of nondominated images for problems at the leaf nodes of the branch-and-bound tree: When these images are added to the nondominated images of previous leaf nodes, the comparison of faces with different dimensions becomes necessary which is a challenging task. Further, the resulting nondominated parts are not necessarily faces but can take all kinds of shapes with open or closed boundaries or both, see Figure 1c. For a detailed and more mathematical description of branch-and-bound algorithms for BOMILPs and especially of the bound sets and fathoming rules, we refer to Belotti et al. (2016). Advantages and disadvantages of branch-and-bound methods carry over from the pure integer case.

Mavrotas and Diakoulaki (1998) have been the first to develop an algorithm that solves a multiobjective mixed-integer problem. Their branch-and-bound algorithm follows the standard procedure of branch-and-bound. More precisely, the algorithm starts with a problem with relaxed binary variables, where it sets a variable at 0 or 1 at each node, and ends as each node is explored or fathomed. If the ideal point of the problem corresponding to a node is dominated by existing (possible) nondominated images, the node is fathomed. At a leaf node, extreme images of the slice problem are computed and compared with previously found images. Later, Mavrotas and Diakoulaki (2005) have noticed that the algorithm outputs dominated images and corrected this issue by implementing an additional dominance check at the end of the algorithm via an auxiliary linear problem. Some minor improvements to the storage of nondominated images and their computation are added. As Vincent et al. (2010, 2013) have presented in their article, the improved algorithm still discards nondominated images while some returned images are dominated. They rectify these issues for the biobjective case via a procedure that compares line segments of different slices and outputs the nondominated parts. Further, they introduce a branching strategy, an initial upper bound set Y_{ESN} and new lower bound sets, that is, the ideal point of both the mixed-binary and the relaxed version and the nondominated set of these two problems.

Belotti et al. (2013) are the first to propose an algorithm for biobjective mixed-integer problems. This branch-and-bound algorithm uses breadth first search, standard branching, and starts with an ε-constraint routine to compute some weakly nondominated images as an initial upper bound set. They implement new fathoming rules based on local nadir points and local nadir sets. Due to the size of the set of nadir points they restrict these to the subset that is weakly

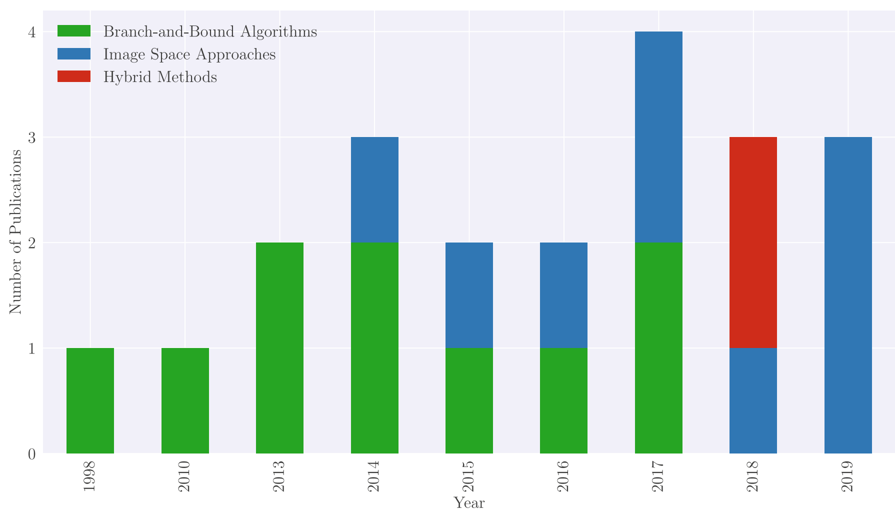

FIGURE 6 The publication history of algorithms for multiobjective mixed-integer linear problems sorted by categories

