The Normal Form

The description of a game can be viewed as a listing of the strategies of the players and the outcome of any set of choices of strategies, without regard to the attitudes of the players toward various outcomes. We now indicate how the final simplification of the game – the normal form – is obtained, by taking into account the preferences of the players.
The result of any set of strategies $f_1, \dots, f_k$ is a probability distribution π_f over the set R of possible outcomes. It would be particularly convenient if a given player could express his/her preference pattern in R by a bounded numerical function u defined on R, such that he or she prefers r_1 to r_2 *iff* $u(r_1) > u(r_2)$. Note that $u(r_1) = u(r_2)$ denotes indifference between r_1 to r_2. Also, the function u is such that if for any probability distgribution ξ over R we define $U(\xi)$ as the expected value of $u(r)$ computed with respect to ξ as

$$U(\xi) = \sum_{r \in R} \xi(r) u(r)$$

the player prefers $\xi_1$to ξ_2 *iff* $U(\xi_1) > U(\xi_2)$.
It is remarkable fact that, under extremely plausible hypothesis concerning the preference pattern such function u exists.

Definition (*utility function*): The function U defined for all probability distributions ξ over R, is called the player's ***utility function***.
U is unique, for a given preference pattern up to a linear transformation. We will assume that each player has such utility function.
The aim of each player in the game is to maximize his/her expected utility. If U_i is the utility function of player i, his/her aim is to make $M_i(f_1, \dots, f_k) = U_i(\pi_f)$ as large as possible where π_f is the probability distribution for fixed $f_1, \dots, f_k$ over R determined by the overall chance move.
We are in a position to give a description of the normal form of a game:

Definition (*normal form of a game*): A game consists of k spaces $F_1, \dots, F_k$ and k bounded numerical functions $M_i(f_1, \dots, f_k)$ defined on the space of all k-tuples $(f_1, \dots, f_k)$, $f_i \in F_i, i = 1, \dots, k$. The game is played as follows: Player i chooses an element f_i of F_i, the k choices being made simultaneously and independently; player i then receives the amount $M_i(f_1, \dots, f_k), i = 1, \dots, k$. The aim of Player i is to make M_i as large as possible. The statement "Player i receives the amount $M_i(f_1, \dots, f_k)$" is shorthand of saying "a situation results whose utility for Player i is $M_i(f_1, \dots, f_k)$".

Example (*two player game involving coin-toss and a number choice*):
Player I moves first and selects one of the two integers $1, 2$. The referee then tosses a coin and if the outcome is "head", he informs player II of player I's choice and not otherwise. Player II then moves and selects one of two integers $3, 4$. The fourth move is again a chance move by the referee and consists of selecting one of three integers $1, 2, 3$ with respective probabilities $0.4, 0.2, 0.4$. The numbers selected in the first, third and the fourth move are added and the amount of dollars is paid by II to I if the sum is even and by I to II if the sum is odd. Note that $|R| = 2 \times 2 \times 2 \times 3 = 24$.

Here are the two strategy spaces:
$F_1 = \{f_1, f_2\}$; $f_1 = (1), f_2 = (2)$

$F^2 = \{f^1, f^2, f^3, f^4, f^5, f^6, f^7, f^8\}$; $f^1 = (3{,}3{,}3), f^2 = (3{,}3{,}4), f^3 = (3{,}4{,}3), f^4 = (3{,}4{,}4), f^5 = (4{,}3{,}3), f^6 = (4{,}3{,}4), f^7 = (4{,}4{,}3), f^8 = (4{,}4{,}4)$
Here the first position of the triple is conditioned upon coin falling *Head* and player I choosing 1, the second position in the triple is conditioned upon coin falling head and player I choosing 2, and the third position of the triple is conditioned upon coin falling *Tail*.

