# Root Cause Analysis for Fulfillment Decisions

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Table of Contents

[Root Cause Analysis for Fulfillment Decisions 1](#_Toc144150987)

[Preliminaries 1](#_Toc144150988)

[Notation 1](#_Toc144150989)

[Assumptions 3](#_Toc144150990)

[Events 3](#_Toc144150991)

[Event Relationships 4](#_Toc144150992)

[Directed Follow Graphs 6](#_Toc144150993)

[Causal association between events 10](#_Toc144150994)

[Directed Causal Graphs 12](#_Toc144150995)

[Problem Statement for Root Cause Analysis of Fulfillment Decisions 12](#_Toc144150996)

[Algorithm For Root Cause Analysis 12](#_Toc144150997)

[Appendix A: Probabilistic Temporal Logic 13](#_Toc144150998)

[Definitions and Review on Probabilistic Temporal Logic in (Kleinberg, Causality, Probability, and Time, 2012) 13](#_Toc144150999)

[Examples of PTL 14](#_Toc144151000)

[Definitions and Review on Probabilistic Real Time Computation Tree Logic (PCTL) in (Hansson & Jonsson, 1994) 17](#_Toc144151001)

[Bibliography 21](#_Toc144151002)

[Downloadable Links for the Bibliography 21](#_Toc144151003)

## Preliminaries

Before we can formulate the problem statement and the algorithm providing a solution, we need to start with a set of notational conventions and definitions.

### Notation

- with capital Latin letters we will denote *scalar quantities* which are either essential algorithm parameters or constants which will not change during the algorithm execution; for example, *number of feasible nodes for the current bundle* (scalar constant) will be denoted with and *inventory for given SKU on given node* (algorithm parameter) will be denoted with . Graphs will also be denoted with capital Latin letters for historical reasons.

– with small Latin letters we will denote *variable/unknown (integral or not) quantities*, not necessarily scalar. For example, with we can denote the number of order-lines fulfilled at a given node.

– with small Greek letters we will denote *variable/unknown (integral or not) quantities*, not necessarily scalar.

– with capital Script letters we will denote a *set* (ordered or unordered) of quantities of the same type; for example, with we will denote the set of SKUs in some bundle of some order

– with capital Greek letters we will denote a *concept*, *logical statement* or a *logical expression* of *logical terms / statements* which is adorned with *semantic meaning*. In case of a logical statement, the latter can be either true or false depending on the context. The capital Epsilon letter will be reserved to denote an event type or event of interest. For instance, will denote the event of type “*an order has been received*”.

- with capital Fraktur letters we will denote a *map* over several arguments where at least one of those arguments is of type logical expression, a logical statement or a set of logical statements. For example, denotes graph representation of the concept by the set of events .

- with double struck Latin capital letters we will denote standard number sets. For example

- the set of complex numbers

- the set of natural numbers

- the set of the real numbers

- the set of integer numbers

Reserved letters for quantities, sets and concepts:

– number of bundles in the order .

– order received at moment .

– the -th bundle of order ; alternatively, denoted as .

or - the set of SKUs for the -th bundle will be denoted with .

- denotes some quantity related to the -th bundle of the -th order.

– denotes some quantity related to the SKU at node e.g., inventory for SKU at node .

- directed graph

- the vertex set of the directed graph

– the arc set of the directed graph

– denotes the *relative frequency of occurrence* of the event given event with the dataset

– denotes *Average Degree Of Causal Significance* (*ADCS*) of event for event given the background contexts

denotes the statement that event *follows in time* event

denotes the statement that event *generally follows* event (either in time or via static association)

denotes the statement that event *precedes in time* event

denotes the statement that event is *generally reachable from* event (either in time or via static association)

denotes the statement that event is *reachable in time* from event

– denotes graph representation of the concept

- denotes complete representation of the concept with the event set

– denotes static dependency map

– denotes static association map

Reserved symbols for relations and operations

- denotes logical conjunction

- denotes logical disjunction

- denotes logical negation

- denotes *follows in time* relation between two concepts

- denotes *generally follows* relation between two concepts

- denotes *is reachable in time* relation between two concepts

- denotes *generally reachable* relation between two concepts

- denotes *static dependency* between two concepts

- denotes *dynamic dependency* between two concepts

- denotes *association (static, dynamic)* between two concepts

– denotes -*spurious cause* relation between two concepts

- denotes *causal association* between two concepts

- *prima facie* causal relation between two concepts

- denotes Eells causal relation between two concepts

- denotes *matching* between directed follow graph (DFG) and a concept

### Assumptions

All orders can be ordered in an increasing sequence of moments in time . That is, we assume that no two orders will arrive at the same moment in time. Thus, the time will take the form of a discrete variable on the natural numbers i.e., . Therefore, any order will be uniquely identified by a subscript .

**Definition**: *Atomic Proposition*

A basic proposition (or *atom*) which cannot be represented as a set of other atoms connected using conjunction , disjunction , negation , implication and equivalence .

### Events

**Definition**: *Event*

The word *Event* will be used to denote a *specific kind* *of* an *event* which is relevant for the causal analysis. *Event* can be viewed as *a template* from which a specific event can be *instantiated*. We will denote each event with capital Greek letter. Where it will be clear from the context, we will use interchangeably the word “*event*” to denote either *Event of specific kind* or an *Event instance*.

**Definition**: *Parameters of Event*

Each event has a *set of parameters* which will be denoted with . The set of parameters of an event together with the semantic description of the event uniquely identify the event. One can think of the semantic description as sort of *“semantic” template* (or *predicate* from some first order logic) identifying this event type. The template parameters will be given obviously with the parameter set which is an ordered set. Thus, each event is defined with the pair . An Event Instance additionally to and is given a specific value for each . We will denote the value space of an Event Instance with . Thus, an event instance is defined with the triplet .

