

Mathematical Models of Neural Systems and Cognitive Functions - Exercise Gjorgjieva II

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June 12, 2024

1 Exercise 1

Consider two presynaptic neurons connected to a single postsynaptic neuron (panel **A**). Their rates are r_1 , r_2 and r_{post} . Neurons 1 and 2 project excitatory connections to the postsynaptic neuron, which also receives an external current h .

- Assuming linear interactions and stable rates, express the steady-state r_{post} (no need to write a differential equation, just the expression for the rate) as a function of the presynaptic rates, the synaptic weights, and the input current.

Answer. $r_{\text{post}} = \sum_{i=1}^2 w_i r_i + h^{\text{input}}$, where w_i denotes the weight from neuron i to the post-synaptic neuron, and h^{input} the input current.

Now consider a Hebbian rate-based plasticity rule, where the synaptic change is proportional to the product of the pre- and post-synaptic rates $\Delta w = \alpha r_{\text{pre}} r_{\text{post}}$. Assume that α is very small, so that the changes in synaptic weights have negligible influences on the neuronal dynamics for the duration of the experiment.

- Assuming both synaptic weights are plastic, write the equation for the change in the weights Δw_1 and for Δw_2 solely as a function of the presynaptic rates, the synaptic weights, and the input current, using your result from (a). Hint: Think of the Hebbian correlation-based rule.

Answer. For the weight between r_1 and r_{post} we have

$$\Delta w_1 = \alpha r_1 r_{\text{post}} \quad (1)$$

$$= \alpha r_1 \sum_{i=1}^2 w_i r_i + \alpha r_1 h^{\text{input}} \quad (2)$$

$$= \alpha w_1 r_1^2 + \alpha w_2 r_2 r_1 + \alpha r_1 h^{\text{input}}. \quad (3)$$

Similarly for the weight between r_1 and r_{post} we have

$$\Delta w_2 = \alpha r_2 r_{post} \quad (4)$$

$$= \alpha r_2 \sum_{i=1}^2 w_i r_i + \alpha r_2 h^{\text{input}} \quad (5)$$

$$= \alpha w_2 r_2^2 + \alpha w_2 r_2 r_1 + \alpha r_2 h^{\text{input}}. \quad (6)$$

Neuron 1 and 2 have a bi-modal response, moving between a high and a low rate. In the figure below each dot is a sample of the joint firing activity of the neurons at a single time interval. We now consider four possibilities. Panel **B**: the two neurons are perfectly correlated: they both fire at high level, or both at low level. Panel **C**: like the previous, except neuron 1 has a higher rate level. Panel **D**: the two neurons are perfectly anti-correlated, when one has high firing, the other has low firing. Panel **E**: the two neurons are independent from each other.

1. Rank the **B,C,D,E** configurations in terms of how much increase in Δw_1 and Δw_2 you would see due to the plasticity rule defined above, explain your answer.
2. Define the Hebbian covariance-based plasticity rule in terms of rates, and explain how it differs from the Hebbian correlation-based rule from (b). Is there any change in the ranking of the four experiments?

Answer. From the previous question we have $\Delta w_1 = \alpha w_1 r_1^2 + \alpha w_2 r_2 r_1 + \alpha r_1 h^{\text{input}}$ and $\Delta w_2 = \alpha w_2 r_2^2 + \alpha w_2 r_2 r_1 + \alpha r_2 h^{\text{input}}$, and we assume that h^{input} is constant in time and in units of Hz, so $\langle h^{\text{input}} \rangle = h^{\text{input}}$. (Angle brackets $\langle \cdot \rangle$ denote averages over time.)

For setting B We consider that half of the time the neurons fire together at 30 Hz and half of the time at 10 Hz. Thus to compute the expected changes in the weights we will compute the expectations of each term in the above equations.

For Δw_1 we have:

- $\langle \alpha w_1 r_1^2 \rangle = \alpha w_1 (0.5 \times 30^2 + 0.5 \times 10^2) \text{Hz}^2 = \alpha w_1 500 \text{ Hz}^2$
- $\langle \alpha w_2 r_2 r_1 \rangle = \alpha w_2 (0.5 \times 30^2 + 0.5 \times 10^2) \text{Hz}^2 = \alpha w_2 500 \text{ Hz}^2$
- $\langle \alpha r_1 h^{\text{input}} \rangle = \alpha 20 h^{\text{input}} \text{ Hz}$

For Δw_2 we have:

- $\langle \alpha w_2 r_2^2 \rangle = \alpha w_2 (0.5 \times 30^2 + 0.5 \times 10^2) \text{Hz}^2 = \alpha w_2 500 \text{ Hz}^2$
- $\langle \alpha w_1 r_2 r_1 \rangle = \alpha w_1 (0.5 \times 30^2 + 0.5 \times 10^2) \text{Hz}^2 = \alpha w_1 500 \text{ Hz}^2$
- $\langle \alpha r_2 h^{\text{input}} \rangle = \alpha 20 h^{\text{input}} \text{ Hz}$

For setting C We consider that half of the time the neurons fire together at 35 and 30 Hz and half of the time at 15 and 10 Hz. Thus to compute the expected changes in the weights we will compute the expectations of each term in the above equations.