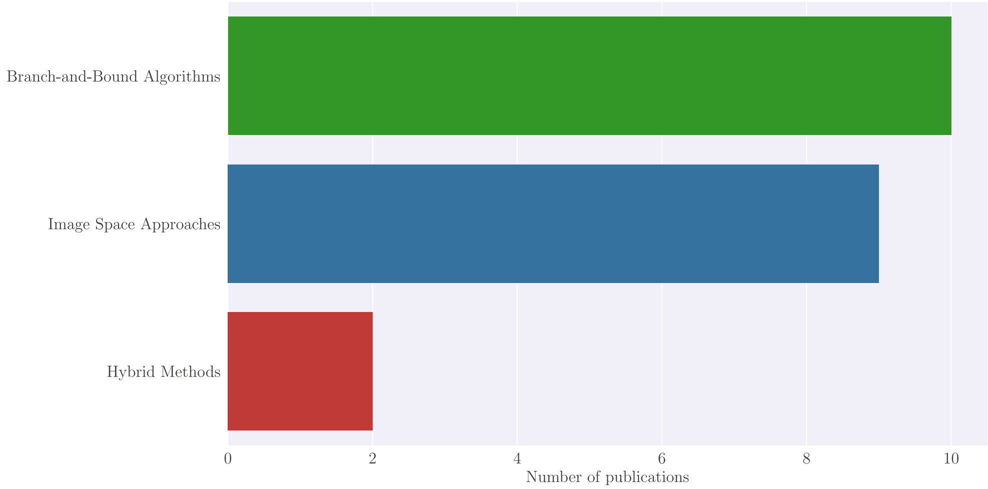

FIGURE 7 The number of articles for each category for multiobjective mixed-integer linear problems

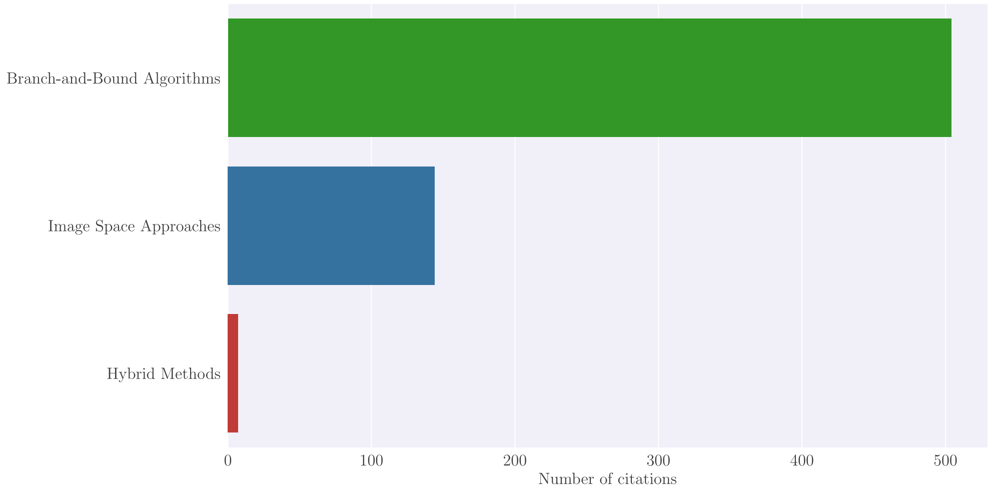

FIGURE 8 The number of citations for each category for multiobjective mixed-integer linear problems

TABLE 2 Assignment of references to the sub-groups for mixed-integer problems

Assignment of references to the sub-groups for mixed-integer problems	
Branch-and-bound algorithms	Adelgren and Gupte (2017), Adelgren et al. (2014, 2018), Belotti et al. (2013, 2016), Mavrotas and Diakoulaki (2005, 1998), Stidsen et al. (2014), Vincent, Seipp, Ruzika, Przybylski, and Gandibleux (2013, 2010)
Image Space Approaches	Boland et al. (2014b, 2015c), Fattahi and Turkay (2018), Perini et al. (2017), Pettersson and Özlen (2019a), Rasmi and Türkay (2019), Rasmi et al. (2017), Rasmi et al. (2019), Soylu and Yıldız (2016)
Hybrid Methods	Soylu (2018), Stidsen and Andersen (2018)

dominated by the ideal point of the relaxed version of the current sub-problem. If this set is empty, the node can be fathomed. Otherwise, the node can be fathomed if each of these local nadir points and sets can be separated by a hyperplane of the nondominated set of the relaxed version of the current sub-problem, which is verified by auxiliary linear problems. In a follow-up article, Adelgren, Berlotti, and Gupte (2018) present a new method to store nondominated images and line segments. Contrary to the previously used dynamic lists, this is based on quad trees which is a balanced tree structure that recursively divides a two dimensional space into four quadrants. Every time a node is added to the tree, all ancestor nodes are checked for dominance and the tree is rebalanced if necessary.