**Note**: Every event additionally to its standard parameter set will have an implicit timestamp parameter which will always be present without regard of the nature of the event. We will not include explicitly the timestamp among the event parameters unless it is necessary in order to define uniquely the event instance.

We define the following *Events* which are relevant for the analysis of Fulfillment decisions causing splits:

*Set of events for analysis of the cause of splits in Fulfillment Decisions*

- order is received. Event parameters:

- the -th bundle of order is being processed. Event parameters: ,

- SKU in the -th bundle of order is being processed. Event parameters: , ,

- node has sufficient inventory for SKU in with order . Event parameters: , , ,

- node has sufficient capacity for SKU in with order . Event parameters: , , ,

- node is shipping eligible for SKU in with order . Event parameters: , , ,

- node is deprioritized; node has SKU in with order . Event parameters: , , ,

- node is turned on; node has SKU in with order . Event parameters: , , ,

- node is turned off; node has SKU in with order . Event parameters: , , ,

- node is soft capacity; node has SKU in with order . Event parameters: , , ,

- service level for node is overridden; node has SKU in with order . Event parameters: , , ,

- carrier for node j is overridden; node j has SKU in with order . Event parameters: , , ,

- backlog days for node is overridden; node has SKU in with order . Event parameters: , , ,

- node is with depleted inventory; node has SKU in with order . Event parameters: , , ,

- node is with depleted capacity; node has SKU in with order . Event parameters: , , ,

- node contains an orphan for SKU which is in with order . Event parameters: , , ,

– splitting decision is made; that is, at least for one bundle the SKUs in are fulfilled by more than one node with order . Event parameters: ,,

We will visualize an event with an ellipse and a Capital Greek letter denoting the event. For example:

We will visualize an Event instance with an ellipsis and will use a Capital Greek letter to denote the specific kind of event which it is an instance of. We will attach a set of labels where each label will represent an *atomic proposition* with pertinent semantic information for this instance. For example, in case of we can have:

Thus, the parameter set of is given with . Since the value space for the parameters of the event instances of is given with .

Let us consider the event : . Since the value space for the parameters of the event instances of is given with .

For we have accordingly and .

For we have accordingly and . Here

For

//TODO: finish this

### Event Relationships

**Definition**: Event *follows in time* Event

We say that event *follows in time* (denoted with ) if event has occurred in time *after* event and there does not exist a third event which occurs *after* and *before* .

The *follows in time* relation implies the timestamp associated with is newer than that associated with i.e. . See the **Note** on the event parameters.

The *follows in time* relation is a strong partial order which satisfies the conditions:

* Irreflexivity:
* Asymmetry: if then
* Transitivity: if and then

Note that the *follows in time* relation is not guaranteed to hold for every pair of event instances – that is, if and are event instances then it can be true that both and . This would be the case if and have the same timestamp or at least one of them is does not have timestamp specified in its parameter set.

**Definition**: Static (*semantic) dependency between events* and

We say that event is static dependent on (denoted with ) if each instance of can exists only *in the context* of some instance of for *any* order data set . That is, removing an instance of in will remove all instances of underneath from the event tree for any chosen .

if there is a static dependency then we can define a map (called *static dependency map*) such that when .

Example of static (semantic) dependency-

Let the event denote the statement that an order with two bundles was received. Let the event denotes the statement that the current bundle being processed is the second bundle for order . Then for the order .

Example of absence of static (semantic) dependency

Let the event denote the statement that an order with two bundles was received.

Let the event denotes the statement that a splitting decision for both bundles in order is made; that is, the SKUs in both bundles are fulfilled by more than one node with order .

Let the event denotes the statement that node contains an orphan for SKU which is in with order .

Then we can write ; that is, the splitting decision for order can be reached only after order has been received. However, there is an absence of static dependency between and . The relevant for instance of event could have been received in an earlier time than that of order ; that is, . Removing the instance of corresponding to order has no impact on the existence of .

**Definition**: Static (semantic) descendant

We say that the event is a static descendant of another event if there exist a chain of events such that:

for some or if .

**Lemma**: *Static (semantic) dependency* defines a directed follow graph of statically associated events

Refer to the DFG shown for **Example 1** as an illustration.

**Definition**: *Dynamic dependency* betweenevents and

We say that event is dynamic dependent on (denoted with ) if all of the following is true:

* removing an instance of *may* trigger the removal of an event which is static dependent of or removal of itself.

//TODO: Finish this

Example of dynamic dependency-

In the previous Example of absence of static (semantic) dependency we considered three events - (order has been received), (node contains an orphan for SKU in the order), (split fulfillment decision has been made). Clearly, there is some kind of dependency between event and as triggering of event at a time earlier than the time the split decision is made can potentially impact the instance. Clearly, this dependency is not static as event at a time later than the time the split decision is made hence this is not a static dependency. Thus, the relation between and matches the definition of dynamic dependency.

**Definition**: Event is *associated (statically, dynamically*) with Event

We say that event is *associated* (*statically*)with (denoted with ) if each instance of is (static, dynamic) descendant of some instance of or vice versa. That is, if the

//TODO: Finish this

In order to find how a set of events are associated statically we can define a map (called *static association*) such that when . The construction and depiction of static association map will be discussed in **Example 1** below.

