II methods

6

NONPARAMETRIC TIME- AND FREQUENCY-DOMAIN METHODS

A linear time-invariant model can be described by its transfer functions or by the corresponding impulse responses, as we found in Chapter 4. In this chapter we shall discuss methods that aim at determining these functions by direct techniques without first selecting a confined set of possible models. Such methods are often also called *nonparametric* since they do not (explicitly) employ a finite-dimensional parameter vector in the search for a best description. We shall discuss the determination of the transfer function $G(q)$ from input to output. Section 6.1 deals with time-domain methods for this, and Sections 6.2 to 6.4 describe frequency-domain techniques of various degrees of sophistication. The determination of $H(q)$ or the disturbance spectrum is discussed in Section 6.5.

It should be noted that throughout this chapter we assume the system to operate in open loop [i.e., $\{u(t)\}$ and $\{v(t)\}$ are independent]. Closed-loop configurations will typically lead to problems for nonparametric methods, as outlined in some of the problems. These issues are discussed in more detail in Chapter 13.

6.1 TRANSIENT-RESPONSE ANALYSIS AND CORRELATION ANALYSIS

Impulse-Response Analysis

If a system that is described by (2.8)

$$y(t) = G_0(q)u(t) + v(t) \tag{6.1}$$

is subjected to a pulse input

$$u(t) = \begin{cases} \alpha, & t = 0 \\ 0, & t \neq 0 \end{cases} \tag{6.2}$$

then the output will be

$$y(t) = \alpha g_0(t) + v(t) \tag{6.3}$$

by definition of G_0 and the impulse response $\{g_0(t)\}$. If the noise level is low, it is thus possible to determine the impulse-response coefficients $\{g_0(t)\}$ from an experiment with a pulse input. The estimates will be

$$\hat{g}(t) = \frac{y(t)}{\alpha} \tag{6.4}$$

and the errors $v(t)/\alpha$. This simple idea is *impulse-response analysis*. Its basic weakness is that many physical processes do not allow pulse inputs of such an amplitude that the error $v(t)/\alpha$ is insignificant compared to the impulse-response coefficients. Moreover, such an input could make the system exhibit nonlinear effects that would disturb the linearized behavior we have set out to model.

Step-Response Analysis

Similarly, a step

$$u(t) = \begin{cases} \alpha, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

applied to (6.1) gives the output

$$y(t) = \alpha \sum_{k=1}^{t} g_0(k) + v(t) \tag{6.5}$$

From this, estimates of $g_0(k)$ could be obtained as

$$\hat{g}(t) = \frac{y(t) - y(t-1)}{\alpha} \tag{6.6}$$

which has an error $[v(t) - v(t-1)]/\alpha$. If we really aim at determining the impulse-response coefficients using (6.6), we would suffer from large errors in most practical applications. However, if the goal is to determine some basic control-related characteristics, such as delay time, static gain, and dominating time constants [i.e., the model (4.50)], step responses (6.5) can very well furnish that information to a sufficient degree of accuracy. In fact, well-known rules for tuning simple regulators such as the Ziegler-Nichols rule (Ziegler and Nichols, 1942) are based on model information reached in step responses.

Based on plots of the step response, some characteristic numbers can be graphically constructed, which in turn can be used to determine parameters in a model of given order. We refer to Rake (1980)for a discussion of such characteristics.

Correlation Analysis

Consider the model description (6.1):

$$y(t) = \sum_{k=1}^{\infty} g_0(k)u(t-k) + v(t) \tag{6.7}$$

If the input is a quasi-stationary sequence [see (2.59)] with

$$\bar{E}u(t)u(t-\tau) = R_u(\tau)$$

and

$$\overline{E}u(t)v(t-\tau) \equiv 0 \qquad \text{(open-loop operation)}$$

then according to Theorem 2.2 (expressed in the time domain)

$$\overline{E}y(t)u(t-\tau) = R_{yu}(\tau) = \sum_{k=1}^{\infty} g_0(k)R_u(k-\tau) \tag{6.8}$$

If the input is chosen as white noise so that

$$R_u(\tau) = \alpha\delta_{\tau 0}$$

then

$$g_0(\tau) = \frac{R_{yu}(\tau)}{\alpha}$$

An estimate of the impulse response is thus obtained from an estimate of $R_{yu}(\tau)$; for example,

$$\hat{R}_{yu}^N(\tau) = \frac{1}{N}\sum_{t=\tau}^{N} y(t)u(t-\tau) \tag{6.9}$$

If the input is not white noise, we may estimate

$$\hat{R}_u^N(\tau) = \frac{1}{N}\sum_{t=\tau}^{N} u(t)u(t-\tau) \tag{6.10}$$

and solve

$$\hat{R}_{yu}^N(\tau) = \sum_{k=1}^{M} \hat{g}(k)\hat{R}_u^N(k-\tau) \tag{6.11}$$

for $\hat{g}(k)$. If the input is open for manipulation, it is of course desirable to choose it so that (6.10) and (6.11) become easy to solve. Equipment for generating such signals and solving for $\hat{g}(k)$ is commercially available. See Godfrey (1980)for a more detailed treatment.

In fact, the most natural way to estimate $g(k)$ when the input is not "exactly white" is to truncate (6.7) at n, and treat it as an n:th order FIR model (4.46) with the parametric (least-squares) methods of Chapter 7. Another way is to filter both inputs and outputs by a prefilter that makes the input as white as possible ("input prewhitening") and then compute the correlation function (6.9) for these filtered sequences.

6.2 FREQUENCY-RESPONSE ANALYSIS

Sine-wave Testing

The fundamental physical interpretation of the transfer function $G(z)$ is that the complex number $G(e^{i\omega})$ bears information about what happens to an input sinusoid [see (2.32) to (2.34)]. We thus have for (6.1) that with

$$u(t) = \alpha\cos\omega t, \qquad t = 0, 1, 2, \ldots \tag{6.12}$$

then

$$y(t) = \alpha|G_0(e^{i\omega})|\cos(\omega t + \varphi) + v(t) + \text{transient} \tag{6.13}$$

where

$$\varphi = \arg G_0(e^{i\omega}) \tag{6.14}$$

This property also gives a clue to a simple way of determining $G_0(e^{i\omega})$:

> With the input (6.12), determine the amplitude and the phase shift of the resulting output cosine signal, and calculate an estimate $\hat{G}_N(e^{i\omega})$ based on that information. Repeat for a number of frequencies in the interesting frequency band.

This is known as *frequency analysis* and is a simple method for obtaining detailed information about a linear system.

Frequency Analysis by the Correlation Method

With the noise component $v(t)$ present in (6.13), it may be cumbersome to determine $|G_0(e^{i\omega})|$ and φ accurately by graphical methods. Since the interesting component of $y(t)$ is a cosine function of known frequency, it is possible to correlate it out from the noise in the following way. Form the sums

$$I_c(N) = \frac{1}{N}\sum_{t=1}^{N} y(t)\cos\omega t, \qquad I_s(N) = \frac{1}{N}\sum_{t=1}^{N} y(t)\sin\omega t \tag{6.15}$$

Inserting (6.13) into (6.15), ignoring the transient term, gives

$$\begin{aligned}
I_c(N) &= \frac{1}{N}\sum_{t=1}^{N}\alpha|G_0(e^{i\omega})|\cos(\omega t + \varphi)\cos\omega t + \frac{1}{N}\sum_{t=1}^{N} v(t)\cos\omega t \\
&= \alpha|G_0(e^{i\omega})|\frac{1}{2}\frac{1}{N}\sum_{t=1}^{N}[\cos\varphi + \cos(2\omega t + \varphi)] \\
&\quad + \frac{1}{N}\sum_{t=1}^{N} v(t)\cos\omega t \qquad\qquad (6.16) \\
&= \frac{\alpha}{2}|G_0(e^{i\omega})|\cos\varphi + \alpha|G_0(e^{i\omega})|\frac{1}{2}\frac{1}{N}\sum_{t=1}^{N}\cos(2\omega t + \varphi) \\
&\quad + \frac{1}{N}\sum_{t=1}^{N} v(t)\cos\omega t
\end{aligned}$$

The second term tends to zero as N tends to infinity, and so does the third term if $v(t)$ does not contain a pure periodic component of frequency ω. If $\{v(t)\}$ is a stationary stochastic process such that

$$\sum_{0}^{\infty} \tau |R_v(\tau)| < \infty$$

then the variance of the third term of (6.16) decays like $1/N$ (Problem 6T.2). Similarly,

$$I_s(N) = -\frac{\alpha}{2}|G_0(e^{i\omega})| \sin\varphi + \alpha|G_0(e^{i\omega})|\frac{1}{2}\frac{1}{N}\sum_{t=1}^{N}\sin(2\omega t + \varphi)$$
$$+ \frac{1}{N}\sum_{t=1}^{N} v(t)\sin\omega t \tag{6.17}$$

These two expressions suggest the following estimates of $|G_0(e^{i\omega})|$ and φ:

$$|\hat{G}_N(e^{i\omega})| = \frac{\sqrt{I_c^2(N) + I_s^2(N)}}{\alpha/2} \tag{6.18a}$$

$$\hat{\varphi}_N = \arg\hat{G}_N(e^{i\omega}) = -\arctan\frac{I_s(N)}{I_c(N)} \tag{6.18b}$$

Rake (1980)gives a more detailed account of this method. By repeating the procedure for a number of frequencies, a good picture of $G_0(e^{i\omega})$ over the frequency domain of interest can be obtained. Equipment that performs such *frequency analysis by the correlation method* is commercially available.

