Game Theory Notes

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# Notation

– partition set of the game’s decision tree vertices into information sets

– information set

– vertex which belongs to specific information set

# The Structure of the Game

In terms of moves, the rules of a game have the following structure. For the first move, the rules specify whether it is to be a personal move or a chance move. If it is a *personal move*, the rules list the available alternatives and specify which player is to make the choice; if it is a *chance move*, the rules list available alternatives and specify the probabilities with which they are to be selected. For moves after the first, say the th move with , the rules specify, as *a function of the choices and outcomes at the first moves* (a) whether the th move is to be a personal move or a chance move, (b) if a chance move, the alternatives and their probabilities of selection, and (c) if a personal move, the alternatives, the player who is to make the choice, and the information concerning the choices and outcomes at the first moves that is given to him before he makes his choice. Finally, the rules specify, as a function of the choices and outcomes at the successive moves, when the game shall terminate and the score, not necessarily numerical, that is to be assigned to each player.

Notes:

( 1 ) For , the characteristics of the th move depend on the results of previous moves.

( 2 ) the information given to a player when he is to make a certain move does not necessarily include the information that was given to him at earlier move

( 3 ) The number of moves occurring in an actual play is not fixed in advance but depends on the successive outcomes and choices. The rules must guarantee that the game will terminate eventually.

( 4 ) The concept of move is to some extent non-unique / non-determinate.

Graphical representation of a game can be given in terms of a *game’s decision tree* as illustrated on Figure 1.

In this figure the vertices represent various positions for the players as the result of the outcomes of sequence of moves. No vertex is occupied by more than one player. Play begins at a specified vertex and each move is a change in position from a vertex to one of its children. When a leaf is reached the play of the game is over. Each non-terminal vertex has a label (in Figure 1 the labels and are for players I and II respectively), specifying who is to move if that vertex is reached, and the number of alternatives is simply the number of branches at the vertex.

The information available to the players is specified by a partition of the vertices into sets , called *information sets*, such that, when a vertex is reached, the player who is to move is told only the set , and not, unless contains only a single element, the exact position of . Certain obvious requirements are imposed on the information partition :

1 ) All must have the same mover and the same number of alternatives

2 ) if and is attainable from then .

Decision Tree of a Game

Figure 1: Decision Tree of a Game

**Definition**: *Game of perfect information*

We have a *Game of perfect information* when *every* information set of a game contains only one element.

# The Concept of a Strategy

Consider a game of chess. A complete set of instructions on how the Whites play should react on each move by the Blacks is informally understood as a *strategy*. Thus, the strategy of the Whites must specify the first move and, for each possible reply by the Blacks, a corresponding next move. In general, for each possible sequence of choices , a choice for his move (the move of the game) among the alternatives available to him as a result of the situation .

The decisions which totality constitutes a strategy are conditional ones in the form:

“If a situation should arise in which I am informed that it is my move, and I am given information about the results from previous moves (both chance and personal) and I am told that the set of alternatives open to me is and my choice would be ”. With the understanding that the information includes a specification of , and an information that it is the receiving player’s turn to move we can define the player’s strategy as i) a listing of all possible and ii) an assignment to each . Here denotes the space of possible ’s. In terms of the game’s decision tree, is the space of all information sets with the player’s label. is the set of alternatives specified by , that is the set of branches leading out of a vertex of the particular information set, the player’s strategy is a function such that for all . That is, is a rule that specifies, for each information set, which branch is selected. The space of all possible strategies for a given player will be designated by .

The number of elements in is given with where is the number of information sets with the player’s label and is the number of alternatives for the th information set.

Example:

Let and be the spaces for White and Black in chess and let and be any strategies for White and Black respectively. A referee, given and , could conduct the entire game without further instructions from either player and announce the result. The information that no previous moves have been made, that it is now White’s turn, and that consists of the 20 legal opening moves for White is a point in , and thus is White’s opening move. The information that White’s opening move was , that no other moves have been made by either player, that it is now Black’s turn to move, and that consists of the 20 legal moves for Black at this stage will be an element of , so that will be Black’s reply, etc. Thus, the outcome of the game is determined uniquely by the strategies and chosen by the players.

The fact noted above for chess, that the outcome of the game is determined once each player has selected a strategy, will clearly continue to hold for any game in which each move is a personal move by some player, but will fail for games where chance moves occur and influence the outcome.

Chance Move Simplification

Let be the set of all possible circumstances in which a chance move would be required. Each includes specification of the nature of the chance move to be performed, i.e. of the set of alternatives among which one is to be selected, and for each of the probability distribution over according to which the selection of is to be made.

