Lecture-22: DTMC: Strong Markov Property

1 Introduction

We are interested in generalizing the Markov property to any random times. For a DTMC, let T be an integer random variable, and we are interested in knowing whether for any historical event $H_{T-1} = \cap_{n=0}^{T-1} \{X_n = x_n\}$ and any state $x, y \in \mathcal{X}$, we have

$$P(\{X_{T+1} = y\} \mid H_{T-1} \cap \{X_T = x\}) = p_{xy}.$$

Example 1.1 (Two-state DTMC). For the two state Markov chain $X \in \{0,1\}^{\mathbb{N}}$. Let $T \in \mathbb{N}$ be an integer random variable defined as

$$T \triangleq \sup \{n \in \mathbb{N} : X_i = 0, \forall i \leqslant n\}.$$

That is, $\{T = n\} = \{X_1 = 0, \dots, X_n = 0, X_{n+1} = 1\}$. Hence, $P(\{X_{T+1} = 1\} \mid H_{T-1} \cap \{X_n = 0\})$ is either zero or one, and not equal to q.

1.1 Stopping Time

For a random sequence $X = (X_n \in \mathcal{X} : n \in \mathbb{N})$, a random variable $T \in \mathbb{N}$ is called a **stopping time** with respect to this random sequence X if $P\{T < \infty\} = 1$ and

$$\{T = n\} \in \sigma(X_1, \dots, X_n), \text{ for all } n \in \mathbb{N}.$$

That is, given the history of the process until time n, we can tell whether the stopping time is n or not. In particular, $P(\{T = n\} \mid \sigma(X_1, \dots, X_n))$ is either one or zero.

Example 1.2 (Simple random walk). Let X be an integer random walk starting at origin and with i.i.d. step-size sequence Z taking values in $\{-1, 1\}$ with probability of positive step being $P\{Z_n = 1\} = \frac{1}{2}$. We define a random variable

$$T \triangleq \inf\{n \in \mathbb{N} : X_n = 0\}.$$

We first examine the event $\{T = n\}$, and write

$$\{T = n\} = \{X_1 \neq 0, X_2 \neq 0, \dots, X_{n-1} \neq 0, X_n = 0\} \in \sigma(X_0, \dots, X_n).$$

Then, we compute the following probability

$$P\{T < \infty\} = \sum_{n \in \mathbb{N}} P\{T = 2n\} = \sum_{n \in \mathbb{N}} \left(\frac{1}{2}\right)^{2n} C_n = 1.$$

Example 1.3 (Two-state DTMC). Let $X \in \{0,1\}^{\mathbb{N}}$ be a two state DTMC with the transition matrix P such that $p_{11} = 1 - p$ and $p_{00} = 1 - q$ for $p, q \in [0,1]$. Then, we can define a random variable

$$T \triangleq \inf\{n \in \mathbb{N} : X_n = 1\}.$$

We first examine the event $\{T = n\}$, and write

$$\{T = n\} = \{X_1 = 0, X_2 = 0, \dots, X_{n-1} = 0, X_n = 1\} \in \sigma(X_0, \dots, X_n).$$

Then, we compute the following probability

$$P_1\{T < \infty\} = \sum_{n \in \mathbb{N}} P_1\{T = n\} = 1 - p + \sum_{n \geqslant 2} p(1-q)^{n-2} q = 1.$$

Similarly, we can compute that $P_0\{T < \infty\} = 1$.

Lemma 1.4 (Wald's Lemma). *Consider a random walk $(X_n : n \in \mathbb{N})$ with i.i.d. step-sizes $(Z_n : n \in \mathbb{N})$ having finite $\mathbb{E}|Z_1|$. Let T be a finite mean stopping time with respect to this random walk. Then,*

$$\mathbb{E}[X_T] = \mathbb{E}[Z_1]\,\mathbb{E}[T].$$

Proof. From the independence of step sizes, it follows that Z_n is independent of $\sigma(X_0, X_1, \dots, X_{n-1})$. Since T is a stopping time with respect to random walk X, we observe that $\{T \geqslant n\} = \{T > n-1\} \in \sigma(X_0, X_1, \dots, X_{n-1})$, and hence $\mathbb{E}[Z_n 1_{\{T \geqslant n\}}] = \mathbb{E}Z_n \mathbb{E}1_{\{T \geqslant n\}}$. Therefore,

