POTENTIAL GAMES FOR DISTRIBUTED PARAMETER ESTIMATION IN NETWORKS WITH AMBIGUOUS MEASUREMENTS

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ABSTRACT

Distributed estimation of a parameter vector in a network of sensor nodes with ambiguous measurements is considered. A multitude of convex sets is considered by each node to model local ambiguities: Any value for the parameter vector in any such set is consistent will local measurements, but not necessarily consistent network-wide. Consensus can be employed to reach an estimate consistent with the measurements of all nodes, assuming that such an estimate exists. Unfortunately, such an approach leads to a non-convex problem. The considered problem is properly decomposed into two sub-problems, where the second is well studied in literature and the first one is modelled as a non-cooperative game, an exact potential function is derived, and an algorithm for its solution is given. Numerical results, consistent with the theoretical findings, demonstrate the efficacy of the proposed approach.

Index Terms— Distributed parameter estimation, non-convex optimization, game theory, potential games, spatial adaptive play

1. INTRODUCTION

The rapid technological progress in electronics and wireless communications has led to an abundance of miniaturized, networked devices that surround us in our everyday lives. Such networks of smart agents give rise to the so-called Internet-of-Things (IoT) that promises to revolutionize our cities [1], the industry [2], the energy grid that obtains a new, smart form [3], and vehicular networks [4].

The increasingly expanding use of networks of small-scale electronic devices along with the continuous improvement of their computational and storage capabilities, drive the demand for efficient distributed algorithms that are able to confront more and more complex problems that arise in this setting [5], [6]. Distributed data processing problems such as parameter estimation, decision making/detection and learning have been studied by many researchers and various

efficient algorithms have been developed. e.g., [7], [8], [9], [10]. Such algorithms alleviate the need for transmitting all the measurements obtained by the devices to a central computer for further processing and, in many cases, they enjoy no loss in performance as compared to centralized approaches. Furthermore, distributed algorithms are in general resilient to several types of device or network failures, they are scalable in the sense that minor or no modifications are required when the network is altered or expanded, and they do not suffer from the "single point of failure" problem, which is inherent in centralized architectures [11].

In this work, the problem of efficiently estimating, in a distributed manner, a parameter vector in a network of sensor nodes with ambiguous measurements is considered. Ambiguities may arise due to interference, poor calibration, model inaccuracies or any other cause. The proposed approach is built upon the set-theoretic estimation methodology [12]. In more detail, the information that a node can infer from its measurements about the parameter vector, is summarized in that the vector should lie inside some set, which due to ambiguities it can be non-convex. Thus, the utilization of the information available to all the nodes in the network, requires that the nodes reach a consensus [13] on a vector at the intersection of all such non-convex sets, assuming that this intersection is non-empty. To tackle this challenging non-convex problem, we focus on a particular case where the non-convex set at each node is given as a union of a finite number of convex sets, and furthermore, we assume that only one such convex set at each node is relevant to the estimation task. Thus, the original problem is decomposed into the sub-problem of identifying the relevant convex sets, followed by the sub-problem of identifying a point at the intersection of a finite number of convex sets, which has been extensively studied in literature [14], [15], [16]. Focusing on the first sub-problem, we model it as a non-cooperative game, we prove that it has a potential function [17], and derive an algorithm for its solution.

1.1. Relation to prior work

In our previous work [18], the problem of distributed estimation from ambiguous measurements was formulated, and an

algorithm that required the organization of the nodes into a circle was studied. Here we drop this requirement, and furthermore follow a different, game-theoretic analysis. Also relevant to this work, are consensus approaches like [16] and [19], but the main difference from these approaches is that, here, each node possesses a non-convex set. In [15], potential games are used to analyse the consensus problem, considering again convex sets. Finally, other researchers have also considered potential games to derive distributed algorithms, but for different problems at hand [20], [21], [22].

