2

TIME-INVARIANT LINEAR SYSTEMS

Time-invariant linear systems no doubt form the most important class of dynamical systems considered in practice and in the literature. It is true that they represent idealizations of the processes encountered in real life. But, even so, the approximations involved are often justified, and design considerations based on linear theory lead to good results in many cases.

A treatise of linear systems theory is a standard ingredient in basic engineering education, and the reader has no doubt some knowledge of this topic. Anyway, in this chapter we shall provide a refresher on some basic concepts that will be instrumental for the further development in this book. In Section 2.1 we shall discuss the impulse response and various ways of describing and understanding disturbances, as well as introduce the transfer-function concept. In Section 2.2 we study frequency-domain interpretations and also introduce the periodogram. Section 2.3 gives a unified setup of spectra of deterministic and stochastic signals that will be used in the remainder of this book. In Section 2.4 a basic ergodicity result is proved. The development in these sections is for systems with a scalar input and a scalar output. Section 2.5 contains the corresponding expressions for multivariable systems.

2.1 IMPULSE RESPONSES, DISTURBANCES, AND TRANSFER FUNCTIONS

Impulse Response

Consider a system with a scalar input signal $u(t)$ and a scalar output signal $y(t)$ (Figure 2.1). The system is said to be *time invariant* if its response to a certain input signal does not depend on absolute time. It is said to be *linear* if its output response to a linear combination of inputs is the same linear combination of the output responses of the individual inputs. Furthermore, it is said to be *causal* if the output at a certain time depends on the input up to that time only.

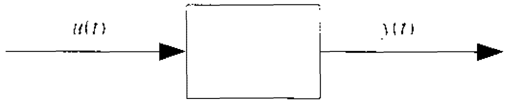

Figure 2.1 The system.

It is well known that a linear, time-invariant, causal system can be described by its *impulse response* (or *weighting function*) $g(\tau)$ as follows:

$$y(t) = \int_{\tau=0}^{\infty} g(\tau)u(t-\tau)d\tau \tag{2.1}$$

Knowing $\{g(\tau)\}_{\tau=0}^{\infty}$ and knowing $u(s)$ for $s \le t$, we can consequently compute the corresponding output $y(s)$, $s \le t$ for any input. *The impulse response is thus a complete characterization of the system.*

Sampling

In this book we shall almost exclusively deal with observations of inputs and outputs in discrete time, since this is the typical data-acquisition mode. We thus assume $y(t)$ to be observed at the *sampling instants* $t_k = kT$, $k = 1, 2, \ldots$:

$$y(kT) = \int_{\tau=0}^{\infty} g(\tau)u(kT-\tau)d\tau \tag{2.2}$$

The interval T will be called the *sampling interval*. It is, of course, also possible to consider the situation where the sampling instants are not equally spread.

Most often, in computer control applications, the input signal $u(t)$ is kept constant between the sampling instants:

$$u(t) = u_k, \qquad kT \le t < (k+1)T \tag{2.3}$$

This is mostly done for practical implementation reasons, but it will also greatly simplify the analysis of the system. Inserting (2.3) into (2.2) gives

$$\begin{aligned} y(kT) &= \int_{\tau=0}^{\infty} g(\tau)u(kT-\tau)d\tau = \sum_{\ell=1}^{\infty} \int_{\tau=(\ell-1)T}^{\ell T} g(\tau)u(kT-\tau)d\tau \\ &= \sum_{\ell=1}^{\infty} \left[\int_{\tau=(\ell-1)T}^{\ell T} g(\tau)d\tau \right] u_{k-\ell} = \sum_{\ell=1}^{\infty} g_T(\ell)u_{k-\ell} \end{aligned} \tag{2.4}$$

where we defined

$$g_T(\ell) = \int_{\tau=(\ell-1)T}^{\ell T} g(\tau)d\tau \tag{2.5}$$

The expression (2.4) tells us what the output will be at the sampling instants. Note that no approximation is involved if the input is subject to (2.3) and that it is sufficient to know the sequence $\{g_T(\ell)\}_{\ell=1}^{\infty}$ in order to compute the response to the input. The relationship (2.4) describes a *sampled-data system*, and we shall call the sequence $\{g_T(\ell)\}_{\ell=1}^{\infty}$ the impulse response of that system.

Even if the input is not piecewise constant and subject to (2.3), the representation (2.4) might still be a reasonable approximation, provided $u(t)$ does not change too much during a sampling interval. See also the following expressions (2.21) to (2.26). Intersample behavior is further discussed in Section 13.3.

We shall stick to the notation (2.3) to (2.5) when the choice and size of T are essential to the discussion. For most of the time, however, we shall for ease of notation assume that T is one time unit and use t to enumerate the sampling instants. We thus write for (2.4)

$$y(t) = \sum_{k=1}^{\infty} g(k)u(t-k), \qquad t = 0, 1, 2, \dots \tag{2.6}$$

For sequences, we shall also use the notation

$$y_s^t = (y(s), y(s+1), \dots, y(t)) \tag{2.7}$$

and for simplicity

$$y_1^t = y^t$$

Disturbances

According to the relationship (2.6), the output can be exactly calculated once the input is known. In most cases this is unrealistic. There are always signals beyond our control that also affect the system. Within our linear framework we assume that such effects can be lumped into an additive term $v(t)$ at the output (see Figure 2.2):

$$y(t) = \sum_{k=1}^{\infty} g(k)u(t-k) + v(t) \tag{2.8}$$

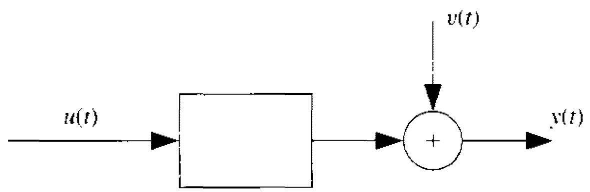

Figure 2.2 System with disturbance.

There are many sources and causes for such a disturbance term. We could list:

- Measurement noise: The sensors that measure the signals are subject to noise and drift.
- Uncontrollable inputs: The system is subject to signals that have the character of inputs, but are not controllable by the user. Think of an airplane, whose movements are affected by the inputs of rudder and aileron deflections, but also by wind gusts and turbulence. Another example could be a room, where the temperature is determined by radiators, whose effect we control, but also by people ($\approx$ 100 W per person) who may move in and out in an unpredictable manner.

The character of the disturbances could also vary within wide ranges. Classical ways of describing disturbances in control have been to study steps, pulses, and sinusoids, while in stochastic control the disturbances are modeled as realizations of stochastic processes. See Figures 2.3 and 2.4 for some typical, but mutually quite different, disturbance characteristics. The disturbances may in some cases be separately measurable, but in the typical situation they are noticeable only via their effect on the output. If the impulse response of the system is known, then of course the actual value of the disturbance $v(t)$ can be calculated from (2.8) at time t.

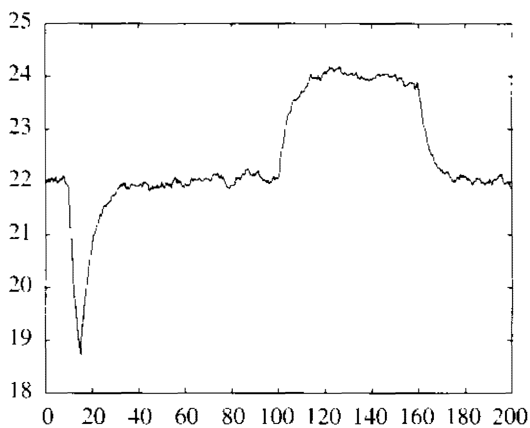

Figure 2.3 Room temperature.

The assumption of Figure 2.2 that the noise enters additively to the output implies some restrictions. Sometimes the measurements of the inputs to the system may also be noise corrupted ("error-in-variable" descriptions). In such cases we take a pragmatic approach and regard the measured input values as the actual inputs $u(t)$ to the process, and their deviations from the true stimuli will be propagated through the system and lumped into the disturbance $v(t)$ of Figure 2.2.

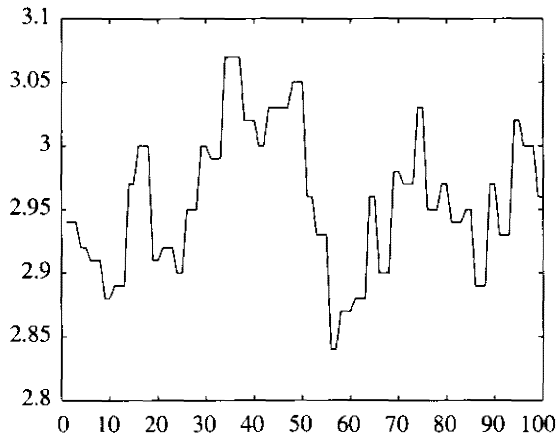

Figure 2.4 Moisture content in paper during paper-making.

Characterization of Disturbances

The most characteristic feature of a disturbance is that *its value is not known beforehand*. Information about past disturbances could, however, be important for making qualified guesses about future values. It is thus natural to employ a probabilistic framework to describe future disturbances. We then put ourselves at time t and would like to make a statement about disturbances at times $t+k$, $k \geq 1$. A complete characterization would be to describe the conditional joint probability density function for $\{v(t+k), k \geq 1\}$, given $\{v(s), s \leq t\}$. This would, however, in most cases be too laborious, and we shall instead use a simpler approach.

Let $v(t)$ be given as

$$v(t) = \sum_{k=0}^{\infty} h(k)e(t-k) \tag{2.9}$$

where $\{e(t)\}$ is *white noise*, i.e., a sequence of independent (identically distributed) random variables with a certain probability density function. Although this description does not allow completely general characterizations of all possible probabilistic disturbances, it is versatile enough for most practical purposes. In Section 3.2 we shall show how the description (2.9) allows predictions and probabilistic statements about future disturbances. For normalization reasons, we shall usually assume that $h(0) = 1$, which is no loss of generality since the variance of e can be adjusted.

It should be made clear that the specification of different probability density functions (PDF) for $\{e(t)\}$ may result in very different characteristic features of the disturbance. For example, the PDF

$$\begin{aligned} e(t) &= 0, \quad \text{with probability } 1-\mu \\ e(t) &= r, \quad \text{with probability } \mu \end{aligned} \tag{2.10}$$

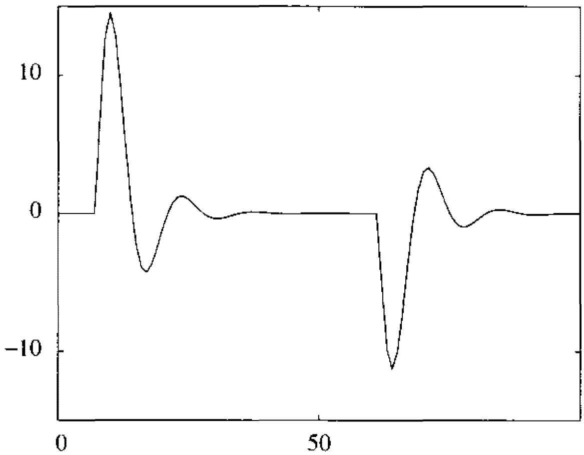

Figure 2.5 A realization of the process (2.9) with e subject to (2.10).

where r is a normally distributed random variable: $r \in N(0, \gamma)$ leads to, if μ is a small number, disturbance sequences with characteristic and "deterministic" profiles occurring at random instants. See Figure 2.5. This could be suitable to describe "classical" disturbance patterns, steps, pulses, sinusoids, and ramps (cf. Figure 2.3!). On the other hand, the PDF

$$e(t) \in N(0, \lambda) \tag{2.11}$$

gives a totally different picture. See Figure 2.6. Such a pattern is more suited to describe measurements noises and irregular and frequent disturbance actions.

Often we only specify the *second-order properties* of the sequence $\{e(t)\}$, that is, the mean and the variances. Note that (2.10) and (2.11) can both be described as "a sequence of independent random variables with zero mean values and variances λ" [$\lambda = \mu\gamma$ for (2.10)], despite the difference in appearance.

Remark. Notice that $\{e(t)\}$ and $\{v(t)\}$ as defined previously are *stochastic processes* (i.e., sequences of random variables). The disturbances that we observe and that are added to the system output as in Figure 2.2 are thus *realizations* of the *stochastic process* $\{v(t)\}$. Strictly speaking, one should distinguish in notation between the process and its realization, but the meaning is usually clear from the context, and we do not here adopt this extra notational burden. Often one has occasion to study signals that are mixtures of deterministic and stochastic components. A framework for this will be discussed in Section 2.3.

