Simple Direct Model Reference Adaptive Control

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***Abstract*—This document outlines the implementation of Direct Model Reference Adaptive Control with Unnormalized Adaptive Laws on a basic dynamic system. It provides insights based on simulations conducted using MATLAB software. The primary objectives are to achieve precise tracking of the system's output to match the reference model's output and to assess the robustness of the employed control systems.**

I. System's Differential Equation

The second-order system under analysis is the below:

$$M\ddot{q} + G\sin q + C\dot{q} = u \tag{1}$$

, where $q \in \mathbb{R}$ is the rotation angle calculated in *rad* and $\dot{q}$ the angular velocity calculated in *rad/s*. The control input is $u \in \mathbb{R}$ calculated in $N \cdot m$. The system's output is the rotation angle q. Lastly, he M, G, C are nonzero positive but unknown constants.

II. Linearization around zero

In this section we will restrict our control analysis around 0, in order to deal with the non-linear component $\sin q$. Even though this will simplify our design, the control law's usage and credibility will be limited in an area around 0. Outside of this area, we cannot guarantee its robustness and efficiency.

A. Plant's analysis

For the system's second order plant (1), we choose the state variables

$$\begin{cases} x_1 = q \\ x_2 = \dot{q} \end{cases} \overset{\frac{d}{dt}}{\Rightarrow} \begin{cases} \dot{x}_1 = \dot{q} = x_2 \\ \dot{x}_2 = \ddot{q} \end{cases} \underset{\text{M}>0}{\overset{(1)}{\Rightarrow}}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{C}{M}\dot{q} - \frac{G}{M}\sin q + \frac{1}{M}u \end{cases} \Rightarrow$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{C}{M}x_2 - \frac{G}{M}\sin x_1 + \frac{1}{M}u \end{cases} \Rightarrow$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{C}{M}x_2 - \frac{G}{M}\sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \tag{2}$$

The Taylor series expansion for $\sin(x_1)$ around 0 is given by:

$$\sin(x_1) = x_1 - \frac{x_1^3}{3!} + \frac{x_1^5}{5!} - \frac{x_1^7}{7!} + \dots$$

For small values of $x_1 = q$, we can approximate $\sin(x_1)$ as:

$$\sin(x_1) \approx x_1$$

By implementing this linearization around zero, we get from (II-A)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{G}{M} & -\frac{C}{M} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \Rightarrow$$

$$\boxed{\begin{cases} \dot{x} = Ax + Bu, \quad x(0) = x_0 \\ y = C^T x \end{cases}} \tag{3}$$

,where $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -\frac{G}{M} & -\frac{C}{M} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Before we continue with our analysis, we have to check whether the (A, B) is *controllable*. In order to assure that, the determinant of the matrix

$$M = \begin{bmatrix} B & AB \end{bmatrix}$$

must be $\neq 0$. This is derived from the need of the system under control, which is in this case of **second** order, to have $rank(M) = n = 2$.

We have

$$\begin{cases} B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \\ AB = \begin{bmatrix} 0 & 1 \\ -\frac{G}{M} & -\frac{C}{M} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \end{cases} \Rightarrow$$

$$\begin{cases} B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \\ AB = \begin{bmatrix} \frac{1}{M} \\ -\frac{C}{M} \end{bmatrix} \end{cases} \Rightarrow M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{M} \\ \frac{1}{M} & -\frac{C}{M} \end{bmatrix}$$

$$\Rightarrow \det M = 0 * (-\frac{C}{M}) - \frac{1}{M} * \frac{1}{M} = \frac{1}{M^2} > 0, \text{for } \forall \text{ M in } \mathbb{R}.$$

So, we conclude that the system is **controllable**, meaning that we shall continue with our control analysis.

B. MRC for SI-SO

We are going to design the Direct MRC for Single Input Single Output, using as feedback only the system's output $y = x_1 = q$.

The transfer function of the linearized system will be

$$(1) \overset{\sin(x_1)\approx x_1}{\underset{x_1=q=y}{\Longrightarrow}} M\ddot{y} + Gy + C\dot{y} = u \overset{\text{Laplace}}{\underset{\text{Transform}}{\Longrightarrow}}$$

$$Ms^2Y - MsY(0) - M\dot{Y}(0) + CsY - CsY(0) + GY = U$$

We assume that $Y(0) = \dot{Y}(0) = 0$ without any noteworthy repercussions and we get

$$Y(Ms^2 + Cs + G) = U \Rightarrow Y = \frac{1}{Ms^2 + Cs + G}U$$

$$\overset{M>0}{\Rightarrow} Y = \frac{\frac{1}{M}}{s^2 + \frac{C}{M}s + \frac{G}{M}}U \overset{Y=Y_p}{\Rightarrow}$$

$$Y_p = G_p(s)U$$

, with $G_p(s)$ expressed in the form

$$G_p(s) = k_p\frac{Z_p(s)}{R_p(s)}$$

, where $k_p(s) = \frac{1}{M}$ the high frequency gain and $Z_p(s) = 1$ alongside with $R_p(s) = s^2 + \frac{C}{M}s + \frac{G}{M}$ are monic polynomials, since the highest order's components is equal to 1.

We observe that the polynomial $Z_p(s)$ has order $m_p = 0$, while the polynomial $R_p(s)$ has order $n_p = 2$, meaning that the system under control is of relative order $n^* = n_p - m_p = 2$.

On that note, the **reference model**

$$Y_m = G_m(s)r = k_m\frac{Z_m}{R_m}r$$

will have the same relative order $n_m^* = n^* = 2$

We refer to the problem of finding the desired **u** to meet the control objective as the MRC problem.

In order to meet the MRC objective with a control law that is implementable, i.e., a control law that is free of differentiators and uses only measurable signals, we assume that the plant and reference model satisfy the following assumptions:

Plant Assumptions

1) $Z_p(s)$ is a monic Hurwitz polynomial of degree m_p
2) An upper bound n of the degree n_p of $R_p(s)$ is known
3) the relative degree $n^* = n_p - m_p$ of $G_p(s)$ is known
4) the sign of the high frequency gain k_p is known

Reference Model Assumptions

1) $Z_m(s)R_m(s)$ are monic Hurwitz polynomials of degree q_m, p_m, respectively, where $p_m \leq n$.
2) The relative degree $n_m^* = p_m - q_m$ of $G_m(s)$ is the same as that of $G_p(s)$, i.e., $n_m^* = n^*$.

