System of Equations

Lecture 1

Justin Stevens

Outline

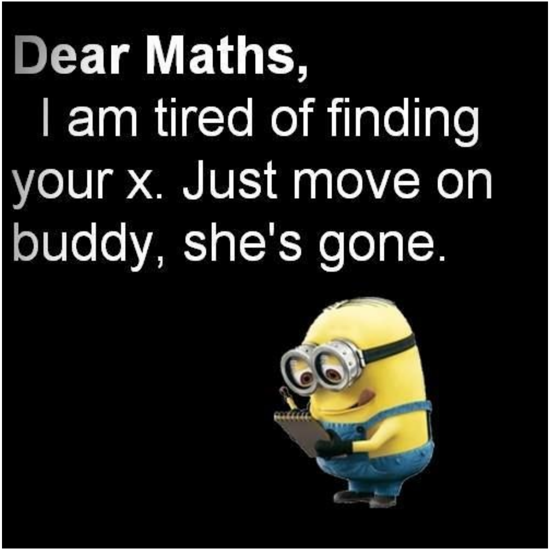
Equations in Three Variables

Example 1. Solve the below system for x, y, z:

$$\begin{cases} x + 2y + 3z = 3 \\ 2x + 5y + 7z = 22 \\ -x + y + z = 6 \end{cases}$$

Example 2. Solve the below system for r, s, t:

$$\begin{cases} r + s + t = 7 \\ 2r + 3s - 5t = 11 \\ 8r + 11s - 13t = 47 \end{cases}$$


Dear Maths,
I am tired of finding your x. Just move on buddy, she's gone.

Example. Solve the below system for x, y, z:

$$\begin{cases} x + 2y + 3z = 3 \\ 2x + 5y + 7z = 22 \\ -x + y + z = 6 \end{cases}$$

Multiplying the first equation by 2 and subtracting gives $y + z = 16$.
Adding the first and third equation gives $3y + 4z = 9$.
Solving this yields $(x, y, z) = (10, 55, -39)$.

1 Solution Dependent on t

Example. Solve the below system for r, s, t:

$$\begin{cases} r + s + t = 7 \\ 2r + 3s - 5t = 11 \\ 8r + 11s - 13t = 47 \end{cases}$$

Multiply the first equation by 2 and second equation by 3 and add these:

$$\begin{aligned} 2\left(r + s + t\right) + 3\left(2r + 3s - 5t\right) &= 2 \cdot 7 + 3 \cdot 11 \\ 8r + 11s - 13t &= 47. \end{aligned}$$

This is exactly the third equation! We consider the first two equations.

We find that $(r, s, t) = (10 - 8t, -3 + 7t, t)$. Substitute and verify!

Follow Up

Example 3. Solve the below system for x, y, z:

$$\begin{cases} x - y + z = 12 \\ 2x + 3y - 4z = 5 \\ 7x + 8y - 11z = 42 \end{cases}$$

Example 4. Solve the below system for r, s, t:

$$\begin{cases} r + 2s + 3t = 7 \\ 2r + 4s + 6t = 14 \\ 3r + 6s + 9t = 21 \end{cases}$$

No Solutions

Example. Solve the below system for x, y, z:

$$\begin{cases} x - y + z = 12 \\ 2x + 3y - 4z = 5 \\ 7x + 8y - 11z = 42 \end{cases}$$

Multiplying the second equation by 3 and add this to the first:

$$\begin{aligned} 3\left(2x + 3y - 4z\right) + \left(x - y + z\right) &= 3 \cdot 5 + 12 \\ 7x + 8y - 11z &= 27. \end{aligned}$$

Hence, this system has no solutions.

2 Basis Solutions

Example. Solve the below system for r, s, t:

$$\begin{cases} r + 2s + 3t = 7 \\ 2r + 4s + 6t = 14 \\ 3r + 6s + 9t = 21 \end{cases}$$

Multiplying the first equation by 2 gives the second equation.

Multiplying the first equation by 3 gives the third equation.

Hence, the solution is $(r, s, t) = (7 - 2s - 3t, s, t)$.

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Outline

1. Equations in Three Variables
2. Symmetry

Symmetry Problems

Example 5. Solve the following system of equations in w, x, y, z:

$$\begin{aligned}3w + x + y + z &= 20\\ w + 3x + y + z &= 6\\ w + x + 3y + z &= 44\\ w + x + y + 3z &= 2.\end{aligned}$$

Example 6. Justin, Lazar, and Daniel are each thinking of a positive number. The product of Justin's and Lazar's is 27. The product of Lazar's and Daniel's is 72. The product of Justin's and Daniel's is 6. Find each person's number.

