Lecture-25: DTMC: Invariant Distribution

1 Invariant Distribution

Let $X = (X_n \in \mathcal{X} : n \in \mathbb{Z}_+)$ be a time-homogeneous Markov chain on state space $\mathcal{X}$ with transition probability matrix P. A probability distribution $\pi = (\pi_x \geqslant 0 : x \in \mathcal{X})$ such that $\sum_{x \in \mathcal{X}} \pi_x = 1$ is said to be **stationary distribution** or invariant distribution for the Markov chain X if $\pi = \pi P$, that is $\pi_y = \sum_{x \in \mathcal{X}} \pi_x p_{xy}$ for all $y \in \mathcal{X}$.

Remark 1. Facts about the invariant distribution π.

i. The global balance equation $\pi = \pi P$ is a matrix equation, that is we have a collection of $|\mathcal{X}|$ equations $\pi_y = \sum_{x \in \mathcal{X}} \pi_x p_{xy}$ for each $y \in \mathcal{X}$.

ii. Balance equation across cuts is $\pi_y(1 - p_{yy}) = \pi_y \sum_{x \neq y} p_{yx} = \sum_{x \neq y} \pi_x p_{xy}$.

iii. The invariant distribution π is left eigenvector of stochastic matrix P with the largest eigenvalue 1. The all ones vector is the right eigenvector of this stochastic matrix P for the eigenvalue 1.

iv. From the Chapman-Kolmogorov equation for initial probability vector π, we have $\pi = \pi P^n$ for $n \in \mathbb{N}$. That is, if $P\{X_0 = x\} = \pi_x$ for each $x \in \mathcal{X}$, then $P_\pi\{X_n = y\} = \pi_y$ for each $y \in \mathcal{X}$ and all $n \in \mathbb{Z}_+$, since $P_\nu\{X_n = y\} = \sum_{x \in \mathcal{X}} \nu(x) p_{xy}^{(n)}$.

v. Resulting process with initial distribution π is stationary, and hence have shift-invariant finite dimensional distributions. For example, for any $k, n \in \mathbb{Z}_+$ and $x_0, \dots, x_n \in \mathcal{X}$, we have

$$P_\pi\{X_0 = x_0, \dots, X_n = x_n\} = P_\pi\{X_k = x_0, \dots, X_{k+n} = x_n\} = \pi_{x_0} p_{x_0 x_1} \dots p_{x_{n-1} x_n}.$$

vi. If the Markov chain is irreducible, with $\pi_x > 0$ for some $x \in \mathcal{X}$. Then for any $y \in \mathcal{X}$, we have $p_{xy}^{(m)} > 0$ for some $m \in \mathbb{N}$. Hence, $\pi_y \geqslant \pi_x p_{xy}^{(m)} > 0$. That is, the entire invariant vector is positive.

vii. Any scaled version of π satisfies the global balance equation. Therefore, $\sum_{x \in \mathcal{X}} \pi_x$ must be finite for positive recurrent Markov chains, to normalize such vectors and get a unique invariant measure.

Theorem 1.1. *An irreducible Markov chain with transition probability matrix P is positive recurrent iff there exists a unique invariant probability measure π on state space $\mathcal{X}$ that satisfies global balance equation $\pi = \pi P$ and $\pi_x = \frac{1}{\mu_{xx}} > 0$ for all $x \in \mathcal{X}$.*

Proof. We will first show that recurrence implies the existence of invariant distribution, and then its converse.

Implication: Let X be a positive recurrent Markov chain on state space $\mathcal{X}$, with initial state $X_0 = x$. We define $N_y(n) \triangleq \sum_{k=1}^{n} 1\{X_k = y\}$ to be the number of visits to state $y \in \mathcal{X}$ in the first n steps of the Markov chain. It follows that $\sum_{y \in \mathcal{X}} N_y(n) = n$ for each $n \in \mathbb{N}$. Let H_x be the first recurrence time to state x, then we have $N_x(H_x) = 1$ and $\sum_{y \in \mathcal{X}} N_y(H_x) = H_x$.

Existence: We denote $v_y \triangleq \mathbb{E}_x[N_y(H_x)]$ for each $y \in \mathcal{X}$. We observe that $v_y \geqslant 0$ for each state $y \in \mathcal{X}$, in particular $v_x = 1$, and $\sum_{y \in \mathcal{X}} v_y = \mathbb{E}_x H_x = \mu_{xx} < \infty$ since X is positive recurrent. We will show that the vector $v = (v_x : x \in \mathcal{X})$ satisfies the global balance equations $v = vP$, and since v is summable, $\pi = \frac{v}{\sum_{x\in\mathcal{X}} v_x}$ is an invariant distribution for the Markov chain X. To see that the vector v satisfies the global balance equations, we observe from the monotone convergence theorem

$$v_y = \mathbb{E}_x N_y(H_x) = \mathbb{E}_x \sum_{n\in\mathbb{N}} 1\{X_n = y, n \leqslant H_x\} = \sum_{n\in\mathbb{N}} P_x\left\{X_n = y, n \leqslant H_x\right\}.$$

