# Notes on Multi-criteria Optimization

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## Orders and Cones

**Definition**: *equivalence relation*

A binary relation on a set is an *equivalence relation* if it is reflexive, symmetric, and transitive.

**Definition**: *preorder*

A binary relation on a set is a *preorder* (*quasi-order*) if it is reflexive and transitive.

Given any preorder two other relations are closely associated with .

**Definition**: *relation*

(1.9)

**Definition**: relation

(1.10)

**Proposition 1.6**: Let be a preorder on . Then the relation defined in (1.9) is irreflexive and transitive and relation defined in (1.10) is an equivalence relation.

**Proposition 1.7**: An asymmetric binary relation is irreflexive.

Notation:

is the *interior* of

is the *relative interior* of

is the *boundary* of

is the *closure* of

is the *convex hull* of

**Definition** *affine set*

The set is affine if

**Definition** *convex set*

The set is convex if

**Definition** *affine hull* or *affine span*

Affine hull of a set is the smallest affine set which contains . Equivalently, it is the intersection of all affine sets containing .

**Definition** *conical combination (weighted sum)*

Given the vectors in real vector space , a conical combination of those is the following element of :

where are non-negative numbers. The conical sum defines a cone in .

**Definition** *conical hull*

The set of all conical combinations for a given set is called the conical hull of and denoted . That is,

By taking , it follows the zero vector (origin) belongs to all conical hulls.

It can be easily shown that the conical hull of a set is a convex set. In fact, it is the intersection of all convex cones containing plus the origin.

Note: If is a compact set (in particular, when it is finite non-empty set of points), then the condition “plus the origin” is unnecessary.

If we discard the origin, we can divide all coefficients by their sum to see that a conical combination is a convex combination scaled by a positive factor.

A graph of a function

Description automatically generated

Figure: in the 2D plane, the conical hull of a circle passing through the origin is the open half-plane defined by the tangent line to the circle at the origin plus the origin.

**Definition** *convex hull*

*Informal definition*: The convex hull of a shape is the smallest convex set that contains the shape.

The convex hull can be defined as the intersection of all convex sets containing any given subset of Euclidean space, or equivalently as the set of all convex combinations of points in the subset. For a bounded subset of the plane, the convex hull may be visualized as the shape enclosed by a rubber band stretched around the subset.

1st definition of convex hull:

The convex hull of a given set is the (unique) minimal convex set containing .

2nd definition of convex hull:

The convex hull of a given set is the intersection of all convex sets containing .

3rd definition of convex hull:

The set of all convex combinations (all coefficients sum up to one and are non-negative) of points in

4th definition of convex hull:

The union of all simplices with vertices in

**Note** on the practical visualization of convex hull

For bounded sets in the Euclidean plane, not all on one line, the boundary of the convex hull is the simple closed curve with minimum perimeter containing . Physical analogy which is often employed for visualizing the boundary is stretching rubber band so that it surrounds the entire set and then releasing it, allowing it to contract; when it becomes taut, itencloses the convex hull of .

However, this definition **does not necessarily generalize to higher dimensions**. For a finite set of points in 3D space, a neighborhood of a spanning tree of the points encloses them with arbitrarily small surface area, smaller than the surface of a convex hull. However, in higher dimensions, variants of [the obstacle problem](https://en.wikipedia.org/wiki/Obstacle_problem) of finding a minimum energy surface above a given shape can have a convex hull as their solution.

A blue triangle with red outline

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Figure 1: the convex hull of the red set is the blue and red convex set.

A diagram of a hexagon with black dots and arrows

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Figure 2: Convex hull of a bounded planar set: rubber band analogy

Equivalence of the definitions of convex hull

The first definition states that there should exist unique minimal convex set containing , for every . It is not obvious why this should be true.

Let us look into the second definition where the intersection of all convex sets containing is well defined – it is a subset of every other convex set that contains , because is included among the sets being intersected. Thus, it is the unique minimal convex set containing . Therefore, the two definitions are equivalent.

Each convex set containing must contain all convex combinations of points in , so the set of all convex combinations is contained in the intersection of all convex sets containing . Conversely, the set of all convex combinations is itself a convex set containing , so it also contains the intersection of all convex sets containing X, and therefore the second and third definition are equivalent.

According to the [Caratheodory’s Theorem](https://en.wikipedia.org/wiki/Carath%C3%A9odory%27s_theorem_(convex_hull)), if is a subset of a -dimensional Euclidean space, every convex combination of finitely many points from is also a convex combination of at most points in . The set of convex combinations of a -tuple of points is a simplex. Therefore, every convex combination of points of X belongs to a simplex whose vertices belong to X, and the third and fourth definitions are equivalent.

