1. (a) We have

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

(which is nonzero), and by computing some dot products along the way we have

$$\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_2) = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\,\mathbf{w}_1 = \mathbf{v}_2 - \frac{-5}{5}\,\mathbf{w}_1 = \mathbf{v}_2 + \mathbf{w}_1.$$

Plugging in $\mathbf{v}_2$ and $\mathbf{w}_1$ then yields

$$\mathbf{w}_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix},$$

which is clearly nonzero and orthogonal to $\mathbf{w}_1$.
Finally, computing some more dot products along the way,

$$\begin{aligned} \mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{V_2}(\mathbf{v}_3) = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_3) - \mathbf{Proj}_{\mathbf{w}_2}(\mathbf{v}_3) &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\,\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\,\mathbf{w}_2 \\ &= \mathbf{v}_3 - \frac{-5}{5}\mathbf{w}_1 - \frac{3}{9}\mathbf{w}_2 \\ &= \mathbf{v}_3 + \mathbf{w}_1 - \frac{1}{3}\mathbf{w}_2. \end{aligned}$$

Now we plug in $\mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2$ to get

$$\mathbf{w}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

The Gram-Schmidt process has produced two nonzero vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ constituting an orthogonal basis for the span of the $\mathbf{v}_j$'s. This span is therefore a plane, with basis $\{\mathbf{w}_1, \mathbf{w}_2\}$. By design, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{w}_1 = \mathbf{v}_2 + \mathbf{v}_1$ and

$$\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{w}_1 - (1/3)\mathbf{w}_2 = \mathbf{v}_3 + \mathbf{v}_1 - (1/3)(\mathbf{v}_2 + \mathbf{v}_1) = (2/3)\mathbf{v}_1 - (1/3)\mathbf{v}_2 + \mathbf{v}_3.$$

(b) By design of $\mathbf{w}_2$ and $\mathbf{w}_3$ we have

$$\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{w}_1, \;\; \mathbf{w}_3 = \mathbf{v}_3 + \mathbf{w}_1 - (1/3)\mathbf{w}_2,$$

so

$$\mathbf{v}_2 = \mathbf{w}_2 - \mathbf{w}_1, \;\; \mathbf{v}_3 = -\mathbf{w}_1 + (1/3)\mathbf{w}_2 + \mathbf{w}_3 = -\mathbf{w}_1 + (1/3)\mathbf{w}_2$$

(the final equality using that $\mathbf{w}_3 = \mathbf{0}$).
Let's confirm the expressions for $\mathbf{v}_2$ and $\mathbf{v}_3$ in terms of the $\mathbf{w}_i$'s by direct computation:

$$\mathbf{w}_2 - \mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \mathbf{v}_2$$

and

$$-\mathbf{w}_1 + (1/3)\mathbf{w}_2 + \mathbf{w}_3 = -\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + (1/3)\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \mathbf{v}_3.$$

(c) The vanishing of $\mathbf{w}_3$ says

$$\mathbf{0} = \mathbf{w}_3 = \mathbf{v}_3 + \mathbf{w}_1 - (1/3)\mathbf{w}_2 = \mathbf{v}_3 + \mathbf{v}_1 - (1/3)(\mathbf{v}_2 + \mathbf{v}_1) = (2/3)\mathbf{v}_1 - (1/3)\mathbf{v}_2 + \mathbf{v}_3.$$

Multiplying through by 3 gives the linear dependence relation

$$2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 = \mathbf{0}$$

as desired. We conclude that $2\mathbf{v}_1 = \mathbf{v}_2 - 3\mathbf{v}_3$ and $3\mathbf{v}_3 = -2\mathbf{v}_1 + \mathbf{v}_2$, so respectively dividing by 2 and 3 yields that

$$\mathbf{v}_1 = \frac{1}{2}\mathbf{v}_2 - \frac{3}{2}\mathbf{v}_3, \ \ \mathbf{v}_3 = -\frac{2}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2.$$

This gives the desired expression as a linear combination for each of $\mathbf{v}_1$ and $\mathbf{v}_3$, and let's now compute each right side directly:

$$\frac{1}{2}\mathbf{v}_2 - \frac{3}{2}\mathbf{v}_3 = \begin{bmatrix} 3/2 \\ 1/2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \mathbf{v}_1$$

as desired and

$$-\frac{2}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2/3 \\ -4/3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \mathbf{v}_3$$

as desired.