For Δw_1 we have:

- $\langle \alpha w_1 r_1^2 \rangle = \alpha w_1 (0.5 \times 35^2 + 0.5 \times 15^2) \text{Hz}^2 = \alpha w_1 725 \text{ Hz}^2$
- $\langle \alpha w_2 r_2 r_1 \rangle = \alpha w_2 (0.5 \times 30 \times 35 + 0.5 \times 10 \times 15) \text{Hz}^2 = \alpha w_2 600 \text{ Hz}^2$
- $\langle \alpha r_1 h^{\text{input}} \rangle = \alpha 25 h^{\text{input}} \text{ Hz}$

For Δw_2 we have:

- $\langle \alpha w_2 r_2^2 \rangle = \alpha w_2 (0.5 \times 30^2 + 0.5 \times 10^2) \text{Hz}^2 = \alpha w_2 500 \text{ Hz}^2$
- $\langle \alpha w_1 r_2 r_1 \rangle = \alpha w_1 (0.5 \times 35 \times 30 + 0.5 \times 10 \times 15) \text{Hz}^2 = \alpha w_1 600 \text{ Hz}^2$
- $\langle \alpha r_2 h^{\text{input}} \rangle = \alpha 20 h^{\text{input}} \text{ Hz}$

For setting D We consider that half of the time the neurons fire together at 30 and 10 Hz and half of the time at 10 and 30 Hz. Thus to compute the expected changes in the weights we will compute the expectations of each term in the above equations.

For Δw_1 we have:

- $\langle \alpha w_1 r_1^2 \rangle = \alpha w_1 (0.5 \times 30^2 + 0.5 \times 10^2) \text{Hz}^2 = \alpha w_1 500 \text{ Hz}^2$
- $\langle \alpha w_2 r_2 r_1 \rangle = \alpha w_2 (0.5 \times 30 \times 10 + 0.5 \times 10 \times 30) \text{Hz}^2 = \alpha w_2 300 \text{ Hz}^2$
- $\langle \alpha r_1 h^{\text{input}} \rangle = \alpha 20 h^{\text{input}} \text{ Hz}$

For Δw_2 we have:

- $\langle \alpha w_2 r_2^2 \rangle = \alpha w_2 (0.5 \times 30^2 + 0.5 \times 10^2) \text{Hz}^2 = \alpha w_2 500 \text{ Hz}^2$
- $\langle \alpha w_1 r_2 r_1 \rangle = \alpha w_1 (0.5 \times 30 \times 10 + 0.5 \times 10 \times 30) \text{Hz}^2 = \alpha w_1 300 \text{ Hz}^2$
- $\langle \alpha r_2 h^{\text{input}} \rangle = \alpha 20 h^{\text{input}} \text{ Hz}$

For setting E We consider that half of the time the neurons fire together at 30 and 10 Hz and half of the time at 10 and 30 Hz. Thus to compute the expected changes in the weights we will compute the expectations of each term in the above equations.

For Δw_1 we have:

- $\langle \alpha w_1 r_1^2 \rangle = \alpha w_1 (0.5 \times 30^2 + 0.5 \times 10^2) \text{Hz}^2 = \alpha w_1 500 \text{ Hz}^2$
- $\langle \alpha w_2 r_2 r_1 \rangle = \alpha w_2 \langle r_2 \rangle \langle r_1 \rangle = \alpha w_2 (0.5 \times 30 + 0.5 \times 10) \times (0.5 \times 30 + 0.5 \times 10) \text{Hz}^2 = \alpha w_2 400 \text{ Hz}^2$
- $\langle \alpha r_1 h^{\text{input}} \rangle = \alpha 20 h^{\text{input}} \text{ Hz}$

For Δw_2 we have:

- $\langle \alpha w_2 r_2^2 \rangle = \alpha w_2 (0.5 \times 30^2 + 0.5 \times 10^2) \text{Hz}^2 = \alpha w_2 500 \text{ Hz}^2$
- $\langle \alpha w_1 r_2 r_1 \rangle = \alpha w_2 \langle r_2 \rangle \langle r_1 \rangle = \alpha w_1 (0.5 \times 30 + 0.5 \times 10) \times (0.5 \times 30 + 0.5 \times 10) \text{Hz}^2 = \alpha w_1 400 \text{ Hz}^2$
- $\langle \alpha r_2 h^{\text{input}} \rangle = \alpha 20 h^{\text{input}} \text{ Hz}$

Assuming that $w_1 = w_2$ for the ranking, the we have C, B, E, D

For question 2: $\Delta w_i = \alpha (r_{\text{pre}} - \langle r_{\text{pre}} \rangle) (r_{\text{post}} - \langle r_{\text{post}} \rangle)$.

