A-Star 2016 Winter Math Camp

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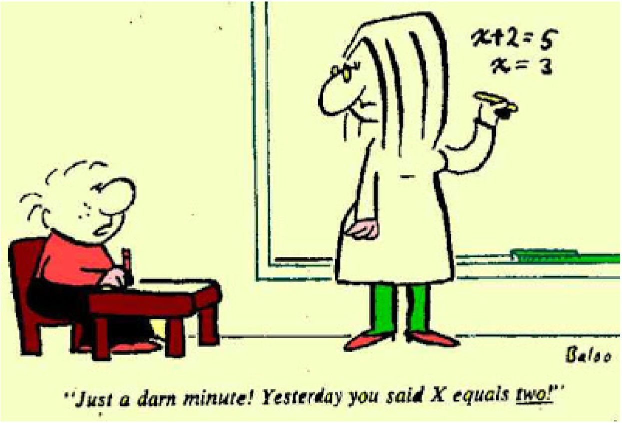
Math Time

1 Algebraic Manipulation



Problem (2000 AMC 12)

If $x, y,$ and z are positive numbers satisfying

$$x + \frac{1}{y} = 4\,, y + \frac{1}{z} = 1\,, \text{and } z + \frac{1}{x} = \frac{7}{3},$$

find the value of xyz.

Problem (AoPS Introduction to Algebra)

Let $A = x + \frac{1}{x}$ and $B = x^2 + \frac{1}{x^2}$. Note that $(x + \frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2}$, therefore, $B = A^2 - 2$. Find formulas for

$$C = x^3 + \frac{1}{x^3}\,, D = x^4 + \frac{1}{x^4}, E = x^5 + \frac{1}{x^5}$$

in terms of A.

1.1 2000 AMC 12

Solution. In order to get the xyz term, we are motivated to multiply the 3 equations together:

$$\begin{aligned}\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right) &= xyz+\frac{1}{xyz}+(x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\\ &= (4)\,(1)\left(\frac{7}{3}\right)=\frac{28}{3}.\end{aligned}$$

What can we do now to simplify this further?

We also add all 3 of the equations:

$$\begin{aligned}\left(x+\frac{1}{y}\right)+\left(y+\frac{1}{z}\right)+\left(z+\frac{1}{x}\right) &= 4+1+\frac{7}{3}=\frac{22}{3}\\ &= (x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).\end{aligned}$$

Therefore, plugging this in to the first equation gives

$$\begin{aligned}xyz+\frac{1}{xyz}+(x+y+z)+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) &= xyz+\frac{1}{xyz}+\frac{22}{3}\\ &= \frac{28}{3}\\ \implies xyz+\frac{1}{xyz} &= 2.\end{aligned}$$

What's xyz equal to then?

Multiply the equation through by xyz and simplify:

$$(xyz)^2 + 1 = 2xyz \implies (xyz)^2 - 2 \cdot xyz + 1 = (xyz - 1)^2 = 0.$$

Therefore, $xyz = \boxed{1}$. □

1.2 Exponent Mayhem

Solution. We begin with C. Inspired by our method for computing B, we attempt to cube $x + \frac{1}{x}$:

$$\begin{aligned}(x + \frac{1}{x})^3 &= x^3 + 3 \cdot x^2 \cdot \frac{1}{x} + 3 \cdot x \cdot \frac{1}{x^2} + \frac{1}{x^3} \\ &= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}.\end{aligned}$$

Therefore,

$$\begin{aligned} A^3 &= x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} = C + 3A \\ \implies C &= A^3 - 3A. \end{aligned}$$

There are two methods for finding D. One of them involves taking $x + \frac{1}{x}$ to the fourth power. In order to continue with this method, however, I must introduce the binomial theorem and Pascal's triangle.

1.3 Pascal's Triangle

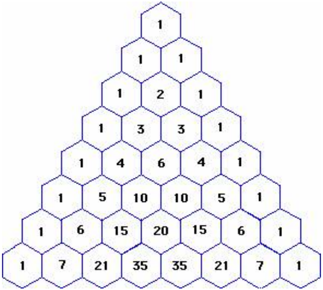


Figure 1: Source: iCoachMath.com

1.4 Binomial-theorem

The binomial theorem states that when we expand $x + y$ to the nth power, the coefficients will be the numbers in the nth row of Pascal's triangle. For instance,

$$(x+y)^4 = \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4.$$

The numbers $1, 4, 6, 4, 1$ make up the 4th row of Pascal's triangle. Furthermore, if you know binomial coefficients, note that

$$\binom{4}{0} = 1, \binom{4}{1} = 4, \binom{4}{2} = 6, \binom{4}{3} = 4, \binom{4}{4} = 1.$$