**Definition**: Event *is (generally) reachable from* Event

We say that event *is (generally) reachable from* (denoted with ) if event has occurred in time *after* event or if there is a static dependency between and ***and*** is reachable from . Any event is reachable from itself, i.e., . Formally,

Note that the *(generally) reachable* relation represents *weak partial order* obeying the reflexivity, asymmetry and transitivity conditions.

**Definition**: Event *is reachable in time from* Event

*Is reachable from* relation (denoted with ) implies that the timestamp associated with is newer than or equal to that associated with i.e., . See the **Note** on the event parameters.

Note that the *reachable in time* relation represents *weak partial order* obeying the reflexivity, asymmetry and transitivity conditions.

**Definition**: (*irreflexive) reachable in time* relation (denoted with ) can be defined similarly to its reflexive counterpart postulating that in order an event to be (irreflexively) reachable in time from the corresponding timestamp must be **newer** than the timestamp of the other event i.e. . The *reachable in time* relation represents *strong partial order* obeying the irreflexivity, asymmetry and transitivity conditions.

We will use both definitions in different occasions.

For example, let the event denotes the node being turned off, and event denotes order received at time . Then can be interpreted as “node was turned off prior to receiving order ”.

**Lemma**:  *reachable in time* relation is the transitive closure of the *follows in time* relation.

**Lemma**: Static (semantic) dependency implies *reachable* relation

That is, . Note that the *reachable* relation between and will hold for all pairs of instances of those events.

### Directed Follow Graphs

We use *Directed Follow Graphs* (DFG) to depict order fulfillment scenarios which we are interested to capture.

Each node of the DFG will represent an *Event instance* of interest. We use an arc (or a directed edge) to denote a *follow-in-time* relation between two Event instances and or static (semantic) dependency between and . Each arc is marked with label which indicates what kind of relation it represents. In this document we draw all arcs which represent *follow-in-time* relation with red color and all arcs which represent *static dependency* with blue color.

Each *follow-in-time* arc has a label with a counter which counts how many times the current arc connecting a pair of events has been seen in the data log given specific dataset.

**Definition**: *Labeled Directed Follow Graph (LDFG)*

We extend the concept of Directed Follow Graph (DFG) by introducing a set of labels to each node and to each arc. Each label represents an atomic proposition which is relevant to the specific node or to specific arc.

We use LDFGs to represent the follow relationships between event types and event instances for a given dataset of orders.

Discussion on how DFG is constructed

Let us consider a given order data set and let us assume we have Fulfillment Optimization engine processing the set of orders sequentially thereby generating order metrics events. Let us assume we have a parsing engine which combs through the order metrics created after the Fulfillment Optimization engine run. This parsing engine parses the events which it is configured to recognize and assembles the DFG instance based on the parsed events data. Let us denote with the events which the parsing engine is configured to recognize. Per our definition of *Parameters of Event* given earlier each event type is represented by the pair where is the template of the event which together with the parameter set uniquely identifies this type of event. Let us denote with the ordered set of values which correspond to each parameter for all instances of generated using . Since each event instance has a timestamp, we can construct DFG from the parsed events. Each *follow-in-time* arc between two event types and will be labeled with the final count showing how many times this pair of events have been seen in a *follow-in-time* relation .

For example:

**Example 1**:

An order for has 3 bundles. The first bundle has three SKUs – , , , the second bundle has two SKUs – and , the third bundle has 4 SKUs – , , ,.

Let us denote with the Event that an order with 3 bundles have been received. Also, we will denote with an instance of event type each of the bundles of the order . We denote with an instance of event type each of the SKUs in each bundle of the order . We will assume the following time sequence of the event instances:

We depict this scenario with the following DFG:

#### Discussion on static association map

How can we define the *static association* map ? The answer can be found in the definition of *Parameters of Event* given earlier. We have the parameter spaces of the two events - and and the value sets and of the corresponding event instances. Note that where . Here denotes the cartesian product of the value sets for each parameter of event ; thus, we have:

.

In general, the map should be defined over the cartesian product of the event tuples .

Let us consider this question from the context of our Fulfillment Decision **Example 1**.

Clearly, we expect that the instance and all instances under the parent instance are statically associated. We expect that each instance and all instances which are children of the current instance are statically associated as well. Let us denote the instance in this example with . Let us denote the three instances of with , , and . Then we obviously we have:

(1)

The relation (1) represents the fact that each child is statically associated to its parent and is statically dependent on the parent event, namely an order with specified number of bundles has been received. This relation is depicted with arrow in (1).

Additionally, we can write:

(2)

The relation (2) represents the fact that the children of the same parent are statically associated.

Similarly, we continue with writing the static association relations involving the instances of

(3)

Additionally, all instances of the same parent are statically associated with each other - we write this as:

and (4)

Here denotes the index set of .

~~Also, all instances are associated with their grandparent which is expressed with:~~

~~and (5)~~

~~Let us construct the map for the sets (1)-(5). We start with (1):~~

//TODO: finish this

**Definition**: *Directed Follow Graph Instance* *(DFGI)*

DFGI is a directed graph in which each node is a specific *event instance (or a token)* of an event type and each red arc denotes *follow-in-time* relation . Each blue arc denotes static (semantic) dependency between the event instances. Static association between nodes is shown via a pair of blue arcs each pointing to the opposite node. Each node (event instance/token) is labeled with the value set of parameters for this event type. For example:

**Definition**: *Aggregated Directed Follow Graph* (*ADFG*)

The ADFG corresponding to DFGI can be obtained by replacing each event instance by its corresponding event type and replacing a multi-set of *follow-in-time* arcs leaving an event instance of type and entering event instance of type with a single arc labeled with the corresponding instance count.

