An Elimination Method for Computing the Generalized Inverse of an Arbitrary Complex Matrix

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An Elimination Method for Computing the Generalized Inverse of an Arbitrary Complex Matrix*

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1. *Introduction*

E. H. Moore [12, 13] and independently Bjerhammar [4] and Penrose [14] showed that the concept of inverses can be generalized. We can render (as in [1]) an equivalent form as follows:

For any $m \times n$ matrix A over the complex field C, there exists an $n \times m$ matrix A^+ over C which is the unique solution of

$$AX = P_{R(A)} \tag{1}$$

$$XA = P_{R(X)} \tag{2}$$

where $R(A)$ is the range of A in E^m and $P_{R(A)}$ is the orthogonal projection on $R(A)$.

A^+ is called the *generalized inverse* (g.i.) of A. If A is nonsingular then $A^+ = A^{-1}$; otherwise A^+ still possesses properties[1] which make it a central concept in matrix theory[2] and in numerical analysis.[3] Methods for computing the g.i. have been given by various authors: [12, 15, 8, 9, 4, 1]. In this paper we present an elimination method for computing A^+. A GATE 20 program for this method is given in [2].

2. *An Elimination Method for Computing A^+*

We recall that [14]

$$A^*AA^+ = A^*. \tag{3}$$

Let E be a nonsingular matrix and P a permutation matrix such that

$$EA^*AP = \left(\frac{I_r \mid \Delta}{0}\right) = \left(\frac{H^*}{0}\right) \tag{4}$$

where[4] $r = \text{rank } A^*A$ and the matrices Δ, $H^* = (I_r \mid \Delta)$ are determined by E, A^*A and P.

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[1] See [14] and the reviews in [7] and [1].

[2] E.g. [14, 9, 10, 11].

[3] For examples and applications see [3, 6, 8, 15, 16].

[4] To avoid trivialities, r is assumed nonzero; i.e. $A \neq 0$.

Equation (3) is rewritten as

$$EA^*APP^*A^+ = EA^* \tag{5}$$

from which we conclude [14] that

$$P^*A^+ = (EA^*AP)^+EA^* + Z \tag{6}$$

where Z is a matrix whose columns lie in $N(EA^*AP)$, the null space of EA^*AP. We will show now that $Z = 0$:

By (4), the columns of Z lie in $N(H^*)$, the null space of H^*. The latter subspace is the orthogonal complement of $R(H) = R((EA^*AP)^*) = R(P^*A^*AE^*)$. Since E is nonsingular and $R(A^*A) = R(A^*) = R(A^+)$, e.g. [10], we verify that $R(H) = R(P^*A^+)$. On the other hand $R(H) = R((EA^*AP)^*) = R((EA^*AP)^+)$. Therefore $R(P^*A^+) = R((EA^*AP)^+) = R(H)$, and Z—whose columns lie in $N(H^*)$—must vanish by (6).

Collecting the above results,

$$P^*A^+ = (EA^*AP)^+EA^* = \left(\frac{H^*}{0}\right)^+ EA^* = (H^{*+} \mid 0)EA^*. \tag{7}$$

From (4) and (5) it follows that the last $(n - r)$ rows of EA^* are zero; from the definition of H^* it therefore follows that the matrix H^*EA^* consists of the first r rows of EA^*. Therefore

$$HH^+EA^* = H^{+*}H^*EA^* = (H^{*+} \mid 0)EA^*. \tag{8}$$

From (7), (8) and the fact that P is a permutation matrix it follows that

$$A^+ = PHH^+EA^*. \tag{9}$$

Finally, if D is an $n \times (n - r)$ matrix such that

$$N(D^*) = R(H) \tag{10}$$

then it is well known that [9]

$$HH^+ = I_n - DD^+, \tag{11}$$

and (9) becomes

$$A^+ = P(I_n - DD^+)EA^*. \tag{12}$$

Given $H^* = (I_r \mid \Delta)$, a natural choice for D is

$$D = \left(\frac{\Delta}{-I_{n-r}}\right). \tag{13}$$

An elimination method for computing the generalized inverse may be based either on equation (7) or on equation (12). Both equations reduce for nonsingular A^*A to $A^+ = EA^*$, and for nonsingular A to the well-known result

$$A^{-1} = EA^*$$

where E is defined by $EA^*A = I_n$. If the matrix A^*A is singular then the method (7) rewritten as[5]

$$A^+ = P\begin{pmatrix} I_r \\ \Delta^* \end{pmatrix}((I_r + \Delta\Delta^*)^{-1} \mid 0)EA^* \tag{7a}$$

requires the inversion of the $r \times r$ matrix $(I_r + \Delta\Delta^*)$. Similarly, if A^*A is singular, the method (12) rewritten as

$$A^+ = P\left(I - \begin{pmatrix} \Delta \\ -I_{n-r} \end{pmatrix}(I_{n-r} + \Delta^*\Delta)^{-1}(\Delta^* \mid -I_{n-r})\right)EA^* \tag{12a}$$

requires the inversion of the $(n - r) \times (n - r)$ matrix $(I_{n-r} + \Delta^*\Delta)$.

Remarks

(i) Zero rows [or columns] in A result in corresponding zero columns [or rows] in A^+. Hence an obvious reduction by working with $\tilde{A}$, a matrix obtained from A by striking all zero rows and columns; computing $\tilde{A}^+$ by either (7a) or (12a), and inserting zero columns and rows to obtain A^+.

(ii) Another possible reduction in computations and space is by working with A^*A if $m \geqq n$ (A is an $m \times n$ matrix), and with AA^* if $m < n$. The latter case results in A^{*+} which must then be transposed to obtain A^+.

(iii) For nonsingular matrices the above methods require more operations than the ordinary inversion methods, due to the formation of A^*A. Thus for the nonsingular case: $m = n = r$ both methods require $(\frac{5}{2}n^3 - 2n^2 + \frac{1}{2}n)$ multiplications, $(\frac{3}{2}n^2 - \frac{n}{2})$ divisions and $(\frac{5}{2}n^3 - 2n^2 + \frac{1}{2}n)$ additions.

(iv) Because the last $(n - r)$ rows of EA^* are zero, in method (12a) one need not compute the last $(n - r)$ columns of the matrix $(I - DD^+)$ [see example below].

(v) As in other elimination methods, the above methods depend critically on the correct determination of the rank, which in turn depends on the approximation and roundoff errors.

(vi) By the fact that $R(H) = R(P^*A^*A) = R(P^*A^*)$, equation (9) can be rewritten as $A^+ = PR_{(P^*A^*)}EA^*$ with E, P as above.

(vii) Equation (7) can be given an alternate proof by using the fact that A^+ is the unique solution of the following extremum problem:

$$\text{Minimize} \quad (\text{trace } X^*X)^{\frac{1}{2}} \qquad \text{subject to} \quad A^*AX = A^*. \tag{13}$$

This fact is an easy corollary of a theorem in [15]. Since E is nonsingular and P is orthogonal, problem (13) has the same solution as the following problem:

$$\begin{aligned} &\text{Minimize} \quad (\text{trace } (X^*PP^*X))^{\frac{1}{2}} = (\text{trace } X^*X)^{\frac{1}{2}} \quad \text{subject to} \\ &\qquad\qquad\qquad\qquad (EA^*AP)P^*X = EA^*. \end{aligned} \tag{14}$$

By [15] the unique solution of (14) is (7):

$$P^*A^+ = (EA^*AP)^+EA^*.$$

[5] Equation (7a) is due to Professor A. Charnes and one of us, e.g. *Notices Amer. Math. Soc. 1*, 1 (1963), 135.

Example

Let

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{pmatrix}.$$

Diagonalize A^*A vs A^*; pivot element circled:

$$\left(\begin{array}{rrrr|rrrrrr} 4 & -2 & -2 & -2 & -1 & -1 & 0 & 0 & 1 & 1 \\ -2 & 4 & -2 & -8 & 0 & 1 & -1 & 1 & -1 & 0 \\ -2 & -2 & 4 & 10 & 1 & 0 & 1 & -1 & 0 & -1 \\ -2 & -8 & 10 & 28 & 2 & -1 & 3 & -3 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{rrrr|rrrrrr} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 3 & -3 & -9 & -\frac{1}{2} & \frac{1}{2} & -1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -3 & 3 & 9 & \frac{1}{2} & -\frac{1}{2} & 1 & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -9 & 9 & 27 & \frac{3}{2} & -\frac{3}{2} & 3 & -3 & -\frac{3}{2} & -\frac{3}{2} \end{array}\right)$$

$$\left(\begin{array}{rr|rr|rrrrrr} 1 & 0 & -1 & -2 & -\frac{2}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{6} \\ 0 & 1 & -1 & -3 & -\frac{1}{6} & \frac{1}{6} & -\frac{2}{6} & \frac{2}{6} & -\frac{2}{6} & \frac{1}{6} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Here:

$$\Delta = \begin{pmatrix} -1 & -2 \\ -1 & -3 \end{pmatrix}$$

$$EA^* = \frac{1}{6}\begin{pmatrix} -2 & -1 & -1 & 1 & 1 & 2 \\ -1 & 1 & -2 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) Using (7a) one proceeds as follows:

$$(I_2 + \Delta\Delta^*) = \begin{pmatrix} 6 & 7 \\ 7 & 11 \end{pmatrix}$$

$$(I_2 + \Delta\Delta^*)^{-1} = \frac{1}{17}\begin{pmatrix} 11 & -7 \\ -7 & 6 \end{pmatrix}$$

$$\begin{pmatrix} I_2 \\ \hline \Delta^* \end{pmatrix} ((I_2 + \Delta\Delta^*)^{-1} \mid 0) = \frac{1}{17}\left(\begin{array}{rr|rr} 11 & -7 & 0 & 0 \\ -7 & 6 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -1 & -4 & 0 & 0 \end{array}\right)$$

$$A^+ = \frac{1}{17}\left(\begin{array}{rr|rr} 11 & -7 & 0 & 0 \\ -7 & 6 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -1 & -4 & 0 & 0 \end{array}\right) \frac{1}{6}\left(\begin{array}{rrrrrr} -2 & -1 & -1 & 1 & 1 & 2 \\ -1 & 1 & -2 & 2 & -1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$= \frac{1}{102}\begin{pmatrix} -15 & -18 & 3 & -3 & 18 & 15 \\ 8 & 13 & -5 & 5 & -13 & -8 \\ 7 & 5 & 2 & -2 & -5 & -7 \\ 6 & -3 & 9 & -9 & 3 & -6 \end{pmatrix}$$