Note on a difference with respect to compactness between convex hull and conical hull

The convex hull of a compact set is also a compact set. This is not true for conical hull. For starters, the conical hull is unbounded. Moreover, it is not necessarily a closed set, here is a counterexample: sphere passing through the origin, with the conical hull being an open half-space plus the origin.

However, if is a non-empty convex compact set which does not contain the origin, then the convex conical hull of is closed set.

**Definition 1.11**: A subset is called a *cone*, if

**Definition 1.13**: A cone in is called

* *nontrivial* or *proper* if ,
* *convex* if for all
* *pointed* if for ,

**Theorem**: *closedness under addition is sufficient for convexity*

A cone is convex if for all we have . In such case the following would also be true: and since is a cone.

**Lemma**: the set is a convex cone if for all and

**Definition**: Set of non-negative elements on by

Given the order relation the set defined as

(1.15)

will be interpreted as “*the set of non-negative elements according to* ”

**Proposition 1.14**: Let be compatible with the scalar multiplication i.e., for all and all it holds that . Then defined in (1.15) is a cone.

*Proof*:

Let . Then for some with . Thus for all . Hence for all .

**Example 1.15**:

Let us consider a weak component-wise order on . Here *iff* for all or for all . Therefore, .

It is interesting to consider the definition (1.15) with fixed, i.e., . If is an order relation, is the set of elements of that is preferred to or that are dominated by .

A natural question to ask is: Under what conditions is the same for all ? In order to answer this question, we need another assumption on order relation .

**Definition**: *binary relation compatible with addition*

is said to be compatible with addition if for all and all .

**Lemma 1.16**: If is compatible with addition and then 0.

*Proof*:

Let . Then there are with such that . Using , compatibility with addition implies or equivalently .

**Lemma 1.16** means that if is compatible with addition then sets do not depend on .

**Theorem 1.17**: Let be a binary relation on which is compatible with scalar multiplication and addition. Then the following statements hold:

1. *iff* is reflexive
2. is pointed *iff* is antisymmetric
3. is convex *iff* is transitive

*Proof*:

*Proof of 2*: Let be reflexive and let such that too. Then there are such that and as well as such that and . Thus, and there must be such that and . Therefore, compatibility with addition implies . Antisymmetry of now yields and therefore , i.e., is pointed.

Proof of 3:

## Appendix

### Affine Space

A diagram of a triangle with arrows

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Figure: the origins from Alice’s and Bob’s perspectives. Vector computation from Alice’s perspective is in red, whereas that from Bob’s is in blue.

Informal Definition of Affine Space:

Affine space is what is left from a vector space after one has forgotten which point is the origin. Imagine that Alice knows that certain point is the actual origin, but Bob believes that another point – call it – is the origin. Two vectors, and , are to be added. Bob draws an arrow from point to point and another arrow from point to point , and completes the parallelogram to find what Bob thinks is , but Alice knows that he actually has computed . Similarly, Alice and Bob may evaluate any linear combination of a and b or of any finite set of vectors and will, generally, get different answers. However, if the sum of the coefficients in a linear combination is 1, then Alice and Bob will arrive at the same answer.

If Alice travels to then Bob can similarly travel to

Under this condition, for all coefficients , Alice and Bob describe the same point with the same linear combination, despite using different origins. While only Alice knows the “linear structure”, both Alice and Bob know the “affine structure” – i.e., the values of affine combinations, defined as linear combinations in which the sum of the coefficients is 1. A set with an affine structure is an affine space.

Definition affine space

//TODO: finish Affine space discussion

### Caratheodory’s Theorem

If a point lies in the convex hull of a set , then x can be written as the convex combination of at most *extremal* points in , as non-extremal points can be removed from without changing the membership of in the convex hull.

**Conical Combination Theorem** (*equivalent to Caratheodory’s Theorem which is for convex hulls*)

If a point lies in the conical hull of a set , then x can be written as a conical combination of at most points.

**Example**:

Caratheodory’s theorem in 2D

We can construct a triangle consisting of points from that encloses any point in the convex hull of .

Let . The convex set of this set is a square. Let in the convex hull of . We can then construct a set , the convex hull of which is a triangle and encloses .

**A blue and pink square with black dots

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Figure: An illustration of Caratheodory’s Theorem for a square in

Proof of Caratheodory’s Theorem:

Note: We will use the fact that is an ordered field, that is a field on which it can be imposed total order. Thus, the theorem can be applied to any field , together with total order.

**Theorem** (*Caratheodory’s Theorem*)

If then

1. is the non-negative sum of at most points of .
2. is the convex sum of at most points of .

We will prove the Caratheodory’s Theorem in the finite case. This reduction to the finite case is possible because is the set of *finite* convex combination of elements of .

**Lemma**:

If then and at most of them are nonzero.

Proof of the Lemma:

When the proof is trivial. If we can prove it for all then by induction, we have proved it for all . Thus,