System of Equations

Lecture 1

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Outline

1. Word Problems
2. Equations in Three Variables
3. Symmetry

Word Problems

Example 1. Two years ago, Gene was nine times as old as Carol. He is now seven times as old as she is. How many years from now will Gene be five times as old as Carol? (*Source: Mandelbrot*)

Example 2. When a bucket is two-thirds full of water, the bucket and water weigh x kilograms. When the bucket is one-half full of water the total weight is y kilograms. In terms of x and y, what is the total weight in kilograms when the bucket is full of water? (*Source: AMC 12*)

Gene and Carol

Example. Two years ago, Gene was nine times as old as Carol. He is now seven times as old as she is. How many years from now will Gene be five times as old as Carol?

Solution. Let the current ages of Gene and Carol be g and c respectively. Currently, we have $g = 7c$. Two years ago, Gene was $g - 2$ years old and Carol was $c - 2$ years old. Therefore, $g - 2 = 9(c - 2)$. We thus have the system of equations:

$$g = 7c$$
$$g - 2 = 9(c - 2).$$

Substituting the first equation into the second gives

$$7c - 2 = 9(c - 2) \implies 7c - 2 = 9c - 18 \implies 2c = 16 \implies c = 8.$$

Similarly, $g = 7 \cdot 8 = 56$. Let the amount of time be t. Therefore,

$$56 + t = 5\,(8 + t) \implies 56 + t = 40 + 5t \implies t = 4 \text{ years.}$$

Bucket Full of Water

Example. When a bucket is two-thirds full of water, the bucket and water weigh x kilograms. When the bucket is one-half full of water the total weight is y kilograms. In terms of x and y, what is the total weight in kilograms when the bucket is full of water?

Let the weight of the bucket without any water be b and the weight of the water be w. Then, we have the system of equations

$$b + 2/3w = x$$
$$b + 1/2w = y.$$

Subtracting these gives $\frac{1}{6}w = x - y \implies w = 6x - 6y$.

Substituting gives $b + (3x - 3y) = y \implies b = 4y - 3x$. Hence,

$$b + w = (4y - 3x) + (6x - 6y) = \boxed{3x - 2y}.$$

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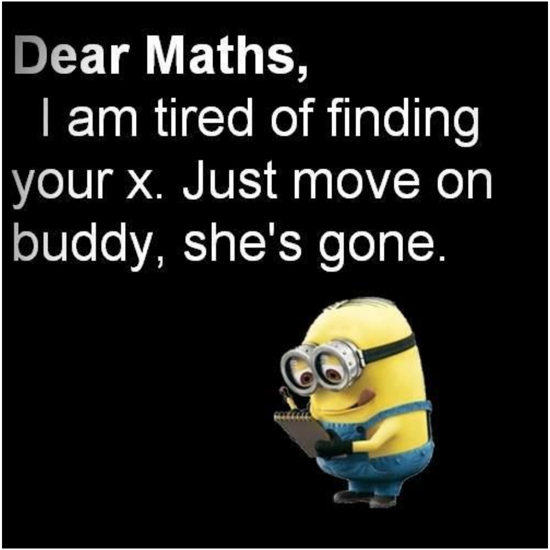
Equations in Three Variables

Example 3. Solve the below system for x, y, z:

$$\begin{cases} x + 2y + 3z = 3 \\ 2x + 5y + 7z = 22 \\ -x + y + z = 6 \end{cases}$$

Example 4. Solve the below system for r, s, t:

$$\begin{cases} r + s + t = 7 \\ 2r + 3s - 5t = 11 \\ 8r + 11s - 13t = 47 \end{cases}$$


Dear Maths,
I am tired of finding your x. Just move on buddy, she's gone.

1 Unique Solution

Example. Solve the below system for x, y, z:

$$\begin{cases} x + 2y + 3z = 3 \\ 2x + 5y + 7z = 22 \\ -x + y + z = 6 \end{cases}$$

Multiplying the first equation by 2 and subtracting gives $y + z = 16$.
Adding the first and third equation gives $3y + 4z = 9$.
Solving this yields $(x, y, z) = (10, 55, -39)$.

1 Solution Dependent on t

Example. Solve the below system for r, s, t:

$$\begin{cases} r + s + t = 7 \\ 2r + 3s - 5t = 11 \\ 8r + 11s - 13t = 47 \end{cases}$$

Multiply the first equation by 2 and second equation by 3 and add these:

$$\begin{aligned} 2\,(r + s + t) + 3\,(2r + 3s - 5t) &= 2 \cdot 7 + 3 \cdot 11 \\ 8r + 11s - 13t &= 47. \end{aligned}$$

This is exactly the third equation! We consider the first two equations.

We find that $(r, s, t) = (10 - 8t, -3 + 7t, t)$. Substitute and verify!

