# Discrete Time Markov Chains

Compiled by D. Gueorguiev 4/21/2024

## Introduction

**Definition**: *Discrete Time Markov Chain (DTMC)*

For a countable set , a discrete valued random sequence is called if for all , all states and any historical event the process satisfies the Markov property

The probability of a DTMC being in state at time from a state at time , is determined by the *transition probability* denoted by

The set is denoted as *the state space* of the Markov chain. The *transition probability matrix* at time is denoted by , such that . We observe that each row is the conditional distribution of given .

For all states , a matrix with non-negative entries is called *sub-stochastic* if the row-sum for all rows . If the above property holds with equality for all rows, then it is denoted as *stochastic matrix*. If matrices and are both stochastic, then the matrix A is called *doubly stochastic*.

i ) all of the entries of a sub-stochastic matrix lie in .

ii ) Each row of the stochastic matrix is probability mass function over the state space .

iii ) Every finite stochastic matrix has a right eigenvector with unit eigenvalue. This can be observed by taking to be an all-one vector of length . Then we see that , since for each

iv ) Every probability transition matrix is a stochastic matrix.

v ) Every finite doubly stochastic matrix has a left and right eigenvector with unit eigenvalue. This follows from the fact that finite stochastic matrices and have a common right eigenvector . It follows that A has a left eigenvector.

**Definition**: *Homogeneous Discrete Time Markov Chain*

Homogeneous DTMC is such chain where the transition probabilities are independent of the index. The linear operator is denoted as the *transition matrix*.

**Proposition**: Conditioned on the initial state, any finite dimensional distribution of a homogeneous Markov chain is stationary

*Proof*: To this end, we compute the transition probabilities for the path taken by the sample path when and by the sample path when . From the homogeneous Markov property of the process , we get:

Similarly, we can write for the sample path given

**Corollary**: The -step transition probabilities are stationary for any homogeneous Markov chain. That is, for any states and , we have

*Proof*: It follows from summing over intermediate steps. Hence, it follows that for a homogeneous Markov chain, we can define -step transition probabilities for and

That is, the row is the conditional distribution of given .

**Theorem**: The -step transition probabilities form a semi-group. That is, for all positive integers

*Proof*: It follows from the Markov property and law of total probability that for any states and positive integers

Since the choice of states were arbitrary, the result follows.

**Corollary**: The -step transition probability matrix is given by for any positive integer .

*Proof*: In particular, we have . Since , we have by induction.

That is, for all states and integers ,

## Representation of DTMC

Consider a Markov chain with countable state space and transition matrix .

### Chapman-Kolmogorov equations

We denote by the initial distribution of the Markov chain, that is . The distribution of is given by , such that for any state we have:

We can write this succinctly in terms of transition probability matrix as . We can alternatively derive this result by the following Lemma.

**Lemma**: The right multiplication of a probability vector with the transition matrix transforms the probability distribution of current state to probability distribution of the next state. That is,