Stidsen et al. (2014) provide an algorithm for biobjective mixed-binary problems, where only one objective function contains continuous variables. They use a special linear problem to obtain nondominated images: the image space is rescaled such that all nondominated images are contained in the unit cube (0,0),(1,0),(0,1),(1,1) and the objective functions are summed up. In their branch-and-bound algorithm, they utilize breadth first search and nodes are fathomed if the objective function value of the auxiliary linear problem is worse than the local nadir points of the images found so far. Besides the standard branching also integer and Pareto branching are performed (cf. Section 3.2): Given that the binary solution part indeed takes binary values, the former one excludes the binary part from further consideration in this branch by no-good constraints, see Soylu and Yıldız (2016) in the following section. The latter branching strategy comes into play, when the current image is dominated but cannot be fathomed. Then, in the branching step new bounds on the objective function value are installed (see also Forget et al., 2020). Additionally, further improvements like image space slicing are presented (see Stidsen & Andersen, 2018 in Section 4.3).

Another algorithm for BOMILPs is presented by Adelgren and Gupte (2017), where they introduce new techniques from single objective branch-and-bound and provide an overview on fathoming rules from a different perspective. Their breadth first search algorithm is equipped with a pre-processing step, in which an upper bound set is computed in the beginning either by a weighted sum or an ε-constraint routine. Further, they introduce pre-solving to simplify the set of feasible solutions. They implement variations of well-known fathoming rules, for example, integer, ideal points, and bound set fathoming. However, they use a different methodology to execute these rules. Their branching strategy consists of a scoring method, where variables that have changing fractional values are branched first. Further improvements include probing to strengthen the bounds, image space fathoming (see Pareto branching [Stidsen et al., 2014]) and locally valid cutting planes.

4.2 | Image space approaches

Contrary to the previous category of algorithms, these approaches solve a given MOMILP in the image space and heavily rely on scalarization problems or use auxiliary single objective problems. Both of these problems can be solved by custom or commercial single objective MIP solvers. As the image space usually has a smaller dimension than the decision space, designers of such algorithms try to exploit this advantage and focus on the analysis of the structure in this space. This category is more general than the category of Section 3.6. Nevertheless, it shares its advantages and disadvantages, that is, these algorithms are illustrative and easy to understand while discarding structural properties that can be obtained in the decision space.

Boland et al. (2014b, 2015c) present the "Triangle Splitting Method" for BOMILPs. Starting with the lexicographically optimal images, a rectangle is built and every local extreme supported nondominated image is found. Then, the rectangle is split into upper rectangular triangles such that the hypotenuse is between two neighbouring images. Via an auxiliary MIP, the hypotenuse is investigated, that is, either the whole hypotenuse is nondominated, otherwise, a nondominated part and an unsupported image is found. Using this new image, the triangle is split into two rectangles and the process is repeated. The splitting direction (horizontal, vertical) is changed between iterations. Further, the authors provide enhancements and a post-processing procedure that obtains a representation of the nondominated set with a minimal number of line segments.

The "ε-Tabu Constraint Algorithm" for BOMILPs by Soylu and Yıldız (2016) starts with a lexicographically optimal image, computes the corresponding slice, and computes the nondominated set of the slice problem via dichotomic search. Then, the line segments of the slice are checked for dominance using an auxiliary problem based on ε-constraints and no-good constraints (Hooker, 2011). The latter constraints exclude all feasible solutions with particular values for the integer or binary variables using the Hamming distance (Hamming, 1950) between the integer or binary parts of two solutions. The constraint can be linearized for both binary Fischetti et al., 2005) and integer variables (Soylu & Yıldız, 2016). If a line segment is dominated (in parts) or the current line segment is the artificial line segment at each end of the slice of the nondominated set, their algorithm switches to the slice that dominates the line segment and starts exploring the new slice. The algorithm terminates, if no line segment is left unexplored.

