Kelly Criterion Theory

Dimitris Markopoulos

Suppose that a coin is flipped and a bet M is made. If the coin lands heads with probability p the payoff is bM. If the coin lands tails with probability $q = 1 - p$ the payoff is $-M$. Therefore, define the random variable Y as follows:

$$Y = \begin{cases} bM \text{ , with probability } p \\ -M \text{ , with probability } q = 1 - p \end{cases}$$

The fair value of the game can be calculated by taking the expectation of Y and then set to zero. Mathematically, this is represented by equation (1).

$$E[Y] = p * (bM) + q * (-M) = 0 \tag{1}$$

Therefore, at fair value the game is just gambling. However, if we change equation (1) to the expectation being larger than 0, the game is bias and should be exploited. Mathematically,

$$\begin{aligned} E[Y] = p * (bM) + q * (-M) > 0 &\Rightarrow pbM > qM \\ &\Rightarrow \frac{p}{q} > \frac{1}{b} \\ &\Rightarrow \frac{q}{p} < b \end{aligned}$$

Suppose we are given $\frac{q}{p} < b$ and a starting wealth of W_0 (with no borrowing) to play the game. Further supposed for some $x \in [0,1]$ each iterative bet is given by some constant x of the current wealth. Let W_1 represent the wealth after one round of betting.

Note, $E[W_0] = W_0$ as W_0 is not a random variable but a fixed constant. Furthermore, intuitively $E[W_1]$ is the wealth before betting plus the expected returns of the bet. We can calculate this mathematically using the total law of expectation by equation (2).

$$E[W_1] = [W_0] + [p(bxW_0) + q(-xW_0)] \tag{2}$$

Equation (2) simplifies as follows:

$$\begin{aligned} E[W_1] &= [W_0] + [p(bxW_0) + q(-xW_0)] \\ &= W_0 + xW_0(bp - q) \\ &= W_0 + xW_0[bp - (1 - p)] \\ &= W_0 + xW_0[p(b + 1) - 1] \end{aligned}$$

Therefore, an expression for the expected change of wealth after one game is given by equation (3).

$$E[W_1 - W_0] = E[W_1] - E[W_0] = W_0 + xW_0[p(b + 1) - 1] - W_0 = xW_0[p(b + 1) - 1] \tag{3}$$

Now we seek to find some x such that the expected difference in wealth is maximized. This is given by equation (4).

$$x_m = \underset{x \in [0,1]}{\operatorname{argmax}} E[W_1 - W_0] = \underset{x \in [0,1]}{\operatorname{argmax}} \{xW_0[p(b + 1) - 1]\} \tag{4}$$

We assume that the game is bias, $b > \frac{q}{p} \Rightarrow p(b + 1) > 1$ therefore, $b > \frac{1-p}{p} \Rightarrow b > \frac{1}{p} - 1 \Rightarrow p(b + 1) > 1$. Now to answer the question, finding x such that the argument in equation (4) is satisfied is simple. The objective function is linear in x therefore the maximum of the argument is attained at the end of the domain of $x \in [0,1] \Rightarrow$

$x_m = 1$. This is intuitive because the more one risks of their initial wealth in the bet, the more they will earn in the payout. Given a single round of a favorable game, betting "all in" will yield the most expected returns. However, this is not ideal. Betting "all in" every round is a sure way to go bankrupt. Therefore, to take a more risk averse position, one should consider optimizing the objective function of the expected value of difference between log returns. This is done below.

Since W_0 is fixed, $\log W_0$ is fixed $\Rightarrow E[\log W_0] = \log W_0$.

Now consider W_1:

$\rightarrow$ The coin is heads w.p. $p \Rightarrow W_1 = W_0 + bxW_0 = W_0(1 + bx)$

$\rightarrow$ The coin is tails w.p. $q = 1 - p \Rightarrow W_1 = W_0 - xW_0 = W_0(1 - x)$

Assumptions: $W_0, b > 0$ & $0 < x < 1$. Therefore, we can apply a $\log W_1$ transformation.

