

MODELS OF LINEAR TIME-INVARIANT SYSTEMS

A model of a system is a description of (some of) its properties, suitable for a certain purpose. The model need not be a true and accurate description of the system, nor need the user have to believe so, in order to serve its purpose.

System identification is the subject of constructing or selecting models of dynamical systems to serve certain purposes. As we noted in Chapter 1, a first step is to determine a class of models within which the search for the most suitable model is to be conducted. In this chapter we shall discuss such classes of models for linear time-invariant systems.

4.1 LINEAR MODELS AND SETS OF LINEAR MODELS

A linear time-invariant model is specified, as we saw in Chapter 2, by the impulse response $\{g(k)\}_1^\infty$, the spectrum $\Phi_v(\omega) = \lambda |H(e^{i\omega})|^2$ of the additive disturbance, and, possibly, the probability density function (PDF) of the disturbance $e(t)$. A complete model is thus given by

$$y(t) = G(q)u(t) + H(q)e(t) \quad (4.1)$$

$f_e(\cdot)$, the PDF of e

with

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k}, \quad H(q) = 1 + \sum_{k=1}^{\infty} h(k)q^{-k} \quad (4.2)$$

A particular model thus corresponds to specification of the three functions G , H , and f_e . It is in most cases impractical to make this specification by enumerating the infinite sequences $\{g(k)\}$, $\{h(k)\}$ together with the function $f_e(x)$. Instead one chooses to work with structures that permit the specification of G and H in terms of a finite number of numerical values. Rational transfer functions and finite-dimensional state-space descriptions are typical examples of this. Also, most often the PDF f_e is

not specified as a function, but described in terms of a few numerical characteristics, typically the first and second moments:

$$\begin{aligned} Ee(t) &= \int xf_e(x) dx = 0 \\ Ee^2(t) &= \int x^2 f_e(x) dx = \lambda \end{aligned} \quad (4.3)$$

It is also common to assume that $e(t)$ is Gaussian, in which case the PDF is entirely specified by (4.3). The specification of (4.1) in terms of a finite number of numerical values, or coefficients, has another and most important consequence for the purposes of system identification. Quite often it is not possible to determine these coefficients *a priori* from knowledge of the physical mechanisms that govern the system's behavior. Instead the determination of all or some of them must be left to estimation procedures. This means that the coefficients in question enter the model (4.1) as *parameters to be determined*. We shall generally denote such parameters by the vector θ , and thus have a model description

$$y(t) = G(q, \theta)u(t) + H(q, \theta)e(t) \quad (4.4a)$$

$f_e(x, \theta)$, the PDF of $e(t)$; $\{e(t)\}$ white noise (4.4b)

The parameter vector θ then ranges over a subset of \mathbf{R}^d , where d is the dimension of θ :

$$\theta \in D_M \subset \mathbf{R}^d \quad (4.5)$$

Notice that (4.4) to (4.5) no longer is a model; it is a *set of models*, and it is for the estimation procedure to select that member in the set that appears to be most suitable for the purpose in question. [One may sometimes loosely talk about "the model (4.4)," but this is abuse of notation from a formal point of view.] Using (3.20), we can compute the one-step-ahead prediction for (4.4). Let it be denoted by $\hat{y}(t|\theta)$ to emphasize its dependence on θ . We thus have

$$\hat{y}(t|\theta) = H^{-1}(q, \theta)G(q, \theta)u(t) + [1 - H^{-1}(q, \theta)] y(t) \quad (4.6)$$

This predictor form does not depend on $f_e(x, \theta)$. In fact, as we stressed in Section 3.3, we could very well arrive at (4.6) by considerations that are not probabilistic. Then the specification (4.4) does not apply. We shall use the term *predictor models* for models that just specify G and H as in (4.4) or in the form (4.6). Similarly, *probabilistic models* will signify descriptions (4.4) that give a complete characterization of the probabilistic properties of the system. A parametrized set of models like (4.6) will be

called a *model structure* and will be denoted by \mathcal{M} . The particular model associated with the parameter value θ will be denoted by $\mathcal{M}(\theta)$. (A formal definition is given in Section 4.5.)

In the following three sections, different ways of describing (4.4) in terms of θ (i.e., different ways of parametrizing the model set) will be discussed. A formalization of the concepts of model sets, parametrizations, model structures, and uniqueness of parametrization will then be given in Section 4.5, while questions of identifiability are discussed in Section 4.6.

4.2 A FAMILY OF TRANSFER-FUNCTION MODELS

Perhaps the most immediate way of parametrizing G and H is to represent them as rational functions and let the parameters be the numerator and denominator coefficients. In this section we shall describe various ways of carrying out such parametrizations. Such model structures are also known as black-box models.

Equation Error Model Structure

Probably the most simple input-output relationship is obtained by describing it as a linear difference equation:

$$\begin{aligned} y(t) + a_1 y(t-1) + \cdots + a_{n_a} y(t-n_a) \\ = b_1 u(t-1) + \cdots + b_{n_b} u(t-n_b) + e(t) \end{aligned} \quad (4.7)$$

Since the white-noise term $e(t)$ here enters as a direct error in the difference equation, the model (4.7) is often called an *equation error model* (structure). The adjustable parameters are in this case

$$\theta = [a_1 \ a_2 \dots a_{n_a} \ b_1 \dots b_{n_b}]^T \quad (4.8)$$

If we introduce

$$A(q) = 1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}$$

and

$$B(q) = b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}$$

we see that (4.7) corresponds to (4.4) with

$$G(q, \theta) = \frac{B(q)}{A(q)}, \quad H(q, \theta) = \frac{1}{A(q)} \quad (4.9)$$

Remark. It may seem annoying to use q as an argument of $A(q)$, being a polynomial in q^{-1} . The reason for this is, however, simply to be consistent with the conventional definition of the z -transform; see (2.17).

We shall also call the model (4.7) an **ARX model**, where **AR** refers to the autoregressive part $A(q)y(t)$ and **X** to the extra input $B(q)u(t)$ (called the exogenous variable in econometrics). In the special case where $n_a = 0$, $y(t)$ is modeled as a finite impulse response (FIR). Such model sets are particularly common in signal-processing applications.

The signal flow can be depicted as in Figure 4.1. From that picture we see that the model (4.7) is perhaps not the most natural one from a physical point of view: the white noise is assumed to go through the denominator dynamics of the system before being added to the output. Nevertheless, the equation error model set has a very important property that makes it a prime choice in many applications: The predictor defines a linear regression.

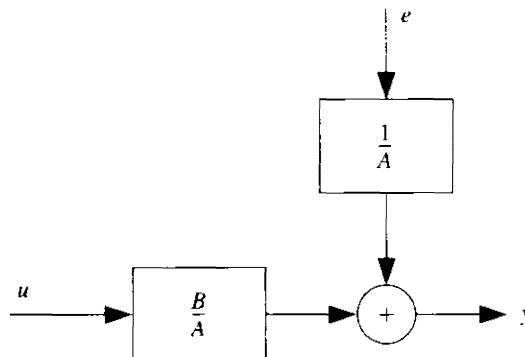


Figure 4.1 The ARX model structure.

Linear Regressions

Let us compute the predictor for (4.7). Inserting (4.9) into (4.6) gives

$$\hat{y}(t|\theta) = B(q)u(t) + [1 - A(q)]y(t) \quad (4.10)$$

Clearly, this expression could have more easily been derived directly from (4.7). Let us reiterate the view expressed in Section 3.3: Without a stochastic framework, the predictor (4.10) is a natural choice if the term $e(t)$ in (4.7) is considered to be “insignificant” or “difficult to guess.” It is thus perfectly natural to work with the expression (4.10) also for “deterministic” models.

Now introduce the vector

$$\varphi(t) = [-y(t-1) \dots -y(t-n_a) \quad u(t-1) \dots u(t-n_b)]^T \quad (4.11)$$

Then (4.10) can be rewritten as

$$\hat{y}(t|\theta) = \theta^T \varphi(t) = \varphi^T(t)\theta \quad (4.12)$$

This is the important property of (4.7) that we alluded to previously. The predictor is a scalar product between a known data vector $\varphi(t)$ and the parameter vector θ . Such a model is called a *linear regression* in statistics, and the vector $\varphi(t)$ is known as the *regression vector*. It is of importance since powerful and simple estimation methods can be applied for the determination of θ .

In case some coefficients of the polynomials A and B are known, we arrive at linear regressions of the form

$$\hat{y}(t|\theta) = \varphi^T(t)\theta + \mu(t) \quad (4.13)$$

where $\mu(t)$ is a known term. See Problem 4E.1 and also (5.67). The estimation of θ in linear regressions will be treated in Section 7.3. See also Appendix II.

ARMAX Model Structure

The basic disadvantage with the simple model (4.7) is the lack of adequate freedom in describing the properties of the disturbance term. We could add flexibility to that by describing the equation error as a moving average of white noise. This gives the model

$$\begin{aligned} y(t) + a_1 y(t-1) + \cdots + a_{n_a} y(t-n_a) &= b_1 u(t-1) + \cdots \\ &+ b_{n_b} u(t-n_b) + e(t) + c_1 e(t-1) + \cdots + c_{n_c} e(t-n_c) \end{aligned} \quad (4.14)$$

With

$$C(q) = 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}$$

it can be rewritten

$$A(q)y(t) = B(q)u(t) + C(q)e(t) \quad (4.15)$$

and clearly corresponds to (4.4) with

$$G(q, \theta) = \frac{B(q)}{A(q)}, \quad H(q, \theta) = \frac{C(q)}{A(q)} \quad (4.16)$$

where now

$$\theta = [a_1 \dots a_{n_a} \ b_1 \dots b_{n_b} \ c_1 \dots c_{n_c}]^T \quad (4.17)$$

In view of the moving average (MA) part $C(q)e(t)$, the model (4.15) will be called ARMAX. The ARMAX model has become a standard tool in control and econometrics for both system description and control design. A version with an enforced integration in the noise description is the ARIMA(X) model (I for integration, with or without the X-variable u), which is useful to describe systems with slow disturbances; see Box and Jenkins (1970). It is obtained by replacing $y(t)$ and $u(t)$ in (4.15) by their differences $\Delta y(t) = y(t) - y(t-1)$ and is further discussed in Section 14.1.

Pseudolinear Regressions

The predictor for (4.15) is obtained by inserting (4.16) into (4.6). This gives

$$\hat{y}(t|\theta) = \frac{B(q)}{C(q)}u(t) + \left[1 - \frac{A(q)}{C(q)}\right]y(t)$$

or

$$C(q)\hat{y}(t|\theta) = B(q)u(t) + [C(q) - A(q)]y(t) \quad (4.18)$$

This means that the prediction is obtained by filtering u and y through a filter with denominator dynamics determined by $C(q)$. To start it up at time $t = 0$ requires knowledge of

$$\begin{aligned} &\hat{y}(0|\theta) \dots \hat{y}(-n_c + 1|\theta) \\ &y(0) \dots y(-n^* + 1), \quad n^* = \max(n_c, n_a) \\ &u(0) \dots u(-n_b + 1) \end{aligned}$$

If these are not available, they can be taken as zero, in which case the prediction differs from the true one with an error that decays as $c \cdot \mu^t$, where μ is the maximum magnitude of the zeros of $C(z)$. It is also possible to start the recursion at time $\max(n^*, n_b)$ and include the unknown initial conditions $\hat{y}(k|\theta)$, $k = 1, \dots, n_c$, in the vector θ .

The predictor (4.18) can be rewritten in formal analogy with (4.12) as follows. Adding $[1 - C(q)]\hat{y}(t|\theta)$ to both sides of (4.18) gives

$$\hat{y}(t|\theta) = B(q)u(t) + [1 - A(q)]y(t) + [C(q) - 1][y(t) - \hat{y}(t|\theta)] \quad (4.19)$$

Introduce the prediction error

$$\varepsilon(t, \theta) = y(t) - \hat{y}(t|\theta)$$

and the vector

$$\begin{aligned} \varphi(t, \theta) = &[-y(t-1) \dots -y(t-n_a) \quad u(t-1) \dots \\ &u(t-n_b) \quad \varepsilon(t-1, \theta) \dots \varepsilon(t-n_c, \theta)]^T \end{aligned} \quad (4.20)$$

Then (4.19) can be rewritten as

$$\hat{y}(t|\theta) = \varphi^T(t, \theta)\theta \quad (4.21)$$

Notice the similarity with the linear regression (4.12). The equation (4.21) itself is, however, no linear regression, due to the nonlinear effect of θ in the vector $\varphi(t, \theta)$. To stress the kinship to (4.12), we shall call it a *pseudolinear regression*.



Other Equation-Error-Type Model Structures

Instead of modeling the equation error in (4.7) as a moving average, as we did in (4.14), it can of course be described as an autoregression. This gives a model set

$$A(q)y(t) = B(q)u(t) + \frac{1}{D(q)}e(t) \quad (4.22)$$

with

$$D(q) = 1 + d_1q^{-1} + \cdots + d_{n_d}q^{-n_d}$$

which, analogously to the previous terminology, could be called ARARX. More generally, we could use an ARMA description of the equation error, leading to an “ARARMAX” structure

$$A(q)y(t) = B(q)u(t) + \frac{C(q)}{D(q)}e(t) \quad (4.23)$$

which of course contains (4.7), (4.15), and (4.22) as special cases. This would thus form the family of equation-error-related model sets, and is depicted in Figure 4.2. The relationship to (4.4) as well as expressions for the predictions are straightforward.

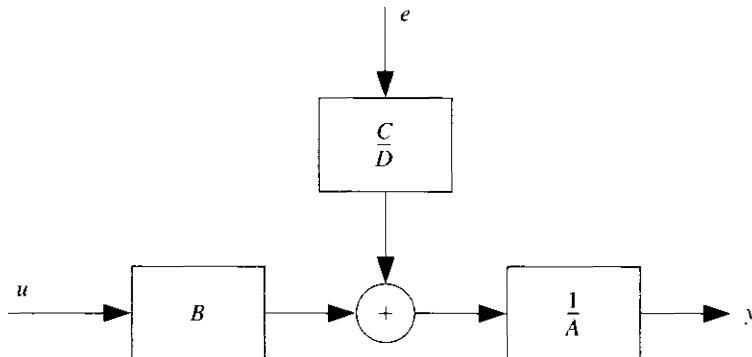


Figure 4.2 The equation error model family: The model structure (4.23).

Output Error Model Structure

The equation error model structures all correspond to descriptions where the transfer functions G and H have the polynomial A as a common factor in the denominators. See Figure 4.2. From a physical point of view it may seem more natural to parametrize these transfer functions independently.

If we suppose that the relation between input and undisturbed output w can be written as a linear difference equation, and that the disturbances consist of white measurement noise, then we obtain the following description:

$$\begin{aligned} w(t) + f_1w(t-1) + \cdots + f_{n_f}w(t-n_f) \\ = b_1u(t-1) + \cdots + b_{n_b}u(t-n_b) \end{aligned} \quad (4.24a)$$

$$y(t) = w(t) + e(t) \quad (4.24b)$$

With

$$F(q) = 1 + f_1 q^{-1} + \cdots + f_{n_f} q^{-n_f}$$

we can write the model as

$$y(t) = \frac{B(q)}{F(q)} u(t) + e(t) \quad (4.25)$$

The signal flow of this model is shown in Figure 4.3.

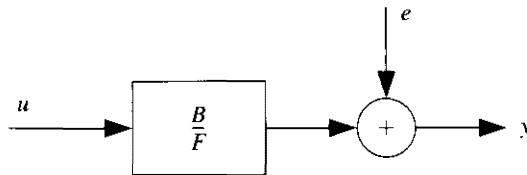


Figure 4.3 The output error model structure.

We call (4.25) an *output error (OE) model (structure)*. The parameter vector to be determined is

$$\theta = [b_1 \ b_2 \dots b_{n_b} \ f_1 \ f_2 \dots f_{n_f}]^T \quad (4.26)$$

Since $w(t)$ in (4.24) is never observed, it should rightly carry an index θ , since it is constructed from u using (4.24a). That is,

$$\begin{aligned} w(t, \theta) &= f_1 w(t-1, \theta) + \cdots + f_{n_f} w(t-n_f, \theta) \\ &= b_1 u(t-1) + \cdots + b_{n_b} u(t-n_b) \end{aligned} \quad (4.27)$$

Comparing with (4.4), we find that $H(q, \theta) = 1$, which gives the natural predictor

$$\hat{y}(t|\theta) = \frac{B(q)}{F(q)} u(t) = w(t, \theta) \quad (4.28)$$

Note that $\hat{y}(t|\theta)$ is constructed from past inputs only. With the aid of the vector

$$\varphi(t, \theta) = [u(t-1) \dots u(t-n_b) \ -w(t-1, \theta) \ \dots \ -w(t-n_f, \theta)]^T \quad (4.29)$$

this can be rewritten as

$$\hat{y}(t|\theta) = \varphi^T(t, \theta) \theta \quad (4.30)$$

which is in formal agreement with the ARMAX-model predictor (4.21). Note that in (4.29) the $w(t-1, \theta)$ are not observed, but, using (4.28), they can be computed: $w(t-k, \theta) = \hat{y}(t-k|\theta)$, $k = 1, 2, \dots, n_f$.

