Monte Carlo – European Call Option Pricing

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The Black-Scholes-Merton (BSM) model gives the theoretical price of a European call option by equation (1).

$$C = S_0 e^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{1}$$

Parameters:
- ⊳ spot price – S_0
- ⊳ strike – K
- ⊳ maturity (years) – T
- ⊳ risk-free rate – r
- ⊳ dividend yield – q
- ⊳ volatility – σ

Notations:
- ⊳ Φ – CDF of $N(0,1)$
- ⊳ C – Call option price

Variables:
- ⊳ $d_1 = \frac{\ln(S_0/K)+(r-q+0.5\sigma^2)T}{\sigma\sqrt{T}}$
- ⊳ $d_2 = d_1 - \sigma\sqrt{T}$

Alternatively, Monte Carlo simulation can be used for valuation. The path for S_T is given by equation (2).

$$S_T = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma Z\sqrt{T}\right\} \text{ where } Z \sim N(0,1) \tag{2}$$

Equation (2) can be used to simulate the stock price at time T. We then calculate the simulated value of a European call option $\hat{C}$ by equation (3) which is expressed intuitively as finding the mean of the discounted payoff.

$$\hat{C} = E[e^{-rT} * \max(S_T - K, 0)] \tag{3}$$

Furthermore, we can estimate the range the theoretical option price is likely to fall within for a certain level of confidence. Assume $\{\hat{C}_i : i \in \mathbb{Z}, 0 \le i \le M\}$. A $(1-\alpha)100\%$ confidence interval for the theoretical price C is given by equation (4) where $\mu = \frac{1}{M}\sum_{i \le M} \hat{C}_i$ and $s^2 = \frac{1}{M-1}\sum_{i \le M}\left(\hat{C}_i - \mu\right)^2$ and $z_{\frac{\alpha}{2}} = \Phi^{-1}(1-\alpha)$.

$$\mu - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{M}} < C < \mu + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{M}} \tag{4}$$

As $M \to \infty$, the distribution of $\hat{\boldsymbol{C}} = [\hat{C}_1 \quad \cdots \quad \hat{C}_M]^T$ converges towards a normal distribution with mean C, the theoretical price. Therefore, we seek methods of convergence of $\hat{\boldsymbol{C}}$ and minimization of variance while accounting for computational costs to the Monte Carlo simulation. A method to reduce variance with lower computational costs is Antithetic Sampling. Antithetic Sampling is a variance reduction technique used in Monte Carlo simulations to improve efficiency and accuracy. The core idea is to generate negatively correlated random variables to reduce variance in simulation estimates. Therefore, two paths for S_T can be returned by generating a single random variable $Z \sim N(0,1)$; this is given by equation (5).

$$\begin{bmatrix} S_{T.1} \\ S_{T.2} \end{bmatrix} = \begin{bmatrix} S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma Z\sqrt{T}\right\} \\ S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma(-Z)\sqrt{T}\right\} \end{bmatrix} \text{ where } Z \sim N(0,1) \tag{5}$$

Citations

Hull, J. C. (2021). *Options, futures, and other derivatives* (Global ed.). Pearson.

Wikipedia contributors. (n.d.). *Itô's lemma*. Wikipedia, The Free Encyclopedia. Retrieved February 8, 2025, from https://en.wikipedia.org/wiki/It%C3%B4%27s_lemma