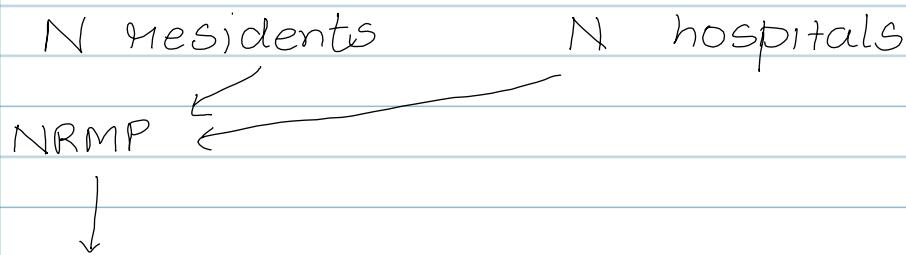


## Matching & Assignment Problems



Formulating as an algorithm

Input: → For each resident  $r_i$  a list of hospitals (a relation  $\succ_{r_i}$  on the set of hospitals)

$h_1 \succ_{r_i} h_2 \iff r_i$  prefers  $h_1$  over  $h_2$

→ For each hospital  $h_j$  a ranked list of residents

$\succ_{h_j}$

Output: Matching

What do we want from the matching?

Need people to listen (abstractly)

↳ Stability condition.

Let Matching be  $M(r_i) = h_j$

Not stable:

$r_1 \sim M(r_1)$        $r_1$  prefers  $M(r_2)$   
 $r_2 \sim M(r_2)$        $M(r_2) \sim r_1$

A pair  $r_1, M(r_2)$

s.t.  $M(r_1) <_{r_1} M(r_2)$

$r_2 <_{M(r_2)} r_1$

is called a blocking pair

A match is stable if there is no blocking pair

Deferred Acceptance Algorithm (DA) Gale-Shapley

At Repeat:

1. Each resident applies down their list.
2. If a hospital receives multiple applications  
it rejects all but its top choice.

Until: All residents are assigned or have exhausted their list

Example:

$r_1, r_2, r_3$			A	B	C
A	A	B	$r_3$	$r_2$	$r_1$
C	B	A	$r_1$	$r_3$	$r_2$
B	C	C	$r_2$	$r_1$	$r_3$

## Execution

A	B	C
$r_1 r_2$	$\cancel{r_3}$	$r_1$
$r_3$	$r_2$	

Claim : DA outputs a stable match.

Pf: We will prove there is no blocking pair  $r-h$

Observation : As the algorithm progresses, the hospital's situation only improves

By contradiction

$r-h$  is a blocking pair, then  $r$

prefers  $h$  to its ultimate outcome

$\Rightarrow r$  had  $\triangleright$  applied to  $h$

$\Rightarrow$  By Observation, this means that  $h$ 's outcome is preferred by it to  $r$

To complete correctness, notice that DA takes  $\leq n^2$  iterations

Cor:  $\exists$  stable match, DA finds it

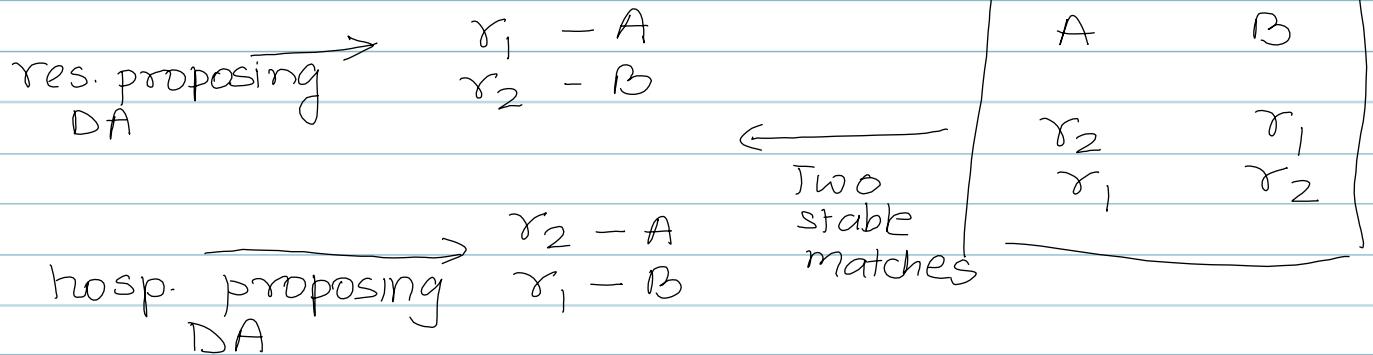
Is the stable match unique?