Box-Jenkins Model Structure

A natural development of the output error model (4.25) is to further model the properties of the output error. Describing this as an ARMA model gives

$$y(t) = \frac{B(q)}{F(q)}u(t) + \frac{C(q)}{D(q)}e(t) \quad (4.31)$$

In a sense, this is the most natural finite-dimensional parametrization, starting from the description (4.4): the transfer functions G and H are independently parametrized as rational functions. The model set (4.31) was suggested and treated in Box and Jenkins (1970). This model also gives us the family of output-error-related models. See Figure 4.4 and compare with Figure 4.2. According to (4.6), the predictor for (4.31) is

$$\hat{y}(t|\theta) = \frac{D(q)B(q)}{C(q)F(q)}u(t) + \frac{C(q) - D(q)}{C(q)}y(t) \quad (4.32)$$

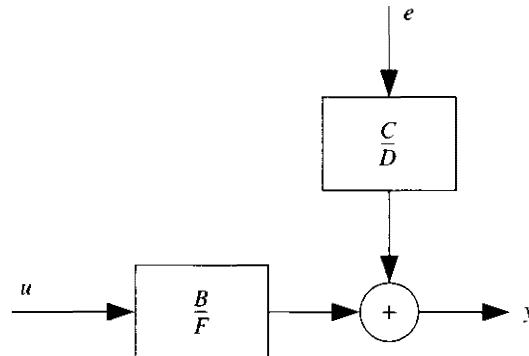


Figure 4.4 The BJ-model structure (4.31).

A General Family of Model Structures

The structures we have discussed in this section actually may give rise to 32 different model sets, depending on which of the five polynomials A , B , C , D , and F are used. (We have, however, only explicitly displayed six of these possibilities here.) Several of these model sets belong to the most commonly used ones in practice, and we have therefore reason to return to them both for explicit algorithms and for analytic results. For convenience, we shall therefore use a generalized model structure

$$A(q)y(t) = \frac{B(q)}{F(q)}u(t) + \frac{C(q)}{D(q)}e(t) \quad (4.33)$$

Sometimes the dynamics from u to y contains a delay of n_k samples, so some leading coefficients of B are zero; that is,

$$B(q) = b_{n_k} q^{-n_k} + b_{n_k+1} q^{-n_k-1} + \cdots + b_{n_k+n_h-1} q^{-n_k-n_h+1} = q^{-n_k} \bar{B}(q), \quad b_{n_k} \neq 0$$

It may then be a good idea to explicitly display this delay by

$$A(q)y(t) = q^{-n_k} \frac{\bar{B}(q)}{F(q)} u(t) + \frac{C(q)}{D(q)} e(t) \quad (4.34)$$

For easier notation we shall, however, here mostly use $n_k = 1$ and (4.33). From expressions for (4.33) we can always derive the corresponding ones for (4.34) by replacing $u(t)$ by $u(t - n_k + 1)$.

The structure (4.33) is too general for most practical purposes. One or several of the five polynomials would be fixed to unity in applications. However, by developing algorithms and results for (4.33), we also cover all the special cases corresponding to more realistic model sets.

From (4.6) we know that the predictor for (4.33) is

$$\hat{y}(t|\theta) = \frac{D(q)B(q)}{C(q)F(q)} u(t) + \left[1 - \frac{D(q)A(q)}{C(q)} \right] y(t) \quad (4.35)$$

The common special cases of (4.33) are summarized in Table 4.1.

TABLE 4.1 Some Common Black-box SISO Models as Special Cases of (4.33)

Polynomials Used in (4.33)	Name of Model Structure
B	FIR (finite impulse response)
AB	ARX
ABC	ARMAX
AC	ARMA
ABD	ARARX
$ABCD$	ARARMAX
BF	OE (output error)
$BFCD$	BJ (Box-Jenkins)

A Pseudolinear Form for (4.35) (*)

The expression (4.35) can also be written as a recursion:

$$C(q)F(q)\hat{y}(t|\theta) = F(q)[C(q) - D(q)A(q)]y(t) + D(q)B(q)u(t) \quad (4.36)$$

From (4.36) we find that the prediction error

$$\varepsilon(t, \theta) = y(t) - \hat{y}(t|\theta)$$

can be written

$$\varepsilon(t, \theta) = \frac{D(q)}{C(q)} \left[A(q)y(t) - \frac{B(q)}{F(q)}u(t) \right] \quad (4.37)$$

It is convenient to introduce the auxiliary variables

$$w(t, \theta) = \frac{B(q)}{F(q)}u(t) \quad (4.38a)$$

and

$$v(t, \theta) = A(q)y(t) - w(t, \theta) \quad (4.38b)$$

Then

$$\varepsilon(t, \theta) = y(t) - \hat{y}(t|\theta) = \frac{D(q)}{C(q)}v(t, \theta) \quad (4.39)$$

Let us also introduce the "state vector"

$$\begin{aligned} \varphi(t, \theta) = & [-y(t-1), \dots, -y(t-n_a), u(t-1), \dots, u(t-n_b), \\ & -w(t-1, \theta), \dots, -w(t-n_f, \theta), \varepsilon(t-1, \theta), \dots, \varepsilon(t-n_c, \theta), \\ & -v(t-1, \theta), \dots, -v(t-n_d, \theta)]^T \end{aligned} \quad (4.40)$$

With the parameter vector

$$\theta = [a_1 \dots a_{n_a} b_1 \dots b_{n_b} f_1 \dots f_{n_f} c_1 \dots c_{n_c} d_1 \dots d_{n_d}]^T \quad (4.41)$$

and (4.40) we can give a convenient expression for the prediction. To find this, we proceed as follows: From (4.38a) and (4.39) we obtain

$$\begin{aligned} w(t, \theta) = & b_1 u(t-1) + \dots + b_{n_b} u(t-n_b) \\ & - f_1 w(t-1, \theta) - \dots - f_{n_f} w(t-n_f, \theta) \end{aligned} \quad (4.42)$$

and

$$\begin{aligned} \varepsilon(t, \theta) = & v(t, \theta) + d_1 v(t-1, \theta) + \dots + d_{n_d} v(t-n_d, \theta) \\ & - c_1 \varepsilon(t-1, \theta) - \dots - c_{n_c} \varepsilon(t-n_c, \theta) \end{aligned} \quad (4.43)$$

Now inserting

$$v(t, \theta) = y(t) + a_1 y(t-1) + \dots + a_{n_a} y(t-n_a) - w(t, \theta)$$

into (4.43) and substituting $w(t, \theta)$ with the expression (4.42), we find that

$$\varepsilon(t, \theta) = y(t) - \theta^T \varphi(t, \theta) \quad (4.44)$$

Hence

$$\hat{y}(t|\theta) = \theta^T \varphi(t, \theta) = \varphi^T(t, \theta) \theta \quad (4.45)$$

The two expressions, (4.36) and (4.45) can both be used for the calculation of the prediction. It should be noticed that the expressions simplify considerably in the special cases of the general model (4.33) that have been discussed in this section.

Other Model Expansions

The FIR model structure

$$G(q, \theta) = \sum_{k=1}^n b_k q^{-k} \quad (4.46)$$

has two important advantages: it is a linear regression (being a special case of ARX) and it is an output error model (being a special case of OE). This means, as we shall see later, that the model can be efficiently estimated and that it is robust against noise. The basic disadvantage is that many parameters may be needed. If the system has a pole close to the unit circle, the impulse response decays slowly, so n has then to be large to approximate the system well. This leads to the question whether it would be possible to retain the linear regression and output error features, while offering better possibilities to treat slowly decaying impulse responses. Generally speaking, such models would look like

$$G(q, \theta) = \sum_{k=1}^n \theta_k L_k(q, \alpha) \quad (4.47)$$

where $L_k(q, \alpha)$ represents a function expansion in the delay operator, which may contain a user-chosen parameter α . This parameter would be treated as fixed in the model structure, in order to make (4.47) a linear regression. A simple choice would be

$$L_k(q, \alpha) = \frac{q^{-k}}{q - \alpha}$$

where α is an estimate of the system pole closest to the unit circle. More sophisticated choices in terms of orthonormal basis expansions, see, e.g., Van den Hof, Heuberger, and Bokor (1995), have attracted wide interest. In particular, *Laguerre polynomials* have been used in this context, (Wahlberg, 1991):

$$L_k(q, \alpha) = \frac{1}{q - \alpha} \left(\frac{1 - \alpha q}{q - \alpha} \right)^{k-1} \quad (4.48)$$

where, again, it is natural to let α be an estimate of the dominating pole (time constant).

Continuous-time Black-box Models (*)

The linear system description could also be parameterized in terms of the continuous-time transfer function (2.22):

$$y(t) = G_c(p, \theta)u(t) \quad (4.49)$$

Adjustments to observed, sampled data could then be achieved either by solving the underlying differential equations or by applying an exact or approximate sampling procedure (2.24). The model (4.49) could also be fitted in the frequency domain, to Fourier transformed band-limited input-output data, as described in Section 7.7.

In addition to obvious counterparts of the structures already discussed, two specific model sets should be mentioned. The first-order system model with a time delay

$$G_c(s, \theta) = \frac{Ke^{-s\tau_c}}{(s\tau + 1)}. \quad \theta = [K, \tau_c, \tau]^T \quad (4.50)$$

has been much used in process industry applications. Orthonormal function series expansions

$$G_c(s, \theta) = \sum_{k=0}^{d-1} a_k f_k(s), \quad \theta = [a_0, \dots, a_{d-1}]^T \quad (4.51)$$

have been discussed in the early literature, and also, e.g., by Belanger (1985). Like for discrete-time models, Laguerre polynomials appear to be a good choice:

$$f_k(s) = \sqrt{2\alpha} \frac{(s - \alpha)^k}{(s + \alpha)^{k+1}}$$

α being a time-scaling factor. Clearly, the model (4.49) can then be complemented with a model for the disturbance effects at the sampling instants as in (2.23).

Multivariable Case: Matrix Fraction Descriptions (*)

Let us now consider the case where the input $u(t)$ is an m -dimensional vector and the output $y(t)$ is a p -dimensional vector. Most of the ideas that we have described in this section have straightforward multivariable counterparts. The simplest case is the generalization of the equation error model set (4.7). We obtain

$$\begin{aligned} y(t) &+ A_1 y(t-1) + \cdots + A_{n_a} y(t-n_a) \\ &= B_1 u(t-1) + \cdots + B_{n_b} u(t-n_b) + e(t) \end{aligned} \quad (4.52)$$

where the A_i are $p \times p$ matrices and the B_i are $p \times m$ matrices.

Analogous to (4.9), we may introduce the polynomials

$$\begin{aligned} A(q) &= I + A_1 q^{-1} + \cdots + A_{n_a} q^{-n_a} \\ B(q) &= B_1 q^{-1} + \cdots + B_{n_b} q^{-n_b} \end{aligned} \quad (4.53)$$

These are now *matrix polynomials* in q^{-1} meaning that $A(q)$ is a matrix whose entries are polynomials in q^{-1} . We note that the system is still given by

$$y(t) = G(q, \theta)u(t) + H(q, \theta)e(t) \quad (4.54)$$

with

$$G(q, \theta) = A^{-1}(q)B(q), \quad H(q, \theta) = A^{-1}(q) \quad (4.55)$$

The inverse $A^{-1}(q)$ of the matrix polynomial is interpreted and calculated in a straightforward way as discussed in connection with (3.34). Clearly, $G(q, \theta)$ will be a $p \times m$ matrix whose entries are rational functions of q^{-1} (or q). The factorization in terms of two matrix polynomials is also called a (left) *matrix fraction description* (MFD). A thorough treatment of such descriptions is given in Chapter 6 of Kailath (1980).

We have not yet discussed the *parametrization* of (4.52) (i.e., which elements of the matrices should be included in the parameter vector θ). This is a fairly subtle issue, which will be further discussed in Appendix 4A. An immediate analog of (4.8) could, however, be noted: Suppose all matrix entries in (4.52) (a total of $n_a \cdot p^2 + n_b \cdot p \cdot m$) are included in θ . We may then define the $[n_a \cdot p + n_b \cdot m] \times p$ matrix

$$\theta = [A_1 A_2 \cdots A_{n_a} B_1 \cdots B_{n_b}]^T \quad (4.56)$$

and the $[n_a \cdot p + n_b \cdot m]$ -dimensional column vector

$$\varphi(t) = \begin{bmatrix} -y(t-1) \\ \vdots \\ -y(t-n_a) \\ u(t-1) \\ \vdots \\ u(t-n_b) \end{bmatrix} \quad (4.57)$$

to rewrite (4.52) as

$$y(t) = \theta^T \varphi(t) + e(t) \quad (4.58)$$

in obvious analogy with the linear regression (4.12). This can be seen as p different linear regressions, written on top of each other, all with the same regression vector.

When additional structure is imposed on the parametrization, it is normally no longer possible to use (4.58), since the different output components will not employ identical regression vectors. Then a d -dimensional column vector θ and a $p \times d$ matrix $\varphi^T(t)$ has to be formed so as to represent (4.52) as

$$y(t) = \varphi^T(t)\theta + e(t) \quad (4.59)$$

See Problems 4G.6 and 4E.12 for some more aspects on (4.58) and (4.59).

In light of the different possibilities for SISO systems, it is easy to visualize a number of variants for the MIMO case, like the vector difference equation (VDE)

$$\begin{aligned} y(t) &+ A_1 y(t-1) + \cdots + A_{n_a} y(t-n_a) \\ &= B_1 u(t-1) + \cdots + B_{n_b} u(t-n_b) \\ &\quad + e(t) + C_1 e(t-1) + \cdots + C_{n_c} e(t-n_c) \end{aligned} \quad (4.60a)$$

or

$$G(q, \theta) = A^{-1}(q)B(q), \quad H(q, \theta) = A^{-1}(q)C(q) \quad (4.60b)$$

which is the natural extension of the ARMAX model. A multivariable Box-Jenkins model takes the form

$$G(q, \theta) = F^{-1}(q)B(q), \quad H(q, \theta) = D^{-1}(q)C(q) \quad (4.61)$$

and so on. The parametrizations of these MFD-descriptions are discussed in Appendix 4A.

4.3 STATE-SPACE MODELS

In the state-space form the relationship between the input, noise, and output signals is written as a system of first-order differential or difference equations using an auxiliary state vector $x(t)$. This description of linear dynamical systems became an increasingly dominating approach after Kalman's (1960) work on prediction and linear quadratic control. For our purposes it is especially useful in that insights into physical mechanisms of the system can usually more easily be incorporated into state-space models than into the models described in Section 4.2.

Continuous-time Models Based on Physical Insight

For most physical systems it is easier to construct models with physical insight in continuous time than in discrete time, simply because most laws of physics (Newton's law of motion, relationships in electrical circuits, etc.) are expressed in continuous time. This means that modeling normally leads to a representation

$$\dot{x}(t) = F(\theta)x(t) + G(\theta)u(t) \quad (4.62)$$

Here F and G are matrices of appropriate dimensions ($n \times n$ and $n \times m$, respectively, for an n -dimensional state and an m -dimensional input). The overdot denotes differentiation with respect to (w.r.t) time t . Moreover, θ is a vector of parameters that typically correspond to unknown values of physical coefficients, material constants, and the like. The modeling is usually carried out in terms of state variables x that have physical significance (positions, velocities, etc.), and then the measured outputs will be known combinations of the states. Let $\eta(t)$ be the measurements that would be obtained with ideal, noise-free sensors:

$$\eta(t) = Hx(t) \quad (4.63)$$

Using p for the differentiation operator, (4.62) can be written

$$[pI - F(\theta)]x(t) = G(\theta)u(t)$$

which means that the transfer operator from u to η in (4.63) is

$$\begin{aligned}\eta(t) &= G_c(p, \theta)u(t) \\ G_c(p, \theta) &= H[pI - F(\theta)]^{-1}G(\theta)\end{aligned}\quad (4.64)$$

We have thus obtained a continuous-time transfer-function model of the system, as in (2.22), that is parametrized in terms of physical coefficients.

In reality, of course, some noise-corrupted version of $\eta(t)$ is obtained, resulting from both measurement imperfections and disturbances acting on (4.62). There are several different possibilities to describe these noise and disturbance effects. Here we first take the simplest approach. Other cases are discussed in (4.84) and (4.96) to (4.99), in Problem 4G.7, and in Section 13.7. Let the measurements be sampled at time instants $t = kT$, $k = 1, 2, \dots$, and the disturbance effects at those time instants be $v_T(kT)$. Hence the measured output is

$$y(kT) = Hx(kT) + v_T(kT) = G_c(p, \theta)u(t) + v_T(kT) \quad (4.65)$$

Sampling the Transfer Function

As we discussed in Section 2.1, there are several ways of transporting $G_c(p, \theta)$ to a representation that is explicitly discrete time. Suppose that the input is constant over the sampling interval T as in (2.3):

$$u(t) = u_k = u(kT), \quad kT \leq t < (k+1)T \quad (4.66)$$

Then the differential equation (4.62) can easily be solved from $t = kT$ to $t = kT + T$, yielding

$$x(kT + T) = A_T(\theta)x(kT) + B_T(\theta)u(kT) \quad (4.67)$$

where

$$A_T(\theta) = e^{F(\theta)T} \quad (4.68a)$$

$$B_T(\theta) = \int_{\tau=0}^T e^{F(\theta)\tau} G(\theta) d\tau \quad (4.68b)$$

(See, e.g., Åström and Wittenmark, 1984.)

Introducing q for the forward shift of T time units, we can rewrite (4.67) as

$$[qI - A_T(\theta)]x(kT) = B_T(\theta)u(kT) \quad (4.69)$$

or

$$\eta(kT) = G_T(q, \theta)u(kT) \quad (4.70)$$

$$G_T(q, \theta) = H[qI - A_T(\theta)]^{-1}B_T(\theta) \quad (4.71)$$

Hence (4.65) can equivalently be given in the sampled-data form

$$y(t) = G_T(q, \theta)u(t) + v_T(t), \quad t = T, 2T, 3T, \dots \quad (4.72)$$

When (4.66) holds, no approximation is involved in this representation. Note, however, that in view of (4.68) $G_T(q, \theta)$ could be quite a complicated function of θ .