An advantage with this method is that a Bode plot of the system can be obtained easily and that one may concentrate the effort to the interesting frequency ranges. The main disadvantage is that many industrial processes do not admit sinusodial inputs in normal operation. The experiment must also be repeated for a number of frequencies which may lead to long experimentation periods.

Relationship to Fourier Analysis

Comparing (6.15) to the definition (2.37),

$$Y_N(\omega) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} y(t)e^{-i\omega t} \tag{6.19}$$

shows that

$$I_c(N) - i I_s(N) = \frac{1}{\sqrt{N}} Y_N(\omega) \tag{6.20}$$

As in (2.46) we find that, for (6.12),

$$U_N(\omega) = \frac{\sqrt{N}\alpha}{2}, \qquad \text{if } \omega = \frac{2\pi r}{N} \text{ for some integer } r \tag{6.21}$$

It is straightforward to rearrange (6.18) as

$$\hat{G}_N(e^{i\omega}) = \frac{\sqrt{N}Y_N(\omega)}{N\alpha/2} \tag{6.22}$$

which, using (6.21), means that

$$\hat{G}_N(e^{i\omega}) = \frac{Y_N(\omega)}{U_N(\omega)} \tag{6.23}$$

Here ω is precisely the frequency of the input signal. Comparing with (2.53), we also find (6.23) a most reasonable estimate (especially since $R_N(\omega)$ in (2.53) is zero for periodic inputs, according to the corollary of Theorem 2.1).

6.3 FOURIER ANALYSIS

Empirical Transfer-function Estimate

We found the expression (6.23) to correspond to frequency analysis with a single sinusoid of frequency ω as input. In a linear system, different frequencies pass through the system independently of each other. It is therefore quite natural to extend the frequency analysis estimate (6.23) also to the case of multifrequency inputs. That is, we introduce the following estimate of the transfer function:

$$\boxed{\hat{\hat{G}}_N(e^{i\omega}) = \frac{Y_N(\omega)}{U_N(\omega)} \qquad (6.24)}$$

with Y_N and U_N defined by (6.19), also for the case where the input is not a single sinusoid. This estimate is also quite natural in view of Theorem 2.1.

We shall call $\hat{\hat{G}}_N(e^{i\omega})$ the *empirical transfer-function estimate* (ETFE), for reasons that we shall discuss shortly. In (6.24) we assume of course that $U_N(\omega) \neq 0$. If this does not hold for some frequencies, we simply regard the ETFE as undefined at those frequencies. We call this estimate empirical, since no other assumptions have been imposed than linearity of the system. In the case of multifrequency inputs, the

ETFE consists of $N/2$ essential points. [Recall that estimates at frequencies intermediate to the grid $\omega = 2\pi k/N$, $k = 0, 1, \dots, N-1$, are obtained by trigonometrical interpolation in (2.37)]. Also, since y and u are real, we have

$$\hat{\hat{G}}_N(e^{2\pi ik/N}) = \overline{\hat{\hat{G}}_N(e^{2\pi i(N-k)/N})} \tag{6.25}$$

[compare (2.40) and (2.41)].

The original data sequence consisting of $2N$ numbers $y(t)$, $u(t)$, $t = 1, 2, \dots, N$, has thus been condensed into the N numbers

$$\text{Re}\hat{\hat{G}}_N(e^{2\pi ik/N}), \qquad \text{Im}\hat{\hat{G}}_N(e^{2\pi ik/N}), \qquad k = 0, 1, \dots, \frac{N}{2} - 1$$

This is quite a modest data reduction, revealing that most of the information contained in the original data y, u still is quite "raw."

In addition to an extension of frequency analysis, the ETFE can be interpreted as a way of (approximately) solving the set of convolution equations

$$y(t) = \sum_{k=1}^{N} g_0(k)u(t-k), \qquad t = 1, 2, \dots, N \tag{6.26}$$

for $g_0(k)$, $k = 1, 2, \dots, N$, using Fourier techniques.

Properties of the ETFE

Assume that the system is subject to (6.1). Introducing

$$V_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} v(t)e^{-i\omega t} \tag{6.27}$$

for the disturbance term, we find from Theorem 2.1 that

$$\hat{\hat{G}}_N(e^{i\omega}) = G_0(e^{i\omega}) + \frac{R_N(\omega)}{U_N(\omega)} + \frac{V_N(\omega)}{U_N(\omega)} \tag{6.28}$$

where the term $R_N(\omega)$ is subject to (2.54) and decays as $1/\sqrt{N}$.

Let us now investigate the influence of the term $V_N(\omega)$ on $\hat{\hat{G}}_N(e^{i\omega})$. Since $v(t)$ is assumed to have zero mean value,

$$EV_N(\omega) = 0, \qquad \forall\omega$$

so that

$$E\hat{\hat{G}}_N(e^{i\omega}) = G_0(e^{i\omega}) + \frac{R_N(\omega)}{U_N(\omega)} \tag{6.29}$$

Here expectation is with respect to $\{v(t)\}$, assuming $\{u(t)\}$ to be a given sequence of numbers.

Let the covariance function $R_v(\tau)$ and the spectrum $\Phi_v(\omega)$ of the process $\{v(t)\}$ be defined by (2.14) and (2.63). Then evaluate

$$\begin{aligned}
EV_N(\omega)V_N(-\xi) &= \frac{1}{N}\sum_{r=1}^{N}\sum_{s=1}^{N} Ev(r)e^{-i\omega r}v(s)e^{+i\xi s} \\
&= \frac{1}{N}\sum_{r=1}^{N}\sum_{s=1}^{N} e^{i(\xi s-\omega r)}R_v(r-s) = [r-s=\tau] \\
&= \frac{1}{N}\sum_{r=1}^{N} e^{i(\xi-\omega)r} \cdot \sum_{\tau=r-N}^{r-1} R_v(\tau)e^{-i\xi\tau}
\end{aligned}$$

Now

$$\sum_{\tau=r-N}^{r-1} R_v(\tau)e^{-i\xi\tau} = \Phi_v(\xi) - \sum_{\tau=-\infty}^{r-N-1} e^{-i\xi\tau}R_v(\tau) - \sum_{\tau=r}^{\infty} e^{-i\xi\tau}R_v(\tau)$$

and

$$\frac{1}{N}\sum_{r=1}^{N} e^{i(\xi-\omega)r} = \begin{cases} 1, & \text{if } \xi = \omega \\ 0, & \text{if } (\xi-\omega) = \dfrac{k2\pi}{N}, \qquad k = \pm 1, \pm 2, \ldots, \pm(N-1) \end{cases}$$

Consider

$$\begin{aligned}
\left| \frac{1}{N}\sum_{r=1}^{N} e^{i(\xi-\omega)r} \sum_{\tau=-\infty}^{r-N-1} e^{-i\xi\tau}R_v(\tau)\right| &\le \frac{1}{N}\sum_{r=1}^{N}\sum_{\tau=-\infty}^{r-N-1} |R_v(\tau)| \\
&\le [\text{change order of summation}] \\
&\le \frac{1}{N}\sum_{\tau=-\infty}^{-1} |\tau| \cdot |R_v(\tau)| \le \frac{C}{N}
\end{aligned}$$

provided

$$\sum_{-\infty}^{\infty} |\tau \cdot R_v(\tau)| < \infty \tag{6.30}$$

Similarly,

$$\left| \frac{1}{N}\sum_{r=1}^{N} e^{i(\xi-\omega)r} \cdot \sum_{\tau=r}^{\infty} e^{-i\xi\tau}R_v(\tau)\right| \le \frac{1}{N}\sum_{\tau=1}^{\infty} \tau \cdot |R_v(\tau)| \le \frac{C}{N}$$

Combining these expressions, we find that

$$E V_N(\omega) V_N(-\xi) = \begin{cases} \Phi_v(\omega) + \rho_2(N), & \text{if } \xi = \omega \\ \rho_2(N), & \text{if } |\xi - \omega| = \frac{k2\pi}{N}, \quad k = 1, 2, \dots N-1 \end{cases} \tag{6.31}$$

with $|\rho_2(N)| \le 2C/N$. These calculations can be summarized as the following result.