Theorem (Binomial Expansion)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Using the expansion for $(x+y)^4$, we see that

$$\begin{aligned}(x+\frac{1}{x})^4 &= x^4 + 4\cdot\left(x^3\cdot\frac{1}{x}\right) + 6\cdot\left(x^2\cdot\frac{1}{x^2}\right) + 4\cdot\left(x\cdot\frac{1}{x^3}\right) + \frac{1}{x^4} \\ &= \left(x^4+\frac{1}{x^4}\right) + 4\left(x^2+\frac{1}{x^2}\right) + 6.\end{aligned}$$

We substitute the formula $B = x^2 + \frac{1}{x^2} = A^2 - 2$ to get:

$$D = x^4 + \frac{1}{x^4} = A^4 - 4(A^2-2) - 6 = A^4 - 4A^2 + 2.$$

A simpler method exists for computing D without the use of the binomial theorem. Note that if we multiply A by C, we get the desired x^4 and $\frac{1}{x^4}$ terms:

$$AC = \left(x + \frac{1}{x}\right)\left(x^3 + \frac{1}{x^3}\right) = x^4 + \left(x^2 + \frac{1}{x^2}\right) + \frac{1}{x^4}.$$

From above, we found $C = A^3 - 3A$. Furthermore, $B = x^2 + \frac{1}{x^2} = A^2 - 2$. Substituting these both in give

$$D = x^4 + \frac{1}{x^4} = A(A^3 - 3A) - (A^2 - 2) = A^4 - 4A^2 + 2.$$

Note this matches the answer above.

We attempt our new method for computing E. Note that if we multiply A by D, we get the desired x^5 and $\frac{1}{x^5}$ terms:

$$AD = \left(x + \frac{1}{x}\right)\left(x^4 + \frac{1}{x^4}\right) = x^5 + \left(x^3 + \frac{1}{x^3}\right) + \frac{1}{x^5}.$$

We substitute $D = A^4 - 4A^2 + 2$ and $C = x^3 + \frac{1}{x^3} = A^3 - 3A$ into the above equation:

$$E = x^5 + \frac{1}{x^5} = A\left(A^4 - 4A^2 + 2\right) - \left(A^3 - 3A\right) = A^5 - 5A^3 + 5A.$$

In general, if $x_n = x^n + \frac{1}{x^n}$, then we can recursively find the next term using the identity

$$x_1 x_{n-1} = x_n + x_{n-2} \implies x_n = x_1 x_{n-1} - x_{n-2}.$$

$\square$

Exponent Mayhem in NIMO

The identity above was a key motivator in a 2015 National Internet Math Olympiad (NIMO) challenge problem I cowrote with Evan Chen!

Problem (Justin Stevens and Evan Chen)
Let a, b, c be reals and p be a prime number. Assume that

$$a^n(b+c)+b^n(a+c)+c^n(a+b)\equiv 8 \pmod{p}$$

for each nonnegative integer n. Let m be the remainder when $a^p+b^p+c^p$ is divided by p, and k the remainder when m^p is divided by p^4. Find the maximum possible value of k.

The solution involves finding similar recursion relations with some number theory tricks as well. The answer is 399; try to figure out why after finishing this course!

2 Symmetry in Systems of Equations

The topic of the next several problems will be **symmetry**.

Problem (Self)

Solve the following system of equations in a, b, c:

$$\begin{aligned} 7a + 5b + c &= 11 \\ a + 4b + 6c &= 18 \\ 2a + b + 3c &= 21. \end{aligned}$$

2.1 Let's Avoid Matrices!

Solution. The matrix version of this equation is

$$\begin{bmatrix} 7 & 5 & 1 \\ 1 & 4 & 6 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \\ 21 \end{bmatrix}.$$

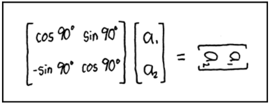


Figure 2: Source: "Matrix Transform", xkcd

Is there a simpler way to do this?

We note that the sum of each column in the matrix is 10. Therefore, we think to sum up the equations:

$$\begin{aligned} 10a + 10b + 10c &= 11 + 18 + 21 = 50 \\ \implies a + b + c &= 5. \end{aligned}$$

We now eliminate to solve for the variables b and c. Subtracting the above relation from the second equation gives

$$\begin{aligned} (a + 4b + 6c) - (a + b + c) &= 18 - 5 = 13 \\ 3b + 5c &= 13. \end{aligned}$$

We also multiply the relation by 2 to eliminate for b and c in the third equation:

$$\begin{aligned} (2a + 2b + 2c) - (2a + b + 3c) &= 2 \cdot 5 - 21 = -11 \\ b - c &= -11. \end{aligned}$$

We multiply the second equation by 5 and add:

$$\begin{aligned}(3b+5c)+(5b-5c) &= 13-5\cdot 11=-42\\ 8b &= -42.\end{aligned}$$

Therefore, $b=\frac{-42}{8}=\frac{-21}{4}$. Substituting this into $b-c=-11$ gives

$$c=11+b=11-\frac{21}{4}=\frac{23}{4}.$$

Finally, plugging both of these into the third equation gives

$$2a=21-b-3c=21+\frac{21}{4}-3\cdot\frac{21}{4}=21-12=9.$$

Hence, $a=\frac{9}{2}$ and $(a,b,c)=\boxed{(\frac{9}{2},\frac{-21}{4},\frac{23}{4})}$.