Let's start with the Plant Assumptions. Firstly, $Z_p(s) = 1$, meaning that it actually is a monic Hurwitz polynomial of degree $m_p = 0$. Secondly, for the $R_p(s)$, $n_p = 2 \leq n$, proving that there is a upper bound. Thirdly, we know the relative degree of $G_p(s)$, being $n^* = 2$. Lastly, $k_p = \frac{1}{M} > 0$, since constant $M > 0$, so we have knowledge of the high frequency gain's sign. So, all the plant assumptions are met.

Before we deal with the Reference Model assumptions, we must first choose its transfer function. This is going to be

$$G_m(s) = k_m\frac{1}{(s + k_o)^2}$$

, $Z_m(s) = 1$ of degree 0, $R_m = (s + k_o)^2$ of degree 2 and k_m, k_o known positive constants of our selection. We choose to have a double pole in $-k_o$, in order to have $\zeta = 1$ damping ratio, meaning that the reference model of relative degree $n_m^* = 2$ will be *critically damped*. In this case, the *natural frequency* of the reference model will be through the matching of the transfer functions

$$G_m(s) = \frac{k_m}{s^2 + 2k_os + k_o^2} = \frac{\omega_n^2}{s^2 + 2\zeta\omega ns + \omega_n^2}$$

$\omega_n = k_o = \sqrt{k_m}$. This selection will also provide output value in the steady state equal to the given input, $y_{ss} = r$.

Lastly, the settling time of the reference model will be

$$\boxed{t_s \approx \frac{4}{\zeta\omega} = \frac{4}{k_o}}$$

which parameter we can modify as we want.

Now, on the Reference model's assumptions. Firstly, $Z_m(s) = 1$ of degree $q_m = 0$, monic and Hurwitz polynomial. Likewise, $R_m = (s+k_o)^2 = s^2+2k_os+k_o^2$ of degree $p_m = 2$, monic since $1s^2$ and Hurwitz, since it has double roots at $-k_o < 0$. We, also, want $p_m \leq n \Rightarrow 2 \leq 2$. So, we are done with the first requirement. Secondly, $n_m^* = p_m - q_m = 2-0 = 2 = n^*$ of the plant, meaning that $G_m(s)$ and $G_p(s)$ have the same relative degree. So, all the reference model assumptions are met.

However, because we are dealing with *MRC SI-SO* of *relative degree 2*, there is one additional assumption we have to meet. In our occasion, we have to show that the

$$G_m^{'}(s) = (s + p_0)G_m(s) = \frac{k_m(s + p_0)}{(s + k_o)^2}, \quad p_0 > 0$$

, of relative degree 1, is strictly proper and *SPR*, because the transfer function $G_m(s)$ cannot be *SPR* for $n^* = 2$.

To prove that $G_m^{'}(s)$ is *SPR*, we have to show that

1) $G_m^{'}(s)$ has derivative for $s = \sigma + j\omega$
2) $Real\{G_m^{'}(j\omega)\} > 0 \ \forall\omega$
3) $\lim_{|\omega|\to\infty} \omega^2 Real\{G_m^{'}(j\omega)\} \in \mathbb{R}$

Firstly, $G_m^{'}(s)$ actually has derivative for $s = \sigma + j\omega$.

Secondly, we have

$$G_m^{'}(j\omega) = \frac{k_m(j\omega + p_0)}{(j\omega)^2 + j2k_o\omega + k_o^2} = \frac{k_m(j\omega + p_0)}{-\omega^2 + j2k_o\omega + k_o^2} =$$

$$= \frac{k_m(j\omega + p_0)(-\omega^2 - j2k_o\omega + k_o^2)}{(\omega^2 + k_o)^2} \overset{Real}{\Rightarrow}$$

$$Real\{G_m^{'}(j\omega)\} = \frac{k_m(\omega^2(2k_o - p_0) + p_0 k_o^2)}{(\omega^2 + k_o)^2}$$

Consequently, to achieve $Real\{G_m^{'}(j\omega)\} > 0, \forall\omega \in \mathbb{R}$, we will have

$$\boxed{\mathbf{k_o \geq \frac{p_0}{2}}}.$$

Lastly, we have

$$\lim_{|\omega|\to\infty} \omega^2 Real\{G_m^{'}(j\omega)\} =$$

$$\lim_{|\omega|\to\infty} \frac{k_m(\omega^4(2k_o - p_0) + p_0 k_o^2 \omega^2)}{(\omega^2 + k_o)^2}$$

$$\overset{\begin{bmatrix}0\\0\end{bmatrix}}{\underset{\text{Hospital}}{\Longrightarrow}} \lim_{|\omega|\to\infty} \frac{k_m(2k_o - p_0)\omega^4}{\omega^4} = k_m(2k_o - p_0) \in \mathbb{R}$$

Therefore, the $G_m^{'}(s) = (s+p_0)G_m(s)$ is *SPR* and we have concluded with the Reference Model assumptions.

In conclusion, we showed that each and every Plant and Reference Model assumption has been met, hence we are able and ready to deploy **Direct MRC for Single Input Single Output for relative degree** $n^* = 2$.

C. Control and Adaptive Law

The *MRAC* Control and Adaptive Law applied, derived from an extensive theoretical analysis, are designed as

$$\boxed{\begin{aligned}
&u = \theta^T\omega + \dot{\theta}^T\phi \\
&\dot{\omega}_1 = F\omega_1 + gu, \quad \omega_1(0) = 0 \\
&\dot{\omega}_2 = F\omega_2 + gy_p, \quad \omega_2(0) = 0 \\
&\dot{\phi} = -p_0\phi + \omega, \quad \phi(0) = 0 \\
&\dot{\theta} = -\Gamma\epsilon\phi \operatorname{sgn}(\frac{k_p}{k_m}) \\
&\omega = \begin{bmatrix}\omega_1^T & \omega_2^T & y & r\end{bmatrix} \\
&\epsilon = y_p - y_m
\end{aligned}}$$

, where $\theta = \begin{bmatrix}\theta_1, \theta_2, \theta_3, c_0\end{bmatrix}^T, \phi \in \mathbb{R}^4$ and $\operatorname{sgn}(\frac{k_p}{k_m}) = 1$. We choose the monic Hurwitz polynomial of $n - 1 = 1$ order containing the $Z_m(s)$

$$\Lambda(s) = \Lambda_0(s)Z_m(s) \overset{Z_m(s)=1}{=} s + \lambda_0, \quad \lambda_0 > 0$$

The F, g components are produced from the coefficients of the $\Lambda(s)$, getting $F = -\lambda_0$ and $g = 1$. Also, the $\Gamma = diag\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \Gamma^T, \gamma_i > 0, \forall i = 1, .., 4$ is the gain positive diagonal definite matrix of the adaptive law.

