For this example, consider the following portfolio (right) using a linear interpolation method (left).

- Given: $z = ay_1 + (1-a)y_2$ for some $a \in [0,1]$.
- Given: $V = Ne^{-zT}$.
- Claim: $\Delta V = w_1(\Delta B_1) + w_2(\Delta B_2)$ for some w_1, w_2 .

Time	T_1	T	T_2
Cashflow	1	N	1
Rates	y_1	z	y_2
Discounted Cashflows	$e^{-y_1T_1}$	Ne^{-zT}	$e^{-y_2T_2}$

2A

Goal - Find w_1 and w_2.

$\Delta V(B_1, B_2) = w_1(\Delta B_1) + w_2(\Delta B_2)$ and using 1st order Talor approximation gives $\Delta V(B_1, B_2) \approx \frac{\partial V}{\partial B_1}\Delta B_1 + \frac{\partial V}{\partial B_2}\Delta B_2$.

(**1**) Calculating $\frac{\partial V}{\partial B_1} = \frac{\partial V}{\partial y_1}\frac{\partial y_1}{\partial B_1}$ is as follows:

$$\frac{\partial V}{\partial y_1} = \frac{\partial}{\partial y_1}\{Ne^{-zT}\} = \frac{\partial}{\partial y_1}\left\{Ne^{-[ay_1+(1-a)y_2]T}\right\} = Ne^{-zT}\frac{\partial}{\partial y_1}\{-[ay_1 + (1-a)y_2]T\} = Ne^{-zT}(-aT) = -Ne^{-zT}aT$$

Invert function to take derivative by taking natural log,

$B_1 = e^{-y_1T_1} \to \ln B_1 = -y_1T_1 \to y_1 = -\frac{1}{T_1}\ln B_1$; $\frac{\partial y_1}{\partial B_1} = -\frac{1}{T_1} * \frac{1}{B_1} = -\frac{1}{T_1B_1}$.

$$\frac{\partial V}{\partial B_1} = \frac{\partial V}{\partial y_1}\frac{\partial y_1}{\partial B_1} = (-Ne^{-zT}aT)\left(-\frac{1}{T_1B_1}\right) = \frac{Ne^{-zT}aT}{T_1B_1} = \frac{Ne^{-zT}aT}{T_1e^{-y_1T_1}}$$

(**2**) Calculating $\frac{\partial V}{\partial B_2} = \frac{\partial V}{\partial y_2}\frac{\partial y_2}{\partial B_2}$ is as follows:

$$\frac{\partial V}{\partial y_2} = \frac{\partial}{\partial y_2}\{Ne^{-zT}\} = \frac{\partial}{\partial y_2}\left\{Ne^{-[ay_1+(1-a)y_2]T}\right\} = Ne^{-zT}\frac{\partial}{\partial y_2}\{-[ay_1 + (1-a)y_2]T\} = -Ne^{-zT}(1-a)T$$

Invert function to take derivative by taking natural log,

$B_2 = e^{-y_2T_2} \to \ln B_2 = -y_2T_2 \to y_2 = -\frac{1}{T_2}\ln B_2$; $\frac{\partial y_2}{\partial B_2} = -\frac{1}{T_2} * \frac{1}{B_2} = -\frac{1}{T_2B_2}$.

$$\frac{\partial V}{\partial B_2} = \frac{\partial V}{\partial y_2}\frac{\partial y_2}{\partial B_2} = (-Ne^{-zT}(1-a)T)\left(-\frac{1}{T_2B_2}\right) = \frac{Ne^{-zT}(1-a)T}{T_2e^{-y_2T_2}}$$

Recall, $\Delta V(B_1, B_2) = w_1(\Delta B_1) + w_2(\Delta B_2)$ and using 1st order Talor approximation gives $\Delta V(B_1, B_2) \approx \frac{\partial V}{\partial B_1}\Delta B_1 + \frac{\partial V}{\partial B_2}\Delta B_2$.

Therefore, $w_1 = \frac{\partial V}{\partial B_1} = \frac{Ne^{-zT}aT}{T_1e^{-y_1T_1}}$ and $w_2 = \frac{\partial V}{\partial B_2} = \frac{Ne^{-zT}(1-a)T}{T_2e^{-y_2T_2}}$.

