Solutions to Math 51 Quiz 1 Practice A

1. (10 points) Abby's data set from her experiment on Monday is

$$(x_1, y_1), (x_2, y_2), \ldots, (x_{50}, y_{50}).$$

The averages $\bar{x}$ and $\bar{y}$ are both 0. The correlation coefficient for Monday's data set is found to be r.

Abby ran the experiment again on Tuesday, and obtained exactly the same data set. What is the correlation coefficient R for both Monday and Tuesday's combined data set

$$(x_1, y_1), (x_2, y_2), \ldots, (x_{50}, y_{50}), (x_1, y_1), (x_2, y_2), \ldots, (x_{50}, y_{50})?$$

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{50} \end{bmatrix}$, $\quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{50} \end{bmatrix}$, then $r = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$.

Let $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{50} \\ x_1 \\ x_2 \\ \vdots \\ x_{50} \end{bmatrix}$, $\quad \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{50} \\ y_1 \\ y_2 \\ \vdots \\ y_{50} \end{bmatrix}$, the the correlation coefficient of two days' worth of dataset is

$$R = \frac{\mathbf{X}\cdot\mathbf{Y}}{\|\mathbf{X}\|\|\mathbf{Y}\|} = \frac{\mathbf{x}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{y}}{\sqrt{\mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{x}}\sqrt{\mathbf{y}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}}} = \frac{2\mathbf{x}\cdot\mathbf{y}}{\sqrt{2}\|\mathbf{x}\|\sqrt{2}\|\mathbf{y}\|} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = r.$$

Indeed, when the data set is repeated, the correlation between x and y variables do not change.
Remark: This is a common early-round interview question for data science and quant jobs

2. (2 points) **True or False:** It is possible to find non-zero vectors $\mathbf{u}$, $\mathbf{v}$ for which $\|\mathbf{u}+2\mathbf{v}\| = \|\mathbf{u}\| - \|2\mathbf{v}\|$.

True: Note that

$$\begin{aligned}
\|\mathbf{u}+2\mathbf{v}\|^2 &= (\mathbf{u}+2\mathbf{v})\cdot(\mathbf{u}+2\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + 4\mathbf{v}\cdot\mathbf{v} + 4\mathbf{u}\cdot\mathbf{v} \\
&= \|\mathbf{u}\|^2 + 4\|\mathbf{v}\|^2 + 4\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \\
(\|\mathbf{u}\| - \|2\mathbf{v}\|)^2 &= \|\mathbf{u}\|^2 + 4\|\mathbf{v}\|^2 - 4\|\mathbf{u}\|\|\mathbf{v}\| \\
\|\mathbf{u}+2\mathbf{v}\|^2 - (\|\mathbf{u}\| - \|2\mathbf{v}\|)^2 &= 4\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta + 4\|\mathbf{u}\|\|\mathbf{v}\| \\
&= 4\|\mathbf{u}\|\|\mathbf{v}\|(1+\cos\theta)
\end{aligned}$$

where $\theta \geq 0$ is the angle between $\mathbf{v}$ and $\mathbf{w}$. If $\theta = \pi$, then $\cos\theta = -1$ and $\|\mathbf{u}+2\mathbf{v}\|^2 = (\|\mathbf{u}\| - \|2\mathbf{v}\|)^2$. But $\theta = \pi$ means that $\mathbf{u}$ is a negative scalar multiple of $2\mathbf{v}$; i.e., $\mathbf{u} = -2k\mathbf{v}$ for some $k > 0$. One can check that letting $k = 1$ is sufficient to satisfy the desired relation, since

$$\|-2\mathbf{v} + 2\mathbf{v}\| = \|\mathbf{0}\| = 0 \quad \text{and} \quad \|-2\mathbf{v}\| - \|2\mathbf{v}\| = 2\|\mathbf{v}\| - 2\|\mathbf{v}\| = 0$$

for any choice of nonzero vector $\mathbf{v}$; so we may respond "True."

(We could also satisfy the relation if k equals 2, or 5, or indeed any scalar value of 1 or more: for, if $k > 0$ and nonzero vector $\mathbf{v}$, then

$$\begin{aligned}\|-2k\mathbf{v} + 2\mathbf{v}\| &= \|(2-2k)\mathbf{v}\| = |2-2k|\|\mathbf{v}\|, \text{ while}\\ \|-2k\mathbf{v}\| - \|2\mathbf{v}\| &= 2k\|\mathbf{v}\| - 2\|\mathbf{v}\| = (2k-2)\|\mathbf{v}\|,\end{aligned}$$

so we have equality if $|2-2k| = 2k-2 = -(2-2k)$, which occurs whenever $2-2k \le 0$, or $k \ge 1$.)