Verifying through Mathematica

The matrix version of the equation is useful to verify through the computer program, Mathematica. I use the two lines of code:

```
m = {{7, 5, 1}, {1, 4, 6}, {2, 1, 3}}
LinearSolve[m, {11, 18, 21}]
```

Output:

```
{9/2, -(21/4), 23/4}
```

This is the same as what we found above!

□

Problem (2014 Purple Comet)

Let x, y, z be positive real numbers satisfying the simultaneous equations

$$\begin{aligned} x(y^2+yz+z^2) &= 3y+10z \\ y(z^2+zx+x^2) &= 21z+24x \\ z(x^2+xy+y^2) &= 7x+28y. \end{aligned}$$

Find $xy+yz+zx$.

2.2 2014 Purple Comet

Solution. We note that the equations on the left hand side are symmetric. We therefore think to **sum** up the equations. I claim that when we do so, the left hand side becomes $(x+y+z)(xy+xz+yz)$. Why?

Expand out, group, and colour the xyz terms:

$$(x+y+z)(xy+xz+yz) = (x^2y+x^2z)+(y^2x+y^2z)+(z^2y+z^2x)+3xyz$$

We distribute and colour the other terms:

$$\begin{aligned} x\left(y^2+yz+z^2\right) &= y^2x+xyz+z^2x \\ y\left(z^2+zx+x^2\right) &= z^2y+xyz+x^2y \\ z\left(x^2+xy+y^2\right) &= x^2z+xyz+y^2z. \end{aligned}$$

Therefore, by proof by colouring, we have showed that summing up the left hand side gives $(x+y+z)(xy+xz+yz)$. Summing up the right hand side gives

$$(3y+10z)+(21z+24x)+(7x+28y)=31\left(x+y+z\right).$$

We equate the two of them, and divide through by $x+y+z$ (since it is positive):

$$(x+y+z)(xy+xz+yz)=31(x+y+z) \implies xy+xz+yz=\boxed{31}.$$

How did I think of the factorization? The motivation behind it came from the problem statement asking to find $xy+yz+zx$! $\square$

We

3 Weird Functions

Problem (2000 AMC 12)
Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

Problem (Mandelbrot)
Let f be a function such that when $a + b = 2^n$ for a, b, n integers, then $f(a) + f(b) = n^2$. What is $f(2002)$?

Problem (Mandelbrot)
Let f be a function which takes 2 inputs as arguments.The value of f is defined recursively: $f(x, y) = x + f(x - 1, x - y)$. If $f(1, 0) = 5$, find $f(5, 2)$.

3.1 2000 AMC 12

The function that we are given is somewhat strange. It takes as input $x/3$ and returns a value based upon this. We can modify the input, however, by setting $x = 9y$ for real y. Plugging this into the definition of the function gives

$$f(3y) = (9y)^2 + 9y + 1.$$

We want the sum of the values z for which

$$\begin{aligned} f(3z) = (9z)^2 + 9y + 1 &= 7 \\ 81z^2 + 9z - 6 &= 0. \end{aligned}$$

If the roots of this quadratic are r_1 and r_2, then, we can rewrite with leading coefficient 81 as:

$$f(3z) = 81(z - r_1)(z - r_2) = 81z^2 + 9z - 6.$$

Equating the z coefficient gives

$$81(-r_1 - r_2) = 9 \implies r_1 + r_2 = \boxed{\frac{-1}{9}}.$$

If you knew Vieta's formula, we could use the fact that the sum of the roots is $-\frac{b}{a} = -\frac{1}{9}$. If you don't know Vieta's formula, don't worry! We'll cover it shortly.

3.2 2002 Mandelbrot

The condition we're given again is odd.

4 Sequences and Series

Problem (2003 AMC 10)

The first four terms in an arithmetic sequence are $x+y, x-y, xy$, and $\frac{x}{y}$, in that order. What is the fifth term?

Problem (USAMTS)

In an attempt to copy down a sequence of six positive integers in arithmetic progression, a student wrote down the five numbers $113, 137, 149, 155, 173$, accidentally omitting one. He later discovered that he also miscopied one of them. Can you help him recover the original sequence?

Problem (1986 AIME)
The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in an incorrect sum of 1986. What was the number of the page that was added twice?

Problem (1994 AHSME)
Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If $x, 2y, 3z$ is an arithmetic sequence, then find the value of r.