2. PROBLEM FORMULATION

Consider a network of devices (hereafter also termed as agents), which are interconnected as represented by a graph $\mathcal{G}(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is the set of nodes of the graph (that represent agents), and $\mathcal{E}$ is the set of edges of the graph, that represent pairs of agents that are able to communicate directly. In the following, it is assumed that the considered graph is connected, i.e., there exists at least one path from any node to any other node. Let $\mathcal{N}_n$ denote the set of agents adjacent to node n, i.e., the set of nodes that can communicate directly with node n including node n itself. Consider also that each agent $n \in \mathcal{N}$ has obtained a vector of measurements $\boldsymbol{y}_n \in \mathbb{R}^M$ which is somehow related to a parameter vector $\boldsymbol{\theta} \in \mathbb{R}^D$. In this work, it is assumed that the connection between the measurements $\boldsymbol{y}_n$ and the parameter vector $\boldsymbol{\theta}$ can be expressed as

$$\boldsymbol{\theta} \in \mathcal{C}_n = \bigcup_{k=1}^{k_n} \mathcal{S}_{n,k} , \qquad (1)$$

where the set $\mathcal{C}_n$ can be non-convex, but given as a union of a finite number (i.e., k_n) of convex sets $\mathcal{S}_{n,k}$. Taking into account the measurements of all agents, the ultimate scope is to compute a point

$$\boldsymbol{\theta} \in \mathcal{C} = \bigcap_{n \in \mathcal{N}} \mathcal{C}_n , \qquad (2)$$

assuming that the intersection $\mathcal{C}$ is non-empty.

To the best of the authors knowledge, the general problem of computing a point in the intersection of a number of non-convex sets has not been solved by the scientific community. To this end, we focus here on a particular case of the above problem. In more detail, we introduce the following assumption.

Assumption A1: The intersection $\mathcal{C}$ is non-empty. Furthermore, there exists exactly one set $\mathcal{S}_{n,l_n}$ for each node n with

$$\mathcal{S}_{n,l_n} \bigcap \mathcal{C} \neq \emptyset$$

In other words, there exists exactly one choice in which each agent n selects one of its convex sets $\mathcal{S}_{n,l_n}$ so that the intersection of all such sets becomes non-empty.

It is easy to verify that, when Assumption A1 is in place, the intersection of the non-convex sets is given by $\mathcal{C} = \bigcap_{n \in \mathcal{N}} \mathcal{S}_{n,l_n}$, and thus $\mathcal{C}$ is a convex set, while in general (i.e., when Assumption A1 does not hold), it may be non-convex. Furthermore, when Assumption A1 holds, the problem (2) is equivalent to solving the following two subproblems: **Sub-problem P1:** Identify the sets $\mathcal{S}_{n,l_n}, n \in \mathcal{N}$, and **Sub-problem P2:** Compute $\theta \in \mathcal{S}_{n,l_n}$.

Clearly, sub-problem P2, namely, that of computing a point in the intersection of a finite number of convex sets, is a problem well studied in literature [14], [15], [16]. In this work, we focus on sub-problem P1, and provide a distributed algorithm for identifying the sets $\mathcal{S}_{n,l_n}$, utilizing the concept of potential games [17].

3. GAME-THEORETIC ANALYSIS

3.1. Game definition

Consider a non-cooperative game in strategic form [23], where the set of players is the set of nodes $\mathcal{N}$ and the set of strategies for each node (i.e., the so-called action sets) are given by $\mathcal{A}_n = \{\mathcal{S}_{n,1}, \mathcal{S}_{n,2}, \ldots, \mathcal{S}_{n,k_n}\}$. That is, a pure (i.e., not mixed, see 3.3) strategy $\alpha_n \in \mathcal{A}_n$ for node/player n is the selection of one of its convex sets. A strategy profile α is a selection of strategies, one for each player. It is useful to define the set of all possible strategy profiles as $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \ldots \mathcal{A}_N$ so that $\alpha \in \mathcal{A}$. Also, we can express any strategy profile α as $\alpha = (\alpha_n, \alpha_{-n})$, where α_n denotes the strategy of node n and α_{-n} denotes the strategies of all players except player n. To complete the definition of the considered game, each node has a utility function $u_n(\alpha)$, which it is defined as

$$u_n(\alpha) = \sum_{k \in \mathcal{N}_n} I(\alpha_n, \alpha_k) , \qquad (3)$$

where $I(\mathcal{S}_a, \mathcal{S}_b)$ is an indicator function defined as

$$I(\mathcal{S}_a, \mathcal{S}_b) = \begin{cases} 1, & \text{if } \mathcal{S}_a \bigcap \mathcal{S}_b \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \qquad (4)$$

In other words, the utility function counts the number of neighbouring nodes that have selected a set that has a non-empty intersection with the set selected by node n. To this end, the considered utility function promotes the selection of sets by node n that have more non-empty bilateral intersections with the sets selected by neighbouring nodes.

