# Notes on Revised Simplex Method for MOLP by J.P. Evans and R.E. Steuer, 1973

Summary:

Revised Simplex Algorithm is presented for the enumeration of the set of extreme efficient points of Multi-objective Linear Programming (MOLP) Problem.

## Introduction

Let and be and matrices, respectively, and let . Define .

**Definition 1.1** *efficient point*

A point is an *efficient point* of iff

Define the set . We want to construct linear multi-objective program to determine for a given , and .

**Lemma 1.2** *characterization of efficient points*

Let and let be a diagonal matrix with

Then iff the system

has no solution for .

## Appendix A: Farkas Lemma

Farkas Lemma:

Let be matrix and . Then exactly one of the following two assertions is true:

1. and .
2. and

To provide a sketch of proof for this important Lemma we need to define few Theorems on *separating hyperplanes*.

**Theorem 1**: Let and be two convex sets in that do not intersect i.e., . Then, there exists , such that for all and for all .

Sketch of proof is supplied on Figure 1:

Figure 1: illustration for proving Theorem 1 on separating hyperplanes.

*Note*: neither equality in the formulation of Theorem 1 can be made strict.

Consider the example shown on Figure 2: The set denotes the right half plane. The ray denoting the positive part of the vertical axis belongs to ; the ray denoting the negative part does not. is the complement of . The two sets A and A are convex and do not intersect. The bisecting inequalities of Theorem 1 hold with and . However, it is easy to show that do not exist such that and .

Figure 2: Illustration on the impossibility of strict inequalities in Theorem 1

In the case of Figure 1 the sets and are *strictly separated*. This means that and .

Strict separation is not always possible even when both and are closed. This is shown on Figure 3 below.

Figure 3: Closed convex sets cannot be strictly separated.

We will prove a special case of Theorem 1 which will be utilized later in furnishing a proof for Farkas Lemma. We will prove strict separation in this special case.

**Theorem 2**: Let and be two closed convex sets in with at least one of them bounded and assume . Then

Sketch of a proof:

Define

The infimum is achieved and is positive. Let and be the points that achieve it. Let

Our separating hyperplane will be a function . We claim that:

Figure 4: Illustration for the sketch of a proof for Theorem 2

In the above construct is chosen to be because

.

We will show that . The proof for the other half will be similar.

Let us assume that the opposite is true – that is, with .

Then . (1)

Define . We claim that is a descent direction for at . Indeed,

where the first equality is obtained as

,

the first inequality is obtained from (1) and the second inequality is implied by the fact that .

Hence,

i.e.

. But this contradicts that was the closest point to . QED.

**Corollary 1**:

Let be a closed convex set and a point not in . Then and can strictly be separated by a hyperplane.

Figure 5: illustration for Corollary 1

**Theorem 4**. *Farkas Lemma*

Let and . Then exactly one of the following sets must be empty:

*Note 1*:

Systems (i) and (ii) are called *strong alternatives*, meaning that exactly one can be feasible. *Weak alternatives* is when at most once of the alternatives can be feasible.

*Note 2*:

This theorem is particularly useful for proving infeasibility of an LP via an explicit and easily verifiable certificate. If somebody gives you a as in (ii), then you are convinced immediately that (i) is infeasible.

Geometric Interpretation of the Farkas Lemma

The geometric interpretation of the Farkas lemma illustrates the connection to the separating hyperplane theorem and makes the proof straightforward.

**Definition 1** (*Cone*)

A set is a cone if .

**Definition 2** (*Conic hull*)

Given a set , the conic hull of , denoted by , is the set of all conic combinations of the points in , i.e.,

Figure 6: Illustration of conic hull

The geometric interpretation of Farkas lemma is then given with the following.

Figure 7: Geometric interpretation of the Farkas lemma

# Bibliography

J.P. Evans, R. S. (1973). A Revised Simplex Method for Linear Multiple Objective Programs. *Mathematical Programming*(5), 54-73.

Stephen Boyd, L. V. (2004). *Convex Optimization.* Cambridge, UK: Cambridge University Press.

Links to Literature

(Stephen Boyd, 2004): [link](https://nike.box.com/s/ziin2snikjadkq5s132jz7lc29doncmx)

(J.P. Evans, 1973): [link](https://nike.box.com/s/b0d6lqcbxv7xc98k5p9scs6kqya0jwjn)