1

Introduction

In this book, we understand the solution of a decision problem as to choose "good" or "best" among a set of "alternatives," where we assume the existence of certain criteria, according to which the quality of the alternatives is measured. In this introductory chapter, we shall first give some examples and distinguish different types of decision problems. Informally, we shall understand optimization problems as mathematical models of decision problems. We introduce the concepts of decision (or variable) and criterion (or objective) space and mention different notions of optimality. Relations and cones are used to formally define optimization problems, and a classification scheme is introduced.

1.1 Optimization with Multiple Criteria

Let us consider the following three examples of decision problems.

Example 1.1. We want to buy a new car and have identified four models we like: a VW Golf, an Opel Astra, a Ford Focus and a Toyota Corolla. The decision will be made according to price, petrol consumption, and power. We prefer a cheap and powerful car with low petrol consumption. In this case, we face a decision problem with four alternatives and three criteria. The characteristics of the four cars are shown in Table 1.1 (data are invented).

How do we decide, which of the four cars is the "best" alternative, when the most powerful car is also the one with the highest petrol consumption, so that we cannot buy a car that is cheap as well as powerful and fuel efficient. However, we observe that with any one of the three criteria alone the choice is easy. □

Table 1.1. Criteria and alternatives in Example 1.1.

		Alternatives			
		VW	Opel	Ford	Toyota
Criteria	Price (1,000 Euros)	16.2	14.9	14.0	15.2
	Consumption ($\frac{l}{100km}$)	7.2	7.0	7.5	8.2
	Power (kW)	66.0	62.0	55.0	71.0

Example 1.2. For the construction of a water dam an electricity provider is interested in maximizing storage capacity while at the same time minimizing water loss due to evaporation and construction cost. A decision must be made on man months used for construction as well as mean radius of the lake, and also it must respect certain constraints such as minimal strength of the dam. Here, the set of alternatives (possible dam designs) allows infinitely many different choices. The criteria are functions of the decision variables to be maximized or minimized. The criteria are clearly in conflict: A dam with big storage capacity will certainly not involve small construction cost, for instance. □

Example 1.3. As a third example, we consider a mathematical problem with two criteria and one decision variable. The criteria or objective functions, which we want to minimize simultaneously over the nonnegative real line, are

$$f_1(x) = \sqrt{x+1} \quad \text{and} \quad f_2(x) = x^2 - 4x + 5 = (x-2)^2 + 1, \tag{1.1}$$

plotted in Figure 1.1. We want to solve the optimization problem

$$\text{“}\min_{x \geq 0}\text{”}(f_1(x), f_2(x)). \tag{1.2}$$

The question is, what are the “minima” and the “minimizers” in this problem? Note that again, for each function individually the corresponding optimization problem is easy: $x_1 = 0$ and $x_2 = 2$ are the (unique) minimizers of f_1 and f_2 on $x \in \mathbb{R} : x \geq 0$, respectively. □

The first two examples allow a first distinction of decision problems. Those decision problems with a countable number of alternatives are called *discrete*, others *continuous*. In this book, we will be concerned with both continuous and discrete problems.

Comparing Examples 1.1 and 1.3, another distinguishing feature of decision problems becomes apparent: In Example 1.1 the alternatives are explicitly

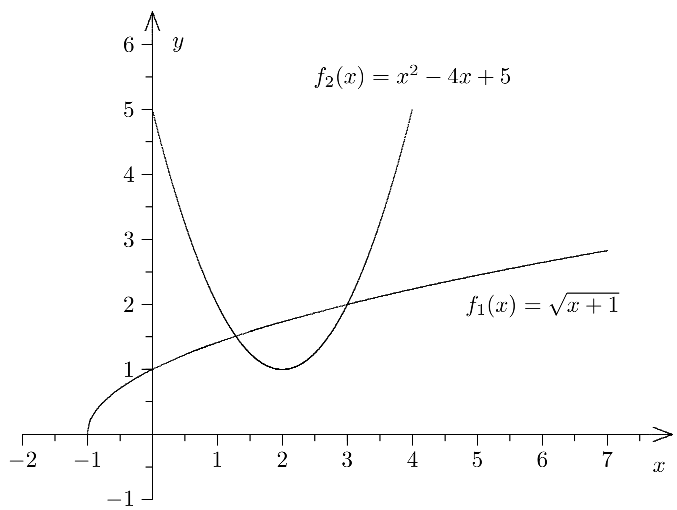

Fig. 1.1. Objective functions of Example 1.3.

given, whereas in 1.3 the alternatives are implicitly described by constraints ($x \geq 0$). Thus, we may distinguish the following types of decision problems, based on the description of the set of alternatives.

- Problems with finitely many alternatives that are explicitly known. The goal is to select a most preferred one. Multicriteria decision aid deals with such problems. We will only have one short section on such problems in this book (Section 8.2).
- Discrete problems where the set of alternatives is described by constraints in the form of mathematical functions. These problems will be covered in Chapters 8 to 10.2.
- Continuous problems. The set of alternatives is generally given through constraints. These are the objects of interest in Chapters 2.

Historically, the first reference to address such situations of conflicting objectives is usually attributed to Pareto (1896) who wrote (the quote is from the 1906 English edition of his book, emphasis added by the author):

> We will say that the members of a collectivity enjoy *maximum ophelimity* in a certain position when it is *impossible to find a way of moving from that position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases*. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity

which certain individuals enjoy, and decreasing that which others enjoy, of being agreeable to some and disagreeable to others.

Applying this concept in our examples, we see that in Example 1.1 all alternatives enjoy “maximum ophelimity,” in Example 1.3 all x in $[0, 2]$, where one of the functions is increasing, the other decreasing. In honor of Pareto, these alternatives are today often called *Pareto optimal solutions* of multiple criteria optimization problems. We will not use that notation, however, and refer to *efficient solutions* instead (see 2.1 for a formal definition). Large parts of this book are devoted to the discussion of the mathematics of efficiency.

1.2 Decision Space and Objective (Criterion) Space

In this section, we informally introduce the fundamental notions of decision (or variable) and criterion (or objective) space, in which the alternatives and their images under the objective function mappings are contained.

Let us consider Example 1.1 again, where – for the moment – we consider price and petrol consumption only for the moment. We can illustrate the criterion values in a two-dimensional coordinate system.

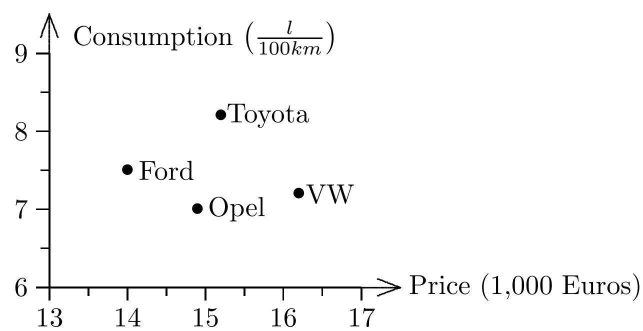

Fig. 1.2. Criterion space in Example 1.1.

From Figure 1.2 it is easy to see that Opel and Ford are the efficient choices. For both there is no alternative that is both cheaper and consumes less petrol. In addition, both Toyota and VW are more expensive and consume more petrol than Opel.

We call $\mathcal{X} = \{\text{VW, Opel, Ford, Toyota}\}$ the *feasible set*, or the set of alternatives of the decision problem. The space, of which the feasible set $\mathcal{X}$ is a subset, is called the *decision space*.

If we denote price by f_1 and petrol consumption by f_2, then the mappings $f_i : \mathcal{X} \to \mathbb{R}$ are criteria or objective functions and the optimization problem can be stated mathematically as in Example 1.3:

$$\text{“}\min_{x \in \mathcal{X}}\text{”}(f_1(x), f_2(x)). \tag{1.3}$$

The image of $\mathcal{X}$ under $f = (f_1, f_2)$ is denoted by $\mathcal{Y} := f(\mathcal{X}) := \{y \in \mathbb{R}^2 : y = f(x) \text{ for some } x \in \mathcal{X}\}$ and referred to as the image of the feasible set, or the feasible set in criterion space. The space from which the criterion values are taken is called the *criterion space*.