This MRAC scheme, since all the required assumptions have been fulfilled, guarantees that

1) All signals in the closed-loop plant are *bounded* and the tracking error ϵ converges to zero asymptotically.
2) If R_p, Z_p are coprime and r is sufficiently rich of order 2n, then the parameter error $\tilde{\theta} = |\theta - \theta^*|$ and the tracking error ϵ converge to zero exponentially fast.

Based on the second point, we will choose

$$r = 1.5\sin(1.3t) + 1.8\cos(1.6t)$$

which is sufficiently rich of 2n=4, to achieve exponential convergence on both parameter and tracking error.

Thus, the two laws have been set and it's now time to run simulations for various values of the free non-zero positive parameters of the MRAC, them being:

1) k_m
2) $k_o, \quad k_m = k_o^2$
3) $p_0 \leq 2k_0$
4) λ_0

D. Simulations

We run the simulations using the ***MATLAB*** software. We gather all the differential equations in one *odefun*, use multiple different hyperparameters' values and record the results. For the system's simulation, we use its original form and not the linearized one, in order to see how our control system would behave in a real application.

We choose to have zero initial conditions for every parameter estimation and see where that will lead us. For the rest of the differential equations, the initial conditions have alreday been defined zero.

We begin our research giving our hyperparameters the values:

1) $k_o = 1$
2) $k_m = 1$
3) $p_0 = 0.1$
4) $\lambda_0 = 0.1$
5) Γ = diag$\{1\ ,\ 1\ ,\ 1\ ,\ 1\}$

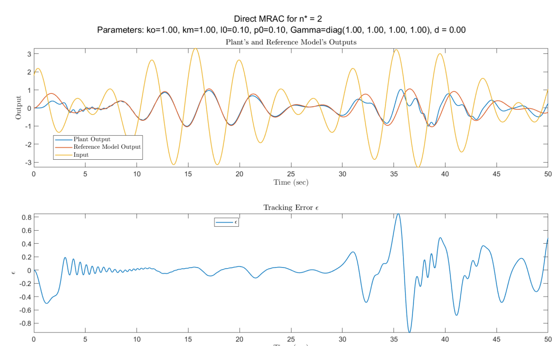

Fig. 1.

We see that the plant achieves in catching the reference model for some time, but, at the same time, they both don't reach the input. The tracking error ϵ isn't converging to zero permanently, something we are going to fix afterwards.

Let's increase the Γ matrix values in order to speed up the process of the convergence of the tracking error to zero. We saw that the most dominant parameter is the c_0 and the follows the θ_1, while the rest are insignificant. So, we will aim on increasing their influence accordingly.

Γ = diag$\{2,\ 1\ ,\ 1\ ,\ 10\}$

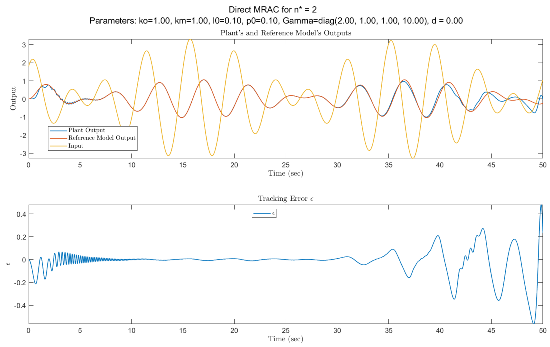

Fig. 2.

We achieved faster convergence. Let's further increase Γ fourth gain.

Γ = diag{5, 1 , 1 , 100}

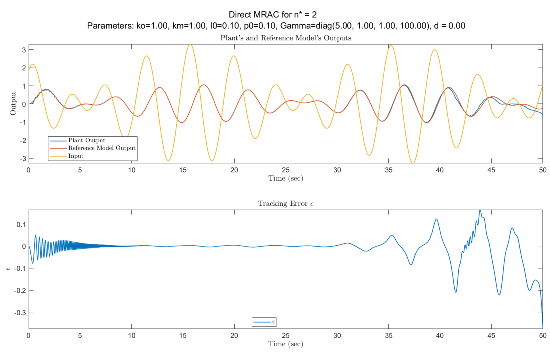

Fig. 3.

Γ = diag{5, 1 , 1 , 1000}

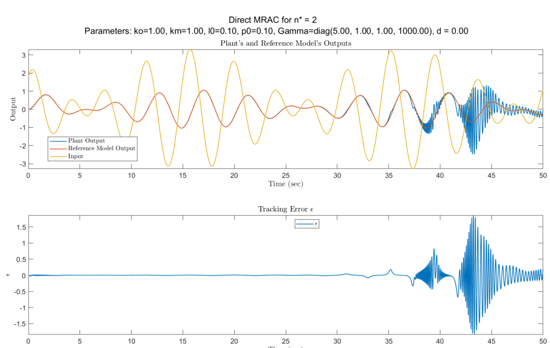

Fig. 4.

By decreasing the p_0, l_0 to

1) $p_0 = 0.01$
2) $\lambda_0 = 0.01$

we will fix the convergence of the tracking error to zero

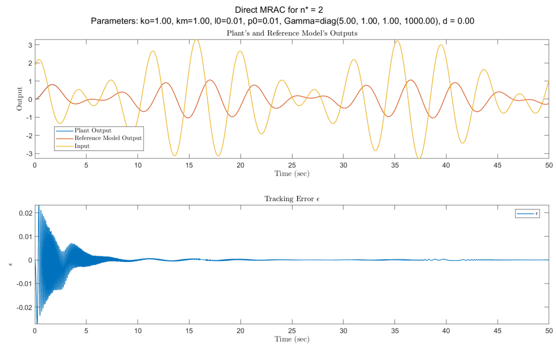

Fig. 5.