$\rightarrow$ The coin is heads w.p. $p \Rightarrow \log W_1 = \log\{W_0(1 + bx)\} = \log W_0 + \log(1 + bx)$

$\rightarrow$ The coin is tails w.p. $q = 1 - p \Rightarrow \log W_1 = \log\{W_0(1 - x)\} = \log W_0 + \log(1 - x)$

Thus,

$$\begin{aligned} E[\log W_1] &= p[\log W_0 + \log(1 + bx)] + (1 - p)[\log W_0 + \log(1 - x)] \\ &= p \log W_0 + p \log(1 + bx) + \log W_0 + \log(1 - x) - p \log W_0 - p \log(1 - x) \\ &= p \log(1 + bx) - p \log(1 - x) + \log(1 - x) + \log W_0 \\ &= p \log(1 + bx) + (1 - p) \log(1 - x) + \log W_0 \end{aligned}$$

$$\begin{aligned} \Rightarrow E[\log W_1 - \log W_0] &= E[\log W_1] - E[\log W_0] \\ &= [p \log(1 + bx) + (1 - p) \log(1 - x) + \log W_0] - \log W_0 \\ &= p \log(1 + bx) + (1 - p) \log(1 - x) \end{aligned}$$

Now we seek to solve equation (5); To find some x such that this expected difference of log returns is maximized.

$$x_K = \underset{x \in [0,1]}{\operatorname{argmax}} E[\log W_1 - \log W_0] = \underset{x \in [0,1]}{\operatorname{argmax}}\{p \log(1 + bx) + (1 - p) \log(1 - x)\} \tag{5}$$

To do this we find the extremum by setting $\frac{\partial}{\partial x} E[\log W_1 - \log W_0] = 0$.

$$\begin{aligned} \frac{\partial}{\partial x}\{p \log(1 + bx) + (1 - p) \log(1 - x)\} &= \frac{p}{1 + bx} * \frac{\partial}{\partial x}\{1 + bx\} + \frac{1 - p}{1 - x} * \frac{\partial}{\partial x}\{1 - x\} \\ &= \frac{p}{1 + bx} * (b) + \frac{1 - p}{1 - x} * (-1) \\ &= \frac{pb}{1 + bx} - \frac{1 - p}{1 - x} \end{aligned}$$

Now set this $= 0$,

$$\begin{aligned} \frac{pb}{1 + bx} - \frac{1 - p}{1 - x} = 0 &\Rightarrow \frac{pb}{1 + bx} = \frac{1 - p}{1 - x} \\ &\Rightarrow pb(1 - x) = (1 - p)(1 + bx) \\ &\Rightarrow pb - pbx = 1 + bx - p - pbx \\ &\Rightarrow bx = pb - 1 + p \\ &\Rightarrow x = \frac{pb - 1 + p}{b}\bigg|_{x = x_K} \\ &\Rightarrow x_K = \frac{p(b + 1) - 1}{b} \end{aligned}$$

Intuitively, x_K is the proportion of initial wealth bet on the game using log returns. This is also known as the Kelly Criterion. We can extend this argument to N rounds of betting where $S = pN$ successful bets occur.

This formula is recursive.

The wealth after one round of betting W_1 is described as follows with 2 unique outcomes:
1 success → The coin is heads w.p. $p \Rightarrow W_1 = W_0(1 + bx)$
1 failure → The coin is tails w.p. $q = 1 - p \Rightarrow W_1 = W_0(1 - x)$

The wealth after another round of betting W_2 is described as follows with 3 unique outcomes:
2 successes → The coin is heads w.p. $p \Rightarrow W_2 = W_1(1 + bx) = W_0(1 + bx)^2$
1 success and 1 failure → The coin is heads w.p. $p \Rightarrow W_2 = W_1(1 + bx) = W_0(1 + bx)^2$
1 failure and 1 success → The coin is heads w.p. $p \Rightarrow W_2 = W_1(1 + bx) = W_0(1 + bx)^2$
2 failures → The coin is tails w.p. $q = 1 - p \Rightarrow W_2 = W_1(1 - x) = W_0(1 - x)^2$

And so on,

The wealth after N round of betting W_2 is described as follows with $N - 1$ unique outcomes:
N successes → The coin is heads w.p. $p \Rightarrow W_2 = W_1(1 + bx) = W_0(1 + bx)^2$
...