Follow Up

Example 5. Solve the below system for x, y, z:

$$\begin{cases} x - y + z = 12 \\ 2x + 3y - 4z = 5 \\ 7x + 8y - 11z = 42 \end{cases}$$

Example 6. Solve the below system for r, s, t:

$$\begin{cases} r + 2s + 3t = 7 \\ 2r + 4s + 6t = 14 \\ 3r + 6s + 9t = 21 \end{cases}$$

No Solutions

Example. Solve the below system for x, y, z:

$$\begin{cases} x - y + z = 12 \\ 2x + 3y - 4z = 5 \\ 7x + 8y - 11z = 42 \end{cases}$$

Multiplying the second equation by 3 and add this to the first:

$$\begin{aligned} 3\left(2x + 3y - 4z\right) + \left(x - y + z\right) &= 3 \cdot 5 + 12 \\ 7x + 8y - 11z &= 27. \end{aligned}$$

Hence, this system has no solutions.

2 Basis Solutions

Example. Solve the below system for r, s, t:

$$\begin{cases} r + 2s + 3t = 7 \\ 2r + 4s + 6t = 14 \\ 3r + 6s + 9t = 21 \end{cases}$$

Multiplying the first equation by 2 gives the second equation.

Multiplying the first equation by 3 gives the third equation.

Hence, the solution is $(r, s, t) = (7 - 2s - 3t, s, t)$.

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Symmetry Problems

Example 7. Solve the following system of equations in w, x, y, z:

$$\begin{aligned} 3w + x + y + z &= 20 \\ w + 3x + y + z &= 6 \\ w + x + 3y + z &= 44 \\ w + x + y + 3z &= 2. \end{aligned}$$

Example 8. Justin, Lazar, and Daniel are each thinking of a positive number. The product of Justin's and Lazar's is 27. The product of Lazar's and Daniel's is 72. The product of Justin's and Daniel's is 6. Find each person's number.

Example 9. If x, y, and z are positive numbers satisfying $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$, and $z + \frac{1}{x} = 7/3$ then what is xyz?

System in 4 variables

Example. Solve the following system of equations in w, x, y, z:

$$\begin{aligned} 3w + x + y + z &= 20 \\ w + 3x + y + z &= 6 \\ w + x + 3y + z &= 44 \\ w + x + y + 3z &= 2. \end{aligned}$$

Adding these equations gives

$$6(w + x + y + z) = 72 \implies w + x + y + z = 12.$$

This yields the solution $(w, x, y, z) = (4, -3, 16, -5)$.

What numbers are they thinking of?

Example. Justin, Lazar, and Daniel are each thinking of a positive number. The product of Justin's and Lazar's is 27. The product of Lazar's and Daniel's is 72. The product of Justin's and Daniel's is 6. Find each person's number.

We have the system of equations

$$\begin{aligned} jl &= 27 \\ ld &= 72 \\ jd &= 6. \end{aligned}$$

Multiplying gives $(jld)^2 = 27 \cdot 72 \cdot 6 = 3^6 \cdot 2^4 \implies jld = 108.$

Therefore, $(j, l, d) = (\frac{3}{2}, 18, 4)$.

More Symmetry

Example. If x, y, and z are positive numbers satisfying $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$, and $z + \frac{1}{x} = 7/3$ then what is xyz?

We want to find the product xyz, therefore, we multiply the three equations:

$$\begin{aligned}\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right) &= xyz + \frac{xy}{x} + \frac{xz}{z} + \frac{x}{zx} + \frac{yz}{y} + \frac{y}{yx} + \frac{z}{yz} + \frac{1}{xyz} \\ &= xyz + (x+y+z) + \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) + \frac{1}{xyz} \\ &= 4 \cdot 1 \cdot \frac{7}{3} = \frac{28}{3}.\end{aligned}$$

Adding the original three equations:

$$\left(x+\frac{1}{y}\right)+\left(y+\frac{1}{z}\right)+\left(z+\frac{1}{x}\right) = 4+1+\frac{7}{3} = \frac{22}{3}$$

How do we now find xyz?

More Symmetry II

Example. If x, y, and z are positive numbers satisfying $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$, and $z + \frac{1}{x} = 7/3$ then what is xyz?

Substituting this into the first equation gives us

$$xyz + \frac{22}{3} + \frac{1}{xyz} = \frac{28}{3} \implies xyz + \frac{1}{xyz} = 2.$$

Finally, multiplying by xyz and re-arranging gives

$$(xyz - 1)^2 = 0 \implies xyz = \boxed{1}.$$

Final Symmetry Problems

Example 10. Let x, y, z be positive real numbers satisfying the simultaneous equations

$$x(y^2 + yz + z^2) = 3y + 10z$$
$$y(z^2 + zx + x^2) = 21z + 24x$$
$$z(x^2 + xy + y^2) = 7x + 28y.$$

Find $xy + yz + zx$. *(Source: 2014 Purple Comet)*

Example 11. Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3,\ \frac{bc}{b+c} = 4,\ \text{and}\ \frac{ca}{c+a} = 5.$$

Compute the value of $\dfrac{abc}{ab+bc+ca}$. *(Source: 2012 Purple Comet)*

2014 Purple Comet I

We note that the equations on the left hand side are symmetric. We therefore think to **sum** up the equations. I claim that when we do so, the left hand side becomes $(x+y+z)(xy+xz+yz)$. Why?

