Notes on Fourier series and Integrals (Dym and McKean, 1972)

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# The Lebesgue Integral

The Lebesgue integral is a generalization of the Riemann’s notion of integral.

Let us consider a positive-continuous function defined on an interval .

To form a Riemann sum we subdivide the interval with the following points of subdivision:

form the Riemann sum

in which is any point between and . It can be verified that this sum approaches the limit – the Riemann integral

as and the biggest of the lengths () tends to 0.

In the Lebesgue integral definition, we subdivide/quantize the range of the function instead of the domain.

Lebesgue’s technique subdivides the vertical axis by a series of points

to form the sum

in which is the sum of the lengths of the subintervals of on which the stated inequality takes place, and finally to verify that this sum approaches the same number as and the biggest of the lengths () tends to 0. The point is that by now extending the idea of *measure* from unions of disjoint subintervals to the wider class of “*measurable*” subsets of the interval , we can integrate a much wider class of functions compared to the Riemann’s definition.

More formal definition of the Lebesgue integral

Fix an interval , which may be bounded (), or a half-line (), or the whole line ().

**Definition**: Borel set

any set that can be formed from open (or equivalently closed) sets through the operations of countable union, countable intersection, and relative complement.

**Definition**: Lebesgue measurable set

i ) the measure of a (countable) union of nonoverlapping intervals is the sum of their lengths.

ii ) the measure of a single point or any countable number of points is 0

iii ) any subset of a Borel set of measure 0 also has measure 0.

**Definition**: Lebesgue measure of an interval

for any measurable set E,

in which the infimum is taken over the class of countable coverings of by means of intervals ; here

**Example 1**

Let us take and . Any open subset of is measurable, since it is the union of a countable number of nonoverlapping open intervals. The natural extension to compact sets K is to use

in which is the (open) complement of relative to . For example, the measure of

…

is equal to

so that its complement is of measure 0, though it is an uncountable set. is the known Cantor set.

Properties of Lebesgue measure

All sets are measurable subsets of . We abbreviate as .

(a)

(b) if

(c)

(d)

(d’) if is empty for

(e) if and

(e’) if and

**Statement 1**: each of (d’), (e), (e’) implies the other two

Proof:

Proof that (d’) implies (e), (e’): Let us assume (d’) is true for all sequences for which the intersection of every pair of elements is empty.

Let us consider a new sequence for which, obviously, . For this sequence we have the following and we want to prove that .

Let us construct the following sequence such that with if and . Thus, we have .

Then for we have . Thus, (e) follows from (d’).

Now to prove that (e) follows from (d’) let us consider another sequence for which, obviously, .

Let us introduce the following sequence such that with .

Then for we have . Thus, (e’) follows from (d’).

Proof that (e) implies (d’), (e’): Let us assume ( e ) is true for all sequences for which the next element is a proper superset of the current one and the union of all elements is equal to .

Let us consider a new sequence such that . We want to prove that

. (E1.1)

We can construct a new sequence such that by setting

Clearly . Also, we set . By (e) we have .

Using

. (E1.2)

Using ( c ) recursively in the expression results in

(E1.3)

From (E1.2) and (E1.3) it follows (E1.1) .

Proving that (e) implies (e’) follows similar logic.

Proving that (e’) implies (d’) and (e) utilizes similar arguments.

**Statement 2**: Let are real continuous functions on . If exists pointwise, then is measurable.

Proof:

Let us consider the following -parametrized interval

. (E2.1)

Clearly, this interval is Lebesgue-measurable for each pair of and values.

Now let us form the following intersection of a sequence formed by freeing in (E2.1). Notice, this set is a Borel set and is also Lebesgue-measurable:

# References

[1] [Fourier Series and Integrals, H. Dym, HP McKean, 1972](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/books/fourier_series_and_integrals_dym_mckean_1972.pdf)

[2] [An Introduction to Abstract Harmonic Analysis, Lynn H. Loomis, Harvard, 1953](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/books/AnIntroductionToAbstractHarmonicAnalysisLoomis1953.pdf)

[3] [Fourier Analysis on Groups, Walter Rudin, 1962](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/books/Fourier_analysis_on_groups_Rudin_1962.pdf)

[4] [Pontryagin Duality and the Structure of Locally Compact Abelian Groups, Sidney A. Morris, 1977](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/books/Pontryagin_Duality_and_The_Structure_of_Locally_Compact_Abelian_Groups_Morris_1977.pdf)

[5] [Poisson summation and the discrete Fourier transform, John Kerl, 2008](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/articles/Poisson_summation_and_the_discrete_Fourier_transform_Kerl_2008.pdf)

[6] [Fourier series, Wikipedia](https://en.wikipedia.org/wiki/Fourier_series)

[7] [Fourier transform, Wikipedia](https://en.wikipedia.org/wiki/Fourier_transform)

[8] [Discrete Fourier Transform, Wikipedia](https://en.wikipedia.org/wiki/Discrete_Fourier_transform)

[9] and spaces, Wikipedia

[10] [Poisson summation formula, Wikipedia](https://en.wikipedia.org/wiki/Poisson_summation_formula)

[11] [Pontryagin duality, Wikipedia](https://en.wikipedia.org/wiki/Pontryagin_duality)

[12] [Abelian group, Wikipedia](https://en.wikipedia.org/wiki/Abelian_group)

[13] [Locally compact Abelian group, Wikipedia](https://en.wikipedia.org/wiki/Locally_compact_abelian_group)

[14] [Quotient group, Wikipedia](https://en.wikipedia.org/wiki/Quotient_group)

[15] [Free Abelian group, Wikipedia](https://en.wikipedia.org/wiki/Free_abelian_group)

[16] [Lattice group, Wikipedia](https://en.wikipedia.org/wiki/Lattice_(group))

[17] [Dual lattice, Wikipedia](https://en.wikipedia.org/wiki/Dual_lattice)

[18] [Dual space, Wikipedia](https://en.wikipedia.org/wiki/Dual_space)