System in 4 variables

Example. Solve the following system of equations in w, x, y, z:

$$\begin{aligned}
3w + x + y + z &= 20 \\
w + 3x + y + z &= 6 \\
w + x + 3y + z &= 44 \\
w + x + y + 3z &= 2.
\end{aligned}$$

Adding these equations gives

$$6\left(w + x + y + z\right) = 72 \implies w + x + y + z = 12.$$

This yields the solution $(w, x, y, z) = (4, -3, 16, -5)$.

What numbers are they thinking of?

Example. Justin, Lazar, and Daniel are each thinking of a positive number. The product of Justin's and Lazar's is 27. The product of Lazar's and Daniel's is 72. The product of Justin's and Daniel's is 6. Find each person's number.

We have the system of equations

$$\begin{aligned} jl &= 27 \\ ld &= 72 \\ jd &= 6. \end{aligned}$$

Multiplying gives $(jld)^2 = 27 \cdot 72 \cdot 6 = 3^6 \cdot 2^4 \implies jld = 108.$

Therefore, $(j, l, d) = (\frac{3}{2}, 18, 4)$.

More Symmetry Problems

Example 7. If x, y, and z are positive numbers satisfying $x + \dfrac{1}{y} = 4$, $y + \dfrac{1}{z} = 1$, and $z + \dfrac{1}{x} = 7/3$ then what is xyz?

Example 8. Let $A = x + \frac{1}{x}$ and $B = x^2 + \frac{1}{x^2}$. Note that $(x + \frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2}$, therefore, $B = A^2 - 2$. Find formulas for

$$C = x^3 + \frac{1}{x^3}, D = x^4 + \frac{1}{x^4}, E = x^5 + \frac{1}{x^5}$$

in terms of A. [a]

[a] Source: AoPS Introduction to Algebra

More Symmetry

Example. If x, y, and z are positive numbers satisfying $x + \dfrac{1}{y} = 4$, $y + \dfrac{1}{z} = 1$, and $z + \dfrac{1}{x} = 7/3$ then what is xyz?

We want to find the product xyz, therefore, we multiply the three equations:

$$\begin{aligned}\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right) &= xyz + \frac{xy}{x} + \frac{xz}{z} + \frac{x}{zx} + \frac{yz}{y} + \frac{y}{yx} + \frac{z}{yz} + \frac{1}{xyz} \\ &= xyz + (x+y+z) + \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) + \frac{1}{xyz} \\ &= 4 \cdot 1 \cdot \frac{7}{3} = \frac{28}{3}.\end{aligned}$$

Adding the original three equations:

$$\left(x+\frac{1}{y}\right)+\left(y+\frac{1}{z}\right)+\left(z+\frac{1}{x}\right) = 4+1+\frac{7}{3} = \frac{22}{3}$$

How do we now find xyz?

More Symmetry II

Example. If x, y, and z are positive numbers satisfying $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$, and $z + \frac{1}{x} = 7/3$ then what is xyz?

Substituting this into the first equation gives us

$$xyz + \frac{22}{3} + \frac{1}{xyz} = \frac{28}{3} \implies xyz + \frac{1}{xyz} = 2.$$

Finally, multiplying by xyz and re-arranging gives

$$(xyz - 1)^2 = 0 \implies xyz = \boxed{1}.$$

Exponent Mayhem

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

To compute C, we cube $x + \frac{1}{x}$:

$$\begin{aligned}(x + \frac{1}{x})^3 &= x^3 + 3 \cdot x^2 \cdot \frac{1}{x} + 3 \cdot x \cdot \frac{1}{x^2} + \frac{1}{x^3} \\ &= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}.\end{aligned}$$

Therefore, substituting $A = x + \frac{1}{x}$ gives

$$A^3 = x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} \ = \ C + 3A$$

$$\implies \ C = A^3 - 3A.$$

Exponent Mayhem II

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

There are two methods for finding D. One of them involves taking $x + \frac{1}{x}$ to the fourth power. In order to continue with this method, however, I must introduce the binomial theorem and Pascal's triangle.

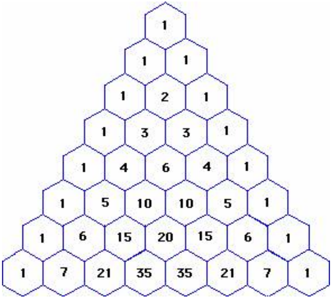

Figure 1: Source: iCoachMath.com

Binomial Theorem

The binomial theorem states that when we expand $x + y$ to the nth power, the coefficients will be the numbers in the nth row of Pascal's triangle. For instance,

$$(x+y)^4 = \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4.$$

The numbers $1, 4, 6, 4, 1$ make up the 4th row of Pascal's triangle.