**Definition**: *Frequency Count* of pair of events and – this is the number of DFG instances in which directly follows in time i.e., .

**Definition**: *DFG Representation* of a concept over an event set

We say that DFG is a representation of if the graph constructed with the events in models *semantically* the internal structure of .

For example, the DFG shown in *Example 1* is DFG representation of the order . The DFG representing will be denoted with or shortly .

**Definition**: *Complete* *Representation* of a concept over an event set

We say that the DFG is a *complete* *representation* of the concept (e.g., fulfillment order, fulfillment decision) from the event set , denoted with , *iff* there does not exist DFG such that with and .

**Definition**: *Order Fulfillment Decision*

The process of fulfilling the order which can be viewed as a set of events relevant to the decision which was made. The events are pairwise related by the *follow-in-time* () relation which is depicted by red-colored arcs. Additionally, we depict semantic dependencies () using blue arcs. The events are represented by DFG over some set of events .

For example:

**Example 2**

Sixty orders with a single bundle and single SKU with unit quantity have been received. Let us define the following event set : The event “order has been received” will be denoted with . The event “The order bundle is being processed” will be denoted with . The event “SKU is being processed” will be denoted with . The event “node has inventory for SKU ” will be denoted with . The event “node has capacity for SKU ” will be denoted with . The event “Reward for node has been calculated” will be denoted with . The event “Fulfilling node has been chosen” will be denoted with .

This is visualized as:

60

60

40

40

20

20

20

40

60

**Definition**: *DFG* *Matching* of an order fulfillment decision

Let denotes the fulfillment decision of order . Let denotes some DFG. We say that the DFG *matches* (denoted with ) if is a representation of over some set of events .

//TODO: Finish this

### Causal association between events

What does it mean that certain event types can be associated causally with each other? Let us consider two event types - and . Per our definition of *Parameters of Event* given earlier the event is characterized with the pair where is the template of the event which together with the parameter set uniquely identifies this type of event. Similarly, we will consider another event type represented by . Now let us consider a given order data set and let us assume we have Fulfillment Optimization engine processing the set of orders sequentially thereby generating order metrics events. Let us denote with the ordered set of values which correspond to each parameter for all instances of generated using . Similarly, with we denote the ordered set of values which correspond to each parameter and generated for all instances of using order data set . For an instance of we will denote the values of the instance parameters with . Thus, for each instance of in (denoted as ) we have . Similarly, for we have .

**Definition**: *Causal association* *between events* and – Given the dataset we say that and are *associated (causally)* if one of the following is true:

* both and are *causes* of another event in //do we need this?
* both and are *caused* by another event in //do we need this?
* either *causes* or *causes*
* there is a semantic association between and

Note: we denote causal association between the events and with the symbol i.e. .

For example, a *prima facie causal association* implies that all causal relationships in its definition are *prima facie causes* (defined in the paragraph below).

**Definition**: *Conditional probability of an event*

Let us consider the event type . Per our definition of *Parameters of Event* given earlier the event is characterized with the pair where is the template of the event which together with the parameter set uniquely identifies this type of event. Similarly, we will consider another event type represented by . Now let us consider a given order data set and let us assume we have Fulfillment Optimization engine processing the set of orders sequentially thereby generating order metrics events.

Let us run the Fulfillment engine with the given order set and we find that in out of the instances in which event has occurred there has been an instance of *associated with* each instance of .

Then given the data set the relative frequency of occurrences of given is obtained as:

(6)

We say that the relative frequency given is an estimate for the conditional probability i.e.

(7)

**Definition**: Event is *prima facie cause* of Event

Given the data set let us denote with the set of instances of which follow the set of instances of , denoted with. That is, .

We say that event is a *prima facie cause* ofevent (denoted with ) *iff*:

the sets and are non-empty

**and**

(8)

**Lemma**: *Prima facie* cause between event and event implies dynamic dependency between the two events

That is, .

**Definition**: Event is -*spurious cause* of an Event

Let us consider the event type given with its template and parameter space .

Let us consider another event type given with its template and parameter space .

Given the data set we denote with the set of all events with which is associated such that .

Let is an event such that:

* it is not necessarily in :
* is reachable from i.e.

Then we say that is-spuriouscause of an Event *iff*

* over
* over

We denote -spuriouscause with

**Definition**: Event is *Suppe’s* cause of an Event (a.k.a. *Suppe’s causality*)

We define and as in the definition of -*spurious cause*.

Given the data set we denote with the set of all events with which is associated such that .

Let is an event such that:

* it is not necessarily in :
* follows i.e.,

Then we say that Suppe’scause of an Event *iff*

* over (*Suppe*’s causal relation hypothesis)

We denote *Suppe*’scausality relation with

**Definition**: Event is *Eells* cause of an Event (a.k.a. *Eells’ causality*)

Let us consider the event type given with its template and parameter space .

Let us consider another event type given with its template and parameter space .

Given the data set we denote with the set of all events with which is causally associated such that .

Additionally, we define the following causal *background contexts* . Those are formed by holding fixed the set of all factors causally associated with . For instance, given a set of three associated with events one possible background context will be

Let is an event such that:

* it is not necessarily in :
* is reachable from i.e.,

Then we say that is *Eells-*caused by Event *iff*

over ( )

We denote *Eells*-causality with

**Definition**: *Average Degree Of Causal Significance* (*ADCS*) of event for event in given context

The *Average Degree Of Causal Significance* (*ADCS*) of event for event given the background contexts is defined as:

We use the Latin capital letter to denote *ADCS* from the Lat. *significatio* (significance).

