ARML: Telescoping Series

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1   Lecture

With certain sums/products, the majority of the terms will cancel which helps to simplify calculations. Notation used throughout the document:

$$\prod_{i=1}^{n} a_i = a_1 \times a_2 \times a_3 \times \cdots \times a_n$$

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

Example 1.1 (Mathcounts)**.** *Evaluate the product*

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{6}\right)\left(1 + \frac{1}{7}\right)$$

Solution. The product is equivalent to

$$\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\left(\frac{6}{5}\right)\left(\frac{7}{6}\right)\left(\frac{8}{7}\right) = \frac{8}{2} = 4$$

after cancellation □

Example 1.2. *Simplify the product*

$$\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right)\cdots\left(1 - \frac{1}{n}\right)$$

Solution.

$$
\begin{aligned}
\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right)\cdots\left(1 - \frac{1}{n}\right) &= \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{n-1}{n}\right) \\
&= \left(\frac{2}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{\cancel{4}}{\cancel{5}}\right)\cdots\left(\frac{\cancel{n-1}}{n}\right) \\
&= \frac{2}{n}
\end{aligned}
$$

$\square$

Example 1.3. *Evaluate* $\displaystyle\prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right)$

Solution.

$$
\begin{aligned}
\prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right) &= \prod_{k=2}^{n}\frac{(k+1)(k-1)}{k^2} \\
&= \left(\prod_{k=2}^{n}\frac{k+1}{k}\right)\left(\prod_{k=2}^{n}\frac{k-1}{k}\right) \\
&= \left(\frac{n+1}{2}\right)\left(\frac{1}{n}\right) = \frac{n+1}{2n}
\end{aligned}
$$

$\square$

Example 1.4 (AMC 12). *Let* $T_n = 1 + 2 + 3 + \cdots + n$ *and*

$$
P_n = \frac{T_2}{T_2 - 1} \cdot \frac{T_3}{T_3 - 1} \cdot \frac{T_4}{T_4 - 1} \cdots \frac{T_n}{T_n - 1}
$$

for $n = 2, 3, 4, \cdots$. *Find* P_{1991}.

Solution. Notice that $T_n = \frac{n(n+1)}{2}$ and

$$
T_n - 1 = \frac{n(n+1)}{2} - 1 = \frac{n^2 + n - 2}{2} = \frac{(n+2)(n-1)}{2}
$$

. Therefore the product which we want to evaluate is equivalent to

$$
\begin{aligned}
P_n &= \left(\prod_{i=2}^{1991}\frac{i}{i+2}\right)\left(\prod_{i=2}^{1991}\frac{i+1}{i-1}\right) \\
&= \frac{2 \times 3}{1992 \times 1993} \times \left(\frac{1991 \times 1992}{1 \times 2}\right) \\
&= \frac{3 \times 1991}{1993} = \frac{5973}{1993}
\end{aligned}
$$

□

Example 1.5. *Evaluate the sum*

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots + \frac{1}{99 \times 100}$$

Solution. Notice that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, therefore the sum is equivalent to

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right) = \frac{1}{1} - \frac{1}{100} = \frac{99}{100}$$

□

2 Problem Solving

Here is a set of problems involving telescoping series. If you have any questions or want hints on any of these questions please feel free to ask me!

Problem 2.1 (AHSME). Find the sum $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)} + \cdots + \dfrac{1}{255 \times 257}$

Problem 2.2. Find the product $\displaystyle\prod_{n=1}^{20}\left(1 + \frac{2n+1}{n^2}\right)$.

Problem 2.3. Consider the sequence $1, -2, 3, -4, 5, -6, \cdots$ whose nth term is $(-1)^{n+1} \cdot n$. What is the average of the first 200 terms of the sequence?

Problem 2.4 (HMMT). Evaluate $1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - 4 \cdot 5 + \cdots + 2001 \cdot 2002$.

Problem 2.5 (Mandelbrot). Calculate

$$\prod_{n=1}^{13} \frac{n(n+2)}{(n+4)^2}$$

Problem 2.6 (AHSME). Calculate

$$T = \frac{1}{3 - \sqrt{8}} - \frac{1}{\sqrt{8} - \sqrt{7}} + \frac{1}{\sqrt{7} - \sqrt{6}} - \frac{1}{\sqrt{6} - \sqrt{5}} + \frac{1}{\sqrt{5} - 2}.$$

Problem 2.7. Find

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}$$

Problem 2.8. Find the sum $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n-1}{n!}$

Problem 2.9 (AIME)**.** Let $x = \dfrac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$.

Problem 2.10. Find the integer closest to $1000 \displaystyle\sum_{n=3}^{1000} \frac{1}{n^2-4}$.

Problem 2.11. Evaluate $\displaystyle\sum_{k=2}^{n} k!(k^2+k+1)$

Problem 2.12 (Mandelbrot)**.** Compute the product

$$\frac{\left(1998^2 - 1996^2\right)\left(1998^2 - 1995^2\right) \cdots \left(1998^2 - 0^2\right)}{\left(1997^2 - 1996^2\right)\left(1997^2 - 1995^2\right) \cdots \left(1997^2 - 0^2\right)}$$

Problem 2.13 (USAMTS)**.** Determine the value of

$$S = \sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} + \sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}} + \cdots + \sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}} + \cdots + \sqrt{1+\frac{1}{1999^2}+\frac{1}{2000^2}}$$

[Hint: This problem is very difficult. Try expressing each of the radicals in term of n]

Problem 2.14. Evaluate the infinite product $\displaystyle\prod_{n=2}^{\infty} \left(\frac{n^3-1}{n^3+1}\right)$ [Hint: Factor and write out the first few terms]