3. (2 points) **True or False:** the point $\mathbf{p} = (0,3,-1)$ is inside the triangle with vertices at $\mathbf{a} = (0,3,6)$, $\mathbf{b} = (-6,0,0)$ and $\mathbf{c} = (2,4,-6)$. (Remark: $\mathbf{p}$ is inside the triangle with vertices $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ when $\mathbf{p}$ is a convex linear combination of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.)

True (i.e., "always true"): We need to find x, y and z where

$$\mathbf{p} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}, \qquad \text{namely} \qquad \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6y+2z \\ 3x+4z \\ 6x-6z \end{bmatrix}, \qquad \begin{cases} -6y+2z &= 0 \\ 3x+4z &= 3 \\ 6x-6z &= -1 \end{cases}.$$

Solving the three equations simultaneously.
From the first equation $-6y + 2z = 0$, we see $z = 3y$.
From the second equation $3x + 4z = 3$, we see that $3x = 3 - 4z$.
Substitute $3x = 3 - 4z$ into the third equation, we have

$$6x - 6z = 2(3-4z) - 6z = 6 - 14z = -1, \qquad 14z = 7, \qquad z = \frac{1}{2.}$$

Since $z = 3y$ and $3z = 3 - 4z$, we see

$$y = \frac{1}{3}z = \frac{1}{6},$$
$$3x = 3 - 4z = 3 - 4\frac{1}{2} = 1, \qquad x = \frac{1}{3}$$

Since $0 \le x, y, z \le 1$ and $x + y + z = 1$, we verified that $\mathbf{p}$ is indeed a convex linear combination of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, so $\mathbf{p}$ is inside the triangle with the vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

4. (3 points) Let $\mathbf{v}$ be a fixed nonzero vector in $\mathbf{R}^3$. Geometrically, the collection of vectors $\mathbf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3$ satisfying the condition

$$\|\mathbf{w} - \mathbf{v}\|^2 - \|\mathbf{w}\|^2 = 1$$

is a

a) line.
b) plane.
c) $\mathbf{R}^3$.
d) sphere.
e) might take different shapes depending on what $\mathbf{v}$ and $\mathbf{w}$ are.

Recall that $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$, we have

$$\begin{aligned}\|\mathbf{w} - \mathbf{v}\|^2 - \|\mathbf{w}\|^2 &= (\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) - \mathbf{w} \cdot \mathbf{w}\\ &= \mathbf{w} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w}\\ &= -2\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{v} \qquad\qquad = 1\end{aligned}$$

We have
$$2\mathbf{v} \cdot \mathbf{w} = 1 + \|\mathbf{v}\|^2.$$
Let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then
$$2\mathbf{v} \cdot \mathbf{w} = 2ax + 2by + 2cz = 1 + a^2 + b^2 + c^2$$
for some fixed numerical values a, b and c gives an equation of a plane.

5. (3 points) Suppose we have two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^n$, where
$$\|\mathbf{v}\| = 2\sqrt{2}, \quad \|\mathbf{w}\| = 2, \quad \mathbf{v} \cdot \mathbf{w} = 4$$
Which of the following must always be true? (Choose only one.)

- $\mathbf{v} + \mathbf{w}$ is perpendicular to $\mathbf{v} - \mathbf{w}$.
- $\|\mathbf{v} - \mathbf{3w}\| = 10$
- $\boxed{\|\mathbf{v} - \mathbf{w}\| = 2}$

First, we have $\mathbf{v} + \mathbf{w}$ perpendicular to $\mathbf{v} - \mathbf{w}$ if and only if
$$0 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2,$$
but in our case the lengths of $\mathbf{v}$ and $\mathbf{w}$ are not the same. Now notice that we can evaluate the other possibilities by computing the squared length of $\mathbf{v} - c\mathbf{w}$ in terms of a scalar c:
$$\begin{aligned} \|\mathbf{v} - c\mathbf{w}\|^2 &= (\mathbf{v} - c\mathbf{w}) \cdot (\mathbf{v} - c\mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} + c^2 \mathbf{w} \cdot \mathbf{w} - c\mathbf{v} \cdot \mathbf{w} - c\mathbf{w} \cdot \mathbf{v} \\ &= \|\mathbf{v}\|^2 + c^2\|\mathbf{w}\|^2 - 2c(\mathbf{v} \cdot \mathbf{w}) \\ &= 8 + 4c^2 - 8c \end{aligned}$$
We may now check that if $c = 3$, this equals $8 + (4)(9) - (8)(3) = 20$, so $\|\mathbf{v} - 3\mathbf{w}\| = \sqrt{20}$; but if $c = 1$, this equals $8 + (4)(1) - 8 = 4$, so $\|\mathbf{v} - \mathbf{w}\| = 2$.