Exponents and Radicals

Lecture 3

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Exponent Mayhem

Example 1. Let $A = x + \frac{1}{x}$ and $B = x^2 + \frac{1}{x^2}$. Note that $(x + \frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2}$, therefore, $B = A^2 - 2$. Find formulas for

$$C = x^3 + \frac{1}{x^3}, D = x^4 + \frac{1}{x^4}, E = x^5 + \frac{1}{x^5}$$

in terms of A. [a]

[a] Source: AoPS Introduction to Algebra

Exponent Mayhem

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

To compute C, we cube $x + \frac{1}{x}$:

$$\begin{aligned}(x + \frac{1}{x})^3 &= x^3 + 3 \cdot x^2 \cdot \frac{1}{x} + 3 \cdot x \cdot \frac{1}{x^2} + \frac{1}{x^3} \\ &= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}.\end{aligned}$$

Therefore, substituting $A = x + \frac{1}{x}$ gives

$$\begin{aligned}A^3 = x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} \ &= \ C + 3A \\ \implies \ C = A^3 - 3A.\end{aligned}$$

Exponent Mayhem II

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

There are two methods for finding D. One of them involves taking $x + \frac{1}{x}$ to the fourth power. In order to continue with this method, however, I must introduce the binomial theorem and Pascal's triangle.

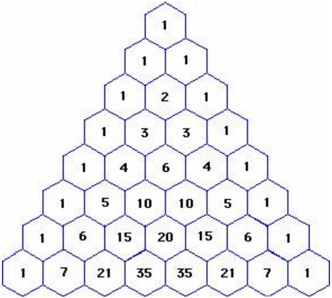

Figure 1: Source: iCoachMath.com

Binomial Theorem

The binomial theorem states that when we expand $x + y$ to the nth power, the coefficients will be the numbers in the nth row of Pascal's triangle. For instance,

$$(x+y)^4 = \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4.$$

The numbers $1, 4, 6, 4, 1$ make up the 4th row of Pascal's triangle.

Furthermore, if you know binomial coefficients, note that

$$\binom{4}{0} = 1, \binom{4}{1} = 4, \binom{4}{2} = 6, \binom{4}{3} = 4, \binom{4}{4} = 1.$$

Theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Exponent Mayhem III

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

Using the expansion for $(x + y)^4$, we see that

$$(x + \frac{1}{x})^4 = x^4 + 4 \cdot \left(x^3 \cdot \frac{1}{x}\right) + 6 \cdot \left(x^2 \cdot \frac{1}{x^2}\right) + 4 \cdot \left(x \cdot \frac{1}{x^3}\right) + \frac{1}{x^4}$$
$$= \left(x^4 + \frac{1}{x^4}\right) + 4\left(x^2 + \frac{1}{x^2}\right) + 6.$$

We substitute the formula $B = x^2 + \frac{1}{x^2} = A^2 - 2$ to get:

$$D = x^4 + \frac{1}{x^4} = A^4 - 4(A^2 - 2) - 6 = A^4 - 4A^2 + 2.$$

Exponent Mayhem IV

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

A simpler method exists for computing D without the use of the binomial theorem. Note that if we multiply A by C, we get the desired x^4 and $\frac{1}{x^4}$ terms:

$$AC = \left(x + \frac{1}{x}\right)\left(x^3 + \frac{1}{x^3}\right) = x^4 + \left(x^2 + \frac{1}{x^2}\right) + \frac{1}{x^4}.$$

From above, we found $C = A^3 - 3A$. Furthermore, $B = x^2 + \frac{1}{x^2} = A^2 - 2$. Substituting these both in give

$$D = x^4 + \frac{1}{x^4} = A(A^3 - 3A) - (A^2 - 2) = A^4 - 4A^2 + 2.$$

Note this matches the answer above.

Exponent Mayhem V

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

We attempt our new method for computing E. Note that if we multiply A by D, we get the desired x^5 and $\frac{1}{x^5}$ terms:

$$AD = \left(x + \frac{1}{x}\right)\left(x^4 + \frac{1}{x^4}\right) = x^5 + \left(x^3 + \frac{1}{x^3}\right) + \frac{1}{x^5}.$$

We substitute $D = A^4 - 4A^2 + 2$ and $C = x^3 + \frac{1}{x^3} = A^3 - 3A$ into the above equation:

$$E = x^5 + \frac{1}{x^5} = A\left(A^4 - 4A^2 + 2\right) - \left(A^3 - 3A\right) = A^5 - 5A^3 + 5A.$$

In general, if $x_n = x^n + \frac{1}{x^n}$, then we can recursively find the next term using the identity

$$x_1 x_{n-1} = x_n + x_{n-2} \implies x_n = x_1 x_{n-1} - x_{n-2}.$$

Exponent Mayhem in NIMO

The identity above was a key motivator in a 2015 National Internet Math Olympiad (NIMO) challenge problem I cowrote with Evan Chen!

Example. (Justin Stevens and Evan Chen) Let a, b, c be reals and p be a prime number. Assume that

$$a^n(b+c)+b^n(a+c)+c^n(a+b)\equiv 8 \pmod{p}$$

for each nonnegative integer n. Let m be the remainder when $a^p+b^p+c^p$ is divided by p, and k the remainder when m^p is divided by p^4. Find the maximum possible value of k.

The solution involves finding similar recursion relations with some number theory tricks as well. The answer is 399; try to figure out why after finishing this course!