Example 4.1 DC Servomotor

In this example we shall study a physical process, where we have some insight into the dynamic properties. Consider the dc motor depicted in Figure 4.5 with a block diagram in Figure 4.6. The input to this system is assumed to be the applied voltage, u , and the output the angle of the motor shaft, η . The relationship between applied voltage u and the resulting current i in the rotor circuit is given by the well-known relationship

$$u(t) = R_a i(t) + L_a \frac{di(t)}{dt} + s(t) \quad (4.73)$$

where $s(t)$ is the back electromotive force, due to the rotation of the armature circuit in the magnetic field:

$$s(t) = k_v \frac{d}{dt} \eta(t)$$

The current i gives a turning torque of

$$T_a(t) = k_a \cdot i(t)$$

on the motor shaft, which is also affected by a torque $T_\ell(t)$ from the load. Newton's law then gives

$$J \frac{d^2}{dt^2} \eta(t) = T_a(t) - T_\ell(t) - f \frac{d}{dt} \eta(t) \quad (4.74)$$

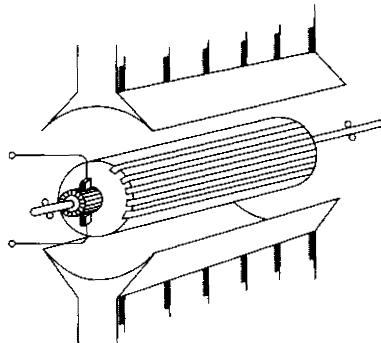


Figure 4.5 The dc motor.

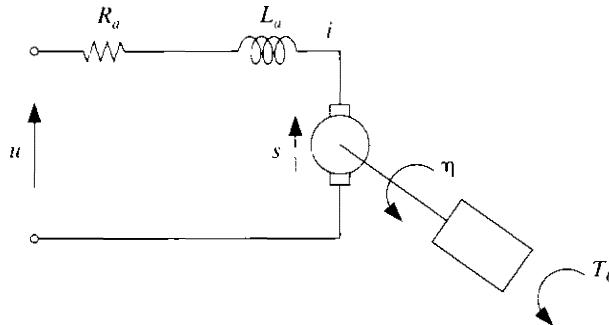


Figure 4.6 Block diagram of the dc motor.

where J is the moment of inertia of the rotor plus load and f represents viscous friction. Assuming that the inductance of the armature circuit can be neglected, $L_a \approx 0$, the preceding equations can be summarized in state-space form as

$$\begin{aligned} \frac{d}{dt}x(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}x(t) + \begin{bmatrix} 0 \\ \beta/\tau \end{bmatrix}u(t) + \begin{bmatrix} 0 \\ \gamma'/\tau \end{bmatrix}T_e(t) \quad (4.75) \\ \eta(t) &= [1 \ 0]x(t) \end{aligned}$$

with

$$\begin{aligned} x(t) &= \begin{bmatrix} \eta(t) \\ \frac{d}{dt}\eta(t) \end{bmatrix} \\ \tau &= \frac{JR_a}{fR_a + k_ak_v}, \quad \beta = \frac{k_a}{fR_a + k_ak_v}, \quad \gamma' = -\frac{R_a}{fR_a + k_ak_v} \end{aligned}$$

Assume now that the torque T_e is identically zero. To determine the dynamics of the motor, we now apply a piecewise constant input and sample the output with the sampling interval T . The state equation (4.75) can then be described by

$$x(t + T) = A_T(\theta)x(t) + B_T(\theta)u(t) \quad (4.76)$$

where

$$\theta = \begin{bmatrix} \tau \\ \beta \end{bmatrix}$$

and, according to (4.68),

$$A_T(\theta) = \begin{bmatrix} 1 & \tau(1 - e^{-T/\tau}) \\ 0 & e^{-T/\tau} \end{bmatrix}, \quad B_T(\theta) = \begin{bmatrix} \beta(\tau e^{-T/\tau} - \tau + T) \\ \beta(1 - e^{-T/\tau}) \end{bmatrix} \quad (4.77)$$

Also assume that $y(t)$, the actual measurement of the angle $\eta(t)$, is made with a certain error $v(t)$:

$$y(t) = \eta(t) + v(t) \quad (4.78)$$

This error is mainly caused by limited accuracy (e.g., due to the winding of a potentiometer) and can be described as a sequence of independent random variables with zero mean and known variance R_2 (computed from the truncation error in the measurement), provided the measurements are not too frequent. We thus have a model

$$y(t) = G_T(q, \theta)u(t) + v(t)$$

with $v(t)$ being white noise. The natural predictor is thus

$$\hat{y}(t|\theta) = G_T(q, \theta)u(t) = [1 \ 0][qI - A_T(\theta)]^{-1}B_T(\theta)u(t) \quad (4.79)$$

This predictor is parametrized using only two parameters β and τ . Notice that if we used our physical insight to conclude only that the system is of second order we would use, say, a second-order ARX or OE model containing *four* adjustable parameters. As we shall see, using fewer parameters has some positive effects on the estimation procedure: the variance of the parameter estimates will decrease. The price is, however, not insignificant. The predictor (4.79) is a far more complicated function of its two parameters than the corresponding ARX or OE model of its four parameters. \square

Equations (4.67) and (4.65) constitute a standard discrete-time state-space model. For simplicity we henceforth take $T = 1$ and drop the corresponding index. We also introduce an arbitrary parametrization of the matrix that relates x to y : $H = C(\theta)$. We thus have

$$x(t+1) = A(\theta)x(t) + B(\theta)u(t) \quad (4.80a)$$

$$y(t) = C(\theta)x(t) + v(t) \quad (4.80b)$$

corresponding to

$$y(t) = G(q, \theta)u(t) + v(t) \quad (4.81)$$

$$G(q, \theta) = C(\theta)[qI - A(\theta)]^{-1}B(\theta) \quad (4.82)$$

Although sampling a time-continuous description is a natural way to obtain the model (4.80), it could also for certain applications be posed directly in discrete time, with the matrices A , B , and C directly parametrized in terms of θ , rather than indirectly via (4.68).

Noise Representation and the Time-invariant Kalman Filter

In the representation (4.80) and (4.81) we could further model the properties of the noise term $\{v(t)\}$. A straightforward but entirely valid approach would be to postulate a noise model of the kind

$$v(t) = H(q, \theta)e(t) \quad (4.83)$$

with $\{e(t)\}$ being white noise with variance λ . The θ -parameters in $H(q, \theta)$ could be partly in common with those in $G(q, \theta)$ or be extra additional noise model parameters.

For state-space descriptions, it is, however, more common to split the lumped noise term $v(t)$ into contributions from *measurement noise* $v(t)$ and *process noise* $w(t)$ acting on the states, so that (4.80) is written

$$\begin{aligned} x(t+1) &= A(\theta)x(t) + B(\theta)u(t) + w(t) \\ y(t) &= C(\theta)x(t) + v(t) \end{aligned} \quad (4.84)$$

Here $\{w(t)\}$ and $\{v(t)\}$ are assumed to be sequences of independent random variables with zero mean values and covariances

$$\begin{aligned} Ew(t)w^T(t) &= R_1(\theta) \\ Ev(t)v^T(t) &= R_2(\theta) \\ Ew(t)v^T(t) &= R_{12}(\theta) \end{aligned} \quad (4.85)$$

The disturbances $w(t)$ and $v(t)$ may often be signals whose physical origins are known. In Example 4.1 the load variation $T_\ell(t)$ was a “process noise,” while the inaccuracy in the potentiometer angular sensor $v(t)$ was the “measurement noise.” In such cases it may of course not always be realistic to assume that these signals are white noises. To arrive at (4.84) and (4.85) will then require extra modeling and extension of the state vector. See Problem 4G.2.

Let us now turn to the problem of predicting $y(t)$ in (4.84). This state-space description is one to which the celebrated Kalman filter applies (see, e.g., Anderson and Moore, 1979, for a thorough treatment). The conditional expectation of $y(t)$, given data $y(s), u(s), s \leq t$ (i.e., from the infinite past up to time $t-1$), is, provided v and w are Gaussian processes, given by

$$\begin{aligned} \hat{x}(t+1, \theta) &= A(\theta)\hat{x}(t, \theta) + B(\theta)u(t) + K(\theta)[y(t) - C(\theta)\hat{x}(t, \theta)] \\ \hat{y}(t|\theta) &= C(\theta)\hat{x}(t, \theta) \end{aligned} \quad (4.86)$$

Here $K(\theta)$ is given as

$$K(\theta) = [A(\theta)\bar{P}(\theta)C^T(\theta) + R_{12}(\theta)][C(\theta)\bar{P}(\theta)C^T(\theta) + R_2(\theta)]^{-1} \quad (4.87a)$$

where $\bar{P}(\theta)$ is obtained as the positive semidefinite solution of the stationary Riccati equation:

$$\begin{aligned} \bar{P}(\theta) &= A(\theta)\bar{P}(\theta)A^T(\theta) + R_1(\theta) - [A(\theta)\bar{P}(\theta)C^T(\theta) + R_{12}(\theta)] \\ &\times [C(\theta)\bar{P}(\theta)C^T(\theta) + R_2(\theta)]^{-1}[A(\theta)\bar{P}(\theta)C^T(\theta) + R_{12}(\theta)]^T \end{aligned} \quad (4.87b)$$

K

The predictor filter can thus be written as

$$\begin{aligned}\hat{y}(t|\theta) &= C(\theta)[qI - A(\theta) + K(\theta)C(\theta)]^{-1} B(\theta)u(t) \\ &\quad + C(\theta)[qI - A(\theta) + K(\theta)C(\theta)]^{-1} K(\theta)y(t)\end{aligned}\quad (4.88)$$

The matrix $\bar{P}(\theta)$ is the covariance matrix of the state estimate error:

$$\bar{P}(\theta) = \bar{E}[x(t) - \hat{x}(t, \theta)][x(t) - \hat{x}(t, \theta)]^T \quad (4.89)$$

Innovations Representation

The prediction error

$$y(t) - C(\theta)\hat{x}(t, \theta) = C(\theta)[x(t) - \hat{x}(t, \theta)] + v(t) \quad (4.90)$$

in (4.86) amounts to that part of $y(t)$ that cannot be predicted from past data: “the innovation.” Denoting this quantity by $e(t)$ as in (3.25), we find that (4.86) can be rewritten as

$$\begin{aligned}\hat{x}(t+1, \theta) &= A(\theta)\hat{x}(t, \theta) + B(\theta)u(t) + K(\theta)e(t) \\ y(t) &= C(\theta)\hat{x}(t, \theta) + e(t)\end{aligned}\quad (4.91a)$$

The covariance of $e(t)$ can be determined from (4.90) and (4.89):

$$Ee(t)e^T(t) = \Lambda(\theta) = C(\theta)\bar{P}(\theta)C^T(\theta) + R_2(\theta) \quad (4.91b)$$

Since $e(t)$ appears explicitly, this representation is known as the *innovations form* of the state-space description. Using the shift operator q , we can clearly rearrange it as

$$y(t) = G(q, \theta)u(t) + H(q, \theta)e(t) \quad (4.92a)$$

$$G(q, \theta) = C(\theta)[qI - A(\theta)]^{-1} B(\theta) \quad (4.92b)$$

$$H(q, \theta) = C(\theta)[qI - A(\theta)]^{-1} K(\theta) + I$$

showing its relationship to the general model (4.4) and to a direct modeling of $v(t)$ as in (4.83). See also Problem 4G.3.

Directly Parametrized Innovations Form

In (4.91) the Kalman gain $K(\theta)$ is computed from $A(\theta)$, $C(\theta)$, $R_1(\theta)$, $R_{12}(\theta)$, and $R_2(\theta)$ in the fairly complicated manner given by (4.87). It is an attractive idea to sidestep (4.87) and the parametrization of the R -matrices by directly parametrizing $K(\theta)$ in terms of θ . This has the important advantage that the predictor (4.88) becomes a much simpler function of θ . Such a model structure we call a *directly parametrized innovations form*.

The R -matrices describing the noise properties contain $\frac{1}{2}n(n + 1) + np + \frac{1}{2}p(p + 1)$ matrix elements (discounting symmetric ones), while the Kalman gain K contains np elements ($p = \dim y$, $n = \dim x$). If we have no prior knowledge about the R -matrices and thus would need many parameters to describe them, it would therefore be a better alternative to parametrize $K(\theta)$, also from the point of view of keeping $\dim \theta$ small. On the other hand, physical insight into (4.84) may entail knowing, for example, that the process noise affects only one state and is independent of the measurement noise, which might have a known variance. Then the parametrization of $K(\theta)$ via (4.85) and (4.87) may be done using less parameters than would be required in a direct parametrization of $K(\theta)$.

Remark. The parametrization in terms of (4.85) also gives a parametrization of the $p(p + 1)/2$ elements of $\Lambda(\theta)$ in (4.91). A direct parametrization of (4.91) would involve extra parameters for Λ , which, however, would not affect the predictor. (Compare also Problems 7E.4 and 8E.2.)

Directly parametrized innovations forms also contain black-box models that are in close relationship to those discussed in Section 4.2.

Example 4.2 Companion Form Parametrizations

In (4.91) let

$$\theta^T = [a_1 \ a_2 \ a_3 \ b_1 \ b_2 \ b_3 \ k_1 \ k_2 \ k_3]$$

and

$$A(\theta) = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix},$$

$$B(\theta) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad K(\theta) = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$C(\theta) = [1 \ 0 \ 0]$$

These matrices are said to be in *companion form* or in observer canonical form (see, e.g., Kailath, 1980). It is easy to verify that with these matrices

$$C(\theta)[qI - A(\theta)]^{-1}B(\theta) = \frac{b_1q^{-1} + b_2q^{-2} + b_3q^{-3}}{1 + a_1q^{-1} + a_2q^{-2} + a_3q^{-3}}$$

and

$$C(\theta)[qI - A(\theta)]^{-1}K(\theta) = \frac{k_1q^{-1} + k_2q^{-2} + k_3q^{-3}}{1 + a_1q^{-1} + a_2q^{-2} + a_3q^{-3}}$$

so that

$$1 + C(\theta) [qI - A(\theta)]^{-1} K(\theta) = \frac{1 + c_1 q^{-1} + c_2 q^{-2} + c_3 q^{-3}}{1 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3}}$$

with

$$c_i \stackrel{\triangle}{=} a_i + k_i, \quad i = 1, 2, 3$$

With this we have consequently obtained a parametrization of the ARMAX model set (4.15) and (4.16) for $n_a = n_b = n_c = 3$. \square

The corresponding parametrization of a multioutput model is more involved and is described in Appendix 4A.

Time-varying Predictors (*)

For the predictor filter (4.86) and (4.87) we assumed all previous data from time minus infinity to be available. If data prior to time $t = 0$ are lacking, we could replace them by zero, thus starting the recursion (4.86) at $t = 0$ with $\hat{x}(0) = 0$, and take the penalty of a suboptimal estimate. This was also our philosophy in Section 3.2.

An advantage with the state-space formulation is that a correct treatment of incomplete information about $t < 0$ can be given at the price of a slightly more complex predictor. If the information about the history of the system prior to $t = 0$ is given in terms of an initial state estimate $x_0(\theta) = \hat{x}(0, \theta)$ and associated uncertainty

$$\Pi_0(\theta) = E[x(0) - x_0(\theta)][x(0) - x_0(\theta)]^T \quad (4.93)$$

then the Kalman filter tells us that the one-step-ahead prediction is given by, (see, e.g., Anderson and Moore, 1979),

$$\hat{x}(t+1, \theta) = A(\theta)\hat{x}(t, \theta) + B(\theta)u(t) + K(t, \theta)[y(t) - C(\theta)\hat{x}(t, \theta)] \quad (4.94)$$

$$\begin{aligned} \hat{y}(t|\theta) &= C(\theta)\hat{x}(t, \theta), \quad \hat{x}(0, \theta) = x_0(\theta) \\ K(t, \theta) &= [A(\theta)P(t, \theta)C^T(\theta) + R_{12}(\theta)] \\ &\times [C(\theta)P(t, \theta)C^T(\theta) + R_2(\theta)]^{-1} \end{aligned} \quad (4.95)$$

$$\begin{aligned} P(t+1, \theta) &= A(\theta)P(t, \theta)A^T(\theta) + R_1(\theta) - K(t, \theta) \\ &\times [C(\theta)P(t, \theta)C^T(\theta) + R_2(\theta)]K^T(t, \theta), \quad P(0, \theta) = \Pi_0(\theta) \end{aligned}$$

Now $K(t, \theta)$ determined by (4.95) converges, under general conditions, fairly rapidly to $K(\theta)$ given by (4.87) (see, e.g., Anderson and Moore, 1979). For many problems it is thus reasonable to apply the limit form (4.86) with (4.87) directly to simplify calculations. For short data records, though, the solution (4.93) to (4.95) gives a useful possibility to deal with the transient properties in a correct way, including possibly a parametrization of the unknown initial conditions $x_0(\theta)$ and $\Pi_0(\theta)$. Clearly, the steady-state approach (4.86) with (4.87) is a special case of (4.94) to (4.95), corresponding to $x_0(\theta) = 0$, $\Pi_0(\theta) = \bar{P}(\theta)$.

Sampling Continuous-time Process Noise (*)

Just as for the systems dynamics, we may have more insight into the nature of the process noise in continuous time. We could then pose a disturbed state-space model

$$\dot{x}(t) = F(\theta)x(t) + G(\theta)u(t) + \bar{w}(t) \quad (4.96)$$

where $\bar{w}(t)$ is formal white noise with covariance function

$$E\bar{w}(t)\bar{w}^T(s) = \bar{R}_1(\theta)\delta(t - s) \quad (4.97)$$

where δ is Dirac's delta function. When the input is piecewise constant as in (4.66), the corresponding discrete-time state equation becomes

$$x(kT + T) = A_T(\theta)x(kT) + B_T(\theta)u(kT) + w_T(kT) \quad (4.98)$$

where A_T and B_T are given by (4.68) and $w_T(kT)$, $k = 1, 2, \dots$ is a sequence of independent random vectors with zero means and covariance matrix

$$Ew_T(kT)w_T^T(kT) = R_1(\theta) = \int_0^T e^{F(\theta)\tau} \bar{R}_1(\theta) e^{F^T(\theta)\tau} d\tau \quad (4.99)$$

See Åström (1970) for a derivation.