**Lemma**:

Static dependency implies *Eells* causality. However, *Eells* causality does not imply static dependency.

That is, if then it is true . However, if it does not necessarily follow that .

Example of *Eells*-causality-

Let the event denote the statement that an order with one bundle was received. Let the event denotes the statement that the capacity feasible nodes for order are node and node .

//TODO: finish this

***Note***: the *caused by* relations do **not** impose a total order; that is, for **every** pair of events and it does **not** follow that either or is true. Therefore, a set of events cannot be visualized as an ordered sequence; instead, we will use *Directed Causal Graph* for the purpose.

### Directed Causal Graphs

**Definition**: *Directed Causal Graph (DSG)*

A directed graph in which each node represents an Event Type, and each arc represents causal relation. Each arc is labeled with causal significant factor, a real number between and , describing how significant is the causal relationship between the two event types.

## Problem Statement for Root Cause Analysis of Fulfillment Decisions

The goal of the RCA algorithm applied to Fulfillment Optimization events is to understand and analyze causal relationship between predefined set of events based on the order metrics payloads. Each detected causal relationship will be assigned a significance factor which will indicate based on the supplied dataset how significant was this causal relationship inferred from the dataset and the configured set of events . Thus, the result of a single RCA algorithm run with a given dataset will be a Directed Causal Graph instance , where the vertex set will be a subset of the events set i.e., . Each arc will represent a causal relationship between the connected events, and it will be labeled with a causal significance factor (abbrev. ).

For instance, for the set of events shown earlier (see *Set of events for analysis of the cause of splits in Fulfillment Decisions*) we can have the following output of the RCA algorithm:

//TODO: finish this

## Algorithm For Root Cause Analysis

*Brief description of the RCA algorithm*

1. Choose a set of events of interest. ,
2. Compile order sequence from the given events dataset
3. Using the given dataset create Directed Follow Graph instances for each order in the dataset.
4. From the created , construct Aggregated Directed Follow Graph with the set of events of interest ,
5. Using Eell’s definition of causality calculate the Average Degree of Causal Significance (ADCS) for each pair of nodes in using the already calculated in 3. frequency counts for each pair of events in .
6. Given a minimum significance level construct Directed Causal Graph (DCG) using and for each pair of events in such that every arc in will have significance factor larger or equal to .

//TODO: finish the algorithm

## Examples

//TODO: finish the examples

## Appendix A: Probabilistic Temporal Logic

### Definitions and Review on Probabilistic Temporal Logic in (Kleinberg, Causality, Probability, and Time, 2012)

*Probabilistic Temporal Logic* (PTL) is a tool for state machine model checking which is a more complete alternative of the Labeled DFG defined earlier. A somewhat reduced subset of Probabilistic Temporal Logic is defined with the help of *Kripke* structures. With randomness introduced *Kripke* structure is roughly equivalent to a Discrete Time Markov Chain, and it is just another tool to validate specific first order logic statements relevant for RCA against our process model.

**Definition A1**: *Kripke structure*

Let be a set of atomic propositions. A *Kripke* structure over is defined as the tuple where

* is a finite set of states
* is the set of initial states
* is a total transition relation, such that
* : is a function which labels each state with a set of atomic propositions that are true within it.

The function (relation) being a total transition function (relation) means that for every state, there is at least one transition from that state (to itself or to another state). The function (relation) maps states to the truth values of propositions at that state. Since there are propositions, there are possible truth values and maps each state to one of these.

A *path* in a *Kripke* structure is an infinite sequence of states. Precisely, a path is a sequence of states () such that for every , . That says that the series of transitions described in the sequence is possible. The notation is used to denote the *subpath suffix* of the path starting with state .

To find the properties that are true in such kind of structures we need a formal method for representing the properties to be tested. There are number of temporal logic systems which express (slightly) different sets of formula. We are going to introduce *Computational Tree Logic* (CTL) system which will be used to build upon later and define PTL.

The formulas in CTL are composed of paired *path quantifiers* and *temporal operators*. Path quantifiers describe whether a property holds ***for all paths*** (denoted with the operator ), or ***for some path*** (denoted with operator ), starting at a given state. The temporal operators describe where along the path the properties will hold. For example, if is some state, then is a valid CTL formula, but is not, since is not paired with one of or . More formally,

* *Finally* () – at some state on the path the property will hold
* *Globally* (G) – the property will hold along the entire path
* *Next* () – the property will hold at the next state of the path
* *Until* () – applies to two properties, the first one holds in every state along the path until at some state the second property holds
* *Weak Until aka Until or Release* (W)

//Finish this paragraph on CTL

As in CTL, in PTL there are two types of formulas: *path formulas* and *state formulas*. State formulas express properties that must hold within a state, such as being labeled with certain atomic propositions, while path formulas refer to sequences of states along which the formula must hold. The formulas are comprised of atomic propositions , propositional logical connectives (such as ), and the modal operators denoting time and probability. The logic syntax tells how valid PTL formulas are constructed:

1. All atomic propositions are state formulas
2. If and are state formulas, so are , , , and

### Examples of PTL

: Event is reachable from event with probability at least p after at least r steps and at most s steps

//Finish the paragraph on PTL Examples

**Definition A2**: *prima facie* cause expressed with PTL formulas

These conditions mean that 1) a state where is true will be reached with non-zero probability and 2) the probability of reaching a state where e is true (within the time bounds) is greater after being in a state where c is true (probability ) than 3) it is by simply starting from initial state of the system (probability ). When making inferences from data that means that must occur at least once, and the conditional probability of given is greater than the marginal probability of (usually calculated from frequencies). Since negative (probability lowering) causes can be defined in terms of their complement (so that if lowers the probability of , raises its probability, the definition here is in terms of positive, probability raising causes.