Covariance Function

In the sequel, we shall assume that $e(t)$ has zero mean and variance λ. With the description (2.9) of $v(t)$, we can compute the mean as

$$Ev(t) = \sum_{k=0}^{\infty} h(k)Ee(t-k) = 0 \tag{2.12}$$

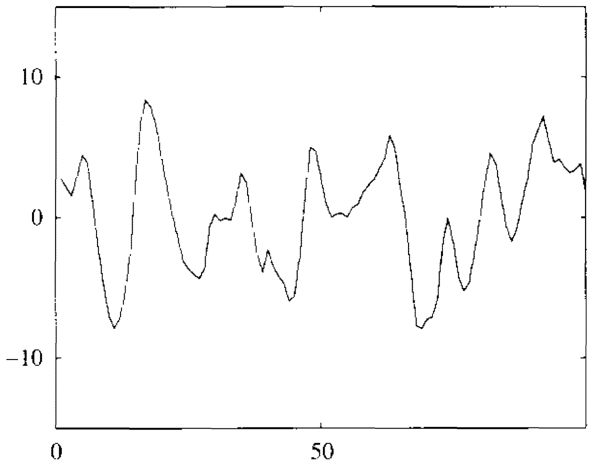

Figure 2.6 A realization of the same process (2.9) as in Figure 2.5, but with e subject to (2.11).

and the covariance as

$$\begin{aligned} Ev(t)v(t-\tau) &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} h(k)h(s)Ee(t-k)e(t-\tau-s) \\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} h(k)h(s)\delta(k-\tau-s)\lambda \\ &= \lambda\sum_{k=0}^{\infty} h(k)h(k-\tau) \end{aligned} \tag{2.13}$$

Here $h(r) = 0$ if $r < 0$. We note that this covariance is independent of t and call

$$R_v(\tau) = Ev(t)v(t-\tau) \tag{2.14}$$

the *covariance function of the process* v. This function, together with the mean, specifies the *second-order properties of* v. These are consequently uniquely defined by the sequence $\{h(k)\}$ and the variance λ of e. Since (2.14) and $Ev(t)$ do not depend on t, the process is said to be *stationary*.

Transfer Functions

It will be convenient to introduce a shorthand notation for sums like (2.8) and (2.9), which will occur frequently in this book. We introduce the *forward shift operator* q by

$$qu(t) = u(t+1)$$

and the *backward shift operator* q^{-1}:

$$q^{-1}u(t) = u(t-1)$$

We can then write for (2.6)

$$\begin{aligned} y(t) &= \sum_{k=1}^{\infty} g(k)u(t-k) = \sum_{k=1}^{\infty} g(k)(q^{-k}u(t)) \\ &= \left[\sum_{k=1}^{\infty} g(k)q^{-k}\right] u(t) = G(q)u(t) \end{aligned} \tag{2.15}$$

where we introduced the notation

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k} \tag{2.16}$$

We shall call $G(q)$ the *transfer operator* or the *transfer function* of the linear system (2.6). Notice that (2.15) thus describes a relation between the sequences u^t and y^t.

Remark. We choose q as argument of G rather than q^{-1} (which perhaps would be more natural in view of the right side) in order to be in formal agreement with z-transform and Fourier-transform expressions. Strictly speaking, the term transfer *function* should be reserved for the z-transform of $\{g(k)\}_1^{\infty}$, that is,

$$G(z) = \sum_{k=1}^{\infty} g(k)z^{-k} \tag{2.17}$$

but we shall sometimes not observe that point. □

Similarly with

$$H(q) = \sum_{k=0}^{\infty} h(k)q^{-k} \tag{2.18}$$

we can write

$$v(t) = H(q)e(t) \tag{2.19}$$

for (2.9). Our basic description for a linear system with additive disturbance will thus be

$$\boxed{y(t) = G(q)u(t) + H(q)e(t) \qquad (2.20)}$$

with $\{e(t)\}$ as a sequence of independent random variables with zero mean values and variances λ.

Continuous-time Representation and Sampling Transfer Functions (*)

For many physical systems it is natural to work with a continuous-time representation (2.1), since most basic relationships are expressed in terms of differential equations. With $G_c(s)$ denoting the Laplace transform of the impulse response function $\{g(\tau)\}$ in (2.1), we then have the relationship

$$Y(s) = G_c(s)U(s) \tag{2.21}$$

between $Y(s)$ and $U(s)$, the Laplace transforms of the output and input, respectively. Introducing p as the differentiation operator, we could then write

$$y(t) = G_c(p)u(t) \tag{2.22}$$

as a shorthand operator form of (2.1) or its underlying differential equation. Now, (2.1) or (2.22) describes the output at all values of the continuous time variable t. If $\{u(t)\}$ is a known function (piecewise constant or not), then (2.22) will of course also serve as a description of the output of the sampling instants. We shall therefore occasionally use (2.22) also as a system description for the sampled output values, keeping in mind that the computation of these values will involve numerical solution of a differential equation. In fact, we could still use a discrete-time model (2.9) for the disturbances that influence our discrete-time measurements, writing this as

$$y(t) = G_c(p)u(t) + H(q)e(t), \qquad t = 1, 2, \ldots \tag{2.23}$$

Often, however, we shall go from the continuous-time representation (2.22) to the standard discrete-time one (2.15) by transforming the transfer function

$$G_c(p) \rightarrow G_T(q) \tag{2.24}$$

T here denotes the sampling interval. When the input is piecewise constant over the sampling interval, this can be done without approximation, in view of (2.4). See Problem 2G.4 for a direct transfer-function expression, and equations (4.67) to (4.71), for numerically more favorable expressions. One can also apply approximate formulas that correspond to replacing the differentiation operator p by a difference approximation. We thus have the Euler approximation

$$G_T(q) \approx G_c\left(\frac{q-1}{T}\right) \tag{2.25}$$

and Tustin's formula

$$G_T(q) \approx G_c\left(\frac{2}{T}\frac{q-1}{1+q}\right) \tag{2.26}$$

See Åström and Wittenmark (1984)for a further discussion.

(*)Denotes sections and subsections that are optional reading; they can be omitted without serious loss of continuity. See Preface.

Some Terminology

The function $G(z)$ in (2.17) is a complex-valued function of the complex variable z. Values β_i, such that $G(\beta_i) = 0$, are called *zeros* of the transfer function (or of the system), while values α_i for which $G(z)$ tends to infinity are called *poles*. This coincides with the terminology for analytic functions (see, e.g., Ahlfors, 1979). If $G(z)$ is a rational function of z, the poles will be the zeros of the denominator polynomial.

We shall say that the transfer function $G(q)$ (or "the system G" or "the filter G") is *stable* if

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k}, \qquad \sum_{k=1}^{\infty} |g(k)| < \infty \tag{2.27}$$

The definition (2.27) coincides with the system theoretic definition of bounded-input, bounded-output (BIBO) stability (e.g. Brockett, 1970): If an input $\{u(t)\}$ to $G(q)$ is subject to $|u(t)| \leq C$, then the corresponding output $z(t) = G(q)u(t)$ will also be bounded, $|z(t)| \leq C'$, provided (2.27) holds. Notice also that (2.27) assures that the (Laurent) expansion

$$G(z) = \sum_{k=1}^{\infty} g(k)z^{-k}$$

is convergent for all $|z| \geq 1$. This means that the function $G(z)$ *is analytic on and outside the unit circle*. In particular, it then has no poles in that area.

We shall often have occasion to consider families of filters $G_\alpha(q)$, $\alpha \in \mathcal{A}$:

$$G_\alpha(q) = \sum_{k=1}^{\infty} g_\alpha(k)q^{-k}, \qquad \alpha \in \mathcal{A} \tag{2.28}$$

We shall then say that such *a family is uniformly stable* if

$$|g_\alpha(k)| \leq g(k), \forall\alpha \in \mathcal{A}, \sum_{k=1}^{\infty} g(k) < \infty \tag{2.29}$$

Sometimes a slightly stronger condition than (2.27) will be required. We shall say that $G(q)$ is *strictly stable* if

$$\sum_{k=1}^{\infty} k|g(k)| < \infty \tag{2.30}$$

Notice that, for a transfer function that is rational in q, stability implies strict stability (and, of course, vice versa). See Problem 2T.3.

Finally, we shall say that a filter $H(q)$ is *monic* if its zeroth coefficient is 1 (or the unit matrix):

$$H(q) = \sum_{k=0}^{\infty} h(k)q^{-k}, \qquad h(0) = 1 \tag{2.31}$$

2.2 FREQUENCY-DOMAIN EXPRESSIONS

Sinusoid Response and the Frequency Function

Suppose that the input to the system (2.6) is a sinusoid:

$$u(t) = \cos \omega t \tag{2.32}$$

It will be convenient to rewrite this as

$$u(t) = \operatorname{Re} e^{i\omega t}$$

with Re denoting "real part." According to (2.6), the corresponding output will be

$$\begin{aligned} y(t) &= \sum_{k=1}^{\infty} g(k) \operatorname{Re} e^{i\omega(t-k)} = \operatorname{Re} \sum_{k=1}^{\infty} g(k) e^{i\omega(t-k)} \\ &= \operatorname{Re} \left\{ e^{i\omega t} \cdot \sum_{k=1}^{\infty} g(k) e^{-i\omega k} \right\} = \operatorname{Re} \left\{ e^{i\omega t} \cdot G(e^{i\omega}) \right\} \qquad (2.33) \\ &= |G(e^{i\omega})| \cos(\omega t + \varphi) \end{aligned}$$

where

$$\varphi = \arg G(e^{i\omega}) \tag{2.34}$$

Here, the second equality follows since the $g(k)$ are real and the fourth equality from the definition (2.16) or (2.17). The fifth equality follows straightforward rules for complex numbers.

In (2.33) we assumed that the input was a cosine since time minus infinity. If $u(t) = 0, t < 0$, we obtain an additional term

$$-\operatorname{Re} \left\{ e^{i\omega t} \sum_{k=t}^{\infty} g(k) e^{-i\omega k} \right\}$$

in (2.33). This term is dominated by

$$\sum_{k=t}^{\infty} |g(k)|$$

and therefore is of transient nature (tends to zero as t tends to infinity), provided that $G(q)$ is stable.

In any case, (2.33) tells us that the output to (2.32) will also be a cosine of the same frequency, but with an amplitude magnified by $|G(e^{i\omega})|$ and a phase shift of $\arg G(e^{i\omega})$ radians. The complex number

$$G(e^{i\omega}) \tag{2.35}$$

which is the transfer function evaluated at the point $z = e^{i\omega}$, therefore gives full information as to what will happen in stationarity, when the input is a sinusoid of frequency ω. For that reason, the complex-valued function

$$G(e^{i\omega}), \qquad -\pi \le \omega \le \pi \tag{2.36}$$

is called the *frequency function* of the system (2.6). It is customary to graphically display this function as $\log|G(e^{i\omega})|$ and $\arg G(e^{i\omega})$ plotted against $\log\omega$ in a *Bode plot.* The plot of (2.36) in the complex plane is called the *Nyquist plot.* These concepts are probably better known in the continuous-time case, but all their basic properties carry over to the sampled-data case.

Periodograms of Signals over Finite Intervals

Consider the finite sequence of inputs $u(t)$, $t = 1, 2, \ldots, N$. Let us define the function $U_N(\omega)$ by

$$U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} u(t) e^{-i\omega t} \tag{2.37}$$

The values obtained for $\omega = 2\pi k/N$, $k = 1, \ldots, N$, form the familiar discrete Fourier transform (DFT) of the sequence u_1^N. We can then represent $u(t)$ by the inverse DFT as

$$u(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} U_N(2\pi k/N) e^{i2\pi kt/N} \tag{2.38}$$

To prove this, we insert (2.37) into the right side of (2.38), giving

$$\begin{aligned}
&\frac{1}{N} \sum_{k=1}^{N} \sum_{s=1}^{N} u(s) \exp\left(-\frac{i2\pi ks}{N}\right) \cdot \exp\left(\frac{i2\pi kt}{N}\right) \\
&= \frac{1}{N} \sum_{s=1}^{N} u(s) \sum_{k=1}^{N} \exp\left(\frac{2\pi ik(t-s)}{N}\right) \\
&= \frac{1}{N} \sum_{s=1}^{N} u(s) N \cdot \delta(t-s) = u(t)
\end{aligned}$$

Here we used the relationship

$$\frac{1}{N} \sum_{k=1}^{N} e^{2\pi irk/N} = \begin{cases} 1, & r = 0 \\ 0, & 1 \le r < N \end{cases} \tag{2.39}$$

From (2.37) we note that $U_N(\omega)$ is periodic with period 2π:

$$U_N(\omega + 2\pi) = U_N(\omega) \tag{2.40}$$

Also, since $u(t)$ is real,

$$U_N(-\omega) = \overline{U_N(\omega)} \tag{2.41}$$

where the overbar denotes the complex conjugate. The function $U_N(\omega)$ is therefore uniquely defined by its values over the interval $[0, \pi]$. It is, however, customary to consider $U_N(\omega)$ for $-\pi \leq \omega \leq \pi$, and in accordance with this, (2.38) is often written

$$u(t) = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} U_N(2\pi k/N) e^{i2\pi kt/N} \tag{2.42}$$

making use of (2.40) and the periodicity of $e^{i\omega}$. In (2.42) and elsewhere we assume N to be even; for odd N analogous summation boundaries apply.

