# ON THE REPRESENTATION BY LINEAR SUPERPOSITIONS <sup>1</sup>

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Abstract. In a number of papers, Y. Sternfeld investigated the problems of representation of continuous and bounded functions by linear superpositions. In particular, he proved that if such representation holds for continuous functions, then it holds for bounded functions. We consider the same problem without involving any topology and establish a rather practical necessary and sufficient condition for representability of an arbitrary function by linear superpositions. In particular, we show that if some representation by linear superpositions holds for continuous functions, then it holds for all functions. This will lead us to the analogue of the well-known Kolmogorov superposition theorem for multivariate functions on the d-dimensional unit cube.

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#### 1. Introduction

Let X be any set and  $h_i: X \to \mathbb{R}, i = 1, ..., r$ , be arbitrarily fixed functions. Consider the set

$$\mathcal{B}(X) = \mathcal{B}(h_1, ..., h_r; X) = \left\{ \sum_{i=1}^r g_i(h_i(x)), \ x \in X, \ g_i : \mathbb{R} \to \mathbb{R}, \ i = 1, ..., r \right\}$$
(1.1)

Members of this set will be called linear superpositions (see [35]). We are going to answer the question: what conditions on X guarantee that each function on X will be in the set  $\mathcal{B}(X)$ ? The simplest case  $X \subset \mathbb{R}^d$ , r = d and  $h_i$  are the coordinate functions has been solved in [16]. See also [5,15] for the case r = 2.

By  $\mathcal{B}_c(X)$  and  $\mathcal{B}_b(X)$  denote the right hand side of (1.1) with continuous and bounded  $g_i : \mathbb{R} \to \mathbb{R}, i = 1, ..., r$ , respectively. Our starting point is the well-known superposition theorem of Kolmogorov [17]. It states that for the unit cube  $\mathbb{I}^d$ ,  $\mathbb{I} = [0, 1], d \geq 2$ , there exists 2d + 1 functions  $\{s_q\}_{q=1}^{2d+1} \subset C(\mathbb{I}^d)$  of the form

$$s_q(x_1, ..., x_d) = \sum_{p=1}^d \varphi_{pq}(x_p), \ \varphi_{pq} \in C(\mathbb{I}), \ p = 1, ..., d, \ q = 1, ..., 2d + 1$$
 (1.2)

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such that each function  $f \in C(\mathbb{I}^d)$  admits the representation

$$f(x) = \sum_{q=1}^{2d+1} g_q(s_q(x)), \ x = (x_1, ..., x_d) \in \mathbb{I}^d, \ g_q \in C(\mathbb{R}).$$
 (1.3)

In our notation, (1.3) means that  $\mathcal{B}_c(s_1, ..., s_{2d+1}; \mathbb{I}^d) = C(\mathbb{I}^d)$ . This surprising and deep result, which solved (negatively) Hilbert's 13-th problem, was improved and generalized in several directions. It was first observed by Lorentz [21] that the functions  $g_q$  can be replaced by a single continuous function g. Sprecher [29] showed that the theorem can be proven with constant multiples of a single function  $\varphi$  and translations. Specifically,  $\varphi_{pq}$  in (1.2) can be chosen as  $\lambda^p \varphi(x_p + \varepsilon q)$ , where  $\varepsilon$  and  $\lambda$  are some positive constants. Fridman [9] succeeded in showing that the functions  $\varphi_{pq}$  can be constructed to belong to the class Lip(1). Vitushkin and Henkin [35] showed that  $\varphi_{pq}$  cannot be taken to be continuously differentiable.

Ostrand [25] extended the Kolmogorov theorem to general compact metric spaces. In particular, he proved that for each compact d-dimensional metric space X there exist continuous real functions  $\{\alpha_i\}_{i=1}^{2d+1} \subset C(X)$  such that  $\mathcal{B}_c(\alpha_1, ..., \alpha_{2d+1}; X) = C(X)$ . Sternfeld [32] showed that the number 2d+1 cannot be reduced for any d-dimensional space X. Thus the number of terms in the Kolmogorov superposition theorem is the best possible.

Some papers of Sternfeld have been devoted to the representation of continuous and bounded functions by linear superpositions. Let C(X) and B(X) denote the space of continuous and bounded functions on some set X respectively (in the first case, X is supposed to be a compact metric space). Let  $F = \{h\}$  be a family of functions on X. F is called a uniformly separating family (u.s.f.) if there exists a number  $0 < \lambda \le 1$  such that for each pair  $\{x_j\}_{j=1}^m$ ,  $\{z_j\}_{j=1}^m$  of disjoint finite sequences in X, there exists some  $h \in F$  so that if from the two sequences  $\{h(x_j)\}_{j=1}^m$  and  $\{h(z_j)\}_{j=1}^m$  in h(X) we remove a maximal number of pairs of points  $h(x_{j_1})$  and  $h(z_{j_2})$  with  $h(x_{j_1}) = h(z_{j_2})$ , there remains at least  $\lambda m$  points in each sequence (or , equivalently, at most  $(1 - \lambda)m$  pairs can be removed). Sternfeld [31] proved that for a finite family  $F = \{h_1, ..., h_r\}$  of functions on X, being a u.s.f is equivalent to the equality  $\mathcal{B}_b(h_1, ..., h_r; X) = B(X)$ , and that in the case where X is a compact metric space and the elements of F are continuous functions on X, the equality  $\mathcal{B}_c(h_1, ..., h_r; X) = C(X)$  implies that F is a u.s.f. Thus, in particular, Sternfeld obtained that the formula (1.3) is valid for all bounded functions, where  $g_q$  are bounded functions depending on f (see also [15, p.21]).