In Example 1.3 the feasible set is

$$\mathcal{X} = \{x \in \mathbb{R} : x \geq 0\} \tag{1.4}$$

and the objective functions are

$$f_1(x) = \sqrt{1+x} \quad \text{and} \quad f_2(x) = x^2 - 4x + 5. \tag{1.5}$$

The decision space is $\mathbb{R}$ because $\mathcal{X} \subset \mathbb{R}$. The criterion space is $\mathbb{R}^2$, as $f(\mathcal{X}) \subset \mathbb{R}^2$. To obtain the image of the feasible set in criterion space we substitute y_1 for $f_1(x)$ and y_2 for $f_2(x)$ to get $x = (y_1)^2 - 1$ (solving $y_1 = \sqrt{1+x}$ for x). Therefore we obtain $y_2 = ((y_1)^2 - 1)^2 + 4 - 4(y_1)^2 + 5 = (y_1)^4 - 6(y_1)^2 + 10$. The graph of this function (shown in Figure 1.3) is the analogue of Figure 1.2 for Example 1.1. Note that $x \geq 0$ translates to $y_1 \geq 1$, so that $\mathcal{Y} := f(\mathcal{X})$ is the part of the graph to the right of the vertical line $y_1 = 1$.

Computing the minimum of y_2 as a function of y_1, we see that the efficient solutions $x \in [0, 2]$ found before correspond to values of $y_1 = f_1(x)$ in $[1, \sqrt{3}]$ and $y_2 = f_2(x) \in [1, 5]$. These points on the graph of $y_2(y_1)$ with $1 \leq y_1 \leq \sqrt{3}$ (and $1 \leq y_2 \leq 5$) will be called *nondominated points*.

In Figure 1.4 we can see how depicting the feasible set $\mathcal{Y}$ in criterion space can help identify nondominated points and – taking inverse images – efficient solutions. The right angle attached to the efficient point $(\hat{y}_1, \hat{y}_2)$ illustrates that there is no other point $y \in f(\mathcal{X})$, $y \neq \hat{y}$ such that $y_1 \leq \hat{y}_1$ and $y_2 \leq \hat{y}_2$. This is true for the image under f of any $x \in [0, 2]$. This observation confirms the definition of nondominated points as the image of the set of efficient points under the objective function mapping.

In the examples, we have seen that we will often have many efficient solutions of a multicriteria optimization problem. Can we consider these as “optimal decisions,” in an application context such as, e.g. the dam construction problem of Example 1.2. Or, in the car selection problem, do we have to buy all four cars after all? Obviously, a final choice has to be made among efficient

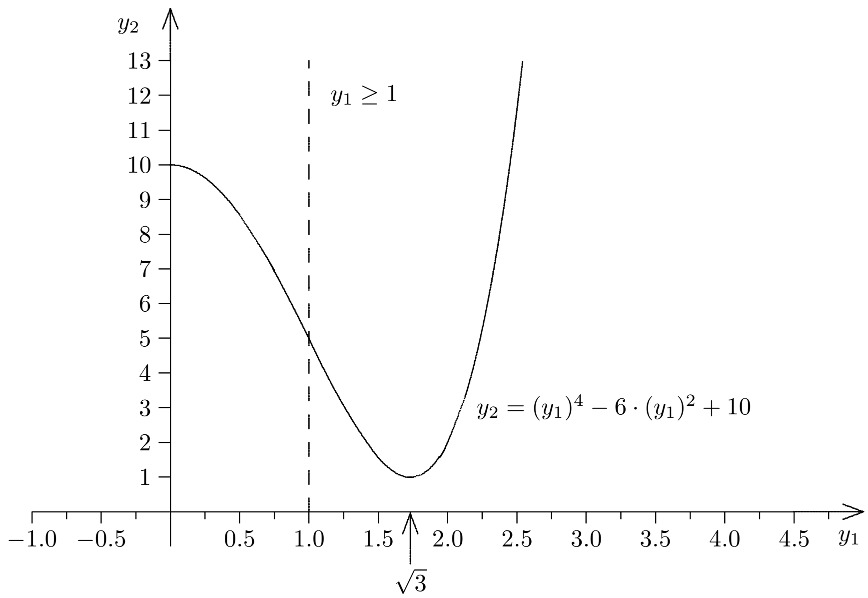

Fig. 1.3. Criterion space in Example 1.3.

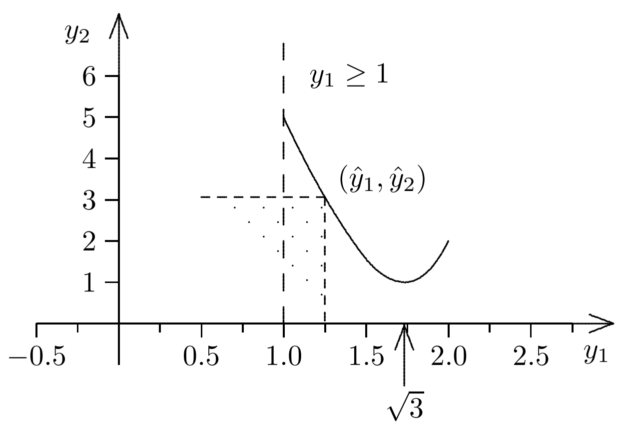

Fig. 1.4. Nondominated points in Example 1.3.

solutions. This aspect of decision making, the support of decision makers in the selection of a final solution from a set of mathematically "equally optimal" solutions, is often referred to as multicriteria decision aid (MCDA), see e.g. the textbooks of Roy (1996), Vincke (1992), or Keeney and Raiffa (1993).

Although finding efficient solutions is the most common form of multicriteria optimization, the field is not limited to that concept. There are other possibilities to cope with multiple conflicting objectives, as we shall see in the following section.

1.3 Notions of Optimality

Up to now we have written the minimization in multicriteria optimization problems in quotation marks –

$$\begin{aligned} &\text{“}\min\text{”}(f_1(x), \ldots, f_p(x)) \\ &\text{subject to } x \in \mathcal{X} \end{aligned} \tag{1.6}$$

– for good reason, since we can easily associate different interpretations with the “min.” In this and the following sections we discuss what minimization means.

The fundamental importance of efficiency (Pareto optimality) is based on the observation that any x which is not efficient cannot represent a most preferred alternative for a decision maker, because there exists at least one other feasible solution $x' \in \mathcal{X}$ such that $f_k(x') \leq f_k(x)$ for all $k = 1, \ldots, p$, where strict inequality holds at least once, i.e., x' should clearly be preferred to x. So for all definitions of optimality we deal with in this text, the relationship with efficiency will always be a topic which needs to be and will be discussed. Some other notions of optimality are informally presented now.

We can imagine situations in which there is a ranking among the objectives. In Example 1.1, price might be more important than petrol consumption, this in turn more important than power. This means that even an extremely good value for petrol consumption cannot compensate for a slightly higher price. Then the criterion vectors $(f_1(x), f_2(x), f_3(x))$ are compared lexicographically (see Table 1.2 for a definition of the lexicographic order and Section 5.1 for more on lexicographic optimization) and we would want to solve

$$\operatorname*{lexmin}_{x \in \mathcal{X}}(f_1(x), f_2(x), f_3(x)). \tag{1.7}$$

In Example 1.1 we should choose the Ford because for this ranking of objectives it is the unique optimal solution (the cheapest).

Let us assume that in Example 1.3 the objective functions measure some negative impacts of a decision (environmental pollution, etc.) to be minimized. We might not want to accept a high value of one criterion for a low value of the other. It is then appropriate to minimize the worst of both objectives. Accordingly we would solve

$$\min_{x \geq 0} \max_{i=1,2} f_i(x). \tag{1.8}$$

This problem is illustrated in Figure 1.5, where the solid line shows the maximum of f_1 and f_2. The optimal solution of the problem is obtained for $x \approx 1.285$, see Figure 1.5.