**Definition A3**: Suppes’ definition of *prima facie* cause

An event is a *prima facie* cause of event *iff*:

We should interpret this as being for all and where . That is, the probability of A occurring at any time after B is greater than the marginal probability of A occurring at any time. Thus, the conditions 1-3 do not refer to specific values of and but rather describe the relationship between and . In some cases, these causes may turn out to be false. Even if something meets the criterion of being a *prima facie* cause, this may be due only to common cause of it and the effect. Suppes introduces two ways in which something may be a false, or spurious cause. In each, the idea is that there is some earlier event than the *prima facie* cause that accounts equally well for the effect, so that his other information is known, the spurious cause does not have any influence on the effect.

**Definition A4**: Suppes’ first definition of *spurious cause*

An event , a prima facie cause of event , is a *spurious cause* in sense one iff and such that:

While is a possible cause of , there may be another, earlier, event that has more explanatory relevance to . However, condition 2 of the definition above is very strong and perhaps counterintuitive. It means that there exists an event that completely eliminates the effectiveness of the cause for predicting the effect. One way of relaxing this condition is to find not individual events but rather kinds of events. In Suppes’ second definition of spurious causes there will be a set of nonempty sets that cover the full sample space, and which are mutually exclusive (pairwise disjoint). Thus, only one of these sets can be true *and* together they cover all possibilities.

**Definition A5**: Suppes’ second definition of *spurious cause*

An event , a prima facie cause of event , is a *spurious cause* in sense two iff there is a partition where and for every in :

Distinction between these two kinds of spuriousness is made with an example given by (Otte, 1982) on pp63:

*For now on I will abbreviate “spurious in sense two” by and “spurious in sense one” by . This definition makes an event if the world can be partitioned in such a way that the above conditions are satisfied. Thus, if we can observe a certain kind of event given by the partition, the observation of the later event is uninformative, which makes it a cause. Suppes proves that if an event is a cause, then it is a cause. The converse of this theorem, however, is not necessarily true: it is possible for an event to be a cause and not be a cause.*

*As an example of a cause, let us take the case of decreasing air pressure causing not only rain but a falling barometer reading. The falling barometer reading is a prima facie cause of rain; given that the barometer reading is dropping, the probability that it will rain rises. Letting denote rain, denote a falling barometer reading, and denote decreasing air pressure, the probability of rain given that the barometer reading, and the air pressure are decreasing, , is equal to the probability of rain given that the air pressure is decreasing, ; thus the second condition of the second definition of spurious cause is satisfied. The third condition is likewise satisfied, since the probability of rain given decreasing air pressure and a falling barometer reading is a least as great as the probability of rain given a falling barometer reading, . Thus, by the second definition a falling barometer reading is a cause of rain. The falling barometer reading is a cause of rain. If we let be our partition (decreasing air pressure, non-decreasing air pressure), then*

*So the falling barometer reading is a cause of the rain.*

**Theorem A1**: Assume there is a *Kripke* structure representing the underlying system governing the occurrences of the events. Then the conditions for causality given in the **Definition A2** for prima facie cause earlier are satisfied if and only if the conditions for causality given by **Definition A3** (**Definition A3**) are satisfied.

*Proof:*

We begin by showing that **Definition A2Definition A3** and then show that **Definition A3Definition A2**.

**Proposition A1.1**: **Definition A2Definition A3**

*Proof:*

Assume that , and there is a *Kripke* structure , representing the underlying system governing the occurrences of these events. Also assume that states in that satisfy and are labeled as such. If in **Definition A3**, we assume that in that satisfy and are labeled as such. If in **Definition A3**, we assume that in there will be at least one transition between an event at and one at . That is, the timescale of is as fine as that of Suppes and vice versa. Further, we assume that the probabilities of Suppes’s formulation and those in come from the same source and this if represented correctly, in **Definition A3** is equal to in **Definition A2**.

*Condition 1*:

By definition of , the probability of occurring at any time is less than . Recall that the probability of a path is the product of the transition probabilities along the path, and the probability of a set of paths is the sum of their individual path probabilities. For a structure to satisfy this formula, the set of paths from the start state that reach a state where holds must be less than , and the probability of reaching a state where holds in this system is less than . Thus,

(A.1)

Now we must show . We now show that this conditional probability is greater than or equal to if:

(A.2)

is satisfied.

The probability of a transition from state to state that labels the edge between them,

,

Is the conditional probability:

(A.3)

The probability of reaching one time unit after . Then, for a path:

,

we can calculate the probability, given , of reaching (via ) within two time units:

(A.4)

and since and are independent conditioned on this becomes:

(A.5)

Note that the probabilities of the righthand side are simply the transition probabilities from to , and to (since there is one time unit between the states, they can only be reached via single transition).

Thus, the conditional probability is precisely the path probability:

(A.6)

Then, if we have a set of paths from to , the conditional probability is the sum of these path probabilities. For example, we may have the following paths:

In which case:

(A.7)

and from eq. (A.6) this becomes:

(A.8)

the sum of the individual path probabilities. Let us now assume that is labeled with and is labeled with , these are the only and states in the system, and there are no other paths between the states taking less than or equal to 2 time units. Then, this probability we have computed is in fact the probability of:

(A.9)

since the probability of reaching , during a window of time simply means looking at the set of paths reaching during that window. Similarly, to find the probability of:

(A.10)

we must consider the set of paths from states labeled with to those labeled with that take at least 1 time unit. Since there can be cycles in our graph, calculating the probability associated with a leads-to formula with an infinite upper time bound requires a slightly different method.

