Stochastic Gradient Descent from Convex Optimization Perspective

compiled by D.Gueorguiev, 3/29/2025

# Gradient Descent

Consider unconstrained smooth convex optimization

That is, is convex and differentiable with . Denote the optimal criterion value by and a solution by .

Gradient descent algorithm

choose initial point

repeat

stop when certain stopping criterion is satisfied

A grid with colored lines

Description automatically generated

A grid with colored lines and dots

Description automatically generated

At each iteration, consider the expansion

Quadratic approximation, replacing usual Hessian by

linear approximation to

proximity term to , with weight

Choose next point to minimize quadratic approximation:

A line with dots and a dotted line

Description automatically generated

Blue point is , red point is . The latter is given with

Question: How do we choose step sizes?

Simply taking for all can diverge if is too big.

A graph of a function

Description automatically generatedConsider , gradient descent after 8 steps:

Can be slow if is too small. Same example, gradient descent after 100 steps:

A graph of a graph of a stringed object

Description automatically generated with medium confidence

Convergence analysis later will give us a precise idea of “just right”

## Backtracking Line Search

One way to adaptively choose the step size is to use backtracking line search:

1 ) First fix parameters and

2 ) At each iteration, start with ,

while iter\_count < max\_iter\_count:

if :

shrink as

else:

perform gradient descent update

Simple and tends to work well in practice (further simplification: take )

A graph of a function

Description automatically generatedBacktracking interpretation

Here

Setting , backtracking picks up roughly the right step size (12 outer steps, 40 steps total).

A diagram of a dotted line

Description automatically generated with medium confidence

## Exact Line Search

We could also choose step to do the best we can along direction of negative gradient, called exact line search:

Approximations to exact line search are typically not as efficient as backtracking and it is typically not worth it.

## Convergence Analysis

Assume that f is convex and differentiable, with , and additionally that is Lipschitz continuous with constant ,

for any

or twice differentiable i.e. .

**Theorem**: Gradient descent with fixed step size satisfies

and same result holds for backtracking, with replaced by .

We say gradient descent has convergence rate . That is, it finds -suboptimal point in iterations.

## Analysis for strong convexity

**Definition**: strong convexity of means is convex for some when twice differentiable i.e for all .

**Theorem**: Assuming Lipschitz gradient as before and strong convexity one can show that gradient descent with fixed step size or with backtracking line search satisfies

where

Rate under strong convexity is , exponentially fast. That is, it finds -suboptimal point in iterations.

The gradient descent convergence is sometimes called linear convergence because the objective function value versus iteration curve looks linear on semi-log plot.

A graph of a function

Description automatically generated

//TODO: finish the Gradient Descent section

# The Sub-Gradient Method

//TODO: finish the Sub-Gradient Method

# Proximal Gradient Descent

//TODO: finish the Proximal Gradient Descent

# Stochastic Gradient Descent

//TODO: finish the Stochastic Gradient Descent

# References

[1] [Convex Optimization, Steven Boyd, Lieven Vandenberghe, 2009](https://github.com/dimitarpg13/optimization_classification_regression/blob/main/literature/books/ConvexOptimization_Boyd_2004.pdf)

[2] [Gradient Descent, Convex Optimization 10-725, CMU, slides, Ryan Tibshirani](https://github.com/dimitarpg13/statistical_learning_and_kernel_methods/blob/main/literature/articles/gradient_descent/grad-descent_Ryan_Tibshirani_slides.pdf)

[3] [Sub-Gradient Method, Convex Optimization 10-725, CMU, slides, Ryan Tibshirani](https://github.com/dimitarpg13/statistical_learning_and_kernel_methods/blob/main/literature/articles/gradient_descent/sub-gradient-method-Ryan_Tibshirani_slides.pdf)

[4] [Proximal Gradient Descent, Convex Optimization 10-725, CMU, slides, Ryan Tibshirani](https://github.com/dimitarpg13/statistical_learning_and_kernel_methods/blob/main/literature/articles/gradient_descent/proximal-gradient-descent_Ryan_Tibshirani_slides.pdf)

[5] [Stochastic Gradient Descent, Convex Optimization 10-725, CMU, slides, Ryan Tibshirani](https://github.com/dimitarpg13/statistical_learning_and_kernel_methods/blob/main/literature/articles/gradient_descent/stochastic-gradient-descent_Ryan_Tibshirani_slides.pdf)

[6] [Theory of Convex Functions, ORF 523, Princeton U., A. Ahmadi, 2015](https://github.com/dimitarpg13/optimization_classification_regression/blob/main/literature/lecture_notes/convexity/Theory_of_convex_functions_ORF523_S16_Lec7_Ahmadi_2015.pdf)

[7] Space

# Appendix

## Cauchy-Schwarz Inequality

Let and are vectors in

Then we have:

## Strong Convexity and Strong Convexity with modulus

**Definition** *Strong Convexity (by Boyd and Vandenberghe)*

a function is strongly convex on iff

(A1.1)

For we have:

for some on the line segment .

By the strong convexity assumption (A1.1) the last term on the RHS is at least so we have the inequality

(A1.2)

When , we recover the basic inequality characterizing convexity; for we obtain a better lower bound on than follows from convexity alone.

We will show that the inequality (A1.2) can be used to bound , which is the suboptimality of the point , in terms of . The righthand side of (A1.2) is a convex function of (for fixed ). Setting the gradient with respect to equal to zero, we find that minimizes the RHS. Therefore, we have

.

Since this holds for any , we have

(A1.3)

This inequality shows that if the gradient is small at a point, then the point is nearly optimal. The inequality (A1.3) can also be interpreted as a condition for *suboptimality* which generalizes the optimality condition (A1.2):

(A1.4)

We can also derive a bound on , the distance between and any optimal point , in terms of :

(A1.5)

To see this, we apply (A1.2) with to obtain

where we use the Cauchy-Schwarz inequality in the second inequality.

**Theorem** *Equivalence of Strong Convexity definitions*

a function is strongly convex of modulus if and we have

or equivalently

Proof:

//TODO: finish the proof for the equivalence of strong convexity definitions