In (2.42) we represent the signal $u(t)$ as a linear combination of $e^{i\omega t}$ for N different frequencies ω. As is further elaborated in Problem 2D.1, this can also be rewritten as sums of $\cos \omega t$ and $\sin \omega t$ for the same frequencies, thus avoiding complex numbers.

The number $U_N(2\pi k/N)$ tells us the "weight" that the frequency $\omega = 2\pi k/N$ carries in the decomposition of $\{u(t)\}_{t=1}^N$. Its absolute square value $|U_N(2\pi k/N)|^2$ is therefore a measure of the contribution of this frequency to the "signal power." This value

$$|U_N(\omega)|^2 \tag{2.43}$$

is known as the *periodogram* of the signal $u(t)$, $t = 1, 2, \ldots, N$.

Parseval's relationship,

$$\sum_{k=1}^{N} |U_N(2\pi k/N)|^2 = \sum_{t=1}^{N} u^2(t) \tag{2.44}$$

reinforces the interpretation that the energy of the signal can be decomposed into energy contributions from different frequencies. Think of the analog decomposition of light into its spectral components!

Example 2.1 Periodogram of a Sinusoid

Suppose that

$$u(t) = A \cos \omega_0 t \tag{2.45}$$

where $\omega_0 = 2\pi/N_0$ for some integer $N_0 > 1$. Consider the interval $t = 1, 2, \ldots, N$, where N is a multiple of N_0 : $N = s \cdot N_0$. Writing

$$\cos \omega_0 t = \frac{1}{2}\left[e^{i\omega_0 t} + e^{-i\omega_0 t}\right]$$

gives

$$U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \frac{A}{2}\left[e^{i(\omega_0-\omega)t} + e^{-i(\omega_0+\omega)t}\right]$$

Using (2.39), we find that

$$|U_N(\omega)|^2 = \begin{cases} N \cdot \frac{A^2}{4}, & \text{if } \omega = \pm\omega_0 = \dfrac{2\pi}{N_0} = \dfrac{2\pi s}{N} \\ 0, & \text{if } \omega = \dfrac{2\pi k}{N}, \quad k \neq s \end{cases} \tag{2.46}$$

The periodogram thus has two spikes in the interval $[-\pi, \pi]$. □

Example 2.2 Periodogram of a Periodic Signal

Suppose $u(t) = u(t + N_0)$ and we consider the signal over the interval $[1, N]$. $N = s \cdot N_0$. According to (2.42), the signal over the interval $[1, N_0]$ can be written

$$u(t) = \frac{1}{\sqrt{N_0}} \sum_{r=-N_0/2+1}^{N_0/2} A_r e^{2\pi i t r/N_0} \tag{2.47}$$

with

$$A_r = \frac{1}{\sqrt{N_0}} \sum_{t=1}^{N_0} u(t) e^{-2\pi i t r/N_0} \tag{2.48}$$

Since u is periodic, (2.47) applies over the whole interval $[1, N]$. It is thus a sum of N_0 sinusoids, and the results of the previous example (or straightforward calculations) show that

$$|U_N(\omega)|^2 = \begin{cases} s \cdot |A_r|^2, & \text{if } \omega = \dfrac{2\pi r}{N_0}, \quad r = 0, \pm 1 \pm \cdots \pm \dfrac{N_0}{2} \\ 0, & \text{if } \omega = \dfrac{2\pi k}{N}, \quad k \neq r \cdot s \end{cases} \tag{2.49}$$

□

The periodograms of Examples 2.1 and 2.2 turned out to be well behaved. For signals that are realizations of stochastic processes, the periodogram is typically a very erratic function of frequency. See Figure 2.8 and Lemma 6.2.

Transformation of Periodograms(*)

As a signal is filtered through a linear system, its periodogram changes. We show next how a signal's Fourier transform is affected by linear filtering. Results for the transformation of periodograms are then immediate.

Theorem 2.1. Let $\{s(t)\}$ and $\{w(t)\}$ be related by the strictly stable system $G(q)$:

$$s(t) = G(q)w(t) \tag{2.50}$$

The input $w(t)$ for $t \leq 0$ is unknown, but obeys $|w(t)| \leq C_w$ for all t. Let

$$S_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} s(t) e^{-i\omega t} \tag{2.51}$$

$$W_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} w(t) e^{-i\omega t} \tag{2.52}$$

Then

$$S_N(\omega) = G(e^{i\omega}) W_N(\omega) + R_N(\omega) \tag{2.53}$$

where

$$|R_N(\omega)| \leq 2C_w \cdot \frac{C_G}{\sqrt{N}} \tag{2.54}$$

with

$$C_G = \sum_{k=1}^{\infty} k|g(k)| \tag{2.55}$$

Proof. We have by definition

$$\begin{aligned} S_N(\omega) &= \frac{1}{\sqrt{N}} \sum_{t=1}^{N} s(t) e^{-it\omega} = \frac{1}{\sqrt{N}} \sum_{k=1}^{\infty} \sum_{t=1}^{N} g(k) w(t-k) e^{-it\omega} \\ &= [\text{change variables: } t - k = \tau] \\ &= \frac{1}{\sqrt{N}} \sum_{k=1}^{\infty} g(k) e^{-ik\omega} \cdot \sum_{\tau=1-k}^{N-k} w(\tau) e^{-i\tau\omega} \end{aligned}$$

Now

$$\begin{aligned} &\left| W_N(\omega) - \frac{1}{\sqrt{N}} \sum_{\tau=1-k}^{N-k} w(\tau) e^{-i\tau\omega} \right| \\ &\leq \left| \frac{1}{\sqrt{N}} \sum_{\tau=1-k}^{0} w(\tau) e^{-i\tau\omega} \right| + \left| \frac{1}{\sqrt{N}} \sum_{N-k+1}^{N} w(\tau) e^{-i\tau\omega} \right| \\ &\leq \frac{2}{\sqrt{N}} \cdot k \cdot C_w \end{aligned} \tag{2.56}$$

Hence

$$\left|S_N(\omega) - G(e^{i\omega})W_N(\omega)\right| = \left|\sum_{k=1}^{\infty} g(k)e^{-ik\omega}\left[\frac{1}{\sqrt{N}}\sum_{\tau=1-k}^{N-k} w(\tau)e^{i\tau\omega} - W_N(\omega)\right]\right|$$

$$\leq \frac{2}{\sqrt{N}}\sum_{k=1}^{\infty}\left|kg(k)C_w e^{-ik\omega}\right| \leq \frac{2C_w \cdot C_G}{\sqrt{N}}$$

and (2.53) to (2.55) follow. □

Corollary. Suppose $\{w(t)\}$ is periodic with period N. Then $R_N(\omega)$ in (2.53) is zero for $\omega = 2\pi k/N$.

Proof. The left side of (2.56) is zero for a periodic $w(\tau)$ at $\omega = 2\pi k/N$. □

2.3 SIGNAL SPECTRA

The periodogram defines, in a sense, the frequency contents of a signal over a finite time interval. This information may, however, be fairly hidden due to the typically erratic behavior of a periodogram as a function of ω. We now seek a definition of a similar concept for signals over the interval $t \in [1, \infty)$. Preferably, such a concept should more clearly demonstrate the different frequency contributions to the signal.

A definition for our framework is, however, not immediate. It would perhaps be natural to define the spectrum of a signal s as

$$\lim_{N\to\infty} |S_N(\omega)|^2 \tag{2.57}$$

but this limit fails to exist for many signals of practical interest. Another possibility would be to use the concept of the spectrum, or spectral density, of a stationary stochastic process as the Fourier transform of its covariance function. However, the processes that we consider here are frequently not stationary, for reasons that are described later. We shall therefore develop a framework for describing signals and their spectra that is applicable to deterministic as well as stochastic signals.

A Common Framework for Deterministic and Stochastic Signals

In this book we shall frequently work with signals that are described as stochastic processes with deterministic components. The reason is, basically, that we prefer to consider the input sequence as deterministic, or at least partly deterministic, while disturbances on the system most conveniently are described by random variables. In this way the system output becomes a stochastic process with deterministic components. For (2.20) we find that

$$Ey(t) = G(q)u(t)$$

so $\{y(t)\}$ is not a stationary process.

To deal with this problem, we introduce the following definition.

Definition 2.1. *Quasi-Stationary Signals* A signal $\{s(t)\}$ is said to be quasi-stationary if it is subject to

$$\begin{aligned} &\text{(i)} \qquad Es(t) = m_s(t), \qquad |m_s(t)| \le C, \quad \forall t \\ &\text{(ii)} \qquad Es(t)s(r) = R_s(t,r), \qquad |R_s(t,r)| \le C \end{aligned} \tag{2.58}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} R_s(t, t-\tau) = R_s(\tau), \qquad \forall \tau \tag{2.59}$$

Here expectation E is with respect to the "stochastic components" of $s(t)$. If $\{s(t)\}$ itself is a deterministic sequence, the expectation is without effect and quasi-stationarity then means that $\{s(t)\}$ is a bounded sequence such that the limits

$$R_s(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} s(t)s(t-\tau)$$

exist. If $\{s(t)\}$ is a stationary stochastic process, (2.58) and (2.59) are trivially satisfied, since then $Es(t)s(t-\tau) \triangleq R_s(\tau)$ does not depend on t.

For easy notation we introduce the symbol $\overline{E}$ by

$$\overline{E}f(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Ef(t) \tag{2.60}$$

with an implied assumption that the limit exists when the symbol is used. Assumption (2.59), which simultaneously is a definition of $R_s(\tau)$, then reads

$$\overline{E}s(t)s(t-\tau) = R_s(\tau) \tag{2.61}$$

Sometimes, with some abuse of notation, we shall call $R_s(\tau)$ the *covariance function* of s, keeping in mind that this is a correct term only if $\{s(t)\}$ is a stationary stochastic process with mean value zero.

Similarly, we say that two signals $\{s(t)\}$ and $\{w(t)\}$ are *jointly quasi-stationary* if they both are quasi-stationary and if, in addition, the *cross-covariance* function

$$R_{sw}(\tau) = \overline{E}s(t)w(t-\tau) \tag{2.62}$$

exists. We shall say that jointly quasi-stationary signals are *uncorrelated* if their cross-covariance function is identically zero.

Definition of Spectra

When limits like (2.61) and (2.62) hold, we define the (power) *spectrum* of $\{s(t)\}$ as

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau)e^{-i\tau\omega} \tag{2.63}$$

and the *cross spectrum* between $\{s(t)\}$ and $\{w(t)\}$ as

$$\Phi_{sw}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{sw}(\tau)e^{-i\tau\omega} \tag{2.64}$$

provided the infinite sums exist. In the sequel, as we talk of a signal's "spectrum," we always implicitly assume that the signal has all the properties involved in the definition of spectrum.

While $\Phi_s(\omega)$ always is real, $\Phi_{sw}(\omega)$ is in general a complex-valued function of ω. Its real part is known as the *cospectrum* and its imaginary part as the *quadrature spectrum*. The argument $\arg \Phi_{sw}(\omega)$ is called the *phase spectrum*, while $|\Phi_{sw}(\omega)|$ is the *amplitude spectrum*.

Note that, by definition of the inverse Fourier transform, we have

$$\overline{E}s^2(t) = R_s(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_s(\omega)d\omega \tag{2.65}$$

Example 2.3 Periodic Signals

Consider a deterministic, periodic signal with period M, i.e., $s(t) = s(t+M)$. We then have $Es(t) = s(t)$ and $Es(t)s(r) = s(t)s(r)$ and:

$$\frac{1}{N}\sum_{t=1}^{N} s(t)s(t-\tau) = \frac{MK}{N}\cdot\frac{1}{K}\sum_{\ell=0}^{K-1}\left\{\frac{1}{M}\sum_{t=1}^{M} s(t+\ell M)s(t-\tau+\ell M)\right\}$$
$$+\frac{1}{N}\sum_{t=KM-1}^{N} s(t)s(t-\tau)$$

where K is chosen as the maximum number of full periods, i.e., $N - MK < M$. Due to the periodicity, the sum within braces in fact does not depend on ℓ. Since there are at most $M-1$ terms in the last sum, this means that the limit as $N\to\infty$ will exist with

$$\overline{E}s(t)s(t-\tau) = R_s(\tau) = \frac{1}{M}\sum_{t=1}^{M} s(t)s(t-\tau)$$

A periodic, deterministic signal is thus quasi-stationary. We clearly also have $R_s(\tau+kM) = R_s(\tau)$.