The set R of possible outcomes for this game where I denotes player I, 0 denotes the referee and II denotes player II is shown below:
I –> 2 – 0 –> Head – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Head – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Head – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^3), (f_2, f^4), (f_2, f^7), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 2 – 0 –> Tail – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$

I –> 2 – 0 –> Head – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
I –> 2 – 0 –> Head – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
I –> 2 – 0 –> Head – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^2), (f_2, f^5), (f_2, f^6)$
I –> 2 – 0 –> Tail – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 2 – 0 –> Tail – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_2, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 2 – 0 –> Tail – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_2, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$

I –> 1 – 0 –> Head – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^5), (f_1, f^6), (f_1, f^7), (f_1, f^8)$
I –> 1 – 0 –> Head – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_1, f^5), (f_1, f^6), (f_1, f^7), (f_1, f^8)$
I –> 1 – 0 –> Head – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^5), (f_1, f^6), (f_1, f^7), (f_1, f^8)$
I –> 1 – 0 –> Tail – II –> 4 – 0 –> 3 = 9, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^2), (f_1, f^4), (f_1, f^6), (f_1, f^8)$
I –> 1 – 0 –> Tail – II –> 4 – 0 –> 2 = 8, probability $P = 0.5 \times 0.2 = 0.1$, strategies $(f_1, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$
I –> 1 – 0 –> Tail – II –> 4 – 0 –> 1 = 7, probability $P = 0.5 \times 0.4 = 0.2$, strategies $(f_1, f^2), (f_2, f^4), (f_2, f^6), (f_2, f^8)$

I –> 1 – 0 –> Head – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies
$(f_1, f^1), (f_1, f^2), (f_1, f^3), (f_1, f^4)$
I –> 1 – 0 –> Head – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies
$(f_1, f^1), (f_1, f^2), (f_1, f^3), (f_1, f^4)$
I –> 1 – 0 –> Head – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies
$(f_1, f^1), (f_1, f^2), (f_1, f^3), (f_1, f^4)$

I –> 1 – 0 –> Tail – II –> 3 – 0 –> 3 = 8, probability $P = 0.5 \times 0.4 = 0.2$, strategies
$(f_1, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 1 – 0 –> Tail – II –> 3 – 0 –> 2 = 7, probability $P = 0.5 \times 0.2 = 0.1$, strategies
$(f_1, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$
I –> 1 – 0 –> Tail – II –> 3 – 0 –> 1 = 6, probability $P = 0.5 \times 0.4 = 0.2$, strategies
$(f_1, f^1), (f_2, f^3), (f_2, f^5), (f_2, f^7)$

In the theory of games it is usual to treat first a special class of games, *the two-person zero-sum games*. The theory of these games is particularly simple and complete and we will consider only such games in our discussion.

Definition (*two-person game*): a game with $k = 2$: we have only two utility functions M_1 and M_2 and two strategy sets F_1 and F_2 for each of the two players.

Definition (*zero-sum game*): A game for which the following holds true:
$\sum_{i=1}^{k} M_i(f_1, \ldots, f_k) = 0$ for all $f_1, \ldots, f_k$
More precisely, since each M_i is unique up to a linear transformation, a game is a **zero-sum** if there is a determination of $M_1, \ldots., M_k$ for which $\sum_{i=1}^{k} M_i(f_1, \ldots, f_k) = 0$ for all $f_1, \ldots, f_k$. Thus a two-person zero-sum game is a game between two players in which their interests are diametrically opposed: one player gains at the expense of the other. Consequently, there is no motive for collusion between the players. It is precisely the fact that collusion is unprofitable that simplifies the theory.

Definition (*constant-sum game*): A **constant-sum game** i.e. one in which $\sum_{i=1}^{k} M_i(f_1, \ldots, f_k) = c$ for all $f_1, \ldots, f_k$ is zero-sum game in the sense defined above, since an alternative choice of utility functions is $M_1^* = M_1 - c$, $M_i^* = M_i$ for $i \neq 1$, and $\sum_{i=1}^{k} M_i^* = 0$. Thus the theory developed for zero sum two person games applies for constant sum two person games.

Since for two-person zero sum game we have $M_2(f_1, f_2) = -M_1(f_1, f_2)$ we need to specify only M_1. We will consider only two-person zero-sum games from now on.

Definition (*game in a normal form*): A **game in a normal form** is a triple (X, Y, M), where X, Y are arbitrary spaces and M is a bounded numerical function defined on the product space $X \times Y$ of pairs (x, y), $x \in X, y \in Y$. The points $x(y)$ are called strategies for player I (II) and the function M is called payoff. The game G is played as follows: I chooses $x \in X$, II chooses $y \in Y$, the choices being made independently and simultaneously. II then pays I the amount $M(x, y)$.