State-space Models

In summary, we have found that state-space models provide us with a spectrum of modeling possibilities: We may use physical modeling in continuous time with or without a corresponding time-continuous noise description to obtain structures with physical parameters θ . We can use physical parametrization of the dynamics part combined with a black-box parametrization of the noise properties, such as in the directly parametrized innovations form (4.91), or we can arrive at a noise model that is also physically parametrized via (4.96) to (4.99). Finally, we can use black-box state-space structures, such as the one of Example 4.2. These have the advantage over the input-output black box that the flexibility in choice of representation can secure better numerical properties of the parametrization (Problem 16E.1).

4.4 DISTRIBUTED PARAMETER MODELS (*)

Models that involve partial differential equations (PDE), directly or indirectly, when relating the input signal to the output signal are usually called *distributed parameter models*. “Distributed” then refers to the state vector, which in general belongs to a function space, rather than \mathbf{R}^n . There are basically two ways to deal with such models. One is to replace the space variable derivative by a difference expression or to truncate a function series expansion so as to approximate the PDE by an ordinary differential equation. Then a “lumped” finite-dimensional model, of the kind we discussed in Section 4.3, is obtained. (“Lumped” refers to the fact that the distributed states are lumped together into a finite collection.) The other approach is to stick to the original PDE for the calculations, and only at the final, numerical, stage introduce approximations to facilitate the computations. It should be noted that this second approach also remains within the general model structure (4.4), provided the underlying PDE is linear and time invariant. This is best illustrated by an example.

Example 4.3 Heating Dynamics

Consider the physical system schematically depicted in Figure 4.7. It consists of a well-insulated metal rod, which is heated at one end. The heating power at time t is the input $u(t)$, while the temperature measured at the other end is the output $y(t)$. This output is sampled at $t = 1, 2, \dots$

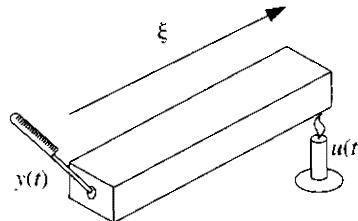


Figure 4.7 The heat-rod system.

Under ideal conditions, this system is described by the heat-diffusion equation. If $x(t, \xi)$ denotes the temperature at time t , ξ length units from one end of the rod, then

$$\frac{\partial x(t, \xi)}{\partial t} = \kappa \frac{\partial^2 x(t, \xi)}{\partial \xi^2} \quad (4.100)$$

where κ is the coefficient of thermal conductivity. The heating at the far end means that

$$\left. \frac{\partial x(t, \xi)}{\partial \xi} \right|_{\xi=L} = K \cdot u(t) \quad (4.101)$$

where K is a heat-transfer coefficient. The near end is insulated so that

$$\left. \frac{\partial x(t, \xi)}{\partial \xi} \right|_{\xi=0} = 0 \quad (4.102)$$

The measurements are

$$y(t) = x(t, 0) + v(t), \quad t = 1, 2, \dots \quad (4.103)$$

where $\{v(t)\}$ accounts for the measurement noise. The unknown parameters are

$$\theta = \begin{bmatrix} \kappa \\ K \end{bmatrix} \quad (4.104)$$

Approximating

$$\frac{\partial^2 x(t, \xi)}{\partial \xi^2} = \frac{x(t, \xi + \Delta L) - 2x(t, \xi) + x(t, \xi - \Delta L)}{(\Delta L)^2}, \quad \xi = k \cdot \Delta L$$

transfers (4.100) to a state-space model of order $n = L/\Delta L$, where the state variables $x(t, k \cdot \Delta L)$ are lumped representatives for $x(t, \xi)$, $k \cdot \Delta L \leq \xi < (k+1) \cdot \Delta L$. This often gives a reasonable approximation of the heat-diffusion equation.

Here we instead retain the PDE (4.100) by Laplace transforming it. Thus let $X(s, \xi)$ be the Laplace transform of $x(t, \xi)$ with respect to t for fixed ξ . Then (4.100) to (4.102) take the form

$$\begin{aligned} sX(s, \xi) &= \kappa X''(s, \xi) \\ X'(s, L) &= K \cdot U(s) \\ X'(s, 0) &= 0 \end{aligned} \quad (4.105)$$

Prime and double prime here denote differentiation with respect to ξ , and $U(s)$ is the Laplace transform of $u(t)$. Solving (4.105) for fixed s gives

$$X(s, \xi) = A(s)e^{-\xi\sqrt{s/\kappa}} + B(s)e^{\xi\sqrt{s/\kappa}}$$

where the constants $A(s)$ and $B(s)$ are determined from the boundary values

$$X'(s, 0) = 0$$

$$X'(s, L) = K \cdot U(s)$$

which gives

$$A(s) = B(s) = \frac{K \cdot U(s)}{\sqrt{s/\kappa}(e^{L\sqrt{s/\kappa}} - e^{-L\sqrt{s/\kappa}})} \quad (4.106)$$

Inserting this into (4.103) gives

$$Y(s) = X(s, 0) + V(s) = G_c(s, \theta)U(s) + V(s) \quad (4.107)$$

$$G_c(s, \theta) = \frac{2K}{\sqrt{s/\kappa}(e^{L\sqrt{s/\kappa}} - e^{-L\sqrt{s/\kappa}})} \quad (4.108)$$

where $V(s)$ is the Laplace transform of the noise $\{v(t)\}$. We have thus arrived at a model parametrization of the kind (4.49). With some sampling procedure and a model for the measurement noise sequence, it can be carried further to the form (4.4). Note that $G_c(s, \theta)$ is an analytic function of s although not rational. All our concepts of poles, zeros, stability, and so on, can still be applied. \square

We can thus include distributed parameter models in our treatment of system identification methods. There is a substantial literature on this subject. See, for example, Banks, Crowley, and Kunisch (1983) and Kubrusly (1977). Not surprisingly, computational issues, choice of basis functions, and the like, play an important role in this literature.

4.5 MODEL SETS, MODEL STRUCTURES, AND IDENTIFIABILITY: SOME FORMAL ASPECTS (*)

In this chapter we have dealt with models of linear systems, as well as with parametrized sets of such models. When it comes to analysis of identification methods, it turns out that certain properties will have to be required from these models and model sets. In this section we shall discuss such formal aspects. To keep notation simple, we treat explicitly only SISO models.

Some Notation

For the expressions we shall deal with in this section, it is convenient to introduce some more compact notation. With

$$T(q) = [G(q) \quad H(q)] \text{ and } \chi(t) = \begin{bmatrix} u(t) \\ e(t) \end{bmatrix} \quad (4.109)$$

we can rewrite (4.1) as

$$y(t) = T(q)\chi(t) \quad (4.110)$$

The model structure (4.4) can similarly be written

$$y(t) = T(q, \theta)\chi(t), \quad T(q, \theta) = [G(q, \theta) \quad H(q, \theta)] \quad (4.111)$$

Given the model (4.110), we can determine the one-step-ahead predictor (3.46), which we can rewrite as

$$\hat{y}(t|t-1) = W(q)z(t) \quad (4.112)$$

with

$$W(q) = \begin{bmatrix} W_u(q) & W_y(q) \end{bmatrix} \quad z(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \quad (4.113)$$

$$W_u(q) = H^{-1}(q)G(q), \quad W_y(q) = [1 - H^{-1}(q)] \quad (4.114)$$

Clearly, (4.114) defines a one-to-one relationship between $T(q)$ and $W(q)$:

$$T(q) \leftrightarrow W(q) \quad (4.115)$$

Remark. Based on (4.110), we may prefer to work with the k -step-ahead predictor (3.31). To keep the link (4.115), we can view (3.31) as the one-step-ahead predictor for the model (3.32).

Models

We noted already in (4.1) that a model of a linear system consists of specified transfer functions $G(z)$ and $H(z)$, possibly complemented with a specification of the prediction error variance λ , or the PDF $f_e(x)$ of the prediction error e . In Sections 3.2 and 3.3, we made the point that what matters in the end is by which expression future outputs are predicted. The one-step-ahead predictor based on the model (4.1) is given by (4.112).

While the predictor (4.112) via (4.115) is in a one-to-one relationship with (4.110), it is useful to relax the link (4.115) and regard (4.112) as the basic model. This will, among other things, allow a direct extension to nonlinear and time-varying models, as shown in Section 5.7. We may thus formally define what we mean by a model:

Definition 4.1. A *predictor model* of a linear, time-invariant system is a stable filter $W(q)$, defining a predictor (4.112) as in (4.113).

Stability, which was defined in (2.27) (applying to both components of $W(q)$) is necessary to make the right side of (4.112) well defined. While predictor models are meaningful also in a deterministic framework without a stochastic alibi, as discussed in Section 3.3, it is useful also to consider models that specify properties of the associated prediction errors (innovations).

Definition 4.2. A *complete probabilistic model* of a linear, time-invariant system is a pair $(W(q), f_e(x))$ of a predictor model $W(q)$ and the PDF $f_e(x)$ of the associated prediction errors.

Clearly, we can also have models where the PDFs are only partially specified (e.g., by the variance of e).

In this section we shall henceforth only deal with predictor models and therefore drop this adjective. The concepts for probabilistic models are quite analogous.

We shall say that two models $W_1(q)$ and $W_2(q)$ are *equal* if

$$W_1(e^{i\omega}) = W_2(e^{i\omega}). \quad \text{almost all } \omega \quad (4.116)$$

A model

$$W(q) = [W_u(q) \quad W_y(q)]$$

will be called a *k -step-ahead predictor model* if

$$W_y(q) = \sum_{\ell=k}^{\infty} w_y(\ell)q^{-\ell}. \quad \text{with } w_y(k) \neq 0 \quad (4.117)$$

and an *output error model* (or a *simulation model*) if $W_y(q) \equiv 0$.

Note that the definition requires the predictors to be stable. This does not necessarily mean that the system dynamics is stable.

Example 4.4 Unstable System

Suppose that

$$G(q) = \frac{bq^{-1}}{1 + aq^{-1}}, \quad \text{with } |a| > 1$$

and

$$H(q) = \frac{1}{1 + aq^{-1}}$$

This means that the model is described by

$$y(t) + ay(t - 1) = bu(t - 1) + e(t)$$

and the dynamics from u to y is unstable. The predictor functions are, however:

$$W_y(q) = -aq^{-1}, \quad W_u(q) = bq^{-1}$$

implying that

$$\hat{y}(t|t - 1) = -ay(t - 1) + bu(t - 1)$$

which clearly satisfies the condition of Definition 4.1. □

Model Sets

Definition 4.1 describes one given model of a linear system. The identification problem is to determine such a model. The search for a suitable model will typically be conducted over a set of candidate models. Quite naturally, we define a *model set* \mathcal{M}^* as

$$\mathcal{M}^* = \{W_\alpha(q) | \alpha \in \mathcal{A}\} \tag{4.118}$$

This is just a collection of models, each subject to Definition 4.1, here “enumerated” with an index α covering an index set \mathcal{A} .

Typical model sets could be

$$\mathcal{M}^* = \mathcal{L}^* = \{\text{all linear models}\}$$

that is, all models that are subject to Definition 4.1, or

$$\mathcal{M}_n^* = \{\text{all models such that } W_y(q) \text{ and } W_u(q) \text{ are polynomials of } q^{-1} \text{ of degree at most } n\} \tag{4.119}$$

or a finite model set

$$\mathcal{M}^* = \{W_1(q), W_2(q), W_3(q)\} \tag{4.120}$$

We say that *two model sets are equal*, $\mathcal{M}_1^* = \mathcal{M}_2^*$, if for any W_1 in \mathcal{M}_1^* there exists a W_2 in \mathcal{M}_2^* such that $W_1 = W_2$ [defined by (4.116)], and vice versa.

Model Structures: Parametrization of Model Sets

Most often a model set of interest is noncountable. Since we have to conduct a search over it for “the best model,” it is then interesting how the indexation is chosen. The basic idea is to parametrize (index) the set “smoothly” over a “nice” area and perform the search over the parameter set (the index set). To put this formally, we let the model be indexed by a d -dimensional vector θ :

$$W(q, \theta)$$

To formalize “smoothly,” we require that for any given z , $|z| \geq 1$, the complex-valued function $W(z, \theta)$ of θ be differentiable:

$$\Psi(z, \theta) = \frac{d}{d\theta} W(z, \theta) \quad (4.121a)$$

Here

$$\begin{aligned} \Psi(z, \theta) &= \begin{bmatrix} \Psi_u(z, \theta) & \Psi_y(z, \theta) \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{d\theta} W_u(z, \theta) & \frac{d}{d\theta} W_y(z, \theta) \end{bmatrix} \end{aligned} \quad (4.121b)$$

is a $d \times 2$ matrix. Thus the gradient of the prediction $\hat{y}(t|\theta)$ is given by

$$\psi(t, \theta) = \frac{d}{d\theta} \hat{y}(t|\theta) = \Psi(q, \theta) z(t) \quad (4.121c)$$

Since the filters Ψ will have to be computed and used when the search is carried out, we also require them to be stable. We thus have the following definition:

Definition 4.3. A *model structure* \mathcal{M} is a differentiable mapping from a connected, open subset $D_{\mathcal{M}}$ of \mathbf{R}^d to a model set \mathcal{M}^* , such that the gradients of the predictor functions are stable.

To put this definition in mathematical notation we have

$$\mathcal{M} : D_{\mathcal{M}} \ni \theta \rightarrow \mathcal{M}(\theta) = W(q, \theta) \in \mathcal{M}^* \quad (4.122)$$

such that the filter Ψ in (4.121) exists and is stable for $\theta \in D_{\mathcal{M}}$. We will thus use $\mathcal{M}(\theta)$ to denote the particular model corresponding to θ and reserve \mathcal{M} for the mapping itself.

Remark. The requirement that $D_{\mathcal{M}}$ should be open is in order for the derivatives in (4.121) to be unambiguously well defined. When using model structures, we may prefer to work with compact sets $D_{\mathcal{M}}$. Clearly, as long as $D_{\mathcal{M}}$ is contained in an open set where (4.121) are defined, no problems will occur. Differentiability can also be defined over more complicated subsets of \mathbf{R}^d than open ones, that is, *differentiable manifolds* (see, e.g., Boothby, 1975). See the chapter bibliography for further comments.

Example 4.5 An ARX Structure

Consider the ARX model

$$y(t) + ay(t-1) = b_1u(t-1) + b_2u(t-2) + e(t)$$

The predictor is given by (4.10), which means that

$$W(q, \theta) = [b_1q^{-1} + b_2q^{-2} \quad -aq^{-1}], \quad \theta = [a \quad b_1 \quad b_2]^T$$

and

$$\Psi(q, \theta) = \begin{bmatrix} 0 & -q^{-1} \\ q^{-1} & 0 \\ q^{-2} & 0 \end{bmatrix}$$

□

The parametrized model sets that we have explicitly studied in this chapter have been in terms of (4.4), that is,

$$y(t) = G(q, \theta)u(t) + H(q, \theta)e(t), \quad \theta \in D_M \quad (4.123)$$

or using (4.111)

$$y(t) = T(q, \theta)\chi(t)$$

It is immediate to verify that, in view of (4.114),

$$\Psi(q, \theta) = \frac{1}{(H(q, \theta))^2} T'(q, \theta) \begin{bmatrix} H(q, \theta) & 0 \\ -G(q, \theta) & 1 \end{bmatrix} \quad (4.124)$$

where $T'(q, \theta)$ is the $d \times 2$ matrix

$$T'(q, \theta) = \frac{d}{d\theta} T(q, \theta) = \left[\frac{d}{d\theta} G(q, \theta) \quad \frac{d}{d\theta} H(q, \theta) \right] \quad (4.125)$$

Differentiability of W is thus assured by differentiability of T .

It should be clear that all parametrizations we have considered in this chapter indeed are model structures in the sense of Definition 4.3. We have, for example:

Lemma 4.1. The parametrization (4.35) together with (4.41) with θ confined to $D_M = \{\theta | F(z) \cdot C(z) \text{ has no zeros on or outside the unit circle}\}$ is a model structure.

Proof. We need only verify that the gradients of

$$W_u(z, \theta) = \frac{B(z)D(z)}{C(z)F(z)}$$

and

$$W_y(z, \theta) = 1 - \frac{D(z)A(z)}{C(z)}$$

with respect to θ are analytical in $|z| \geq 1$ for $\theta \in D_{\mathcal{M}}$. But this is immediate since, for example

$$\frac{\partial}{\partial c_k} W_u(z, \theta) = -\frac{B(z)D(z)z^{-k}}{[C(z)]^2 F(z)}$$
□

Lemma 4.2. Consider the state-space parametrization (4.91). Assume that the entries of the matrices $A(\theta)$, $B(\theta)$, $K(\theta)$, and $C(\theta)$ are differentiable with respect to θ . Suppose that $\theta \in D_{\mathcal{M}}$, with

$$D_{\mathcal{M}} = \{\theta | \text{all eigenvalues of } A(\theta) - K(\theta)C(\theta) \text{ are inside the unit circle}\}$$

Then the parametrization of the corresponding predictor is a model structure.

Proof. See Problem 4D.1. □

Notice that when $K(\theta)$ is obtained as the solution of (4.87), then by a standard Kalman filter property (see Anderson and Moore, 1979),

$$D_{\mathcal{M}} = \{\theta | [A(\theta), R_1(\theta)] \text{ stabilizable and } [A(\theta), C(\theta)] \text{ detectable}\} \quad (4.126)$$

When relating different model structures, we shall use the following concept.