Lemma 6.1. Consider a strictly stable system

$$y(t) = G_0(q)u(t) + v(t) \tag{6.32}$$

with a disturbance $\{v(t)\}$ being a stationary stochastic process with spectrum $\Phi_v(\omega)$ and covariance function $R_v(\tau)$, subject to (6.30). Let $\{u(t)\}$ be independent of $\{v(t)\}$ assume that $|u(t)| \le C$ for all t. Then with $\hat{\hat{G}}_N(e^{i\omega})$ defined by (6.24), we have

$$E\hat{\hat{G}}_N(e^{i\omega}) = G_0(e^{i\omega}) + \frac{\rho_1(N)}{U_N(\omega)} \tag{6.33a}$$

where

$$|\rho_1(N)| \le \frac{C_1}{\sqrt{N}} \tag{6.33b}$$

and

$$E[\hat{\hat{G}}_N(e^{i\omega}) - G_0(e^{i\omega})][\hat{\hat{G}}_N(e^{-i\xi}) - G_0(e^{-i\omega})]$$

$$= \begin{cases} \frac{1}{|U_N(\omega)|^2}[\Phi_v(\omega) + \rho_2(N)], & \text{if } \xi = \omega \\ \frac{\rho_2(N)}{U_N(\omega)U_N(-\xi)}, & \text{if } |\xi - \omega| = \frac{2\pi k}{N}, \quad k = 1, 2, \dots, N-1 \end{cases} \tag{6.34a}$$

where

$$|\rho_2(N)| \le \frac{C_2}{N} \tag{6.34b}$$

Here U_N is defined by (2.37), and we restrict ourselves to frequencies for which $\hat{\hat{G}}_N$ is defined. According to Theorem 2.1 and (6.30), the constants can be taken as

$$C_1 = \left(2\sum_{k=1}^{\infty} |kg_0(k)|\right) \cdot \max|u(t)| \tag{6.35a}$$

$$C_2 = C_1^2 + \sum_{k=-\infty}^{\infty} |\tau R_v(\tau)| \tag{6.35b}$$

If $\{u(t)\}$ is periodic, then according to the Corollary of Theorem 2.1 $\rho_1(N) = 0$ at $\omega = 2\pi k/N$, so we can take $C_1 = 0$.

Remark. Note that the input is regarded as a given sequence. Probabilistic quantities, such as E, "bias," and "variance" refer to the probability space of $\{v(t)\}$. This does not, of course, exclude that the input may be generated as a realization of a stochastic process independent of $\{v(t)\}$.

The properties of the ETFE are closely related to those of periodogram estimates of spectra. See (2.43) and (2.74). We have the following result.

Lemma 6.2. Let $v(t)$ be given by

$$v(t) = H(q)e(t)$$

where $\{e(t)\}$ is a white-noise sequence with variance λ and fourth moment μ^2, and H is a strictly stable filter. Let $V_N(\omega)$ be defined by (6.27), and let $\Phi_v(\omega)$ be the spectrum of $v(t)$. Then

$$E|V_N(\omega)|^2 = \Phi_v(\omega) + \rho_3(N) \tag{6.36}$$

$$E(|V_N(\omega)|^2 - \Phi_v(\omega))(|V_N(\xi)|^2 - \Phi_v(\xi))$$
$$= \begin{cases} [\Phi_v(\omega)]^2 + \rho_4(N), & \text{if } \xi = \omega \quad \omega \neq 0, \pi \\ \rho_4(N), & \text{if } |\xi - \omega| = \dfrac{2\pi k}{N}, \quad k = 1, 2, \ldots N-1 \end{cases} \tag{6.37}$$

where

$$|\rho_3(N)| \leq \frac{C}{N}, \qquad |\rho_4(N)| \leq \frac{C}{N}$$

Proof. Equation (6.36) is a restatement of (6.31). A simple proof of (6.37) is outlined in Problem 6D.2 under somewhat more restrictive conditions. A full proof can be given by direct evaluation of (6.37). See, for example, Brillinger (1981), Theorem 5.2.4, for that. See Problem 6G.5 for ideas on how the bias term can be improved by the use of data tapering. □

These lemmas, together with the results of Section 2.3, tell us the following:

Case 1. *The input is periodic.* When the input is periodic and N is a multiple of the period, we know from Example 2.2 that $|U_N(\omega)|^2$ increases like const $\cdot N$ for some ω and is zero for others [see (2.49)]. The number of frequencies $\omega = 2\pi k/N$ for which $|U_N(\omega)|^2$ is nonzero, and hence for which the ETFE is defined, is fixed and no more than the period length of the signal. We thus find that

- The ETFE $\hat{\hat{G}}_N(e^{i\omega})$ is defined only for a fixed number of frequencies.
- At these frequencies the ETFE is unbiased and its variance decays like $1/N$.

We note that the results (6.16) on frequency analysis by the correlation method are obtained as a special case.

Case 2. *The input is a realization of a stochastic process.* Lemma 6.2 shows that the periodogram $|U_N(\omega)|^2$ is an erratic function of ω, which fluctuates around $\Phi_u(\omega)$ which we assume to be bounded. Lemma 6.1 thus tells us that

- The ETFE is an asymptotically unbiased estimate of the transfer function at increasingly (with N) many frequencies.
- The variance of the ETFE does not decrease as N increases, and it is given as the noise-to-signal ratio at the frequency in question.
- The estimates at different frequencies are asymptotically uncorrelated.

It follows from this discussion that, in the case of a periodic input signal, the ETFE will be of increasingly good quality at the frequencies that are present in the input. However, when the input is not periodic, the variance does not decay with N, but remains equal to the noise-to-signal ratio at the corresponding frequency. This latter property makes the empirical estimate a very crude estimate in most cases in practice.

It is easy to understand the reason why the variance does not decrease with N. We determine as many independent estimates as we have data points. In other words, we have no feature of data and information compression. This in turn is due to the fact that we have only assumed linearity about the true system. Consequently, the system's properties at different frequencies may be totally unrelated. From this it also follows that the only possibility to increase the information per estimated parameter is to assume that the system's behavior at one frequency is related to that at another. In the subsequent section, we shall discuss one approach to how this can be done.

6.4 SPECTRAL ANALYSIS

Spectral analysis for determining transfer functions of linear systems was developed from statistical methods for spectral estimation. Good accounts of this method are given in Chapter 10 in Jenkins and Watts (1968) and in Chapter 6 in Brillinger (1981), and the method is widely discussed in many other textbooks on time series analysis. In this section we shall adopt a slightly non-standard approach to the subject by deriving the standard techniques as a smoothed version of the ETFE.

Smoothing the ETFE

We mentioned at the end of the previous section that the only way to improve on the poor variance properties of the ETFE is to assume that the values of the true transfer function at different frequencies are related. We shall now introduce the rather reasonable prejudice that

$$\text{The true transfer function } G_0(e^{i\omega}) \text{ is a smooth function of } \omega. \tag{6.38}$$

If the frequency distance $2\pi/N$ is small compared to how quickly $G_0(e^{i\omega})$ changes, then

$$\hat{\hat{G}}_N(e^{2\pi ik/N}), \qquad k \text{ integer}, \qquad 2\pi k/N \approx \omega \tag{6.39}$$

are uncorrelated, unbiased estimates of roughly the same constant $G_0(e^{i\omega})$, each with a variance of

$$\frac{\Phi_v(2\pi k/N)}{|U_N(2\pi k/N)|^2}$$

according to Lemma 6.1. Here we neglected terms that tend to zero as N tends to infinity.

If we assume $G_0(e^{i\omega})$ to be constant over the interval

$$\frac{2\pi k_1}{N} = \omega_0 - \Delta\omega < \omega < \omega_0 + \Delta\omega = \frac{2\pi k_2}{N} \tag{6.40}$$

then it is well known that the best (in a minimum variance sense) way to estimate this constant is to form a weighted average of the "measurements" (6.39) for the frequencies (6.40), each measurement weighted according to its inverse variance [compare Problem 6E.3, and Lemma II.2, (II.65), in Appendix II:

$$\hat{G}_N(e^{i\omega_0}) = \frac{\sum_{k=k_1}^{k_2} \alpha_k \hat{\hat{G}}_N(e^{2\pi ik/N})}{\sum_{k=k_1}^{k_2} \alpha_k} \tag{6.41a}$$

$$\alpha_k = \frac{|U_N(2\pi k/N)|^2}{\Phi_v(2\pi k/N)} \tag{6.41b}$$

For large N we could with good approximation work with the integrals that correspond to the (Riemann) sums in (6.41):

$$\hat{G}_N(e^{i\omega_0}) = \frac{\int_{\xi=\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \alpha(\xi)\hat{\hat{G}}_N(e^{i\xi})d\xi}{\int_{\xi=\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \alpha(\xi)d\xi}, \quad \alpha(\xi) = \frac{|U_N(\xi)|^2}{\Phi_v(\xi)} \tag{6.42}$$

If the transfer function G_0 is not constant over the interval (6.40) it is reasonable to use an additional weighting that pays more attention to frequencies close to ω_0:

$$\hat{G}_N(e^{i\omega_0}) = \frac{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)\alpha(\xi)\hat{\hat{G}}_N(e^{i\xi})d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)\alpha(\xi)d\xi} \tag{6.43}$$

Here $W_\gamma(\xi)$ is a function centered around $\xi = 0$ and γ is a "shape parameter," which we shall discuss shortly.

Clearly, (6.42) corresponds to

$$W_\gamma(\xi) = \begin{cases} 1, & |\xi| < \Delta\omega \\ 0, & |\xi| > \Delta\omega \end{cases} \tag{6.44}$$

Now, if the noise spectrum $\Phi_v(\omega)$ is known, the estimate (6.43) can be realized as written. If $\Phi_v(\omega)$ is not known we could argue as follows: Suppose that the noise spectrum does not change very much over frequency intervals corresponding to the "width" of the weighting function $W_\gamma(\xi)$:

$$\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0) \left| \frac{1}{\Phi_v(\xi)} - \frac{1}{\Phi_v(\omega_0)} \right| d\xi = \text{"small"} \tag{6.45}$$

Then $\alpha(\xi)$ in (6.42) can be replaced by $\alpha(\xi) = |U_N(\xi)|^2/\Phi_v(\omega_0)$, which means that the constant $\Phi_v(\omega_0)$ cancels when (6.43) is formed. Under (6.45) the estimate

$$\boxed{\hat{G}_N(e^{i\omega_0}) = \frac{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)|U_N(\xi)|^2 \hat{\hat{G}}_N(e^{i\xi})d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)|U_N(\xi)|^2 d\xi}} \tag{6.46}$$

is thus a good approximation of (6.42) and (6.43).