$\diamond$

2. (a) As usual, $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix}$. Next, computing dot products along the way,

$$\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_2) = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \mathbf{v}_2 - \frac{42}{14}\mathbf{w}_1 = \mathbf{v}_2 - 3\mathbf{w}_1.$$

Plugging in $\mathbf{v}_2$ and $\mathbf{w}_1$ gives

$$\mathbf{w}_2 = \begin{bmatrix} -2 \\ 10 \\ 7 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 10-6 \\ 7-9 \\ -1-3(-1) \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix},$$

which is nonzero (and as a safety check it is orthogonal to $\mathbf{w}_1$ since $0 + 8 - 6 - 2 = 0$).
Finally, again computing dot products along the way, we have

$$\begin{aligned} \mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_3) - \mathbf{Proj}_{\mathbf{w}_2}(\mathbf{v}_3) &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 \\ &= \mathbf{v}_3 - \frac{14}{14}\mathbf{w}_1 - \frac{-56}{28}\mathbf{w}_2 \\ &= \mathbf{v}_3 - \mathbf{w}_1 + 2\mathbf{w}_2. \end{aligned}$$

Plugging in $\mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2$ gives

$$\mathbf{w}_3 = \begin{bmatrix} 10 \\ -6 \\ 10 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \\ 7 \\ 5 \end{bmatrix} + \begin{bmatrix} -4 \\ 8 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

This is also nonzero, so the process ends with 3 nonzero vectors. Thus, the span V of the three $\mathbf{v}_i$'s has dimension 3, so the $\mathbf{v}_i$'s are linearly independent.

(b) From the work in (a) we have $\mathbf{w}_1 = \mathbf{v}_1$, $\mathbf{w}_2 = \mathbf{v}_2 - 3\mathbf{w}_1 = \mathbf{v}_2 - 3\mathbf{v}_1$, and

$$\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{w}_1 + 2\mathbf{w}_2 = \mathbf{v}_3 - \mathbf{v}_1 + 2(\mathbf{v}_2 - 3\mathbf{v}_1) = -7\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3.$$

Likewise, we have $\mathbf{v}_1 = \mathbf{w}_1$, $\mathbf{v}_2 = \mathbf{w}_2 + 3\mathbf{w}_1$, and $\mathbf{v}_3 = \mathbf{w}_3 + \mathbf{w}_1 - 2\mathbf{w}_2$.
We now directly compute the expressions thereby obtained for each of $\mathbf{v}_3$ and $\mathbf{w}_3$:

$$\mathbf{v}_2 - 3\mathbf{v}_1 = \begin{bmatrix} -2 \\ 10 \\ 7 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 10-6 \\ 7-9 \\ -1+3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \\ 2 \end{bmatrix} = \mathbf{w}_2$$

as desired, and

$$-7\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \begin{bmatrix} 0 \\ -14 \\ -21 \\ 7 \end{bmatrix} + \begin{bmatrix} -4 \\ 20 \\ 14 \\ -2 \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ 10 \\ 4 \end{bmatrix} = \begin{bmatrix} -4+10 \\ -14+20-6 \\ -21+14+10 \\ 7-2+4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \\ 9 \end{bmatrix} = \mathbf{w}_3$$

as desired.

(c) The determination of the $\mathbf{w}_i$'s gives their lengths as

$$\|\mathbf{w}_1\| = \sqrt{0+4+9+1} = \sqrt{14}, \;\; \|\mathbf{w}_2\| = 2\sqrt{1+4+1+1} = 2\sqrt{7}, \;\; \|\mathbf{w}_3\| = 3\sqrt{4+0+1+9} = 3\sqrt{14}.$$

Hence, the associated unit vectors $\mathbf{w}_i' = \mathbf{w}_i/\|\mathbf{w}_i\|$ are

$$\mathbf{w}_1' = \begin{bmatrix} 0 \\ 2/\sqrt{14} \\ 3/\sqrt{14} \\ -1/\sqrt{14} \end{bmatrix}, \;\; \mathbf{w}_2' = \begin{bmatrix} -1/\sqrt{7} \\ 2/\sqrt{7} \\ -1/\sqrt{7} \\ 1/\sqrt{7} \end{bmatrix}, \;\; \mathbf{w}_3' = \begin{bmatrix} 2/\sqrt{14} \\ 0 \\ 1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}.$$

◇

3. (a) We expand the left side to rewrite it as as $a_1\mathbf{v}_1 - 3t\mathbf{v}_1 + a_2\mathbf{v}_2 + 7t\mathbf{v}_2 + a_3\mathbf{v}_3 - 5t\mathbf{v}_3 + a_4\mathbf{v}_4 + 2t\mathbf{v}_4$ and then collect the t-parts to rewrite this as

$$(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4) + t(-3\mathbf{v}_1 + 7\mathbf{v}_2 - 5\mathbf{v}_3 + 2\mathbf{v}_4).$$

The assumed linear dependence relation gives that the t-part vanishes, leaving us with $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4$, as desired.

