

# Lecture-24: DTMC: Irreducibility and Aperiodicity

## 1 Communicating classes

For states  $x, y \in \mathcal{X}$ , it is said that state  $y$  is **accessible** from state  $x$  if  $p_{xy}^{(n)} > 0$  for some  $n \in \mathbb{Z}_+$ , and denoted by  $x \rightarrow y$ . If two states  $x, y \in \mathcal{X}$  are accessible to each other, they are said to **communicate** with each other, denoted by  $x \leftrightarrow y$ .

1. A set of states that communicate are called a **communicating class**.
2. A communicating class  $\mathcal{C}$  is called **closed** if no edges leave this class. That is, for all  $x \in \mathcal{C}$  and  $y \notin \mathcal{C}$ , we have  $p_{xy} = 0$ .
3. An **open** communicating class is not closed.

**Proposition 1.1.** *Communication is an equivalence relation.*

*Proof.* Relation on state space  $\mathcal{X}$  is a subset of product of sets  $\mathcal{X} \times \mathcal{X}$ . Communication is a relation on state space  $\mathcal{X}$ , as it relates two states  $x, y \in \mathcal{X}$ . To show equivalence, we have to show reflexivity, symmetry, and transitivity of the relation.

Symmetry: Further, if  $x \leftrightarrow y$ , then we know that  $x \rightarrow y$  and  $y \rightarrow x$  and hence  $y \leftrightarrow x$ . Hence, the symmetry of the relation follows.

Transitivity: For transitivity, suppose  $x \leftrightarrow y$  and  $y \leftrightarrow z$ . Let  $m, n \in \mathbb{Z}_+$  such that  $p_{xy}^{(m)} > 0$  and  $p_{yz}^{(n)} > 0$ . Then by Chapman Kolmogorov equation, we have

$$p_{xz}^{(m+n)} = \sum_{w \in \mathcal{X}} p_{xw}^{(m)} p_{wz}^{(n)} \geq p_{xy}^{(m)} p_{yz}^{(n)} > 0.$$

This implies  $x \rightarrow z$ , and using similar arguments one can show that  $z \rightarrow x$ , and the transitivity follows.

Reflexivity: If this relation has a single element, then it is obvious. If not, then for  $x \leftrightarrow y$ , we have Since  $p_{xy}^{(n)} > 0$  and  $p_{yx}^{(m)} > 0$  for some  $m, n \in \mathbb{Z}_+$ . Therefore,  $p_{xx}^{(n+m)} \geq p_{xy}^{(n)} p_{yx}^{(m)} > 0$  and hence we have  $x \leftrightarrow x$ , implying the reflexivity of the relation.

□

Hence the communication relation partitions state space  $\mathcal{X}$  into equivalence classes. Each equivalence class is called a **communicating class**. A property of states is said to be a **class property** if for each communicating class  $\mathcal{C}$ , either all states in  $\mathcal{C}$  have the property, or none do.

### 1.1 Irreducibility and periodicity

A Markov chain with a single class is called an **irreducible** Markov chain. That is, for any two states  $x, y \in \mathcal{X}$ , there exists an integer  $n \in \mathbb{N}$  such that  $p_{xy}^{(n)} > 0$ . That is, any state  $y$  can be reached from any state  $x$  using transitions of positive probability.

Let  $\mathcal{T}(x) \triangleq \{n \in \mathbb{N} : p_{xx}^{(n)} > 0\}$  be the set of times when the chain can possibly return to the initial state  $x$ . The **period** of any state  $x \in \mathcal{X}$  is defined as

$$d(x) \triangleq \gcd \mathcal{T}(x) = \gcd\{n \in \mathbb{N} : p_{xx}^{(n)} > 0\}.$$

We define  $d(x) = \infty$ , if  $p_{xx}^{(n)} = 0$  for all  $n \in \mathbb{N}$ . A state  $x \in \mathcal{X}$  is called **aperiodic** if the period  $d(x)$  is 1.

**Proposition 1.2.** If  $x \leftrightarrow y$ , then  $d(x) = d(y)$ . That is, periodicity is a class property.

*Proof.* Let  $m, n \in \mathbb{N}$  be such that  $p_{xy}^{(m)} p_{yx}^{(n)} > 0$ . Suppose  $s \in \mathcal{T}(x)$ , that is  $p_{xx}^{(s)} > 0$ . Then

$$p_{yy}^{(n+m)} \geq p_{yx}^{(n)} p_{xy}^{(m)} > 0, \quad p_{yy}^{(n+s+m)} \geq p_{yx}^{(n)} p_{xx}^{(s)} p_{xy}^{(m)} > 0.$$

Hence  $d(y)|n+m$  and  $d(y)|n+s+m$ , and hence  $d(y)|s$  for any  $s \in \mathcal{T}(x)$ . In particular, it implies that  $d(y)|d(x)$ . By symmetrical arguments, we get  $d(x)|d(y)$ . Hence  $d(x) = d(y)$ .  $\square$

For an irreducible chain, the period of the chain is defined to be the period which is common to all states. An irreducible Markov chain is called **aperiodic** if the single communicating class is aperiodic.