Equivalent Games

If, in a given game, one relabels the strategies of either player, the new game is essentially not different than the old. Every statement about either game can be translated into a corresponding statement about the other and we wish to consider the two games equivalent.

Another simple transformation which does not alter the essential character of the game is the deletion of duplicate strategies. In other words, if a player I has two strategies x_1, x_2 such that $M(x_1, y) = M(x_2, y)$ for all y, the deletion of x_2 from X is an inessential change in the game, even though it might, for example destroy such properties as symmetry.

Definition (*reduction of game*): Let $G_1 = (X_1, Y_1, M_1)$ and $G_2 = (X_2, Y_2, M_2)$ be two games. Then G_2 is a reduction of G_2, written $G_2\ r\ G_1$, if either:

(a) $X_2 = X_1$, and there is a function f from Y_1 onto Y_2 such that $M_1(x, y) = M_2(x, f(y))$ for all $x \in X_1$, $y \in Y_1$, or
(b) $Y_2 = Y_1$, and there is a function g from X_1 onto X_2 such that $M_1(x, y) = M_2(g(x), y)$ for all $x \in X_1$, $y \in Y_1$

If f is a $1-1$ transformation, G_2 is obtained from $G_1$by relabeling of strategies; if f is not $1-1$, G_2 is obtained from G_1 by deletion of certain duplicated strategies and relabeling.

Definition (*equivalent games*): Two games G and G' are called **equivalent**, written $G \sim G'$, *iff* there is a finite sequence of games $G_0, G_1, \dots, G_n$ with $G_0 = G, G_n = G'$, and for each $i = 1, \dots, n$ either $G_{i-1}\ r\ G_i$ or $G_i\ r\ G_{i-1}$.

Example (*equivalent games*): Let $G = (X, Y, M)$ be a game, where $X = (x_1, \dots, x_N)$, a set of N real numbers, $Y = (y_1, \dots, y_R)$, also a set of R real numbers; and for $x \in X, y \in Y$,

M(x,y)=
//TODO: finish this

Illustrative Examples

Game G_1: *Matching Pennies*
Players I and II simultaneously place coins on the table. If the coins agree, i.e. both show heads or both tails II pays I one unit. If not I pays II one unit.
Clearly each player player has two strategies – *heads* and *tails*. The game is equivalent to one with the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Game G_2: *Matching Pennies with Spying*
This game is like *Matching Pennies*, except that I is required to place his coin first, and II is permitted to see the result before placing his own coin. I still has two strategies – *heads* and *tails*. A strategy for II specifies his choice when he sees heads and his choice when he sees tails, so that II has four strategies. Denoting heads by 1, tails by 2, and by (i, j) the strategy that chooses i when I chooses 1, and j when I chooses 2, we obtain the matrix:

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \end{array}$$

Game G_3: *Matching Pennies with Imperfect Spying*
After I makes his choice, a coin is tossed that has probability p of showing I's choice and $1-p$ of showing the opposite. The result of the toss is revealed to II, who then makes his choice. Again, I has two strategies and II has four; the matrix is:

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{pmatrix} 1 & 2p-1 & 1-2p & -1 \\ -1 & 2p-1 & 1-2p & 1 \end{pmatrix} \end{array}$$

where (i,j) now denotes the strategy "II chooses i when 1 is announced and j when 2 is announced".

Game $G_4(k,N)$: *Addition*
I and II alternatively choose integers, each choice being one of the integers $1,\dots,k$ and each choice made with the knowledge of all preceding choices. As soon as the sum of the chosen integers exceeds N, the last player to choose pays his opponent one unit.
The situation at which player I finds himself at his rth move is described by a sequence $s_r = (i_1, i_2, \dots, i_{2r-2})$ with each i_j being one of the integers $1,\dots,k$ and

$$\sum_{j=1}^{2r-2} i_j \le N$$

Denote by S_r the set of possible sequences s_r where $r = 2, \dots, \left[\frac{N}{2}\right] + 1$ and $[z]$ denotes the closest integer which does not exceed z. A strategy x for I consists of a set of $\left[\frac{N}{2}\right] + 1$ functions $f_1, \dots, f_{\left[\frac{N}{2}\right]+1}$, where f_r is a function defined on S_r assuming only values $1, 2, \dots, k$: f_r specifies I's rth move when the previous history of the play is s_r. Similarly, a strategy y for II is a set of $\left[\frac{(N+1)}{2}\right]$ functions $g_1, \dots, g_{\left[\frac{(N+1)}{2}\right]}$, where g_r is defined for the set T_r of all sequences $t_r = (i_1, \dots, i_{2r-1})$ with each i_j being one of the integers $1,2,\dots,k$ and