2B

Goal – Check if $V = w_1B_1 + w_2B_2$.

$$w_1B_1 + w_2B_2 = \left(\frac{Ne^{-zT}aT}{T_1e^{-y_1T_1}}\right)e^{-y_1T_1} + \left(\frac{Ne^{-zT}(1-a)T}{T_2e^{-y_2T_2}}\right)e^{-y_2T_2}$$

$$= \frac{Ne^{-zT}aT}{T_1} + \frac{Ne^{-zT}(1-a)T}{T_2}$$
$$= Ne^{-zT} * T\left(\frac{a}{T_1} + \frac{1-a}{T_2}\right)$$

Therefore, $V = Ne^{-zT}$ is proportional to $w_1B_1 + w_2B_2$ up to a constant $\gamma = T\left(\frac{a}{T_1} + \frac{1-a}{T_2}\right)$, i.e., $V \propto w_1B_1 + w_2B_2$.

2C

Goal – Repeat using $-zT = a\log B_1 + (1-a)\log B_2$

$\Delta V(B_1, B_2) = w_1(\Delta B_1) + w_2(\Delta B_2)$ and using 1st order Talor approximation gives $\Delta V(B_1, B_2) \approx \frac{\partial V}{\partial B_1}\Delta B_1 + \frac{\partial V}{\partial B_2}\Delta B_2$.

$$\frac{\partial V}{\partial B_1} = \frac{\partial}{\partial B_1}\{Ne^{-zT}\} = \frac{\partial}{\partial B_1}\left\{Ne^{a\log B_1 + (1-a)\log B_2}\right\} = \frac{\partial}{\partial B_1}\left\{Ne^{\log(B_1^a) + \log\left(B_2^{(1-a)}\right)}\right\} = N\frac{\partial}{\partial B_1}\left\{B_1^aB_2^{(1-a)}\right\} = NB_2^{(1-a)}\frac{\partial}{\partial B_1}\{B_1^a\}$$

$$\rightarrow \frac{\partial V}{\partial B_1} = NB_2^{(1-a)}aB_1^{a-1}$$

$$\frac{\partial V}{\partial B_2} = \frac{\partial}{\partial B_2}\{Ne^{-zT}\} = \frac{\partial}{\partial B_2}\left\{Ne^{a\log B_1 + (1-a)\log B_2}\right\} = \frac{\partial}{\partial B_2}\left\{Ne^{\log(B_1^a) + \log\left(B_2^{(1-a)}\right)}\right\} = N\frac{\partial}{\partial B_2}\left\{B_1^aB_2^{(1-a)}\right\} = NB_1^a\frac{\partial}{\partial B_2}\left\{B_2^{(1-a)}\right\}$$

$$\rightarrow \frac{\partial V}{\partial B_2} = NB_1^a(1-a)B_2^{-a}$$

Therefore, $w_1 = \frac{\partial V}{\partial B_1} = NB_2^{(1-a)}aB_1^{a-1}$ and $w_2 = \frac{\partial V}{\partial B_2} = NB_1^a(1-a)B_2^{-a}$.

$$\begin{aligned} w_1B_1 + w_2B_2 &= \left(NB_2^{(1-a)}aB_1^{a-1}\right)B_1 + (NB_1^a(1-a)B_2^{-a})B_2 \\ &= NB_2^{1-a}aB_1^a + NB_1^a(1-a)B_2^{1-a} \\ &= aNB_1^aB_2^{1-a} + NB_1^aB_2^{1-a} - aNB_1^aB_2^{1-a} \\ &= NB_1^aB_2^{1-a} \end{aligned}$$

$$V = Ne^{-zT} = Ne^{[a\log B_1 + (1-a)\log B_2]} = NB_1^aB_2^{1-a}$$

Therefore, $V = w_1B_1 + w_2B_2$; Proved. ■

I would recommend log-linear interpolation instead of linear interpolation on the zero rates in computing the VaR of this portfolio because the log-linear interpolation better captures the curvature in rates; rates exhibiting exponential behavior is common in fixed-income markets and thus log-linear interpolation is generally preferable for this portfolio.

Notes

Reference Formulas

$\Delta f(x_1, \dots, x_n) \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n = \left(x_1\frac{\partial f}{\partial x_1}\right)\left(\frac{\Delta x_1}{x_1}\right) + \cdots + \left(x_n\frac{\partial f}{\partial x_n}\right)\left(\frac{\Delta x_n}{x_n}\right)$ s.t. $\frac{\Delta x_i}{x_i} = \frac{\Delta S_i}{S_i} \approx r_i$ daily return.

$\Delta P_t \approx \sum_{i=1}^n S_i\frac{\partial f}{\partial S_i}r_i = \sum_{i=1}^n w_ir_i$ s.t. $w_i = S_i\frac{\partial f}{\partial S_i}$.

$$\Delta f(x_1, \dots, x_n) \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n = \left(x_1\frac{\partial f}{\partial x_1}\right)\left(\frac{\Delta x_1}{x_1}\right) + \cdots + \left(x_n\frac{\partial f}{\partial x_n}\right)\left(\frac{\Delta x_n}{x_n}\right)$$