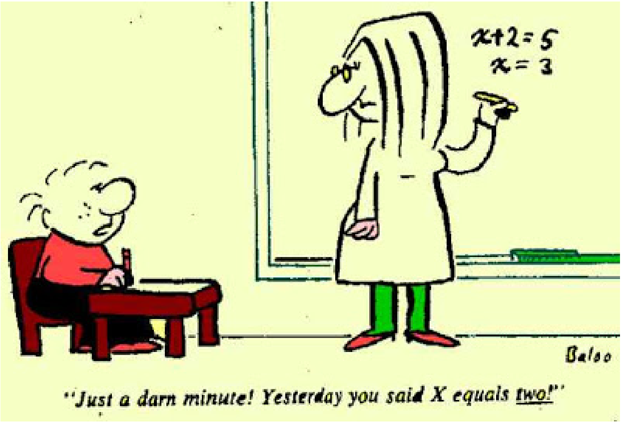
A-Star 2016 Winter Math Camp

Justin Stevens

26th December, 2016

Math Time

1 Algebraic Manipulation



Problem (2000 AMC 12)

If $x, y,$ and z are positive numbers satisfying

$$x + \frac{1}{y} = 4\,, y + \frac{1}{z} = 1\,, \text{and } z + \frac{1}{x} = \frac{7}{3},$$

find the value of xyz.

Problem (AoPS Introduction to Algebra)

Let $A = x + \frac{1}{x}$ and $B = x^2 + \frac{1}{x^2}$. Note that $(x + \frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2}$, therefore, $B = A^2 - 2$. Find formulas for

$$C = x^3 + \frac{1}{x^3}\,, D = x^4 + \frac{1}{x^4}, E = x^5 + \frac{1}{x^5}$$

in terms of A.

1.1 2000 AMC 12

Solution. In order to get the xyz term, we are motivated to multiply the 3 equations together:

$$\begin{aligned}\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right) &= xyz+\frac{1}{xyz}+(x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\\ &= (4)\,(1)\left(\frac{7}{3}\right)=\frac{28}{3}.\end{aligned}$$

What can we do now to simplify this further?

We also add all 3 of the equations:

$$\begin{aligned}\left(x+\frac{1}{y}\right)+\left(y+\frac{1}{z}\right)+\left(z+\frac{1}{x}\right) &= 4+1+\frac{7}{3}=\frac{22}{3}\\ &= (x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).\end{aligned}$$

Therefore, plugging this in to the first equation gives

$$\begin{aligned}xyz+\frac{1}{xyz}+(x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) &= xyz+\frac{1}{xyz}+\frac{22}{3}\\ &= \frac{28}{3}\\ \implies xyz+\frac{1}{xyz} &= 2.\end{aligned}$$

What's xyz equal to then?

Multiply the equation through by xyz and simplify:

$$(xyz)^2 + 1 = 2xyz \implies (xyz)^2 - 2 \cdot xyz + 1 = (xyz - 1)^2 = 0.$$

Therefore, $xyz = \boxed{1}$. □

1.2 Exponent Mayhem

Solution. We begin with C. Inspired by our method for computing B, we attempt to cube $x + \frac{1}{x}$:

$$\begin{aligned} (x + \frac{1}{x})^3 &= x^3 + 3 \cdot x^2 \cdot \frac{1}{x} + 3 \cdot x \cdot \frac{1}{x^2} + \frac{1}{x^3} \\ &= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}. \end{aligned}$$

Therefore,

$$\begin{aligned} A^3 &= x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} = C + 3A \\ \Longrightarrow C &= A^3 - 3A. \end{aligned}$$

There are two methods for finding D. One of them involves taking $x + \frac{1}{x}$ to the fourth power. In order to continue with this method, however, I must introduce the binomial theorem and Pascal's triangle.

1.3 Pascal's Triangle

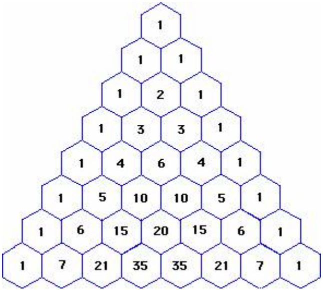


Figure 1: Source: iCoachMath.com

1.4 Binomial-theorem

The binomial theorem states that when we expand $x+y$ to the nth power, the coefficients will be the numbers in the nth row of Pascal's triangle. For instance,

$$(x+y)^4 = \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4.$$

The numbers $1, 4, 6, 4, 1$ make up the 4th row of Pascal's triangle. Furthermore, if you know binomial coefficients, note that

$$\binom{4}{0} = 1, \binom{4}{1} = 4, \binom{4}{2} = 6, \binom{4}{3} = 4, \binom{4}{4} = 1.$$

