# Examples using Policy and Value Functions

(from *Sutton and Barto*)

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## Gridworld

The cells of the grid correspond to the states of the environment. At each cell, four actions are possible: ***north***, ***south***, ***east***, and ***west*** which deterministically cause the agent to move one cell in the respective direction on the grid. Actions that would take the agent off the grid leave its position unchanged, but also result in a reward of .

Other actions result in a reward of 0, except those that move the agent out of the special states and . From state , all four actions yield a reward of and take the agent to . From state , all actions yield a reward of and take the agent to .

A black background with a black square

Description automatically generated with medium confidence

Figure 1: Gridworld Example

Suppose the agent selects all four actions with equal probability in all states. Figure 2 below shows the value function, , for this policy, for the discounted reward case of .



Figure 2: State-value function for Gridworld

Now let us denote with and we set .

Then represents the state of the position in the grid. Thus .

We use the derived in the Appendix expression for (A.18)-(A.24):

(A.18)

(A.19)

(A.20)

For convenience we abbreviate:

, , (A.21)

Using (A.16)-(A.21) in (3) leads to :

(A.22)

(A.22) represents a linear system of equations with respect to the unknowns .

Let us denote with the column vector of the unknowns .

We denote with the matrix formed by the elements and the vector as shown below:

, (A.23)

Then (A.22) in matrix form:

(A.24)

Let us assume random (equiprobable) policy

## Appendix

### Notation and Definitions from Sutton and Barto’s RL book

**Distribution of the Dynamics of the MDP**: defined through the following 4 arguments function:

which is the probability to get from state to state with action and with reward .

**Distribution of the state-transition probabilities**: defined through the following 3 arguments function:

**Markov Decision Process** (abbrev *MDP*): a 5-tuple with

* is a set of states (finite or infinite, discrete, or continuous)
* is a set of actions (finite or infinite, discrete, or continuous)
* is the function which describes the MDP dynamics i.e. probability to get from state to state with action and with reward .
* defines a *reward function*
* is the discount factor which determines to what extent the focus is on the most recent rewards. with there is no focus on the most recent rewards only.

Note: There is another equivalent definition of Markov process which uses the *state-transition probabilities distribution* represented by the three-argument function . With this definition the Markov Decision Process is defined as a 5-tuple , where:

* is a set of states (finite or infinite, discrete, or continuous)
* is a set of actions (finite or infinite, discrete, or continuous)
* is the distribution of the *state-transition probabilities* i.e. the probability to get from state to state with action .
* defines a *reward function*
* is the discount factor

Note 2: A more detailed definition of MDP involves specifying the initial state distribution and augments either of the MDP definitions as:

Markov Decision Process with specified *initial state* is a 6-tuple , where:

* is a set of states (finite or infinite, discrete, or continuous)
* is a set of actions (finite or infinite, discrete, or continuous)
* is the probability to get from state to state with action .
* defines a *reward function*
* defines the initial state distribution
* is the discount factor

**Learning Policy** (or just *Policy*): function which represents mapping from states to probabilities of selecting each possible action.

If the agent is following policy at time , then is the probability that if . Note that is an ordinary function which defines a probability distribution over for each .

We would like to modify the policy with training or experience.

### State-Value and State-Action Functions

Let us assume that the current state is , and actions are selected according to a stochastic policy . Then we would like to derive an expression for the expectation of in terms of and .

Recall, the function defines the dynamics of the MDP and is given as:

for all (A.1)

Then we can write:

(A.2)

Here denotes the reward of going from state to state taking action is given by MDP’s function: .

**State-Value Function for Policy** (or simply *Value function;* aka *function*): the value function of a state under a policy , denoted with , is the expected return when starting in and following thereafter. For MDPs, we can define formally by

for all (A.3)

where denotes the expected value of a random variable given that the agent follows policy , and is any time step. Note that the value of the terminal state, if any, is always zero.

**Action-Value Function for Policy** (aka *function*):

We define the value of taking action in state under a policy , denoted , as expected return starting from , taking the action , and thereafter following policy :

for all and (A.4)

Let us express in terms of and . Given a state s, the state value function , given with (A.3), is equal to the expected cumulative return from that state given a distribution of actions . The action value function is the expectation of the return given state , and taking action as a starting point, and following policy thereafter. Therefore, given a state the action-value function is the weighted sum of the action-values over all relevant actions weighted by the policy weight:

(A.5)

Given a state and an action let us express the action-value function in terms of the state value function and the function defining the MDP dynamics . Recall, given a state and an action , the action value function is given by the mathematical expectation of the discounted future rewards i.e. return . The return is the discounted sequence of rewards after the time step and it can be written as:

(A.6)

It is important to recognize that

. (A.7)

The first term on the right-hand side of (A.7) can be expressed as:

. (A.8)

As before, denotes the reward of going from state to state taking action is given by MDP’s function: .