For the spectrum we then have:

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau)e^{-i\omega\tau} = \sum_{\ell=-\infty}^{\infty}\sum_{\tau=0}^{M-1} R_s(\tau+\ell M)e^{-i\omega\tau}e^{-i\omega\ell M}$$
$$= \Phi_s^p(\omega)\sum_{\ell=-\infty}^{\infty} e^{-i\ell M\omega} = \Phi_s^p(\omega)F(\omega, M)$$

where

$$\Phi_s^p(\omega) = \sum_{\tau=0}^{M-1} R_s(\tau)e^{i\omega\tau}, \quad F(\omega, M) = \sum_{\ell=-\infty}^{\infty} e^{-i\ell M\omega}$$

The function F is not well defined in the usual sense, but using the Dirac delta-function, it is well known (and well in line with (2.39)) that:

$$F(\omega, M) = \frac{2\pi}{M}\sum_{k=1}^{M} \delta(\omega - 2\pi k/M), \quad 0 \le \omega < 2\pi \tag{2.66}$$

This means that we can write:

$$\Phi_s(\omega) = \frac{2\pi}{M}\sum_{k=0}^{M-1} \Phi_s^p(2\pi k/M)\delta(\omega - 2\pi k/M), \quad 0 \le \omega < 2\pi \tag{2.67}$$

In $\Phi_s^p(2\pi k/M)$ we recognize the k:th Fourier coefficient of the periodic signal $R_s(\tau)$. Recall that the spectrum is periodic with period 2π, so different representations of (2.67) can be given.

The spectrum of a signal that is periodic with period M has thus (at most) M delta spikes, at the frequencies $2\pi k/M$, and is zero elsewhere. Although the limit (2.63) does not exist in a formal sense in this case, it is useful to extend the spectrum concept for signals with periodic and constant components in this way. □

Example 2.4 Spectrum of a Sinusoid

Consider again the signal (2.45), now extended to the interval $[1, \infty)$. We have

$$\frac{1}{N}\sum_{k=1}^{N} Eu(k)u(k-\tau) = \frac{1}{N}\sum_{k=1}^{N} A^2\cos(\omega_0 k)\cos(\omega_0(k-\tau)). \tag{2.68}$$

(Expectation is of no consequence since u is deterministic.) Now

$$\cos(\omega_0 k)\cos(\omega_0(k-\tau)) = \frac{1}{2}(\cos(2\omega_0 k - \omega_0\tau) + \cos\omega_0\tau)$$

which shows that

$$\bar{E}u(t)u(t-\tau) = \frac{A^2}{2}\cos\omega_0\tau = R_u(\tau)$$

The spectrum now is

$$\Phi_u(\omega) = \sum_{\tau=-\infty}^{\infty} \frac{A^2}{2}\cos(\omega_0\tau)e^{-i\omega\tau} = \frac{A^2}{4}(\delta(\omega-\omega_0) + \delta(\omega+\omega_0))\cdot 2\pi \tag{2.69}$$

This result fits well with the finite interval expression (2.46) and the general expression (2.67) for periodic signals. □

Example 2.5 Stationary Stochastic Processes

Let $\{v(t)\}$ be a stationary stochastic process with covariance function (2.14). Since (2.59) then equals (2.14), our definition of spectrum coincides with the conventional one. Suppose now that the process v is given as (2.9). Its covariance function is then given by (2.13). The spectrum is

$$\Phi_v(\omega) = \sum_{\tau=-\infty}^{\infty} \lambda e^{-i\tau\omega} \sum_{k=\max(0,\tau)}^{\infty} h(k)h(k-\tau)$$

$$= \lambda \sum_{\tau=-\infty}^{\infty} \sum_{k=\max(0,\tau)}^{\infty} h(k)e^{-ik\omega}h(k-\tau)e^{i(k-\tau)\omega}$$

$$= [k-\tau = s] = \lambda \sum_{s=0}^{\infty} h(s)e^{is\omega} \sum_{k=0}^{\infty} h(k)e^{-ik\omega} = \lambda \left|H(e^{i\omega})\right|^2$$

using (2.18). This result is very important for our future use:

> The stochastic process described by $v(t) = H(q)e(t)$, where $\{e(t)\}$ is a sequence of independent random variables with zero mean values and covariances λ, has the spectrum
>
> $$\Phi_v(\omega) = \lambda \left|H(e^{i\omega})\right|^2 \qquad (2.70)$$

This result, which was easy to prove for the special case of a stationary stochastic process, will be proved in the general case as Theorem 2.2 later in this section. Figure 2.7 shows the spectrum of the process of Figures 2.5 and 2.6, while the periodogram of the realization of Figure 2.6 is shown in Figure 2.8. □

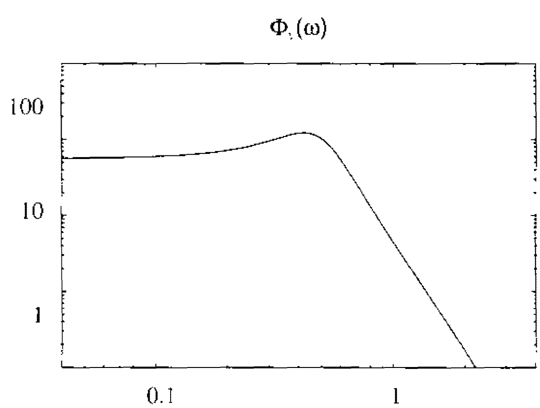

Figure 2.7 The spectrum of the process $v(t) - 1.5v(t-1) + 0.7v(t-2) = e(t) + 0.5e(t-1)$, $\{e(t)\}$ being white noise.

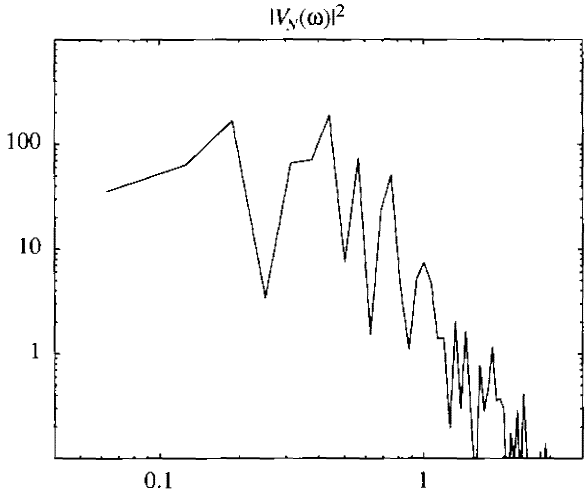

Figure 2.8 The periodogram of the realization of Figure 2.6.

Example 2.6 Spectrum of a Mixed Deterministic and Stochastic Signal

Consider now a signal

$$s(t) = u(t) + v(t) \tag{2.71}$$

where $\{u(t)\}$ is a deterministic signal with spectrum $\Phi_u(\omega)$ and $\{v(t)\}$ is a stationary stochastic process with zero mean value and spectrum $\Phi_v(\omega)$. Then

$$\begin{aligned} \overline{E}s(t)s(t-\tau) &= \overline{E}u(t)u(t-\tau) + \overline{E}u(t)v(t-\tau) \\ &\quad + \overline{E}v(t)u(t-\tau) + \overline{E}v(t)v(t-\tau) \\ &= R_u(\tau) + R_v(\tau) \end{aligned} \tag{2.72}$$

since $\overline{E}v(t)u(t-\tau) = 0$. Hence

$$\Phi_s(\omega) = \Phi_u(\omega) + \Phi_v(\omega) \tag{2.73}$$

□

Connections to the Periodogram

While the original idea (2.57) does not hold, a conceptually related result can be proved; that is, the expected value of the periodogram *converges weakly* to the spectrum:

$$E\,|S_N(\omega)|^2 \underset{w}{\rightarrow} \Phi_s(\omega) \tag{2.74}$$

By this is meant that

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} E\,|S_N(\omega)|^2\, \Psi(\omega)\, d\omega = \int_{-\pi}^{\pi} \Phi_s(\omega)\Psi(\omega)\, d\omega \tag{2.75}$$

for all sufficiently smooth functions $\Psi(\omega)$. We have

Lemma 2.1. Suppose that $\{s(t)\}$ is quasi-stationary with spectrum $\Phi_s(\omega)$. Let

$$S_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} s(t) e^{-it\omega}$$

and let $\Psi(\omega)$ be an arbitrary function for $|\omega| \le \pi$ with Fourier coefficients a_τ, such that

$$\sum_{\tau=-\infty}^{\infty} |a_\tau| < \infty$$

Then (2.75) holds.

Proof.

$$\begin{aligned} E\,|S_N(\omega)|^2 &= \frac{1}{N} \sum_{k=1}^{N} \sum_{\ell=1}^{N} Es(k)s(\ell) e^{i\omega(k-\ell)} \\ &= [\ell - k = \tau] = \sum_{\tau=-(N-1)}^{N-1} R_N(\tau) e^{-i\omega\tau} \end{aligned} \tag{2.76}$$

where

$$R_N(\tau) = \frac{1}{N} \sum_{k=1}^{N} Es(k)s(k-\tau) \tag{2.77}$$

with the convention that $s(k)$ is taken as zero outside the interval $[1, N]$. Multiplying (2.76) by $\Psi(\omega)$ and integrating over $[-\pi, \pi]$ gives

$$\int_{-\pi}^{\pi} E\,|S_N(\omega)|^2\, \Psi(\omega)\, d\omega = \sum_{\tau=-(N-1)}^{N-1} R_N(\tau) a_\tau$$

by the definition of a_τ. Similarly, allowing interchange of summation and integration, we have

$$\int_{-\pi}^{\pi} \Phi_s(\omega)\Psi(\omega)\,d\omega = \sum_{\tau=-\infty}^{\infty} R_s(\tau)a_\tau$$

Hence

$$\int_{-\pi}^{\pi} E\,|S_N(\omega)|^2\,\Psi(\omega)\,d\omega - \int_{-\pi}^{\pi} \Phi_s(\omega)\Psi(\omega)\,d\omega$$

$$= \sum_{\tau=-(N-1)}^{N} a_\tau\,[R_N(\tau) - R_s(\tau)] + \sum_{|\tau|\geq N} a_\tau R_s(\tau)$$

Problem 2D.5 now completes the proof. □

Notice that for stationary stochastic processes the result (2.74) can be strengthened to "ordinary" convergence (see Problem 2D.3). Notice also that, in our framework, results like (2.74) can be applied also to realizations of stochastic processes simply by ignoring the expectation operator. We then view the realization in question as a given "deterministic" sequence, and will then, of course, have to require that the conditions (2.58) and (2.59) hold for this particular realization [disregard "E" also in (2.58) and (2.59)].

Transformation of Spectra by Linear Systems

As signals are filtered through linear systems, their properties will change. We saw how the periodogram was transformed in Theorem 2.1 and how white noise created stationary stochastic processes in (2.70). For spectra we have the following general result.

Theorem 2.2. Let $\{w(t)\}$ be a quasi-stationary signal with spectrum $\Phi_w(\omega)$, and let $G(q)$ be a stable transfer function. Let

$$s(t) = G(q)w(t) \tag{2.78}$$

Then $\{s(t)\}$ is also quasi-stationary and

$$\Phi_s(\omega) = \left|G(e^{i\omega})\right|^2 \Phi_w(\omega) \tag{2.79}$$

$$\Phi_{sw}(\omega) = G(e^{i\omega})\Phi_w(\omega) \tag{2.80}$$

Proof. The proof is given in Appendix 2A. □

Corollary. Let $\{y(t)\}$ be given by

$$y(t) = G(q)u(t) + H(q)e(t) \tag{2.81}$$

where $\{u(t)\}$ is a quasi-stationary, deterministic signal with spectrum $\Phi_u(\omega)$, and $\{e(t)\}$ is white noise with variance λ. Let G and H be stable filters. Then $\{y(t)\}$ is quasi-stationary and

$$\Phi_y(\omega) = \left|G(e^{i\omega})\right|^2 \Phi_u(\omega) + \lambda \left|H(e^{i\omega})\right|^2 \tag{2.82}$$

$$\Phi_{yu}(\omega) = G(e^{i\omega})\Phi_u(\omega) \tag{2.83}$$

Proof. The corollary follows from the theorem using Examples 2.5 and 2.6. □

Spectral Factorization

Typically, the transfer functions $G(q)$ and $H(q)$ used here are rational functions of q. Then results like (2.70) and Theorem 2.2 describe spectra as real-valued rational functions of $e^{i\omega}$ (which means that they also are rational functions of $\cos\omega$).