Therefore,

$$W_0(1 + bx)^S(1 - x)^{N-S} = W_0(1 + bx)^{pN}(1 - x)^{N-pN} = W_0(1 + bx)^{pN}(1 - x)^{N(1-p)}$$

The formula for the final wealth after $S = pN$ successful bets (W_n) is given by equation (6).

$$W_0(1 + bx)^{pN}(1 - x)^{N(1-p)} \tag{6}$$

Now to find the value of x such that equation (5) is maximized,

$$x_{K'} = \underset{x \in [0,1]}{\operatorname{argmax}} \left\{W_0(1 + bx)^{pN}(1 - x)^{N(1-p)}\right\}$$

Find the extremum by setting $\frac{\partial}{\partial x}\left\{W_0(1 + bx)^{pN}(1 - x)^{N(1-p)}\right\} = 0$.

$$\begin{aligned}
&\frac{\partial}{\partial x}\left\{W_0(1 + bx)^{pN}(1 - x)^{N(1-p)}\right\} \\
&= \frac{\partial}{\partial x}\{W_0(1 + bx)^{pN}\} * (1 - x)^{N(1-p)} + W_0(1 + bx)^{pN} * \frac{\partial}{\partial x}\left\{(1 - x)^{N(1-p)}\right\} \\
&= W_0 pN(1 + bx)^{pN-1} * \frac{\partial}{\partial x}\{1 + bx\} * (1 - x)^{N(1-p)} + W_0(1 + bx)^{pN} * N(1 - p)(1 - x)^{N(1-p)-1} * \frac{\partial}{\partial x}\{1 - x\} \\
&= W_0 pN(1 + bx)^{pN-1} b(1 - x)^{N(1-p)} - W_0(1 + bx)^{pN} N(1 - p)(1 - x)^{N(1-p)-1}
\end{aligned}$$

Now set this $= 0$,

$$\begin{aligned}
&\Rightarrow W_0 pN(1 + bx)^{pN-1} b(1 - x)^{N(1-p)} - W_0(1 + bx)^{pN} N(1 - p)(1 - x)^{N(1-p)-1} = 0 \\
&\Rightarrow W_0 pN(1 + bx)^{pN-1} b(1 - x)^{N(1-p)} = W_0(1 + bx)^{pN} N(1 - p)(1 - x)^{N(1-p)-1} \\
&\Rightarrow pb(1 - x)^1 = (1 + bx)^1(1 - p) \\
&\Rightarrow pb - pbx = 1 - p + bx(1 - p) \\
&\Rightarrow -pbx - bx(1 - p) = 1 - p - pb \\
&\Rightarrow pbx + bx(1 - p) = -1 + p + pb \\
&\Rightarrow x(pb + b - pb) = -1 + p + pb \\
&\Rightarrow x(b) = -1 + p + pb
\end{aligned}$$

$$\Rightarrow x = \left.\frac{p(b+1)-1}{b}\right|_{x=x_{K'}}$$

$$\Rightarrow x_{K'} = \frac{p(b+1)-1}{b}$$

Therefore, we have proved that,

$$x_K = x_{K'} = \frac{p(b+1)-1}{b} \qquad \blacksquare$$

The Kelly Criterion $\frac{p(b+1)-1}{b}$ is theoretically viable.

Citations

Wikipedia contributors. (n.d.). Kelly criterion. Wikipedia, The Free Encyclopedia. Retrieved February 5, 2025, from https://en.wikipedia.org/wiki/Kelly_criterion

DeGroot, M. H., & Schervish, M. J. (2011). *Probability and statistics* (4th ed.). Addison Wesley.