Definition 4.4. A model structure \mathcal{M}_1 is said to be *contained in* \mathcal{M}_2 ,

$$\mathcal{M}_1 \subset \mathcal{M}_2 \quad (4.127)$$

if $D_{\mathcal{M}_1} \subset D_{\mathcal{M}_2}$ and the mapping \mathcal{M}_1 is obtained by restricting \mathcal{M}_2 to $\theta \in D_{\mathcal{M}_1}$. The archetypical situation for (4.127) is when \mathcal{M}_2 defines n th-order models and \mathcal{M}_1 defines m th-order models, $m < n$. One could think of \mathcal{M}_1 as obtained from \mathcal{M}_2 by fixing some parameters (typically to zero).

The following property of a model structure is sometimes useful:

Definition 4.5. A model structure \mathcal{M} is said to have an *independently parametrized transfer function and noise model* if

$$\theta = \begin{bmatrix} \rho \\ \eta \end{bmatrix}, \quad D_{\mathcal{M}} = D_{\rho} \times D_{\eta}, \quad \rho \in D_{\rho}, \quad \eta \in D_{\eta} \quad (4.128)$$

$$T(q, \theta) = \begin{bmatrix} G(q, \rho) & H(q, \eta) \end{bmatrix}$$

We note that in the family (4.33) the special cases with $A(q) \equiv 1$ correspond to independent parametrizations of G and H .

Remark On “Finite Model Structures”: Sometimes the set of candidate models is finite as in (4.120). It may still be desirable to index it using a parameter vector θ , now ranging over a finite set of points. Although such a construction does not qualify as a “model structure” according to Definition 4.3, it should be noted that the estimation procedures of Sections 7.1 to 7.4, as well as the convergence analysis of Sections 8.1 to 8.5, still make sense in this case.

Model Set as a Range of a Model Structure

A model structure will clearly define a model set by its range:

$$\mathcal{M}^* = \mathcal{R}(\mathcal{M}) = \text{Range } \mathcal{M} = \{\mathcal{M}(\theta) | \theta \in D_{\mathcal{M}}\}$$

An important problem for system identification is to find a model structure whose range equals a given model set. This may sometimes be an easy problem and sometimes highly nontrivial.

Example 4.6 Parametrizing \mathcal{M}_3^*

Consider the set \mathcal{M}_n^* defined by (4.119) with $n = 3$. If we take

$$\theta = [a_1 \ a_2 \ a_3 \ b_1 \ b_2 \ b_3]^T, \quad d = 6$$

$$D_{\mathcal{M}} = \mathbf{R}^6$$

and

$$W_y(q, \theta) = -a_1q^{-1} - a_2q^{-2} - a_3q^{-3}$$

$$W_u(q, \theta) = b_1q^{-1} + b_2q^{-2} + b_3q^{-3}$$

we have obviously constructed a model structure whose range equals \mathcal{M}_3^* \square

A given model set can typically be described as the range of several different model structures (see Problems 4E.6 and 4E.9).

Model Set as a Union of Ranges of Model Structures

In the preceding example it was possible to describe the desired model set as the range of a model structure. We shall later encounter model sets for which this is not possible, at least not with model structures with desired identifiability properties. The remedy for these problems is to describe the model set as a union of ranges of different model structures:

$$\mathcal{M}^* = \bigcup_{i=1}^{\beta} \mathcal{R}(\mathcal{M}_i) \tag{4.129}$$

This idea has been pursued in particular for representing linear multioutput systems. We shall give the details of this procedure in Appendix 4A. Let us here only remark that model sets described by (4.129) are useful also for working with models of different orders, and that they are often used, at least implicitly, when the order of a suitable model is unknown and is to be determined.

Identifiability Properties

Identifiability is a concept that is central in identification problems. Loosely speaking, the problem is whether the identification procedure will yield a unique value of the parameter θ , and/or whether the resulting model is equal to the true system. We shall deal with the subject in more detail in the analysis chapter (see Sections 8.2 and 8.3). The issue involves aspects on whether the data set (the experimental conditions) is informative enough to distinguish between different models as well as properties of the model structure itself: If the data are informative enough to distinguish between nonequal models, then the question is whether different values of θ can give equal models. With our terminology, the latter problem concerns the *invertibility of the model structure \mathcal{M}* (i.e., whether \mathcal{M} is injective). We shall now discuss some concepts related to such invertibility properties. Remember that these are only one leg of the identifiability concept. They are to be complemented in Sections 8.2 and 8.3.

Definition 4.6. A model structure \mathcal{M} is *globally identifiable at θ^** if

$$\mathcal{M}(\theta) = \mathcal{M}(\theta^*), \quad \theta \in D_{\mathcal{M}} \Rightarrow \theta = \theta^* \quad (4.130)$$

Recall that model equality was defined in (4.116), requiring the predictor transfer functions to coincide. According to (4.115), this means that the underlying transfer functions G and H coincide.

Once identifiability at a point is defined, we proceed to properties of the whole set.

Definition 4.7. A model structure \mathcal{M} is *strictly globally identifiable* if it is globally identifiable at all $\theta^* \in D_{\mathcal{M}}$.

This definition is quite demanding. As we shall see, it is difficult to construct model structures that are strictly globally identifiable. The difficulty for linear systems, for example, is that global identifiability may be lost at points on hyper-surfaces corresponding to lower-order systems. Therefore, we introduce a weaker and more realistic property:

Definition 4.8. A model structure \mathcal{M} is *globally identifiable* if it is globally identifiable at almost all $\theta^* \in D_{\mathcal{M}}$.

Remark. This means that \mathcal{M} is globally identifiable at all $\theta^* \in \tilde{D}_{\mathcal{M}} \subset D_{\mathcal{M}}$, where

$$\delta D_{\mathcal{M}} = \left\{ \theta \mid \theta \in D_{\mathcal{M}}, \theta \notin \tilde{D}_{\mathcal{M}} \right\}$$

is a set of Lebesgue measure zero in \mathbf{R}^d (recall that $D_{\mathcal{M}}$ and hence $\delta D_{\mathcal{M}}$ is a subset of \mathbf{R}^d).

For corresponding local properties, the most natural definition of local identifiability of \mathcal{M} at θ^* would be to require that there exists an ε such that

$$\mathcal{M}(\theta) = \mathcal{M}(\theta^*), \quad \theta \in \mathcal{B}(\theta^*, \varepsilon) \Rightarrow \theta = \theta^* \quad (4.131)$$

where $\mathcal{B}(\theta^*, \varepsilon)$ denotes an ε -neighborhood of θ^* .

(Strict) local identifiability of a model structure can then be defined analogously to Definitions 4.7 and 4.8. See also Problem 4G.4.

Use of the Identifiability Concept

The identifiability concept concerns the unique representation of a given system description in a model structure. Let

$$S : y(t) = G_0(q)u(t) + H_0(q)e(t) \quad (4.132)$$

be such a description. We could think of it as a “true” or “ideal” description of the actual system, but such an interpretation is immaterial for the moment. Let \mathcal{M} be a model structure based on one-step-ahead predictors for

$$y(t) = G(q, \theta)u(t) + H(q, \theta)e(t) \quad (4.133)$$

Then define the set $D_T(S, \mathcal{M})$ as those θ -values in $D_{\mathcal{M}}$ for which $S = \mathcal{M}(\theta)$. We can write this as

$$D_T(S, \mathcal{M}) = \left\{ \theta \in D_{\mathcal{M}} \mid G_0(z) = G(z, \theta), H_0(z) = H(z, \theta) \text{ almost all } z \right\} \quad (4.134)$$

This set is empty in case $S \notin \mathcal{M}$. (Here, with abuse of notation, \mathcal{M} also denotes the range of the mapping \mathcal{M} .)

Now suppose that $S \in \mathcal{M}$ so that $S = \mathcal{M}(\theta_0)$ for some value θ_0 . Furthermore, suppose that \mathcal{M} is globally identifiable at θ_0 . Then

$$D_T(S, \mathcal{M}) = \{\theta_0\} \quad (4.135)$$

One aspect of the choice of a good model structure is to select \mathcal{M} so that (4.135) holds for the given description S . Since S is unknown to the user, this will typically involve tests of several different structures \mathcal{M} . The identifiability concepts will then provide useful guidance in finding an \mathcal{M} such that (4.135) holds.

4.6 IDENTIFIABILITY OF SOME MODEL STRUCTURES

Definition 4.6 and (4.116) together imply that a model structure is globally identifiable at θ^* if and only if

$$\begin{aligned} G(z, \theta) &\equiv G(z, \theta^*) \text{ and } H(z, \theta) \equiv H(z, \theta^*) \\ \text{for almost all } z \Rightarrow \theta &= \theta^* \end{aligned} \quad (4.136)$$

For local identifiability, we consider only θ confined to a sufficiently small neighborhood of θ^* . A general approach to test local identifiability is given by the criterion in Problem 4G.4.

Global identifiability is more difficult to deal with in general terms. In this section we shall only briefly discuss identifiability of physical parameters and give some results for general black-box SISO models. Black-box multivariable systems are dealt with in Appendix 4A.

Parametrizations in Terms of Physical Parameters

Modeling physical processes typically leads to a continuous-time state-space model (4.62) to (4.63), summarized as (4.65) ($T = 1$):

$$y(t) = G_c(p, \theta)u(t) + v(t) \quad (4.137)$$

For proper handling we should sample G_c , and include a noise model H so that (4.136) can be applied for identifiability tests. A simpler test to apply is

$$G_c(s, \theta) = G_c(s, \theta^*) \text{ almost all } s \Rightarrow \theta = \theta^*? \quad (4.138)$$

It is true that this is not identical to (4.136): When sampling G_c , ambiguities may occur; two different G_c can give the same G_T [cf. (2.24)]. Equation (4.138) is thus not sufficient for (4.136) to hold. However, with a carefully selected sampling interval, this ambiguity should not cause any problems. Also, a θ -parametrized noise model may help in resolving (4.138). This condition is thus not necessary for (4.136) to hold. However, in most applications the noise characteristics are not so significant that they indeed bear information about the physical parameters. All this means that (4.138) is a reasonable test for global identifiability of the corresponding model structure at θ^* .

Now, (4.138) is a difficult enough problem. Except for special structures there are no general techniques available other than brute-force solution of the equations underlying (4.138). See Problems 4E.5 and 4E.6 for some examples. A comprehensive treatment of (4.138) for state-space models is given by Walter (1982), and Godfrey (1983) discusses the same problem for compartmental models. See also Godfrey and Distefano (1985). A general approach based on differential algebra is described in Ljung and Glad (1994b).

SISO Transfer-function Model Structures

We shall now aim at an analysis of the general black-box SISO model structure (4.33) together with (4.41). Let us first illustrate the character of the analysis with two simple special cases.

Consider the ARX model structure (4.7) together with (4.9):

$$\begin{aligned} G(z, \theta) &= \frac{B(z)}{A(z)}, & H(z, \theta) &= \frac{1}{A(z)} \\ \theta &= [a_1 \dots a_{n_a} b_1 \dots b_{n_b}]^T \end{aligned} \quad (4.139)$$

Equality for H in (4.136) implies that the A -polynomials coincide, which in turn implies that the B -polynomials must coincide for the G to be equal. It is thus immediate to verify that (4.136) holds for all θ^* in the model structure (4.139). Consequently, *the structure (4.139) is strictly globally identifiable*.

Let us now turn to the OE model structure (4.25) with orders n_b and n_f . At $\theta = \theta^*$ we have

$$\begin{aligned} G(z, \theta^*) &= \frac{B^*(z)}{F^*(z)} = \frac{b_1^* z^{-1} + \dots + b_{n_b}^* z^{-n_b}}{1 + f_1^* z^{-1} + \dots + f_{n_f}^* z^{-n_f}} \\ &= z^{n_f - n_b} \frac{b_1^* z^{n_b - 1} + \dots + b_{n_b}^*}{z^{n_f} + f_1^* z^{n_f - 1} + \dots + f_{n_f}^*} = z^{n_f - n_b} \frac{z^{n_b} B^*(z)}{z^{n_f} F^*(z)} \end{aligned} \quad (4.140)$$

We shall work with the polynomial $\tilde{F}^*(z) = z^{n_f} F^*(z)$ in the variable z , rather than with $F^*(z)$, which is a polynomial in z^{-1} . The reason is that $z^{n_f} F^*(z)$ always has degree n_f regardless of whether $f_{n_f}^*$ is zero. Let $\tilde{B}^*(z) = z^{n_b} B^*(z)$, and let θ be an arbitrary parameter value. We can then write (4.136),

$$G(z, \theta^*) = G(z, \theta) = \frac{B(z)}{F(z)} = z^{n_f - n_b} \frac{\tilde{B}(z)}{\tilde{F}(z)}$$

as

$$\tilde{F}(z) \tilde{B}^*(z) - \tilde{F}^*(z) \tilde{B}(z) \equiv 0 \quad (4.141)$$

Since $\tilde{F}^*(z)$ is a polynomial of degree n_f , it has n_f zeros:

$$\tilde{F}^*(\alpha_i) = 0, \quad i = 1, \dots, n_f$$

Suppose that $\tilde{B}^*(\alpha_i) \neq 0$, $i = 1, \dots, n_f$; that is, $\tilde{B}^*(z)$ and $\tilde{F}^*(z)$ are *coprime* (have no common factors). Then (4.141) implies that

$$\tilde{F}(\alpha_i) = 0, \quad i = 1, \dots, n_f$$

[if a zero α_i has multiplicity n_i , then differentiate (4.141) $n_i - 1$ times to conclude that it is a zero of the same multiplicity to $\tilde{F}(z)$]. Consequently, we have $\tilde{F}(z) \equiv \tilde{F}^*(z)$, which in turn implies that $\tilde{B}(z) = \tilde{B}^*(z)$ so that $\theta = \theta^*$. If, on the other hand, \tilde{F}^* and \tilde{B}^* do have a common factor so that

$$\tilde{F}^*(z) = \gamma(z)\tilde{F}_1^*(z), \quad \tilde{B}^*(z) = \gamma(z)\tilde{B}_1^*(z)$$

then all θ , such that

$$\tilde{F}(z) = \beta(z)\tilde{F}_1^*(z), \quad \tilde{B}(z) = \beta(z)\tilde{B}_1^*(z)$$

for arbitrary $\beta(z)$ will yield equality in (4.141). Hence the model structure is neither globally nor locally identifiable at θ^* [$\beta(z)$ can be chosen arbitrarily close to $\gamma(z)$]. We thus find that *the OE structure (4.25) is globally and locally identifiable at θ^* if and only if the corresponding numerator and denominator polynomials $z^{n_f} F^*(z)$ and $z^{n_b} B^*(z)$ are coprime.*

The generalization to the black-box SISO structure (4.33) is now straightforward:

Theorem 4.1. Consider the model structure \mathcal{M} corresponding to

$$A(q)y(t) = \frac{B(q)}{F(q)}u(t) + \frac{C(q)}{D(q)}e(t) \quad (4.142)$$

with θ , given by (4.41), being the coefficients of the polynomials involved. The degrees of the polynomials are n_a , n_b , and so on. This model structure is globally identifiable at θ^* if and only if all of (i) to (vi) hold:

- i. There is no common factor to all $z^{n_a} A^*(z)$, $z^{n_b} B^*(z)$, and $z^{n_c} C^*(z)$.
- ii. There is no common factor to $z^{n_b} B^*(z)$ and $z^{n_f} F^*(z)$.
- iii. There is no common factor to $z^{n_c} C^*(z)$ and $z^{n_d} D^*(z)$.
- iv. If $n_a \geq 1$, then there must be no common factor to $z^{n_f} F^*(z)$ and $z^{n_d} D^*(z)$.
- v. If $n_d \geq 1$, then there must be no common factor to $z^{n_a} A^*(z)$ and $z^{n_b} B^*(z)$.
- vi. If $n_f \geq 1$, then there must be no common factor to $z^{n_a} A^*(z)$ and $z^{n_c} C^*(z)$.

The starred polynomials correspond to θ^* .

Notice that several of the conditions (i) to (vi) will be automatically satisfied in the common special cases of (4.142). Notice also that any of the conditions (i) to (vi) can be violated only for “special” θ^* , placed on hyper-surfaces in \mathbf{R}^d . We thus have the following corollary:

Corollary. The model structure given by (4.142) is globally identifiable.

Looking for a “True” System Within Identifiable Structures

We shall now illustrate the usefulness of Theorem 4.1 by applying it to the problem of finding an \mathcal{M} such that (4.135) holds for a given S . Suppose that S is given by

$$S : G_0(q) = \frac{B_0(q)}{A_0(q)F_0(q)}, \quad H_0(q) = \frac{C_0(q)}{A_0(q)D_0(q)} \quad (4.143)$$

with orders n_a^0, n_b^0 , and so on (after all possible cancellations of common factors). This system belongs to the model structure \mathcal{M} in (4.142) provided all the model orders are at least as large as the true ones:

$$n_a \geq n_a^0, \quad n_b \geq n_b^0, \text{ etc.} \quad (4.144)$$

When (4.144) holds, let θ_0 be a value that gives the description (4.143):

$$S = \mathcal{M}(\theta_0) \quad (4.145)$$

Now, clearly, \mathcal{M} will be globally identifiable at θ_0 and (4.135) will hold if we have equality in all of (4.144). The true orders n_a^0, \dots are, however, typically not known, and it would be quite laborious to search for all combinations of model orders until equalities in (4.144) were obtained. The point of Theorem 4.1 is that such a search is not necessary; the structure \mathcal{M} is globally identifiable at θ_0 under weaker conditions.