In practice, the converse of such results is of major interest: Given a spectrum $\Phi_v(\omega)$, can we then find a transfer function $H(q)$ such that the process $v(t) = H(q)e(t)$ has this spectrum with $\{e(t)\}$ being white noise? It is quite clear that this is not possible for all positive functions $\Phi_v(\omega)$. For example, if the spectrum is zero on an interval, then the function $H(z)$ must be zero on a portion of the unit circle. But since by necessity $H(z)$ should be analytic outside and on the unit circle for the expansion (2.18) to make sense, this implies that $H(z)$ is zero everywhere and cannot match the chosen spectrum.

The exact conditions under which our question has a positive answer are discussed in texts on stationary processes, such as Wiener (1949)and Rozanov (1967). For our purposes it is sufficient to quote a simpler result, dealing only with spectral densities $\Phi_v(\omega)$ that are rational in the variable $e^{i\omega}$ (or $\cos\omega$).

Spectral factorization: Suppose that $\Phi_v(\omega) > 0$ is a rational function of $\cos\omega$ (or $e^{i\omega}$). Then there exists a monic rational function of z, $R(z)$, with no poles and no zeros on or outside the unit circle such that

$$\Phi_v(\omega) = \lambda \left|R(e^{i\omega})\right|^2$$

The proof of this result consists of a straightforward construction of R, and it can be found in standard texts on stochastic processes or stochastic control (e.g., Rozanov, 1967; Åström, 1970).

Example 2.7 ARMA Processes

If a stationary process $\{v(t)\}$ has rational spectrum $\Phi_v(\omega)$, we can represent it as

$$v(t) = R(q)e(t) \tag{2.84}$$

where $\{e(t)\}$ is white noise with variance λ. Here $R(q)$ is a rational function

$$R(q) = \frac{C(q)}{A(q)}$$

$$C(q) = 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}$$

$$A(q) = 1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}$$

so that we may write

$$\begin{aligned} v(t) + a_1 v(t-1) + \cdots + a_{n_a} v(t-n_a) \\ = e(t) + c_1 e(t-1) + \cdots + c_{n_c} e(t-n_c) \end{aligned} \tag{2.85}$$

for (2.84). Such a representation of a stochastic process is known as an *ARMA model.* If $n_c = 0$, we have an autoregressive (AR) model:

$$v(t) + a_1 v(t-1) + \cdots + a_{n_a} v(t-n_a) = e(t) \tag{2.86}$$

And if $n_a = 0$, we have a moving average (MA) model:

$$v(t) = e(t) + c_1 e(t-1) + \cdots + c_{n_c} e(t-n_c) \tag{2.87}$$

□

The spectral factorization concept is important since it provides a way of representing the disturbance in the standard form $v = H(q)e$ from information about its spectrum only. The spectrum is usually a sound engineering way of describing properties of signals: "The disturbances are concentrated around 50 Hz" or "We are having low-frequency disturbances with little power over 1 rad/s." Rational functions are able to approximate functions of rather versatile shapes. Hence the spectral factorization result will provide a good modeling framework for disturbances.

Second-order Properties

The signal spectra, as defined here, describe the *second-order properties* of the signals (for stochastic processes, their second-order statistics, i.e., first and second moments). Recall from Section 2.1 that stochastic processes may have very different looking realizations even if they have the same covariance function (see Figures 2.5 and 2.6)! The spectrum thus describes only certain aspects of a signal. Nevertheless, it will turn out that many properties related to identification depend only on the spectra of the involved signals. This motivates our detailed interest in the second-order properties.

2.4 SINGLE REALIZATION BEHAVIOR AND ERGODICITY RESULTS (*)

All the results of the previous section are also valid, as we pointed out, for the special case of a given deterministic signal $\{s(t)\}$. Definitions of spectra, their transformations (Theorem 2.2) and their relationship with the periodogram (Lemma 2.1) hold unchanged; we may just disregard the expectation E and interpret $\overline{E} f(t)$ as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} f(t)$$

There is a certain charm with results like these that do not rely on a probabilistic framework: we anyway observe just one realization, so why should we embed this observation in a stochastic process and describe its average properties taken over an ensemble of potential observations? There are two answers to this question. One is that such a framework facilitates certain calculations. Another is that it allows us to deal with the question of what would happen if the experiment were repeated.

Nevertheless, it is a valid question to ask whether the spectrum of the signal $\{s(t)\}$, as defined in a probabilistic framework, differs from the spectrum of the actually observed, single realization were it to be considered as a given, deterministic signal. This is the problem of *ergodic* theory, and for our setup we have the following fairly general result.

Theorem 2.3. Let $\{s(t)\}$ be a quasi-stationary signal. Let $Es(t) = m(t)$. Assume that

$$s(t) - m(t) = v(t) = \sum_{k=0}^{\infty} h_t(k)e(t-k) = H_t(q)e(t) \tag{2.88}$$

where $\{e(t)\}$ is a sequence of independent random variables with zero mean values, $Ee^2(t) = \lambda_t$, and bounded fourth moments, and where $\{H_t(q), t = 1, 2, \dots, \}$ is a uniformly stable family of filters. Then, with probability 1 as N tends to infinity,

$$\frac{1}{N}\sum_{t=1}^{N} s(t)s(t-\tau) \to \bar{E}s(t)s(t-\tau) = R_s(\tau) \tag{2.89a}$$

$$\frac{1}{N}\sum_{t=1}^{N} [s(t)m(t-\tau) - Es(t)m(t-\tau)] \to 0 \tag{2.89b}$$

$$\frac{1}{N}\sum_{t=1}^{N} [s(t)v(t-\tau) - Es(t)v(t-\tau)] \to 0 \tag{2.89c}$$

The proof is given in Appendix 2B.

The theorem is quite important. It says that, provided the stochastic part of the signal can be described as filtered white noise as in (2.88), then

> *the spectrum of an observed single realization of $\{s(t)\}$, computed as for a deterministic signal, coincides, with probability 1, with that of the process $\{s(t)\}$, defined by ensemble averages (E) as in (2.61).*

This de-emphasizes the distinction between deterministic and stochastic signals when we consider second-order properties only. A signal $\{s(t)\}$ whose spectrum is $\Phi_s(\omega) \equiv \lambda$ may, for all purposes related to second-order properties, be regarded as a realization of white noise with variance λ.

The theorem also gives an answer to the question of whether our "theoretical" spectrum, defined in (2.63) using the physically unrealizable concepts of E and lim, relates to the actually observed periodogram (2.43). According to Theorem 2.3 and Lemma 2.1, "smoothed" versions of $|S_N(\omega)|^2$ will look like $\Phi_s(\omega)$ for large N. Compare Figures 2.7 and 2.8. This link between our theoretical concepts and the real data is of course of fundamental importance. See Section 6.3.

2.5 MULTIVARIABLE SYSTEMS (*)

So far, we have worked with systems having a scalar input and a scalar output. In this section we shall consider the case where the output signal has p components and the input signal has m components. Such systems are called *multivariable*. The extra work involved in dealing with models of multivariable systems can be split up into two parts:

1. The easy part: mostly notational changes, keeping track of transposes, and noting that certain scalars become matrices and might not commute.
2. The difficult part: multioutput models have a much richer internal structure, which has the consequence that their parametrization is nontrivial. See Appendix 4A. (Multiple-input, single-output, MISO, models do not expose these problems.)

Let us collect the p components of the output signal into a p-dimensional column vector $y(t)$ and similarly construct an m-dimensional input vector $u(t)$. Let the disturbance $e(t)$ also be a p-dimensional column vector. The basic system description then looks just like (2.20):

$$y(t) = G(q)u(t) + H(q)e(t) \tag{2.90}$$

where now $G(q)$ is a transfer function matrix of dimension $p \times m$ and $H(q)$ has dimension $p \times p$. This means that the i, j entry of $G(q)$, denoted by $G_{ij}(q)$, is the scalar transfer function from input number j to output number i. The sequence $\{e(t)\}$ is a sequence of independent random p-dimensional vectors with zero mean values and covariance matrices $Ee(t)e^T(t) = \Lambda$.

Now, all the development in this chapter goes through with proper interpretation of matrix dimensions. Note in particular the following:

- The impulse responses $g(k)$ and $h(k)$ will be $p \times m$ and $p \times p$ matrices, respectively, with norms

$$\|g(k)\| = \left(\sum_{i,j} |g_{ij}|^2\right)^{1/2} \tag{2.91}$$

  replacing absolute values in the definitions of stability.
- The definitions of covariances become [cf. (2.59)]

$$\bar{E}s(t)s^T(t-\tau) = R_s(\tau) \tag{2.92}$$

$$\bar{E}s(t)w^T(t-\tau) = R_{sw}(\tau) \tag{2.93}$$

  These are now matrices, with norms as in (2.91).
- Definitions of spectra remain unchanged, while the counterpart of Theorem 2.2 reads

$$\Phi_s(\omega) = G(e^{i\omega})\Phi_w(\omega)G^T(e^{-i\omega}) \tag{2.94}$$

This result will then also handle how cross spectra transform. If we have

$$y(t) = G(q)u(t) + H(q)w(t) = \begin{bmatrix} G(q) & H(q) \end{bmatrix} \begin{bmatrix} u(t) \\ w(t) \end{bmatrix}$$

where u and w are jointly quasi-stationary, scalar sequences, we have:

$$\begin{aligned}
\Phi_y(\omega) &= \begin{bmatrix} G(e^{i\omega}) & H(e^{i\omega}) \end{bmatrix} \begin{bmatrix} \Phi_u(\omega) & \Phi_{uw}(\omega) \\ \Phi_{uw}(-\omega) & \Phi_w(\omega) \end{bmatrix} \begin{bmatrix} G(e^{-i\omega}) \\ H(e^{-i\omega}) \end{bmatrix} \\
&= \left|G(e^{i\omega})\right|^2 \Phi_u(\omega) + \left|H(e^{i\omega})\right|^2 \Phi_w(\omega) \\
&\quad + G(e^{i\omega})\Phi_{uw}(\omega)H(e^{-i\omega}) + H(e^{i\omega})\Phi_{uw}(-\omega)G(e^{-i\omega}) \\
&= \left|G(e^{i\omega})\right|^2 \Phi_u(\omega) + \left|H(e^{i\omega})\right|^2 \Phi_w(\omega) \\
&\quad + 2\mathrm{Re}\left(G(e^{i\omega})\Phi_{uw}(\omega)H(e^{-i\omega})\right)
\end{aligned} \tag{2.95}$$

where we used that $G(e^{i\omega})$ and $G(e^{-i\omega})$ are complex conjugates as well as $\Phi_{uw}(\omega)$ and $\Phi_{uw}(-\omega)$. The counterpart of the corollary to Theorem 2.2 for multivariable systems reads

$$\Phi_y(\omega) = G(e^{i\omega})\Phi_u(\omega)G^T(e^{-i\omega}) + H(e^{i\omega})\Lambda H^T(e^{-i\omega}) \tag{2.96}$$

- The spectral factorization result now reads: Suppose that $\Phi_v(\omega)$ is a $p \times p$ matrix that is positive definite for all ω and whose entries are rational functions of $\cos\omega$ (or $e^{i\omega}$). Then there exists a $p \times p$ monic matrix function $H(z)$ whose entries are rational functions of z (or z^{-1}) such that the (rational) function $\det H(z)$ has no poles and no zeros on or outside the unit circle. (For a proof, see Theorem 10.1 in Rozanov, 1967).
- The formulation of Theorem 2.3 carries over without changes. (In fact, the proof in Appendix 2B is carried out for the multivariable case).

2.6 SUMMARY

We have established the representation

$$y(t) = G(q)u(t) + H(q)e(t) \tag{2.97}$$

as the basic description of a linear system subject to additive random disturbances. Here $\{e(t)\}$ is a sequence of independent random variables with zero mean values and variances λ (in the multioutput case, covariance matrices Λ). Also,

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k}$$

$$H(q) = 1 + \sum_{k=1}^{\infty} h(k)q^{-k}$$

The filter $G(q)$ is *stable* if

$$\sum_{k=1}^{\infty} |g(k)| < \infty$$

As the reader no doubt is aware of, other particular ways of representing linear systems, such as state-space models and difference equations, are quite common in practice. These can, however, be viewed as particular ways of representing the sequences $\{g(k)\}$ and $\{h(k)\}$, and they will be dealt with in some detail in Chapter 4.