We may remark that, if (6.45) does not hold, it might be better to include a procedure where $\Phi_v(\omega)$ is estimated and use that estimate in (6.43).

Connection with the Blackman-Tukey Procedure (*)

Consider the denominator of (6.46). It is a weighted average of the periodogram $|U_N(\xi)|^2$. Using the result (2.74), we find that, as $N \to \infty$,

$$\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)|U_N(\xi)|^2 d\xi \to \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)\Phi_u(\xi) d\xi \tag{6.47}$$

where $\Phi_u(\omega)$ is the spectrum of $\{u(t)\}$, as defined by (2.61) to (2.63). If, moreover,

$$\int_{-\pi}^{\pi} W_\gamma(\xi) d\xi = 1$$

and the weighting function $W_\gamma(\xi)$ is concentrated around $\xi = 0$ with a width over which $\Phi_u(\omega)$ does not change much, then the right side of (6.47) is close to $\Phi_u(\omega_0)$. We may thus interpret the left side as an estimate of this quantity:

$$\boxed{\hat{\Phi}_{yu}^N(\omega_0) = \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)|U_N(\xi)|^2 d\xi} \tag{6.48}$$

Similarly, since

$$|U_N(\xi)|^2 \hat{\hat{G}}_N(e^{i\xi}) = |U_N(\xi)|^2 \frac{Y_N(\xi)}{U_N(\xi)} = Y_N(\xi)\overline{U}_N(\xi) \tag{6.49}$$

we have that the numerator of (6.46)

$$\hat{\Phi}_{yu}^N(\omega_0) = \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0) Y_N(\xi) \overline{U}_N(\xi) d\xi \tag{6.50}$$

is an estimate of the cross spectrum between output and input. The transfer function estimate (6.46) is thus the ratio of two spectral estimates:

$$\boxed{\hat{G}_N(e^{i\omega_0}) = \frac{\hat{\Phi}_{yu}^N(\omega_0)}{\hat{\Phi}_u^N(\omega_0)} \tag{6.51}}$$

which makes sense, in view of (2.80). The spectral estimates (6.48) and (6.50) are the standard estimates, suggested in the literature, for spectra and cross spectra as smoothed periodograms. See Blackman and Tukey (1958), Jenkins and Watts (1968), or Brillinger (1981).

An alternative way of expressing these estimates is common. The Fourier coefficients for the periodogram $|U_N(\omega)|^2$ are

$$\hat{R}_u^N(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |U_N(\omega)|^2 e^{i\tau\omega} d\omega = \frac{1}{N} \sum_{t=1}^{N} u(t) u(t - \tau) \tag{6.52}$$

[For this expression to hold exactly, the values $u(s)$ outside the interval $1 \leq s \leq N$ have to be interpreted by periodic continuation; i.e., $u(s) = u(s - N)$ if $s > N$; see Problem 6D.1.]

Similarly, let the Fourier coefficients of the function $2\pi W_\gamma(\xi)$ be

$$w_\gamma(\tau) = \int_{-\pi}^{\pi} W_\gamma(\xi) e^{i\xi\tau} d\xi \tag{6.53}$$

Since the integral (6.48) is a convolution, its Fourier coefficients will be the product of (6.52) and (6.53), so a Fourier expansion of (6.48) gives

$$\hat{\Phi}_u^N(\omega) = \sum_{\tau=-\infty}^{\infty} w_\gamma(\tau) \hat{R}_u^N(\tau) e^{-i\tau\omega} \tag{6.54}$$

The idea is now that the nice, smooth function $W_\gamma(\xi)$ is chosen so that its Fourier coefficients vanish for $|\tau| > \delta_\gamma$, where typically $\delta_\gamma \ll N$. It is consequently sufficient to form (6.52) (using the rightmost expression) for $|\tau| \leq \delta_\gamma$, and then take

$$\hat{\Phi}_u^N(\omega) = \sum_{\tau=-\delta_\gamma}^{\delta_\gamma} w_\gamma(\tau) \hat{R}_u^N(\tau) e^{-i\tau\omega} \tag{6.55}$$

This is perhaps the most convenient way of forming the spectral estimate. The expressions for $\hat{\Phi}_{yu}^N(\omega)$ are of course analogous.

Weighting Function $W_\gamma(\xi)$: The Frequency Window

Let us now discuss the weighting function $W_\gamma(\xi)$. In spectral analysis, it is often called the *frequency window*. [Similarly, $w_\gamma(\tau)$ is called the *lag window*.] If this window is "wide," then many different frequencies will be weighted together in (6.40). This should lead to a small variance of $\hat{G}_N(e^{i\omega_0})$. At the same time, a wide window will involve frequency estimates farther away from ω_0, with expected values that may differ considerably from $G_0(e^{i\omega_0})$. This will cause large bias. The width of the window will thus control the trade-off between bias and variance. To make this trade-off a bit more formal, we shall use the scalar γ to describe the width, so a large value of γ corresponds to a narrow window.

We shall characterize the window by the following numbers

$$\int_{-\pi}^{\pi} W_\gamma(\xi)d\xi = 1, \quad \int_{-\pi}^{\pi} \xi W_\gamma(\xi)d\xi = 0, \quad \int_{-\pi}^{\pi} \xi^2 W_\gamma(\xi)d\xi = M(\gamma) \tag{6.56a}$$

$$\int_{-\pi}^{\pi} |\xi|^3 W_\gamma(\xi)d\xi = C_3(\gamma), \quad \int_{-\pi}^{\pi} W_\gamma^2(\xi)d\xi = \frac{1}{2\pi}\overline{W}(\gamma) \tag{6.56b}$$

As γ increases (and the frequency window gets more narrow), the number $M(\gamma)$ decreases, while $\overline{W}(\gamma)$ increases.

Some typical windows are given in Table 6.1. [See, also, Table 3.3.1 in Brillinger (1981)for a more complete collection of windows.] Notice that the scaling quantity

TABLE 6.1 Some Windows for Spectral Analysis

	$2\pi W_\gamma(\omega)$	$w_\gamma(\tau), \quad 0 \le \vert\tau\vert \le \gamma$
Bartlett	$\frac{1}{\gamma}\left(\frac{\sin \gamma\omega/2}{\sin \omega/2}\right)^2$	$1 - \frac{\vert\tau\vert}{\gamma}$
Parzen	$\frac{4(2+\cos\omega)}{\gamma^3}\left(\frac{\sin \gamma\omega/4}{\sin \omega/2}\right)^4$	$\begin{cases} 1 - \frac{6\tau^2}{\gamma^2}\left(1 - \frac{\vert\tau\vert}{\gamma}\right), & 0 \le \vert\tau\vert \le \frac{\gamma}{2} \\ 2\left(1 - \frac{\vert\tau\vert}{\gamma}\right)^3, & \frac{\gamma}{2} \le \vert\tau\vert \le \gamma \end{cases}$
Hamming	$\frac{1}{2}D_\gamma(\omega) + \frac{1}{4}D_\gamma(\omega - \pi/\gamma)$ $+\frac{1}{4}D_\gamma(\omega + \pi/\gamma)$, where $D_\gamma(\omega) = \frac{\sin(\gamma + \frac{1}{2})\omega}{\sin \omega/2}$	$\frac{1}{2}\left(1 + \cos\frac{\pi\tau}{\gamma}\right)$

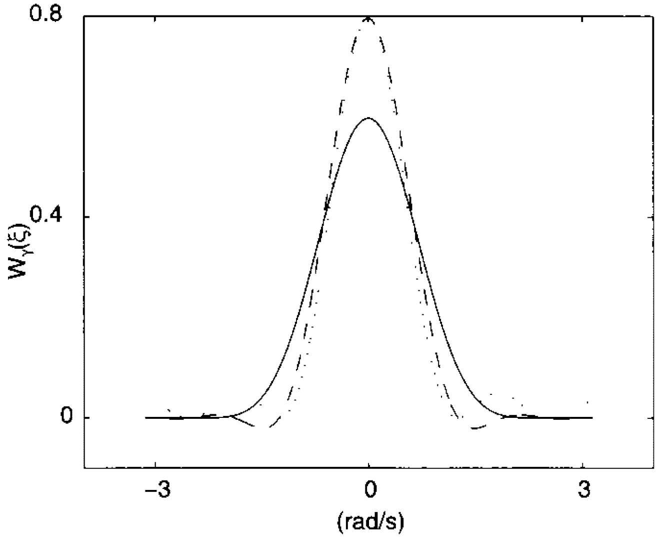

Figure 6.1 Some common frequency windows. Solid line: Parzen; dashed line: Hamming; dotted line: Bartlett, $\gamma = 5$.