# The Normal Form

The description of a game can be viewed as a listing of the strategies of the players and the outcome of any set of choices of strategies, without regard to the attitudes of the players toward various outcomes. We now indicate how the final simplification of the game – the normal form – is obtained, by taking into account the preferences of the players.

The result of any set of strategies is a probability distribution over the set of possible outcomes. It would be particularly convenient if a given player could express his/her preference pattern in by a bounded numerical function defined on , such that he or she prefers to *iff* . Note that denotes indifference between to . Also, the function is such that if for any probability distgribution over we define as the expected value of computed with respect to as

the player prefers to *iff* .

It is remarkable fact that, under extremely plausible hypothesis concerning the preference pattern such function exists.

***Definition*** (*utility function*): The function defined for all probability distributions over , is called the player’s ***utility function***.

is unique, for a given preference pattern up to a linear transformation. We will assume that each player has such utility function.

The aim of each player in the game is to maximize his/her expected utility. If is the utility function of player , his/her aim is to make as large as possible where is the probability distribution for fixed over determined by the overall chance move.

We are in a position to give a description of the normal form of a game:

***Definition*** (*normal form of a game*): A game consists of spaces and bounded numerical functions defined on the space of all -tuples , . The game is played as follows: Player chooses an element of , the choices being made simultaneously and independently; player then receives the amount . The aim of Player is to make as large as possible. The statement “Player receives the amount ” is shorthand of saying “a situation results whose utility for Player is ”.

***Example*** (*two player game involving coin-toss and a number choice*):

Player moves first and selects one of the two integers . The referee then tosses a coin and if the outcome is “head”, he informs player of player ’s choice and not otherwise. Player then moves and selects one of two integers . The fourth move is again a chance move by the referee and consists of selecting one of three integers with respective probabilities . The numbers selected in the first, third and the fourth move are added, and the amount of dollars is paid by to if the sum is even and by to if the sum is odd. Note that .

Here are the two strategy spaces:

; ,

; ,

Here the first position of the triple is conditioned upon coin falling *Head* and player choosing 1, the second position in the triple is conditioned upon coin falling head and player choosing 2, and the third position of the triple is conditioned upon coin falling *Tail*.

The set of possible outcomes for this game where I denotes player , 0 denotes the referee and II denotes player is shown below:

I –> 2 – 0 –> Head – II –> 4 – 0 –> 3 = 9, probability , strategies

I –> 2 – 0 –> Head – II –> 4 – 0 –> 2 = 8, probability , strategies

I –> 2 – 0 –> Head – II –> 4 – 0 –> 1 = 7, probability , strategies

I –> 2 – 0 –> Tail – II –> 4 – 0 –> 3 = 9, probability , strategies

I –> 2 – 0 –> Tail – II –> 4 – 0 –> 2 = 8, probability , strategies

I –> 2 – 0 –> Tail – II –> 4 – 0 –> 1 = 7, probability , strategies

I –> 2 – 0 –> Head – II –> 3 – 0 –> 3 = 8, probability , strategies

I –> 2 – 0 –> Head – II –> 3 – 0 –> 2 = 7, probability , strategies

I –> 2 – 0 –> Head – II –> 3 – 0 –> 1 = 6, probability , strategies

I –> 2 – 0 –> Tail – II –> 3 – 0 –> 3 = 8, probability , strategies

I –> 2 – 0 –> Tail – II –> 3 – 0 –> 2 = 7, probability , strategies

I –> 2 – 0 –> Tail – II –> 3 – 0 –> 1 = 6, probability , strategies

I –> 1 – 0 –> Head – II –> 4 – 0 –> 3 = 9, probability , strategies

I –> 1 – 0 –> Head – II –> 4 – 0 –> 2 = 8, probability , strategies

I –> 1 – 0 –> Head – II –> 4 – 0 –> 1 = 7, probability , strategies

I –> 1 – 0 –> Tail – II –> 4 – 0 –> 3 = 9, probability , strategies

I –> 1 – 0 –> Tail – II –> 4 – 0 –> 2 = 8, probability , strategies

I –> 1 – 0 –> Tail – II –> 4 – 0 –> 1 = 7, probability strategies

I –> 1 – 0 –> Head – II –> 3 – 0 –> 3 = 8, probability strategies

I –> 1 – 0 –> Head – II –> 3 – 0 –> 2 = 7, probability strategies

I –> 1 – 0 –> Head – II –> 3 – 0 –> 1 = 6, probability strategies

I –> 1 – 0 –> Tail – II –> 3 – 0 –> 3 = 8, probability , strategies

I –> 1 – 0 –> Tail – II –> 3 – 0 –> 2 = 7, probability , strategies

I –> 1 – 0 –> Tail – II –> 3 – 0 –> 1 = 6, probability , strategies

In the theory of games it is usual to treat first a special class of games, *the two-person zero-sum games*. The theory of these games is particularly simple and complete and we will consider only such games in our discussion.