Perini et al. (2017) generalize the "Balanced Box Method" (see Boland et al. (2015a), in Section 3.6) to mixed-integer problems, where rectangles (initialized by the lexicographically optimal images) are split into two rectangles by solving two MIPs. In case that a nondominated image y^* is found that lies on the splitting line, the nondominated line segment corresponding to y^* is computed. First, the line segment with the same integer part as y^* is found by solving a MOLP. Then, parts of this segment that are dominated are successively excluded using auxiliary MIPs with ε-constraints and a weighted sum objective. The authors prove that only polynomially many, with respect to the number of line segments, single objective problems have to be solved. Further, they provide improvements like solution harvesting and information about common integer parts.

The "One Direction Search Method" for BOMILPs[4] by Fattahi and Turkay (2018) starts with one lexicographically optimal image and explores the slice problem for the fixed binary part of this image. This is done by a variant of the dichotomic search method such that the extreme points of this MOLP are computed in a decreasing manner with respect to the objective function value of the first objective. For each line segment between two extreme points, dominated parts of the segment are identified via an auxiliary constraint problem. If a line segment is (partially) dominated by an image with different binary part, the new binary part is explored next. Otherwise, the line segment is nondominated. If all line segments of a slice are explored, the binary part is excluded via no-good constraints and a new slice is found by a lexicographic search until the other lexicographically optimal image is reached.

The algorithm by Rasmi and Türkay (2019) returns all nondominated images for biobjective problems and for problems with more than two objectives a superset containing all facets that have at least one nondominated image (plus some unnecessary facets). They use variants of Benson's method together with cone and no-good constraints (see Soylu & Yıldız, 2016, above) to find all integer solution parts that contain at least one nondominated image. Then, they fix the integer solution part and solve the corresponding continuous problem to obtain all possible nondominated images for this integer part. Via a line search and Benson's method, they detect the nondominated parts of edges and check if a facet has at least one nondominated image. In particular, this idea is applied to TOMILPs by

Rasmi et al. (2017), Rasmi et al. (2019) which is the first algorithm designed for triobjective mixed-integer linear problems. They use an iterative process starting with one lexicographically optimal solution. They fix the value of one objective function to the one of the lexicographically optimal solution and consider the resulting two dimensional plane. Here, they find all integer variables that have at least one nondominated image in the corresponding slice by iteratively solving lexicographic problems, where integer variable values and areas are excluded, in particular by no-good constraints. Non-dominance is checked via Benson's method. If there are no unexplored slices on this plane with at least one nondominated image left, a nondominated image of another slice with larger value of the previously fixed objective function is found by a lexicographic problem and all (partially) nondominated faces of the previously excluded integer variable values are computed. This is repeated until infeasibility is reached. At last, partially nondominated faces of the slices are examined by pairwise comparison of two faces of different slices.

Pettersson and Özlen (2019a) provide the first algorithm for MOMILPs with an arbitrary number of objectives. Using a multiobjective pure integer algorithm (improved version of the algorithm provided by Özlen et al. (2014), see Section 3.5, with no-good constraints (Soylu & Yıldız, 2016, above) and slices), they find integer solutions and subsequently polytopes of feasible integer parts via Benson's method that contain the integer solutions and cover the nondominated set. Then, this set of polytopes is modified such that no two polytopes have a non-empty intersection. Finally, the polytopes are compared and nondominated parts and dominated parts of the polytopes are identified. Unfortunately, this is only a short proceedings article. We are awaiting the full paper to follow and check the algorithm in detail.

4.3 | Hybrid methods

Hybrid methods are a mixture between image space and decision space algorithms and are developed to overcome drawbacks from pure decision or pure image space approaches on the one hand and exploit advantages on the other hand. The proposed algorithms of the category are image space approaches that use methods from decision space algorithms (branch-and-bound algorithms) or vice versa.

