Notes on Wavelets and related Transforms

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# Introduction

## Time-invariant and Transient Phenomena

Classical signal processing has devoted much of its efforts to the design of time-invariant and space-invariant operators that modify stationary signal properties. With most notable representative the Fourier transform the classical approach does not address many information-processing scenarios and the related use cases.

The search of the optimal basis in absolute terms is a hopeless quest; instead, we have a multitude of different transforms and bases each having their strengths and weaknesses , therefore each suitable in specific subset of contexts.

The Fourier transform is used in time-invariant signal processing because sinusoidal waves are eigenvectors of linear time-invariant operators. A linear time-invariant operator is entirely specified by the eigenvalues :

(1)

To compute , a signal is decomposed as sum of sinusoidal eigenvectors :

(2)

If has finite energy, one can show that the amplitude of each sinusoidal wave is the Fourier transform of :

(3)

Applying the operator to f in (2) and inserting the eigenvector expression (1) gives

(4)

The operator amplifies or attenuates each sinusoidal component of by . It represents a *frequency filtering* of .

The Fourier transform provides solution to a variety of problems in which linear time-invariant operator is finding the relevant for the problem at hand pattern. However, if we are interested in transient phenomena the Fourier transform cannot find the relevant patterns for those problems at hand. The Fourier coefficient is obtained in (3) by correlating with a sinusoidal wave . Since the support of covers the whole real line, depends on the values for all times . This globally encoded information makes it difficult to analyze any local property of from .

## Focusing on the Transients

The uncertainty principle states that the energy spread of a function and its Fourier transform cannot be simultaneously arbitrary small. Motivated by quantum mechanics, in 1946 the physicist Gabor ([3]) defined elementary time frequency atoms as waveforms that have minimal spread in a time-frequency plane. To measure time-frequency information content, he proposed decomposing signals over these elementary atomic waveforms.

By showing that such decompositions are closely related to our sensitivity to sounds, and that they exhibit important structures in speech and music recordings Gabor demonstrated the importance of localized time-frequency signal processing.

### Windowed Fourier Transform

Gabor atoms are constructed by translating in time and frequency a time window :

(5)

The energy of is concentrated in the neighborhood of over an interval of size , measured by the standard deviation of . Its Fourier transform is a translation of of the Fourier transform of :

(6)

A diagram of a diagram of a graph

Description automatically generated with medium confidenceThe energy of is therefore localized near the frequency , over an interval of size , which measures the domain where is non-negligible. In a time-frequency plane , the energy spread of the atom is represented by the Heisenberg rectangle illustrated by Figure 1.

Figure 1: Time-frequency boxes aka *Heisenberg rectangles* representing the energy spread of two Gabor atoms

This rectangle is centered at and has a time width and a frequency width . The uncertainty principle proves that its area satisfies

(7)

This area is minimum when is a Gaussian, in which case the atoms are called *Gabor functions*.

The windowed Fourier transform defined by Gabor correlates a signal with each atom :

(8)

(8) can be interpreted as Fourier integral localized in the neighborhood of by the window .

# References

[1] [Wavelet Tour of Signal Processing, Stephane Mallat, 2004](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/articles/wavelets/Mallat_Wavelet-Tour-of-Signal-Processing.pdf)

[2] [Ten Lectures of Wavelet, Ingrid Daubechies, Bell Labs, Rutgers U., 1992](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/articles/wavelets/Ten_Lectures_of_Wavelets.pdf)

[3] [Theory of Communication, D Gabor, 1946](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/articles/Theory_of_Communication_Gabor_1946.pdf)

# Appendix

## Parseval’s Theorem

Let and are two complex-valued functions on of period that are square integrable (with respect to Lebesgue measure) over intervals of period length, with Fourier series

(A1.1)

and

(A1.2)

respectively. Then

(A1.3)

Plugging (A1.1) and (A1.2) in (A1.3) gives

//TODO: presentation and derivation of Parseval formula