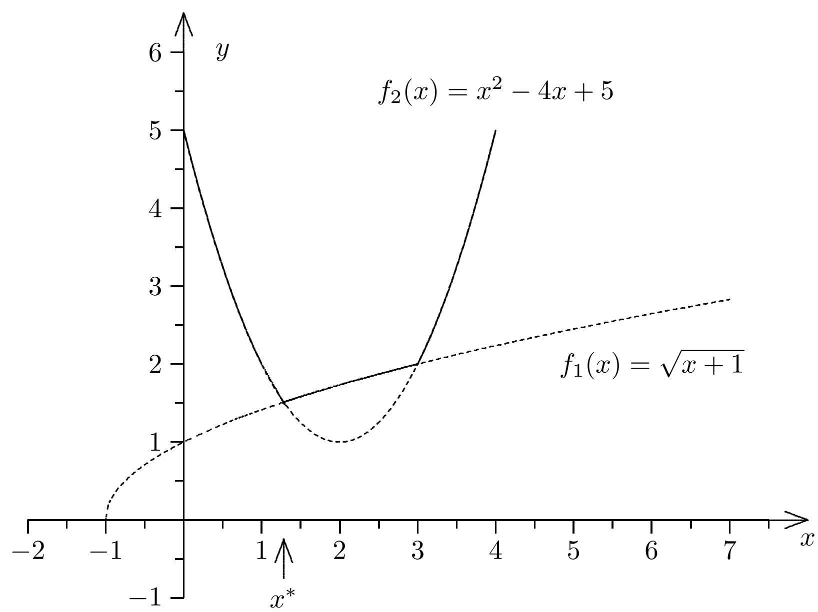

Fig. 1.5. Min-max solution of Example 1.3.

In both examples, we got unique optimal solutions, and there are no incomparable values. And indeed, in the min-max example one could think of this problem as a single objective optimization problem. However, both have to be considered as multicriteria problems, because the multiple objectives are in the formulation of the problems. Thus, in order to define the meaning of "min," we have to define how objective function vectors $(f_1(x), \ldots, f_p(x))$ have to be compared for different alternatives $x \in \mathcal{X}$. The different possibilities to do that arise from the fact that for $p \geq 2$ there is no canonical order on $\mathbb{R}^p$ as there is on $\mathbb{R}$. Therefore weaker definitions of orders have to be used.

1.4 Orders and Cones

In this section we will first introduce binary relations and some of their properties to define several classes of orders. The second main topic is cones, defining sets of nonnegative elements of $\mathbb{R}^p$. We will prove the equivalence of properties of orders and geometrical properties of cones. An indication of the relationship between orders and cones has already been shown in Figure 1.4, where we used a cone (the negative orthant of $\mathbb{R}^2$) to confirm that $\hat{y}$ is nondominated.

Let $\mathcal{S}$ be any set. A *binary relation* on $\mathcal{S}$ is a subset $\mathcal{R}$ of $\mathcal{S} \times \mathcal{S}$. We introduce some properties of binary relations.

Definition 1.4. *A binary relation $\mathcal{R}$ on $\mathcal{S}$ is called*

- reflexive *if $(s, s) \in \mathcal{R}$ for all $s \in \mathcal{S}$,*
- irreflexive *if $(s, s) \notin \mathcal{R}$ for all $s \in \mathcal{S}$,*
- symmetric *if $(s^1, s^2) \in \mathcal{R} \Longrightarrow (s^2, s^1) \in \mathcal{R}$ for all $s^1, s^2 \in \mathcal{S}$,*
- asymmetric *if $(s^1, s^2) \in \mathcal{R} \Longrightarrow (s^2, s^1) \notin \mathcal{R}$ for all $s^1, s^2 \in \mathcal{S}$,*
- antisymmetric *if $(s^1, s^2) \in \mathcal{R}$ and $(s^2, s^1) \in \mathcal{R} \Longrightarrow s^1 = s^2$ for all $s^1, s^2 \in \mathcal{S}$,*
- transitive *if $(s^1, s^2) \in \mathcal{R}$ and $(s^2, s^3) \in \mathcal{R} \Longrightarrow (s^1, s^3) \in \mathcal{R}$ for all $s^1, s^2, s^3 \in \mathcal{S}$,*
- negatively transitive *if $(s^1, s^2) \notin \mathcal{R}$ and $(s^2, s^3) \notin \mathcal{R} \Longrightarrow (s^1, s^3) \notin \mathcal{R}$ for all $s^1, s^2, s^3 \in \mathcal{S}$,*
- connected *if $(s^1, s^2) \in \mathcal{R}$ or $(s^2, s^1) \in \mathcal{R}$ for all $s^1, s^2 \in \mathcal{S}$ with $s^1 \neq s^2$,*
- strongly connected *(or* total*) if $(s^1, s^2) \in \mathcal{R}$ or $(s^2, s^1) \in \mathcal{R}$ for all $s^1, s^2 \in \mathcal{S}$.*

Definition 1.5. *A binary relation $\mathcal{R}$ on a set $\mathcal{S}$ is*

- *an* equivalence relation *if it is reflexive, symmetric, and transitive,*
- *a* preorder (quasi-order) *if it is reflexive and transitive.*

Instead of $(s^1, s^2) \in \mathcal{R}$ we shall also write $s^1 \mathcal{R} s^2$. In the case of $\mathcal{R}$ being a preorder the pair $(\mathcal{S}, \mathcal{R})$ is called a *preordered set*. In the context of (pre)orders yet another notation for the relation $\mathcal{R}$ is convenient. We shall write $s^1 \preceq s^2$ as shorthand for $(s^1, s^2) \in \mathcal{R}$ and $s^1 \not\preceq s^2$ for $(s^1, s^2) \notin \mathcal{R}$ and indiscriminately refer to the relation $\mathcal{R}$ or the relation $\preceq$. This notation can be read as "preferred to."

Given any preorder $\preceq$, two other relations are closely associated with $\preceq$. We define them as follows:

$$s^1 \prec s^2 :\Longleftrightarrow s^1 \preceq s^2 \text{ and } s^2 \not\preceq s^1, \tag{1.9}$$

$$s^1 \sim s^2 :\Longleftrightarrow s^1 \preceq s_2 \text{ and } s^2 \preceq s_1. \tag{1.10}$$

Actually, $\prec$ and $\sim$ can be seen as the strict preference and equivalence (or indifference) relation, respectively, associated with the preference defined by preorder $\preceq$.

Proposition 1.6. *Let $\preceq$ be a preorder on $\mathcal{S}$. Then relation $\prec$ defined in (1.9) is irreflexive and transitive and relation $\sim$ defined in (1.10) is an equivalence relation.*

Proof. We consider $\sim$ first. This relation is reflexive because $\preceq$ is. Furthermore $\sim$ is symmetric by definition. Now let $s^1, s^2, s^3 \in \mathcal{S}$ be such that $s^1 \sim s^2$ and $s^2 \sim s^3$. Then using transitivity of $\preceq$

$$\left.\begin{array}{l} s^1 \preceq s^2 \preceq s^3 \implies s^1 \preceq s^3 \\ s^3 \preceq s^2 \preceq s^1 \implies s^3 \preceq s^1 \end{array}\right\} \implies s^1 \sim s^3. \tag{1.11}$$

For $\prec$, note that $\prec$ is irreflexive by definition. Suppose there are $s^1, s^2, s^3 \in \mathcal{S}$ such that $s^1 \prec s^2$ and $s^2 \prec s^3$. Then $s^1 \preceq s^2 \preceq s^3$ and from transitivity of $\preceq$, $s^1 \preceq s^3$. To show that $s^1 \prec s^3$, assume $s^3 \preceq s^1$. But since $s^1 \preceq s^2$ we get $s^3 \preceq s^2$ from transitivity of $\preceq$. This contradiction implies $s^3 \not\preceq s^1$, i.e., $s^1 \prec s^3$. □

Another easily seen result concerns asymmetry and irreflexivity of binary relations.

Proposition 1.7. *An asymmetric binary relation is irreflexive. A transitive, irreflexive binary relation is asymmetric.*

Proof. The proof is left to the reader, see Exercise 1.4 □

Definition 1.8. *A binary relation $\preceq$ on $\mathcal{S}$ is*

- *a* total preorder *if it is reflexive, transitive and connected,*
- *a* total order *if it is an antisymmetric total preorder,*
- *a* strict weak order *if it is asymmetric and negatively transitive.*

From total preorders, strict weak orders can be obtained and vice versa, as Proposition 1.9 shows.