The "Search and Remove Algorithm" by Soylu (2018) is a graphical approach for BOMILPs that is equipped with bound sets. In every step, the extreme supported images of a sub-problem (the first subproblem is the original one) are computed and it is checked via bound sets whether the nondominated set of the sub-problem has potential nondominated images for the original problem. More precisely, the lower bound sets are computed by the ideal point and the extreme points of the sub-problem and its LP-relaxations. In contrast, the upper bound set is computed by local nadir points. Then, the slices corresponding to the ESN images are excluded using no-good constraints (see Soylu & Yıldız, 2016, in Section 4.2). This is repeated until the sub-problem does not have any potential nondominated images or it is infeasible. Next, the nondominated sets of all found slices are computed and the nondominated parts of their line segments are computed by dividing the image space into consecutive, disjoint subregions and finding the lower line segment in each region.

Stidsen and Andersen (2018) pursue the idea of slicing in the image space introduced by Stidsen et al. (2014). Image space slices divide the image space into distinct areas by rays shooting from the origin. They may give a possibility to apply parallelization and are a (possibly faster) alternative to lower bounds in the image space. Here, Stidsen and Andersen focus on a procedure to compute these slices and test it on the algorithm presented by Stidsen et al. (2014). In particular, they provide two procedures: If a set of images of the problem is given, for example, by a heuristic, they install the slices such that the area with possible nondominated images in these slices is reasonably small. This is done by a heuristic that solves a shortest path problem on a network with images as nodes and arcs representing the size of the rectangle spanned by the images. If no set of images is provided, they split the image space evenly and use a lexicographic problem to find the upper left most and bottom right most image in each slice.

5 | SUMMARY

This article has surveyed over 100 articles in the field of multiobjective linear optimization with integer and mixed-integer variables providing a thorough overview on exact algorithms for this problem category. It is intended to serve as both a reference for established researchers in the field of multiobjective (mixed-)integer optimization and an entry point for young researchers. In addition to the logical division into integer and mixed-integer multiobjective problems, further groups were formed in order to present solution procedures more clearly. These categories are as mutually exclusive as possible and only as similar in content as necessary.

For multiobjective integer linear problems, branch-and-bound and branch-and-cut algorithms have been one of the first solution approaches available. In the last decades, algorithms have been enhanced by more sophisticated bound sets among others, thus, this category denotes still an active area of research. Although less articles on epsilon constraint methods have been published, it is the most cited category. In particular, these methods are used for application purposes due to its simplicity and speed. Hence, they can be seen as the "working horses" for solving multiobjective linear problems. Recently, image space decomposition methods have been on the rise. These methods not only return all nondominated images but provide a better understanding of the structure of the nondominated set as well. Also, algorithms with different scalarization methods or more exotic algorithms like the ones based on algebraic programming have been published.

For multiobjective mixed-integer linear problems, literature is more scarce than for the pure integer case due to the shorter publication history. Similar to the integer case, branch-and-bound algorithms are the dominant solution methods, while algorithms that work in the image space have been published later. More recently, researchers have developed hybrid methods that exploit the advantages of both solution and image space approaches.

In the last 15 years, the interest in solving multiobjective problems has increased. This is certainly also to be explained by the simultaneous increase in computing power making the more computationally demanding multiobjective problems available for modelling and solving real-world problems. Overall, a wide variety of algorithmic approaches exists. For the future, we expect that further examining the structure of the image space and its connection to the decision space may lead to new algorithmically exploitable insights. Likewise, results in single objective integer programming can be adapted for the multiobjective case as this has been done via branch-and-bound and, lately, via the adaption of algebraic programming. Also, it may be interesting to investigate, whether and to what extent state-of-the-art techniques like machine learning and quantum computing can be adapted to obtain the whole nondominated set. We are positive that pursuing these new and also existing directions of research while providing multiobjective solvers in easy-to-use software packages will make multiobjective optimization *the* standard approach for mathematical modelling of real-world optimization problems.

ACKNOWLEDGEMENTS

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ENDNOTES

[1] https://github.com/vOptSolver

[2] https://voptsolver.github.io/vOptLib/

[3] We apologize to all authors of articles who are unintentionally not mentioned in this survey despite a careful literature search.

[4] This algorithm may work also for integer variables, but the authors do not recommend it due to the complexity of the no-good constraints.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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