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| Distribution | pmfpdf | Mean | Variance | Moment Generating Function: | Etc. |
|  |  |  |  |  | binary event w/ |
|  |  |  |  |  | of successes in trials. |
|  |  |  |  |  | of events occurring in a fixed time-period with a known rate *.* |
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|  |  |  |  |  | Memoryless Distribution: |
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| CHI-SQUARE DIST.  degrees of freedom. |  |  |  |  |  |
| t-DISTRIBUTION |  |  |  |  | and : |

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| **JACOBIAN TRANSFORMATION** : Given , for which the joint pdf is . Define as and where we assume the functions and are one-to-one. Let the inverse of this transformation be given by and . Then the joint pdf of , is where is the determinant: and denotes the absolute value of the determinant . Thus, the joint pdf is obtained by starting with the joint pdf replacing each value by its expression in terms of , and then multiplying the result by . is called the Jacobian of the transformation. |

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| **Theorem** : .  **PROOF** : Let . . We know that and . Finally, we know . Therefore,  **RESULTS – RB ;** The following results show that for any unbiased estimator of a parameter , we can find another unbiased estimator (a function of a sufficient statistic only) with a smaller variance. In summary, let be an unbiased, , let be a sufficient statistic for . Then the random variable has and a smaller variance.  **RAO BLACKWELL** : Let . Let be a sufficient statistic for and let (not a function of alone) be another unbiased estimator of . Then, the random variable is an unbiased estimator of and .  **PROOF** : Theorem . Let and . As Variance , .  Therefore, in our search for “best” estimators of we can and should restrict our attention to functions of the sufficient statistic.  **COMPLETE FAMILY**: The family of pdf’s is said to be complete if for every requires is zero except on a set of points with probability zero.  **CRLB** : (UNBIASED ESTIMATOR) Cramer-Rao Lower Bound: . where . (BIASED ESTIMATOR) .  If is an unbiased estimator of and attains the CRLB, then is a **MVUE** of .  **MINIMUM VARIANCE UNBIASED ESTIMATOR (MVUE)**: Let be an exponential family. Let be a sufficient statistic for . Then is complete and is said to be a complete sufficient statistic for . |

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| **CONFIDENCE INTERVALS** : meaning = range of values that you expect your estimate to fall between a certain percentage of the time if you run your experiment again.  . If is known and we want a CI for : .  as increases thus . If is unknown use : CI for : .  where are unknown. Construct CI for . . Since  After observing , this yields a CI for : . |

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| If ; . **PROOF** : . As . Thus,  We are given from above . We know chi-square distribution is a sum of iid standard normal random variables squared: ; ( iid) . Therefore, we can use MGF as follows to derive distribution of : for . Therefore, let . Since MGF uniquely describes a distribution. is the MGF given by . Therefore, . |

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| **THEOREM** : Suppose that . Then the sample mean and sample variance are independent random variables where and .  **PROOF** : WLOG (FOR CASE: ; , ) : Show . Firstly, and . Let and . We will apply Jacobian transformation and show that the joint pdf for , factors. Finding the inverse is done with which yields . Therefore, our inverse functions are given by,  and . Since their joint pdf is the product of standard normal pdf given by . Therefore, . Note that . Hence this simplifies to . Now to compute the Jacobian. : . Thus, . We can now factor this into a product of marginals to prove independence between and . . NOTE: AND . Hence, and are independent random variables where and .  **REMARKS**: Using the fact {if , are independent, then , are independent for any functions , not depending on or }, we also proved . |

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| **Theorem** : If , then the statistic .  **Proof** : First, note that since . Therefore, . Denoting these random variables as ; ; and using the fact that and thus we know . Now to get a relationship between , , we see: . Since the moment generating function is factorable, i.e., . Therefore, since we know and , . This is the MGF of . Therefore, . |

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| **Theorem** : has the same pdf as the ratio of: divided by where and and are independent . **Proof** : .  **t-distribution** : If given ; and , then (t-distribution with degrees of freedom). This result implies that the pdf is given by (*see chart*) .  **Proof** : . Therefore, the Jacobian is given by . The pdf of and is given by the product of their marginals as . Therefore, where . Therefore, . As a result, plugging in all values: . Thus, which simplifies to . Now we want to find the pdf of so we integrate out the values . . Let and . ; we see this is the which sums to . Therefore, . Plugging in , , yields: . Finally, . Proved . |

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| **Theorem** (sufficiency w/o factorization criteria): Let for some , then is a sufficient statistic for if does not depend on .  **Example (1)**: Given ; . What is ? . Suppose if , then , therefore, if , . Therefore, does not depend on . Therefore, is a sufficient statistic.  **Example (2)**: Given and ( find this using iid product of MGF). . Finding . Therefore, does not depend on and is sufficient.  **Theorem**  (Factorization Criterion-Jointly Sufficient): Let be functions of real variables. The statistics for , are jointly sufficient statistics for if and only if the joint pdf can be factored as follows for all values of and all values of : ####### ############# .Here the functions and are nonnegative, the function may depend on but does not depend on , and the function will depend on but depends on only through the functions . |

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| Relevant Midterm Predictions Problems: (On sufficiency)  **Example (1)**: *Assuming comes from an exponential family, i.e., . What is a sufficient statistic?* ANSWER: . This simplifies to . Therefore, is a sufficient statistic for by factorization criterion.  **Example (2)**: *Assuming parameter exponential family, . Sample . Show is jointly sufficient*  . ANSWER: , this simplifies to . Therefore, is jointly sufficient .  **Example (3)**: *Suppose where are unknown.* *Prove and are jointly sufficient.* ANSWER : Since we can equivalently prove and are jointly sufficient as is just a function of and . Therefore, . Simplifies to . Jointly sufficient . |

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| If an unbiased estimator has the smallest variance among all other unbiased estimators. We call such an estimator a MVUE.  **MVUE and Sufficient Statistics**: . Let be an unbiased estimator of , . It is desirable to find an unbiased estimator of with smallest variance (MVUE) .  **Question:** How do we find an unbiased estimator of parameter with minimum variance? (MVUE) : . Find a sufficient statistic of . . Minimum variance estimators are functions of the sufficient statistic. . Assuming the family of distributions is a complete family of distributions, then is an unbiased estimator with minimum variance.  **Theorem** : Let , , a statistic is called a “best statistic” if is an unbiased estimator of and for all other estimators with .  **Example**: Given , is a better unbiased estimator of then because |

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| Important Theorems/Properties/Etc .  **Factorization Theorem** (sufficiency/joint sufficiency) : where for .  **Invariance Property of MLE** ( is one-to-one) : If is the MLE of , then for any function , the MLE of is .  e.g. let . . If , then the MLE for is .  **Method of Moments**: Let be a random sample from a population. Method of moment estimation (MOME): Equate sample moments to population moments. If the population has parameters, the MOME consists of solving the system of equations , for the 𝑟 parameters, where , i.e., (kth sample moment) (kth population moment).  **Result** : If , then and . |