Braess' Paradox in a Loss Network

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Abstract

Braess' paradox is said to occur in a network if the addition of an extra link leads to worse performance. It has been shown to occur in transportation networks (such as road networks) and also in queueing networks. Here, we show that it can occur in loss networks.

Loss Networks; Braess' Paradox; Blocking Probabilities

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1 Introduction

Consider a road network in which each user chooses its route so as to minimize its expected delay. It would seem intuitively clear that the addition of an extra link (and hence route choice) could only reduce the delay (or at worst not affect it). This is true for an uncongested network, where the delay on a link is independent of the volume of traffic. However, for a congested network it may not be true. In 1968, Braess [2] presented an example of a network for which the addition of the extra link increases the expected delay for all drivers.

This example might be seen as nothing more than an interesting curiosity. However, there is evidence to suggest that it has occurred in actual road networks. The following quotation is taken from Murchland [17, page 394].

> *Knödel [14] remarks that Braess' example may seem contrived, but a recent experience in Stuttgart shows that it can occur in reality. Major road investments in the city centre, in the vicinity of the Schlossplatz,*

failed to yield the benefits expected. They were only obtained when a cross street, the lower part of Königstrasse, was subsequently withdrawn from traffic use.

Since 1968, many authors have investigated Braess' paradox in traffic networks, see for example [17, 16, 7, 20, 8, 19]. Also it has been shown that similar paradoxes can occur in, for example, mechanical and electrical networks [5], water-supply networks [3] and queueing networks under fixed [6] and dynamic [4] routing schemes. General discussions of the paradox may be found in [10] and [13]. A related queueing example where the efficiency decreases as the waiting capacity is increased is discussed in Whitt [22]. The resolution of the paradox is well understood, and is often couched in terms of the distinction between a user-determined equilibrium and a system optimum, see Bell and Stidham [1]. Simple examples of the paradox are especially useful in demonstrating this point, and, in the context of communication networks, alerting practitioners to the subtlety of interaction between routing schemes and capacity expansion procedures. In the context of road networks, Steinberg and Zangwill [19] have argued that the occurrence of Braess' paradox is not a rare phenomenon. We believe similar comments can be made with respect to loss networks – for the paradox to occur, a few simple rules need to be obeyed by the link capacities and the arrival rate needs to be in the appropriate interval.

In this paper we show that Braess' paradox can occur in loss networks. The reader is referred to Kelly [12] for a precise definition of a loss network. Loss networks are used to model many multi-resource access problems where requests for access that cannot be fully met are denied and lost. The classical example (from which the terminology is derived) is the circuit-switched telephone network.

In earlier network routing examples of Braess' paradox, the performance measure is additive across the links on a route and takes a different functional form on different links. In a loss network the performance measures of interest are not additive and a single parameter, its capacity, suffices to define each link. Nevertheless, it has been known for a long time that alternative routeing in loss networks, if not properly controlled, can cause a reduction in overall performance, see for example [18, 15, 9]. Wroe, Cope and Whitehead [23] have noted the possibility of degradation of performance in a loss network when capacity is increased, although their paper does not specify the network or give a quantitative analysis.

In this paper we present two simple explicitly analysed examples of the occurrence of Braess' paradox in loss networks. The first example, described in Section 2, is for a network operating under fixed routing, while the second example, described in Section 3, concerns a network in which alternative routing is allowed.

2 Fixed Routing

Consider the network shown in Figure 1. The only origin-destination pair (or traffic stream) is A-D. Connections AB and CD consist of a single link of capacity C_1 circuits, whereas connections AC and BD consist of n separate links of capacity C_2. We consider two forms of the network, the *reduced network* in which link BC is not included, and where a call from A to D chooses to travel along ABD with probability p and along ACD with probability $1-p$, and the *augmented network* in which the single link BC is included and calls travel along ABD, ACD and ABCD with probabilities $p(1-q)$, $(1-p)(1-q)$ and q, respectively. Such a configuration is not unrealistic: for example, nodes A and D may represent subnetworks, within which little blocking occurs, connected via multiple links to gateway nodes B and C. At a user equilibrium the probabilities attached to different routes are such that the blocking probabilities are the same along all routes used, and not greater than that which would be incurred on an unused route. A routing scheme obeying such a rule is said to follow Wardrop's principle [21].

There are $2n$ possible routes in the reduced network, corresponding to the n routes via ABD and the n routes via ACD. Label these $1, 2, \ldots, n, n+1, \ldots, 2n$. The symmetry of the reduced network implies that $p = \frac{1}{2}$ at user equilibrium, and further, that a call chooses to travel along a given routes $i = 1, 2, \ldots, 2n$ with probability $\frac{1}{2n}$.