We battled the problem with the tracking error. However, in order to conquer the control objective, we have to make both plant and reference model to follow the input r. To accomplish that, we will tune the k_o, k_m hyperparameters. We increase them to

1) $k_o = 10$
2) $k_m = 100$

and we get

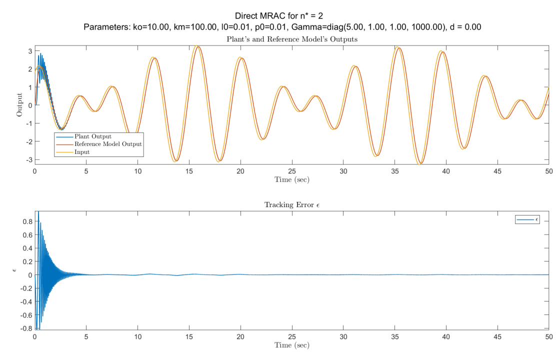

Fig. 6.

We see that even though the two outputs are trying to reach the input far better now, the convergence isn't perfect. However, if we increase k_m, k_o one order of magnitude, the initial overshoot of the error will increase even more. So, we are satisfied with this outcome.

E. Conclusions on Linearized System

We ended up with the hyperparameter values :

1) $k_o = 10$
2) $k_m = 100$
3) $p_0 = 0.01$
4) $\lambda_0 = 0.01$
5) Γ = diag{5 , 1 , 1 , 1000}

, with which we drove the plant to follow the reference model and , subsequently, them to follow the sufficiently reach input r with exponential rate. With $k_o = 100 \Rightarrow t_s = \frac{4}{100} = 0.04s$ the settling time of the reference model and with sampling period used in the odefun $t_{\text{sampling}} = 0.01s$

sufficiently short for our under control system, we drive the model and the plant wherever we like fastly.

Some issues would be the high gain values and the significant oscillations of the c_0 estimation and the plant's output. If the real system could possibly broke down due to these observations, then we would work with lower values, meaning higher convergence times with less precision.

III. Without Linearization

In this part, we will occupy ourselves with the system itself as defined in (1) without applying any kind of linearization to deal with the non-linear component $\sin q$. To succeed in this task, we are going to implement *Direct MRC* with full-state measurement, where, the control objective is

1) all signals in the closed-loop plant are bounded
2) plant state x follows the state x_m of the reference model.

A. MRC for Vector-Case

We have from the previous plant analysis before applying linearization around zero

$$\text{(II-A)} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{C}{M} x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} (u - G\sin x_1) \Rightarrow$$

$$\boxed{\dot{x} = Ax + B\Lambda(u + f(x))}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{C}{M} \end{bmatrix}$, $\Lambda = \frac{1}{M}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $f(x) = -Gsin(x_1) = -G\phi(x), \quad \mathbb{R}^2 \Rightarrow \mathbb{R}$

The Reference model's state must be also $\mathbb{R}^2$ with specific ζ and ω_n. According to that, we choose its transfer function

$$G_m(s) = \frac{k_m \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where k_m an extra hyperparameter for any further needed tuning.

Then, we have,

$$x_m(s) = G_m(s)r(s) \Rightarrow s^2 x_m(s) + 2\zeta\omega_n x_m(s)s + \omega_n^2 x_m(s) = k_m\omega_n^2 r(s)$$

$$\overset{\text{Inverse Laplace}}{\underset{\text{Transform}}{\Longrightarrow}} \ddot{x}_m + 2\zeta\omega_n\dot{x}_m + \omega_n^2 x_m = k_m\omega_n^2 r$$

We have the state variables

$$\begin{cases} x_1 = x_m \\ x_2 = \dot{x}_m \end{cases} \overset{\frac{d}{dt}}{\Rightarrow} \begin{cases} \dot{x}_1 = \dot{x}_m = x_2 \\ \dot{x}_2 = \ddot{x}_m \end{cases} \Rightarrow$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2 + k_m\omega_n^2 r \end{cases}$$

We finally get

$$\boxed{\dot{x}_m = A_m x_m + B_m r}$$

where $x_m = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A_m = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$ and $B_m = \begin{bmatrix} 0 \\ k_m\omega_n^2 \end{bmatrix}$

B. Control and Adaptive Law

We are ready to begin with our control analysis. We have already proved that the system (A,BΛ) is controllable. We will start by assuming that A, B are known matrices and then, based on that assumptions, we shall continue with the given plant with unknown ones.

1) (A, BΛ) known: We choose the control law

$$u = -K^*x - L^*r - N^*\phi(x)$$

where $K^* = \begin{bmatrix} k_1^* & k_2^* \end{bmatrix}, \in \mathbb{R}^{1x2}$, $L^* = \lambda^* \in \mathbb{R}^{1x1}$ and $N^* = n^* \in \mathbb{R}^{1x1}$ and the plant will be

$$\dot{x} = Ax + B\Lambda(u + f(x)) \Rightarrow$$
$$\dot{x} = Ax + B\Lambda(-K^*x - L^*r - N^*\phi(x) + f(x)) \Rightarrow$$

$$\begin{cases} \dot{x} = (A - B\Lambda K^*)x - B\Lambda L^* r + B\Lambda(-G\phi(x) - N^*\phi(x)) \\ \dot{x}_m = A_m x_m + B_m r \end{cases}$$

In order to make state x and x_m equal for the same input r, we will have to match the two differential equations for the same initial conditions

We want

$$A - B\Lambda K^* = A_m \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & -\frac{C}{M} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{M} \begin{bmatrix} k_1^* & k_2^* \end{bmatrix} =$$
$$= \begin{bmatrix} 0 & 1 \\ -\frac{k_1^*}{M} & -\frac{C+k_2^*}{M} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$$

By matching the two parts of the equation, we get

$$\boxed{\begin{aligned} k_1^* &= M\omega_n^2 \\ k_2^* &= 2M\zeta\omega_n - C \end{aligned}}$$

Moreover, we want

$$-B\Lambda L^* = B_m \Rightarrow -\begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{M} \lambda^* = \begin{bmatrix} 0 \\ k_m\omega_n^2 \end{bmatrix}$$

By matching the two parts of the equation, we get

$$\boxed{\lambda^* = -Mk_m\omega_n^2}$$

Lastly, we want

$$-G\phi(x) - N^*\phi(x) = 0 \Rightarrow -G - n^* = 0 \Rightarrow$$
$$\boxed{n^* = -G}$$

Ultimately, for perfect matching we desire

$$\boxed{\begin{aligned} A - B\Lambda K^* &= A_m \\ -B\Lambda L^* &= B_m \end{aligned}}$$

and, at the same time, we also acquired the desired parameters of the control law.

