Wave train decomposition

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This writeup contains an improved algorithm to replace the 2007 version.

Formulation

Let $x(t) \in \mathbb{R}^N$ with $t = 1, \ldots, T$ denote a short (usually 0.5–10 s) epoch of a discretized N-channel EEG recording.

We aim to decompose $x(t)$ into a sum of K trains of identical short N-channel waveforms $w_k(\tau) \in \mathbb{R}^N$ defined on the interval $-d \le \tau \le +d$:

$$\begin{aligned} x(t) &= \sum_{k=1}^{K} \sum_{\tau=-d}^{+d} u_k(t-\tau) w_k(\tau) + \varepsilon(t) \\ &\equiv \sum_{k=1}^{K} \sum_{t'=t-d}^{t+d} u_k(t') w_k(t'-t) + \varepsilon(t) \\ &\equiv \sum_{k=1}^{K} (u_k * w_k)(t) + \varepsilon(t) \end{aligned} \tag{1}$$

where $u_k(t) \in \{0, 1\}$ is the occurrence indicator function comprising 1s at time points when kth waveform occurs in the recording and 0s otherwise; the operator $*$ denotes convolution.

Algorithm

The algorithm is initiated by assigning random uniformly distributed numbers from $[0, 1]$ to the occurrence function $u_k(t)$. Until the algorithm converges, the occurrence function is allowed to assume fractional values between 0 and 1. We define the loss function with a penalty term for the nonbinarity of $u_k(t)$:

$$\mathcal{L} = \sum_{t=1}^{T} \|\varepsilon(t)\|^2 + \lambda \sum_{t=1}^{T} \sum_{k=1}^{K} (u_k(t) - u_k^2(t)) \tag{2}$$

The gradient of the loss function with respect to the waveform coefficients is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial w_k(\tau)} &= -2 \sum_{t=1}^{T} \varepsilon(t) u_k(t-\tau) \\ &\equiv -2(\varepsilon \star u_k)(\tau) \end{aligned} \tag{3}$$

where $\star$ denotes correlation.

Similarly, the gradient with respect to the occurrence indicator function is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u_k(t)} &= -2 \sum_{t'=1}^{T} \varepsilon(t') w_k(t-t') + 2\lambda(1 - 2u_k(t)) \\ &= -2(\varepsilon * w_k)(t) + 2\lambda(1 - 2u_k(t)) \end{aligned} \tag{4}$$

These gradients provide the update rules for gradient descent:

$$\Delta w_k(\tau) = \gamma \cdot (\varepsilon \star u_k)(\tau) \tag{5}$$

$$\Delta u_k(t) = \gamma \left[(\varepsilon * w_k)(t) - \lambda(1 - 2u_k(t)) \right] \tag{6}$$

with bounded $u_k(t) \in [0, 1]$. The update coefficients γ decreases with the convergence of the algorithm while the nonbinarity penalty coefficient λ increases.