Proposition 1.9. *If $\preceq$ is a total preorder on $\mathcal{S}$, then the associated relation $\prec$ is a strict weak order. If $\prec$ is a strict weak order on $\mathcal{S}$, then $\preceq$ defined by*

$$s^1 \preceq s^2 \iff \text{either } s^1 \prec s^2 \text{ or } (s^1 \not\prec s^2 \text{ and } s^2 \not\prec s^1) \tag{1.12}$$

is a total preorder.

Proof. Let $\preceq$ be a total preorder on $\mathcal{S}$. Then $\prec$ is irreflexive and transitive by Proposition 1.6 and hence asymmetric by Proposition 1.7. For negative transitivity we show that $s^1 \not\prec s^2, s^2 \not\prec s^3$ implies $s^1 \not\prec s^3$ for all $s^1, s^2, s^3 \in \mathcal{S}$. So let $s^1, s^2, s^3 \in \mathcal{S}$ such that $s^1 \not\prec s^2$ and $s^2 \not\prec s^3$ and assume $s^1 \prec s^3$. From

$s^1 \not\prec s^2$ we have $s^2 \prec s^1$ or $s^2 \preceq s^1$ because $\preceq$ is conected. In both cases it follows that $s^2 \prec s^3$, contradicting the assumption.

Let $\prec$ be a strict weak order on $\mathcal{S}$. The relation $\preceq$ is reflexive by definition. For transitivity consider the following cases for $s^1, s^2, s^3 \in \mathcal{S}$ with $s^1 \preceq s^2$ and $s^2 \preceq s^3$:

1. $s^1 \prec s^2$, $s^2 \not\prec s^3$ and $s^3 \not\prec s^2$. Then $s^1 \prec s^3$, because otherwise $s^1 \not\prec s^3$ and $s^3 \not\prec s^2$ imply $s^1 \not\prec s^2$, a contradiction.
2. $s^1 \not\prec s^2$, $s^2 \not\prec s^1$ and $s^2 \prec s^3$. Then $s^1 \prec s^3$ because otherwise $s^1 \not\prec s^3$ and $s^2 \not\prec s^1$ imply $s^2 \not\prec s^3$, again a contradiction.
3. $s^1 \not\prec s^2$, $s^2 \not\prec s^1$, $s^2 \not\prec s^3$, $s^3 \not\prec s^2$. Then $s^1 \not\prec s^3$ and $s^3 \not\prec s^1$ (from negative transitivity) imply $s^1 \preceq s^3$.
4. $s^1 \prec s^2$ and $s^2 \prec s^3$. We get $s^2 \not\prec s^1$ from asymmetry and from $s^1 \prec s^2$. Thus, if $s^1 \not\prec s^3$, negative transitivity implies $s^2 \not\prec s^3$, a contradiction.

In all cases we can conclude $s^1 \preceq s^3$, as desired. Finally, for connectedness let $s^1, s^2 \in \mathcal{S}$, $s^1 \neq s^2$. Then $s^1 \prec s^2$ or $s^2 \prec s^1$ or ($s^1 \not\prec s^2$ and $s^2 \not\prec s^1$) and therefore $s^1 \preceq s^2$ or $s^2 \preceq s^1$. $\square$

The most important classes of relations in multicriteria optimization – partial orders and strict partial orders – are introduced now.

Definition 1.10. *A binary relation $\preceq$ is called*

- partial order *if it is reflexive, transitive and antisymmetric,*
- strict partial order *if it is asymmetric and transitive (or, equivalently, if it is irreflexive and transitive).*

Throughout this book, we use several orders on the Euclidian space $\mathbb{R}^p$ which we define now. Please note that these notations are not unique in multicriteria optimization literature and always check definitions when consulting another source. Let $y^1, y^2 \in \mathbb{R}^p$, and if $y^1 \neq y^2$ let $k^* := \min\{k : y^1_k \neq y^2_k\}$. We shall use the notations and names given in Table 1.2 for the most common ((strict) partial) orders on $\mathbb{R}^p$ appearing in this text.

With the (weak, strict) componentwise orders, we define subsets of $\mathbb{R}^p$ as follows:

- $\mathbb{R}^p_{\geqq} := \{y \in \mathbb{R}^p : y \geqq 0\}$, the nonnegative orthant of $\mathbb{R}^p$;
- $\mathbb{R}^p_{\geq} := \{y \in \mathbb{R}^p : y \geq 0\} = \mathbb{R}^p_{\geqq} \setminus \{0\}$;
- $\mathbb{R}^p_{>} := \{y \in \mathbb{R}^p : y > 0\} = \operatorname{int} \mathbb{R}^p_{\geqq}$, the positive orthant of $\mathbb{R}^p$.

Note that for $p = 1$ we have $\mathbb{R}_{\geq} = \mathbb{R}_{>}$.

We can now proceed to show how the definition of a set of nonnegative elements in $\mathbb{R}^p$ ($\mathbb{R}^2$ for purposes of illustration) can be used to derive a geometric interpretation of properties of orders. These equivalent views on orders

Table 1.2. Some orders on $\mathbb{R}^p$.

Notation	Definition	Name
$y^1 \leqq y^2$	$y^1_k \leq y^2_k \quad k = 1, \ldots, p$	weak componentwise order
$y^1 \leq y^2$	$y^1_k \leq y^2_k \quad k = 1, \ldots, p;\ y^1 \neq y^2$	componentwise order
$y^1 < y^2$	$y^1_k < y^2_k \quad k = 1, \ldots, p$	strict componentwise order
$y^1 \leq_{\text{lex}} y^2$	$y^1_{k^*} < y^2_{k^*}$ or $y^1 = y^2$	lexicographic order
$y^1 \leq_{MO} y$	$\max\limits_{k=1,\ldots,p} y^1_k \leq \max\limits_{k=1,\ldots,n} y^2_k$	max-order

will be extremely useful in multicriteria optimization. But first we need the definition of a cone.

Definition 1.11. *A subset* $\mathcal{C} \subseteq \mathbb{R}^p$ *is called a* cone, *if* $\alpha d \in \mathcal{C}$ *for all* $d \in \mathcal{C}$ *and for all* $\alpha \in \mathbb{R}, \alpha > 0$.

Example 1.12. The left drawing in Figure 1.6 shows the cone $\mathcal{C} = \{d \in \mathbb{R}^2 : d_k \geq 0, k = 1, 2\} = \mathbb{R}^2_{\geqq}$. This is the cone of nonnegative elements of the weak componentwise order. The right drawing shows a smaller cone $\mathcal{C} \subset \mathbb{R}^2_{\geqq}$.

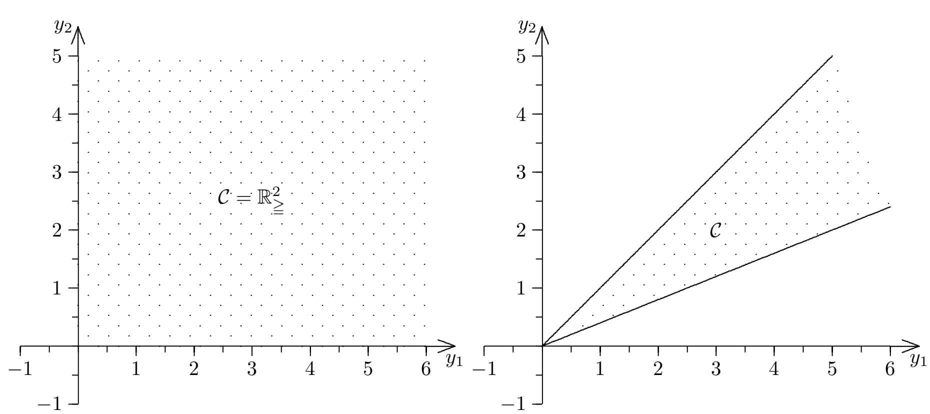

Fig. 1.6. Illustration of two cones.