Furthermore, if you know binomial coefficients, note that

$$\binom{4}{0} = 1, \binom{4}{1} = 4, \binom{4}{2} = 6, \binom{4}{3} = 4, \binom{4}{4} = 1.$$

Theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Exponent Mayhem III

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

Using the expansion for $(x+y)^4$, we see that

$$\begin{aligned}(x + \frac{1}{x})^4 &= x^4 + 4 \cdot \left(x^3 \cdot \frac{1}{x}\right) + 6 \cdot \left(x^2 \cdot \frac{1}{x^2}\right) + 4 \cdot \left(x \cdot \frac{1}{x^3}\right) + \frac{1}{x^4} \\ &= \left(x^4 + \frac{1}{x^4}\right) + 4\left(x^2 + \frac{1}{x^2}\right) + 6.\end{aligned}$$

We substitute the formula $B = x^2 + \frac{1}{x^2} = A^2 - 2$ to get:

$$D = x^4 + \frac{1}{x^4} = A^4 - 4(A^2 - 2) - 6 = A^4 - 4A^2 + 2.$$

Exponent Mayhem IV

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

A simpler method exists for computing D without the use of the binomial theorem. Note that if we multiply A by C, we get the desired x^4 and $\frac{1}{x^4}$ terms:

$$AC = \left(x + \frac{1}{x}\right)\left(x^3 + \frac{1}{x^3}\right) = x^4 + \left(x^2 + \frac{1}{x^2}\right) + \frac{1}{x^4}.$$

From above, we found $C = A^3 - 3A$. Furthermore, $B = x^2 + \frac{1}{x^2} = A^2 - 2$. Substituting these both in give

$$D = x^4 + \frac{1}{x^4} = A(A^3 - 3A) - (A^2 - 2) = A^4 - 4A^2 + 2.$$

Note this matches the answer above.

Exponent Mayhem V

Example. Find formulas for $C = x^3 + \frac{1}{x^3}$, $D = x^4 + \frac{1}{x^4}$, $E = x^5 + \frac{1}{x^5}$ in terms of $A = x + \frac{1}{x}$.

We attempt our new method for computing E. Note that if we multiply A by D, we get the desired x^5 and $\frac{1}{x^5}$ terms:

$$AD = \left(x + \frac{1}{x}\right)\left(x^4 + \frac{1}{x^4}\right) = x^5 + \left(x^3 + \frac{1}{x^3}\right) + \frac{1}{x^5}.$$

We substitute $D = A^4 - 4A^2 + 2$ and $C = x^3 + \frac{1}{x^3} = A^3 - 3A$ into the above equation:

$$E = x^5 + \frac{1}{x^5} = A\left(A^4 - 4A^2 + 2\right) - \left(A^3 - 3A\right) = A^5 - 5A^3 + 5A.$$

In general, if $x_n = x^n + \frac{1}{x^n}$, then we can recursively find the next term using the identity

$$x_1 x_{n-1} = x_n + x_{n-2} \implies x_n = x_1 x_{n-1} - x_{n-2}.$$

Exponent Mayhem in NIMO

The identity above was a key motivator in a 2015 National Internet Math Olympiad (NIMO) challenge problem I cowrote with Evan Chen!

Example. (Justin Stevens and Evan Chen) Let a, b, c be reals and p be a prime number. Assume that

$$a^n(b+c)+b^n(a+c)+c^n(a+b) \equiv 8 \pmod{p}$$

for each nonnegative integer n. Let m be the remainder when $a^p+b^p+c^p$ is divided by p, and k the remainder when m^p is divided by p^4. Find the maximum possible value of k.

The solution involves finding similar recursion relations with some number theory tricks as well. The answer is 399; try to figure out why after finishing this course!

Final Symmetry Problems

Example 9. Let x, y, z be positive real numbers satisfying the simultaneous equations

$$x(y^2 + yz + z^2) = 3y + 10z$$
$$y(z^2 + zx + x^2) = 21z + 24x$$
$$z(x^2 + xy + y^2) = 7x + 28y.$$

Find $xy + yz + zx$. *(Source: 2014 Purple Comet)*

Example 10. Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3,\ \frac{bc}{b+c} = 4,\ \text{and}\ \frac{ca}{c+a} = 5.$$

Compute the value of $\dfrac{abc}{ab+bc+ca}$. *(Source: 2012 Purple Comet)*

We note that the equations on the left hand side are symmetric. We therefore think to **sum** up the equations. I claim that when we do so, the left hand side becomes $(x+y+z)(xy+xz+yz)$. Why?