Let X be a compact metric space. The family  $F = \{h\} \subset C(X)$  is said to be a measure separating family (m.s.f.) if there exists a number  $0 < \lambda \le 1$  such that for any measure  $\mu$  in  $C(X)^*$ , the inequality  $\|\mu \circ h^{-1}\| \ge \lambda \|\mu\|$  holds for some  $h \in F$ . Sternfeld [33] proved that  $\mathcal{B}_c(h_1, ..., h_r; X) = C(X)$  if and only if the family  $\{h_1, ..., h_r\}$  is a m.s.f. In [31], it has been shown that if r = 2, then the properties u.s.f. and m.s.f. are equivalent. Therefore, the equality  $\mathcal{B}_b(h_1, h_2; X) = B(X)$  is equivalent to  $\mathcal{B}_c(h_1, h_2; X) = C(X)$ . But for r > 2, these two properties are no longer equivalent. That is,  $\mathcal{B}_b(h_1, ..., h_r; X) = B(X)$  does not always imply  $\mathcal{B}_c(h_1, ..., h_r; X) = C(X)$  (see [33]).

Our purpose is to consider the above mentioned problem of representation by linear

superpositions without involving any topology (that of continuity or boundedness). More precisely, we want to characterize those sets X for which  $\mathcal{B}(h_1, ..., h_r; X) = T(X)$ , where T(X) is the space of all functions on X. This will be done in terms of closed paths, the explicit and practically convenient objects. We show that nonexistence of closed paths in X is equivalent to the equality  $\mathcal{B}(X) = T(X)$ . In particular, we will obtain that  $\mathcal{B}_c(X) = C(X)$  implies  $\mathcal{B}(X) = T(X)$ . Therefore, the formula (1.3) is valid for all multivariate functions over the unite cube  $\mathbb{I}^d$ , where  $g_q$  are univariate functions depending on f. We will also present an example due to Khavinson [15] showing that even in the case r = 2, the equality  $\mathcal{B}(h_1, h_2; X) = T(X)$  does not imply  $\mathcal{B}_c(h_1, h_2; X) = C(X)$ . At the end we will make some observations around the problems of representation and interpolation by ridge functions, which are widely used in multivariate approximation theory.

### 2. Closed paths

In the sequel, by  $\delta_A$  we will denote the characteristic function of a set  $A \subset \mathbb{R}$ . That is,

$$\delta_A(y) = \begin{cases} 1, & \text{if } y \in A \\ 0, & \text{if } y \notin A. \end{cases}$$

The following definition is based on the ideas set forth in the works [1] and [16].

**Definition 2.1.** Given a set X and functions  $h_i: X \to \mathbb{R}$ , i = 1, ..., r. A set of points  $\{x_1, ..., x_n\} \subset X$  is called to be a closed path with respect to the functions  $h_1, ..., h_r$  (or, concisely, a closed path if there is no confusion), if there exists a vector  $\lambda = (\lambda_1, ..., \lambda_n)$  with the nonzero real coordinates  $\lambda_i$ , i = 1, ..., n, such that

$$\sum_{j=1}^{n} \lambda_j \delta_{h_i(x_j)} = 0, \ i = 1, ..., r.$$
(2.1)

Let for i=1,...,r, the set  $\{h_i(x_j),\ j=1,...,n\}$  have  $k_i$  different values. Then it is not difficult to see that Eq. (2.1) stands for a system of  $\sum_{i=1}^r k_i$  homogeneous linear equations in unknowns  $\lambda_1,...,\lambda_n$ . If this system has any solution with the nonzero components, then the given set  $\{x_1,...,x_n\}$  is a closed path. In the last case, the system has also a solution  $m=(m_1,...,m_n)$  with the nonzero integer components  $m_i,\ i=1,...,n$ . Thus, in Definition 2.1, the vector  $\lambda=(\lambda_1,...,\lambda_n)$  can be replaced by a vector  $m=(m_1,...,m_n)$  with  $m_i\in\mathbb{Z}\backslash\{0\}$ .

For example, the set  $l = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (1,1,1)\}$  is a closed path in  $\mathbb{R}^3$  with respect to the functions  $h_i(z_1, z_2, z_3) = z_i$ , i = 1, 2, 3. The vector  $\lambda$  in Definition 2.1 can be taken as (-2, 1, 1, 1, -1).

In the case r=2, the picture of closed path becomes more clear. Let, for example,  $h_1$  and  $h_2$  be the coordinate functions on  $\mathbb{R}^2$ . In this case, a closed path is the union of some sets  $A_k$  with the property: each  $A_k$  consists of vertices of a closed broken line with the sides parallel to the coordinate axis. These objects (sets  $A_k$ ) have been exploited in

practically all works devoted to the approximation of bivariate functions by univariate functions, although under the different names (see, for example, [15, chapter 2]). If X and the functions  $h_1$  and  $h_2$  are arbitrary, the sets  $A_k$  can be described as a trace of some point traveling alternatively in the level sets of  $h_1$  and  $h_2$ , and then returning to its primary position. It should be remarked that in the case r > 2, closed paths do not admit such a simple geometric description. We refer the reader to Braess and Pinkus [1] for the description of closed paths when r = 3 and  $h_i(\mathbf{x}) = \mathbf{a}^i \cdot \mathbf{x}$ ,  $\mathbf{x} \in \mathbb{R}^2$ ,  $\mathbf{a}^i \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ , i = 1, 2, 3.