□

For the following discussion it will be useful to have the operations of the multiplication of a set with a scalar and the sum of two sets. Let $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^p$ and $\alpha \in \mathbb{R}$. We denote by

$$\alpha\mathcal{S} := \{\alpha s : s \in \mathcal{S}\}, \tag{1.13}$$

especially $-\mathcal{S} = \{-s : s \in \mathcal{S}\}$. Furthermore, the (algebraic, Minkowski) sum of $\mathcal{S}_1$ and $\mathcal{S}_2$ is

$$\mathcal{S}_1 + \mathcal{S}_2 := \{s^1 + s^2 : s^1 \in \mathcal{S}_1, s^2 \in \mathcal{S}_2\}. \tag{1.14}$$

If $\mathcal{S}_1 = \{s\}$ is a singleton, we also write $s + \mathcal{S}_2$ instead of $\{s\} + \mathcal{S}_2$. Note that these are just simplified notations that do not involve any set arithmetic, e.g. $2\mathcal{S} \neq \mathcal{S} + \mathcal{S}$ in general.

This is also the appropriate place to introduce some further notation used throughout the book. For $\mathcal{S} \subseteq \mathbb{R}^n$ or $\mathcal{S} \subseteq \mathbb{R}^p$

- $\mathrm{int}(\mathcal{S})$ is the interior of $\mathcal{S}$,
- $\mathrm{ri}(\mathcal{S})$ is the relative interior of $\mathcal{S}$,
- $\mathrm{bd}(\mathcal{S})$ is the boundary of $\mathcal{S}$,
- $\mathrm{cl}(\mathcal{S}) = \mathrm{int}(\mathcal{S}) \cup \mathrm{bd}(\mathcal{S})$ is the closure of $\mathcal{S}$,
- $\mathrm{conv}(\mathcal{S})$ is the convex hull of $\mathcal{S}$.

The parentheses might be omitted for simplification of expressions when the argument is clear.

Definition 1.13. *A cone $\mathcal{C}$ in $\mathbb{R}^p$ is called*

- nontrivial *or* proper *if $\mathcal{C} \neq \emptyset$ and $\mathcal{C} \neq \mathbb{R}^n$,*
- convex *if $\alpha d^1 + (1-\alpha)d^2 \in \mathcal{C}$ for all $d^1, d^2 \in \mathcal{C}$ and for all $0 < \alpha < 1$,*
- pointed *if for $d \in \mathcal{C}, d \neq 0$, $-d \notin \mathcal{C}$, i.e., $\mathcal{C} \cap (-\mathcal{C}) \subseteq \{0\}$.*

Due to the definition of a cone, $\mathcal{C}$ is convex if for all $d^1, d^2 \in \mathcal{C}$ we have $d^1 + d^2 \in \mathcal{C}$, too: $\alpha d^1 \in \mathcal{C}$ and $(1-\alpha)d^2 \in \mathcal{C}$ because $\mathcal{C}$ is a cone. Therefore, closedness of $\mathcal{C}$ under addition is sufficient for convexity. Then, using the algebraic sum, we can say that $\mathcal{C} \subset \mathbb{R}^p$ is a convex cone if $\alpha\mathcal{C} \subseteq \mathcal{C}$ for all $\alpha > 0$ and $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$. We will only consider nontrivial cones throughout the book.

Given an order relation $\mathcal{R}$ on $\mathbb{R}^p$, we can define a set

$$\mathcal{C}_{\mathcal{R}} := \{y^2 - y^1 : y^1 \mathcal{R} y^2\}, \tag{1.15}$$

which we would like to interpret as the set of nonnegative elements of $\mathbb{R}^p$ according to $\mathcal{R}$. We will now prove some relationships between the properties of $\mathcal{R}$ and $\mathcal{C}_{\mathcal{R}}$.

Proposition 1.14. *Let $\mathcal{R}$ be compatible with scalar multiplication, i.e., for all $(y^1, y^2) \in \mathcal{R}$ and all $\alpha \in \mathbb{R}_>$ it holds that $(\alpha y^1, \alpha y^2) \in \mathcal{R}$. Then $\mathcal{C}_{\mathcal{R}}$ defined in (1.15) is a cone.*

Proof. Let $d \in \mathcal{C}_\mathcal{R}$. Then $d = y^2 - y^1$ for some $y^1, y^2 \in \mathbb{R}^p$ with $(y^1, y^2) \in \mathcal{R}$. Thus $(\alpha y^1, \alpha y^2) \in \mathcal{R}$ for all $\alpha > 0$. Hence $\alpha d = \alpha(y^2 - y^1) = \alpha y^2 - \alpha y^1 \in \mathcal{C}_\mathcal{R}$ for all $\alpha > 0$. □

Example 1.15. Let us consider the weak componentwise order on $\mathbb{R}^p$. Here $y^1 \leqq y^2$ if and only if $y^1_k \leq y^2_k$ for all $k = 1, \ldots, p$ or $y^2_k - y^1_k \geq 0$ for all $k = 1, \ldots, p$. Therefore $\mathcal{C}_\leqq = \{d \in \mathbb{R}^p : d_k \geq 0, k = 1, \ldots, p\} = \mathbb{R}^p_\geqq$. □

It is interesting to consider the definition (1.15) with $y^1 \in \mathbb{R}^p$ fixed, i.e., $\mathcal{C}_\mathcal{R}(y^1) = \{y^2 - y^1 : y^1 \mathcal{R} y^2\}$. If $\mathcal{R}$ is an order relation, $y^1 + \mathcal{C}_\mathcal{R}(y^1)$ is the set of elements of $\mathbb{R}^p$ that y^1 is preferred to or that are dominated by y^1.

A natural question to ask is: Under what conditions is $\mathcal{C}_\mathcal{R}(y)$ the same for all $y \in \mathbb{R}^p$? In order to answer that question, we need another assumption on order relation $\mathcal{R}$. $\mathcal{R}$ is said to be compatible with addition if $(y^1 + z, y^2 + z) \in \mathcal{R}$ for all $z \in \mathbb{R}^p$ and all $(y^1, y^2) \in \mathcal{R}$.

Lemma 1.16. *If $\mathcal{R}$ is compatible with addition and $d \in \mathcal{C}_\mathcal{R}$ then $0 \mathcal{R} d$.*

Proof. Let $d \in \mathcal{C}_\mathcal{R}$. Then there are $y^1, y^2 \in \mathbb{R}^p$ with $y^1 \mathcal{R} y^2$ such that $d = y^2 - y^1$. Using $z = -y^1$, compatibility with addition implies $(y^1 + z)\mathcal{R}(y^2 + z)$ or $0 \mathcal{R} d$. □

Lemma 1.16 means that if $\mathcal{R}$ is compatible with addition, the sets $\mathcal{C}_\mathcal{R}(y)$, $y \in \mathbb{R}^p$, do not depend on y. In this book, we will be mainly concerned with this case. For relations that are compatible with addition, we obtain further results.

Theorem 1.17. *Let $\mathcal{R}$ be a binary relation on $\mathbb{R}^p$ which is compatible with scalar multiplication and addition. Then the following statements hold.*

1. *$0 \in \mathcal{C}_\mathcal{R}$ if and only if $\mathcal{R}$ is reflexive.*
2. *$\mathcal{C}_\mathcal{R}$ is pointed if and only if $\mathcal{R}$ is antisymmetric.*
3. *$\mathcal{C}_\mathcal{R}$ is convex if and only if $\mathcal{R}$ is transitive.*

Proof.
1. Let $\mathcal{R}$ be reflexive and let $y \in \mathbb{R}^p$. Then $y \mathcal{R} y$ and $y - y = 0 \in \mathcal{C}_\mathcal{R}$. Let $0 \in \mathcal{C}_\mathcal{R}$. Then there is some $y \in \mathbb{R}^p$ with $y \mathcal{R} y$. Now let $y' \in \mathbb{R}^p$. Then $y' = y + z$ for some $z \in \mathbb{R}^p$. Since $y \mathcal{R} y$ and $\mathcal{R}$ is compatible with addition we get $y' \mathcal{R} y'$.
2. Let $\mathcal{R}$ be antisymmetric and let $d \in \mathcal{C}_\mathcal{R}$ such that $-d \in \mathcal{C}_\mathcal{R}$, too. Then there are $y^1, y^2 \in \mathbb{R}^p$ such that $y^1 \mathcal{R} y^2$ and $d = y^1 - y^1$ as well as $y^3, y^4 \in \mathbb{R}^p$ such that $y^3 \mathcal{R} y^4$ and $-d = y^4 - y^3$. Thus, $y^2 - y^1 = y^3 - y^4$ and there must be $y \in \mathbb{R}^p$ such that $y^2 = y^3 + y$ and $y^1 = y^4 + y$. Therefore compatibility with addition implies $y^2 \mathcal{R} y^1$. Antisymmetry of $\mathcal{R}$ now yields $y^2 = y^1$ and therefore $d = 0$, i.e., $\mathcal{C}_\mathcal{R}$ is pointed.