We have also discussed the frequency function $G(e^{i\omega})$, bearing information about how an input sinusoid of frequency ω is transformed by the system. Frequency-domain concepts in terms of the frequency contents of input and output signals were also treated. The Fourier transform of a finite-interval signal was defined as

$$U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} u(t) e^{-i\omega t} \tag{2.98}$$

A signal $s(t)$ such that the limits

$$\bar{E}s(t)s(t-\tau) = R_s(\tau)$$

exist is said to be *quasi-stationary*. Here

$$\bar{E}f(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} Ef(t)$$

Then the *spectrum* of $s(t)$ is defined as

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau) e^{-i\omega\tau} \tag{2.99}$$

For y generated as in (2.97) with $\{u(t)\}$ and $\{e(t)\}$ independent, we then have

$$\Phi_y(\omega) = \left|G(e^{i\omega})\right|^2 \Phi_u(\omega) + \lambda \left|H(e^{i\omega})\right|^2$$

2.7 BIBLIOGRAPHY

The material of this chapter is covered in many textbooks on systems and signals. For a thorough elementary treatment, see Oppenheim and Willsky (1983). A discussion oriented more toward signals as time series is given in Brillinger (1981), which also contains several results of the same character as our Theorems 2.1 and 2.2.

A detailed discussion of the sampling procedure and connections between the physical time-continuous system and the sampled-data description (2.6) is given in Chapter 4 of Åström and Wittenmark (1984). Chapter 6 of that book also contains an illuminating discussion of disturbances and how to describe them mathematically. The idea of describing stochastic disturbances as linearly filtered white noise goes back to Wold (1938).

Fourier techniques in the analysis and description of signals go back to the Babylonians. See Oppenheim and Willsky (1983), Section 4.0, for a brief historical account. The periodogram was evidently introduced by Schuster (1894)to study periodic phenomena without having to consider relative phases. The statistical properties of the periodogram were first studied by Slutsky (1929). See also Brillinger (1983). Concepts of spectra are intimately connected to the harmonic analysis of time series, as developed by Wiener (1930), Wold (1938), Kolmogorov (1941), and others. Useful textbooks on these concepts (and their estimation) include Jenkins and Watts (1968) and Brillinger (1981). Our definition of the Fourier transform (2.37) with summation from 1 to N and a normalization with $1/\sqrt{N}$ suits our purposes, but is not standard. The placement of 2π in the definition of the spectrum or in the inverse transform, as we have it in (2.65), varies in the literature. Our choice is based on the wish to let white noise have a constant spectrum whose value equals the variance of the noise. The particular framework chosen here to accommodate mixtures of stochastic processes and deterministic signals is apparently novel, but has a close relationship to the classical concepts.

The result of Theorem 2.2 is standard when applied to stationary stochastic processes. See, for example, James, Nichols, and Phillips (1947)or Åström (1970). The extension to quasi-stationary signals appears to be new.

Spectral factorization turned out to be a key issue in the prediction of time series. It was formulated and solved by Wiener (1949)and Paley and Wiener (1934). The multivariable version is treated in Youla (1961). The concept is now standard in textbooks on stationary processes (see, e.g., Rozanov, 1967).

The topic of single realization behavior is a standard problem in probability theory. See, for example, Ibragimov and Linnik (1971), Billingsley (1965), or Chung (1974)for general treatments of such problems.

2.8 PROBLEMS[†]

2G.1 Let $s(t)$ be a p-dimensional signal. Show that

$$\overline{E}\,|s(t)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr}(\Phi_s(\omega))\, d\omega$$

2G.2 Let $\Phi_s(\omega)$ be the (power) spectrum of a scalar signal defined as in (2.63). Show that

i. $\Phi_s(\omega)$ is real.

ii. $\Phi_s(\omega) \geq 0\, \forall\omega$.

iii. $\Phi_s(-\omega) = \Phi_s(\omega)$.

2G.3 Let $s(t) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}$ and let its spectrum be

$$\Phi_s(\omega) = \begin{bmatrix} \Phi_y(\omega) & \Phi_{yu}(\omega) \\ \Phi_{uy}(\omega) & \Phi_u(\omega) \end{bmatrix}$$

[†] See the Preface for an explanation of the numbering system.

Show that $\Phi_s(\omega)$ is a Hermitian matrix: that is,

$$\Phi_s(\omega) = \Phi_s^*(\omega)$$

where * denotes transpose and complex conjugate. What does this imply about the relationships between the cross spectra $\Phi_{yu}(\omega)$, $\Phi_{uy}(\omega)$, and $\Phi_{yu}(-\omega)$?

2G.4 Let a continuous time system representation be given by

$$y(t) = G_c(p)u(t)$$

The input is constant over the sampling interval T. Show that the sampled input-output data are related by

$$y(t) = G_T(q)u(t)$$

where

$$G_T(q) = \int_{s=-i\infty}^{i\infty} G_c(s)\frac{e^{sT}-1}{s}\frac{1}{q-e^{sT}}\,ds$$

Hint: Use (2.5).

2E.1 A stationary stochastic process has the spectrum

$$\Phi_v(\omega) = \frac{1.25 + \cos\omega}{1.64 + 1.6\cos\omega}$$

Describe $\{v(t)\}$ as an ARMA process.

2E.2 Suppose that $\{\eta(t)\}$ and $\{\xi(t)\}$ are two mutually independent sequences of independent random variables with

$$E\eta(t) = E\xi(t) = 0, \qquad E\eta^2(t) = \lambda_\eta, \qquad E\xi^2(t) = \lambda_\xi$$

Consider

$$w(t) = \eta(t) + \xi(t) + \gamma\xi(t-1)$$

Determine a MA(1) process

$$v(t) = e(t) + ce(t-1)$$

where $\{e(t)\}$ is white noise with

$$Ee(t) = 0, \qquad Ee^2(t) = \lambda_e$$

such that $\{w(t)\}$ and $\{v(t)\}$ have the same spectra: that is, find c and λ_e so that $\Phi_v(\omega) \equiv \Phi_w(\omega)$.

2E.3 **(a)** In Problem 2E.2 assume that $\{\eta(t)\}$ and $\{\xi(t)\}$ are jointly Gaussian. Show that if $\{e(t)\}$ also is chosen as Gaussian then the joint probability distribution of the process $\{w(t)\}$ [i. e., the joint PDFs of $w(t_1), w(t_2), \ldots, w(t_p)$ for any collection of time instances t_i] coincides with that of the process $\{v(t)\}$. Then, for all practical purposes, *the processes $\{v(t)\}$ and $\{w(t)\}$ are indistinguishable.*

(b) Assume now that $\eta(t) \in N(0, \lambda_\eta)$, while

$$\xi(t) = \begin{cases} 1, & \text{w.p.} \quad \dfrac{\lambda_\xi}{2} \\ -1, & \text{w.p.} \quad \dfrac{\lambda_\xi}{2} \\ 0, & \text{w.p.} \quad 1 - \lambda_\xi \end{cases}$$

Show that, although v and w have the same spectra, we cannot find a distribution for $e(t)$ so that they have the same joint PDFS. *Consequently the process $w(t)$ cannot be represented as an MA(1) process, although it has a second-order equivalent representation of that form.*

2E.4 Consider the "state-space description"

$$x(t+1) = fx(t) + w(t)$$

$$y(t) = hx(t) + v(t)$$

where x, f, h, w, and v are scalars. $\{w(t)\}$ and $\{v(t)\}$ are mutually independent white Gaussian noises with variances R_1 and R_2, respectively. Show that $y(t)$ can be represented as an ARMA process:

$$y(t) + a_1 y(t-1) + \cdots + a_n y(t-n) = e(t) + c_1 e(t-1) + \cdots + c_n e(t-n)$$

Determine n, a_i, c_i, and the variance of $e(t)$ in terms of f, h, R_1, and R_2. What is the relationship between $e(t)$, $w(t)$, and $v(t)$?

2E.5 Consider the system

$$y(t) = G(q)u(t) + v(t)$$

controlled by the regulator

$$u(t) = -F_2(q)y(t) + F_1(q)r(t)$$

where $\{r(t)\}$ is a quasi-stationary reference signal with spectrum $\Phi_r(\omega)$. The disturbance $\{v(t)\}$ has spectrum $\Phi_v(\omega)$. Assume that $\{r(t)\}$ and $\{v(t)\}$ are uncorrelated and that the resulting closed-loop system is stable. Determine the spectra $\Phi_y(\omega)$, $\Phi_u(\omega)$, and $\Phi_{yu}(\omega)$.

2E.6 Consider the system

$$\frac{d}{dt}y(t) + ay(t) = u(t) \tag{2.100}$$

Suppose that the input $u(t)$ is piecewise constant over the sampling interval

$$u(t) = u_k, \qquad kT \le t < (k+1)T$$

(a) Derive a sampled-data system description for u_k, $y(kT)$.

(b) Assume that there is a time delay of T seconds so that $u(t)$ in (2.100) is replaced by $u(t-T)$. Derive a sampled-data system description for this case.

(c) Assume that the time delay is $1.5T$ so that $u(t)$ is replaced by $u(t-1.5T)$. Then give the sampled-data description.

2E.7 Consider a system given by

$$y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1)$$

where $\{u(t)\}$ and $\{e(t)\}$ are independent white noises, with variances μ and λ, respectively. Follow the procedure suggested in Appendix 2C to multiply the system description by $e(t)$, $e(t-1)$, $u(t)$, $u(t-1)$, $y(t)$, and $y(t-1)$, respectively, and take expectation to show that

$$R_{ye}(0) = \lambda, \qquad R_{ye}(1) = (c-a)\lambda$$

$$R_{yu}(0) = 0, \qquad R_{yu}(1) = b\mu$$

$$R_y(0) = \frac{b^2\mu + \lambda + c^2\lambda - 2ac\lambda}{1-a^2},$$

$$R_y(1) = \frac{\lambda(c - a + a^2c - ac^2) - ab^2\mu}{1-a^2}$$

2T.1 Consider a continuous time system (2.1):

$$y(t) = \int_{\tau=0}^{\infty} g(\tau)u(t-\tau)\,d\tau$$

Let $g_T(\ell)$ be defined by (2.5), and assume that $u(t)$ is not piecewise constant, but that

$$\left|\frac{d}{dt}u(t)\right| \le C_1$$

Let $u_k = u\left((k+\frac{1}{2})T\right)$ and show that

$$y(kT) = \sum_{\ell=1}^{\infty} g_T(\ell)u_{k-\ell} + r_k$$

where

$$|r_k| \le C_2 \cdot T^2$$

Give a bound for C_2.

2T.2 If the filters $R_1(q)$ and $R_2(q)$ are (strictly) stable, then show that $R_1(q)R_2(q)$ is also (strictly) stable (see also Problem 3D.1).

2T.3 Let $G(q)$ be a rational transfer function; that is,

$$G(q) = \frac{b_1q^{n-1} + \cdots + b_n}{q^n + a_1q^{n-1} + \cdots + a_n}$$

Show that if $G(q)$ is stable, then it is also strictly stable.

2T.4 Consider the time-varying system

$$x(t+1) = F(t)x(t) + G(t)u(t)$$

$$y(t) = H(t)x(t)$$

Write

$$y(t) = \sum_{k=1}^{t} g_t(k)u(t-k)$$

[take $g_t(k) = 0$ for $k > t$]. Assume that

$$F(t) \to \overline{F}, \qquad \text{as } t \to \infty$$

where $\overline{F}$ has all eigenvalues inside the unit circle. Show that the family of filters $\{g_t(k), t = 1, 2, \ldots\}$ is uniformly stable.

2D.1 Consider $U_N(\omega)$ defined by (2.37). Show that $U_N(2\pi - \omega) = U_N(\omega) = \overline{U}_N(\omega)$ and rewrite (2.38) in terms of real-valued quantities only.

2D.2 Establish (2.39).

2D.3 Let $\{u(t)\}$ be a stationary stochastic process with $R_u(\tau) = Eu(t)u(t-\tau)$, and let $\Phi_u(\omega)$ be its spectrum. Assume that

$$\sum_{1}^{\infty} |\tau R_u(\tau)| < \infty$$

Let $U_N(\omega)$ be defined by (2.37). Prove that

$$E\,|U_N(\omega)|^2 \to \Phi_u(\omega), \qquad \text{as } N \to \infty$$

This is a strengthening of Lemma 2.1 for stationary processes.

2D.4 Let $G(q)$ be a stable system. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} k\,|g(k)| = 0$$

Hint: Use Kronecker's lemma: Let a_k, b_k be sequences such that a_k is positive and decreasing to zero. Then $\Sigma a_k b_k < \infty$ implies

$$\lim_{N\to\infty} a_N \sum_{1}^{N} b_k = 0$$

(see, e.g., Chung, 1974, for a proof of Kronecker's lemma).