Let T(X) denote the set of all functions on X. With each pair  $\langle p, \lambda \rangle$ , where  $p = \{x_1, ..., x_n\}$  is a closed path in X and  $\lambda = (\lambda_1, ..., \lambda_n)$  is a vector known from Definition 2.1, we associate the functional

$$G_{p,\lambda}: T(X) \to \mathbb{R}, \quad G_{p,\lambda}(f) = \sum_{j=1}^{n} \lambda_j f(x_j).$$

In the following, such pairs  $\langle p, \lambda \rangle$  will be called *closed path-vector pairs* of X. It is clear that the functional  $G_{p,\lambda}$  is linear. Besides,  $G_{p,\lambda}(g) = 0$  for all functions  $g \in \mathcal{B}(h_1, ..., h_r; X)$ . Indeed, assume that (2.1) holds. Given  $i \leq r$ , let  $z = h_i(x_j)$  for some j. Hence,  $\sum_{j \ (h_i(x_j)=z)} \lambda_j = 0$  and  $\sum_{j \ (h_i(x_j)=z)} \lambda_j g_i(h_i(x_j)) = 0$ . A summation yields  $G_{p,\lambda}(g_i \circ h_i) = 0$ . Since  $G_{p,\lambda}$  is linear, we obtain that  $G_{p,\lambda}(\sum_{i=1}^r g_i \circ h_i) = 0$ .

The following lemma characterizes the set  $\mathcal{B}(h_1,...,h_r;X)$  under some restrictions and will be used in the proof of Theorem 2.5 below.

**Lemma 2.2.** Let X have closed paths  $h_i(X) \cap h_j(X) = \emptyset$ , for all  $i, j \in \{1, ..., r\}$ ,  $i \neq j$ . Then a function  $f: X \to \mathbb{R}$  belongs to the set  $\mathcal{B}(h_1, ..., h_r; X)$  if and only if  $G_{p,\lambda}(f) = 0$  for any closed path-vector pair  $\langle p, \lambda \rangle$  of X.

*Proof.* The necessity is obvious, since the functional  $G_{p,\lambda}$  annihilates all members of  $\mathcal{B}(h_1,...,h_r;X)$ . Let us prove the sufficiency. Introduce the notation

$$Y_i = h_i(X), i = 1, ..., r;$$
  
 $\Omega = Y_1 \cup ... \cup Y_r.$ 

Consider the following subsets of  $\Omega$ :

$$\mathcal{L} = \{Y = \{y_1, ..., y_r\} : \text{if there exists } x \in X \text{ such that } h_i(x) = y_i, \ i = 1, ..., r\}$$
 (2.2)

Note that  $\mathcal{L}$  is a set of some certain subsets of  $\Omega$ . Each element of  $\mathcal{L}$  is a set  $Y = \{y_1, ..., y_r\} \subset \Omega$  with the property that there exists  $x \in X$  such that  $h_i(x) = y_i$ , i = 1, ..., r. In what follows, all the points x associated with Y by (2.2) will be called (\*)-points of Y. It is clear that the number of such points depends on Y as well as on the functions  $h_1, ..., h_r$ , and may be greater than 1. But note that if any two points  $x_1$  and  $x_2$  are (\*)-points of Y, then necessarily the set  $\{x_1, x_2\}$  forms a closed path with the associated vector  $\lambda = (1; -1)$ . In this case, by the condition of the sufficiency,  $f(x_1) = f(x_2)$ . Let now  $Y^*$  be the set of all (\*)-points of Y. Since we have already known that  $f(Y^*)$  is a single number, we can define the function

$$t: \mathcal{L} \to \mathbb{R}, \ t(Y) = f(Y^*).$$

Or, equivalently, t(Y) = f(x), where x is an arbitrary (\*)-point of Y. Consider now a class S of functions of the form  $\sum_{j=1}^{k} r_j \delta_{D_j}$ , where k is a positive integer,  $r_j$  are real numbers and  $D_j$  are elements of  $\mathcal{L}, \ j=1,...,k$ . We fix neither the numbers  $k, r_j$ , nor the sets  $D_j$ . Clearly,  $\mathcal{S}$  is a linear space. Over  $\mathcal{S}$ , we define the functional

$$F: \mathcal{S} \to \mathbb{R}, \ F\left(\sum_{j=1}^k r_j \delta_{D_j}\right) = \sum_{j=1}^k r_j t(D_j).$$

First of all, we must show that this functional is well defined. That is, the equality

$$\sum_{j=1}^{k_1} r'_j \delta_{D'_j} = \sum_{j=1}^{k_2} r''_j \delta_{D''_j}$$

always implies the equality

$$\sum_{j=1}^{k_1} r'_j t(D'_j) = \sum_{j=1}^{k_2} r''_j t(D''_j).$$

In fact, this is equivalent to the implication

$$\sum_{j=1}^{k} r_j \delta_{D_j} = 0 \Longrightarrow \sum_{j=1}^{k} r_j t(D_j) = 0, \text{ for all } k \in \mathbb{N}, r_j \in \mathbb{R}, D_j \subset \mathcal{L}.$$
 (2.3)

Suppose that the left-hand side of the implication (2.3) be satisfied. Each set  $D_j$  consists of r real numbers  $y_1^j, ..., y_r^j, j = 1, ..., k$ . By the hypothesis of the lemma, all these numbers are different. Therefore,

$$\delta_{D_j} = \sum_{i=1}^r \delta_{y_i^j}, \ j = 1, ..., k.$$
 (2.4)

Eq. (2.4) together with the left-hand side of (2.3) gives

$$\sum_{i=1}^{r} \sum_{j=1}^{k} r_j \delta_{y_i^j} = 0. {(2.5)}$$

Since the sets  $\{y_i^1, y_i^2, ..., y_i^k\}$ , i = 1, ..., r, are pairwise disjoint, we obtain from (2.5) that

$$\sum_{j=1}^{k} r_j \delta_{y_i^j} = 0, \ i = 1, ..., r.$$
 (2.6)

Let now  $x_1, ..., x_k$  be some (\*)-points of the sets  $D_1, ..., D_k$  respectively. Since by (2.2),  $y_i^j = h_i(x_j)$ , for i = 1, ..., r and j = 1, ..., k, it follows from (2.6) that the set  $\{x_1, ..., x_k\}$  is a closed path. Then by the condition of the sufficiency,  $\sum_{j=1}^{k} r_j f(x_j) = 0$ . Hence  $\sum_{j=1}^{k} r_j t(D_j) = 0$ . We have proved the implication (2.3) and hence the functional F is well defined.