Let $y^1, y^2 \in \mathbb{R}^p$ with $y^1 \mathcal{R} y^2$ and $y^2 \mathcal{R} y^1$. Thus, $d = y^2 - y^1 \in \mathcal{C}_\mathcal{R}$ and $-d = y^1 - y^2 \in \mathcal{C}_\mathcal{R}$. If $\mathcal{C}_\mathcal{R}$ is pointed we know that $\{d, -d\} \subset \mathcal{C}$ implies $d = 0$ and therefore $y^1 = y^2$, i.e., $\mathcal{R}$ is antisymmetric.

3. Let $\mathcal{R}$ be transitive and let $d^1, d^2 \in \mathcal{C}_\mathcal{R}$. Since $\mathcal{R}$ is compatible with scalar multiplication, $\mathcal{C}_\mathcal{R}$ is a cone and we only need to show $d^1 + d^2 \in \mathcal{C}_\mathcal{R}$. By Lemma 1.16 we have $0 \mathcal{R} d^1$ and $0 \mathcal{R} d^2$. Compatibility with addition implies $d^1 \mathcal{R} (d^1 + d^2)$, transitivity yields $0 \mathcal{R} (d^1 + d^2)$, from which $d^1 + d^2 \in \mathcal{C}_\mathcal{R}$.
   Let $\mathcal{C}_\mathcal{R}$ be convex and let $y^1, y^2, y^3 \in \mathbb{R}^p$ be such that $y^1 \mathcal{R} y^2$ and $y^2 \mathcal{R} y^3$. Then $d^1 = y^2 - y^1 \in \mathcal{C}_\mathcal{R}$ and $d^2 = y^3 - y^2 \in \mathcal{C}_\mathcal{R}$. Because $\mathcal{C}_\mathcal{R}$ is convex, $d^1 + d^2 = y^3 - y^1 \in \mathcal{C}_\mathcal{R}$. By Lemma 1.16 we get $0 \mathcal{R} (y^3 - y^1)$ and by compatibility with addition $y^1 \mathcal{R} y^3$. □

Example 1.18. 1. The weak componentwise order $\leqq$ is compatible with addition and scalar multiplication. $\mathcal{C}_\leqq = \mathbb{R}^p_\geqq$ contains 0, is pointed, and convex.

2. The max-order $\leq_{MO}$ is compatible with scalar multiplication, but not with addition (e.g. (-3,2) $\leq_{MO}$ (3,1), but this relation is reversed when adding (0,3)). Furthermore, $\leq_{MO}$ is reflexive, transitive, but not antisymmetric (e.g. (1,0) $\leq_{MO}$ (1,1) and (1,1) $\leq_{MO}$ (1,0)).

□

We have defined cone $\mathcal{C}_\mathcal{R}$ given relation $\mathcal{R}$. We can also use a cone to define an order relation. Let $\mathcal{C}$ be a cone. Define $\mathcal{R}_\mathcal{C}$ by

$$y^1 \mathcal{R}_\mathcal{C} y^2 \iff y^2 - y^1 \in \mathcal{C}. \tag{1.16}$$

Proposition 1.19. *Let $\mathcal{C}$ be a cone. Then $\mathcal{R}_\mathcal{C}$ defined in (1.16) is compatible with scalar multiplication and addition in $\mathbb{R}^p$.*

Proof. Let $y^1, y^2 \in \mathbb{R}^p$ be such that $y^1 \mathcal{R}_\mathcal{C} y^2$. Then $d = y^2 - y^1 \in \mathcal{C}$. Because $\mathcal{C}$ is a cone $\alpha d = \alpha(y^2 - y^1) = \alpha y^2 - \alpha y^1 \in \mathcal{C}$. Thus $\alpha y^1 \mathcal{R}_\mathcal{C} \alpha y^2$ for all $\alpha > 0$. Furthermore, $(y^2 + z) - (y^1 + z) \in \mathcal{C}$ and $(y^1 + z) \mathcal{R}_\mathcal{C} (y^2 + z)$ for all $z \in \mathbb{R}^p$. □

Theorem 1.20. *Let $\mathcal{C}$ be a cone and let $\mathcal{R}_\mathcal{C}$ be as defined in (1.16). Then the following statements hold.*

1. *$\mathcal{R}_\mathcal{C}$ is reflexive if and only if $0 \in \mathcal{C}$.*
2. *$\mathcal{R}_\mathcal{C}$ is antisymmetric if and only if $\mathcal{C}$ is pointed.*
3. *$\mathcal{R}_\mathcal{C}$ is transitive if and only if $\mathcal{C}$ is convex.*

Proof. 1. Let $0 \in \mathcal{C}$ and $y \in \mathbb{R}^p$. Thus, $y - y \in \mathcal{C}$ and $y \mathcal{R}_\mathcal{C} y$ for all $y \in \mathbb{R}^p$.
   Let $\mathcal{R}_\mathcal{C}$ be reflexive. Then we have $y \mathcal{R}_\mathcal{C} y$ for all $y \in \mathbb{R}^p$, i.e., $y - y = 0 \in \mathcal{C}$.

2. Let $d \in \mathcal{C}$ and $-d \in \mathcal{C}$. Thus $0\mathcal{R}_{\mathcal{C}}d$ and $0\mathcal{R}_{\mathcal{C}} - d$. Adding d to the latter relation, compatibility with addition yields $d\mathcal{R}_{\mathcal{C}}0$. Then asymmetry implies $d = 0$.
   Let $y^1, y^2 \in \mathbb{R}^p$ be such that $y^1\mathcal{R}_{\mathcal{C}}y^2$ and $y^2\mathcal{R}_{\mathcal{C}}y^1$. Thus, $d = y^2 - y^1$ and $-d = y^1 - y^2 \in \mathcal{C}$. Since $\mathcal{C}$ is pointed, $d = 0$, i.e. $y^1 = y^2$.
3. Let $y^1, y^2, y^3 \in \mathbb{R}^p$ such that $y^1\mathcal{R}_{\mathcal{C}}y^2$ and $y^2\mathcal{R}_{\mathcal{C}}y^3$. Therefore $d^1 = y^2 - y^1 \in \mathcal{C}$ and $d^2 = y^3 - y^2 \in \mathcal{C}$. Because $\mathcal{C}$ is convex, $d^1 + d^2 = y^3 - y^1 \in \mathcal{C}$ and $y^1\mathcal{R}_{\mathcal{C}}y^3$.
   If $d^1, d^2 \in \mathcal{C}$ we have $0\mathcal{R}_{\mathcal{C}}d^1$ and $0\mathcal{R}_{\mathcal{C}}d^2$. Because $\mathcal{R}_{\mathcal{C}}$ is compatible with addition, we get $d^1\mathcal{R}_{\mathcal{C}}(d^1 + d^2)$. By transitivity $0\mathcal{R}_{\mathcal{C}}(d^1 + d^2)$ and $d^1 + d^2 \in \mathcal{C}$. □

Note that Theorem 1.20 does not need the assumption of compatibility with addition since it is a consequence of the definition of $\mathcal{R}_{\mathcal{C}}$. The relationships between cones and binary relations are further investigated in the exercises.

With Theorems 1.17 and 1.20 we have shown equivalence of some partial orders and pointed convex cones containing 0. Since (partial) orders can be used to define "minimization," these results make it possible to analyze multicriteria optimization problems geometrically.

1.5 Classification of Multicriteria Optimization Problems

By the choice of an order $\preceq$ on $\mathbb{R}^p$, we can finally define the meaning of "min" in the problem formulation

$$\text{“}\min_{x \in \mathcal{X}}\text{”} f(x) = \text{“}\min_{x \in \mathcal{X}}\text{”} (f_1(x), \ldots, f_p(x)). \tag{1.17}$$

The different interpretations of "min" pertaining to different orders are the foundation of a classification of multicriteria optimization problems. We only briefly mention it here. A more detailed development can be found in Ehrgott (1997) and Ehrgott (1998).