Theorem (Binomial Expansion)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Using the expansion for $(x+y)^4$, we see that

$$\begin{aligned}(x+\frac{1}{x})^4 &= x^4 + 4\cdot\left(x^3\cdot\frac{1}{x}\right) + 6\cdot\left(x^2\cdot\frac{1}{x^2}\right) + 4\cdot\left(x\cdot\frac{1}{x^3}\right) + \frac{1}{x^4} \\ &= \left(x^4+\frac{1}{x^4}\right) + 4\left(x^2+\frac{1}{x^2}\right) + 6.\end{aligned}$$

We substitute the formula $B = x^2 + \frac{1}{x^2} = A^2 - 2$ to get:

$$D = x^4 + \frac{1}{x^4} = A^4 - 4(A^2-2) - 6 = A^4 - 4A^2 + 2.$$

A simpler method exists for computing D without the use of the binomial theorem. Note that if we multiply A by C, we get the desired x^4 and $\frac{1}{x^4}$ terms:

$$AC = \left(x + \frac{1}{x}\right)\left(x^3 + \frac{1}{x^3}\right) = x^4 + \left(x^2 + \frac{1}{x^2}\right) + \frac{1}{x^4}.$$

From above, we found $C = A^3 - 3A$. Furthermore, $B = x^2 + \frac{1}{x^2} = A^2 - 2$. Substituting these both in give

$$D = x^4 + \frac{1}{x^4} = A(A^3 - 3A) - (A^2 - 2) = A^4 - 4A^2 + 2.$$

Note this matches the answer above.

We attempt our new method for computing E. Note that if we multiply A by D, we get the desired x^5 and $\frac{1}{x^5}$ terms:

$$AD = \left(x + \frac{1}{x}\right)\left(x^4 + \frac{1}{x^4}\right) = x^5 + \left(x^3 + \frac{1}{x^3}\right) + \frac{1}{x^5}.$$

We substitute $D = A^4 - 4A^2 + 2$ and $C = x^3 + \frac{1}{x^3} = A^3 - 3A$ into the above equation:

$$E = x^5 + \frac{1}{x^5} = A\left(A^4 - 4A^2 + 2\right) - \left(A^3 - 3A\right) = A^5 - 5A^3 + 5A.$$

In general, if $x_n = x^n + \frac{1}{x^n}$, then we can recursively find the next term using the identity

$$x_1 x_{n-1} = x_n + x_{n-2} \implies x_n = x_1 x_{n-1} - x_{n-2}.$$

□

Exponent Mayhem in NIMO

The identity above was a key motivator in a 2015 National Internet Math Olympiad (NIMO) challenge problem I cowrote with Evan Chen!

Problem (Justin Stevens and Evan Chen)
Let a, b, c be reals and p be a prime number. Assume that

$$a^n(b+c)+b^n(a+c)+c^n(a+b)\equiv 8 \pmod p$$

for each nonnegative integer n. Let m be the remainder when $a^p+b^p+c^p$ is divided by p, and k the remainder when m^p is divided by p^4. Find the maximum possible value of k.

The solution involves finding similar recursion relations with some number theory tricks as well. The answer is 399; try to figure out why after finishing this course!

2 Functions

Problem (2000 AMC 12)
Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

Problem (Mandelbrot)
Let f be a function such that when $a + b = 2^n$ for a, b, n integers, then $f(a) + f(b) = n^2$. What is $f(2002)$?

Problem (Mandelbrot)
Let f be a function which takes 2 inputs as arguments.The value of f is defined recursively: $f(x, y) = x + f(x - 1, x - y)$. If $f(1, 0) = 5$, find $f(5, 2)$.

3 Sequences and Series

Problem (2003 AMC 10)

The first four terms in an arithmetic sequence are $x+y, x-y, xy$, and $\frac{x}{y}$, in that order. What is the fifth term?

Problem (USAMTS)

In an attempt to copy down a sequence of six positive integers in arithmetic progression, a student wrote down the five numbers $113, 137, 149, 155, 173$, accidentally omitting one. He later discovered that he also miscopied one of them. Can you help him recover the original sequence?

Problem (1986 AIME)
The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in an incorrect sum of 1986. What was the number of the page that was added twice?

Problem (1994 AHSME)
Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If $x, 2y, 3z$ is an arithmetic sequence, then find the value of r.