We have the following reformulation of Theorem 4.1:

Theorem 4.2. Consider the system description S in (4.143) with true polynomial orders n_a^0, n_b^0 , and so on, as defined in the text. Consider model structure \mathcal{M} of Theorem 4.1. Then $S \in \mathcal{M}$ and corresponds to a globally identifiable θ -value if and only if

- i. $\min(n_a - n_a^0, n_b - n_b^0, n_c - n_c^0) = 0$.
- ii. $\min(n_b - n_b^0, n_f - n_f^0) = 0$.
- iii. $\min(n_c - n_c^0, n_d - n_d^0) = 0$.
- iv. If $n_a \geq 1$, then also $\min(n_f - n_f^0, n_d - n_d^0) = 0$.
- v. If $n_d \geq 1$, then also $\min(n_a - n_a^0, n_b - n_b^0) = 0$.
- vi. If $n_f \geq 1$, then also $\min(n_a - n_a^0, n_c - n_c^0) = 0$.

With Theorem 4.2, the search for a true system within identifiable model structures is simplified. If, for example, S can be described in ARMAX form with finite orders n_a^0, n_b^0 and n_c^0 , then we may take $n_a = n_b = n_c = n$ ($n_f = n_d = 0$) in \mathcal{M} , giving a model structure, say, \mathcal{M}_n . By increasing n one unit at a time, we will sooner or later strike a structure where (i) holds and thus S can be uniquely represented.

SISO State-space Models

Consider now a state-space model structure (4.91). It is quite clear that the matrices $A(\theta)$, $B(\theta)$, $C(\theta)$, and $K(\theta)$ cannot be “filled” with parameters, since the corresponding input-output description (4.92) is defined by $3n$ parameters only ($n = \dim x$). To obtain identifiable structures, it is thus natural to seek parametrizations of the matrices that involve $3n$ parameters; the coefficients of the two $(n - 1)$ th order numerator polynomials and the coefficients of the common, monic n th order denominator polynomial or some transformation of these coefficients. One such parametrization is the observer canonical form of Example 4.2, which we can write in symbolic form as

$$\begin{aligned} x(t + 1, \theta) &= A(\theta)x(t, \theta) + B(\theta)u(t) + K(\theta)e(t) \\ y(t) &= C(\theta)x(t, \theta) + e(t) \end{aligned} \quad (4.146a)$$

$$A(\theta) = \left[\begin{array}{c|cc} \times & & \\ \times & & I_{n-1} \\ \vdots & \hline & \\ \times & & 0 \dots 0 \end{array} \right], \quad B(\theta) = \begin{bmatrix} \times \\ \times \\ \vdots \\ \times \end{bmatrix}, \quad K(\theta) = \begin{bmatrix} \times \\ \times \\ \vdots \\ \times \end{bmatrix} \quad (4.146b)$$

$$C(\theta) = [1 \ 0 \dots 0]$$

Here I_{n-1} is the $(n - 1) \times (n - 1)$ unit matrix, while \times marks an adjustable parameter. This representation is observable by construction.

According to Example 4.2, this structure is in one-to-one correspondence with an ARMAX structure with $n_a = n_b = n_c = n$. From Theorem 4.1 we know that this is identifiable at θ^* , provided the corresponding polynomials do not all have a common factor, meaning that the model could be represented using a smaller value of n . It is well known that for state-space models this can only happen if the model is uncontrollable and/or unobservable. Since (4.146) is observable by construction, we thus conclude that *this structure is globally and locally identifiable at θ^* if and only if the two-input system $\{A(\theta^*), [B(\theta^*) \ K(\theta^*)]\}$ is controllable*. Note that this result applies to the particular state-space structure (4.146) only.

4.7 SUMMARY

In this chapter we have studied sets of predictors of the type

$$\hat{y}(t|\theta) = W_u(q, \theta)u(t) + W_y(q, \theta)y(t), \quad \theta \in D_M \subset \mathbf{R}^d \quad (4.147)$$

These are in one-to-one correspondence with model descriptions

$$y(t) = G(q, \theta)u(t) + H(q, \theta)e(t), \quad \theta \in D_M \quad (4.148)$$

with $\{e(t)\}$ as white noise, via

$$\begin{aligned} W_u(q, \theta) &= H^{-1}(q, \theta)G(q, \theta) \\ W_y(q, \theta) &= [1 - H^{-1}(q, \theta)] \end{aligned}$$

When choosing models it is usually most convenient to go via (4.148), even if (4.147) is the “operational” version.

We have denoted parametrized model sets, or *model structures* by \mathcal{M} , while a particular model corresponding to the parameter value θ is denoted by $\mathcal{M}(\theta)$. Such a parametrization is instrumental in conducting a search for “best models.” Two different philosophies may guide the choice of parametrized model sets:

1. *Black-box model structures*: The prime idea is to obtain flexible model sets that can accommodate a variety of systems, without looking into their internal structures. The input-output model structures of Section 4.2, as well as canonically parametrized state-space models (see Example 4.2), are of this character.
2. *Model structures with physical parameters*: The idea is to incorporate physical insight into the model set so as to bring the number of adjustable parameters down to what is actually unknown about the system. Continuous-time state-space models are typical representatives for this approach.

We have also in this chapter introduced formal requirements on the predictor filters $W_u(q, \theta)$ and $W_y(q, \theta)$ (Definition 4.3) and discussed concepts of parameter identifiability (i.e., whether the parameter θ can be uniquely determined from the predictor filters). These properties were investigated for the most typical black-box model structures in Section 4.6 and Appendix 4A. The bottom line of these results is that identifiability can be secured, provided certain orders are chosen properly. The number of such orders to be chosen typically equals the number of outputs.

4.8 BIBLIOGRAPHY

The selection of a parameterized set of models is, as we have noted, vital for the identification problem. This is the link between system identification and parameter estimation techniques. Most articles and books on system identification thus contain material on model structures, even if not presented in as explicit terms as here.

The simple equation error model (4.7) has been widely studied in many contexts. See, for example, Åström (1968), Hsia (1977), Mendel (1973), and Unbehauen, Göhring, and Bauer (1974) for discussions related to identification. Linear models like (4.12) are prime objects of study in statistics; see, for example, Rao (1973) or Draper and Smith (1981). The ARMAX model was introduced into system identification in Åström and Bohlin (1965) and is since then a basic model. The ARARX model structure was introduced into the control literature by Clarke (1967), but was apparently first used in a statistical framework by Cochrane and Orcutt (1949). The term pseudo-linear regression for the representation (4.21) was introduced by

Solo (1978). Output error models are treated, for example, in Dugard and Landau (1980) and Kabaila and Goodwin (1980). The general family (4.33) was first discussed in Ljung (1979). It was used in Ljung and Söderström (1983). Multivariable MFDs are discussed in Kailath (1980). When no input is present, the corresponding model structures reduce to AR, MA, and ARMA descriptions. These are discussed in many textbooks on time series, e.g., Box and Jenkins (1970); Hannan (1970); and Brillinger (1981).

Black-box continuous transfer function models of the type (4.50) have been used in many cases oriented toward control applications. Ziegler and Nichols (1942) determine parameters in such models from step responses and self-oscillatory modes (see Section 6.1).

State-space models in innovations forms as well as the general forms are treated in standard textbooks on control (e.g., Åström and Wittenmark, 1984). The use of continuous-time representations for estimation using discrete data has been discussed, for example, in Mehra and Tyler (1973) and Åström and Källström (1976). The continuous-time model structure is usually arrived at after an initial modeling step. See, for example, Wellstead (1979), Nicholson (1981), Ljung and Glad (1994a) and Cellier (1990) for general modeling techniques and examples. Direct identification of continuous-time systems is discussed in Unbehauen and Rao (1987).

Distributed parameter models and their estimation are treated in, for example, Banks, Crowley, and Kunisch (1983), Kubrusly (1977), Qureshi, Ng, and Goodwin (1980) and Polis and Goodson (1976). Example 4.3 is studied experimentally in Leden, Hamza, and Sheirah (1976).

The prediction aspect of models was emphasized in Ljung (1974) and Ljung (1978). Identifiability is discussed in many contexts. A survey is given in Nguyen and Wood (1982). Often identifiability is related to convergence of the parameter estimates. Such definitions are given in Åström and Bohlin (1965), Staley and Yue (1970), and Tse and Anton (1972). Identifiability definitions in terms of the model structure only was introduced by Bellman and Åström (1970), who called it "structural identifiability." Identifiability definitions in terms of the set $D_T(S, \mathcal{M})$ [defined by (4.134)] were given in Gustavsson, Ljung, and Söderström (1977). The particular definitions of the concept of model structure and identifiability given in Section 4.5 are novel. In Ljung and Glad (1994b) identifiability is treated from an algebraic perspective. It is shown that any globally identifiable structure can be rearranged as a linear regression.

A more general model structure concept than Definition 4.3 would be to let $D_{\mathcal{M}}$ be a differentiable manifold (see, e.g., Byrnes, 1976). However, in our treatment that possibility is captured by letting a model set be described as a union of (overlapping) ranges of model structures as in (4.129). This manifold structure for linear systems was first described by Kalman (1974), Hazewinkel and Kalman (1976) and Clark (1976).

The identifiability of multivariable model structures has been dealt with in numerous articles. See, for example, Kailath (1980), Luenberger (1967), Glover and Willems (1974), Rissanen (1974), Ljung and Rissanen (1976), Guidorzi (1981), Gevers and Wertz (1984), Van Overbeek and Ljung (1982), and Correa and Glover (1984).

In addition to the parameterizations described in the appendix, approaches based on balanced realizations are described in Maciejowski (1985), Ober (1987), and Hanzon and Ober (1997). Parameterizations that are not identifiable, but may still have numerical advantages, are discussed by McKelvey (1994) and McKelvey and Helmersson (1996).

4.9 PROBLEMS

- 4G.1** Consider the predictor (4.18). Show that the effect from an erroneous initial condition in $\hat{y}(s|\theta)$, $s \leq 0$, is bounded by $c \cdot \mu^t$, where μ is the maximum magnitude of the zeros of $C(z)$.
- 4G.2** *Colored measurement noise:* Suppose that a state-space representation is given as

$$\begin{aligned} x(t+1) &= A_1(\theta)x(t) + B_1(\theta)u(t) + w_1(t) \\ y(t) &= C_1(\theta)x(t) + v(t) \end{aligned} \quad (4.149)$$

where $\{w_1(t)\}$ is white with variance $\bar{R}_1(\theta)$, but the measurement noise $\{v(t)\}$ is not white. A model for $v(t)$ can, however, be given as

$$v(t) = H(q, \theta)v(t) \quad (4.150)$$

with $\{v(t)\}$ being white noise with variance $R_2(\theta)$ and $H(q, \theta)$ monic. Introduce a state-space representation for (4.150):

$$\begin{aligned} \xi(t+1) &= A_2(\theta)\xi(t) + K(\theta)v(t) \\ v(t) &= C_2(\theta)\xi(t) + v(t) \end{aligned} \quad (4.151)$$

Combine (4.149) and (4.150) into a single representation that complies with the structure (4.84) to (4.85). Determine $R_1(\theta)$, $R_{12}(\theta)$, and $R_2(\theta)$. Note that if $w_1(t)$ is zero then the new representation will be directly in the innovations form (4.91).

- 4G.3** *Verification of the Steady-State Kalman Filter:* The state-space model (4.84) can be written (suppressing the argument θ and assuming $\dim y = 1$)

$$y(t) = G(q)u(t) + v_1(t)$$

where

$$G(q) = C(qI - A)^{-1}B$$

$$v_1(t) = C(qI - A)^{-1}w(t) + v(t)$$

Let $R_{12} = 0$. The spectrum of $\{v_1(t)\}$ then is

$$\Phi_1(\omega) = C(e^{i\omega} \cdot I - A)^{-1}R_1(e^{-i\omega} \cdot I - A^T)^{-1}C^T + R_2$$

using Theorem 2.2. The innovations model (4.91) can be written

$$y(t) = G(q)u(t) + v_2(t)$$

$$v_2(t) = H(q)e(t), \quad H(q) = C(qI - A)^{-1}K + 1$$

The spectrum of $\{v_2(t)\}$ thus is

$$\Phi_2(\omega) = \lambda [C(e^{i\omega} \cdot I - A)^{-1}K + 1] [C(e^{-i\omega} \cdot I - A)^{-1}K + 1]^T$$

where λ is the variance of $e(t)$.

(a) Show by direct calculation that

$$\Phi_1(\omega) = \Phi_2(\omega) \equiv 0$$

utilizing the expressions (4.87) and (4.91b). The two representations thus have the same second-order properties, and if the noises are Gaussian, they are indistinguishable in practice (see Problem 2E.3).

(b) Show by direct calculation that

$$\begin{aligned} 1 - H^{-1}(q) &= 1 - [1 + C(qI - A)^{-1}K]^{-1} \\ &= C(qI - A + KC)^{-1}K \end{aligned}$$

and

$$\begin{aligned} H^{-1}(q)G(q) &= [1 + C(qI - A)^{-1}K]^{-1}C(qI - A)^{-1}B \\ &= C(qI - A + KC)^{-1}B \end{aligned}$$

(c) Note that the predictor (4.86) can be written as (4.88):

$$\hat{y}(t|\theta) = C(qI - A + KC)^{-1}Bu(t) + C(qI - A + KC)^{-1}Ky(t)$$

and thus that (a) and (b) together with (3.20) constitute a derivation of the steady-state Kalman filter.

4G.4 Consider a model structure \mathcal{M} , with predictor function gradient $\Psi(z, \theta)$ defined in (4.121). Define the $d \times d$ matrix

$$\Gamma_1(\theta) = \int_{-\pi}^{\pi} \Psi(e^{i\omega}, \theta) \Psi^T(e^{-i\omega}, \theta) d\omega$$

(a) Show that \mathcal{M} is locally identifiable at θ if $\Gamma_1(\theta)$ is nonsingular.

(b) Let $\mathbf{T}'(z, \theta)$ be defined by (4.125), and let

$$\Gamma_2(\theta) = \int_{-\pi}^{\pi} \mathbf{T}'(e^{i\omega}, \theta) [\mathbf{T}'(e^{-i\omega}, \theta)]^T d\omega$$

Use (4.124) to show that $\Gamma_2(\theta)$ is nonsingular if and only if $\Gamma_1(\theta)$ is. [Note that by assumption $H(q)$ has no zeros on the unit circle.] $\Gamma_2(\theta)$ can thus be used to test local identifiability.

4G.5 Consider an output error structure with several inputs

$$y(t) = \frac{B_1(q)}{F(q)}u_1(t) + \cdots + \frac{B_m(q)}{F(q)}u_m(t) + e(t)$$

Show that this structure is globally identifiable at a value θ^* if and only if there is no common factor to all of the $m+1$ polynomials

$$z^{n_f} F^*(z), \quad z^{n_b} B_i^*(z), \quad i = 1, \dots, m$$

$$n_f = \text{degree } F^*(z), \quad n_b = \max \text{ degree } B_i^*(z)$$

θ^* here corresponds to the starred polynomials.

4G.6 The Kronecker product of an $m \times n$ matrix $A = (a_{ij})$ and a $p \times r$ matrix $B = (b_{ij})$ is defined as (see, e.g., Barnett, 1975)

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \dots a_{1n}B \\ a_{21}B & a_{22}B \dots a_{2n}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B \dots a_{mn}B \end{bmatrix}$$

This is an $mp \times nr$ matrix. Define the operator “col” as the operation to form a column vector out of a matrix by stacking its columns on top of each other:

$$\text{col } B = \begin{bmatrix} B^1 \\ B^2 \\ \vdots \\ B^r \end{bmatrix}, \quad (rp \times 1 \text{ vector})$$

where B^j is the j th column of B .

Consider (4.56) to (4.59). Show that (4.58) can be transformed into (4.59) with

$$\theta = \text{col } \theta^T$$

$$\varphi(t) = \varphi(t) \otimes I_p$$

where I_p is the $p \times p$ unit matrix. Are other variants of θ and φ also possible?

4G.7 Consider the continuous-time state-space model (4.96) to (4.97). Assume that the measurements are made in wideband noise with high variance, idealized as

$$\bar{y}(t) = Hx(t) + \bar{v}(t)$$

where $\bar{v}(t)$ is formal continuous-time white noise with covariance function

$$E\bar{v}(t)\bar{v}^T(s) = \bar{R}_2(\theta)\delta(t-s)$$

Assume that $\bar{v}(t)$ is independent of $\bar{w}(t)$. Let the output be defined as

$$y((k+1)T) = y_{k+1} = \frac{1}{T} \int_{t=kT}^{(k+1)T} \bar{y}(t) dt$$

Show that the sampled-data system can be represented as (4.98) and (4.99) but with

$$y(kT) = C_T(\theta)x(kT) + D_T(\theta)u(kT - T) + v_T(kT)$$

$$C_T(\theta) = \frac{1}{T}H\Phi_T(\theta)$$

$$Ew_T(kT)v_T^T(kT) = R_{12}(\theta) = \frac{1}{T} \int_0^T e^{F(\theta)\tau} \bar{R}_1(\theta)\Phi_{T-\tau}^T(\theta)H^T d\tau$$

$$Ev_T(kT)v_T^T(kT) = R_2(\theta) = \frac{1}{T}\bar{R}_2(\theta) + \frac{1}{T^2} \int_0^T H\Phi_{T-\tau}(\theta)\bar{R}_1(\theta)\Phi_{T-\tau}^T(\theta)H^T d\tau$$

$$\Phi_T(\theta) = \int_0^T e^{F(\theta)\tau} d\tau; \quad D_T(\theta) = -\frac{1}{T} \int_0^T H\Phi_\tau(\theta) d\tau$$

- 4G.8** Consider the ARX model (4.7). Introduce the δ -operator

$$\delta = 1 - q^{-1}$$

and reparametrize the models in terms of coefficients of powers of δ . Work out the details of a second-order example. Such a parametrization has the advantage of being less sensitive to numerical errors when the sampling interval is short. Middleton and Goodwin (1990).