With the multiple objective functions we can evaluate objective value vectors $(f_1(x), \ldots, f_p(x))$. However, we have seen that these vectors $y = f(x)$, $x \in \mathcal{X}$, are not always compared in objective space, i.e., $\mathbb{R}^p$, directly.

In Example 1.3 we have formulated the optimization problem

$$\min_{x \in \mathcal{X}} \max_{i=1,2} f_i(x). \tag{1.18}$$

That is, we have used a mapping $\theta : \mathbb{R}^2 \to \mathbb{R}$ from objective space $\mathbb{R}^2$ to $\mathbb{R}$, where the min in (1.18) is actually defined by the canonical order on $\mathbb{R}$.

In general, the objective function vectors are mapped from $\mathbb{R}^p$ to an ordered space, e.g. $(\mathbb{R}^P, \preceq)$, where comparisons are made using the order relation $\preceq$. This mapping is called the *model map.*

With the model map, we can now summarize the elements of a multicriteria optimization problem (MOP). These are

- the feasible set $\mathcal{X}$,
- the objective function vector $f = (f_1, \ldots, f_p) : \mathcal{X} \longrightarrow \mathbb{R}^p$,
- the objective space $\mathbb{R}^p$,
- the ordered set $(\mathbb{R}^P, \preceq)$,
- the model map θ.

Feasible set, objective function vector f, and objective space are the *data* of the MOP. The model map provides the link between objective space and ordered set, in which, finally, the meaning of the minimization is defined. Thus with the three main aspects data, model map, and ordered set the classification $(\mathcal{X}, f, \mathbb{R}^p)/\theta/(\mathbb{R}^P, \preceq)$ completely describes a multicriteria optimization problem.

Example 1.21. Let us look at a problem of finding efficient solutions,

$$\min_{x \geq 0}(\sqrt{x+1},\ x^2 - 4x + 1). \tag{1.19}$$

Here $\mathcal{X} = \{x : x \geq 0\} = \mathbb{R}_{\geqq}$ is the feasible set, $f = (f_1, f_2) = (\sqrt{x+1}, x^2 - 4x + 1)$ is the objective function vector, and $\mathbb{R}^p = \mathbb{R}^2$ is the objective space defining the data. Because we compare objective function vectors componentwise, the model map is given by $\theta(y) = y$ and denoted id, the *identity mapping*, henceforth. The ordered set is then $(\mathbb{R}^P, \preceq) = (\mathbb{R}^2, \leq)$. The problem (1.19) is classified as

$$(\mathbb{R}_{\geqq}, f, \mathbb{R}^2)/\mathrm{id}/(\mathbb{R}^2, \leq). \tag{1.20}$$

□

Example 1.22. If we have a ranking of objectives as described in the second example in Section 1.3, we compare objective vectors lexicographically. Let $y^1, y^2 \in \mathbb{R}^p$. Then $y^1 \leq_{\text{lex}} y^2$ if there is some $k^*, 1 \leq k^* \leq p$ such that $y^1_k = y^2_k \quad k = 1, \ldots, k^* - 1$ and $y^1_{k^*} < y^2_{k^*}$ or $y^1 = y^2$. In the car selection Example 1.1, $\mathcal{X} = \{\text{VW, Opel, Ford, Toyota}\}$ is the set of alternatives (feasible set), f_1 is price, f_2 is petrol consumption, and f_3 is power. We define $\theta(y) = (y_1, y_2, -y_3)$ (note that more power is preferred to less). The problem is then classified as

$$(\mathcal{X}, f, \mathbb{R}^3)/\theta/(\mathbb{R}^3, \leq_{\text{lex}}) \tag{1.21}$$

□

Lexicographic optimality is one of the concepts we cover in Chapter 5.

At the end of this chapter, we formally define optimal solutions and optimal values of multicriteria optimization problems.

Definition 1.23. *A feasible solution* $x^* \in \mathcal{X}$ *is called an* optimal solution *of a multicriteria optimization problem* $(\mathcal{X}, f, \mathbb{R}^p)/\theta/(\mathbb{R}^P, \preceq)$ *if there is no* $x \in \mathcal{X}$, $x \neq x^*$ *such that*

$$\theta(f(x)) \preceq \theta(f(x^*)). \tag{1.22}$$

For an optimal solution $x^*, \theta(f(x^*))$ *is called an* optimal value *of the MOP. The* set of optimal solutions *is denoted by* $\mathrm{Opt}((\mathcal{X}, f, \mathbb{R}^p)/\theta/(\mathbb{R}^P, \preceq))$. *The* set of optimal values *is* $\mathrm{Val}((\mathcal{X}, f, \mathbb{R}^p)/\theta/(\mathbb{R}^P, \preceq))$.

Some comments on this definition are necessary. First, since we are often dealing with orders which are not total, a positive definition of optimality, like $\theta(f(x^*)) \preceq \theta(f(x))$ for all $x \in \mathcal{X}$, is not possible in general. Second, for specific choices of θ and $(\mathbb{R}^P, \preceq)$, specific names for optimal solutions and values are commonly used, such as efficient solutions or lexicographically optimal solutions.

In the following chapters we will introduce shorthand notations for optimal sets, usually $\mathcal{X}$ with an index identifying the problem class, such as $\mathcal{X}_E := \{x \in \mathcal{X} : \text{ there is no } x' \in \mathcal{X} \text{ with } f(x') \leq f(x)\}$ for the set of efficient solutions.

We now check the definition 1.23 with Examples 1.21 and 1.22.

Example 1.24. With the problem $(\mathbb{R}_{\geqq}, f, \mathbb{R}^2)/\mathrm{id}/(\mathbb{R}^2, \leq)$ the optimality definition reads: There is no $x \in \mathcal{X}, x \neq x^*$, such that $f(x) \leq f(x^*)$, i.e., $f_k(x) \leq f_k(x^*)$ for all $k = 1, \ldots, p$, and $f(x) \neq f(x^*)$. This is indeed efficiency as we know it. □

Example 1.25. For $(\mathcal{X}, f, \mathbb{R}^3)/\theta/(\mathbb{R}^3, \leq_{\mathrm{lex}})$ with $\theta(y) = (y_1, y_2, -y_3)$, $x^* \in \mathcal{X}$ is an optimal solution if there is no $x \in \mathcal{X}, x \neq x^*$, such that

$$(f_1(x), f_2(x), -f_3(x)) \leq_{\mathrm{lex}} (f_1(x^*), f_2(x^*), -f_3(x^*)). \tag{1.23}$$

□

Quite often, we will discuss multicriteria optimization problems in the sense of efficiency or lexicographic optimality in general, not referring to specific problem data, and derive results which are independent of problem data. For this purpose it is convenient to introduce classes of multicriteria optimization problems.

Definition 1.26. *A* multicriteria optimization class (MCO class) *is the set of all MOPs with the same model map and ordered set and is denoted by*

$$\bullet/\theta/(\mathbb{R}^P, \preceq). \tag{1.24}$$

For instance, $\bullet/\mathrm{id}/(\mathbb{R}^p, \leq)$ will denote the class of all MOPs, where optimality is understood in the sense of efficiency.

1.6 Notes

Roy (1990) portrays multicriteria decision making and multicriteria decision aid as complementary fundamental attitudes for addressing decision making problems. Multicriteria decision making includes areas such as multiattribute utility theory (Keeney and Raiffa, 1993) and multicriteria optimization (Ehrgott and Gandibleux, 2002b). Multicriteria decision aid, on the other hand, includes research on the elicitation of preferences from decision makers, structuring the decision process, and other more "subjective" aspects. The reader is referred to Figueira *et al.* (2005) for a collection of up-to-date surveys on both multicriteria decision making and aid.

Yu (1974) calls $\{\mathcal{C}_{\mathcal{R}}(y^1) : y^1 \in \mathcal{Y}\}$ a *structure of domination*. Results on structures of domination can also be found in Sawaragi *et al.* (1985). If $\mathcal{C}_{\mathcal{R}}(y^1)$ is independent of y^1, the domination structure is called constant. A cone therefore implies a constant domination structure.