$$\sum_{j=1}^{2r-1} i_j \le N$$

Define $i_1(x,y) = f_1$ and inductively for $j > 0$,
$i_{2j}(x,y) = g_j\left(i_1(x,y), \dots, i_{2j-1}(x,y)\right)$
$i_{2j+1}(x,y) = f_{j+1}\left(i_1(x,y), \dots, i_{2j}(x,y)\right)$
(this induction describes the manner in which a referee would carry out the instructions of the players) and let $j^*(x,y)$ be the largest j for which $i_j(x,y)$ is defined. Then

$$M(x,y) = \begin{cases} 1 \text{ if } j^*(x,y) \text{ is even} \\ -1 \text{ if } j^*(x,y) \text{ is odd} \end{cases}$$

Lower and Upper Pure Value

In a game $G = (X, Y, M)$, the consequences of strategy x_0 are described by the function $M(x_0, y)$. Using x_0, player I is certain to receive at least

$$\Lambda_G(x_0) = \inf_{y \in Y} M(x_0, y)$$

and cannot be certain of any definite larger amount. Thus, the number

$$\lambda_G^* = \sup_{x \in X} \Lambda_G(x)$$

is the upper limit to the amount I can guarantee getting: for every $\varepsilon > 0$, the player can, simply by choosing a suitable x, be certain of $\lambda_G^* - \varepsilon$, and for no $\varepsilon > 0$ is there an x which makes the player certain to receive at least $\lambda_G^* + \varepsilon$ against all y. Similarly, we define

$$\Upsilon_G(y_0) = \sup_{x \in X} M(x, y_0), \;\; \upsilon_G^* = \inf_{y \in Y} \Upsilon_G(y)$$

by selecting a y suitably, player II can with certainty restrict his/her loss to $\upsilon_G^* + \epsilon$ but not to $\upsilon_G^* - \epsilon$ for any $\epsilon > 0$. For subsequent reference these statements are stated formally:

Definition (*Capital lambda of x as game infimum*): If $G = (X, Y, M)$ is a game, then, for $x_0 \in X$, $\Lambda_G(x_0) = \inf_{y \in Y} M(x_0, y)$.

Definition (*Capital upsilon of y as game supremum*): If $G = (X, Y, M)$ is a game, then, for $y_0 \in Y$, $\Upsilon_G(y_0) = \sup_{x \in X} M(x, y_0)$

Definition (*Lower pure value as game supremum of infimum*): If $G = (X, Y, M)$ is a game, then the lower pure value of G is the number
$\lambda_G^* = \sup_{x \in X} \Lambda_G(x) = \sup_{x \in X} \inf_{y \in Y} M(x, y)$

Definition (*Upper pure value as game infimum of supremum*): If $G = (X, Y, M)$ is a game, then the upper pure value of G is the number
$\upsilon_G^* = \inf_{y \in Y} \Upsilon_G(x) = \inf_{y \in Y} \sup_{x \in X} M(x, y)$

Theorem (*Inequality between pure lower value and pure upper value*): If $G = (X, Y, M)$ is a game, then, for $x_0 \in X$ and $y_0 \in Y$,
$\Lambda_G(x_0) \le \Upsilon_G(y_0)$ and $\lambda_G^* \le \upsilon_G^*$
Proof: $\Lambda_G(x_0) \le M(x_0, y_0) \le \Upsilon_G(y_0)$. Thus $\lambda_G^* \le \Upsilon_G(y_0)$ for all $y_0 \in Y$ and $\lambda_G^* \le \upsilon_G^*$.