2) (A, BΛ) unknown: In this case, which we study, we will choose the control law

$$u = -K(t)x - L(t)r - N(t)\phi(x)$$

where $K(t), L(t)$ and $N(t)$ are the estimations of the unknown control parameters. Now, we have to guarantee the robustness and stability of the control law when applied on our plant and, simultaneously, to generate the estimations on-line by designing an adaptive law.

We add and subtract in the plant the desired input produced for known (A, BΛ) matrices multiplied by BΛ and we get

$$\begin{aligned}
&\dot{x} = Ax + B\Lambda(u + f(x)) \pm B\Lambda(-K^*x - L^*r - N^*\phi(x)) \Rightarrow \\
&\dot{x} = (A - B\Lambda K^*)x - B\Lambda L^*r + B\Lambda(-G\phi(x) - N^*\phi(x)) \\
&\quad + B\Lambda(u + K^*x + L^*r + N^*\phi(x)) \\
&\overset{\text{(III-B1)}}{\Rightarrow} \boxed{\dot{x} = A_m x + B_m r + B\Lambda(u + K^*x + L^*r + N^*\phi(x))}
\end{aligned}$$

We define the tracking error between the two states as

$$e = x - x_m$$

and we get

$$\begin{aligned}
&e = x - x_m \Rightarrow \dot{e} = \dot{x} - \dot{x}_m \overset{Replacing}{\Rightarrow} \\
&\dot{e} = A_m x + B_m r + B\Lambda(u + K^*x + L^*r + N^*\phi(x)) \\
&\quad - (A_m x_m + B_m r) \\
&\Rightarrow \dot{e} = A_m e + B\Lambda(u + K^*x + L^*r + N^*\phi(x)) \overset{\text{Replacing u}}{\Rightarrow} \\
&\dot{e} = A_m e + B\Lambda[-(K - K^*)x - (L - L^*)r - (N - N^*)\phi(x)]
\end{aligned}$$

We define as the parameter errors

$$\begin{aligned}
\tilde{K} &= K - K^* \Rightarrow \dot{\tilde{K}} = \dot{K} \\
\tilde{L} &= L - L^* \Rightarrow \dot{\tilde{L}} = \dot{L} \\
\tilde{N} &= N - N^* \Rightarrow \dot{\tilde{N}} = \dot{N}
\end{aligned}$$

and the above differential equation becomes

$$\dot{e} = A_m e + B\Lambda(-\tilde{K}x - \tilde{L}r - \tilde{N}\phi(x))$$

We assume that $\Gamma^{-1} = L^* sgn(\lambda^*) = -L^*$, because $\lambda^* = -Mk_m w_n^2 < 0$ and $B\Lambda = -B_m L^{*-1}$ This way we get

$$\boxed{\dot{e} = A_m e - B_m L^{*-1}(-\tilde{K}x - \tilde{L}r - \tilde{N}\phi(x))}$$

We propose the following Lyapunov function candidate

$$\begin{aligned}
V(e, \tilde{K}, \tilde{L}, \tilde{N}) =& e^T P e + tr\{\frac{\tilde{K}^T \Gamma \tilde{K}}{\gamma_1}\} + tr\{\frac{\tilde{L}^T \Gamma \tilde{L}}{\gamma_2}\} + \\
&+ tr\{\frac{\tilde{N}^T \Gamma \tilde{N}}{\gamma_3}\} \geq 0
\end{aligned}$$

where $P^T = P > 0$ satisfies the Lyapunov equation $PA_m + A_m^T P = -Q$ for some $Q^T = Q > 0$.

We derive the function and we get

$$\begin{aligned}
\dot{V} =& \dot{e}^T P e + e^T P \dot{e} + 2tr\{\frac{\tilde{K}^T \Gamma \dot{\tilde{K}}}{\gamma_1}\} + 2tr\{\frac{\tilde{L}^T \Gamma \dot{\tilde{L}}}{\gamma_2}\} + \\
&+ 2tr\{\frac{\tilde{N}^T \Gamma \dot{\tilde{N}}}{\gamma_3}\} = \\
&= e^T(A_m^T P + P A_m)e - (B_m L^{*-1}(-\tilde{K}x - \tilde{L}r - \tilde{N}\phi(x)))^T P e - \\
&- e^T P(B_m L^{*-1}(-\tilde{K}x - \tilde{L}r - \tilde{N}\phi(x))) + 2tr\{\frac{\tilde{K}^T \Gamma \dot{\tilde{K}}}{\gamma_1}\} + \\
&+ 2tr\{\frac{\tilde{L}^T \Gamma \dot{\tilde{L}}}{\gamma_2}\} + 2tr\{\frac{\tilde{N}^T \Gamma \dot{\tilde{N}}}{\gamma_3}\} = \\
&= -e^T Q e - 2e^T P B_m L^{*-1}(-\tilde{K}x - \tilde{L}r - \tilde{N}\phi(x)) + \\
&+ 2tr\{\frac{\tilde{K}^T \Gamma \dot{\tilde{K}}}{\gamma_1}\} + 2tr\{\frac{\tilde{L}^T \Gamma \dot{\tilde{L}}}{\gamma_2}\} + 2tr\{\frac{\tilde{N}^T \Gamma \dot{\tilde{N}}}{\gamma_3}\}
\end{aligned}$$

We have

$$\begin{aligned}
&- 2e^T P B_m L^{*-1}(-\tilde{K}x) = 2tr\{x e^T P B_m L^{*-1} \tilde{K}\} = \\
&= -2tr\{\tilde{K}^T \Gamma B_m^T P e x^T\}
\end{aligned}$$

and likewise

$$\begin{aligned}
&2e^T P B_m L^{*-1}(-\tilde{L}r) = -2tr\{\tilde{L}^T \Gamma B_m^T P e r\} \\
&2e^T P B_m L^{*-1}(-\tilde{N}\phi(x)) = -2tr\{\tilde{N}^T \Gamma B_m^T P e \phi(x)\}
\end{aligned}$$