In the augmented network, q represents the probability that a call chooses the route ABCD (labelled $2n+1$). Then, at user equilibrium, for $q \in [0,1]$, routes $1, 2, \ldots, 2n$ have arrival rates $\frac{\lambda(1-q)}{2n}$ and route $2n+1$ has arrival rate $q\lambda$. The choice $q = 0$ corresponds to the routing pattern of the reduced network, but this routing pattern may not be a user equilibrium in the augmented network, since the blocking probability on route ABCD may be lower than that on the routes that are in use.

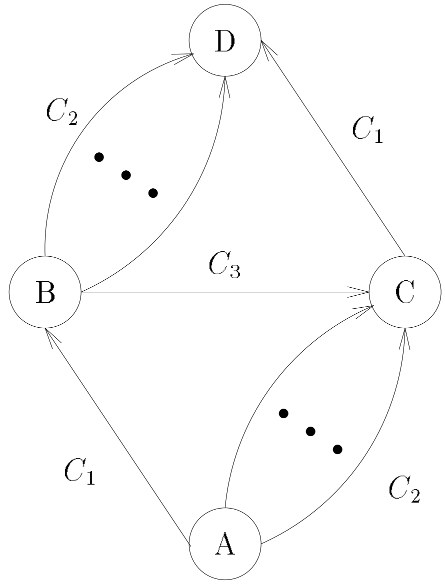

Figure 1: The loss network

Consider the special case where $\lambda = 4$, $n = 2$, $C_1 = 2$, $C_2 = 1$ and $C_3 = 2$. The blocking probabilities are listed in Table 1. In this augmented network, user equilibrium occurs when $q = 0.1401$. Thus, blocking probabilities have increased from 0.50 in the reduced network to 0.5108 in the augmented network.

Exact Analysis	$q = 0$	$q = 0.1401$
$B_{ABD} = B_{ACD}$	0.50	0.5108
B_{ABCD}	0.4375	0.5108

Table 1: Exact blocking probabilities

Observe that, in the reduced network, blocking between nodes A and B is lower than blocking between nodes A and C: but the effect of equalizing these blockings, as is achieved in the augmented network, is to increase network blocking for all users. We would expect this effect to become more pronounced as n gets larger, while preserving nC_2. This has been confirmed with numerical experiments.

In much larger networks an exact analysis will generally be impractical, and reliance may have to be placed on approximation procedures. For example, the Erlang fixed point approximation is used in [11] to provide a framework for the analysis of routing and capacity allocation decisions.

3 Alternative Routing

In this section we consider a loss network of the form shown in Figure 2. It is similar to that shown in Figure 1. We allow a very simple and natural form of alternative routing and provide an example of Braess' paradox under alternative routing.

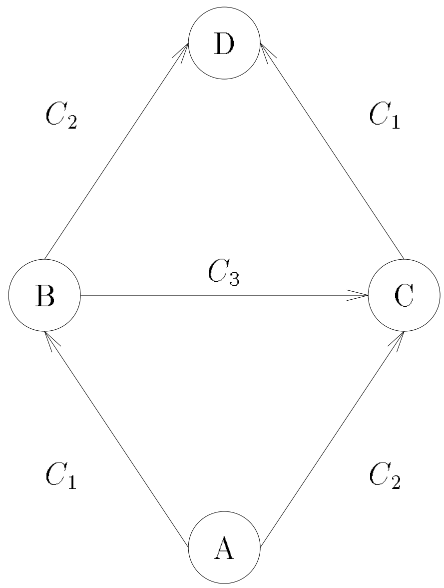

Figure 2: The loss network

We let $\lambda = 4$ and set $C_1 = 2$, $C_2 = 2$ and $C_3 = 2$. This is then the identical network to the one used in Section 2 except that the two parallel links AC are replaced by a single link of the same total capacity, and similarly for the links BD.

The alternative routing rule we use is that a call will randomly choose a route to attempt in the network. If that route is blocked then it will randomly choose an alternative route from the remaining routes in the network and attempt that route. The call will repeat this process until it is accepted or, after it has tried all possible routes, it is lost.

The analysis of the reduced network is very simple. Due to the alternative routing rule (and the symmetry of the network) the network can be thought to collapse down to a single link of capacity 4. This represents a very simple Markov process and the probability that a call is blocked is given by 0.3107.

The analysis of the augmented network is rather more complex. The arrival process to each route is no longer Poisson, but instead may depend on the state of the network. Therefore, we must define the underlying Markov process that describes the behaviour of the network. The state space consists of 14 states and calls are blocked in only two states. The process is governed by a transition matrix of size 14×14, which we do not present, but which the reader can easily reproduce, if required. The equilibrium distribution can easily be found and the blocking probability is 0.3553.

Thus the addition of the extra link (and hence route choice) has increased the blocking probability across the network from 0.3107 to 0.3553.

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