(b) By (a) with $a_1 = 2$, $a_2 = 4$, $a_3 = 3$, and $a_4 = -5$ we see that for any t at all,

$$(-2-3t)\mathbf{v}_1 + (4+7t)\mathbf{v}_2 + (3-5t)\mathbf{v}_3 + (-5+2t)\mathbf{v}_4 = -2\mathbf{v}_1 + 4\mathbf{v}_2 + 3\mathbf{v}_3 - 5\mathbf{v}_4.$$

By using two different nonzero values for t, we get two different 4-tuples of coefficients yielding the same linear combination of the $\mathbf{v}_i$'s as desired. For instance, by using $t = 1$ and then $t = -2$ we get

$$(b_1, b_2, b_3, b_4) = (-2-34+7, 3-5, -5+2) = (-5, 11, -2, -3)$$

and

$$(c_1, c_2, c_3, c_4) = (-2+6, 4-14, 3+10, -5-4) = (4, -10, 13, -9).$$

(c) By subtracting the right side from the left side and combining $\mathbf{w}_i$-terms for each i, we get

$$(a_1 - b_1)\mathbf{w}_1 + (a_2 - b_2)\mathbf{w}_2 + (a_3 - b_3)\mathbf{w}_3 + (a_4 - b_4)\mathbf{w}_4 = \mathbf{0}.$$

But one of the features of linear independence is that the *only* way a linear combination can yield the zero vector is when the coefficients all vanish. Hence, each coefficient $a_i - b_i$ must equal 0, which is the same as saying $a_i = b_i$ for all i, as desired.

◇

4. (a) We compute

$$\mathbf{w}_1 - 2\mathbf{w}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{e}_1$$

and

$$-2\mathbf{w}_1 + 5\mathbf{w}_2 = -2\begin{bmatrix} 5 \\ 2 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{e}_2.$$

By definition of B-coordinates for $B = \{\mathbf{w}_1, \mathbf{w}_2\}$, for any $\mathbf{w} \in \mathbf{R}^2$ we have $[\mathbf{w}]_B = (a_1, a_2)$ where $\mathbf{w} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2$. Hence, the expressions we just verified for $\mathbf{e}_1$ and $\mathbf{e}_2$ as linear combinations of the $\mathbf{w}_i$'s say that

$$[\mathbf{e}_1]_B = (1, -2), \;\; [\mathbf{e}_2]_B = (-2, 5).$$

(b) We have

$$\mathbf{w} = 3\mathbf{e}_1 - 5\mathbf{e}_2 = 3(\mathbf{w}_1 - 2\mathbf{w}_2) - 5(-2\mathbf{w}_1 + 5\mathbf{w}_2) = 3\mathbf{w}_1 - 6\mathbf{w}_2 + 10\mathbf{w}_1 - 25\mathbf{w}_2 = 13\mathbf{w}_1 - 31\mathbf{w}_2,$$
$$\mathbf{w}' = \mathbf{e}_1 + 2\mathbf{e}_2 = (\mathbf{w}_1 - 2\mathbf{w}_2) + 2(-2\mathbf{w}_1 + 5\mathbf{w}_2) = \mathbf{w}_1 - 2\mathbf{w}_2 - 4\mathbf{w}_1 + 10\mathbf{w}_2 = -3\mathbf{w}_1 + 8\mathbf{w}_2,$$
$$\mathbf{w}'' = -2\mathbf{e}_1 + 3\mathbf{e}_2 = -2(\mathbf{w}_1 - 2\mathbf{w}_2) + 3(-2\mathbf{w}_1 + 5\mathbf{w}_2) = -2\mathbf{w}_1 + 4\mathbf{w}_2 - 6\mathbf{w}_1 + 15\mathbf{w}_2 = -8\mathbf{w}_1 + 19\mathbf{w}_2,$$

so

$$[\mathbf{w}]_B = (13, -31), \ \ [\mathbf{w}']_B = (-3, 8), \ \ [\mathbf{w}'']_B = (-8, 19).$$

(c) The meaning of saying that $[\mathbf{v}]_{\mathcal{B}} = (a_1, \dots, a_k)$ and $[\mathbf{v}']_{\mathcal{B}} = (a'_1, \dots, a'_k)$ is

$$\mathbf{v} = \sum_{j=1}^{k} a_j \mathbf{v}_j, \ \ \mathbf{v}' = \sum_{j=1}^{k} a'_j \mathbf{v}_j.$$