Of course, other utility functions are possible. As an example, utility functions that yield a higher value for local sets that have joint (i.e., not bilateral) non-empty intersection with more sets of neighboring players, would offer a trade-off between complexity of computation and performance. However, as such a choice makes the potential function analysis more difficult, such an approach is not considered here and will be the subject of future work.

3.2. Potential function analysis

Consider the function $\phi : \mathcal{A} \to \mathbb{R}$ defined as

$$\phi(\alpha) = \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{N}_n} \frac{I(\alpha_n, \alpha_k)}{2} . \qquad (5)$$

We will prove that this function constitutes a so-called *exact potential function* for the game defined in the previous. An exact potential function, associated with a non-cooperative game, is a function with the property that whenever a player changes its strategy and thus changes its utility function, the potential function changes by the exact same amount. Thus, games for which a bounded potential function exists (i.e., *potential games*), have the interesting property that any sequence of selfish, greedy steps, in which each player increases its utility function, will converge to a so-called *Nash equilibrium* of the game, in which no player has an incentive to change their strategy unilaterally.

To prove that the function in (5) is an exact potential function for the game defined in the previous, consider any fixed player $l \in \mathcal{N}$, that changes its strategy from $\alpha_l^{(1)}$ to $\alpha_l^{(2)}$. Consider also the respective strategy profiles, where the strategies of all other players remain fixed, as $\alpha^{(1)} = \left(\alpha_l^{(1)}, \alpha_{-l} \right)$ and $\alpha^{(2)} = \left(\alpha_l^{(2)}, \alpha_{-l} \right)$. Assume that when player l follows strategy $\alpha_l^{(1)}$ its set has bilateral intersections with neighboring nodes in the set $\mathcal{N}_l^{(1)} \subseteq \mathcal{N}_l$ and, respectively, when it follows $\alpha_l^{(2)}$ the set is $\mathcal{N}_l^{(2)} \subseteq \mathcal{N}_l$. Thus, the utility function of player l, before and after the change of strategy, is given by

$$u_l\left(\alpha^{(1)}\right) = \left|\mathcal{N}_l^{(1)}\right| \quad \text{and} \quad u_l\left(\alpha^{(2)}\right) = \left|\mathcal{N}_l^{(2)}\right| . \qquad (6)$$

Consider now the change of the function in (5) as player l switches its strategy from $\alpha_l^{(1)}$ to $\alpha_l^{(2)}$. We define $\Delta = \phi\left(\alpha^{(2)}\right) - \phi\left(\alpha^{(1)}\right)$ and consider that we can limit the outer summation from $n \in \mathcal{N}$ to $n \in \mathcal{N}_l$, since the players that are not neighbors of player l do not change their bilateral intersections, to get

$$\Delta = \sum_{n \in \mathcal{N}_l} \sum_{k \in \mathcal{N}_n} \left(\frac{I\left(\alpha_n^{(2)}, \alpha_k^{(2)}\right)}{2} - \frac{I\left(\alpha_n^{(1)}, \alpha_k^{(1)}\right)}{2} \right) . \qquad (7)$$

Separating the outer summation $n \in \mathcal{N}_l$ into $n = l$ and $n \in \mathcal{N}_l \setminus \{l\}$, we have

$$\begin{aligned} \Delta &= \sum_{k \in \mathcal{N}_l} \left(\frac{I\left(\alpha_l^{(2)}, \alpha_k^{(2)}\right)}{2} - \frac{I\left(\alpha_l^{(1)}, \alpha_k^{(1)}\right)}{2} \right) \\ &+ \sum_{n \in \mathcal{N}_l \setminus \{l\}} \sum_{k \in \mathcal{N}_n} \left(\frac{I\left(\alpha_n^{(2)}, \alpha_k^{(2)}\right)}{2} - \frac{I\left(\alpha_n^{(1)}, \alpha_k^{(1)}\right)}{2} \right) \end{aligned}$$

