A-Star 2016 Winter Math Camp

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Math Time

1 Algebraic Manipulation

In this section, we will explore several of my favourite problems involving algebraic manipulations.

Problem (2000 AMC 12)

If $x, y,$ and z are positive numbers satisfying

$$x + \frac{1}{y} = 4\,, y + \frac{1}{z} = 1\,, \text{and } z + \frac{1}{x} = \frac{7}{3},$$

find the value of xyz.

Problem (AoPS Introduction to Algebra)

Let $A = x+\frac{1}{x}$ and $B = x^2+\frac{1}{x^2}$. Note that $(x+\frac{1}{x})^2 = x^2+2+\frac{1}{x^2}$, therefore, $B = A^2 - 2$. Find formulas for

$$C = x^3 + \frac{1}{x^3}\,, D = x^4 + \frac{1}{x^4}, E = x^5 + \frac{1}{x^5}$$

in terms of A.

1.1 2000 AMC 12

Solution. In order to get the xyz term, we are motivated to multiply the 3 equations together:

$$\begin{aligned}\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right) &= xyz+\frac{1}{xyz}+(x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\\ &= (4)\,(1)\left(\frac{7}{3}\right)=\frac{28}{3}.\end{aligned}$$

What can we do now to simplify this further?

We also add all 3 of the equations:

$$\begin{aligned}\left(x+\frac{1}{y}\right)+\left(y+\frac{1}{z}\right)+\left(z+\frac{1}{x}\right) &= 4+1+\frac{7}{3}=\frac{22}{3}\\ &= (x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).\end{aligned}$$

Therefore, plugging this in to the first equation gives

$$\begin{aligned}xyz+\frac{1}{xyz}+(x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) &= xyz+\frac{1}{xyz}+\frac{22}{3}\\ &= \frac{28}{3}\\ \implies xyz+\frac{1}{xyz} &= 2.\end{aligned}$$

What's xyz equal to then?

Multiply the equation through by xyz and simplify:

$$(xyz)^2 + 1 = 2xyz \implies (xyz)^2 - 2 \cdot xyz + 1 = (xyz - 1)^2 = 0.$$

Therefore, $xyz = \boxed{1}$. $\square$

1.2 Exponent Mayhem

Solution. We begin with C. Inspired by our method for computing B, we attempt to cube $x + \frac{1}{x}$:

$$\begin{aligned}(x + \frac{1}{x})^3 &= x^3 + 3 \cdot x^2 \cdot \frac{1}{x} + 3 \cdot x \cdot \frac{1}{x^2} + \frac{1}{x^3} \\ &= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}.\end{aligned}$$

Therefore,

$$\begin{aligned} A^3 &= x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} = C + 3A \\ \implies C &= A^3 - 3A. \end{aligned}$$

There are two methods for finding D. One of them involves taking $x + \frac{1}{x}$ to the fourth power. In order to continue with this method, however, I must introduce the binomial theorem and Pascal's triangle.

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