The expectation in the second term on the right-hand side of (A.8) can be expressed as:

. (A.9)

This is the expectation of the return starting at the next time step following the policy given the current state and the action , chosen according to .

Substituting (A.8) and (A.9) into (A.7) gives us:

. (A.10)

Thus, the action-value function given state s and action following policy is expressed as the sum of the next reward and discounted state-value weighted by probability distribution over the possible next states and next rewards from the given action and state .

### Bellman’s Equations for State-Value and State-Action Functions

#### Bellman’s equation for state values

The value functions satisfy recursive relationships, and this property of the former will prove quite useful.

For any policy and for any state , the following consistency condition holds between the value of and the value of its successor states. Starting with (A.6) applied to the definition of :

(A.11)

Using (5) the last equation becomes:

for all (A.12)

where it is implicit that the actions, , are taken from the set , that the next states, , are taken from the set , and that the rewards, , are taken from the set . Here the reward of going from state to state taking action is given by MDP’s function: . Note that the right-hand side of (A.12) is interpreted as an expected value obtained as a sum over the values of the triplet , , and . For each triplet the quantity is weighed by its probability, .

Eq. (A.12) is known as the *Bellman equation* for . It expresses a relationship between the value of a state and the values of its successor states.

Figure A.1: Backup diagram for

This relationship is expressed by the *Backup diagram* shown on Figure A.1. Each open circle, which will be denoted as *reward-state node* so forth, colored in red represents a state and the reward, which is associated with this state For instance, the root node shown on Figure 1 has associated reward and state . Each solid circle, colored in blue represents a state-action pair and will be denoted as *state-action node* so forth. The specific state on the rightmost state-action node is shown as . Each directed blue edge connects state node with state-action node and represents application of the policy to the root reward-state node . Each directed red edge emanating from a state-action node ends in a possible reward-state node corresponding to specific probable pair of reward and new state . Thus, each directed red edge represents the application of the function of the MDP dynamics. The Bellman equation (A.12) averages over all of the possibilities weighing each possibility represented by a path from the root of the Backup diagram on Figure A.1 to a leaf by its probability of occurring. It states that the value of the start state must equal the discounted value of the expected next state plus the reward expected along the way. The value function is the unique solution to its Bellman equation. Various methods exist to compute exactly, approximate, or learn the value function.

#### Derivation of action value system of equations for discrete finite state

Let us assume that we are dealing with *discrete finite state* – that is we have . Here we have denoted .

Then from (A.3) we have:

for all (A.13)

We also use (A.12) rewritten as:

for all (A.14)

hence

(A.15)

The left-hand side of (A2) can be rewritten as:

(A.16)

The right-hand side of (A.15) are rearranged as :

(A.17)

we denote with and the following expressions:

(A.18)

(A.19)

(A.20)

For convenience we abbreviate:

, , (A.21)

Using (A.16)-(A.21) in (3) leads to :

(A.22)

(A.22) represents a linear system of equations with respect to the unknowns .

Let us denote with the column vector of the unknowns .

We denote with the matrix formed by the elements and the vector as shown below:

, (A.23)

Then (A.22) in matrix form:

(A.24)

#### Bellman’s equation for state-action values

Let us derive a similar recursive relation with respect to the state-action value function. That is, we will find out what is the relation between the action value and that for the possible successors to the state-action pair - . The derivation follows from the Backup diagram shown on Figure A.2 below.

Figure A.2: Backup diagram for

From (A.7) we can write:

(A.25)

The expectation of the reward on the right-hand side can be rewritten as:

(A.26)

Here using (A.7) again we denote with the expression for .

Thus we get the Bellman’s equation with respect :

(A.27)

#### Expressing the current state values in terms of the next action values

It is instructive to compare Eq (A.5) which we derived earlier with Eq (A.12) and Eq (A.27).

Eq (A.5) deserves its own Backup diagram shown on Figure 3:

(A.5)

Figure A.3: Backup diagram for relation between and

Eq (A.5) tells us how the value of a state depends on the values of the actions possible in that state and on how likely each action is to be taken under the current policy. The state value which corresponds to the state-value node at the root is obviously and the action values which corresponds to its children are . The probability with which each action is taken is given by the policy i.e. .

#### Expressing the current action values in terms of the next state values

The value of an action, , depends on the expected next reward and the expected sum of the remaining rewards. Expressed as a Backup diagram we arrive at Figure A.4 shown below.

Figure A.4: Backup diagram expressing the dependence of the current action value on the expected next reward-state values.

Formally expressed this relation becomes:

From Eq (A.7) we have

Clearly, where

Eq. (A.9) states that the expectation in the second term of (A.7) can be written as:

Combining the last two results we obtain:

(A.28)

## Bibliography

[Reinforcement Learning, Richard S. Sutton, Andrew G. Barto, second edition, 2020](https://github.com/dimitarpg13/reinforcement_learning_and_game_theory/blob/main/books/ReinforcementLearningSuttonSecondEdition2020.pdf)