Finally, limiting the inner summation from $k \in \mathcal{N}_n$ to $k = l$, since all other terms do not contribute to the outer summation, we have

$$\begin{aligned} \Delta &= \frac{\left|\mathcal{N}_l^{(2)}\right|}{2} - \frac{\left|\mathcal{N}_l^{(1)}\right|}{2} + \frac{\left|\mathcal{N}_l^{(2)}\right| - 1}{2} - \frac{\left|\mathcal{N}_l^{(1)}\right| - 1}{2} \\ &= \left|\mathcal{N}_l^{(2)}\right| - \left|\mathcal{N}_l^{(1)}\right| \end{aligned} \qquad (8)$$

which concludes our proof that the function in (5) is an exact potential function for the considered game. Furthermore, it is easy to see that the potential function is bounded. In particular, $\phi(\alpha) \leq |\mathcal{E}|, \quad \forall \alpha \in \mathcal{A}$, attaining its maximum value when all players select sets with non-empty bilateral intersections with the sets selected by all their neighbors.

It is easy to note that, when each player n selects the strategy $\alpha_n = \mathcal{S}_{n,l_n}$, considered in Assumption A1, then the potential function attains its maximum value. This selection corresponds to a Nash equilibrium for the considered game, however other Nash equilibria may exist. A learning algorithm known as *Spatial Adaptive Play* (SAP) [24], exists that guarantees that the players will reach the Nash equilibrium that maximizes the potential function asymptotically with arbitrary high probability. This approach is detailed in the following.

3.3. Solution via spatial adaptive play

Modeling a problem as a potential game offers the possibility of using any of the many available learning algorithms, with guaranteed results [17], [24], [25], [26]. While most algorithms guarantee that the player behavior will converge to a Nash equilibrium, in this work we employ SAP [24], that guarantees asymptotic convergence to some optimal Nash equilibrium, i.e., where the potential function attains its maximum value.

SAP considers the so-called *mixed strategies*, that is, each player assigns probabilities to its strategies, and selects each strategy with the given probability. Consider that player n utilizes a probability mass function (p.m.f.) at time t, given as $p_n(t)$. According to SAP, at each time $t > 0$, one player n is randomly chosen (with equal probability for each player) and it tries to update its strategy from $\alpha_n(t-1)$ to $\alpha_n(t)$. The rest of the players do not alter their strategies at this time t, that is $\alpha_{-n}(t) = \alpha_{-n}(t-1)$. Player n randomly selects a strategy from its action set $\mathcal{A}_n$, according to $p_n(t)$, whose α_n-th entry ($\alpha_n \in \mathcal{A}_n$) is given by

$$p_n(t, \alpha_n) = \frac{\exp\left(\beta u_n(\alpha_n, \alpha_{-n}(t-1))\right)}{\sum_{\alpha_n' \in \mathcal{A}_n} \exp\left(\beta u_n(\alpha_n', \alpha_{-n}(t-1))\right)} , \qquad (9)$$

where $\beta \geq 0$ is the so-called *exploration parameter*, that controls how likely the players are to select a suboptimal strategy. At the extreme case where $\beta = 0$, the p.m.f.s become uniform, and at the other extreme where $\beta \to \infty$, the p.m.f.s
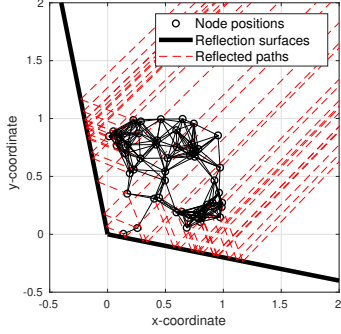
Fig. 1. An Illustration of the AoA Estimation Example

Fig. 2. Simulation Results

give positive probabilities only to those strategies that maximize the utility functions, a set of strategies known as best-response strategies, and the learning algorithm reduces to the so-called *best response dynamics*.