To figure out the $\mathcal{B}$-coordinates of $5\mathbf{v} - 7\mathbf{v}'$, we need to express this as a linear combination of the $\mathbf{v}_j$'s and then read off the coefficients. Since we have such expressions for $\mathbf{v}$ and $\mathbf{v}'$, we plug those in:

$$\begin{aligned} 5\mathbf{v} - 7\mathbf{v}' &= 5(a_1\mathbf{v}_1 + \dots + a_k\mathbf{v}_k) - 7(a'_1\mathbf{v}_1 + \dots + a'_k\mathbf{v}_k) \\ &= (5a_1)\mathbf{v}_1 + \dots + (5a_k)\mathbf{v}_k + (-7a'_1)\mathbf{v}_1 + \dots + (-7a'_k)\mathbf{v}_k \\ &= (5a_1 - 7a'_1)\mathbf{v}_1 + \dots (5a_k - 7a'_k)\mathbf{v}_k \end{aligned}$$

(where the final equality is obtained by combining $\mathbf{v}_i$-terms for each i. The coefficient of $\mathbf{v}_i$ at the end is $5a_i - 7a'_i$, so

$$[5\mathbf{v} - 7\mathbf{v}']_{\mathcal{B}} = (5a_1 - 7a'_1, \dots, 5a_k - 7a'_k)$$

as desired.

In the preceding calculation, nothing special was used about 5 and -7; they could have been any scalars at all. So coming back to (b), since $\mathbf{w} = 3\mathbf{e}_1 - 5\mathbf{e}_2$ we have

$$[\mathbf{w}]_B = 3[\mathbf{e}_1]_B - 5[\mathbf{e}_2]_B = 3(1, -2) - 5(-2, 5) = (3, -6) + (10, -25) = (13, -31),$$
$$[\mathbf{w}']_B = [\mathbf{e}_1]_B + 2[\mathbf{e}_2]_B = (1, -2) + 2(-2, 5) = (1, -2) + (-4, 10) = (-3, 8),$$

and

$$[\mathbf{w}'']_B = -2[\mathbf{e}_1]_B + 3[\mathbf{e}_2]_B = -2(1, -2) + 3(-2, 5) = (-2, 4) + (-6, 15) = (-8, 19).$$

◇

5. (a) The first column of A must be $\mathbf{x} = (1/3)\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. The other columns together with $\mathbf{x}$ need to form an orthonormal basis. There are many choices, and we'll try to find one having an especially simple form:

$$A = \begin{bmatrix} 2/3 & a & b \\ 2/3 & -a & b \\ 1/3 & 0 & c \end{bmatrix}$$

for a, b, c to be found; arranging the second column as it is written guarantees its orthogonality to the first column and to the third column (regardless of the values of a and b), and we need the second column to be a unit vector. This unit-vector condition says $a^2 + a^2 = 1$, or equivalently $2a^2 = 1$, so we use $a = 1/\sqrt{2}$.

Finally, we need the third column to be orthogonal to the first and to be a unit vector. Orthogonality to the first says

$$0 = 2b/3 + 2b/3 + c/3 = (1/3)(4b + c),$$

so $c = -4b$. It remains to find b, for which we use the unit-vector requirement on the third column, which says

$$1 = b^2 + b^2 + c^2 = 2b^2 + 16b^2 = 18b^2,$$

so we use $b = 1/\sqrt{18}$ and $c = -4b = -4/\sqrt{18}$. This yields the answer

$$A = \begin{bmatrix} 2/3 & -1/\sqrt{2} & 1/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & -4/\sqrt{18} \end{bmatrix}$$

(b) We can use

$$B = A^{-1} = A^\top = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & 1/\sqrt{18} & -4\sqrt{18} \end{bmatrix}$$

(c) Similarly, we know that the first column of C must be $\mathbf{y} = (1/\sqrt{3}) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Seeking a C in the same basic shape as the answer to (a), we arrive at the choice

$$C = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

One choice for M is

$$M = CA^{-1} = CB = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & 1/\sqrt{18} & -4\sqrt{18} \end{bmatrix}.$$

◇

6. (a) Let $B = \frac{1}{2}(A + A^\top)$ and let b_{ij} be the entry in the ith row and jth column of B. Then $b_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$. If we look at b_{ji}, we get $b_{ji} = \frac{1}{2}(a_{ji} + a_{ij}) = \frac{1}{2}(a_{ij} + a_{ji}) = b_{ij}$. Alternatively, we can compute

$$B^\top = \frac{1}{2}(A + A^\top)^\top = \frac{1}{2}(A^\top + A^{\top\top}) = \frac{1}{2}(A^\top + A) = \frac{1}{2}(A + A^\top) = B,$$

so B is symmetric.