γ has been chosen so that $\delta_\gamma = \gamma$ in (6.55). The frequency windows are shown graphically in Figure 6.1. For these windows, we have

$$\begin{aligned} &\text{Bartlett:} \quad M(\gamma) = \frac{2.78}{\gamma} \quad \overline{W}(\gamma) \approx 0.67\gamma \\ &\text{Parzen:} \quad M(\gamma) = \frac{12}{\gamma^2} \quad \overline{W}(\gamma) \approx 0.54\gamma \\ &\text{Hamming:} \quad M(\gamma) = \frac{\pi^2}{2\gamma^2} \quad \overline{W}(\gamma) \approx 0.75\gamma \end{aligned} \tag{6.57}$$

The expressions are asymptotic for large γ but are good approximations for $\gamma \gtrsim 5$. See also Problem 6T.1 for a further discussion of how to scale windows.

Asymptotic Properties of the Smoothed Estimate

The estimate (6.46) has been studied in several treatments of spectral analysis. Results that are asymptotic in both N and γ can be derived as follows (see Appendix 6A). Consider the estimate (6.46), and suppose that the true system obeys the assumptions of Lemma 6.1. We then have

Bias

$$\begin{aligned} E\hat{G}_N(e^{i\omega}) - G_0(e^{i\omega}) = M(\gamma) \cdot \left[\frac{1}{2}G_0''(e^{i\omega}) + G_0'(e^{i\omega})\frac{\Phi_u'(\omega)}{\Phi_u(\omega)}\right] \\ + \underset{\gamma\to\infty}{\mathbf{O}(C_3(\gamma))} + \underset{N\to\infty}{\mathbf{O}(1/\sqrt{N})} \end{aligned} \tag{6.58}$$

Here $\mathbf{O}(x)$ is ordo x. See notational conventions at the beginning of the book. Prime and double prime denote differentiation with respect to ω, one and twice respectively.

Variance

$$E|\hat{G}_N(e^{i\omega}) - E\hat{G}_N(e^{i\omega})|^2 = \frac{1}{N} \cdot \overline{W}(\gamma) \cdot \frac{\Phi_v(\omega)}{\Phi_u(\omega)} + \underset{\substack{\gamma\to\infty \\ N\to\infty,\gamma/N\to 0}}{\mathrm{o}(\overline{W}(\gamma)/N)} \tag{6.59}$$

We repeat that expectation here is with respect to the noise sequence $\{v(t)\}$ and that the input is supposed to be a deterministic quasi-stationary signal.

Let us use the asymptotic expressions to evaluate the mean-square error (MSE):

$$E|\hat{G}_N(e^{i\omega}) - G_0(e^{i\omega})|^2 \sim M^2(\gamma)|R(\omega)|^2 + \frac{1}{N}\overline{W}(\gamma)\frac{\Phi_v(\omega)}{\Phi_u(\omega)} \tag{6.60}$$

Here

$$R(\omega) = \frac{1}{2}G_0''(e^{i\omega}) + G_0'(e^{i\omega})\frac{\Phi_u'(\omega)}{\Phi_u(\omega)} \tag{6.61}$$

Some additional results can also be shown (see Brillinger, 1981, Chapter 6, and Problems 6D.3 and 6D.4).

- The estimates $\mathrm{Re}\,\hat{G}_N(e^{i\omega})$ and $\mathrm{Im}\,\hat{G}_N(e^{i\omega})$ are asymptotically uncorrelated and each have a variance equal to half that in (6.59). (6.62)
- The estimates $\hat{G}_N(e^{i\omega})$ at different frequencies are asymptotically uncorrelated. (6.63)
- The estimates $\mathrm{Re}\,\hat{G}_N(e^{i\omega_k})$, $\mathrm{Im}\,\hat{G}_N(e^{i\omega_k})$, $k = 1, 2, \ldots, M$, at an arbitrary collection of frequencies are asymptotically jointly normal distributed with means and covariances given by (6.58) to (6.63). (6.64)
- For a translation to properties of $|\hat{G}_N(e^{i\omega})|$, $\arg \hat{G}_N(e^{i\omega})$, see Problem 9G.1.

From (6.60) we see that a desired property of the window is that both M and $\overline{W}$ should be small. We may also calculate the value of the width parameter γ that minimizes the MSE. Suppose that both γ and N tend to infinity and γ/N tends to zero, so that the asymptotic expressions are applicable. Suppose also that (6.57) holds with $M(\gamma) = M/\gamma^2$ and $\overline{W}(\gamma) = \gamma \cdot \overline{W}$ Then (6.60) gives

$$\gamma_{\text{opt}} = \left(\frac{4M^2|R(\omega)|^2\Phi_u(\omega)}{\overline{W}\Phi_v(\omega)}\right)^{1/5} \cdot N^{1/5} \tag{6.65}$$

This value can of course not be realized by the user, since the constant contains several unknown quantities. We note, however, that in any case it increases like $N^{1/5}$, and it should, in principle, be allowed to be frequency dependent. The frequency window consequently should get more narrow when more data are available, which is a very natural result.

The optimal choice of γ leads to a mean-square error that decays like

$$\text{MSE} \sim C \cdot N^{-4/5} \tag{6.66}$$

In practical use the trade-off (6.65) and (6.66) cannot be reached in formal terms. Instead, a typical procedure would be to start by taking $\gamma = N/20$ (see Table 6.1) and then compute and plot the corresponding estimates $\hat{G}_N(e^{i\omega})$ for various values of γ. As γ is increased, more and more details of the estimate will appear. These will be due to decreased bias (true resonance peaks appearing more clearly and the like), as well as to increased variance (spurious, random peaks). The procedure will be stopped when the user feels that the emerging details are predominately spurious.

Actually, as we noted, (6.65) points to the fact that the optimal window size should be frequency dependent. This can easily be implemented in (6.46), but not in (6.55), and most procedures do not utilize this feature.

Example 6.1 A Simulated System

The system

$$y(t) - 1.5y(t-1) + 0.7y(t-2) = u(t-1) + 0.5u(t-2) + e(t) \tag{6.67}$$

where $\{e(t)\}$ is white noise with variance 1 was simulated with the input as a PRBS signal (see Section 13.3) over 1000 samples. Part of the resulting data record is shown in Figure 6.2. The corresponding ETFE is shown in Figure 6.3a. An estimate $\hat{G}_N(e^{i\omega})$ was formed using (6.46), with $W_\gamma(\xi)$ being a Parzen window with various values of γ. Figure 6.3bcd shows the results for $\gamma = 10, 50$, and 200. Here $\gamma = 50$ appears to be a reasonable choice of window size. □

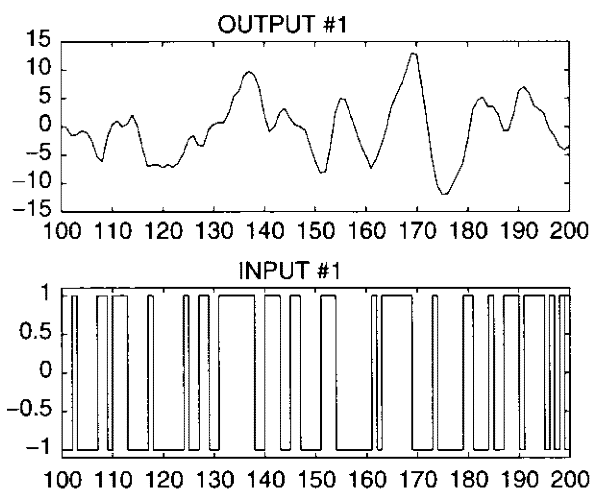

Figure 6.2 The simulated data from (6.67).