Note that the functional F is linear (this can be easily seen from its definition). Let F' be a linear extension of F to the following space larger than S:

$$\mathcal{S}' = \left\{ \sum_{j=1}^k r_j \delta_{\omega_j} \right\},$$

where  $k \in \mathbb{N}$ ,  $r_j \in \mathbb{R}$ ,  $\omega_j \subset \Omega$ . As in the above, we do not fix the parameters k,  $r_j$  and  $\omega_j$ . Define the functions

$$g_i: Y_i \to \mathbb{R}, \ g_i(y_i) \stackrel{def}{=} F'(\delta_{u_i}), \ i = 1, ..., r.$$

Let x be an arbitrary point in X. Obviously, x is a (\*)-point of some set  $Y = \{y_1, ..., y_r\} \subset \mathcal{L}$ . Thus,

$$f(x) = t(Y) = F(\delta_Y) = F\left(\sum_{i=1}^r \delta_{y_i}\right) = F'\left(\sum_{i=1}^r \delta_{y_i}\right) = \sum_{i=1}^r F'(\delta_{y_i}) = \sum_{i=1}^r g_i(y_i) = \sum_{i=1}^r g_i(h_i(x)).$$

**Definition 2.3.** A closed path  $p = \{x_1, ..., x_n\}$  is said to be minimal if p does not contain any closed path as its proper subset.

For example, the set  $l = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (1,1,1)\}$  considered above is a minimal closed path with respect to the functions  $h_i(z_1, z_2, z_3) = z_i$ , i = 1, 2, 3. Adding the point (0,1,1) to l, we will have a closed path, but not minimal. The vector  $\lambda$  associated with  $l \cup \{(0,1,1)\}$  can be taken as (3,-1,-1,-2,2,-1).

A minimal closed path  $p = \{x_1, ..., x_n\}$  has the following obvious properties:

- (a) The vector  $\lambda$  associated with p by Eq. (2.1) is unique up to multiplication by a constant;
- (b) If in (2.1),  $\sum_{j=1}^{n} |\lambda_j| = 1$ , then all the numbers  $\lambda_j$ , j = 1, ..., n, are rational.

Thus, a minimal closed path p uniquely (up to a sign) defines the functional

$$G_p(f) = \sum_{j=1}^n \lambda_j f(x_j), \quad \sum_{j=1}^n |\lambda_j| = 1.$$

**Lemma 2.4.** The functional  $G_{p,\lambda}$  is a linear combination of functionals  $G_{p_1},...,G_{p_k}$ , where  $p_1,...,p_k$  are minimal closed paths in p.

*Proof.* Let  $\langle p, \lambda \rangle$  be a closed path-vector pair of X, where  $p = \{x_1, ..., x_n\}$  and  $\lambda = (\lambda_1, ..., \lambda_n)$ . Assume that p is not minimal. Let  $p_1 = \{y_1, ..., y_s\}$  be a minimal closed path in p and

$$G_{p_1}(f) = \sum_{j=1}^{s} \nu_j f(y_j), \quad \sum_{j=1}^{s} |\nu_j| = 1.$$

To prove the lemma, it is enough to show that  $G_{p,\lambda}$  is a linear combination of  $G_{p_1}$  and some functional  $G_{l,\theta}$ , where l is a closed path in X and a proper subset of p. Without loss of generality, we may assume that  $y_1 = x_1$ . Put

$$t_1 = \frac{\lambda_1}{\nu_1}.$$

Then the functional  $G_{p,\lambda} - t_1 G_{p_1}$  has the form

$$G_{p,\lambda} - t_1 G_{p_1} = \sum_{j=1}^k \theta_j f(z_j),$$

where  $z_j \in p$ ,  $\theta_j \neq 0$ , j = 1, ..., k. Clearly, the set  $l = \{z_1, ..., z_k\}$  is a closed path with the associated vector  $\theta = (\theta_1, ..., \theta_k)$ . Thus, we obtain that  $G_{l,\theta} = G_{p,\lambda} - t_1 G_{p_1}$ . Note that since  $x_1 \notin l$ , the closed path l is a proper subset of p.

**Theorem 2.5.** 1) Let X have closed paths. A function  $f: X \to \mathbb{R}$  belongs to the space  $\mathcal{B}(h_1, ..., h_r; X)$  if and only if  $G_p(f) = 0$  for any minimal closed path  $p \subset X$  with respect to the functions  $h_1, ..., h_r$ .

2) Let X has no closed paths. Then  $\mathcal{B}(h_1,...,h_r;X)=T(X)$ .

*Proof.* 1) The necessity is clear. Let us prove the sufficiency. On the strength of Lemma 2.4, it is enough to prove that if  $G_{p,\lambda}(f) = 0$  for any closed path-vector pair  $\langle p, \lambda \rangle$  of X, then  $f \in \mathcal{B}(X)$ .

Consider a system of intervals  $\{(a_i, b_i) \subset \mathbb{R}\}_{i=1}^r$  such that  $(a_i, b_i) \cap (a_j, b_j) = \emptyset$  for all the indices  $i, j \in \{1, ..., r\}, i \neq j$ . For i = 1, ..., r, let  $\tau_i$  be one-to-one mappings of  $\mathbb{R}$  onto  $(a_i, b_i)$ . Introduce the following functions on X:

$$h'_{i}(x) = \tau_{i}(h_{i}(x)), i = 1, ..., r.$$

It is clear that any closed path with respect to the functions  $h_1, ..., h_r$  is also a closed path with respect to the functions  $h'_1, ..., h'_r$ , and vice versa. Besides,  $h'_i(X) \cap h'_j(X) = \emptyset$ , for all  $i, j \in \{1, ..., r\}$ ,  $i \neq j$ . Then by Lemma 2.2,

$$f(x) = g'_1(h'_1(x)) + \dots + g'_r(h'_r(x)),$$

where  $g'_1, ..., g'_r$  are univariate functions depending on f. From the last equality we obtain that

$$f(x) = g'_1(\tau_1(h_1(x))) + \dots + g'_r(\tau_r(h_r(x))) = g_1(h_1(x)) + \dots + g_r(h_r(x)).$$

That is,  $f \in \mathcal{B}(X)$ .