(b) We have $\|B\mathbf{x}\|^2 = (B\mathbf{x}) \cdot (B\mathbf{x}) = (B\mathbf{x})^\top (B\mathbf{x}) = \mathbf{x}^\top B^\top B\mathbf{x}$, so we use

$$M = B^\top B$$

(which is clearly $n \times n$). This is symmetric since $M^\top = (B^\top B)^\top = B^\top (B^\top)^\top = B^\top B = M$.

◇

7. (a) It meets the xy-plane in the curve given by setting z to be 0 in the equation of S_+, which is the circle $x^2 + y^2 = 1$ in the xy-plane. It meets the xz-plane in the curve given by setting y to be 0 in the equation of S_+, which is the hyperbola $x^2 - z^2 = 1$ in the xz-plane. Similarly it meets yz-plane in the hyperbola $y^2 - z^2 = 1$. The curves along which S_+ meets each of the coordinate planes is shown in Figure 1 .

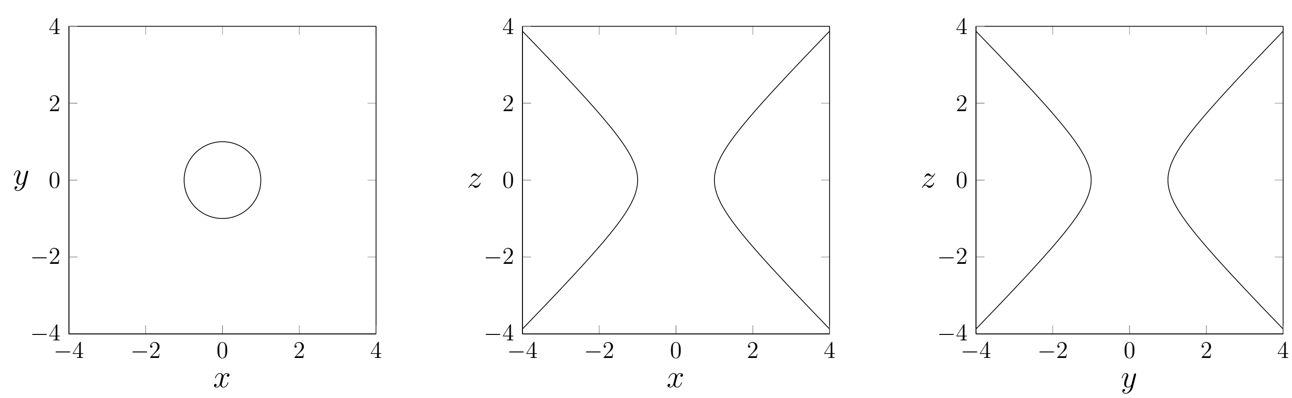


Figure 1: The curves where S_+ meets the xy-plane (left), xz-plane (center), and yz-plane (right).

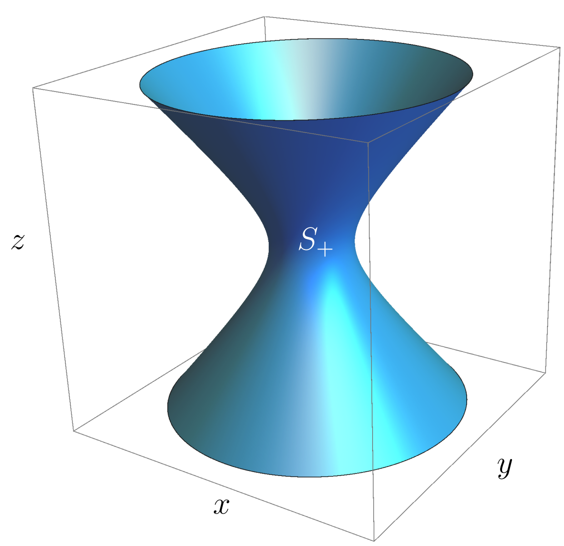
(b) By setting z to be a in the equation of S_+, we get the equation $x^2 + y^2 = 1 + a^2$. This is a circle with radius $\sqrt{1 + a^2}$ having center $(0, 0, a)$ on the z-axis. Rotating around the z-axis has the effect in each plane $z = a$ of rotating around the origin in that plane. But that origin corresponds to the point $(0,0,a)$ that is the center of the circle $x^2+y^2=1+a^2$ in that plane, so such rotation carries the circle into itself. Since S_+ is assembled from where it meets each plane (every point in S_+ lies in exactly one such plane, namely the one for which a is the third coordinate of the point), this information for every a tells us that every such rotation carries S_+ into itself.