(ii) By method (12a) the steps are:

$$D = \begin{pmatrix} \Delta^* \\ \hline -I_2 \end{pmatrix} = \left(\begin{array}{rr} -1 & -2 \\ -1 & -3 \\ \hline -1 & 0 \\ 0 & -1 \end{array}\right)$$

$$D^*D = \begin{pmatrix} 3 & 5 \\ 5 & 14 \end{pmatrix}$$

$$(D^*D)^{-1} = \frac{1}{17}\begin{pmatrix} 14 & -5 \\ -5 & 3 \end{pmatrix}$$

$$D(D^*D)^{-1}D^* = \frac{1}{17}\left(\begin{array}{rr|rr} 6 & 7 & 4 & 1 \\ 7 & 11 & -1 & 4 \\ 4 & -1 & 14 & -5 \\ 1 & 4 & -5 & 3 \end{array}\right)$$

It is actually unnecessary
to compute these columns.

$$I - D(D^*D)^{-1}D^* = \frac{1}{17}\left(\begin{array}{rr|cc} 11 & -7 & & \\ -7 & 6 & & \\ -4 & 1 & & \\ -1 & -4 & & \end{array}\right)$$

$$A^+ = \frac{1}{17}\left(\begin{array}{rr|cc} 11 & -7 & & \\ -7 & 6 & & \\ -4 & 1 & & \\ -1 & -4 & & \end{array}\right)\frac{1}{6}\left(\begin{array}{rrrrrr} -2 & -1 & -1 & 1 & 1 & 2 \\ -1 & 1 & -2 & 2 & -1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$= \frac{1}{102}\begin{pmatrix} -15 & -18 & 3 & -3 & 18 & 15 \\ 8 & 13 & -5 & 5 & -13 & -8 \\ 7 & 5 & 2 & -2 & -5 & -7 \\ 6 & -3 & 9 & -9 & 3 & -6 \end{pmatrix}$$

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REFERENCES

1. Ben-Israel, A., and Charnes, A. Contributions to the theory of generalized inverses. *J. SIAM* (1963).
2. Ben-Israel, A., and Ijiri, Y. A report on the machine calculation of the generalized inverse. Carnegie Inst. Tech., Pittsburgh, Penn., Mar. 1963.
3. Bjerhammar, A. Application of calculus of matrices to method of least squares; with special reference to geodetic calculations. *Trans. Roy. Inst. Tech. Stockholm* 49 (1951), 1–86.
4. ——. Rectangular reciprocal matrices with special reference to geodetic calculations. *Bull. Geodesique* (1951), 188–220.

5. DEN BROEDER, G. G., JR., AND CHARNES, A. Contributions to the theory of a generalized inverse for matrices. Purdue University, Lafayette, Ill., 1957). Republished as ONR Research Memo No. 39 Northwestern Univ., The Tech. Inst., Evanston, Ill., 1962.
6. CLINE, R. E. On the computation of the generalized inverse A^+, of an arbitrary matrix A, and the use of certain associated eigenvectors in solving the allocation problem. Preliminary report, Purdue Univ., Statistical and Computing Lab., Lafayette, Ill., (1958).
7. GREVILLE, T. N. E. The pseudo inverse of a rectangular or singular matrix and its applications to the solution of system of linear equations. *SIAM Rev. 1* (1959) 38–43.
8. ——. Some applications of the pseudo inverse of a matrix. *SIAM Rev. 2*, 1 (1960) 15–22.
9. HESTENES, M. R. Inversion of matrices by biorthogonalization and related results. *J. SIAM 6* (1958) 84.
10. ——. Relative Hermitian matrices. *Pacific J. Math. 11*, 1 (1961) 225–245.
11. ——. A ternary algebra with applications to matrices and linear transformations. *Arch. Rat. Mech. Anal. 11*, 2 (1962), 138–194.
12. MOORE, E. H. *Bull. Amer. Math. Soc. 26* (1920), 394–395.
13. ——. *General Analysis*, Pt I. *Memoir Amer. Philos. Soc. I* (1935).
14. PENROSE, R. A generalized inverse for matrices. *Proc. Camb. Philos. Soc. 51*, 3 (1955) 406–413.
15. ——. On best approximate solution of linear matrix equations. *Proc. Camb. Philos. Soc. 52*, 1 (1956), 17–19.
16. PYLE, L. D. A gradient projection method for solving programming problems using the generalized inverse A^+ and the eigenvectors $e(1), \cdots, e(n)$ of $I - A^+A$. Preliminary Report, Purdue Univ. Statistical and Computing Lab., Lafayette, Ind., 1958.