2) Let  $f: X \to \mathbb{R}$  be an arbitrary function. First suppose that  $h_i(X) \cap h_j(X) = \emptyset$ , for all  $i, j \in \{1, ..., r\}$ ,  $i \neq j$ . In this case, the proof is similar to and even simpler than that of Lemma 2.2. Indeed, the set of all (\*)-points of Y consists of a single point, since otherwise we would have a closed path with two points, which contradicts the hypothesis of the 2-nd part of our theorem. Further, the well definition of the functional F becomes obvious, since the left-hand side of (2.3) also contradicts the nonexistence of closed paths. Thus, as in the proof of Lemma 2.2, we can extend F to the space  $\mathcal{S}'$  and then obtain the desired representation for the function f. Since f is arbitrary,  $T(X) = \mathcal{B}(X)$ .

Using the techniques from the proof of the 1-st part of our theorem, one can easily generalize the above argument to the case when the functions  $h_1, ..., h_r$  have arbitrary ranges.

**Theorem 2.6.** 
$$\mathcal{B}(h_1,...,h_r;X)=T(X)$$
 if and only if X has no closed paths.

Proof. The sufficiency immediately follows from Theorem 2.5. To prove the necessity, assume that X has a closed path  $p = \{x_1, ..., x_n\}$ . Let  $\lambda = (\lambda_1, ..., \lambda_n)$  be a vector associated with p by Eq. (2.1). Consider a function  $f_0$  on X with the property:  $f_0(x_i) = 1$ , for indices i such that  $\lambda_i > 0$  and  $f_0(x_i) = -1$ , for indices i such that  $\lambda_i < 0$ . For this function,  $G_{p,\lambda}(f_0) \neq 0$ . Then by Theorem 2.5,  $f_0 \notin \mathcal{B}(X)$ . Hence  $\mathcal{B}(X) \neq T(X)$ . The contradiction shows that X does not admit closed paths.

The condition whether X have closed paths or not, depends both on X and the functions  $h_1, ..., h_r$ . In the following sections, we see that if  $h_1, ..., h_r$  are "nice" functions (smooth functions with the simple structure. For example, ridge functions) and  $X \subset \mathbb{R}^d$  is a "rich" set (for example, the set with interior points), then X has always closed paths. Thus the representability by linear combinations of univariate functions with the fixed "nice" multivariate functions requires at least that X should not possess interior points. The picture is quite different when the functions  $h_1, ..., h_r$  are not "nice". Even in the case when they are continuous, we will see that many sets of  $\mathbb{R}^d$  (the unite cube, any compact subset of that, or even the whole space  $\mathbb{R}^d$  itself) may have no closed paths. If disregard the continuity, there exists even one function h such that every multivariate function is representable as  $g \circ h$  over any subset of  $\mathbb{R}^d$ .

# 3. The analogue of the Kolmogorov superposition theorem for multivariate functions

Let X be a set and  $h_i: X \to \mathbb{R}$ , i = 1, ..., r, be arbitrarily fixed functions. Consider a class A(X) of functions on X with the property: for any minimal closed path  $p \subset X$  with respect to the functions  $h_1, ..., h_r$  (if it exists), there is a function  $f_0$  in A(X) such

that  $G_p(f_0) \neq 0$ . Such classes will be called "permissible" function classes. Clearly, C(X) and B(X) are both permissible function classes (in case of C(X), X is considered to be a normal topological space).

**Theorem 3.1.** Let A(X) be a permissible function class. If  $\mathcal{B}(h_1, ..., h_r; X) = A(X)$ , then  $\mathcal{B}(h_1, ..., h_r; X) = T(X)$ .

The proof is simple and based on the material of the previous section. Assume for a moment that X admit a closed path p. The functional  $G_p$  annihilates all members of the set  $B(h_1, ..., h_r; X)$ . By the above definition of permissible function classes, A(X) contains a function  $f_0$  such that  $G_p(f_0) \neq 0$ . Therefore,  $f_0 \notin B(h_1, ..., h_r; X)$ . We see that the equality  $B(h_1, ..., h_r; X) = A(X)$  is impossible if X has a closed path. Thus X has no closed paths. Then by Theorem 2.6,  $\mathcal{B}(h_1, ..., h_r; X) = T(X)$ .

**Remark.** In the "if part" of Theorem 3.1, instead of  $\mathcal{B}(h_1,...,h_r;X)$  and A(X) one can take  $\mathcal{B}_c(h_1,...,h_r;X)$  and C(X) (or  $\mathcal{B}_b(h_1,...,h_r;X)$  and B(X)) respectively.

The main advantage of Theorem 3.1 is that we need not check directly if the set X has no closed paths, which in many cases may turn out to be very tedious task. Using this theorem, we can extend free-of-charge the existing superposition theorems for classes B(X) or C(X) (or some other permissible function classes) to all functions defined on X. For example, this theorem allows us to obtain the analogue of the Kolmogorov superposition theorem for all multivariate functions defined on the unit cube.

Corollary 3.2. Let  $d \geq 2$  and  $s_q$ , q = 1, ..., 2d + 1, be the Kolmogorov functions (1.2). Then each function  $f : \mathbb{I}^d \to \mathbb{R}$  can be represented by the formula (1.3), where  $g_q$  are univariate functions depending on f.