(c) We claim that every point P in space can be rotated around the z-axis to wind up in the half-plane $x \geq 0$ in the xz-plane. This can be seen by drawing a 3-dimensional picture, but here is another way. If a is the third coordinate of P and we work inside the plane $z=a$ through P that meets the xz-plane along the "x-axis" of points $(x,0,a)$ in that plane, this becomes the assertion that any point in a plane can be rotated around the origin to wind up on the non-negative x-axis, which we see by staring at points in a plane. Hence, anything in space that is rotationally symmetric around the z-axis is obtained by taking the part of it in the half-plane $x \geq 0$ in the xz-plane and rotating that around the z-axis.

Now consider the region in S_+ where it meets the half-plane $x \geq 0$ in the xz-plane $y=0$ meets S_+. This is given by setting y to be 0 and requiring $x \geq 0$ in the equation of S_+. The equation is $Q(x,0,z)=1$, which is to say the hyperbola $x^2-z^2=1$, and by demanding $x \geq 0$ it is exactly $x=\sqrt{1+z^2}$. This is one of the two connected parts (or "branches") of the hyperbola $x^2-z^2=1$. Hence, S_+ is obtained by rotating the curve $x=\sqrt{1+z^2}$ (one "branch" of $x^2-z^2=1$) around the z-axis, as shown on the left in Figure 2 .

(d) It meets each plane $z=a$ in the region given by the equation $x^2+y^2=a^2-1$. This is a circle centered at the origin when $|a|>1$, it is the point $(0,0,a)$ when $a=\pm 1$, and it is empty when $|a|<1$. In all cases, rotation around the origin in the plane $z=a$ carries this into itself, so once again S_- is rotationally symmetric around the z-axis.

(e) The part of S_- in the xz-plane is given by setting y to be 0 in the equation of S_-; this is $Q(x,0,z)=-1$, or equivalently $x^2-z^2=-1$. This is again a hyperbola (written equivalently as $z^2-x^2=1$). Imposing the condition $x \geq 0$ yields the curve $x=\sqrt{z^2-1}$, which only makes sense when $|z| \geq 1$, and for $z \geq 1$ and for $z \leq -1$ it yields two curves that *do not touch*: one curve C with $z>0$ and another curve C' with $z<0$. Applying rotation around the z-axis to this curve consisting of two parts recovers the entirety of S_- by (d), so this level set is a surface consisting of two parts (each obtained from rotating one of the two curves C, C' around the z-axis). This surface is shown on the right in Figure 2 .



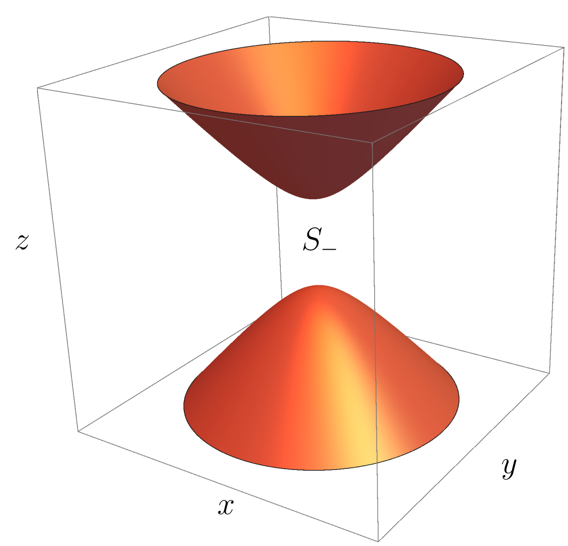


Figure 2: The surfaces S_+ (left) and S_- (right).

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8. (a) We compute

$$\begin{aligned} H_{\mathbf{v}}^\top = (\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top)^\top = \mathrm{I}_n^\top - \frac{2}{\|\mathbf{v}\|^2}(\mathbf{v}\mathbf{v}^\top)^\top &= \mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}(\mathbf{v}^\top)^\top\mathbf{v}^\top \\ &= \mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top \\ &= H_{\mathbf{v}}. \end{aligned}$$

(b) We compute

$$\begin{aligned} H_{\mathbf{v}}^\top H_{\mathbf{v}} = H_{\mathbf{v}}H_{\mathbf{v}} &= (\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top)(\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top) \\ &= \mathrm{I}_n^2 - \frac{4}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top + \frac{4}{\|\mathbf{v}\|^4}\mathbf{v}(\mathbf{v}^\top)\mathbf{v}\mathbf{v}^\top \\ &= \mathrm{I}_n - \frac{4}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top + \frac{4}{\|\mathbf{v}\|^4}\mathbf{v}(\|\mathbf{v}\|^2)\mathbf{v}^\top \\ &= \mathrm{I}_n - \frac{4}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top + \frac{4}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top \\ &= \mathrm{I}_n. \end{aligned}$$