Expand out, group, and colour the xyz terms:

$$(x+y+z)(xy+xz+yz) = \left(x^2y+x^2z\right) + \left(y^2x+y^2z\right) + \left(z^2y+z^2x\right) + 3xyz$$

We distribute and colour the other terms:

$$\begin{aligned} x\left(y^2+yz+z^2\right) &= y^2x + xyz + z^2x \\ y\left(z^2+zx+x^2\right) &= z^2y + xyz + x^2y \\ z\left(x^2+xy+y^2\right) &= x^2z + xyz + y^2z. \end{aligned}$$

2014 Purple Comet II

Therefore, by proof by colouring, we have showed that summing up the left hand side gives $(x+y+z)(xy+xz+yz)$. Summing up the right hand side gives

$$(3y+10z)+(21z+24x)+(7x+28y)=31\left(x+y+z\right).$$

We equate the two of them, and divide through by $x+y+z$ (since it is positive):

$$(x+y+z)(xy+xz+yz)=31(x+y+z) \implies xy+xz+yz=\boxed{31}.$$

How did I think of the factorization? The motivation behind it came from the problem statement asking to find $xy+yz+zx$!

Fractional System

Example. Let a, b, and c be non-zero real numbers such that

$$\frac{ab}{a+b} = 3, \ \frac{bc}{b+c} = 4, \text{ and } \frac{ca}{c+a} = 5.$$

Compute the value of $\dfrac{abc}{ab+bc+ca}$. *(Source: 2012 Purple Comet)*

Let's take the **reciprocal** of all of the equations:

$$\frac{a+b}{ab} = \frac{1}{3}, \ \frac{b+c}{bc} = \frac{1}{4}, \text{ and } \frac{c+a}{ca} = \frac{1}{5}.$$

Note that we can simplify each of these expressions! For instance, $\frac{a+b}{ab} = \frac{1}{b} + \frac{1}{a}$. Then, the system of equations becomes

$$\frac{1}{b} + \frac{1}{a} = \frac{1}{3}, \ \frac{1}{c} + \frac{1}{b} = \frac{1}{4}, \text{ and } \frac{1}{a} + \frac{1}{c} = \frac{1}{5}.$$

Fractional System II

We continue by taking the reciprocal of the desired expression, $\frac{abc}{ab+bc+ca}$:

$$\frac{ab+bc+ca}{abc} = \frac{1}{c} + \frac{1}{a} + \frac{1}{b}.$$

Aha! We've turned this into something we know how to work with. We sum up the three equations from the previous slide to get

$$2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}.$$

Therefore, $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{47}{120}$. We have to take the reciprocal, however, to get

$$\frac{abc}{ab+bc+ca} = \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} = \boxed{\frac{120}{47}}.$$

Substitution Problem

Example 11. Let x and y be real numbers with $x > y$. Find x if

$$x^2y^2 + x^2 + y^2 + 2xy = 40 \text{ and } xy + x + y = 8.$$

(Source: 2013 HMMT)

2013 HMMT

Example. Let x and y be real numbers with $x > y$. Find x if

$$x^2y^2 + x^2 + y^2 + 2xy = 40 \text{ and } xy + x + y = 8.$$

We substitute $a = xy$ and $b = x + y$. The system then becomes:

$$a^2 + b^2 = 40\,,\ a + b = 8.$$

We square the second equation to arrive at $(a+b)^2 = a^2 + 2ab + b^2 = 64$.

Subtracting from the first equation: $2ab = 64 - 40 = 24 \implies ab = 12$.

Solving this system, we now see that $(a, b) = (2, 6), (6, 2)$.

2013 HMMT II

Example. Let x and y be real numbers with $x > y$. Find x if

$$x^2y^2 + x^2 + y^2 + 2xy = 40 \text{ and } xy + x + y = 8.$$

If $(a, b) = (6, 2)$, then we have $xy = 6$ and $x + y = 2$. We use the identity

$$(z - x)(z - y) = z^2 - z(x + y) + xy.$$

This case gives the quadratic $z^2 - 2z + 6$ with discriminant $\Delta = 2^2 - 4 \cdot 6 = -20$. Hence, there are no positive real solutions.

On the other hand, when $(a, b) = (2, 6)$, we have $xy = 2$ and $x + y = 6$. The quadratic for this case is $z^2 - 6z + 2$. Completing the square:

$$z^2 - 6z + 2 = (z - 3)^2 - 7 \implies z = 3 \pm \sqrt{7}.$$

Since $x > y$, we have $x = \boxed{3 + \sqrt{7}}$.