It should be remarked that Sternfeld [31], in particular, obtained that the formula (1.3) is valid for functions  $f \in B(\mathbb{I}^d)$  provided that  $g_q$  are bounded functions depending on f (see [15, chapter 1] for more detailed information and interesting discussions).

Let X be a compact metric space and  $h_i \in C(X)$ , i = 1, ..., r. The result of Sternfeld (see Introduction) and Theorem 3.1 give us the implications

$$\mathcal{B}_c(h_1, ..., h_r; X) = C(X) \Rightarrow \mathcal{B}_b(h_1, ..., h_r; X) = B(X) \Rightarrow \mathcal{B}(h_1, ..., h_r; X) = T(X).$$
 (3.1)

The first implication is invertible when r = 2 (see [31]). We want to show that the second is not invertible even in the case r = 2. The following interesting example is due to Khavinson [15, p.67].

Let  $X \subset \mathbb{R}^2$  consists of a broken line whose sides are parallel to the coordinate axis and whose vertices are

$$(0;0),(1;0),(1;1),(1+\frac{1}{2^2};1),(1+\frac{1}{2^2};1+\frac{1}{2^2}),(1+\frac{1}{2^2}+\frac{1}{3^2};1+\frac{1}{2^2}),...$$

We add to this line the limit point of the vertices  $(\frac{\pi^2}{6}, \frac{\pi^2}{6})$ . Let r=2 and  $h_1, h_2$  be the coordinate functions. Then the set X has no closed paths with respect to  $h_1$  and  $h_2$ . By Theorem 2.6, every function f on X is of the form  $g_1(x_1) + g_2(x_2)$ ,  $(x_1, x_2) \in X$ . Now construct a function  $f_0$  on X as follows. On the link joining (0;0) to (1;0)  $f_0(x_1, x_2)$  continuously increases from 0 to 1; on the link from (1;0) to (1;1) it continuously decreases from 1 to 0; on the link from (1;1) to  $(1+\frac{1}{2^2};1)$  it increases from 0 to  $\frac{1}{2}$ ; on the link from  $(1+\frac{1}{2^2};1)$  to  $(1+\frac{1}{2^2};1+\frac{1}{2^2})$  it decreases from  $\frac{1}{2}$  to 0; on the next link it increases from 0 to  $\frac{1}{3}$ , etc. At the point  $(\frac{\pi^2}{6}, \frac{\pi^2}{6})$  set the value of  $f_0$  equal to 0. Obviously,  $f_0$  is a continuous functions and by the above argument,  $f_0(x_1, x_2) = g_1(x_1) + g_2(x_2)$ . But  $g_1$  and  $g_2$  cannot be chosen as continuous functions, since they get unbounded as  $x_1$  and  $x_2$  tends to  $\frac{\pi^2}{6}$ . Thus,  $\mathcal{B}(h_1, h_2; X) = T(X)$ , but at the same time  $\mathcal{B}_c(h_1, h_2; X) \neq C(X)$  (or, equivalently,  $\mathcal{B}_b(h_1, h_2; X) \neq B(X)$ ).

We have seen that the unit cube in  $\mathbb{R}^d$  has no closed paths with respect to some 2d+1continuous functions (namely, the Kolmogorov functions  $s_q$  (1.2)). From the result of Ostrand [25] (see Introduction) it follows that d-dimensional compact sets X also have no closed paths with respect to some 2d+1 continuous functions on X. One may ask if there exists a finite family of functions  $\{h_i: \mathbb{R}^d \to \mathbb{R}\}_{i=1}^n$  such that any subset of  $\mathbb{R}^d$  does not admit closed paths with respect to this family? The answer is positive. This follows from the result of Demko [7]: there exist 2d+1 continuous functions  $\varphi_1,...,\varphi_{2d+1}$  defined on  $\mathbb{R}^d$ such that every bounded continuous function on  $\mathbb{R}^d$  is expressible in the form  $\sum_{i=1}^{2d+1} g \circ \varphi_i$ for some  $g \in C(\mathbb{R})$ . This theorem together with Theorem 2.6 yield that every function on  $\mathbb{R}^d$  is expressible in the form  $\sum_{i=1}^{2d+1} g_i \circ \varphi_i$  for some  $g_i : \mathbb{R} \to \mathbb{R}$ , i = 1, ..., 2d + 1. We do not yet know if  $g_i$  here or in Corollary 3.2 can be replaced by a single univariate function. We also don't know if the number 2d + 1 can be reduced so that the whole space of  $\mathbb{R}^d$  (or any d-dimensional compact subset of that, or at least the unit cube  $\mathbb{I}^d$ ) has no closed paths with respect to some continuous functions  $\varphi_1, ..., \varphi_k : \mathbb{R}^d \to \mathbb{R}$ , where k < 2d + 1. One of the basic results of Sternfeld [32] says that the dimension of a compact metric space X equals d if and only if there exist functions  $\varphi_1,...,\varphi_{2d+1} \in C(X)$  such that  $\mathcal{B}_c(\varphi_1,...,\varphi_{2d+1};X) = C(X)$  and for any family  $\{\psi_i\}_{i=1}^k \subset C(X), k < 2d+1$ , we have  $\mathcal{B}_c(\psi_1,...,\psi_k;X) \neq C(X)$ . In particular, from this result it follows that the number of terms in the Kolmogorov superposition theorem cannot be reduced. But since the equalities  $\mathcal{B}_c(X) = C(X)$  and  $\mathcal{B}(X) = T(X)$  are not equivalent, the above question on the nonexistence of closed paths in  $\mathbb{R}^d$  with respect to less than 2d+1 continuous functions is far from trivial.

If disregard the continuity, one can construct even one function  $\varphi: \mathbb{R}^d \to \mathbb{R}$  such that the whole space  $\mathbb{R}^d$  will not possess closed paths with respect to  $\varphi$  and therefore, every function  $f: \mathbb{R}^d \to \mathbb{R}$  will admit the representation  $f = g \circ \varphi$  with some univariate g depending on f. Our argument easily follows from Theorem 2.6 and the result of Sprecher [30]: for any natural number  $d, d \geq 2$ , there exist functions  $h_p: \mathbb{I} \to \mathbb{R}, p = 1, ..., d$ , such that every function  $f \in C(\mathbb{I}^d)$  can be represented in the form

$$f(x_1, ..., x_d) = g\left(\sum_{p=1}^d h_p(x_p)\right),$$

where g is a univariate (generally discontinuous) function depending on f.

### 4. Ridge functions

The set  $\mathcal{B}(h_1,...,h_r;X)$ , where  $h_i$ , i=1,...,r, are linear functionals over  $\mathbb{R}^d$ , or more precisely, the set

$$\mathcal{R}(X) = \mathcal{R}\left(\mathbf{a}^{1}, ..., \mathbf{a}^{r}; X\right) = \left\{\sum_{i=1}^{r} g_{i}\left(\mathbf{a}^{i} \cdot \mathbf{x}\right) : \mathbf{x} \in X \subset \mathbb{R}^{d}, \ g_{i} : \mathbb{R} \to \mathbb{R}, i = 1, ..., r\right\}$$
(4.1)

appears in many works (see, for example, [1,2,8,12,14,19,20,27]). Here,  $\mathbf{a}^i$ , i=1,...,r, are fixed vectors (directions) in  $\mathbb{R}^d \setminus \{\mathbf{0}\}$  and  $\mathbf{a}^i \cdot \mathbf{x}$  stands for the usual inner product of  $\mathbf{a}^i$  and  $\mathbf{x}$ . The functions  $g_i(\mathbf{a}^i \cdot \mathbf{x})$  involved in (4.1) are *ridge* functions. Such functions are used in the theory of PDE (where they are called *plane waves*, see, e.g., [13]), in statistics (see, e.g., [3,11]), in computerized tomography (see, e.g., [14,20]), in neural networks (see, e.g., [28] and a great deal of references therein). In modern approximation theory, ridge functions are widely used to approximate complicated multivariate functions (see, e.g., [4,6,10,18,19,22,23,24,26,27,34,36]). In this section, we are going to make some remarks on the representation of multivariate functions by sums of ridge functions.

The problem of representation of multivariate functions by functions in  $\mathcal{R}(\mathbf{a}^1,...,\mathbf{a}^r;X)$  is not new. Braess and Pinkus [1] considered the partial case of this problem: characterize a set of points  $(\mathbf{x}^1,...,\mathbf{x}^k) \subset \mathbb{R}^d$  such that for any data  $\{\alpha_1,...,\alpha_k\} \subset \mathbb{R}$  there exists a function  $g \in \mathcal{R}(\mathbf{a}^1,...,\mathbf{a}^r;\mathbb{R}^d)$  satisfying  $g(\mathbf{x}^i) = \alpha_i$ , i = 1,...,k. In connection with this problem, they introduced the notion of the NI-property (non interpolation property) and MNI-property (minimal non interpolation property) of a finite set of points as follows:

MNI-property (minimal non interpolation property) of a finite set of points as follows: Given directions  $\{\mathbf{a}^j\}_{j=1}^r \subset \mathbb{R}^d \setminus \{\mathbf{0}\}$ , we say that a set of points  $\{\mathbf{x}^i\}_{i=1}^k \subset \mathbb{R}^d$  has the NI-property with respect to  $\{\mathbf{a}^j\}_{j=1}^r$ , if there exists  $\{\alpha_i\}_{i=1}^k \subset \mathbb{R}$  such that we cannot find a function  $g \in \mathcal{R}\left(\mathbf{a}^1,...,\mathbf{a}^r;\mathbb{R}^d\right)$  satisfying  $g(\mathbf{x}^i) = \alpha_i, i = 1,...,k$ . We say that the set  $\{\mathbf{x}^i\}_{i=1}^k \subset \mathbb{R}^d$  has the MNI-property with respect to  $\{\mathbf{a}^j\}_{j=1}^r$ , if  $\{\mathbf{x}^i\}_{i=1}^k$  but no proper subset thereof has the NI-property.

It follows from Theorem 2.6 that a set  $\{\mathbf{x}^i\}_{i=1}^k$  has the NI-property if and only if  $\{\mathbf{x}^i\}_{i=1}^k$  contains a closed path with respect to the functions  $h_i = \mathbf{a}^i \cdot \mathbf{x}$ , i = 1, ..., r (or, simply, to the directions  $\mathbf{a}^i$ , i = 1, ..., r) and the MNI-property if and only if the set  $\{\mathbf{x}^i\}_{i=1}^k$  itself is a minimal closed path with respect to the given directions. Taking into account this argument and Definitions 2.1 and 2.3, we obtain that the set  $\{\mathbf{x}^i\}_{i=1}^k$  has the NI-property if and only if there is a vector  $\mathbf{m} = (m_1, ..., m_k) \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$  such that

$$\sum_{j=1}^{k} m_j g(\mathbf{a}^i \cdot \mathbf{x}^j) = 0,$$

for i = 1, ..., r and all functions  $g : \mathbb{R} \to \mathbb{R}$ . This set has the MNI-property if and only if the vector  $\mathbf{m}$  has the additional properties: it is unique up to multiplication by a constant and all its components are different from zero. This special corollary of Theorem 2.6 was proved in [1].

Since ridge functions are nice functions of simple structure, representation of every multivariate function by linear combinations of such functions may not be possible over many sets in  $\mathbb{R}^d$ . The following remark indicates the class of sets having interior points.

**Remark.** Let  $X \subset \mathbb{R}^d$  have nonempty interior. Then  $\mathcal{R}(\mathbf{a}^1,...,\mathbf{a}^r;X) \neq T(X)$ .