- 1) Any execution of DA above produces the same matches

Resident - proposing DA  
 Hospital proposing DA

Try with 2

$r_1, r_2$	$r_1, r_2$	A      B
A      A      X	A      B	$r_1$ $r_2$
B      B	B      A	$r_2$ $r_1$



## Theorem

Resident - proposing - DA results in a pointwise best outcome for residents and pointwise worst for hospitals

If  $M_{RDA}$  is the matching and  $M$  is any stable match

$$M_{RDA}(r) \geq_r M(r) \quad \forall r$$

$$M_{RDA}^{-1}(h) \leq_h M^{-1}(h) \quad \forall h$$

## Algorithm to Mechanism

↑  
runs on simmgs

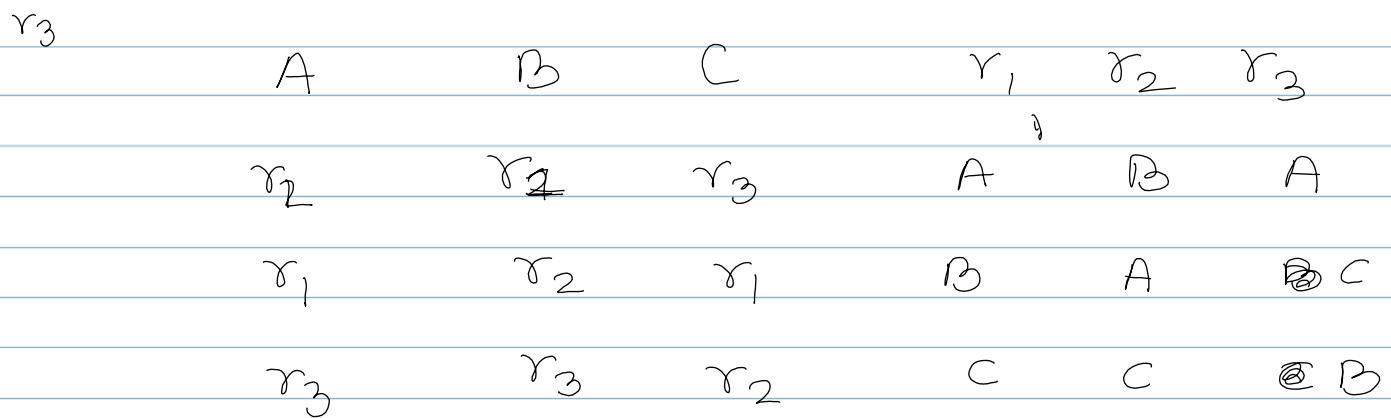
↑  
runs on inputs from people  
(will need incentives)

A mechanism is truthful if a player  
can't improve their outcome by misreporting  
preferences.

Is rDA truthful?

Yes for residents

No for hospitals

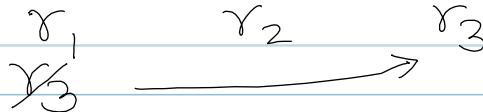


Normal

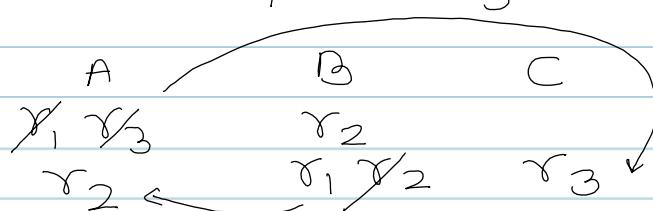
A

B

C



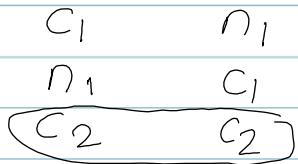
Now A switches  $r_1$  and  $r_3$



## 1970 + Couple Constraints

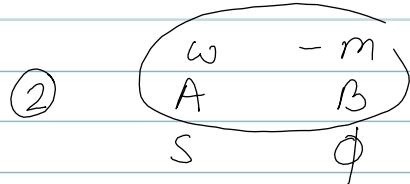
$$r_1 - r_2$$

individually

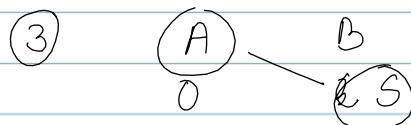


There might not be a stable match :-

$w - m$	$\frac{s}{A}$	$\frac{A}{w} \quad \frac{B}{s}$
$A - B$	$A$	$w \quad s$
$B$		$s \quad m$



No stable match !!



"Large market hypothesis"

Mechanism design without money

→ school matches : grade-schools

Schools might not have preferences.  
Only students have preferences

Lotteries

input.

1	2	3
A	B	A
B	C	B
C	A	C

output → A distribution of matches

Random Serial Dictatorship:

Pick a random permutation of player  
( $\pi$ )

For  $i = 1$  to  $n$

$\pi(i)$  picks its favourite remaining item

RSD is truthful : why?

When it's your turn, you're the dictator.

Conj<sup>??</sup>: Only truthful assignment with some desirable properties.

Example

1	2	3	4
$\frac{1}{2}$ A	$\frac{1}{2}$ A	$\frac{1}{2}$ B	$\frac{1}{2}$ B
B	B	A	A
$\frac{1}{4}$ C	$\frac{1}{4}$ C	$\frac{1}{4}$ C	$\frac{1}{4}$ C
$\frac{1}{4}$ D	$\frac{1}{4}$ D	$\frac{1}{4}$ D	$\frac{1}{4}$ D

ideal mechanism

$$\begin{array}{l} \text{RSD} \\ \text{D} \rightarrow \frac{1}{4} \\ \text{C} \rightarrow \frac{1}{4} \\ \text{B} \rightarrow \frac{1}{12} \\ \text{A} \rightarrow \frac{5}{12} \end{array}$$