Notes on the Theory of Super-Resolution

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# Introduction

## Uncertainty Principles and Signal Recovery

Uncertainty Principle for Continuous Functions:

if a function is zero everywhere outside of interval of length and its Fourier transform is zero outside of interval of length then

(1)

A more general principle utilizing measure set theory holds:

Measure Set Uncertainty Principle:

If a function is zero almost everywhere outside of a measurable set and is zero almost everywhere outside a measurable set , then

(2)

where and denote the measures of the sets and , and is a small number dependent on the measure of the almost-everywhere-zero set.

In words, and cannot be both highly concentrated no matter what sets and we pick in which the concentration occurs.

Uncertainty Principle for Discrete Functions

Let be a sequence of length and let be its discrete Fourier transform.

Let is not zero at points and is not zero at points. Then

(3)

The inequality (3) holds on all kinds of sets where and are nonzero: these may be intervals or any other sets.

Principles (2) and (3) have applications in signal recovery.

The continuous-time principle shows that missing segments of a bandlimited function can be restored robustly in the presence of noise if

The discrete-time principle indicates that a wide-band signal can be reconstructed from narrow-band data provided the wide-band signal to be recovered is *sparse* (aka *impulsive*).

Application of the continuous-time principle:

a bandlimited function corrupted by noise of unknown properties can be restored perfectly, without error, if the noise is sparse: zero outside an (unknown) set of measure . This is known as the *Logan’s phenomenon* (discovered by B.F. Logan).

The version of Logan’s phenomenon (i.e. the version of the Logan’s phenomenon for discrete time) can be used in the study of an -algorithm for restoring a sparse signal from narrowband data. It shows that the -algorithm recovers a wideband signal perfectly from noiseless narrowband data, provided that the signal is sufficiently sparse.

## The Discrete-Time Uncertainty Principle

Let be a sequence of length and let be its discrete Fourier transform

(4)

The inverse obviously is

(5)

As before we denote with and the number of non-zero entries in and accordingly.

**Theorem 1**: bound for the time-bandwidth product

(6)

**Corollary 1**: bound for the total number of non-zero elements

(7)

Example attaining the limits (6) and (7)

Let us consider the sequence . Let us assume that admits factorization . Then, the sequence

(8)

has equally-spaced nonzero elements. is the indicator function of a subgroup of and its discrete Fourier transform is, up to constant factor, the indicator function of the dual subgroup.

Thus, we have:

(9)

Here ,

(10)

if is multiple of then clearly , but if is not multiple of then . Thus (10) has only equally spaced (with period ) non-zero values .

Thus, for the sequence (8) we have and and hence .

Moreover, using (5) we can write

We will show that apart from simple modifications these are the only pairs of sequences to attain the bound ; the extremal functions for this uncertainty principle are periodic spike trains with an integral number of periods in the length .

Let us modify the definition of and so these are defined on the discrete circle which wraps around so that and are consecutive sites. The wraparound convention is equivalent to interpreting subscripts modulo ;

# References

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[2] [Super-Resolution via Sparsity Constraints, David L. Donoho, UC Berkeley, 1990](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/articles/compressed_sensing/SuperResolutionViaSparsityConstraintsDonoho.pdf)

[3] [Decoding by Linear Programming, Emmanuel Candes, Terence Tao, Caltech, 2004](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/articles/compressed_sensing/DecodingByLinearProgrammingCandes2005.pdf)

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[5] [Toward a Mathematical Theory of Super-Resolution, Emmanuel J. Candes, Carlos Fernandez-Granda, 2012](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/articles/compressed_sensing/TowardMathematicalTheoryOfSuperresolutionCandes.pdf)

[6] [Logan’s phenomenon: Uncertainty Principle and Robust Reconstruction, Journey into Randomness blog, 2011](https://linbaba.wordpress.com/tag/logan-phenomenon/)

[7] [Poisson summation and the discrete Fourier transform, John Kerl, 2008](https://github.com/dimitarpg13/spectral_analysis/blob/main/literature/articles/Poisson_summation_and_the_discrete_Fourier_transform_Kerl_2008.pdf)

[8] and spaces, Wikipedia

[9] [Poisson summation formula, Wikipedia](https://en.wikipedia.org/wiki/Poisson_summation_formula)

[10] [Discrete Fourier transform, Wikipedia](https://en.wikipedia.org/wiki/Discrete_Fourier_transform)

# Appendix

## Poisson Summation and Periodization of a function

Let represents smooth complex valued function defined on which decays at infinity with all derivates. Then the following identity, known as *the Poisson summation formula*, holds:

(1)

where is the Fourier transform of , i.e. .

## Periodization of a Function

Consider the periodic functions

and , (2)

Using the discrete Fourier transform definition with unitary normalization factor ([9]) we write: