Rate Problems

Lecture 2

Justin Stevens

Rate, Distance, and Time

Theorem. For rate problems, Distance $=$ Rate $\cdot$ Time.

Example 1. To arrive at AStar Summer Math Camp, Justin drives for one hour at 30 mph and for the second hour at 20 mph. What is his average speed for the trip?

Example 2. To arrive at AStar Summer Math Camp, Justin drives for a total *distance* of 120 miles. Suppose he drives the first half of the distance at 30 mph and the second half of the distance at 20 mph. What is his average speed for the trip?

Arithmetic Mean

Example. To arrive at AStar Summer Math Camp, Justin drives for one hour at 30 mph and for the second hour at 20 mph. What is his average speed for the trip?

Solution. The distance I drive in each of the blocks are given by

$$d_1 = 30 \text{ mph} \cdot 1 \text{ h} = 30 \text{ miles}$$
$$d_2 = 20 \text{ mph} \cdot 1 \text{ h} = 20 \text{ miles}.$$

The sum of these distances is $d_1 + d_2 = 50$ miles. The total time taken is 2 hours, therefore, the average speed is

$$\text{Rate} = \frac{\text{Distance}}{\text{Time}} = \frac{50 \text{ miles}}{2 \text{ hours}} = 25 \text{ mph}.$$

Definition. The arithmetic mean of two numbers a and b is $\frac{a+b}{2}$.

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Harmonic Mean

Example. To arrive at AStar Summer Math Camp, Justin drives for a total *distance* of 120 miles. Suppose he drives the first half of the distance at 30 mph and the second half of the distance at 20 mph. What is his average speed for the trip?

Solution. The distance driven in each half if 60 miles. Therefore,

$$60 = 30 \cdot t_1 \implies t_1 = 2 \text{ hr}$$
$$60 = 20 \cdot t_2 \implies t_2 = 3 \text{ hr}.$$

Therefore, the total time I drive is given by $t = t_1 + t_2 = 2 + 3 = 5$ hr. Therefore, my average speed is given by

$$\text{Rate} = \frac{\text{Distance}}{\text{Time}} = \frac{120 \text{ miles}}{5 \text{ hr}} = 24 \text{ mph}.$$

Definition. The harmonic mean of two numbers a and b is $\frac{2}{\frac{1}{a}+\frac{1}{b}}$.

Clock Problems

Example 3. What is the first time after 10 o'clock at which the minute hand and the hour hand of a clock point in the exact same direction? *

Example 4. Cassandra sets her watch to the correct time at noon. At the actual time of 1 : 00 PM, she notices that her watch reads 12 : 57 and 36 seconds. Assume that her watch loses time at a constant rate. What will be the actual time when her watch first reads 10 : 00 PM? †

* Source: AoPS Introduction to Algebra

† Source: 2003 AMC 12B

Minute Hand and Hour Hand

Example. What is the first time after 10 o'clock at which the minute hand and the hour hand of a clock point in the exact same direction? ‡

The hour hand travels 5 minutes in the time the minute hand travels 60:

$$\frac{\text{Hour}}{\text{Minute}} = \frac{5}{60} \implies \text{Hour} = \frac{1}{12}\text{Minute}.$$

The hour hand begins at 50 and the minute hand begins at 0. Therefore,

$$50 + \frac{1}{12}m = m \implies m = 50\left(\frac{12}{11}\right) = \frac{600}{11} = 54\frac{6}{11}.$$

Hence, the time is $10 : 54 : \frac{6}{11}$.

‡ Source: AoPS Introduction to Algebra

Cassandra's Watch

Example. Cassandra sets her watch to the correct time at noon. At the actual time of 1 : 00 PM, she notices that her watch reads 12 : 57 and 36 seconds. Assume that her watch loses time at a constant rate. What will be the actual time when her watch first reads 10 : 00 PM?

Actual Time	Cassandra Time
12:00 PM	12:00 PM
1:00 PM	12:57:36 PM
?	10:00 PM

Let Cassandra's time be t_c and the actual time be t_t. We see that

$$t_t - t_c = 2 \cdot 60 + 24 = 144 \text{ seconds.}$$

Since $t_t = 3600$ s, we see that

$$\frac{t_t - t_c}{t_t} = \frac{144}{3600} = \frac{1}{25} \implies \frac{t_c}{t_t} = \frac{24}{25}.$$

Cassandra's Watch II

Example. Cassandra sets her watch to the correct time at noon. At the actual time of 1 : 00 PM, she notices that her watch reads 12 : 57 and 36 seconds. Assume that her watch loses time at a constant rate. What will be the actual time when her watch first reads 10 : 00 PM?

Let Cassandra's time be t_c and the actual time be t_t.

Rearranging $t_t = \frac{25}{24} t_c \implies t_t = \frac{25}{24} t_c$.

Hence, at time $t_c = 10 \cdot 60$ minutes, we have

$$t_t = \frac{25}{24} (600 \text{ minutes}) = 625 \text{ minutes} \implies t_t = \boxed{10 : 25 \text{ pm}}.$$

Three Stooges

Example 5. Moe, Larry, and Curly work together to build a house. When Moe and Larry work together, it takes them 10 hours. When Larry and Curly work together, it takes 12 hours. When Moe and Curly work together, it takes 15 hours. How long does it take all three working together?

Moe, Larry, and Curly

Example. Moe, Larry, and Curly work together to build a house. When Moe and Larry work together, it takes them 10 hours. When Larry and Curly work together, it takes 12 hours. When Moe and Curly work together, it takes 15 hours. How long does it take all three?

Let the rate it takes Moe be r_m, the rate it takes Larry be r_l and the rate it takes Curly be r_c. Therefore, we have the system of equations

$$\begin{aligned} 1 &= (r_m + r_l)\,10 \\ 1 &= (r_l + r_c)\,12 \\ 1 &= (r_m + r_c)\,15. \end{aligned}$$

Dividing these, we see that $r_m + r_l = \frac{1}{10}$, $r_l + r_c = \frac{1}{12}$, and $r_m + r_c = \frac{1}{15}$:

$$2\left(r_m + r_l + r_c\right) = \frac{1}{10} + \frac{1}{12} + \frac{1}{15} = \frac{1}{4} \implies r_m + r_l + r_c = \frac{1}{8}.$$

Hence, it takes all three of them $\boxed{\text{8 hours}}$ working together.

⋆ Paula the Painter

Example 6. Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?
(A) 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 60

⋆ Paula the Painter Solution

Let Paula's rate be p and her helper's rate be h. Let the length of the lunch break be L. We thus have the system of equations:

$$\begin{aligned} (8-L)(p+h) &= 50 \\ (6.2-L)\,h &= 24 \\ (11.2-L)\,p &= 26. \end{aligned}$$

Expanding these gives:

$$\begin{aligned} 8p+8h-Lp-Lh &= 50 \\ 6.2h-Lh &= 24 \\ 11.2p-Lp &= 26. \end{aligned}$$

Adding the last two equations gives

$$6.2h+11.2p-Lh-Lp = 50.$$

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Equating the values that are 50 gives

$$\begin{aligned} 8p + 8h - Lp - Lh &= 6.2h + 11.2p - Lh - Lp \\ 1.8h &= 3.2p \\ h &= \frac{16}{9}p. \end{aligned}$$

Substituting this into the bottom two equations gives:

$$\begin{aligned} (6.2 - L)\,\frac{16}{9}p &= 24 \\ (11.2 - L)\,p &= 26. \end{aligned}$$

Rearranging the top equation gives $(6.2 - L)\,p = 24 \cdot \frac{9}{16} = \frac{27}{2}$.

Subtracting from the bottom gives $5p = 26 - \frac{27}{2} = \frac{25}{2} \implies p = \frac{5}{2}$.

Finally, we have $L = \frac{4}{5} \implies L = \boxed{48 \text{ minutes}}$.

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Track Problems

Example 7. Brenda and Sally run in opposite directions on a circular track, starting at diametically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters? §

(A) 250 (B) 300 (C) 350 (D) 400 (E) 500

Example 8. In an h-meter race, Sam is exactly d meters ahead of Walt when Sam finishes the race. The next time they race, Sam sportingly starts d meters behind Walt, who is at the original start line. Both runners run at the same constant speed as they did in the first race. How many meters ahead is the winner of the second race when the winner crosses the finish line? ¶

(A) $\frac{d}{h}$ (B) 0 (C) $\frac{d^2}{h}$ (D) $\frac{h^2}{d}$ (E) $\frac{d^2}{h-d}$

§ Source: 2004 AMC12

¶ Source: 1998 AHSME

Brenda and Sally

Let the length of the track be L. Then, the distance ran is:

	Brenda	Sally
First Meeting (t_1)	100	$L/2 - 100$
Second Meeting (t_2)	$L - 150$	150

Let Brenda's rate be r_b and Sally's rate be r_s. From the distance formula:

$$\begin{aligned} 100 &= r_b t_1, \quad L/2 - 100 = r_s t_1 \\ L - 150 &= r_b t_2, \quad 150 \qquad\quad = r_s t_2. \end{aligned}$$

Dividing these equations, we arrive at the ratios

$$\frac{t_1}{t_2} = \frac{100}{L-150} = \frac{L/2 - 100}{150} \implies L = \boxed{350 \text{ meters}}.$$

Brenda and Sally II

Example. Brenda and Sally run in opposite directions on a circular track, starting at diametically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

The girls combined run half the length of the track in their first meeting and the entire length in their second meeting:

$$\begin{aligned} L/2 &= (r_b + r_s)\, t_1 \\ L &= (r_b + r_s)\, t_2. \end{aligned}$$

Therefore, $t_2 = 2t_1$. For the first meeting, Brenda runs $r_b t_1 = 100$ meters.

Hence, in the second meeting, Brenda runs $r_b\,(2t_1) = 200$ meters.

Therefore, the length of the track in meters is $200 + 150 = \boxed{350 \text{ meters}}$.

Sam and Walt

Let Sam's rate be r_s and Walt's be r_w. Let the first race take time t_1:

$$\begin{cases} r_s t_1 &= h \\ r_w t_1 &= h - d \end{cases} \implies r_s = r_w \cdot \frac{h}{h-d}.$$

Let the time it takes Sam to complete the second race be t_s. Therefore,

$$r_s t_s = h + d$$
$$\left(r_w \cdot \frac{h}{h-d}\right) t_s = h + d$$
$$r_w t_s = \frac{(h+d)(h-d)}{h}.$$

In time t_s, Walt runs a distance $d_w = r_w t_s = \frac{h^2 - d^2}{h} = h - \frac{d^2}{h}$.

Therefore, Sam wins the second race and the answer is $\boldsymbol{\frac{d^2}{h}}$.

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