2D.5 Let $b_N(\tau)$ be a doubly indexed sequence such that, $\forall\tau$,

$$b_N(\tau) \to b(\tau), \qquad \text{as } N \to \infty$$

(but not necessarily uniformly in τ). Let a_τ be an infinite sequence, and assume that

$$\sum_{1}^{\infty} |a_\tau| < \infty, \qquad |b(\tau)| \le C \qquad \forall\tau$$

Show that

$$\lim_{N\to\infty} \left[\sum_{\tau=-N}^{N} a_\tau\,(b_N(\tau) - b(\tau)) + \sum_{|\tau|>N} a_\tau b(\tau) \right] = 0$$

Hint: Study Appendix 2A.

APPENDIX 2A: PROOF OF THEOREM 2.2

We carry out the proof for the multivariate case. Let $w(s) = 0$ for $s \leq 0$, and consider

$$\begin{aligned} R_s^N(\tau) &= \frac{1}{N}\sum_{t=1}^{N} Es(t)s^T(t-\tau) \\ &= \frac{1}{N}\sum_{t=1}^{N}\sum_{k=0}^{t}\sum_{\ell=0}^{t-\tau} g(k)Ew(t-k)w^T(t-\tau-\ell)g^T(\ell) \end{aligned} \tag{2A.1}$$

With the convention that $w(s) = 0$ if $s \notin [0, N]$, we can write

$$R_s^N(\tau) = \sum_{k=0}^{N}\sum_{\ell=0}^{N} g(k)\frac{1}{N}\sum_{t=1}^{N} Ew(t-k)w^T(t-\tau-\ell)g^T(\ell) \tag{2A.2}$$

If $w(s) \neq 0$, $s \leq 0$, then $s(t)$ gets the negligible contribution

$$\bar{s}(t) = \sum_{k=t}^{\infty} g(k)w(t-k)$$

Let

$$R_w^N(\tau) = \frac{1}{N}\sum_{t=1}^{N} Ew(t)w^T(t-\tau)$$

We see that $R_w^N(\tau + \ell - k)$ and the inner sum in (2A.2) differ by at most $\max(k, |\tau + \ell|)$ summands, each of which are bounded by C according to (2.58). Thus

$$\begin{aligned} &\left| R_w^N(\tau + \ell - k) - \frac{1}{N}\sum_{t=1}^{N} Ew(t-k)w^T(t-\tau-\ell)\right| \\ &\quad \leq C\frac{\max(k, |\tau + \ell|)}{N} \leq \frac{C}{N}(k + |\tau + \ell|) \end{aligned} \tag{2A.3}$$

Let us define

$$R_s(\tau) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} g(k)R_w(\tau + \ell - k)g^T(\ell) \tag{2A.4}$$

Then

$$R_s(\tau) - R_s^N(\tau) \le \sum_{k=N+1}^{\infty} \sum_{\ell=N+1}^{\infty} |g(k)||g(\ell)||R_{u'}(\tau + \ell - k)|$$

$$+ \sum_{k=0}^{N} \sum_{\ell=0}^{N} |g(k)||g(\ell)||R_{u'}(\tau + \ell - k) - R_{u'}^N(\tau + \ell - k)|$$

$$+ \frac{C}{N} \sum_{k=0}^{N} k\,|g(k)| \cdot \sum_{\ell=0}^{N} |g(\ell)|$$

$$+ \frac{C}{N} \sum_{\ell=0}^{N} |\tau + \ell||g(\ell)| \cdot \sum_{k=0}^{N} |g(k)| \tag{2A.5}$$

The first sum tends to zero as $N \to \infty$ since $|R_{u'}(\tau)| \le C$ and $G(q)$ is stable. It follows from the stability of $G(q)$ that

$$\frac{1}{N} \sum_{k=0}^{N} k\,|g(k)| \to 0, \qquad \text{as } N \to \infty \tag{2A.6}$$

(see Problem 2D.4). Hence the last two sums of (2A.5) tend to zero as $N \to \infty$. Consider now the second sum of (2A.5). Select an arbitrary $\varepsilon > 0$, and choose $N = N_\varepsilon$ such that

$$\sum_{k=N_\varepsilon+1}^{\infty} |g(k)| < \frac{\varepsilon}{[C \cdot C_1]} \quad \text{where} \quad C_1 = \sum_{k=0}^{\infty} |g(k)| \tag{2A.7}$$

This is possible since G is stable. Then select N'_ε such that

$$\max_{\substack{1 \le \ell \le N_\varepsilon \\ 1 \le k \le N_\varepsilon}} \left| R_{u'}(\tau + \ell - k) - R_{u'}^N(\tau + \ell - k) \right| < \frac{\varepsilon}{C_1^2}$$

for $N > N'_\varepsilon$. This is possible since

$$R_{u'}^N(\tau) \to R_{u'}(\tau), \qquad \text{as } N \to \infty \tag{2A.8}$$

(w is quasi-stationary) and since only a finite number of $R_w(s)$'s are involved (no uniform convergence of (2A.8) is necessary). Then, for $N > N'_\varepsilon$, we have that the second sum of (2A.5) is bounded by

$$\sum_{k=0}^{N_\varepsilon}\sum_{\ell=0}^{N_\varepsilon}|g(k)||g(\ell)|\cdot\frac{\varepsilon}{C_1^2} + \sum_{k=N_\varepsilon+1}^{\infty}\sum_{\ell=0}^{\infty}|g(k)||g(\ell)|\cdot 2C$$
$$+\sum_{k=0}^{N_\varepsilon}\sum_{\ell=N_\varepsilon+1}^{\infty}|g(k)||g(\ell)|\cdot 2C$$

which is less than 5ε, according to (2A.7). Hence, also, the second sum of (2A.5) tends to zero as $N\to\infty$, and we have proved that the limit of (2A.5) is zero and hence that $s(t)$ is quasi-stationary.

The proof that $\bar{E}s(t)w^T(t-\tau)$ exists is analogous and simpler.

For $\Phi_s(\omega)$ we now find that

$$\begin{aligned}
\Phi_s(\omega) &= \sum_{\tau=-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}g(k)R_w(\tau+\ell-k)g^T(\ell)\right)e^{-i\tau\omega}\\
&= \sum_{\tau=-\infty}^{\infty}\sum_{k=0}^{\infty}g(k)e^{-ik\omega}\sum_{\ell=0}^{\infty}R_w(\tau+\ell-k)e^{-i(\tau+\ell-k)\omega}g^T(\ell)e^{i\ell\omega}\\
&= [\tau+\ell-k=s]\\
&= \sum_{k=0}^{\infty}g(k)e^{-ik\omega}\cdot\sum_{s=-\infty}^{\infty}R_w(s)e^{-is\omega}\cdot\sum_{\ell=0}^{\infty}g^T(\ell)e^{i\ell\omega}\\
&= G(e^{i\omega})\Phi_s(\omega)G^T(e^{-i\omega})
\end{aligned}$$

Hence (2.79) is proved. The result (2.80) is analogous and simpler.

For families of linear filters we have the following result:

Corollary. Let $\{G_\theta(q),\ \theta\in D\}$ be a uniformly stable family of linear filters, and let $\{w(t)\}$ be a quasi-stationary sequence. Let $s_\theta(t) = G_\theta(q)w(t)$ and $R_s(\tau,\theta) = \bar{E}s_\theta(t)s_\theta^T(t-\tau)$. Then:

$$\sup_{\theta\in D}\left\|\frac{1}{N}\sum_{t=1}^{N}s_\theta(t)s_\theta^T(t-\tau) - R_s(\tau,\theta)\right\| \to 0 \quad \text{as } N\to\infty$$

Proof. We only have to establish that the convergence in (2A.5) is uniform in $\theta \in D$. In the first step all the $g(k)$-terms carry an index θ : $g_\theta(k)$. Interpreting

$$g(k) = \sup_{\theta \in D} |g_\theta(k)|$$

(2A.5) will of course still hold. Since the family $G_\theta(q)$ is uniformly stable, the sum over $g(k)$ will be convergent, which was the only property used to prove that (2A.5) tends to zero. This completes the proof of the corollary. □

APPENDIX 2B: PROOF OF THEOREM 2.3

In this appendix we shall show a more general variant of Theorem 2.3, which will be of value for the convergence analysis of Chapter 8. We also treat the multivariable case.

Theorem 2B.1. Let $\{G_\theta(q), \theta \in D_\theta\}$ and $\{M_\theta(q), \theta \in D_\theta\}$ be uniformly stable families of filters, and assume that the deterministic signal $\{w(t)\}$, $t = 1, 2, \ldots,$ is subject to

$$|w(t)| \leq C_w, \qquad \forall t \tag{2B.1}$$

Let the signal $s_\theta(t)$ be defined, for each $\theta \in D_\theta$, by

$$s_\theta(t) = G_\theta(q)v(t) + M_\theta(q)w(t) \tag{2B.2}$$

where $\{v(t)\}$ is subject to the conditions of Theorem 2.3 (see (2.88) and let $Ee(t)e^T(t) = \Lambda_t$). Then

$$\sup_{\theta \in D_\theta} \left\| \frac{1}{N} \sum_{t=1}^{N} \left[s_\theta(t) s_\theta^T(t) - E s_\theta(t) s_\theta^T(t) \right] \right\| \to 0 \text{ w.p.1, as } N \to \infty \tag{2B.3}$$

Remark. We note that with $\dim s = 1$, $D_\theta = \{\theta^*\}$ (only one element), $G_\theta^*(q) = 1$, $M_\theta^*(q) = 1$ and $w(t) = m(t)$, then (2B.3) implies (2.89). With

$$s_\theta(t) = \begin{bmatrix} s(t) \\ m(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} v(t) + \begin{bmatrix} w(t) \\ w(t) \\ 0 \end{bmatrix} \tag{2B.4}$$

the different cross products in (2B.3) imply all the results (2.89).

To prove Theorem 2B.1, we first establish two lemmas.

Lemma 2B.1. Let $\{v(t)\}$ obey the conditions of Theorem 2.3 and let

$$C_H = \sum_{k=1}^{\infty} \sup_t |h_t(k)|, \qquad C_l = \sup_t E\,|e(t)|^4, \qquad C_w = \sup_t |w(t)|$$

Then, for all r, N, k, and l,

$$E\left\|\sum_{t=r}^{N}[v(t-k)v^T(t-\ell) - Ev(t-k)v^T(t-\ell)]\right\|^2$$

$$\leq 4 \cdot C_e \cdot C_H^4 \cdot (N-r) \tag{2B.5}$$

$$E\left\|\sum_{t=r}^{N} v(t-k)w^T(t-l)\right\|^2$$

$$\leq 4 \cdot C_e \cdot C_H^2 \cdot C_u^2 \cdot (N-r) \tag{2B.6}$$

Proof of Lemma 2B.1. With no loss of generality, we may take $k = l = 0$. We then have

$$S_r^N \triangleq \sum_{t=r}^{N} v(t)v^T(t) - Ev(t)v^T(t) = \sum_{t=r}^{N}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} h_t(k)\alpha(t,k,\ell)h_t^T(\ell) \tag{2B.7}$$

where

$$\alpha(t,k,\ell) = e(t-k)e^T(t-\ell) - \Lambda_{t-\ell}\delta_{k\ell} \tag{2B.8}$$

For the square of the i, j entry of the matrix (2B.7), we have

$$(S_r^N(i,j))^2 = \sum_{t=r}^{N}\sum_{s=r}^{N}\sum_{k_1=0}^{\infty}\sum_{\ell_l=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{\ell_2=0}^{\infty} \gamma(t,s,k_1,k_2,\ell_1,\ell_2)$$

with

$$\gamma(t,s,k_1,k_2,\ell_1,\ell_2)$$

$$= h_t^{(i)}(k_1)\alpha(t,k_1,\ell_1)\left[h_t^{(j)}(\ell_1)\right]^T h_s^{(i)}(k_2)\alpha(s,k_2,\ell_2)\left[h_s^{(j)}(\ell_2)\right]^T$$

Superscript (i) indicates the ith row vector. Since $\{e(t)\}$ is a sequence of independent variables, the expectation of γ is zero, unless at least some of the time indexes involved in $\alpha(t,k_1,\ell_1)$ and $\alpha(s,k_2,\ell_2)$ coincide, that is, unless

$$t-k_1 = s-k_2 \quad \text{or} \quad t-k_1 = s-\ell_2 \quad \text{or} \quad t-\ell_1 = s-k_2 \quad \text{or} \quad t-\ell_1 = s-\ell_2$$

For given values of t, k_1, k_2, ℓ_1, and ℓ_2 this may happen for at most four values of s. For these we also have

$$E\gamma(t,s,k_1,k_2,\ell_1,\ell_2) \le C_e \cdot |h(k_1)| \cdot |h(k_2)| \cdot |h(\ell_1)| \cdot |h(\ell_2)|$$