Expand out, group, and colour the xyz terms:

$$(x+y+z)(xy+xz+yz) = \left(x^2y + x^2z\right) + \left(y^2x + y^2z\right) + \left(z^2y + z^2x\right) + 3xyz$$

We distribute and colour the other terms:

$$\begin{aligned} x\left(y^2+yz+z^2\right) &= y^2x + xyz + z^2x \\ y\left(z^2+zx+x^2\right) &= z^2y + xyz + x^2y \\ z\left(x^2+xy+y^2\right) &= x^2z + xyz + y^2z. \end{aligned}$$

2014 Purple Comet II

Therefore, by proof by colouring, we have showed that summing up the left hand side gives $(x+y+z)(xy+xz+yz)$. Summing up the right hand side gives

$$(3y+10z)+(21z+24x)+(7x+28y)=31\left(x+y+z\right).$$

We equate the two of them, and divide through by $x+y+z$ (since it is positive):

$$(x+y+z)(xy+xz+yz)=31(x+y+z) \implies xy+xz+yz=\boxed{31}.$$

How did I think of the factorization? The motivation behind it came from the problem statement asking to find $xy+yz+zx$!

Fractional System

Example. Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3,\ \frac{bc}{b+c} = 4, \text{ and } \frac{ca}{c+a} = 5.$$

Compute the value of $\dfrac{abc}{ab+bc+ca}$. *(Source: 2012 Purple Comet)*

Let's take the **reciprocal** of all of the equations:

$$\frac{a+b}{ab} = \frac{1}{3},\ \frac{b+c}{bc} = \frac{1}{4},\ \text{and } \frac{c+a}{ca} = \frac{1}{5}.$$

Note that we can simplify each of these expressions! For instance, $\frac{a+b}{ab} = \frac{1}{b} + \frac{1}{a}$. Then, the system of equations becomes

$$\frac{1}{b} + \frac{1}{a} = \frac{1}{3},\ \frac{1}{c} + \frac{1}{b} = \frac{1}{4},\ \text{and } \frac{1}{a} + \frac{1}{c} = \frac{1}{5}.$$

Fractional System II

We continue by taking the reciprocal of the desired expression, $\frac{abc}{ab+bc+ca}$:

$$\frac{ab+bc+ca}{abc} = \frac{1}{c} + \frac{1}{a} + \frac{1}{b}.$$

Aha! We've turned this into something we know how to work with. We sum up the three equations from the previous slide to get

$$2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}.$$

Therefore, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{47}{120}$. We have to take the reciprocal, however, to get

$$\frac{abc}{ab+bc+ca} = \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} = \boxed{\frac{120}{47}}.$$

Substitution Problem

Example 12. Let x and y be real numbers with $x > y$. Find x if

$$x^2y^2 + x^2 + y^2 + 2xy = 40 \text{ and } xy + x + y = 8.$$

(Source: 2013 HMMT)

2013 HMMT

Example. Let x and y be real numbers with $x > y$. Find x if

$$x^2y^2 + x^2 + y^2 + 2xy = 40 \text{ and } xy + x + y = 8.$$

We substitute $a = xy$ and $b = x + y$. The system then becomes:

$$a^2 + b^2 = 40\,,\; a + b = 8.$$

We square the second equation to arrive at $(a + b)^2 = a^2 + 2ab + b^2 = 64$.

Subtracting from the first equation: $2ab = 64 - 40 = 24 \implies ab = 12$.

Solving this system, we now see that $(a, b) = (2, 6), (6, 2)$.

2013 HMMT II

Example. Let x and y be real numbers with $x > y$. Find x if

$$x^2y^2 + x^2 + y^2 + 2xy = 40 \text{ and } xy + x + y = 8.$$

If $(a, b) = (6, 2)$, then we have $xy = 6$ and $x + y = 2$. We use the identity

$$(z - x)(z - y) = z^2 - z(x + y) + xy.$$

This case gives the quadratic $z^2 - 2z + 6$ with discriminant $\Delta = 2^2 - 4 \cdot 6 = -20$. Hence, there are no positive real solutions.

On the other hand, when $(a, b) = (2, 6)$, we have $xy = 2$ and $x + y = 6$. The quadratic for this case is $z^2 - 6z + 2$. Completing the square:

$$z^2 - 6z + 2 = (z - 3)^2 - 7 \implies z = 3 \pm \sqrt{7}.$$

Since $x > y$, we have $x = \boxed{3 + \sqrt{7}}$.