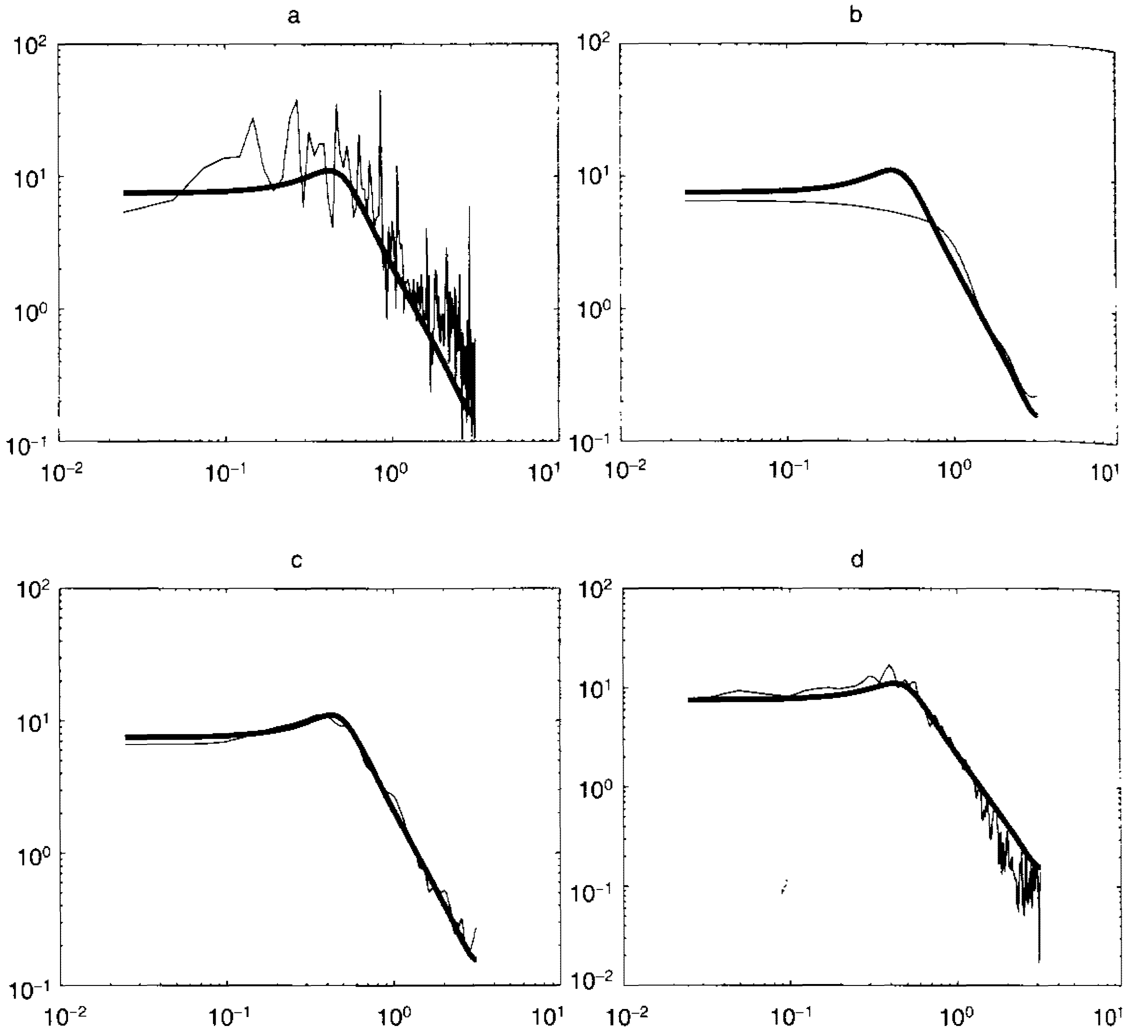

Figure 6.3 Amplitude plots of the estimate $\hat{G}_N(e^{i\omega})$. a: ETFE. b: $\gamma = 10$. c: $\gamma = 50$. d: $\gamma = 200$. Thick lines: true system; thin lines: estimate.

Another Way of Smoothing the ETFE (∗)

The guiding idea behind the estimate (6.46) is that the ETFEs at neighboring frequencies are asymptotically uncorrelated, and that hence the variance could be reduced by averaging over these. The ETFEs obtained over different data sets will also provide uncorrelated estimates, and another approach would be to form averages over these. Thus, split the data set Z^N into M batches, each containing R data ($N = R \cdot M$). Then form the ETFE corresponding to the kth batch:

$$\hat{G}_R^{(k)}(e^{i\omega}), \qquad k = 1, 2, \ldots, M \tag{6.68}$$

The estimate can then be formed as a direct average

$$\hat{G}_N(e^{i\omega}) = \frac{1}{M}\sum_{k=1}^{M}\hat{G}_R^{(k)}(e^{i\omega}) \tag{6.69}$$

or one that is weighted according to the inverse variances:

$$\hat{G}_N(e^{i\omega}) = \frac{\sum_{k=1}^{M}\beta_R^{(k)}(\omega)\cdot\hat{G}_R^{(k)}(e^{i\omega})}{\sum_{k=1}^{M}\beta_R^{(k)}(\omega)} \tag{6.70}$$

with

$$\beta_R^{(k)}(\omega) = |U_R^{(k)}(\omega)|^2 \tag{6.71}$$

being the periodogram of the kth subbatch. The inverse variance of $\hat{G}_R^{(k)}(e^{i\omega})$ is $\beta_R^{(k)}(\omega)/\Phi_v(\omega)$, but the factor $\Phi_v(\omega)$ cancels when (6.70) is formed.

An advantage with the estimate (6.70) is that the fast Fourier transform (FFT) can be efficiently used when Z^N can be decomposed so that R is a power of 2. Compare Problem 6G.4. The method is known as Welch's method. Welch (1967).

6.5 ESTIMATING THE DISTURBANCE SPECTRUM (*)

Estimating Spectra

So far we have described how to estimate G_0 in a relationship (6.1):

$$y(t) = G_0(q)u(t) + v(t) \tag{6.72}$$

We shall now turn to the problem of estimating the spectrum of $\{v(t)\}$, $\Phi_v(\omega)$. Had the disturbances $v(t)$ been available for direct measurement, we could have used (6.48):

$$\hat{\Phi}_v^N(\omega) = \int_{-\pi}^{\pi} W_\gamma(\xi - \omega)|V_N(\xi)|^2 d\xi \tag{6.73}$$

Here $W_\gamma(\cdot)$ is a frequency window of the kind described earlier.

It is entirely analogous to the analysis of the previous section to calculate the properties of (6.73). We have:

Bias:

$$E\hat{\Phi}_v^N(\omega) - \Phi_v(\omega) = \tfrac{1}{2}M(\gamma)\cdot\Phi_v''(\omega) + \underset{\gamma\to\infty}{O}(C_1(\gamma)) + \underset{N\to\infty}{O}(1/\sqrt{N}) \tag{6.74}$$

Variance:

$$\operatorname{Var}\hat{\Phi}_v^N(\omega) = \frac{\overline{W}(\gamma)}{N} \cdot \Phi_v^2(\omega) + \underset{N\to\infty}{\mathbf{o}(1/N)}, \qquad \omega \neq 0, \pm\pi \tag{6.75}$$

Moreover, estimates at different frequencies are asymptotically uncorrelated.

The Residual Spectrum

Now the $v(t)$ in (6.72) are not directly measurable. However, given an estimate $\hat{G}_N$ of the transfer function, we may replace v in the preceding expression by

$$\hat{v}(t) = y(t) - \hat{G}_N(q)u(t) \tag{6.76}$$

which gives the estimate

$$\boxed{\hat{\Phi}_v^N(\omega) = \int_{-\pi}^{\pi} W_\gamma(\xi - \omega)|Y_N(\xi) - \hat{G}_N(e^{i\xi})U_N(\xi)|^2 d\xi \qquad (6.77)}$$

If $\hat{G}_N(e^{i\xi})$ is formed using (6.46) with the same window $W_\gamma(\cdot)$, this expression can be rearranged as follows [using (6.48) to (6.51)]:

$$\begin{aligned}
&\int_{-\pi}^{\pi} W_\gamma(\xi - \omega)|Y_N(\xi)|^2 d\xi + \int_{-\pi}^{\pi} W_\gamma(\xi - \omega)|U_N(\xi)|^2 |\hat{G}_N(e^{i\xi})|^2 d\xi \\
&\quad - 2\operatorname{Re}\int_{-\pi}^{\pi} W_\gamma(\xi - \omega)\hat{G}_N(e^{i\xi})U_N(\xi)\overline{Y_N(\xi)}d\xi \\
&\approx \int_{-\pi}^{\pi} W_\gamma(\xi - \omega)|Y_N(\xi)|^2 d\xi + |\hat{G}_N(e^{i\omega})|^2 \int_{-\pi}^{\pi} W_\gamma(\xi - \omega)|U_N(\xi)|^2 d\xi \\
&\quad - 2\operatorname{Re}\hat{G}_N(e^{i\omega}) \int_{-\pi}^{\pi} W_\gamma(\xi - \omega)U_N(\xi)\overline{Y_N(\xi)}d\xi \\
&= \hat{\Phi}_y^N(\omega) + \frac{|\hat{\Phi}_{yu}^N(\omega)|^2}{(\hat{\Phi}_u^N(\omega))^2} \cdot \hat{\Phi}_u^N(\omega) - 2\operatorname{Re}\frac{\hat{\Phi}_{yu}^N(\omega)}{\hat{\Phi}_u^N(\omega)} \cdot \overline{\hat{\Phi}_{yu}^N(\omega)}
\end{aligned}$$

Here the approximate equality follows from replacing the smooth function $\hat{G}_N(e^{i\xi})$ over the small interval around $\xi = \omega$ with its value at ω. Hence we have

$$\boxed{\hat{\Phi}_v^N(\omega) = \hat{\Phi}_y^N(\omega) - \frac{|\hat{\Phi}_{yu}^N(\omega)|^2}{\hat{\Phi}_u^N(\omega)} \qquad (6.78)}$$

Asymptotically, as $N \to \infty$ and $\gamma \to \infty$, so that $\hat{G}_N(e^{i\omega}) \to G_0(e^{i\omega})$ according to (6.60), we find that the estimate (6.77) tends to (6.73). The asymptotic properties (6.74) and (6.75) will also hold for (6.77) and (6.78). In addition to the properties already listed, we may note that the estimates $\hat{\Phi}_v^N(\omega)$ are asymptotically uncorrelated with $\hat{G}_N(e^{i\omega})$. Moreover $\hat{\Phi}_v^N(\omega_k)$, $\hat{G}_N(e^{i\omega_k})$, $k = 1, 2, \ldots, r$, are asymptotically jointly normal random variables with mean and covariances given by (6.58) to (6.64) and (6.74) to (6.75). A detailed account of the asymptotic theory is given in Chapter 6 of Brillinger (1981).