- 4E.1** Consider the ARX model structure

$$\begin{aligned} y(t) + a_1 y(t-1) + \cdots + a_{n_a} y(t-n_a) \\ = b_1 u(t-1) + \cdots + b_{n_b} u(t-n_b) + e(t) \end{aligned}$$

where b_1 is known to be 0.5. Write the corresponding predictor in the linear regression form (4.13).

- 4E.2** Consider the continuous-time model (4.75) of the dc servo with $T_i(t) \equiv 0$. Apply the Euler approximation (2.25) to obtain an approximate discrete-time transfer function that is a simpler function of θ .
- 4E.3** Consider the small network of tanks in Figure 4.8. Each tank holds 10 volume units of fluid. Through the pipes A and E flows 1 volume unit per second, through the pipe B, α units, and through C and D, $1 - \alpha$ units per second. The concentration of a certain substance in the fluid is u in pipe A (the input) and y in pipe E (the output). Write down a structured state-space model for this system. Assume that each tank is perfectly mixed (i.e. the substance has the same concentration throughout the tank). (Models of this character are known as *compartmental models* and are very common in chemical and biological applications: see Godfrey, 1983.)

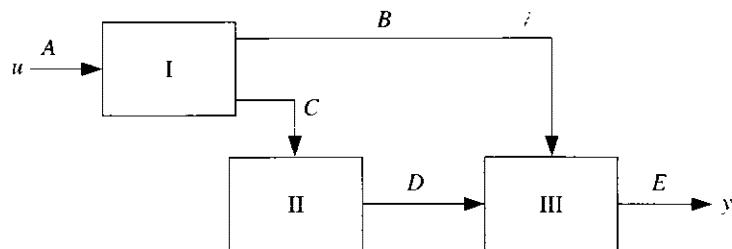


Figure 4.8 A network of tanks.

- 4E.4** Consider the RLC circuit in Figure 4.9 with ideal voltage source $u_1(t)$ and ideal current source $u_i(t)$. View this circuit as a linear time-invariant system with two inputs

$$u(t) = \begin{bmatrix} u_1(t) \\ u_i(t) \end{bmatrix}$$

and one output: the voltage $y(t)$. R , L , and C are unknown constants. Discuss several model set parametrizations that could be feasible for this system and describe their advantages and disadvantages.

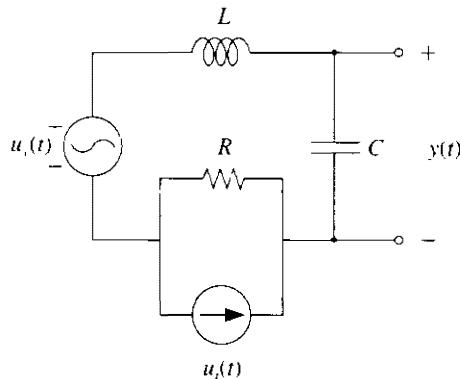


Figure 4.9 A simple circuit.

Hint: The basic equations for this circuit are

$$u_i(t) = L \frac{di(t)}{dt} + y(t) + R[i(t) + u_r(t)]$$

$$y(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$$

4E.5 A state-space model of ship-steering dynamics can be given as follows:

$$\frac{d}{dt} \begin{bmatrix} v(t) \\ r(t) \\ h(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ r(t) \\ h(t) \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \\ 0 \end{bmatrix} u(t)$$

where $u(t)$ is the rudder angle, $v(t)$ the sway velocity, $r(t)$ the turning rate, and $h(t)$ the heading angle.

- (a) Suppose only $u(t)$ and $y(t) = h(t)$ are measured. Show that the six parameters a_{ij} , b_{ij} are not identifiable.
- (b) Try also to show that if $u(t)$ and $y(t) = \begin{bmatrix} v(t) \\ h(t) \end{bmatrix}$ are measured then all six parameters are globally identifiable at values such that the model is controllable. If you cannot complete the calculations, indicate how you would approach the problem (reference: Godfrey and DiStefano, 1985).

4E.6 Consider the model structure (4.91) with

$$A(\theta) = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix}, \quad B(\theta) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

$$C(\theta) = [1 \ 0], \quad K(\theta) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\theta = [a_1 \ a_2 \ b_1 \ b_2 \ k_1 \ k_2]^T, \quad \theta \in D_1 \subset \mathbf{R}^6$$

and another structure

$$A(\eta) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad B(\eta) = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$C(\eta) = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}, \quad K(\eta) = \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}$$

$$\eta = [\lambda_1 \ \lambda_2 \ \mu_1 \ \mu_2 \ \gamma_1 \ \gamma_2 \ \kappa_1 \ \kappa_2]^T. \quad \eta \in D_2 \subset \mathbf{R}^8$$

Determine D_1 and D_2 so that the two model structures determine the same model set. What about identifiability properties?

- 4E.7** Consider the heated metal rod of Example 4.3. Introduce a five-state lumped approximation and write down the state-space model explicitly.
- 4E.8** Consider the OE model structure with $n_b = 2$, $n_f = 1$, and b_1 fixed to unity:

$$y(t) = \frac{q^{-1} + b_2 q^{-2}}{1 + f_1 q^{-1}} u(t) + e(t). \quad \theta = [b_2 \ f_1]^T$$

Determine $\Gamma_2(\theta)$ of Problem 4G.4 explicitly. When is it singular?

- 4E.9** Consider the model structures

$$\mathcal{M}_1 : y(t) = -ay(t-1) + bu(t-1)$$

$$\theta = \begin{bmatrix} a \\ b \end{bmatrix}, \quad D_{\mathcal{M}_1} = \{|a| \leq 1, b > 0\}$$

and

$$\mathcal{M}_2 : y(t) = -(\cos \alpha)y(t-1) + e^\beta u(t-1)$$

$$\eta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad D_{\mathcal{M}_2} = \{0 \leq \alpha \leq \pi, -\infty < \beta < \infty\}$$

Show that $\mathcal{R}(\mathcal{M}_1) = \mathcal{R}(\mathcal{M}_2)$. Discuss possible advantages and disadvantages with the two structures.

- 4E.10** Consider the dc-motor model (4.75). Assume that the torque T_t can be seen as a white-noise zero mean disturbance with variance σ^2 (i.e. the variations in T_t are random and fast compared to the dynamics of the motor). Apply (4.97) to (4.99) to determine $R_1(\theta)$ and $R_{12}(\theta)$ in a sampled model (4.84) and (4.85) of the motor, with $A(\theta)$ and $B(\theta)$ given by (4.77) and

$$\theta = \begin{bmatrix} \tau \\ \beta \\ \gamma \end{bmatrix}, \quad \gamma = \gamma' \cdot \sigma$$

As an alternative, we could use a directly parametrized innovations form (4.91) with $A(\theta)$ and $B(\theta)$ again given by (4.77), but

$$K(\theta) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \text{ and } \theta = [\tau \ \beta \ k_1 \ k_2]^T$$

Discuss the advantages and disadvantages of these two parametrizations.

4E.11 Consider the system description

$$\begin{aligned}x(t+1) &= ax(t) + bu(t) + \xi(t) \\y(t) &= x(t) + e(t)\end{aligned}$$

where $e(t)$ is white Gaussian noise and $\xi(t)$ has the distribution

$$\begin{aligned}\xi(t) &= 0, & w.p. & 1 - \lambda \\ \xi(t) &= +1, & w.p. & \lambda/2 \\ \xi(t) &= -1, & w.p. & \lambda/2\end{aligned}$$

The coefficients a , b , and λ are adjustable parameters. Can this description be cast into the form (4.4)? If so, at the expense of what approximations?

4E.12 Consider a multivariable ARX model set

$$y(t) + A_1 y(t-1) + A_2 y(t-2) = B_1 u(t-1) + e(t)$$

where $\dim y = p = 2$, $\dim u = m = 1$, and where the matrices are parametrized as

$$A_1 = \begin{bmatrix} \times & \times \\ \alpha & \times \end{bmatrix}, \quad A_2 = \begin{bmatrix} \times & \times \\ 0 & \times \end{bmatrix}, \quad B_1 = \begin{bmatrix} \beta \\ \times \end{bmatrix}$$

where α and β are known values and \times indicates a parameter to be estimated. Write the predictor in the form

$$\hat{y}(t|\theta) = \varphi^T(t)\theta + \mu(t)$$

with $\mu(t)$ as a known term and give explicit expressions for φ and θ . Can this predictor be written in the form (4.58)?

- 4T.1** Determine the k -step-ahead predictor for the ARMAX model (4.15).
- 4T.2** Give an expression for the k -step-ahead predictor for (4.91).
- 4T.3** Suppose that $W_u(q)$ and $W_y(q)$ are given functions, known to be determined as k -step-ahead predictors for the system description

$$y(t) = G(q)u(t) + H(q)e(t)$$

Can $G(e^{i\omega})$ and $H(e^{i\omega})$ be uniquely computed from $W_u(e^{i\omega})$ and $W_y(e^{i\omega})$? What if G and H are known to be of the ARMAX structure

$$G(q) = \frac{B(q)}{A(q)}, \quad H(q) = \frac{C(q)}{A(q)}$$

where A , B , and C have known (and suitable) orders?

- 4D.1** Prove Lemma 4.2.

APPENDIX 4A: IDENTIFIABILITY OF BLACK-BOX MULTIVARIABLE MODEL STRUCTURES

The topic of multivariable model structures and canonical forms for multivariable systems is often regarded as difficult, and there is an extensive literature in the field. We shall here give a no-frills account of the problem, and the reader is referred to the literature for more insights and deeper results. See the bibliography.

The issue still is whether (4.136) holds at a given θ . Our development parallels the one in Section 4.6. We start by discussing polynomial parametrizations or MFDS, such as (4.55) to (4.61), and then turn to state-space models. Throughout the section, p denotes the number of outputs and m the number of inputs.

Matrix Fraction Descriptions (MFD)

Consider first the simple multivariable ARX structure (4.52) or (4.56). This uses

$$G(z, \theta) = A^{-1}(z)B(z), \quad H(z, \theta) = A^{-1}(z) \quad (4A.1)$$

with θ comprising all the coefficients of the matrix polynomials (in $1/z$) $A(z)$ and $B(z)$. These could be of arbitrary orders. Just as for the SISO case (4.139), it is immediate to verify that (4.136) holds for all θ^* . Hence the model structure given by the MFD (4A.1) is *strictly globally identifiable*.

Let us now turn to the output error model structure

$$G(z, \theta) = F^{-1}(z)B(z), \quad H(z, \theta) = I \quad (4A.2)$$

It should be noted that the analysis of (4A.2) contains also the analysis of the multivariable ARMAX structure and multivariable Box-Jenkins models. See the corollary to Theorem 4A.1, which follows.

The matrix polynomial $F(z)$ is here a $p \times p$ matrix

$$F(z) = \begin{bmatrix} F_{11}(z) & F_{12}(z) \dots F_{1p}(z) \\ F_{21}(z) & F_{22}(z) \dots F_{2p}(z) \\ \vdots & \vdots & \vdots \\ F_{p1}(z) & F_{p2}(z) \dots F_{pp}(z) \end{bmatrix} = F^{(0)} + F^{(1)}z^{-1} + \dots + F^{(v)}z^{-v} \quad (4A.3)$$

whose entries are polynomials in z^{-1} :

$$F_{ij}(z) = f_{ij}^{(0)} + f_{ij}^{(1)}z^{-1} + \dots + f_{ij}^{(v_{ij})}z^{-v_{ij}} \quad (4A.4)$$

The degree of the F_{ij} polynomial will thus be denoted by v_{ij} and $v = \max v_{ij}$. Similarly, $B(z)$ is a $p \times m$ matrix polynomial. Let the degrees of its entries be denoted by μ_{ij} .

The structure issue is really to select the orders v_{ij} and μ_{ij} [i.e., $p(p+m)$ integers]. This will give a staggering amount of possible model structures. Some special cases discussed in the literature are

$$1. v_{ij} = n, \mu_{ij} = r \quad (4A.5)$$

$$2. v_{ij} = 0, i \neq j; \quad v_{ii} = n_i, \mu_{ij} = r_i \quad (4A.6)$$

$$3. v_{ij} = n_j, \text{ all } i; \quad \mu_{ij} = r_j, \text{ all } i \quad (4A.7)$$

In all these cases we fix the leading matrix to be a unit matrix:

$$F^{(0)} = I; \quad \text{i.e., } f_{ij}^{(0)} = \delta_{ij} \quad (4A.8)$$

The form (4A.5) is called the “full polynomial form” in Söderström and Stoica (1983). It clearly is a special case of (4A.7). It is used and discussed in Hannan (1969, 1976), Kashyap and Rao (1976), Jakeman and Young (1979), and elsewhere.

The form (4A.6) gives a diagonal F -matrix and has been used, for example, in Kashyap and Nasburg (1974), Sinha and Caines (1977), and Gauthier and Landau (1978).

The structure (4A.7) where the different columns are given different orders is discussed, for example, in Guidorzi (1975), Gauthier and Landau (1978), and Gevers and Wertz (1984).

Remark. In the literature, especially the one discussing canonical forms rather than identification applications, often the polynomials

$$\bar{F}(z) = z^v F(z) = F^{(0)} z^v + F^{(1)} z^{v-1} + \cdots + F^{(v)} \quad (4A.9)$$

in the variable z are considered instead of $F(z)$ (just as we did the SISO case). Canonical representations of $F(z)$ [such as the “Hermite form”; see Dickinson, Kailath, and Morf, 1974; Hannan, 1971a; or Kailath, 1980] will then typically involve singular matrices $F^{(0)}$. Such representations are not suitable for our purposes since $y(t)$ cannot be solved for explicitly in terms of past data.

The identifiability properties of the diagonal form (4A.6) can be analyzed by SISO arguments. For the others we need some theory for matrix polynomials.

Some Terminology for Matrix Polynomials

Kailath (1980), Chapter 6, gives a detailed account of various concepts and properties of matrix polynomials. We shall here need just a few:

A $p \times p$ matrix polynomial $P(x)$ is said to be *unimodular* if $\det P(x) = \text{constant}$. Then $P^{-1}(x)$ is also a matrix polynomial. Two polynomials $P(x)$ and $Q(x)$ with the same number of rows have a *common left divisor* if there exists a matrix polynomial $L(x)$ such that

$$P(x) = L(x) \tilde{P}(x)$$

$$Q(x) = L(x) \tilde{Q}(x)$$

for some matrix polynomials $\tilde{P}(x)$ and $\tilde{Q}(x)$.

$P(x)$ and $Q(x)$ are said to be *left coprime* if all common left divisors are unimodular. This is a direct extension of the corresponding concept for scalar polynomials. A basic theorem says that if $P(x)$ and $Q(x)$ are left coprime then there exist matrix polynomials $A(x)$ and $B(x)$ such that

$$P(x)A(x) + Q(x)B(x) = I \text{ (identity matrix)} \quad (4A.10)$$

Loss of Identifiability in Multivariable MFD Structures

We can now state the basic identifiability result.

Theorem 4A.1. Consider the output error MFD model structure (4A.2) with the polynomial degrees chosen according to the scheme (4A.7). Let θ comprise all the coefficients in the resulting matrix polynomials, and let $F_*(z)$ and $B_*(z)$ be the polynomials in $1/z$ that correspond to the value θ^* . Let

$$\begin{aligned} D_p(z) &= \text{diag}(z^{n_1}, \dots, z^{n_p}) \\ D_m(z) &= \text{diag}(z^{r_1}, \dots, z^{r_m}) \end{aligned}$$

be diagonal matrices, with n_i and r_i defined in (4A.7), and define $\tilde{F}_*(z) = F_*(z)D_p(z)$, $\tilde{B}_*(z) = B_*(z)D_m(z)$ as polynomials in z . Then the model structure in question is globally and locally identifiable at θ^* if and only if

$$\tilde{F}_*(z) \text{ and } \tilde{B}_*(z) \text{ are left coprime} \quad (4A.11)$$

Proof. Let θ correspond to $F(z)$ and $B(z)$, and assume that

$$G(z, \theta) = G(z, \theta^*) = F^{-1}(z)B(z) = F_*^{-1}(z)B_*(z)$$

This can also be written as

$$D_p(z)\tilde{F}^{-1}(z)\tilde{B}(z)D_m^{-1}(z) = D_p(z)\tilde{F}_*^{-1}(z)\tilde{B}_*(z)D_m^{-1}(z)$$

where \tilde{F} and \tilde{B} are defined analogously to \tilde{F}_* and \tilde{B}_* . This gives

$$\tilde{B}_*(z) = \tilde{F}_*(z)\tilde{F}^{-1}(z)\tilde{B}(z) \quad (4A.12)$$

When \tilde{B}_* and \tilde{F}_* are left coprime there exist, according to (4A.10), matrix polynomials $X(z)$ and $Y(z)$ such that

$$\tilde{F}_*(z)X(z) + \tilde{B}_*(z)Y(z) = I$$

Inserting (4A.12) into this expression gives

$$\tilde{F}_*(z)\tilde{F}^{-1}(z)\left[\tilde{F}(z)X(z) + \tilde{B}(z)Y(z)\right] = I$$

or

$$\tilde{F}(z)X(z) + \tilde{B}(z)Y(z) = \tilde{F}(z)\tilde{F}_*^{-1}(z) \stackrel{\Delta}{=} U(z)$$

Since the left side is a matrix polynomial in z , so is $U(z)$. We have

$$\tilde{F}(z) = U(z)\tilde{F}_*(z) \tag{4A.13}$$

Note that, by (4A.8),

$$I = \lim_{z \rightarrow \infty} F(z) = \lim_{z \rightarrow \infty} \tilde{F}(z)D_p^{-1}(z) = \lim_{z \rightarrow \infty} \tilde{F}_*(z)D_p^{-1}(z)$$

Hence, multiplying (4A.13) by $D_p^{-1}(z)$ gives

$$I = \lim_{z \rightarrow \infty} U(z)$$

which since $U(z)$ is a polynomial in z , shows that $U(z) \equiv I$, and hence $F(z) \equiv F_*(z)$, which in turn implies that $B(z) = B_*(z)$, and the if-part of the theorem has been proved. If (4A.11) does not hold, a common, nonunimodular, left factor $U_*(z)$ can be pulled out from $F_*(z)$ and $B_*(z)$ and be replaced by an arbitrary matrix with the same orders as $U_*(z)$ [subject to the constraint (4A.8)]. This proves the only-if-part of the theorem. \square

The theorem can immediately be extended to a model structure

$$G(z, \theta) = F^{-1}(z)B(z), \quad H(z, \theta) = D^{-1}(z)C(z) \tag{4A.14}$$

with F and D subject to the degree structure (4A.7). It can also be extended to the multivariable ARMAX structure:

$$G(z, \theta) = A^{-1}(z)B(z), \quad H(z, \theta) = A^{-1}(z)C(z) \tag{4A.15}$$

Corollary 4A.1. Consider the ARMAX model structure (4A.15) with the degrees of the polynomial $A(z)$ subject to (4A.7). Let $\tilde{A}_*(z)$ and $\tilde{\beta}_*(z) = [\tilde{B}_*(z) \quad \tilde{C}_*(z)]$, a $p \times (m+p)$ matrix polynomial, be the polynomials that correspond to θ^* , as described in the theorem. Then the structure is identifiable at θ^* if and only if

$\tilde{A}_*(z)$ and $\tilde{\beta}_*(z)$ are left coprime

The usefulness of these identifiability results lies in the fact that only p orders (the column degrees) have to be chosen with care to find a suitable identifiable structure, despite the fact that $p \cdot m$ [or even $p \cdot (m + p)$ in the ARMAX case] different transfer functions are involved.