By substituting the above in the derivative of the Lyapunov function we get

$$\begin{aligned}
\dot{V} =& - e^T Q e + 2tr\{\tilde{K}^T \Gamma(\frac{\dot{K}}{\gamma_1} - B_m^T P e x^T)\} + \\
&+ 2tr\{\tilde{L}^T \Gamma(\frac{\dot{L}}{\gamma_2} - B_m^T P e r)\} + 2tr\{\tilde{N}^T \Gamma(\frac{\dot{N}}{\gamma_3} - B_m^T P e \phi(x))\}
\end{aligned}$$

By choosing the ***Adaptive Laws***

$$\boxed{\begin{aligned}
\dot{K} &= \gamma_1 B_m^T P e x^T \\
\dot{L} &= \gamma_2 B_m^T P e r \\
\dot{N} &= \gamma_3 B_m^T P e \phi(x)
\end{aligned}}$$

We get

$$\dot{V} = -e^T Q e \leq 0$$

We showed that

$$\begin{cases} V > 0 \\ \dot{V} \leq 0 \end{cases} \Rightarrow \lim_{t \to \infty} V(e, \tilde{K}, \tilde{L}, \tilde{N}) = V_\infty < \infty$$

$$\Rightarrow \begin{cases} (\mathbf{e}, \tilde{K}, \tilde{L}, \tilde{N}) \in L_\infty \\ x_m \in L_\infty \\ r \in L_\infty \\ K^*, L^*, N^* \in L_\infty \end{cases} \Rightarrow \begin{cases} x = e + x_m \in L_\infty \\ K, L, N \in L_\infty \\ \phi(x) \in L_\infty \end{cases}$$

$$\Rightarrow u \in L_\infty$$

Moreover, we have from integration

$$\begin{aligned}
&- \int_\infty^0 \dot{V}(\tau) d\tau = - \int_\infty^0 (e^T Q e) \tau d\tau \\
&\Rightarrow V(0) - V_\infty = - \int_\infty^0 (e^T Q e) \tau d\tau \\
&\Rightarrow \boxed{e \in L_2}
\end{aligned}$$

Also, we deduce that

$$\begin{cases} \dot{e} = A_m e + B\Lambda(-\tilde{K}x - \tilde{L}r - \tilde{N}\phi(x)) \\ (e, \tilde{K}, \tilde{L}, \tilde{N}) \in L_\infty \\ (x, r, \phi(x)) \in L_\infty \end{cases} \Rightarrow \boxed{\dot{e} \in L_\infty}$$

Finally, from all the above, by using the ***Barbalat's Lemma***, we acquire

$$\begin{cases} e \in L_2 \cup L_\infty \\ \dot{e} \in L_\infty \end{cases} \Rightarrow \begin{cases} \boxed{\lim_{t\to\infty} e(t) = 0} \\ \dot{K} = \gamma_1 B_m^T P e x^T \\ \dot{L} = \gamma_2 B_m^T P e r \\ \dot{N} = \gamma_3 B_m^T P e \phi(x) \end{cases} \Rightarrow \boxed{\begin{cases} \lim_{t\to\infty} \dot{K} = 0 \\ \lim_{t\to\infty} \dot{L} = 0 \\ \lim_{t\to\infty} \dot{N} = 0 \end{cases}}$$

meaning that the error converges to zero eventually and the rate of the parameters' estimations declines, driving the estimations into constant values.

In conclusion, we showed that the below control and adaptive law altogether

$$\boxed{\begin{aligned} &u = -Kx - Lr - N\phi(x) \\ &\dot{K} = \gamma_1 B_m^T P e x^T \\ &\dot{L} = \gamma_2 B_m^T P e r \\ &\dot{N} = \gamma_3 B_m^T P e \phi(x) \\ &\dot{x} = Ax + B\Lambda(u + f(x)) \\ &\dot{x}_m = A_m x_m + B_m r \\ &e = x - x_m \end{aligned}}$$

is capable of achieving the control objective, while also keep all the signals inside the close-loop upper-bounded. We also guaranteed that the estimations of the control law's parameters will converge into a constant value. In order to converge them into the desired values found before, them being :

$$\boxed{\begin{aligned} &k_1^* = M\omega_n^2 \\ &k_2^* = 2M\zeta\omega_n - C \\ &\lambda^* = -Mk_m\omega_n^2 \\ &n^* = -G \end{aligned}}$$

we have to provide to our plant and reference model a sufficiently rich input of order 2n = 4,to achieve exponential convergence on both parameter and tracking error. This can be

$$r = 1.5\sin(1.3t) + 1.8\cos(1.6t)$$

Lastly, we prefer the matrix P to be diagonal. Let's say that we want it to be

$$P = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

where q is the hyperparameter we will be able to tune

We have in the Lyapunov equation

$$PA_m + A_m^T P = -Q \Rightarrow -Q = P = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} + \begin{bmatrix} 0 & -\omega_n^2 \\ 1 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} 0 & q \\ -q\omega_n^2 & -2q\zeta\omega_n \end{bmatrix} + \begin{bmatrix} 0 & -q\omega_n^2 \\ q & -2q\zeta\omega_n \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & q(\omega_n^2 - 1) \\ q(\omega_n^2 - 1) & 4q\zeta\omega_n \end{bmatrix}$$

So, we select the Q matrix as described above in order to get the desired diagonal P matrix from the Lyapunov equation, which will provide that each state's control law will not be affected by the other states' errors.

C. Simulations

For the simulations, we will use

$$\omega_n = 1$$
$$\zeta = 0.7$$

given by the assignment. For these values, the reference model's transfer function will be

$$G_m(s) = \frac{k_m}{s^2 + 1.4s + 1}$$

By opting for zero initial conditions across all parameters, including those of the plant and reference model states, we initiate our exploration of attaining the control objective.

It's important to note that the convergence of parameter estimations to their desired values is not mandatory for achieving the control objective, and as such, we will neglect it.