Let $\lambda_{xy}^{(n)} \triangleq P_x\left\{X_n = y, n \leqslant H_x\right\}$. Observe that $\lambda_{xy}^{(1)} = p_{xy}$ for each $y \in \mathcal{X}$. For $n \geqslant 2$, we have

$$\begin{aligned}\lambda_{xy}^{(n)} &= \sum_{\ell\neq x} P_x\left\{X_n = y, X_{n-1} = \ell, n \leqslant H_x\right\} = \sum_{\ell\neq x} P(\{X_n = y\} \mid \{X_{n-1} = \ell, n \leqslant H_x, X_0 = x\}) P_x\left\{X_{n-1} = \ell, n \leqslant H_x\right\}\\ &= \sum_{\ell\neq x} P(\{X_n = y\} \mid \{X_{n-1} = \ell\}) P_x\left\{X_{n-1} = \ell, n-1 \leqslant H_x\right\} = \sum_{\ell\neq x} \lambda_{x\ell}^{(n-1)} p_{\ell y}.\end{aligned}$$

Hence, we have for each $y \in \mathcal{X}$, $v_y = \sum_{n\in\mathbb{N}} \lambda_{xy}^{(n)}$. Therefore,

$$v_y = p_{xy} + \sum_{n\geqslant 2}\sum_{\ell\neq x} \lambda_{x\ell}^{(n-1)} p_{\ell y} = p_{xy} + \sum_{\ell\neq x} p_{\ell y} \sum_{n\in\mathbb{N}} P_x\left\{X_n = \ell, n \leqslant H_x\right\} = v_x p_{xy} + \sum_{\ell\neq x} v_\ell p_{\ell y} = \sum_{x\in\mathcal{X}} v_x p_{xy}.$$

Hence, $\pi = \frac{v}{\sum_{x\in\mathcal{X}} v_x}$ is an invariant measure of the transition matrix P, and $\pi_x = \frac{v_x}{\sum_{y\in\mathcal{X}} v_y} = \frac{1}{\mu_{xx}} > 0$.

Uniqueness: Next, we show that this is a unique invariant measure independent of the initial state x, and hence $\pi_y = \frac{1}{\mu_{yy}} > 0$ for all $y \in \mathcal{X}$. For uniqueness, we observe from the Chapman-Kolmogorov equations and invariance of π that $\pi = \frac{1}{n}\pi(P + P^2 + \cdots + P^n)$. Hence, $\pi_y = \sum_{x\in\mathcal{X}} \pi_x \frac{1}{n}\sum_{k=1}^n p_{xy}^{(k)}$ for any $y \in \mathcal{X}$. Taking limit $n \to \infty$ on both sides, and exchanging limit and summation on right hand side using bounded convergence theorem for summable series π, we get for all $y \in \mathcal{X}$

$$\pi_y = \frac{1}{\mu_{yy}} \sum_{x\in\mathcal{X}} \pi_x = \frac{1}{\mu_{yy}} > 0.$$

Converse: Let π be the positive invariant distribution of Markov chain X. Then, if the Markov chain was transient or null recurrent, we would have $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n p_{xy}^{(k)} = 0$. Since π is an invariant vector, we get $\pi = \pi P^k$ for each $k \in \mathbb{N}$ and hence $\pi = \pi\frac{1}{n}\sum_{k=1}^n P^k$. Taking limit on both sides, we have $\pi = 0$, yielding a contradiction for its positivity.

□

Corollary 1.2. *An irreducible Markov chain on a finite state space $\mathcal{X}$ has a unique and positive stationary distribution π.*

An irreducible, aperiodic, positive recurrent Markov chain is called **ergodic**.

Remark 2. Additional remarks about the stationary distribution π.

i. For a Markov chain with multiple positive recurrent communicating classes $\mathcal{C}_1, \dots, \mathcal{C}_m$, one can find the positive equilibrium distribution for each class, and extend it to the entire state space $\mathcal{X}$ denoting it by π_k for class $k \in [m]$. It is easy to check that any convex combination $\pi = \sum_{k=1}^m \alpha_k \pi_k$ satisfies the global balance equation $\pi = \pi P$, where $\alpha_k \geqslant 0$ for each $k \in [m]$ and $\sum_{k=1}^m \alpha_k = 1$. Hence, a Markov chain with multiple positive recurrent classes have a convex set of invariant probability measures, with the individual invariant distribution π_k for each positive recurrent class $k \in [m]$ being the extreme points.

ii. Let $\mu(0) = e_x$, that is let the initial state of the positive recurrent Markov chain be $X_0 = x$. Then, we know that

$$\pi_y = \frac{1}{\mu_{yy}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n p_{xy}^{(k)} = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_x N_y(n).$$

That is, π_y is limiting average of number of visits to state $y \in \mathcal{X}$.

iii. If a positive recurrent Markov chain is aperiodic, then limiting probability of being in a state y is its invariant probability, that is $\pi_y = \lim_{n\to\infty} p_{xy}^{(n)}$.