Consider now any game G. No method of play for I can guarantee him more than υ_G^* since II can restrict his loss to υ_G^* and no method of play for II can with certainty reduce his loss below λ_G^* since I can guarantee this amount. Thus, if $\lambda_G^* = \upsilon_G^* = \upsilon$ no method of play can guarantee either player any improvement over υ and we have seen that each player can attain υ (more precisely, approximate υ as closely as the player wishes). Thus, for such games, choosing an x_0 with $\Lambda_G(x_0) = \upsilon$ is an unimprovable method of play for I in the sense that no method of play can guarantee more, and similarly for II. This situation leads to the following definitions

Definition (*pure value*): If $G = (X, Y, M)$ is a game and if $\lambda_G^* = \upsilon_G^* = \upsilon_G$ then the number υ_G is called the pure value of G.

Definition (*optimal strategy of game with pure value*): If $G = (X, Y, M)$ is a game and if υ_G is the pure value of G, then a good strategy for player I in G is any $x_0 \in X$ with $\Lambda_G(x_0) = \upsilon_G$ and good strategy for player II in G is any $y_0 \in Y$ with $\Upsilon_G(y_0) = \upsilon_G$.

Theorem (*pure upper and lower value of equivalent games*): If two games $G_1 = (X_1, Y_1, M_1)$ and $G_2 = (X_2, Y_2, M_2)$ are equivalent, then $\lambda_{G_1}^* = \lambda_{G_2}^*$ and $\upsilon_{G_1}^* = \upsilon_{G_2}^*$.
Proof: It is sufficient to prove the theorem in the special case where one of the games is a reduction of the other. Suppose for definiteness that G_2 is reduction of G_1 and that f is a function mapping X_1 onto X_2. Since, for all $x \in X_1$ and all $y \in Y_1 (= Y_2)$,
$M_1(x, y) = M_2(f(x), y)$
we have
$\inf_{y \in Y_1} M_1(x, y) = \inf_{y \in Y_2} M_2(f(x), y)$
Hence, for all $x \in X_1$,
$\Lambda_{G_1}(x) = \Lambda_{G_2}(f(x))$
so that
$\lambda_{G_1}^* = \sup_{x \in X_1} \Lambda_{G_1}(x) = \sup_{x \in X_1} \Lambda_{G_2}(f(x)) = \lambda_{G_2}^*$
The proof that $\upsilon_{G_1}^* = \upsilon_{G_2}^*$ is similar.

Problem (*opposite player strategies yielding constant return*): If there are strategies x_0, y_0 such that $M(x_0, y) = c_1$ for all $y \in Y$, $M(x, y_0) = c_2$ for all $x \in X$, then $c_1 = c_2 = \lambda_G^* = \upsilon_G^*$.
Solution:
We have $\Lambda_G(x_0) = \inf_{y \in Y} M(x_0, y) = M(x_0, y_0) = c_1(x_0)$. Similarly, $\Upsilon_G(y_0) = \sup_{x \in X} M(x, y_0) = M(x_0, y_0) = c_2(y_0)$. Hence $c_1(x_0) = c_2(y_0) = c$ where c does not depend neither on x nor on y. Therefore c is the game pure value.

Problem (*opposite player strategies and a number between them*): If there are strategies x_0, y_0 and a number υ such that $M(x_0, y) \geq \upsilon \geq M(x, y_0)$ for all x, y, then $\lambda_G^* = \upsilon_G^* = \upsilon$ and x_0, y_0 are good strategies for *I, II*.

Perfect Information Games

Among the games that do have a pure value are the *perfect information games* of which chess, checkers and tic-tac-toe are examples.
Essentially, a game of perfect information is one that can be described in terms of successive moves in such a way that, at each personal move, the mover knows the choices and the outcomes of all preceding personal and chance moves. Perfect information game is a game in which every information set is a unit set. It is intuitively clear that this condition is equivalent to the requirement that every branch of the tree of the game also be a tree of some game. The latter condition leads to an inductive definition for games in normal form. In this definition the order of a perfect information game intuitively corresponds to the maximum number of moves in that game.