Coherency Spectrum

Denote

$$\hat{\kappa}_{yu}^N(\omega) = \sqrt{\frac{|\hat{\Phi}_{yu}^N(\omega)|^2}{\hat{\Phi}_y^N(\omega)\hat{\Phi}_u^N(\omega)}} \tag{6.79}$$

Then

$$\hat{\Phi}_v^N(\omega) = \hat{\Phi}_y^N(\omega)[1 - (\hat{\kappa}_{yu}^N(\omega))^2] \tag{6.80}$$

The function $\kappa_{yu}(\omega)$ is called the *coherency spectrum* (between y and u) and can be viewed as the (frequency dependent) correlation coefficient between the input and output sequences. If this coefficient is 1 at a certain frequency, then there is perfect correlation between input and output at that frequency. There is consequently no noise interfering at that frequency, which is confirmed by (6.80).

6.6 SUMMARY

In this chapter we have shown how simple techniques of transient and frequency response can give valuable insight into the properties of linear systems. We have introduced the empirical transfer-function estimate (ETFE)

$$\hat{\hat{G}}_N(e^{i\omega}) = \frac{Y_N(\omega)}{U_N(\omega)} \tag{6.81}$$

based on data over the interval $1 \le t \le N$. Here

$$Y_N(\omega) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} y(t)e^{-it\omega}, \qquad U_N(\omega) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} u(t)e^{-it\omega},$$

The ETFE has the property (see Lemma 6.1) that it is asymptotically unbiased, but has a variance of $\Phi_v(\omega)/|U_N(\omega)|^2$.

We showed how smoothing the ETFE leads to the spectral analysis estimate

$$\hat{G}_N(e^{i\omega}) = \frac{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega)|U_N(\xi)|^2\hat{\hat{G}}_N(e^{i\xi})d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega)|U_N(\xi)|^2 d\xi} \tag{6.82}$$

A corresponding estimate of the noise spectrum is

$$\hat{\Phi}_v^N(\omega) = \int_{-\pi}^{\pi} W_\gamma(\xi - \omega)|Y_N(\xi) - \hat{G}_N(e^{i\xi})U_N(\xi)|^2 d\xi \tag{6.83}$$

The properties of these estimates were summarized in (6.58) to (6.64) and (6.74) to (6.75).

These properties depend on the parameter γ, which describes the width of the associated frequency window W_γ. A narrow such window (large γ) gives small bias but high variance for the estimate, while the converse is true for wide windows.

6.7 BIBLIOGRAPHY

Section 6.1: Wellstead (1981)gives a general survey of nonparametric methods for system identification. A survey of transient response methods is given in Rake (1980). Several ways of determining numerical characteristics from step responses are discussed in Schwarze (1964). Correlation techniques are surveyed in Godfrey (1980).

Section 6.2: Frequency analysis is a classical identification method that is described in many textbooks on control. For detailed treatments, see Rake (1980)which also contains several interesting examples.

Section 6.3: General Fourier techniques are also discussed in Rake (1980). The term "empirical transfer function estimate" for $\hat{\hat{G}}$ is introduced in this chapter, but the estimate as such is well known.

Sections 6.4 and 6.5: Spectral analysis is a standard subject in textbooks on time series. See, for example, Grenander and Rosenblatt (1957)(Chapters 4 to 6), Anderson (1971)(Chapter 9), and Hannan (1970)(Chapter V). Among books devoted entirely to spectral analysis, we could point to Kay (1988), Marple (1987), and Stoica and Moses (1997). These texts deal primarily with estimation of power (auto-) spectra. Among specific treatments of frequency-domain techniques, including estimation of transfer functions, we note Brillinger (1981)for a thorough analytic study, Jenkins and Watts (1968)for a more leisurely discussion of both statistical properties and application aspects, and Bendat and Piersol (1980)for an application-oriented approach. Another extensive treatment is Priestley (1981). Overviews of different frequency-domain techniques are given in Brillinger and Krishnaiah (1983), and a control-oriented survey is given by Godfrey (1980). The treatment given here is based on Ljung (1985a). The first reference to the idea of smoothing the periodogram to obtain a better spectral estimate appears to be Daniell (1946). A comparative discussion of windows for spectral analysis is given in Gecklini and Yavuz (1978)and Papoulis (1973).

In addition to direct frequency-domain methods for estimating spectra, many efficient methods are based on parametric fit, such as those to be discussed in the following chapter. So called maximum entropy methods (MEM) have found wide use in signal-processing applications. See Burg (1967)for the first idea and Marple (1987)for a comparative survey of different approaches.

6.8 PROBLEMS

6G.1 Consider the system

$$y(t) = G_0(q)u(t) + v(t)$$

controlled by the regulator

$$u(t) = -F(q)y(t) + r(t)$$

where $r(t)$ is an external reference signal. r and v are independent and their spectra are $\Phi_r(\omega)$ and $\Phi_v(\omega)$, respectively. The usual spectral analysis estimate of G_0 is given in (6.51) as well as (6.46). Show that as N and γ tend to infinity then $\hat{G}_N(e^{i\omega})$ will converge to

$$G_*(e^{i\omega}) = \frac{G_0(e^{i\omega})\Phi_r(\omega) - F(e^{-i\omega})\Phi_v(\omega)}{\Phi_r(\omega) + |F(e^{i\omega})|^2\Phi_v(\omega)}$$

What happens in the two special cases $\Phi_r \equiv 0$ and $F \equiv 0$, respectively? *Hint:* Compare Problem 2E.5.

6G.2 *Prefiltering.* Prefilter inputs and outputs:

$$u_F(t) = L_u(q)u(t), \qquad y_F(t) = L_y(q)y(t)$$

If (6.32) holds, then the filtered variables obey

$$y_F(t) = G_0^F(q)u_F(t) + v_F(t)$$

$$G_0^F(q) = \frac{L_y(q)}{L_u(q)}G_0(q), \qquad v_F(t) = L_y(q)v(t)$$

Apply spectral analysis to u_F, y_F, thus forming an estimate $\hat{G}_N^F(e^{i\omega})$. The estimate of the original transfer function then is

$$\hat{G}_N(e^{i\omega}) = \frac{L_u(e^{i\omega})}{L_y(e^{i\omega})}\hat{G}_N^F(e^{i\omega})$$

Determine the asymptotic properties of $\hat{G}_N(e^{i\omega})$ and discuss how L_u and L_y can be chosen for smallest MSE (cf. Ljung, 1985a).

6G.3 In Figure 6.3 the amplitude of the ETFE appears to be systematically larger than the true amplitude, despite the fact that the ETFE is unbiased according to Lemma 6.1. However, $\hat{\hat{G}}$ being an unbiased estimate of G_0 does not imply that $|\hat{\hat{G}}|$ is an unbiased estimate of $|G_0|$. In fact, prove that

$$E|\hat{\hat{G}}_N(e^{i\omega})|^2 = |G_0(e^{i\omega})|^2 + \frac{\Phi_v(\omega)}{|U_N(\omega)|^2}$$

asymptotically for large N, under the assumptions of Lemma 6.1.

6G.4 The Cooley-Tukey spectral estimate for a process $\{v(t)\}$ is defined as

$$\hat{\Phi}_v^N(\omega) = \frac{1}{M}\sum_{k=1}^{M}|V_R^{(k)}(\omega)|^2$$

where $|V_R^{(k)}(\omega)|^2$ is the periodogram estimate of the kth subbatch of data:

$$V_R^{(k)}(\omega) = \frac{1}{\sqrt{R}}\sum_{\ell=1}^{R} v((k-1)R+\ell)e^{-i\omega\ell})$$

See Cooley and Tukey (1965)or Hannan (1970), Chapter V. The cross-spectral estimate is defined analogously. This estimate has the advantage that the FFT (fast Fourier transform) can be applied (most efficiently if R is a power of 2). Show that the estimate (6.70) is the ratio of two appropriate Cooley-Tukey spectral estimates.

6G.5 "*Tapers*" or "*faders*." The bias term $\rho_3(N)$ in (6.36) can be reduced if tapering is introduced: Let $V_N^{(T)}(\omega)$ be defined by

$$V_N^{(T)}(\omega) = \sum_{t=1}^{N} h_t v(t)e^{-it\omega}$$

where $\{h_t\}_1^N$ is a sequence of numbers (*a tapering function*) such that

$$\sum_{t=1}^{N} h_t^2 = 1$$

Let

$$H_N(\omega) = \sum_{t=1}^{N} h_t e^{-it\omega}$$

Show that, under the conditions of Lemma 6.2,

$$E|V_N^{(T)}(\omega)|^2 = \int_{-\pi}^{\pi}|H_N(\omega-\xi)|^2\Phi_v(\xi)d\xi$$

Show that our standard periodogram estimate, which uses $h_t \equiv 1/\sqrt{N}$, gives

$$|H_N(\omega)|^2 = \frac{1}{N}\left[\frac{\sin N\omega/2}{\sin \omega/2}\right]^2$$

Other tapering coefficients (or "faders" or "convergence factors") may give functions $|H_N(\omega)|^2$ that are more "δ-function like" than in the preceding equation (see, e.g., Table 6.1). The tapered periodogram can of course also be used to obtain smoothed spectra. They will typically lead to decreased bias and (slightly) increased variance (Brillinger, 1981, Theorem 5.2.3 and Section 5.8).

6E.1 Determine an estimate for $G_0(e^{i\omega})$ based on the impulse-response estimates (6.4). Show that this estimate coincides with the ETFE (6.24).