//Finish the paragraph the leads-to formula with lower and upper bound

*Leads-to with Both Lower and Upper Time Bounds*

This paragraph deals with evaluation of *Leads-To* with applied window of time in which leads to . We assume a minimum time after is true before which is true. Here it is shown that it is possible to add such a lower bound. By Definition:

(A.11)

where . Thus, we are only adding a minimum time to the consequent of the conditional. If we can label states where is true, then we can proceed as in the algorithm of (Hansson & Jonsson, 1994).

//Finish the paragraph on Leads-to with Both Lower and Upper Bounds taken from the B.2 of (Kleinberg, Causality, Probability, and Time, 2012)

### Definitions and Review on Probabilistic Real Time Computation Tree Logic (PCTL) in (Hansson & Jonsson, 1994)

#### Notation

Assume a finite set of *atomic propositions*. We use , , etc. to denote atomic propositions. Formulas in PCTL are built from atomic propositions, propositional logic connectives and operators for expressing time and probabilities. The set of PCTL formulas is divided into *path formulas* and *state formulas*. Their syntax is defined inductively as follows:

* Each atomic proposition is a state formula
* If and are state formulas, then so are , , ,
* If and are state formulas, and is a nonnegative integer or , then and are path formulas,
* If is a path formula and is a real number with , then and are state formulas.

We shall use , , etc. to range over PCTL formulas. Intuitively, state formulas represent properties of states and path formulas represent properties of paths (i.e., sequences of states). The propositional connectives , , , and have their usual meanings. The operator is the *(strong) until* operator, and is the *unless* (or *weak until*) operator. For a given state , the formulas and express that holds for a path from with a probability of at least and greater than , respectively.

We shall use as a shorthand for , and use as a shorthand for . Intuitively, means that there is at least a probability that both will become true within time units and that will be true from now on until becomes true. means that there is at least a probability that either will remain true for at least time units, or that both will become true within time units and that will be true from now on until becomes true.

PCTL formulas are interpreted over structures that are discrete time Markov chains. A specified initial state is associated with the structure. In addition, for each state there is an assignment of truth values to atomic propositions appearing in a given formula. Formally, a structure is a quadruple , where

is a finite set of states, ranged over by , , etc.,

is an *initial state*,

is a *transition probability function*, , such that for all in we have

,

is a labeling function assigning atomic propositions to states, i.e.,

Intuitively, each transition is considered to require one *time unit*. We will display structures as transition diagrams, where states (circles) are labeled with atomic propositions and transitions with non-zero probability are represented as arrows labeled with their probabilities (e.g., the arrow going from state to state is labelled with ). The initial state is indicated with an extra arrow.

A path from a state in a structure is an infinite sequence

of states with as the first state. The :th state () of is denoted , and the prefix of of length is denoted , i.e.,

For each structure and state we define a probability measure on the set of paths from . is defined on the probability space , where is the set of paths starting in and is a sigma-algebra on generated by sets

Of paths with a common finite prefix . The measure is defined as follows: for each finite sequence of states,

i.e., the measure of the set of paths for which is equal to the product . For we define . This uniquely defines the measure on all sets of paths in the sigma-algebra .

We define the truth of PCTL formulas for a structure by a satisfaction relation:

which means that the state formula is true at state in the structure . In order to define the satisfaction relation for states, it is helpful to use another relation

which means that the path satisfies the path formula in . The relations and are inductively defined as follows:

iff

iff not

iff and

iff or

iff or

iff there exist an such that and

iff or

iff the -measure of the set of paths starting in for which is at least .

iff the -measure of the set of paths starting in for which is greater than .

We define

where is the initial state of .

#### Properties expressible in PCTL

We will present examples of properties that can be expressed in PCTL. First, we discuss some of the facilities of PCTL which makes it suitable for specification of soft and hard deadlines.

The main difference between PCTL and branching time temporal logics such as CTL, is the quantification over paths and the ability to specify quantitative time. CTL allows universal () and existential () quantification over paths, i.e., one can state that a property should hold for all computations (paths) or that it should hold for some computations (paths). It is not possible to state that a property should hold for a certain portion of the computations, e.g., for at least 50% of the computations. In PCTL, on the other hand, arbitrary probabilities can be assigned to path formulas, thus obtaining a more general quantification over paths. An analogy to universal and existential quantification can in PCTL be defined as:

Quantitative time allows us to specify time-critical properties that relate the occurrence of events of a system in real-time. In PCTL it is possible to state that a property will hold continuously during a specific time interval, or that a property will hold sometime during a time interval. Combining this with the above quantification we can define

means that the formula holds continuously for time units with a probability of at least , and means that the formula holds within time units with a probability of at least .

An important requirement on most real-time and distributed systems is that they should be continuously operating, e.g., every time the controller receives an alarm signal from a sensor the controller should take the appropriate action. We can express such requirements with the following PCTL operators:

means that is always (in all states that can be reached with non-zero probability), means that a state where is will eventually be reached with probability 1, means that there is a non-zero probability for to be continuously true, and means that there exists a state where holds which can be reached with non-zero probability.

**Definition** (*Owicki & Lamport, 1982*)[[1]](#footnote-1): (unquantified) *leads-to* operator ()

Whenever a becomes true, b will eventually hold.

**Definition**: PCTL quantified *leads-to* operator ():

means that whenever holds there is a probability of at least p that will hold within time units.