***Definition*** (*two-person game*): a game with : we have only two utility functions and and two strategy sets and for each of the two players.

***Definition*** (*zero-sum game*): A game for which the following holds true:

for all

More precisely, since each is unique up to a linear transformation, a game is a **zero-sum** if there is a determination of for which for all . Thus a two-person zero-sum game is a game between two players in which their interests are diametrically opposed: one player gains at the expense of the other. Consequently, there is no motive for collusion between the players. It is precisely the fact that collusion is unprofitable that simplifies the theory.

***Definition*** (*constant-sum game*): A **constant-sum game** i.e. one in which for all is zero-sum game in the sense defined above, since an alternative choice of utility functions is , for , and . Thus the theory developed for zero sum two person games applies for constant sum two person games.

Since for two-person zero sum game we have we need to specify only . We will consider only two-person zero-sum games from now on.

***Definition*** (*game in a normal form*): A **game in a normal form** is a triple , where are arbitrary spaces and is a bounded numerical function defined on the product space of pairs . The points are called strategies for player and the function is called payoff. The game G is played as follows: chooses , II chooses , the choices being made independently and simultaneously. then pays the amount .

# Equivalent Games

If, in a given game, one relabels the strategies of either player, the new game is essentially not different than the old. Every statement about either game can be translated into a corresponding statement about the other and we wish to consider the two games equivalent.

Another simple transformation which does not alter the essential character of the game is the deletion of duplicate strategies. In other words, if a player has two strategies such that for all , the deletion of from is an inessential change in the game, even though it might, for example destroy such properties as symmetry.

***Definition*** (*reduction of game*): Let and be two games. Then is a reduction of , written , if either:

1. , and there is a function from onto such that for all , , or
2. , and there is a function from onto such that

for all ,

If is a transformation, is obtained from by relabeling of strategies; if is not , is obtained from by deletion of certain duplicated strategies and relabeling.

***Definition*** (*equivalent games*): Two games and are called **equivalent**, written , *iff* there is a finite sequence of games with , and for each either or .

***Example*** (*equivalent games*): Let be a game, where , a set of real numbers, , also a set of real numbers; and for ,

M(x,y)=

//TODO: finish this

# Illustrative Examples

Game : *Matching Pennies*

Players and simultaneously place coins on the table. If the coins agree, i.e. both show heads or both tails pays I one unit. If not pays one unit.

Clearly each player player has two strategies – *heads* and *tails*. The game is equivalent to one with the matrix .

Game : *Matching Pennies with Spying*

This game is like *Matching Pennies*, except that is required to place his coin first, and is permitted to see the result before placing his own coin. still has two strategies – *heads* and *tails*. A strategy for specifies his choice when he sees heads and his choice when he sees tails, so that has four strategies. Denoting heads by , tails by , and by the strategy that chooses when chooses , and when chooses , we obtain the matrix:

Game : *Matching Pennies with Imperfect Spying*

After makes his choice, a coin is tossed that has probability of showing ’s choice and of showing the opposite. The result of the toss is revealed to , who then makes his choice. Again, has two strategies and has four; the matrix is:

where now denotes the strategy “ chooses when is announced and when is announced”.

Game : *Addition*

and alternatively choose integers, each choice being one of the integers and each choice made with the knowledge of all preceding choices. As soon as the sum of the chosen integers exceeds , the last player to choose pays his opponent one unit.

The situation at which player finds himself at his th move is described by a sequence with each being one of the integers and

Denote by the set of possible sequences where and denotes the closest integer which does not exceed . A strategy for consists of a set of functions , where is a function defined on assuming only values : specifies ’s th move when the previous history of the play is . Similarly, a strategy for is a set of functions , where is defined for the set of all sequences with each being one of the integers and

Define and inductively for ,

(this induction describes the manner in which a referee would carry out the instructions of the players) and let be the largest for which is defined. Then

# Lower and Upper Pure Value

In a game , the consequences of strategy are described by the function . Using , player is certain to receive at least

and cannot be certain of any definite larger amount. Thus, the number

is the upper limit to the amount can guarantee getting: for every , the player can, simply by choosing a suitable , be certain of , and for no is there an which makes the player certain to receive at least against all . Similarly, we define

,

by selecting a suitably, player can with certainty restrict his/her loss to but not to for any . For subsequent reference these statements are stated formally:

***Definition*** (*Capital lambda of as game infimum*): If is a game, then, for , .

***Definition*** (*Capital upsilon of as game supremum*): If is a game, then, for ,

***Definition*** (*Lower pure value as game supremum of infimum*): If is a game, then the lower pure value of is the number

***Definition*** (*Upper pure value as game infimum of supremum*): If is a game, then the upper pure value of is the number

***Theorem*** (*Inequality between pure lower value and pure upper value*): If is a game, then, for and ,

and

*Proof*: . Thus for all and .