Theorem 1.3. *For an ergodic Markov chain X with invariant distribution π, and nth step distribution $\mu(n)$, we have $\lim_{n\to\infty}\mu(n) = \pi$ in the total variation distance.*

Proof. Consider independent time homogeneous Markov chains $X = (X_n \in \mathcal{X} : n \in \mathbb{Z}_+)$ and $Y = (Y_n \in \mathcal{X} : n \in \mathbb{Z}_+)$ each with transition matrix P. The initial state of Markov chain X is assumed to be $X_0 = x$, whereas the Markov chain Y is assumed to have an initial distribution π. It follows that Y is a stationary process, while X is not. In particular,

$$\mu_y(n) = P_x\{X_n = y\} = p_{xy}^{(n)}, \qquad P_\pi\{Y_n = y\} = \pi_y.$$

Let $\tau = \inf\{n \in \mathbb{Z}_+ : X_n = Y_n\}$ be the first time that two Markov chains meet, called the **coupling time**.

Finiteness: First, we show that the coupling time is almost surely finite. To this end, we define a a new Markov chain on state space $\mathcal{X}\times\mathcal{X}$ with transition probability matrix Q such that $q((x,w),(y,z)) = p_{xy}p_{wz}$ for each pair of states $(x,w),(y,z) \in \mathcal{X}\times\mathcal{X}$. The n-step transition probabilities for this couples Markov chain are given by

$$q^{(n)}((x,w),(y,z)) \triangleq p_{xy}^{(n)}p_{wz}^{(n)}.$$

Ergodicity: Since the Markov chain X with transition probability matrix P is irreducible and aperiodic, for each $x,y,w,z \in \mathcal{X}$ there exists an $n_0 \in \mathbb{Z}_+$ such that $q^{(n)}((x,w),(y,z)) = p_{xy}^{(n)}p_{wz}^{(n)} > 0$ for all $n \geqslant n_0$ from a previous Lemma on aperiodicity. Hence, the irreducibility and aperiodicity of this new **product** Markov chain follows.

Invariant: It is easy to check that $\theta(x,w) = \pi_x\pi_w$ is the invariant distribution for this product Markov chain, since $\theta(x,w) > 0$ for each $(x,w) \in \mathcal{X}\times S$, $\sum_{x,w\in\mathcal{X}}\theta(x,w) = 1$, and for each $(y,z) \in \mathcal{X}\times S$, we have

$$\sum_{x,w\in\mathcal{X}}\theta(x,w)q((x,w),(y,z)) = \sum_{x\in\mathcal{X}}\pi_x p_{xy}\sum_{w\in\mathcal{X}}\pi_w p_{wz} = \pi_y\pi_z = \theta(y,z).$$

Recurrence: This implies that the product Markov chain is positive recurrent, and each state $(x,x) \in \mathcal{X}\times\mathcal{X}$ is reachable with unit probability from any initial state $(y,w) \in \mathcal{X}\times\mathcal{X}$.

In particular, the coupling time is almost surely finite.

Coupled process: Second, we show that from the coupling time onwards, the evolution of two Markov chains is identical in distribution. That is, for each $y \in \mathcal{X}$ and $n \in \mathbb{Z}_+$,

$$P_{X_\tau}\{X_n = y, n \geqslant \tau\} = P_{Y_\tau}\{Y_n = y, n \geqslant \tau\}.$$

This follows from the strong Markov property for the joint process where τ is stopping time for the joint process $((X_n, Y_n) : n \in \mathbb{Z}_+)$ such that $X_\tau = Y_\tau$, and both marginals have the identical transition matrix.

Limit: For any $y \in \mathcal{X}$, we can write the difference as

$$\left|p_{xy}^{(n)} - \pi_y\right| = |P_x\{X_n = y, n < \tau\} - P_\pi\{Y_n = y, n < \tau\}| \leqslant P_{\delta_x,\pi}(\tau > n).$$

Since the coupling time is almost surely finite for each initial state $x,y \in \mathcal{X}$, we have $\sum_{n\in\mathbb{N}} P_{\delta_x,\pi}\{\tau = n\} = 1$ and the tail-sum $P_{\delta_x,\pi}\{\tau > n\}$ goes to zero as n grows large, and the result follows.

□