# Notes on Multi-criteria Optimization

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## Orders and Cones

**Definition**: *equivalence relation*

A binary relation on a set is an *equivalence relation* if it is reflexive, symmetric, and transitive.

**Definition**: *preorder*

A binary relation on a set is a *preorder* (*quasi-order*) if it is reflexive and transitive.

Given any preorder two other relations are closely associated with .

**Definition**: *relation*

(1.9)

**Definition**: relation

(1.10)

**Proposition 1.6**: Let be a preorder on . Then the relation defined in (1.9) is irreflexive and transitive and relation defined in (1.10) is an equivalence relation.

**Proposition 1.7**: An asymmetric binary relation is irreflexive.

Notation:

is the *interior* of

is the *relative interior* of

is the *boundary* of

is the *closure* of

is the *convex hull* of

**Definition** *affine set*

The set is affine if

**Definition** *convex set*

The set is convex if

**Definition** *affine hull* or *affine span*

Affine hull of a set is the smallest affine set which contains . Equivalently, it is the intersection of all affine sets containing .

**Definition** *convex hull*

*Informal definition*: The convex hull of a shape is the smallest convex set that contains the shape.

The convex hull can be defined as the intersection of all convex sets containing any given subset of Euclidean space, or equivalently as the set of all convex combinations of points in the subset. For a bounded subset of the plane, the convex hull may be visualized as the shape enclosed by a rubber band stretched around the subset.

1st definition of convex hull:

The convex hull of a given set is the (unique) minimal convex set containing .

2nd definition of convex hull:

The convex hull of a given set is the intersection of all convex sets containing .

3rd definition of convex hull:

A blue triangle with red outline

Description automatically generated

Figure 1: the convex hull of the red set is the blue and red convex set.

A diagram of a hexagon with black dots and arrows

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Figure 2: Convex hull of a bounded planar set: rubber band analogy

**Definition 1.11**: A subset is called a *cone*, if

**Definition 1.13**: A cone in is called

* *nontrivial* or *proper* if ,
* *convex* if for all
* *pointed* if for ,

## Appendix

### Affine Space

A diagram of a triangle with arrows

Description automatically generated

Figure: the origins from Alice’s and Bob’s perspectives. Vector computation from Alice’s perspective is in red, whereas that from Bob’s is in blue.

Informal Definition of Affine Space:

Affine space is what is left from a vector space after one has forgotten which point is the origin. Imagine that Alice knows that certain point is the actual origin, but Bob believes that another point – call it – is the origin. Two vectors, and , are to be added. Bob draws an arrow from point to point and another arrow from point to point , and completes the parallelogram to find what Bob thinks is , but Alice knows that he actually has computed . Similarly, Alice and Bob may evaluate any linear combination of a and b or of any finite set of vectors and will, generally, get different answers. However, if the sum of the coefficients in a linear combination is 1, then Alice and Bob will arrive at the same answer.

If Alice travels to then Bob can similarly travel to

Under this condition, for all coefficients , Alice and Bob describe the same point with the same linear combination, despite using different origins. While only Alice knows the “linear structure”, both Alice and Bob know the “affine structure” – i.e., the values of affine combinations, defined as linear combinations in which the sum of the coefficients is 1. A set with an affine structure is an affine space.