State-space Model Structures

For a multivariable state-space model (4.146), we introduce a parametric structure, analogous to (4.146):

$$A(\theta) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \times & \times \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \times & \times \end{bmatrix}, \quad B(\theta) = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}$$

$$K(\theta) = \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \end{bmatrix}, \quad C(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (4A.16)$$

The number of rows with \times 's in $A(\theta)$ equals the number of outputs. We have thus illustrated the structure for $n = 9$, $p = 3$, $m = 2$. In words, the general structure can be defined as:

Let $A(\theta)$ initially be a matrix filled with zeros and with ones along the superdiagonal. Let then row numbers r_1, r_2, \dots, r_p , where $r_p = n$, be filled with parameters. Take $r_0 = 0$ and let $C(\theta)$ be filled with zeros, and then let row i have a one in column $r_{i-1} + 1$. Let $B(\theta)$ and $K(\theta)$ be filled with parameters. (4A.17)

The parametrization is uniquely characterized by the p numbers r_i that are to be chosen by the user. We shall also use

$$v_i = r_i - r_{i-1}$$

and call

$$\bar{v}_n = \{v_1, \dots, v_p\} \quad (4A.18)$$

the *multiindex* associated with (4A.17). Clearly,

$$n = \sum_{i=1}^p v_i \quad (4A.19)$$

By a *multiindex* \bar{v}_n we henceforth understand a collection of p numbers $v_i \geq 1$ subject to (4A.19). For given n and p , there exist $\binom{n-1}{p-1}$ different multi-indexes. Notice that the structure (4A.17) contains $2np + mn$ parameters regardless of \bar{v}_n .

The key property of a “canonical” parametrization like (4A.16) is that the corresponding state vector $x(t, \theta)$ can be interpreted in a pure input-output context. This can be seen as follows. Fix time t , and assume that $u(s) = e(s) \equiv 0$ for $s \geq t$. Denote the corresponding outputs that are generated by the model for $s \geq t$ by $\hat{y}_\theta(s|t-1)$. We could think of them as projected outputs for future times s as calculated at time $t-1$. The state-space equations give directly

$$\begin{aligned} \hat{y}_\theta(t|t-1) &= C(\theta)x(t, \theta) \\ \hat{y}_\theta(t+1|t-1) &= C(\theta)A(\theta)x(t, \theta) \\ &\vdots \\ \hat{y}_\theta(t+n-1|t-1) &= C(\theta)A^{n-1}(\theta)x(t, \theta) \end{aligned} \quad (4A.20)$$

With

$$O(\theta) = \begin{bmatrix} C(\theta) \\ C(\theta)A(\theta) \\ \vdots \\ C(\theta)A^{n-1}(\theta) \end{bmatrix} \quad (4A.21)$$

(the $np \times n$ observability matrix) and

$$\hat{Y}_n^\theta(t) = \begin{bmatrix} \hat{y}_\theta(t|t-1) \\ \vdots \\ \hat{y}_\theta(t+n-1|t-1) \end{bmatrix}$$

We can write (4A.20) as

$$\hat{Y}_n^\theta(t) = O_n(\theta)x(t, \theta) \quad (4A.22)$$

It is straightforward to verify that (4A.17) has a fundamental property: The $np \times n$ observability matrix $O_n(\theta)$ will have n rows that together constitute the unit matrix, regardless of θ . The reader is invited to verify that row number $kp+i$ of O_n will be

$$[0 \ 0 \dots 0 \ 1 \ 0 \dots 0]$$

with 1 in position $r_{i-1} + k + 1$. This holds for $1 \leq i \leq p, 0 \leq k < v_i$. Thus (4A.22) implies that the state variables corresponding to the structure (4A.17) are

$$x_{r_{i-1}+k+1}(t, \theta) = \hat{y}_\theta^{(i)}(t+k|t-1), \quad i = 1, \dots, p, \quad 0 \leq k < v_i \quad (4A.23)$$

Here superscript (i) denotes the i th component of \hat{y} . This interpretation of state variables as predictors is discussed in detail in Akaike (1974b) and Rissanen (1974). By the relation (4A.23), n rows are picked out from the np vector $\hat{Y}_n^\theta(t)$ in (4A.22). The indexes of these rows are uniquely determined by the multiindex \bar{v}_n . Let them be denoted by

$$I_{\bar{v}_n} = \{(k-1)p+i: 1 \leq k \leq v_i; 1 \leq i \leq p\} \quad (4A.24)$$

The key relationship is (4A.23). It shows that the state variables depend only on the input-output properties of the associated model.

Consider now two values θ^* and θ that give the same input-output properties of (4A.17). Then $\hat{y}_\theta(t+k|t-1) = \hat{y}_{\theta^*}(t+k|t-1)$, since these are computed from input-output properties only. Thus $x(t, \theta) = x(t, \theta^*)$. Now, if θ^* corresponds to a minimal realization, so must θ , and Theorem 6.2.4 of Kailath (1980) gives that there exists an invertible matrix T such that

$$\begin{aligned} A(\theta^*) &= TA(\theta)T^{-1}, & B(\theta^*) &= TB(\theta) \\ K(\theta^*) &= TK(\theta), & C(\theta^*) &= C(\theta)T^{-1} \end{aligned} \quad (4A.25)$$

corresponding to the change of basis

$$x(t, \theta^*) = Tx(t, \theta) \quad (4A.26)$$

But (4A.26) together with our earlier observation that $x(t, \theta^*) = x(t, \theta)$ shows that $T = I_n$ and hence that $\theta^* = \theta$.

We have now proved the major part of the following theorem:

Theorem 4A.2. Consider the state-space model structure (4A.17). This structure is globally and locally identifiable at θ^* if and only if $\{A(\theta^*), [B(\theta^*) \ K(\theta^*)]\}$ is controllable.

Proof. The if-part was proved previously. To show the only-if-part, we find that if θ^* does not give a controllable system then its input-output properties can be described by a lower-dimensional model with an additional, arbitrary, noncontrollable model. This can be accomplished by infinitely many different θ 's. \square

It follows from the theorem that the parametrization (4A.17) is globally identifiable, and as such is a good candidate to describe systems of order n . What is not clear yet is whether *any* n th-order linear system can be represented in the form (4A.17) for *an arbitrary choice* of multiindex \bar{v}_n . That is the question we now turn to.

Hankel-Matrix Interpretation

Consider a multivariable system description

$$y(t) = G_0(q)u(t) + H_0(q)e(t) = T_0(q)\chi(t) \quad (4A.27)$$

with

$$T_0(q) = [G_0(q) \quad H_0(q)], \quad \chi(t) = \begin{bmatrix} u(t) \\ e(t) \end{bmatrix}$$

Assume that $T_0(q)$ has full row rank [i.e., $L T_0(q)$ is not identically zero for any nonzero $1 \times p$ vector L]. Let

$$T_0(q) = [0 \quad I] + \sum_{k=1}^{\infty} H_k q^{-k} \quad (4A.28)$$

be the impulse response of the system. The matrices H_k are here $p \times (p+m)$. Define the matrix

$$\mathcal{H}_{r,s} = \begin{bmatrix} H_1 & H_2 & \dots & H_s \\ H_2 & H_3 & \dots & H_{s+1} \\ H_3 & H_4 & \dots & H_{s+2} \\ \vdots & \vdots & & \vdots \\ H_r & H_{r+1} & \dots & H_{r+s-1} \end{bmatrix} \quad (4A.29)$$

This structure with the same block elements along antidiagonals is known as a *block Hankel matrix*. Consider the semifinite matrix $\mathcal{H}_r = \mathcal{H}_{r,\infty}$. For this matrix we have the following two fundamental results.

Lemma 4A.1. Suppose that the n rows $I_{\bar{v}_n}$ [see (4A.24)] of \mathcal{H}_n span all the rows of \mathcal{H}_{n+1} . Then the system (4A.27) can be represented in the state-space form (4A.17) corresponding to the multiindex \bar{v}_n .

The proof consists of an explicit construction and is given at the end of this appendix.

Lemma 4A.2. Suppose that

$$\text{rank } \mathcal{H}_{n+1} \leq n \quad (4A.30)$$

Then there exists a multiindex \bar{v}_n such that the n rows $I_{\bar{v}_n}$ span \mathcal{H}_{n+1} . The proof of this lemma is also given at the end of this appendix.

It follows from the two lemmas that (4A.30) is a sufficient condition for (4A.27) to be an n -dimensional linear system (i.e., to admit a state-space representation of

order n). It is, however, well known that this is also a necessary condition. (\mathcal{H} is obtained as the product of the observability and controllability matrices.) We thus conclude:

Any linear system that can be represented in state-space form of order n can also be represented in the particular form (4A.17) for some multiindex \bar{v}_n (4A.31)

When (4A.30) holds, we thus find that the np rows of \mathcal{H}_n span an n -dimensional (or less) linear space. The *generic* situation is then that the same space is spanned by *any* subset of n rows of \mathcal{H}_n . (By this term we mean if we randomly pick from a uniform distribution np row vectors to span an n -dimensional space the probability is 1 that any subset of n vectors will span the same space.) We thus conclude:

A state-space representation in the form (4A.17) for a particular multiindex \bar{v}_n is capable of describing almost all n -dimensional linear systems. (4A.32)

Overlapping Parametrizations

Let $\mathcal{M}_{\bar{v}_n}$ denote the model structure (4A.17) corresponding to \bar{v}_n . The result (4A.31) then implies that the model set

$$\overline{\mathcal{M}} = \bigcup_{\bar{v}_n} \mathcal{R}(\mathcal{M}_{\bar{v}_n}) \quad (4A.33)$$

(union over all possible multiindices \bar{v}_n) covers all linear n -dimensional systems. We have thus been able to describe the set of all linear n -dimensional systems as the union of ranges of identifiable structures [cf. (4.129)]. From (4A.32), it follows that the ranges of $\mathcal{M}_{\bar{v}_n}$ overlap considerably. This is no disadvantage for identification: on the contrary, one may then change from one structure to another without losing information. The practical use of such overlapping parametrizations for identification is discussed in van Overbeek and Ljung (1982). Using a topological argument, Delchamps and Byrnes (1982) give estimates on the number of overlapping structures needed in (4A.33). See also Hannan and Kavalieris (1984).

Connections Between Matrix Fraction and State-Space Descriptions

In the SISO case the connection between a state-space model in observability form and the corresponding ARMAX model is simple and explicit (see Example 4.2). Unfortunately, the situation is much more complex in the multivariable case. We refer to Gevers and Wertz (1984), Guidorzi (1981), and Beghelli and Guidorzi (1983) for detailed discussions.

We may note, though, the close connection between the indexes v_i used in (4A.17) and the column degrees n_i in (4A.7). Both determine the number of time shifts of the i th component of y that are explicitly present in the representations. The shifts are, however, forward for the state space and backward for the MFD. The relationship between the v_i and the observability indexes is sorted out in the proof of Lemma 4A.2.

A practical difference between the two representations is that the state-space representation naturally employs the state $x(t)$ (n variables) as a memory vector for simulation and other purposes. When (4A.2) is simulated in a straightforward fashion, the different delayed components of y and u are stored, a total number of $np + m \cdot \sum r_i$ variables. This is of course not necessary, but an efficient organization of the variables to be stored amounts to a state-space representation. There are consequently several advantages associated with state-space representations for multivariable systems.

Proofs of Lemmas 4A.1 and 4A.2

It now remains only to prove Lemmas 4A.1 and 4A.2.

Proof of Lemma 4A.1. Let

$$S(t) = \begin{bmatrix} \chi(t-1) \\ \chi(t-2) \\ \vdots \end{bmatrix}$$

Let [cf. (4A.20) to (4A.22)]

$$\hat{y}_0(t|t-k) = \sum_{\ell=k}^{\infty} H_{\ell} \chi(t-\ell) \quad (4A.34)$$

and

$$\hat{Y}_N(t) = \begin{bmatrix} \hat{y}_0(t|t-1) \\ \vdots \\ \hat{y}_0(t+N-1|t-1) \end{bmatrix}$$

Then, from (4A.28) and (4A.29),

$$\hat{Y}_N(t) = \mathcal{H}_N S(t) \quad (4A.35)$$

Now enumerate the row indexes i_r of $I_{\bar{v}_n}$ in (4A.24) as follows:

$$\begin{aligned} i_1 &= 1, & i_2 &= p+1, \dots, i_{v_1} &= (v_1-1) \cdot p + 1 \\ i_{v_1+1} &= 2, & i_{v_1+2} &= p+2, \dots, i_{r_2} &= (v_2-1) \cdot p + 2 \\ &\vdots &&& \\ i_{r_{p-1}+1} &= p, & i_{r_{p-1}+2} &= p+p, \dots, i_{r_p} &= (v_p-1) \cdot p + p \end{aligned} \quad (4A.36)$$

Recall that

$$r_k = \sum_1^k v_j$$

Now construct the n -vector $x(t)$ by taking its r th component to be the i_r th component of $\hat{Y}_N(t)$. Let us now focus on the components $i_1 + p, i_2 + p, \dots, i_n + p$ of

(4A.35). Collect these components into a vector $\xi(t + 1)$. They all correspond to rows of \mathcal{H}_{n+1} . But this matrix is spanned by $x(t)$ by the assumption of the lemma. Hence

$$\xi(t + 1) = Fx(t) \quad (4A.37)$$

for some matrix F . Now several of the components of $\xi(t + 1)$ will also belong to $x(t)$, as shown in (4A.36). The corresponding rows of F will then be zeros everywhere except for a 1 in one position. A moment's reflection on (4A.36) shows that the matrix F will in fact have the structure (4A.17). Also, with H given by (4A.17),

$$y(t) = Hx(t) + e(t) \quad (4A.38)$$

Let us now return to (4A.37). Consider component r of $x(t + 1)$, which by definition equals row i_r of $\hat{Y}_N(t + 1)$. This row is given as $\hat{y}_0^{(j)}(t + k|t)$ for some values j and k that depend on i_r . But, according to (4A.34), we have

$$\hat{y}_0(t + k|t) = \hat{y}_0(t + k|t - 1) + H_k \chi(t) \quad (4A.39)$$

Hence

$$x_r(t + 1) = \hat{y}_0^{(j)}(t + k|t) = \hat{y}_0^{(j)}(t - 1 + (k + 1)|t - 1) + [H_k \chi(t)]_j$$

But the first term of the right side equals component number $i_r + p$ of $\hat{Y}_N(t)$ [i.e., $\xi_r(t + 1)$]. Hence

$$x(t + 1) = \xi(t + 1) + M\chi(t) \quad (4A.40)$$

for some matrix M . Equations (4A.37), (4A.38), and (4A.40) now form a state-space representation of (4A.27) within the structure (4A.17) and the lemma is proved. \square

Proof of Lemma 4A.2. The defining property of the Hankel matrix \mathcal{H}_N in (4A.29) means that the same matrix is obtained by either deleting the first block column (and the last block row) or by deleting the first block row. This implies that, if row i of block row k [i.e., row $(k - 1)p + i$] lies in the linear span of all rows above it, then so must row i of block $k + 1$.

Now suppose that

$$\text{rank } \mathcal{H}_{n+1} = n$$

and let us search the rows from above for a set of linearly independent ones. A row that is not linearly dependent on the ones above it is thus included in the basis; the others are rejected. When the search is finished, we have selected n rows from \mathcal{H}_{n+1} . The observation mentioned previously implies that, if row $kp + i$ is included in this basis for $k \geq 1$, then so is row $(k - 1)p + i$. Hence the row indexes will obey the structure

$$1, \quad p + 1, \quad 2p + 1, \dots, (\sigma_1 - 1)p + 1$$

$$2, \quad p + 2, \quad 2p + 2, \dots, (\sigma_2 - 1)p + 2$$

\vdots

$$p, \quad p + p, \quad 2p + p, \dots, (\sigma_p - 1)p + p$$

for some numbers $\{\sigma_i\}$ that are known as the *observability indexes* of the system. Since the total number of selected rows is n , we have

$$\sum_1^P \sigma_i = n$$

The rows thus correspond to the multiindex $\bar{\sigma}_n$ as in (4A.24) and the lemma is proved. Notice that several other multiindexes may give a spanning set of rows; one does not have to look for the first linearly independent rows. \square