The hyper-parameters which we will proceed to tune are going to have the initial values

1) $k_m = 1$
2) $q = 1$
3) $\gamma_1 = 1$
4) $\gamma_2 = 1$
5) $\gamma_3 = 1$

We have

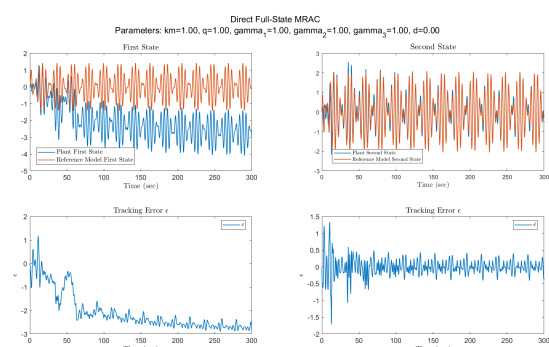

Fig. 7.

We can see that the reference model can't precisely track the input due to the low value of ω_n and due to $\zeta = 0.7 \neq 1$, which means that we have overshoot.

Moreover, we observe that even though the second state's error converges, the first one tends to walk away from the reference model. To fix this, we will increase the q for faster convergence of the adaptive laws.

1) $q = 10$

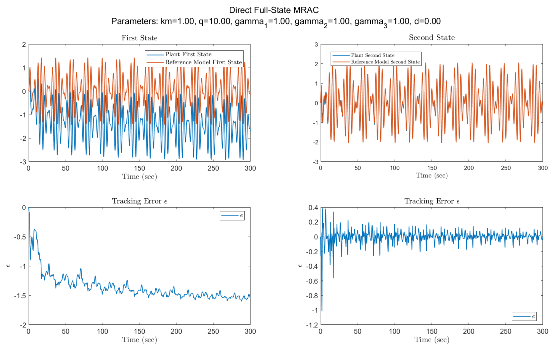

Fig. 8.

The e still doesn't converge into zero. We shall increase it even more.

1) $q = 100$

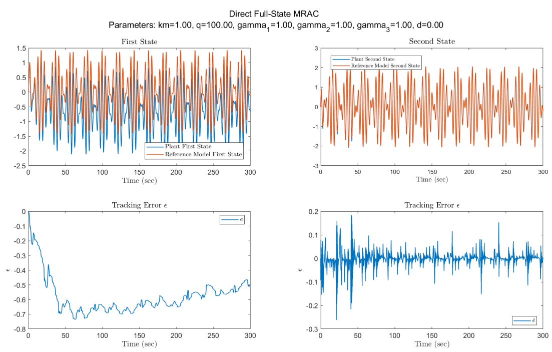

Fig. 9.

We see that the first state error tends to zero. Let's see what will occur when we increase the γ hyper-parameters.

1) $\gamma_1 = 10$
2) $\gamma_2 = 10$
3) $\gamma_3 = 10$

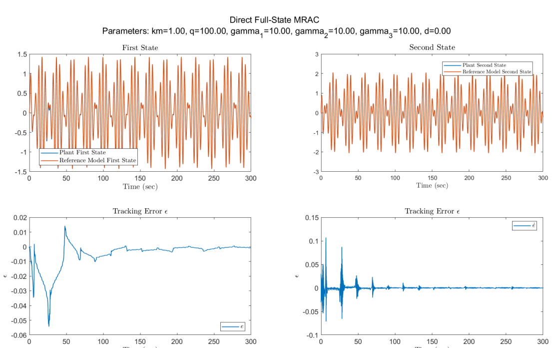

Fig. 10.

We can finally conclude that both errors converge into zero, meaning that the plant state x follows the state $x_m \in \mathbb{R}^2$ of the reference model.

D. Conclusions on non-Linearized System

We ended up with the hyperparameter values:

1) $k_m = 1$
2) $q = 100$
3) $\gamma_1 = 10$
4) $\gamma_2 = 10$
5) $\gamma_3 = 10$

Even though we didn't use the previous approach of assuming that $\sin q = q$, which battled each and every non-linearity in the plant, we found a control and adaptive law capable enough of achieving the same control objective, although it took much more time to settle (10x).

IV. Outside Disturbance

We shall conclude our study upon the Direct MRAC method by adding outside disturbance into both linearized and not plants, after the balance has been restored, meaning that the errors are near zero. The plant will then be

$$M\ddot{q} + G\sin q + C\dot{q} = u + d(t)$$

where d(t) the disturbance in the form of a *pulse* with *5 seconds* duration. Its height will be varying, in order to observe how the control law and the system will behave for different disturbances. We perform this test in order to examine the *robustness* of our two control systems.

We can assume that the balance is restored at t=10s in the linearized system and at t=100s in the non-linearized one. Based on that, we will apply the disturbance pulse for t=[100, 105]s and t=[10, 15]s accordingly.

$d(t) = 1$

Linearized System

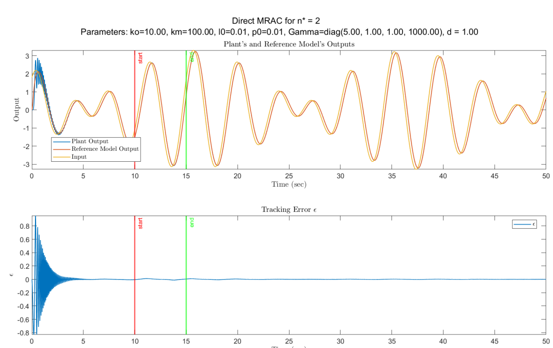

Fig. 11.

Non-Linearized System

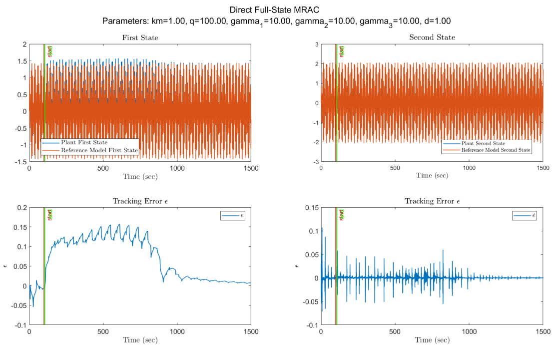

Fig. 12.

For a pulse of such low magnitude, the linearized system exhibits no discernible change, maintaining its original characteristics. In contrast, the non-linearized system is adversely impacted, leading to a difficulty of the error to converge to zero. Nevertheless, it occurs long after the end of the pulse.