$$\mathbb{E}\sum_{n=1}^{T} Z_n = \mathbb{E}\sum_{n \in \mathbb{N}} Z_n 1_{\{T \geqslant n\}} = \sum_{n \in \mathbb{N}} \mathbb{E}Z_n \mathbb{E}\left[1_{\{T \geqslant n\}}\right] = \mathbb{E}Z_1 \mathbb{E}\left[\sum_{n \in \mathbb{N}} 1_{\{T \geqslant n\}}\right] = \mathbb{E}[Z_1]\mathbb{E}[T].$$

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it since the summand is positive and we apply monotone convergence theorem. $\square$

Corollary 1.5. *Consider the stopping time $T_i = \min\{n \in \mathbb{N} : X_n = i\}$ for an integer random walk X with xxd steps Z. Then, the mean of stopping time $\mathbb{E}T_i = i/\mathbb{E}Z_1$.*

1.2 Strong Markov property (SMP)

Let T be an integer valued stopping time with respect to a random sequence X. Then for all states $x, y \in \mathcal{X}$ and the event $H_{T-1} = \cap_{n=0}^{T-1}\{X_n = x_n\}$, the process X satisfies the **strong Markov property** if

$$P(\{X_{T+1} = y\} \mid \{X_T = x\} \cap H_{T-1}) = P(\{X_{T+1} = y\} \mid \{X_T = x\}).$$

Lemma 1.6. *Markov chains satisfy the strong Markov property.*

Proof. Let $X \in \mathcal{X}^{\mathbb{Z}_+}$ be a DTMC with transition matrix P. We take any historical event $H_{T-1} = \cap_{n=0}^{T-1}\{X_n = x_n\}$, and $x, y \in \mathcal{X}$. Then, we have

$$\begin{aligned} P(\{X_{T+1} = y\} \mid H_{T-1} \cap \{X_T = x\}) &= \frac{\sum_{n \in \mathbb{Z}_+} P(\{X_{T+1} = y, X_T = x\} \cap H_{T-1} \cap \{T = n\})}{P(\{X_T = x\} \cap H_{T-1})} \\ &= \sum_{n \in \mathbb{Z}_+} P(\{X_{n+1} = y\} \mid \{X_n = x\} \cap H_{n-1} \cap \{T = n\}) P(\{T = n\} \mid \{X_T = x\} \cap H_{T-1}) \\ &= p_{xy} \sum_{n \in \mathbb{Z}_+} P(\{T = n\} \mid \{X_T = x\} \cap H_{T-1}) = p_{xy}. \end{aligned}$$

This equality follows from the fact that $\{T = n\}$ is completely determined by $\{X_0, \dots, X_n\}$ $\square$

Remark 1. We have already seen an example where SMP doesn't hold.

i. As an exercise, if we try to use the Markov property on arbitrary random variable T, the SMP may not hold. For example, define a non-stopping time T for $y \in \mathcal{X}$

$$T = \inf\{n \in \mathbb{Z}_+ : X_{n+1} = y\}.$$

In this case, we have

$$P(\{X_{T+1} = y\} \mid \{X_T = x, \dots, X_0 = x_0\}) = 1_{\{p_{xy} > 0\}} \neq P(\{X_1 = y\} \mid \{X_0 = x\}) = p_{xy}.$$

ii. A useful application of the strong Markov property is as follows. Let $x_0 \in \mathcal{X}$ be a fixed state and $\tau_0 = 0$. Let τ_n denote the stopping times at which the Markov chain visits x_0 for the nth time. That is,

$$\tau_n \triangleq \inf\{n > \tau_{n-1} : X_n = x_0\}.$$

Then $(X_{\tau_n + m} : m \in \mathbb{Z}_+)$ is a stochastic replica of $(X_m : m \in \mathbb{Z}_+)$ with $X_0 = x_0$.