In terms of the relationships between orders and cones, Noghin (1997) performs a similar analysis to Theorems 1.17 and 1.20. He calls a relation $\mathcal{R}$ a cone order, if there *exists* a cone $\mathcal{C}$ such that $y^1 \mathcal{R} y^2$ if and only if $y^2 - y^1 \in \mathcal{C}$. He proves that $\mathcal{R}$ is irreflexive, transitive, compatible with addition and scalar multiplication if and only if $\mathcal{R}$ is a cone relation with a pointed convex cone $\mathcal{C}$ not containing 0.

Exercises

1.1. Consider the problem

$$\text{“}\min\text{”}(f_1(x), f_2(x)) \text{ subject to } x \in [-1, 1],$$

where

$$f_1(x) = \sqrt{5 - x^2}, \quad f_2(x) = \frac{x}{2}.$$

Illustrate the problem in decision and objective space and determine the the nondominated set $\mathcal{Y}_N := \{y \in \mathcal{Y} : \text{there is no } y' \in \mathcal{Y} \text{ with } y' \leq y\}$ and the efficient set $\mathcal{X}_E := \{x \in \mathcal{X} : f(x) \in \mathcal{Y}_N\}$.

1.2. Consider the following binary relations on $\mathbb{R}^p$ (see Table 1.2):

$$\begin{aligned} y^1 \leqq y^2 &\iff y^1_k \leq y^2_k \; k = 1, \ldots, p; \\ y^1 \leq y^2 &\iff y^1 \leqq y^2 \text{ and } y^1 \neq y^2; \\ y^1 < y^2 &\iff y^1_k < y^2_k \; k = 1, \ldots, p. \end{aligned}$$

Which of the properties listed in Definition 1.4 do these relations have?

1.3. Solve the problem of Exercise 1.1 as max-ordering and lexicographic problems:

$$\begin{aligned} \min_{x \in [-1,1]} \; & \max_{i=1,2} f_i(x), \\ \operatorname{lexmin}_{x \in [-1,1]} \; & \left(f_1(x), f_2(x)\right), \\ \operatorname{lexmin}_{x \in [-1,1]} \; & \left(f_2(x), f_1(x)\right). \end{aligned}$$

Compare the optimal solutions with efficient solutions. What do you observe?

1.4. Prove the following statements.

1. An asymmetric relation is irreflexive.
2. A transitive and irreflexive relation is asymmetric.
3. A negatively transitive and asymmetric relation is transitive.
4. A transitive and connected relation is negatively transitive.

1.5. This exercise is about cones and orders.

1. Determine the cones related to the (strict and weak) componentwise order and the lexicographic order on $\mathbb{R}^2$.
2. Find and illustrate $\mathcal{C}_{\leq_{MO}}(y)$ for $y = 0, y = (2, 1)$ and $y = (-1, 3)$.
3. Give an example of a non-convex cone $\mathcal{C}$ and list the properties of the related order $\mathcal{R}_{\mathcal{C}}$.

1.6. A cone $\mathcal{C}$ is called acute, if there exists an open halfspace $H_a = \{x \in \mathbb{R}^p : \langle x, a \rangle > 0\}$ such that $\mathrm{cl}(\mathcal{C}) \subset H_a \cup \{0\}$. Is a pointed cone always acute? What about a convex cone?

1.7. Consider the order relations $\leqq, \leq, <, \leq_{lex}$, and $\leq_{MO}$ on $\mathbb{R}^p$ and determine their relationships, i.e., statements of the form

$$y^1 \mathcal{R}_a y^2 \Longrightarrow y^1 \mathcal{R}_b y^2,$$

where $\mathcal{R}_a, \mathcal{R}_b \in \{\leqq, \leq, <, \leq_{lex}, \leq_{MO}\}$. What do these statements imply for the related cones $\mathcal{C}_{\mathcal{R}}$?

1.8. Let $|| \, || : \mathbb{R}^p \longrightarrow \mathbb{R}_{\geqq}$ be a norm. Define $y^1 \leq_{|| \, ||} y^2 \Longleftrightarrow ||y^1|| \leq ||y^2||$. Is $\leq_{|| \, ||}$ a partial order? Is it connected? Determine $\mathcal{C}_{\leq_{|| \, ||}}$ for some norm $|| \, ||$ of your choice.

1.9. A cone $\mathcal{C}$ in some vector space $\mathcal{V}$ is called *generating* if $\mathcal{V} = \mathcal{C} - \mathcal{C}$ (loosely speaking, every $v \in \mathcal{V}$ can be written as the difference of two nonnegative elements).

Consider $\mathcal{V} = C[0,1]$, the vector space of all continuous functions $f : [0,1] \longrightarrow \mathbb{R}$. Show that

$$\mathcal{C} := \{f \in C[0,1] : f(x) \geq 0 \text{ for all } x \in [0,1]\}$$

is a cone that defines a partial order $\mathcal{R}_{\mathcal{C}}$, and that $C[0,1] = \mathcal{C} - \mathcal{C}$, i.e., for all $f \in C[0,1]$ there are $f^1, f^2 \in \mathcal{C}$ such that $f = f^1 - f^2$. Can you give an example of a cone $\mathcal{C} \subset \mathbb{R}^p$ with $\mathcal{C} - \mathcal{C} \neq \mathbb{R}^p$ and find a relationship between the cone property "generating" and a property of the order $\mathcal{R}_{\mathcal{C}}$?

1.10. In this exercise, the relationships between cones and relations are further developed.

1. Let $\mathcal{R}$ be a relation. Define $\mathcal{C}_{\mathcal{R}}$ as in (1.15). Define $\mathcal{R}_{\mathcal{C}_{\mathcal{R}}}$ as in (1.16) with $\mathcal{C} = \mathcal{C}_{\mathcal{R}}$. Under what conditions is $\mathcal{R}_{\mathcal{C}_{\mathcal{R}}} = \mathcal{R}$, i.e., $y^1 \mathcal{R}_{\mathcal{C}_{\mathcal{R}}} y^2 \Longleftrightarrow y^1 \mathcal{R} y^2$?
2. Let $\mathcal{C}$ be a cone. Define $\mathcal{R}_{\mathcal{C}}$ as in (1.16). Define $\mathcal{C}_{\mathcal{R}_{\mathcal{C}}}$ as in (1.15) with $\mathcal{R} = \mathcal{R}_{\mathcal{C}}$. Is $\mathcal{C}_{\mathcal{R}_{\mathcal{C}}} = \mathcal{C}$ always, i.e., $d \in \mathcal{C}_{\mathcal{R}_{\mathcal{C}}} \Longleftrightarrow d \in \mathcal{C}$?

1.11. Generalize the definition of $\mathcal{R}_{\mathcal{C}}$ for the case where $\mathcal{C}$ is an arbitrary set. Derive relationships between properties of $\mathcal{C}$ and $\mathcal{R}_{\mathcal{C}}$.

2

Efficiency and Nondominance

This chapter covers the fundamental concepts of efficiency and nondominance. We first present some fundamental properties of nondominated points and several existence results for nondominated points and efficient solutions in Section 2.1. Section 2.2 introduces ideal and nadir points as bounds on the set of nondominated solutions. Then we briefly review weakly and strictly efficient solutions in Section 2.3. The same section also includes a geometric characterization of the three optimality concepts, with some extensions for the case of weakly efficient solutions. Finally, in Section 2.4 we introduce several definitions of properly efficient solutions, important subsets of efficient solutions from a computational point of view and in applications, and investigate their relationships.

Most of the material in this chapter can be found in the two books Göpfert and Nehse (1990) and Sawaragi *et al.* (1985), where the results are presented in more generality. We will also refer to the original publications for the main results.

2.1 Efficient Solutions and Nondominated Points

In this chapter we consider multicriteria optimization problems of the class $\bullet/\mathrm{id}/(\mathbb{R}^p, \leq)$:

$$\begin{aligned} \min\ &(f_1(x), \ldots, f_p(x)) \\ &\text{subject to } x \in \mathcal{X}. \end{aligned} \tag{2.1}$$

The image of the feasible set $\mathcal{X}$ under the objective function mapping f is denoted as $\mathcal{Y} := f(\mathcal{X})$. Let us formally repeat the definition of efficient solutions and nondominated points. Definition 2.1 also introduces the notion of dominance.