Consider now any game . No method of play for can guarantee him more than since can restrict his loss to and no method of play for can with certainty reduce his loss below since can guarantee this amount. Thus, if no method of play can guarantee either player any improvement over and we have seen that each player can attain (more precisely, approximate as closely as the player wishes). Thus, for such games, choosing an with is an unimprovable method of play for in the sense that no method of play can guarantee more, and similarly for . This situation leads to the following definitions

***Definition*** (*pure value*): If is a game and if then the number is called the pure value of .

***Definition*** (*optimal* *strategy of game with pure value*): If is a game and if is the pure value of , then a good strategy for player in is any with and good strategy for player in is any with .

***Theorem*** (*pure upper and lower value of equivalent games*): If two games and are equivalent, then and .

*Proof*: It is sufficient to prove the theorem in the special case where one of the games is a reduction of the other. Suppose for definiteness that is reduction of and that is a function mapping onto . Since, for all and all ,

we have

Hence, for all ,

so that

The proof that is similar.

***Problem*** (*opposite player strategies yielding constant return*): If there are strategies , such that for all , for all , then .

*Solution*:

We have . Similarly, . Hence where does not depend neither on nor on . Therefore is the game pure value.

***Problem*** (*opposite player strategies and a number between them*): If there are strategies , and a number such that for all , then and , are good strategies for *I, II*.

# Perfect Information Games

Among the games that do have a pure value are the *perfect information games* of which chess, checkers and tic-tac-toe are examples.

Essentially, a game of perfect information is one that can be described in terms of successive moves in such a way that, at each personal move, the mover knows the choices and the outcomes of all preceding personal and chance moves. Perfect information game is a game in which every information set is a unit set. It is intuitively clear that this condition is equivalent to the requirement that every branch of the tree of the game also be a tree of some game. The latter condition leads to an inductive definition for games in normal form. In this definition the order of a perfect information game intuitively corresponds to the maximum number of moves in that game.

***Definition*** (*perfect information game*): A game is a perfect information game of order *iff* is constant. A Game is a perfect information game of order *iff* there is a set and a class of games for , such that each is a perfect information game of order , and such that either:

*Case 1.* consists of all pairs with , , consists of all functions y defined on A with for all , and

*or*

*Case 2.* consists of all pairs with , , consists of all functions defined on with for all , and

*or*

*Case 3.* consist of all functions defined on with , for all , and

where

A game is called perfect information game if is perfect information game of order for some .

Our inductive description corresponds to the fact that the result of the first move in perfect information game with moves is another perfect information game with moves, so that the first move can be considered as a choice of one of a given collection of perfect information games with moves, with the three possible cases corresponding to the cases in which the first move – the choice of – is a personal move of , a personal move of or a chance move. For any perfect information game of order , the first moves of may be considered as a game of perfect information, whose outcome is not a number, but a game of perfect information of order , so that the first moves of may be regarded as a struggle to determine which game with moves shall be played.

***Theorem 1.7.1*** Every perfect information game has a pure value. Moreover, if is a perfect information game of order n+1, and is the class of perfect information games , as required by the last Definition, then corresponding to the three cases for in the aforementioned Definition the pure value of is given by either:

*Case 1*. or

*Case 2*. or

*Case 3*.

where is the pure value of . In addition, if there is a such that and for every there are good strategies in , then good strategies exist in and corresponding to the three cases in the last Definition are given by either:

*Case 1*. , for all

*Case 2*. , for all

Case 3. , for all

Proof: The theorem is obvious for and we suppose that the theorem holds true for all perfect information games of order less than .

*Case 1*. Let . For any , choose and such that

Then for all ,



and for all ,



Thus has a pure value .

Furthermore, if the supremum of is attained by some and there are good strategies , for every , then, since equations (1) and (2) are valid for all , the choices are good strategies in . The proofs for Cases 2 and 3 are similar.

The inductive description of the value and good strategies in perfect information games can be used to solve such games. We illustrate the method to solve the game Addition

//TODO: finish the discussion on games normal form

References

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