Hence

$$E\left(S_r^N(i,j)\right)^2 \le \sum_{k_1=0}^{\infty} |h(k_1)| \cdot \sum_{k_2=0}^{\infty} |h(k_2)| \cdot \sum_{\ell_1=0}^{\infty} |h(\ell_1)|$$
$$\cdot \sum_{\ell_2=0}^{\infty} |h(\ell_2)| \cdot \sum_{t=r}^{N} 4 \cdot C_e \le 4 \cdot C_e C_H^4 (N-r)$$

which proves (2B.5) of the lemma. The proof of (2B.6) is analogous and simpler. $\square$

Corollary to Lemma 2B.1. Let

$$w(t) = \sum_{k=0}^{\infty} \alpha_t(k) e(t-k), \quad v(t) = \sum_{k=0}^{\infty} \beta_t(k) e(t-k)$$

Then

$$E\left\| \sum_{t=r}^{N} w(t)v(t) - Ew(t)v(t) \right\|^2 \le C \cdot C_w^2 \cdot C_v^2 \cdot (N-r)$$

$$C_w = \sum_{k=0}^{\infty} \sup_t |\alpha_t(k)|, \quad C_v = \sum_{k=0}^{\infty} \sup_t |\beta_t(k)|$$

Lemma 2B.2. Let

$$R_r^N = \sup_{\theta \in D_\theta} \left\| \sum_{t=r}^{N} s_\theta(t) s_\theta^T(t) - E s_\theta(t) s_\theta^T(t) \right\| \tag{2B.9}$$

Then

$$E(R_r^N)^2 \le C(N-r) \tag{2B.10}$$

Proof of Lemma 2B.2. First note the following fact: If

$$\varphi = \sum_{k=0}^{\infty} a(k) z(k) \tag{2B.11}$$

where $\{a(k)\}$ is a sequence of deterministic matrices, such that

$$\sum_{k=0}^{\infty} \|a(k)\| \le C_a$$

and $\{z(k)\}$ is a sequence of random vectors such that $E\,|z(k)|^2 \le C_z$ then:

$$\begin{aligned} E|\varphi|^2 &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \operatorname{tr}\left[a(k)Ez(k)z^T(\ell)a^T(\ell)\right] \\ &\le \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \|a(k)\| \cdot \left[E\,|z(k)|^2\right]^{1/2} \cdot \left[E\,|z(\ell)|^2\right]^{1/2} \cdot \|a(\ell)\| \qquad (2B.12) \\ &\le C_z \cdot \left[\sum_{k=0}^{\infty} \|a(k)\|\right]^2 \le C_z \cdot C_a^2 \end{aligned}$$

Here the first inequality is Schwarz's inequality. We now have

$$\begin{aligned} R_\theta(N, r) &= \sum_{t=r}^{N} \left[s_\theta(t)s^T\theta(t) - Es_\theta(t)s_\theta^T(t)\right] \\ &= \sum_{t=r}^{N}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} g_\theta(k)\left[v(t-k)v^T(t-\ell) - Ev(t-k)v^T(t-\ell)\right]g_\theta^T(\ell) \\ &\quad + \sum_{t=r}^{N}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} g_\theta(k)v(t-k)w^T(t-\ell)m_\theta^T(\ell) \\ &\quad + \sum_{t=r}^{N}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} m_\theta(k)w(t-k)v^T(t-\ell)g_\theta^T(\ell) \qquad (2B.13) \end{aligned}$$

This gives

$$\begin{aligned} \sup_\theta \|R_\theta(N, r)\| &\le \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \sup_\theta \|g_\theta(k)\| \cdot \sup_\theta \|g_\theta(\ell)\| \cdot \left\|S_r^N(k, \ell)\right\| \\ &\quad + 2\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \sup_\theta \|g_\theta(k)\| \sup_\theta \|m_\theta(\ell)\| \cdot \left\|\tilde{S}_r^N(k, \ell)\right\| \qquad (2B.14) \end{aligned}$$

with S_r^N and $\tilde{S}_r^N$ defined by

$$S_r^N(k, \ell) = \sum_{t=r}^{N} \left[v(t-k)v^T(t-\ell) - Ev(t-k)v^T(t-\ell)\right]$$

$$\tilde{S}_r^N(k, \ell) = \sum_{t=r}^{N} v(t-k)w^T(t-\ell)$$

Since $G_\theta(q)$ and $M_\theta(q)$ are uniformly stable families of filters,

$$\sup_\theta \|g_\theta(k)\| \leq \bar{g}(k), \quad \sum_{k=1}^{\infty} \bar{g}(k) = C_G < \infty,$$

$$\sup_\theta \|m_\theta(k)\| \leq \bar{m}(k), \quad \sum_{k=1}^{\infty} \bar{m}(k) = C_M < \infty$$

Applying (2B.11) and (2B.12) together with Lemma 2B.1 to (2B.14) gives

$$E\left[\sup_\theta \|R_\theta(N, r)\|\right]^2 \leq 2 \cdot C_G^4 \cdot 4 \cdot C_e \cdot C_H^4 \cdot (N - r)$$
$$+ \ 8 \cdot C_G^2 \cdot C_M^2 \cdot 4 \cdot C_w^2 \cdot C_H^2 \cdot (N - r) \leq C \cdot (N - r)$$

which proves Lemma 2B.2. □

We now turn to the proof of Theorem 2B.1. Denote

$$r(t, \theta) = s_\theta(t) s_\theta^T(t) - E s_\theta(t) s_\theta^T(t) \tag{2B.15}$$

and let

$$R_r^N = \sup_{\theta \in D_\mathcal{M}} \|R_\theta(N, r)\| \tag{2B.16}$$

with $R_\theta(N, r)$ defined by (2B.13). According to Lemma 2B.2

$$E\left(\frac{1}{N^2} R_1^{N^2}\right)^2 \leq \left(\frac{1}{N^2}\right)^2 \cdot C \cdot N^2 \leq \frac{C}{N^2}$$

Chebyshev's inequality (I.19) gives

$$P\left(\frac{1}{N^2} R_1^{N^2} > \varepsilon\right) \leq \frac{1}{\varepsilon^2} E\left(\frac{R_1^{N^2}}{N^2}\right)^2$$

Hence:

$$\sum_{k=1}^{\infty} P\left(\frac{1}{k^2} R_1^{k^2} > \varepsilon\right) \leq \frac{C}{\varepsilon^2} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

which, via Borel-Cantelli's lemma [see (I.18)], implies that

$$\frac{1}{k^2} R_1^{k^2} \to 0, \qquad \text{w.p. 1} \qquad \text{as } k \to \infty \tag{2B.17}$$

Now suppose that

$$\sup_{N^2 \leq k \leq (N+1)^2} \frac{1}{k} R_1^k$$

is obtained for $k = k_N$ and $\theta = \theta_N$. Hence

$$\begin{aligned}
\sup_{N^2 \le k \le (N+1)^2} \frac{1}{k} R_1^k &= \frac{1}{k_N} \left| \sum_{t=1}^{k_N} r(t, \theta_N) \right| \\
&\le \frac{1}{k_N} \left| \sum_{t=1}^{N^2} r(t, \theta_N) \right| + \frac{1}{k_N} \left| \sum_{t=N^2+1}^{k_N} r(t, \theta_N) \right| \qquad \text{(2B.18)} \\
&\le \frac{1}{k_N} \cdot R_1^{N^2} + \frac{1}{k_N} \cdot R_{N^2+1}^{k_N}
\end{aligned}$$

Since $k_N \ge N^2$, the first term on the right side of (2B.18) tends to zero w.p.1 in view of (2B.17). For the second one we have, using Lemma 2B.2,

$$\begin{aligned}
E \left| \frac{1}{k_N} R_{N^2+1}^{k_N} \right|^2 &\le \frac{1}{N^4} \cdot E \max_{N^2 \le k \le (N+1)^2} \left| R_{N^2+1}^k \right|^2 \\
&\le \frac{1}{N^4} \sum_{k=N^2+1}^{(N+1)^2} E \left| R_{N^2+1}^k \right|^2 \le \frac{1}{N^4} \sum_{k=N^2+1}^{(N+1)^2} C \cdot (k - N^2) \le \frac{C}{N^2}
\end{aligned}$$

which using Chebyshev's inequality (I.19) and the Borel-Cantelli lemma as before, shows that also the second term of (2B.18) tends to zero w.p.1. Hence

$$\sup_{N^2 \le k \le (N+1)^2} \frac{1}{k} R_1^k \to 0, \quad \text{w.p.1}, \quad \text{as } N \to \infty \qquad \text{(2B.19)}$$

which proves the theorem.

Corollary to Theorem 2B.1. Suppose that the conditions of the theorems hold, but that (2.88) is weakened to

$$E\left[e(t) | e(t-1), \ldots, e(0)\right] = 0, \quad E\left[e^2(t) | e(t-1), \ldots, e(0)\right] = \lambda$$

$$E\, |e(t)|^4 \le C$$

Then the theorem still holds. [That is: $\{e(t)\}$ need not be white noise, it is sufficient that it is a martingale difference.]

Proof. Independence was used only in Lemma 2B.1. It is easy to see that this lemma holds also under the weaker conditions. $\square$

APPENDIX 2C: COVARIANCE FORMULAS

For several calculations we need expressions for variances and covariances of signals in ARMA descriptions. These are basically given by the inverse formulas

$$R_s(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_s(\omega)e^{i\omega\tau}\,d\omega \tag{2C.1a}$$

$$R_{sv}(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{sv}(\omega)e^{i\omega\tau}\,d\omega \tag{2C.1b}$$

With the expressions according to Theorem 2.2 for the spectra, (2C.1) takes the form

$$\begin{aligned} R_s(\tau) &= \frac{\lambda}{2\pi}\int_{-\pi}^{\pi}\left|\frac{C(e^{i\omega})}{A(e^{i\omega})}\right|^2 e^{i\omega\tau}\,d\omega = [z = e^{i\omega}] \\ &= \frac{\lambda}{2\pi}\oint \frac{C(z)C(1/z)}{A(z)A(1/z)} z^{\tau-1}\,dz \end{aligned} \tag{2C.2}$$

for an ARMA process. The last integral is a complex integral around the unit circle, which could be evaluated using residue calculus. Åström, Jury, and Agniel (1970)(see also Åström, 1970, Ch. 5) have derived an efficient algorithm for computing (2C.2) for $\tau = 0$. It has the following form:

$$A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad C(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_n$$

Let $a_i^n = a_i$ and $c_i^n = c_i$ and define a_i^k, c_i^k recursively by

$$a_i^{n-k} = \frac{a_0^{n-k+1}a_i^{n-k+1} - a_{n-k+1}^{n-k+1}a_{n-k+1-i}^{n-k+1}}{a_0^{n-k+1}}$$

$$c_i^{n-k} = \frac{a_0^{n-k+1}c_i^{n-k+1} - c_{n-k+1}^{n-k+1}a_{n-k+1-i}^{n-k+1}}{a_0^{n-k+1}}$$

$$i = 0, 1, \ldots, n-k, \qquad k = 1, 2, \ldots, n$$

Then for (2C.2)

$$R_s(0) = \frac{1}{a_0}\sum_{k=0}^{n}\frac{(c_k^k)^2}{a_0^k} \tag{2C.3}$$

An explicit expression for the variance of a second-order ARMA process

$$\begin{aligned} y(t) + a_1 y(t-1) + a_2 y(t-2) &= e(t) + c_1 e(t-1) + c_2 e(t-2) \\ Ee^2(t) &= 1 \end{aligned} \tag{2C.4}$$

is

$$\operatorname{Var} y(t) = \frac{(1+a_2)\left(1+(c_1)^2+(c_2)^2\right) - 2a_1c_1(1+c_2) - 2c_2\left(a_2-(a_1)^2+(a_2)^2\right)}{(1-a_2)(1-a_1+a_2)(1+a_1+a_2)} \tag{2C.5}$$

To find $R_y(\tau)$ and the cross covariances $R_{ye}(\tau)$ by hand calculations in simple examples, the easiest approach is to multiply (2C.4) by $e(t)$, $e(t-1)$, $e(t-2)$, $y(t)$, $y(t-1)$, and $y(t-2)$ and take expectation. This gives six equations for the six variables $R_{ye}(\tau)$, $R_y(\tau)$, or $\tau = 0, 1, 2$. Note that $R_{ye}(\tau) = 0$ for $\tau < 0$.

For numerical computation in MATLAB it is easiest to represent the ARMA process in state-space form. with

$$[\, y(t-1) \quad \dots \quad y(t-n) \quad e(t-1) \quad \dots \quad e(t-n) \,]^T$$

as states, and then use **dlyap** to compute the state covariance matrix. This will contain all variances and covariances of interest.