6E.2 Consider the system

$$y(t) = G_0(q)u(t) + v(t)$$

This system is controlled by output proportional feedback

$$u(t) = -Ky(t)$$

Let the ETFE $\hat{\hat{G}}_N(e^{i\omega})$ be computed in the straightforward way (6.24). What will this estimate be? Compare with Lemma 6.1.

6E.3 Let w_k, $k = 1, \dots, M$, be independent random variables, all with mean values 1 and variances $E(w_k - 1)^2 = \lambda_k$. Consider

$$w = \sum_{k=1}^{M} \alpha_k w_k$$

Determine α_k, $k = 1, \dots, M$, so that

(a) $Ew = 1$.

(b) $E(w - 1)^2$ is minimized.

6T.1 A general approach to treat the relationships between the scaling parameter γ and the lag and frequency windows $w_\gamma(\tau)$, and $W_\gamma(\omega)$ [see (6.53)] can be given as follows. Choose an even function $w(x)$ such that $w(0) = 1$ and $w(x) = 0$, $|x| > 1$, with Fourier transform

$$W(\lambda) = \int_{-\infty}^{\infty} w(x)e^{-ix\lambda}dx$$

Let

$$\overline{W} = 2\pi \int_{-\infty}^{\infty} W^2(\lambda)d\lambda, \quad = \int_{-\infty}^{\infty} \lambda^2 W(\lambda)d\lambda$$

Then define the lag window

$$w_\gamma(\tau) = w(\tau/\gamma)$$

This gives a frequency window

$$W_\gamma(\omega) = \sum_{\tau=-\gamma}^{\gamma} w_\gamma(\tau)e^{-i\tau\omega}$$

Show that, for large γ,

(a) $W_\gamma(\omega) \approx \gamma \cdot W(\gamma\omega)$

(b) $M(\gamma) \approx M/\gamma^2$

(c) $\overline{W}(\gamma) \approx \overline{W} \cdot \gamma$

where $M(\gamma)$ and $\overline{W}(\gamma)$ are defined by (6.56). Moreover, compute and compare $W_\gamma(\omega)$ and $\gamma \cdot W(\gamma \cdot \omega)$ for $w(x) = 1 - |x|$, $|x| \le 1$ (the Bartlett window). [Compare (6.57). See also Hannan (1970), Section V.4.]

6T.2 Let $\{v(t)\}$ be a stationary stochastic process with zero mean value and covariance function $R_v(\tau)$, such that

$$\sum_{-\infty}^{\infty} |\tau R_v(\tau)| < \infty$$

Let

$$S_N = \frac{1}{N} \sum_{t=1}^{N} \alpha_t v(t), \qquad |\alpha_t| \leq C_1$$

Show that

$$E|S_N|^2 \leq \frac{C_2}{N}$$

for some constant C_2.

6D.1 Prove (6.52) with the proper interpretation of values outside the interval $1 \leq t \leq N$.

6D.2 Prove a relaxed version of Lemma 6.2 with $|\rho_k(N)| \leq C/\sqrt{N}$ by a direct application of Theorem 2.1 and the properties of periodograms of white noise.

6D.3 Prove (6.63) by using expressions analogous to (6A.3) and (6A.4).

6D.4 Prove (6.62) by using (6.63) and

$$\text{Re}\hat{G}(e^{i\omega}) = \frac{\hat{G}(e^{i\omega}) + \overline{\hat{G}(e^{-i\omega})}}{2}$$

$$\text{Im}\hat{G}(e^{i\omega}) = \frac{\hat{G}(e^{i\omega}) - \overline{\hat{G}(e^{-i\omega})}}{2i}$$

APPENDIX 6A: DERIVATION OF THE ASYMPTOTIC PROPERTIES OF THE SPECTRAL ANALYSIS ESTIMATE

Consider the transfer function estimate (6.46). In this appendix we shall derive the asymptotic properties (6.58) and (6.59). In order not to get too technical, some elements of the derivation will be kept heuristic. Recall that $\{u(t)\}$ here is regarded as a deterministic quasi-stationary sequence, and, hence, such that (6.47) holds.

We then have

$$E\hat{G}_N(e^{i\omega_0}) = \frac{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)|U_N(\xi)|^2 \left[G_0(e^{i\xi}) + \rho_1(N)/U_N(\xi)\right] d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)|U_N(\xi)|^2 d\xi}$$

$$\approx \frac{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)\Phi_u(\xi)G_0(e^{i\xi})d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)\Phi_u(\xi)d\xi} \tag{6A.1}$$

using first Lemma 6.1 and then (6.47), neglecting the decaying term $\rho_1(N)$.

Now, expanding in Taylor series (prime denoting differentiation with respect to ω),

$$G_0(e^{i\xi}) \approx G_0(e^{i\omega_0}) + (\xi - \omega_0)G_0'(e^{i\omega_0}) + \frac{1}{2}(\xi - \omega_0)^2 G_0''(e^{i\omega_0})$$

$$\Phi_0(\xi) \approx \Phi_u(\omega_0) + (\xi - \omega_0)\Phi_u'(\omega_0) + \frac{1}{2}(\xi - \omega_0)^2 \Phi_u''(\omega_0)$$

and noting that, according to (6.56),

$$\int_{-\pi}^{\pi} (\xi - \omega_0) W_\gamma(\xi - \omega_0) d\xi = 0$$

$$\int_{-\pi}^{\pi} (\xi - \omega_0)^2 W_\gamma(\xi - \omega_0) d\xi = M(\gamma)$$

we find that the numerator of (6A.1) is approximately

$$G_0(e^{i\omega_0})\Phi_u(\omega_0) + M(\gamma)\left[\frac{1}{2}\Phi_u'' G_0 + \frac{1}{2}G_0''\Phi_u + \Phi_u' G_0'\right]$$

and the denominator

$$\Phi_u(\omega_0) + \frac{1}{2}M(\gamma)[\Phi_u'']$$

where we neglect effects that are of order $C_3(\gamma)$ [an order of magnitude smaller than $M(\gamma)$ as $\gamma \to \infty$; see (6.56)]. Equation (6A.1) thus gives

$$E\hat{G}_N(e^{i\omega_0}) \approx G_0(e^{i\omega_0}) + M(\gamma)\left[\frac{1}{2}G_0''(e^{i\omega_0}) + G_0'(e^{i\omega_0})\frac{\Phi_u'(\omega_0)}{\Phi_u(\omega_0)}\right]$$

which is (6.58).

For the variance expression, we first have from (6.28) and (6.46) that

$$\hat{G}_N(e^{i\omega_0}) - E\hat{G}_N(e^{i\omega_0}) \approx \frac{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)|U_N(\xi)|^2[V_N(\xi)/U_N(\xi)]d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)|U_N(\xi)|^2 d\xi} \quad (6A.2)$$

Let us study the numerator of this expression. We write this, approximately, as a Riemann sum [see (6.41); we could have kept it discrete all along]:

$$\int_{-\pi}^{\pi} W_\gamma(\xi - \omega_0)\overline{U}_N(\xi)V_N(\xi)d\xi \approx A_N$$

$$\triangleq \frac{2\pi}{N}\sum_{k=-(N/2)+1}^{N/2} W_\gamma\left(\frac{2\pi k}{N} - \omega_0\right)\overline{U}_N\left(\frac{2\pi k}{N}\right)V_N\left(\frac{2\pi k}{N}\right) \quad (6A.3)$$

We have, with summation from $1 - N/2$ to $N/2$,

$$
\begin{aligned}
EA_N\overline{A}_N &= \frac{4\pi^2}{N^2}\sum_k\sum_\ell W_\gamma\left(\frac{2\pi k}{N} - \omega_0\right) W_\gamma\left(\frac{2\pi \ell}{N} - \omega_0\right)\overline{U}_N\left(\frac{2\pi k}{N}\right) \\
&\quad \times \overline{U}_N\left(-\frac{2\pi \ell}{N}\right) EV_N\left(\frac{2\pi k}{N}\right) V_N\left(-\frac{2\pi \ell}{N}\right) \qquad \text{(6A.4)} \\
&\approx \frac{4\pi^2}{N^2}\sum\left[W_\gamma\left(\frac{2\pi k}{N} - \omega_0\right)\right]^2 \cdot \left|U_N\left(\frac{2\pi k}{N}\right)\right|^2 \Phi_v\left(\frac{2\pi k}{N}\right)
\end{aligned}
$$

using (6.31) and neglecting the term $\rho_2(N)$.

Returning to the integral form, we thus have, using (6.47)

$$
EA_N\overline{A}_N \approx \frac{2\pi}{N}\int_{-\pi}^{\pi} W_\gamma^2(\xi - \omega_0)\Phi_u(\xi)\Phi_v(\xi)d\xi \approx \frac{1}{N}\overline{W}(\gamma)\Phi_u(\omega_0)\Phi_v(\omega_0)
$$

using (6.56) and the fact that, for large γ, $W_\gamma(\xi)$ is concentrated around $\xi = 0$.

The denominator of (6A.2) approximately equals $\Phi_u(\omega_0)$ for the same reason.

We thus find that

$$
\text{Var}[\hat{G}_N(e^{i\omega_0})] \approx \frac{(1/N)\overline{W}(\gamma)\Phi_u(\omega_0)\Phi_v(\omega_0)}{[\Phi_u(\omega_0)]^2}
$$

and (6.59) has been established.