**Proposition 1.3.** If the transition matrix  $P$  is aperiodic and irreducible, then there is an integer  $r_0$  such that  $p_{xy}^{(r)} > 0$  for all  $x, y \in \mathcal{X}$  and  $r \geq r_0$ .

## 1.2 Transient and recurrent states

**Proposition 1.4.** Transience and recurrence are class properties.

*Proof.* Let us start with proving recurrence is a class property. Let  $x$  be a recurrent state and let  $x \leftrightarrow y$ . Hence there exist some  $m, n > 0$ , such that  $p_{xy}^{(m)} > 0$  and  $p_{yx}^{(n)} > 0$ . As a consequence of the recurrence,  $\sum_{s \in \mathbb{N}} p_{xx}^{(s)} = \infty$ . It follows that  $y$  is recurrent by observing

$$\sum_{s \in \mathbb{N}} p_{yy}^{(m+n+s)} \geq \sum_{s \in \mathbb{N}} p_{yx}^{(n)} p_{xx}^{(s)} p_{xy}^{(m)} = \infty.$$

Now, if  $x$  were transient instead, we conclude that  $y$  is also transient by the following observation

$$\sum_{s \in \mathbb{N}} p_{yy}^{(s)} \leq \frac{\sum_{s \in \mathbb{N}} p_{xx}^{(m+n+s)}}{p_{yx}^{(n)} p_{xy}^{(m)}} < \infty.$$

$\square$

**Corollary 1.5.** If  $y$  is recurrent, then for any state  $x$  such that  $y \rightarrow x$ , then  $x \rightarrow y$  and  $f_{xy} = 1$ .

*Proof.* Since  $y \rightarrow x$ , there exists an integer  $n \in \mathbb{Z}_+$  such that probability of hitting state  $x$  starting from state  $y$  in  $n$ -steps without revisiting state  $y$  is positive. That is,

$$a_{yx}^{(n)} \triangleq P_y \{X_n = x, X_{n-1} \neq y, \dots, X_1 \neq y\} = P_y \{H_y > n, X_n = x\} > 0.$$

Suppose  $f_{xy} < 1$ , then we have

$$1 - f_{yy} = P_y \{H_y = \infty\} \geq P_y \{H_y = \infty, X_n = x\} = P_x \{H_y = \infty\} P_y \{H_y > n, X_n = x\} = a_{yx}^{(n)} (1 - f_{xy}) > 0.$$

This is a contradiction since state  $y$  is recurrent. This implies that  $f_{xy} = 1$  and hence  $x \rightarrow y$ .  $\square$

**Corollary 1.6.** Let  $x, y \in \mathcal{X}$  be in the same communicating class and the state  $y$  is recurrent. Then,  $\lim_{n \in \mathbb{N}} \frac{\sum_{k=1}^n p_{xy}^{(k)}}{n} = \frac{1}{\mu_{yy}}$ . Furthermore, if the state  $y$  is aperiodic, then  $\lim_{n \in \mathbb{N}} p_{xy}^{(n)} = \frac{1}{\mu_{yy}}$ .

*Proof.* Since  $y$  is recurrent and  $y \rightarrow x$ , it follow that  $f_{xy} = 1$  from the previous Lemma. From the Theorem 1.7 in previous lecture, it follows that  $\lim_{n \in \mathbb{N}} \frac{\sum_{k=1}^n p_{xy}^{(k)}}{n} = \frac{1}{\mu_{yy}}$ .  $\square$

**Theorem 1.7.** The states in a communicating class are of one of the following types; all transient, or all null recurrent, or all positive recurrent.

*Proof.* It suffices to show that if  $x, y$  belong to the same communicating class and  $y$  is null recurrent, then  $x$  is null recurrent as well. We take  $r, s \in \mathbb{N}$ , such that  $p_{yx}^{(r)} p_{xy}^{(s)} > 0$ . It follows that  $p_{yy}^{r+\ell+s} \geq p_{yx}^{(r)} p_{xx}^{(\ell)} p_{xy}^{(s)}$  for all  $\ell \in \mathbb{N}$ . Hence, for any  $n > r + s$ , we have

$$\frac{1}{n} \sum_{k=1}^n p_{yy}^{(k)} \geq \frac{1}{n} \sum_{k=r+s+1}^n p_{yy}^{(k)} \geq \left( \frac{n-r-s}{n} \right) \left( \frac{1}{n-r-s} \sum_{\ell=1}^{n-r-s} p_{xx}^{(\ell)} \right) p_{yx}^{(r)} p_{xy}^{(s)}.$$

Since  $y$  is null recurrent LHS goes to zero as  $n$  increases, which implies  $\lim_{n \in \mathbb{N}} \frac{1}{n} \sum_{\ell=1}^n p_{xx}^{(\ell)} = 0$ . Hence,  $x$  is null recurrent as well.  $\square$