(c) We compute

$$\begin{aligned} H_{\mathbf{v}}(c\mathbf{v}) = (\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top)c\mathbf{v} = c\mathbf{v} - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}(\mathbf{v}^\top c\mathbf{v}) &= c\mathbf{v} - c\frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\|\mathbf{v}\|^2 \\ &= c\mathbf{v} - 2c\mathbf{v} \\ &= -c\mathbf{v}. \end{aligned}$$

(d) We compute

$$H_{\mathbf{v}}(\mathbf{u}) = (\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top)\mathbf{u} = \mathbf{u} - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}(\mathbf{v}^\top\mathbf{u}),$$

and the final term being subtracted is $\mathbf{0}$ since $\mathbf{v}^\top\mathbf{u}$ is the 1×1 matrix whose unique entry is $\mathbf{v}\cdot\mathbf{u} = 0$ (by hypothesis). Thus, we are left with just $\mathbf{u}$ (as desired).

(e) A normal vector to the plane is $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, and the corresponding Householder reflection matrix is given by

$$\begin{aligned} H_{\mathbf{v}} = \mathrm{I}_3 - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top = \mathrm{I}_3 - \frac{2}{9}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1/9 & 4/9 & -8/9 \\ 4/9 & 7/9 & 4/9 \\ -8/9 & 4/9 & 1/9 \end{bmatrix}. \end{aligned}$$

(By inspection this is symmetric, as it has to be, and it is orthogonal since the columns are seen by inspection to be mutually perpendicular – for that we can ignore the common factor of $1/9$ everywhere – and all have length 1 because the numerators constitute a matrix whose columns all have squared-length equal to $81 = 9^2$.)

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9. (a) Consider a vector $\mathbf{x} \in \mathbf{R}^n$. Then $A''\mathbf{x} = \begin{bmatrix} A\mathbf{x} \\ \mathbf{v}'\mathbf{x} \end{bmatrix} = \begin{bmatrix} A\mathbf{x} \\ \mathbf{v}\cdot\mathbf{x} \end{bmatrix}$. In particular, $\mathbf{x} \in N(A'')$ precisely when $A\mathbf{x}$ and $\mathbf{v}\cdot\mathbf{x}$ both vanish, which is exactly the condition that $\mathbf{x} \in N(A)$ and $\mathbf{x}$ is orthogonal to $\mathbf{v}$.

(b) Let $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$, so $N(A)$ is the y-axis in $\mathbf{R}^2$. If $\mathbf{v} = \mathbf{0}$ then $A'' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and so $N(A'')$ is again the y-axis (so $N(A'') = N(A)$). If $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A'' = I_2$, which is invertible. In particular, $N(A'') = \{0\} \subsetneq N(A)$ in that case.

(c) Suppose $\sum_{j=1}^{k} c_j \mathbf{a}''_{i_j}$ vanishes in $\mathbf{R}^{m+1}$; we want to show the c_j's all vanish. By staring at the first m vector entries, we see that $\sum_{j=1}^{k} c_j \mathbf{a}_{i_j}$ vanishes in $\mathbf{R}^m$. Thus, the coefficients $c_1, \dots, c_k$ vanish by linear independence of $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}$. Every linearly independent subset of a linear subspace of $\mathbf{R}^m$ can be extended to a basis of the subspace, so those k linearly independent columns of A'' belong to a basis of $C(A'')$ and hence $\dim C(A'') \geq k = \dim C(A)$ as desired.

(d) Suppose $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Then $\dim C(A) = 1$. If $\mathbf{v} = \mathbf{0}$ then $A'' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and so $C(A'')$ is a line; hence, $\dim C(A'') = \dim C(A) = 1$. If instead $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A'' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. In that case, $\dim C(A'') = 2 > \dim C(A)$.

$\diamond$

10. (a) An orthogonal basis is $\{\mathbf{a}_1, \mathbf{a}'_2\}$ for

$$\mathbf{a}'_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1}\mathbf{a}_1 = \mathbf{a}_2 - \frac{44}{11}\mathbf{a}_1 = \mathbf{a}_2 - 4\mathbf{a}_1 = \begin{bmatrix} 2 \\ 13 \\ -3 \end{bmatrix} - \begin{bmatrix} 4 \\ 12 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

The given relations $\mathbf{a}_3 = -3\mathbf{a}_1 + 2\mathbf{a}_2$ and $\mathbf{a}_4 = \mathbf{a}_1 - \mathbf{a}_2$ (easily verified directly) can be written as linear dependence relations among columns:

$$-3\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}, \quad \mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_4 = \mathbf{0},$$

whose left sides are respectively $A\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ via the relationship between $A\mathbf{x}$ and a linear combination of columns of A. This explains why $\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ belong to $N(A)$.