Many modal operators can be derived from the basic PCTL operators. We can for instance define an operator that corresponds to the CTL operator (E.M. Clarke, 1983) as follows:

As an example, we will specify a mutual exclusion property. Consider two processes ( and ) using the same criticial section. The atomic propositions , , and indicates that is in its non-critical, trying, and critical regions, respectively. The mutual exclusion property can be expressed as:

This is not sufficient for most *real-time systems* since the property only states that simultaneous access to the critical section must be avoided always under all circumstances. To capture a specific real-time behavior, we can specify that whenever enters its trying region, it will enter its critical region within 4 time units. This can in PCTL be expressed as:

For some systems, it might be sufficient that the deadline is almost always met (e.g. in 99% of the cases). The relaxed property can be expressed as:

Relaxing the timing requirement might enable a less costly implementation that still shows acceptable behavior. To be on the safe side we could add a strict upper limit to the relaxed property, combining the hard and soft deadlines above. If we assume that we want to always enter its critical region within 10 time units, and almost always within 4 time units we get the property:

#### Model Checking in PCTL

In this section we present a model checking algorithm, which given a structure and a PCTL formula determines whether . The algorithm is based on the algorithm for model checking CTL (E.M. Clarke, 1983). It is designed so that when it finishes each state will be labeled with the set of subformulas of that are in the state. One can then conclude that if the initial state () is labeled with .

For each state of the structure, the algorithm uses a variable to indicate the subformulas that are in state . Initially, each state is labeled with the atomic propositions that are in , i.e., . The labeling is then performed starting with the smallest subformula of that has not yet been labeled and ending with labeling states with itself. Composite formulas are labeled based on the labeling of their parts. Assuming that we have performed the labeling of and , the labeling corresponding to negation () and propositional connectives () is straightforward, i.e.

if ,

if ,

if or

if or

where in addition the new formula must be a subformula of . The next section presents two algorithms for labeling states with the modal subformulas of PCTL. After that, in the subsequent section, we discuss labeling in cases with extreme parameter values (e.g., , , and ).

#### Labeling states with the modal subformulas of PCTL

We shall give an algorithm for the labeling of states for the formula assuming that we have done the labeling for formulas and , and that .

//Appendix: Finish the paragraph on PTL Theory

### Review on Computation Tree Logic and Specification Language in (E.M. Clarke, 1983)

In this Appendix section we will review an efficient procedure for verifying that a finite state concurrent system meets a specification expressed in a (propositional) branching-time temporal logic. The reviewed algorithm has linear complexity in both the size of the specification and the size of the global transition. The global state graph can be viewed as a finite Kripke structure and an efficient algorithm can be given to determine whether a given structure is a model of a particular formula. The algorithm, which we call a *model checker*, is similar to the global flow analysis algorithms used in compiler optimization and has complexity linear in both the size of the structure and the size of the specification.

#### The Specification Language

The syntax for CTL is given below. is the underlying set of *atomic propositions*.

1. Every atomic proposition is a CTL formula
2. If and are CTL formulae, the so are , , , , , .

The symbols and have their usual meanings. is the *nexttime* operator; the formulae () intuitively means that holds in every (in some) immediate successor of the current program state. is the *until* operator; The formula () intuitively means that for every computation path (for some computation path), there exists an initial prefix of the path such that holds at the last state of the prefix and holds at all other states along the prefix.

The semantics of CTL formulae with respect to a labeled state-transition graph is defined below. Formally, a CTL structure is a triple where

1. is a finite set of states
2. is a binary relation on i.e., . It gives the possible transitions between states and must be total i.e., .
3. is an assignment of atomic propositions to states i.e.,

A *path* is an infinite sequence of states () such that . For any structure and state , there is an *infinite computation tree* with root labeled such that is an arc in the tree *iff* .

We use the standard notation to indicate truth in a structure: means that formula holds at state in structure . When the structure is understood, we simply write . The relation is defined inductively as follows:

iff

iff not

iff and

iff for all states t such that ,

iff for all states t such that ,

iff for all paths () s.t.

iff for some path () s.t.

#### Model Checker

Assume that we wish to determine whether formula is true in the finite structure . When the algorithm finishes, each state will be labelled with the set of subformulae true in the state. We let denote this set for state . Consequently, iff at termination. We first consider the case in which each state is currently labelled with the *immediate* subformulae of which are true in that state. We will use the following primitives for manipulating formulas and accessing the labels associated with states:

* and give the first and second arguments of a two-argument formula such as
* will return true (false) if state is (is not) labelled with formula .
* adds formula f to the current label of state .

The state labeling algorithm (procedure ) must be able to handle seven cases depending on whether f is atomic or has one of the following forms: ,,. We will only consider the case in which here since all other cases are either straightforward or similar. For the case the algorithm uses depth first search to explore the state graph. The bit array is used to indicate which states have been visited by the search algorithm. The algorithm also uses stack ST to keep track of those states which require additional processing before the truth or falsity of can be determined. The Boolean procedure will determine (in constant true) whether state is currently on the stack .

def label\_graph(f, b):

ST = empty\_stack

for s in S:

marked(s) = false

for s in S:

if not marked(s):

au(f,s,b)

def au(f,s,b):

//Appendix: Finish the paragraph on CTL algorithms

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## Downloadable Links for the Bibliography

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(Eells, 1991): [here](https://github.com/dimitarpg13/root_cause_analysis_and_model_checking/blob/main/literature/books/eells_probabilistic_causality_1991.pdf)

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1. In (Hansson & Jonsson, 1994) and (Owicki & Lamport, 1982) the symbol is used to denote the *leads-to* operator. In this document we use to denote *prima facie* cause. [↑](#footnote-ref-1)