In a game where all players utilize SAP, it is known that the stationary distribution is given by [24]

$$p(\alpha) = \frac{\exp\left(\beta\phi(\alpha)\right)}{\sum_{\alpha'\in\mathcal{A}}\exp\left(\beta\phi(\alpha')\right)} \ . \qquad (10)$$

In other words, if the parameter β is sufficiently large, positive probabilities will be given only to the joint strategies that maximize the potential function. In other words, in this case, all players will select sets with non-empty intersections with the sets selected by all their neighboring players asymptotically with arbitrary high probability.

It is interesting to note, however, that in general, ensuring that all bilateral set intersections are non-empty, at each node, may not be adequate for the joint intersection of the selected sets to exist. In other words, the algorithm considered here may converge to solutions that give positive probabilities to sets that are not the correct ones. Still, since the correct sets will be also given positive probabilities, it could be possible that another procedure could be employed to identify the correct sets, possibly by considering joint intersections at each node. This approach is the subject of future work.

Furthermore, considering the so-called *Helly's theorem* [27] (that states that in a finite collection of convex sets $\mathcal{X}_1, \mathcal{X}_2, \cdots \mathcal{X}_N$, that are all subsets of $\mathbb{R}^d$, with $N > d$, if all the $\binom{N}{d+1}$ collections of $d+1$ sets have non-empty intersections, then the joint intersection of all sets is non-empty) we can deduce that in the case where the graph $\mathcal{G}(\mathcal{N}, \mathcal{E})$ is the complete graph, and the sets $\mathcal{S}_{n,k}$ are one dimensional, then the bilateral intersections guarantee the existence of the joint intersection, and thus, the proposed approach is guaranteed to solve problem P1.

4. NUMERICAL RESULTS

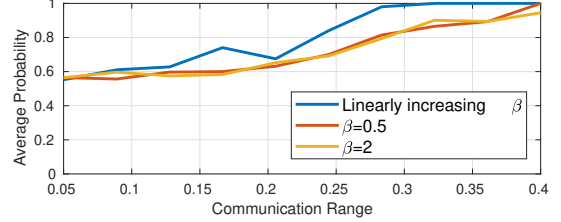To demonstrate the efficacy of the derived scheme, an Angle-of-Arrival (AoA) estimation example is considered, where a network of nodes is interested in estimating the angle of the direct path in a scenario where signal reflections generate ambiguities. An example of the considered scenario can be seen in Fig. 1.

In the adopted scenario, the source signal is received by each node n from three different paths; directly and via two reflections, with AoAs denoted as $\theta_{n,0}$, $\theta_{n,1}$, $\theta_{n,2}$ and received powers denoted as $P_{n,0}$, $P_{n,1}$, $P_{n,2}$, respectively. Each node n, estimates the three AoAs and the corresponding received powers, and associates a convex set only for the AoAs for which the respective power exceeds a predefined threshold. The convex sets are defined using a parameter δ, namely, they are of the form $[\theta - \delta, \theta + \delta]$. This parameter designates the uncertainty of the estimation procedure used for estimating the AoAs and it is set to $\delta = 0.2$ for this experiment. Thus, in this setting, the relevant convex sets considered in Assumption A1 are those that correspond to the direct path.

We simulate a network of $N = 200$ nodes, uniformly deployed in the unit square, and place the source that emits the signal at $(35, 35)$. We perform 10 Monte-Carlo runs, for various sensor locations, and examine the effect of the node communication range and the exploration parameter β. Also, we perform 6000 Iterations (strategy changes) in the SAP algorithm, for each run. The results are given in Fig. 2, where the average (across all nodes, and all runs) probability for selecting the correct sets is given, as a function of the sensor communication range. We can see that, in all cases, the probability reaches the value 1, when the communication range is high enough. Interestingly, the best results were obtain for a approach that linearly increases β, letting the algorithm first search, and then converge.

5. CONCLUSIONS

In this work, a distributed parameter estimation problem that takes into account ambiguous measurements was considered. Following a set-theoretic estimation approach, the problem was modelled as that of computing a point at the intersection of non-convex sets. Under proper assumptions, the problem was decomposed into two sub-problems. While the second problem is well studied in literature, the first was modelled as a potential game and a suitable algorithm for its confrontation was derived.

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