$d(t) = 10$

Linearized System

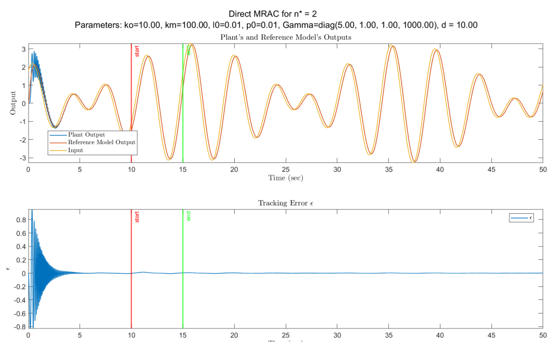

Fig. 13.

Non-Linearized System

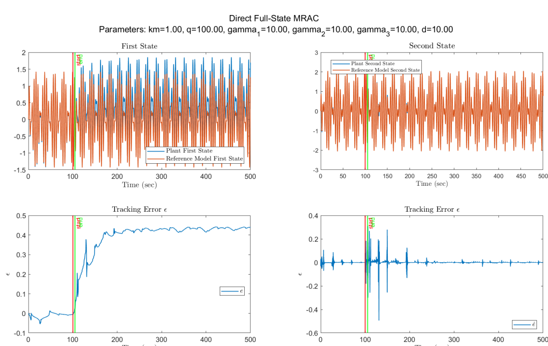

Fig. 14.

The pulse now is of higher magnitude. The linearized system performs the same as before. In contrast, the non-linearized system is affected significantly. Even though it doesn't converge the first state error into zero, it converges it into a small constant value, making the error at least bounded.

$d(t) = 100$

Linearized System

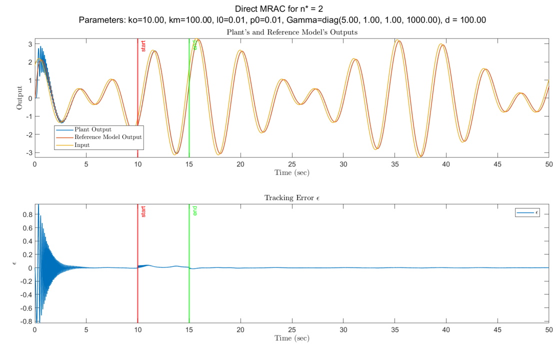

Fig. 15.

Non-Linearized System

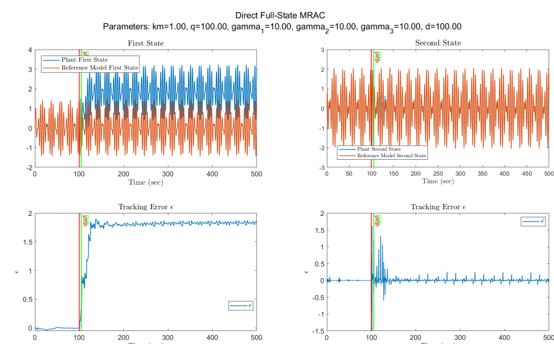

Fig. 16.

Once more, the linearized plant exhibited no alteration, persisting unaffected despite a significant disturbance. In contrast, the control law applied to the non-linearized counterpart failed to converge the first state error to zero; instead, it converges into a small constant value, a little bit higher than before.

After these experiments, we can conclude the partial robustness of the non-linearized control system, which kept the signals in the close looped bounded in spite of not converging both errors into zero after applying the disturbance.

We will continue the logarithmic increase of the pulse to check further the robustness of the linearized system.

$d(t) = 1000$

Linearized System

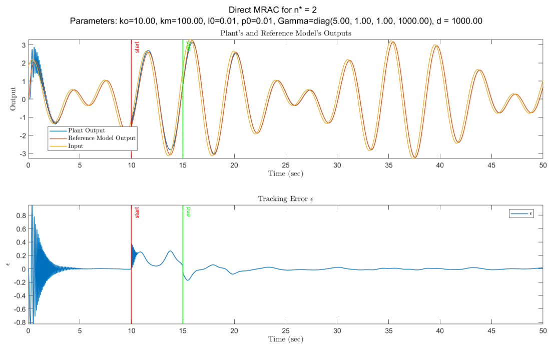

Fig. 17.

$d(t) = 10000$
Linearized System

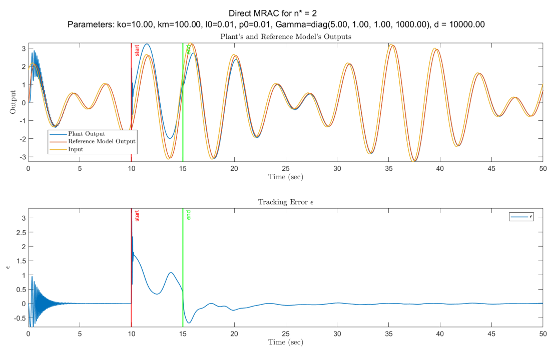

Fig. 18.

$d(t) = 100000$
Linearized System

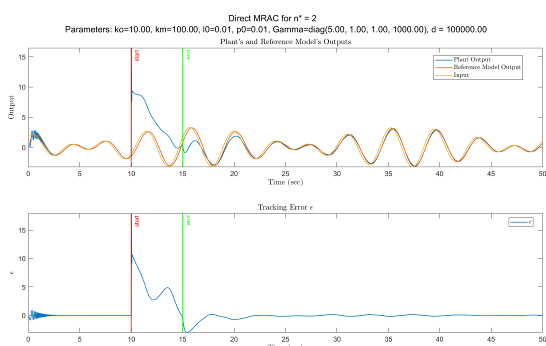

Fig. 19.

We observe that a substantial and unrealistic amplification of the disturbance pulse magnitude prompted noticeable changes in the linearized system, albeit temporarily. These changes were swiftly mitigated once the pulse concluded.

From the aforementioned, we can conclude the absolute robustness of the linearized system. Regardless of the magnitude of the disturbance pulse, the system consistently maintained bounded signals in the closed-loop, and errors swiftly converged to zero immediately after the pulse concluded.

V. Final Conclusions

In this assignment, we successfully implemented *Direct Model Reference Adaptive Control* for both Single Input-Single Output and Full-State vector cases, applying it to both linearized and non-linearized plants LTI. Our findings during this exploration unveiled a crucial insight: while linearization provides a simplified representation of a system, it may not be entirely trustworthy, and its outcomes should be viewed as indicative rather than fully reflective of the true, more complex system behavior.