Definition (*perfect information game*): A game $G = (X, Y, M)$ is a perfect information game of order 0 *iff* $M(x, y)$ is constant. A Game $G = (X, Y, M)$ is a perfect information game of order $n + 1$ *iff* there is a set A and a class $\mathcal{G}_A$ of games $G_a = (X_a, Y_a, M_a)$ for $a \in A$, such that each G_a is a perfect information game of order n, and such that either:

Case 1. X consists of all pairs $x = (a, z)$ with $a \in A$, $z \in X_a$, Y consists of all functions y defined on A with $y(a) \in Y_a$ for all a, and

$M\big((a,z), y\big) = M_a\big(z, y(a)\big)$ *or*

Case 2. Y consists of all pairs $y = (a, z)$ with $a \in A$, $z \in Y_a$, X consists of all functions x defined on A with $x(a) \in X_a$ for all a, and

$M\big(x, (a,z)\big) = M_a(x(a), z)$ *or*

Case 3. X, Y consist of all functions x, y defined on A with $x(a) \in X_a$, $y(a) \in Y_a$ for all a, and

$M(x,y) = \sum_{a\in A} p(a) M_a\big(x(a), y(a)\big)$

where $p(a) \geq 0$, $\sum_{a\in A} p(a) = 1$
A game G is called perfect information game if G is perfect information game of order n for some n.

Our inductive description corresponds to the fact that the result of the first move in perfect information game with n moves is another perfect information game with $n-1$ moves, so that the first move can be considered as a choice of one of a given collection of perfect information games with $n-1$ moves, with the three possible cases corresponding to the cases in which the first move – the choice of a – is a personal move of I , a personal move of II or a chance move. For any perfect information game G of order n, the first k moves of G may be considered as a game of perfect information, whose outcome is not a number, but a game of perfect information of order $n - k$, so that the first k moves of G may be regarded as a struggle to determine which game with $n-k$ moves shall be played.

Theorem 1.7.1 Every perfect information game has a pure value. Moreover, if $G = (X, Y, M)$ is a perfect information game of order n+1, and $\mathfrak{G}_A$is the class of perfect information games $G_a = (X_a, Y_a, M_a)$, $a \in A$ as required by the last Definition, then corresponding to the three cases for $\mathfrak{G}_A$ in the aforementioned Definition the pure value v_G of G is given by either:

Case 1. $v_G = \sup\limits_{a\in A} v_G(a)$ or
Case 2. $v_G = \inf\limits_{a\in A} v_G(a)$ or
Case 3. $v_G = \sum_{a\in A} p(a)\, v_G(a)$

where $v_G(a)$ is the pure value of G_a. In addition, if there is a $b \in A$ such that $v_G(b) = v_G$ and for every $a \in A$ there are good strategies $x_a^* \in X_a, y_a^* \in Y_a$ in G_a, then good strategies $x^* \in X, y^* \in Y$ exist in G and corresponding to the three cases in the last Definition are given by either:

Case 1. $x^* = (b, x_b^*)$, $y^*(a) = y_a^*$ for all $a \in A$
Case 2. $y^* = (b, y_b^*)$, $x^*(a) = x_a^*$ for all $a \in A$
Case 3. $x^*(a) = x_a^*$, $y^*(a) = y_a^*$ for all $a \in A$

Proof: The theorem is obvious for $n = 0$ and we suppose that the theorem holds true for all perfect information games of order less than $n + 1$.

Case 1. Let $v = \sup_{a \in A} v_G(a)$. For any $\epsilon > 0$, choose $x^* = (b, z)$ and y^* such that
$v_G(b) > v - \epsilon$
$\Lambda_{G_b}(z) > v_G(b) - \epsilon$
$\Upsilon_{G_a}(y^*(a)) < v_G(a) + \epsilon$

Then for all $y \in Y$,

(1) $M(x^*, y) = M_b(z, y(a)) \geq \Lambda_{G_b}(z) > v - 2\epsilon$

and for all $x \in X$,

(2) $M(x, y^*) = M_a(x_a, y^*(a)) \leq \Upsilon_{G_a}(y^*(a)) < v_G(a) + \epsilon \leq v + \epsilon$

Thus G has a pure value $v_G = v$.
Furthermore, if the supremum of $v_G(a)$ is attained by some $b \in A$ and there are good strategies $x_a^* \in X_a$, $y_a^* \in Y_a$ for every G_a, then, since equations (1) and (2) are valid for all $\epsilon > 0$, the choices $x^* \in X$, $y^* \in Y$ are good strategies in G. The proofs for Cases 2 and 3 are similar.

The inductive description of the value and good strategies in perfect information games can be used to solve such games. We illustrate the method to solve the game Addition

//TODO: finish this