Indeed, let **y** be a point in the interior of X. Consider vectors  $\mathbf{b}^i$ , i = 1, ..., r, with sufficiently small coordinates such that  $\mathbf{a}^i \cdot \mathbf{b}^i = 0$ , i = 1, ..., r. Note that the vectors  $\mathbf{b}^i$ , i = 1, ..., r, can be chosen pairwise linearly independent. With each vector  $\varepsilon = (\varepsilon_1, ..., \varepsilon_r)$ ,  $\varepsilon_i \in \{0, 1\}$ , i = 1, ..., r, we associate the point

$$\mathbf{x}_{arepsilon} = \mathbf{y} + \sum_{i=1}^{r} arepsilon_{i} \mathbf{b}^{i}.$$

Since the coordinates of  $\mathbf{b}^i$  are sufficiently small, we may assume that all the points  $\mathbf{x}_{\varepsilon}$  are in the interior of X. We correspond each point  $\mathbf{x}_{\varepsilon}$  to the number  $(-1)^{|\varepsilon|}$ , where  $|\varepsilon| = \varepsilon_1 + \cdots + \varepsilon_r$ . One may easily verify that the pair  $\langle \{\mathbf{x}_{\varepsilon}\}, \{(-1)^{|\varepsilon|}\} \rangle$  is a closed path-vector pair of X. Therefore, by Theorem 2.6,  $\mathcal{R}(\mathbf{a}^1, ..., \mathbf{a}^r; X) \neq T(X)$ .

It should be noted that the above method of construction of the set  $\{\mathbf{x}_{\varepsilon}\}$  is due to Lin and Pinkus [19].

Let us now give some examples of sets over which the representation by linear combinations of ridge functions is possible.

- (1) Let r=2 and X be the union of two parallel lines not perpendicular to the given directions  $\mathbf{a}^1$  and  $\mathbf{a}^2$ . Then X has no closed paths with respect to  $\{\mathbf{a}^1, \mathbf{a}^2\}$ . Therefore, by Theorem 2.6,  $\mathcal{R}(\mathbf{a}^1, \mathbf{a}^2; X) = T(X)$ .
- (2) Let r=2,  $\mathbf{a}^1=(1,1)$ ,  $\mathbf{a}^2=(1,-1)$  and X be the graph of the function  $y=\arcsin(\sin x)$ . Then X has no closed paths and hence  $\mathcal{R}(\mathbf{a}^1,\mathbf{a}^2;X)=T(X)$ .
- (3) Let now given r directions  $\{\mathbf{a}^j\}_{j=1}^r$  and r+1 points  $\{\mathbf{x}^i\}_{i=1}^{r+1} \subset \mathbb{R}^d$  such that

$$\mathbf{a}^1 \cdot \mathbf{x}^i = \mathbf{a}^1 \cdot \mathbf{x}^j \neq \mathbf{a}^1 \cdot \mathbf{x}^2, \text{ for } 1 \leq i, j \leq r+1, i, j \neq 2$$
  
 $\mathbf{a}^2 \cdot \mathbf{x}^i = \mathbf{a}^2 \cdot \mathbf{x}^j \neq \mathbf{a}^2 \cdot \mathbf{x}^3, \text{ for } 1 \leq i, j \leq r+1, i, j \neq 3$   
......

$$\mathbf{a}^r \cdot \mathbf{x}^i = \mathbf{a}^r \cdot \mathbf{x}^j \neq \mathbf{a}^r \cdot \mathbf{x}^{r+1}, \text{ for } 1 \leq i, j \leq r.$$

The simplest data realizing these equations are the basis directions in  $\mathbb{R}^d$  and the points (0,0,...,0), (1,0,...,0), (0,1,...,0), ..., (0,0,...,1). From the first equation we obtain that  $\mathbf{x}^2$  cannot be a point of any closed path in  $X = {\mathbf{x}^1,...,\mathbf{x}^{r+1}}$ . Sequentially, from the second, third, ..., r-th equations it follows that the points  $\mathbf{x}^3, \mathbf{x}^4, ..., \mathbf{x}^{r+1}$  also cannot be points of closed paths in X respectively. Thus the set X does not contain closed paths at all. By Theorem 2.6,  $\mathcal{R}(\mathbf{a}^1, ..., \mathbf{a}^r; X) = T(X)$ .

(4) Let given directions  $\{\mathbf{a}^j\}_{j=1}^r$  and a curve  $\gamma$  in  $\mathbb{R}^d$  such that for any  $c \in \mathbb{R}$ ,  $\gamma$  has at most one common point with at least one of the hyperplanes  $\mathbf{a}^j \cdot \mathbf{x} = c, \ j = 1, ..., r$ . By Definition 2.1, the curve  $\gamma$  has no closed paths and hence  $\mathcal{R}(\mathbf{a}^1, ..., \mathbf{a}^r; \gamma) = T(\gamma)$ .

At the end we want to draw the reader's attention to one more problem concerning the set  $\mathcal{R}(\mathbf{a}^1,...,\mathbf{a}^r;X)$ . The problem is to determine if a given function  $f:X\to\mathbb{R}$  belongs to this set. One solution is proposed by Theorem 2.5: consider all minimal closed paths p of X and check if  $G_p(f)=0$ . This problem was considered by some other authors too. For example, Lin and Pinkus [19] characterized the set  $\mathcal{R}_c(\mathbf{a}^1,...,\mathbf{a}^r;\mathbb{R}^d)$  in terms of the ideal of polynomials vanishing at all points  $\lambda \mathbf{a}^i \in \mathbb{R}^d$ , i=1,...,r,  $\lambda \in \mathbb{R}$ . Two more characterizations of  $\mathcal{R}_c(\mathbf{a}^1,...,\mathbf{a}^r;\mathbb{R}^d)$  may be found in Diaconis and Shahshahani [8].

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