(b) For any 3-vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ we have

$$\begin{aligned} \mathbf{Proj}_{C(A)}(\mathbf{b}) = \mathbf{Proj}_{\mathbf{a}_1}(\mathbf{b}) + \mathbf{Proj}_{\mathbf{a}'_2}(\mathbf{b}) &= \frac{\mathbf{b} \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{b} \cdot \mathbf{a}'_2}{\mathbf{a}'_2 \cdot \mathbf{a}'_2}\mathbf{a}'_2 \\ &= \frac{b_1 + 3b_2 - b_3}{11}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \frac{-2b_1 + b_2 + b_3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Plugging in $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$ respectively yields

$$\mathbf{Proj}_{C(A)}(\mathbf{b}_1) = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} \neq \mathbf{b}_1,$$

$$\mathbf{Proj}_{C(A)}(\mathbf{b}_2) = -\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix} = \mathbf{b}_2,$$

$$\mathbf{Proj}_{C(A)}(\mathbf{b}_3) = -\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} = \mathbf{b}_3.$$

(Of course, these projections could be computed directly without first going through a formula for general $\mathbf{b}$ as we did above.) Hence, $A\mathbf{x} = \mathbf{b}_1$ does not have a solution whereas $A\mathbf{x} = \mathbf{b}_2$ and $A\mathbf{x} = \mathbf{b}_3$ each have a solution.

The work we did to verify that $A\mathbf{x} = \mathbf{b}_2$ has a solution yields an expression for $\mathbf{b}_2$ in terms of the first and second columns of A:

$$\mathbf{b}_2 = -\mathbf{a}_1 + 2\mathbf{a}_2' = -\mathbf{a}_1 + 2(\mathbf{a}_2 - 4\mathbf{a}_1) = -9\mathbf{a}_1 + 2\mathbf{a}_2,$$

so this says $\mathbf{b}_2 = A\begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix}$. In other words, one solution to $A\mathbf{x} = \mathbf{b}_2$ is $\mathbf{x}_0 = \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix}$. (There are many other solutions, as we will see in part (c).) This is readily verified to really be a solution:

$$A\mathbf{x}_0 = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 13 & 17 & -10 \\ -1 & -3 & -3 & 2 \end{bmatrix} \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -9+4 \\ -27+26 \\ 9-6 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix} = \mathbf{b}_2.$$

Likewise, work we did to verify that $A\mathbf{x} = \mathbf{b}_3$ has a solution yields an expression for $\mathbf{b}_3$ in terms of the first and second columns of A:

$$\mathbf{b}_3 = -\mathbf{a}_1 - \mathbf{a}_2' = -\mathbf{a}_1 - (\mathbf{a}_2 - 4\mathbf{a}_1) = 3\mathbf{a}_1 - \mathbf{a}_2,$$

so this says $\mathbf{b}_3 = A\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. In other words, one solution to $A\mathbf{x} = \mathbf{b}_3$ is $\mathbf{x}_0' = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. (There are many other solutions, as we will see in part (c).) This is readily verified to really be a solution:

$$A\mathbf{x}_0' = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 13 & 17 & -10 \\ -1 & -3 & -3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 9-13 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} = \mathbf{b}_3.$$

(c) The solutions to $A\mathbf{x} = \mathbf{b}_2$ are the vectors of the form $\mathbf{x}_0 + \mathbf{v}$ for $\mathbf{x}_0 = \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} \in N(A)$, where we know from (a) that $N(A)$ has as a basis $\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$. Hence, solutions are given by the parametric form

$$\begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t'\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -9-3t+t' \\ 2+2t-t' \\ -t \\ -t' \end{bmatrix}$$

for $t, t' \in \mathbf{R}$.

The solutions to $A\mathbf{x} = \mathbf{b}_3$ are the vectors of the form $\mathbf{x}_0' + \mathbf{v}$ for $\mathbf{x}_0' = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} \in N(A)$, where we know from (a) that $N(A)$ has as a basis $\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$. Hence, solutions are given by the parametric form

$$\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t'\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3-3t+t' \\ -1+2t-t' \\ -t \\ -t' \end{bmatrix}$$

for $t, t' \in \mathbf{R}$.

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11. (a) Since $C(A)$ is a linear subspace of $\mathbf{R}^2$, its dimension is at most 2, hence is either 0, 1, or 2. The same applies to $N(A)$. But $C(A)$ is nonzero since some column is nonzero, so its dimension is 1 or 2; this says that $C(A)$ is either a line through the origin (dimension 1) or the entirety of $\mathbf{R}^2$ (the only n-dimensional linear subspace of $\mathbf{R}^n$ is itself). Likewise, $N(A)$ can't exhaust $\mathbf{R}^2$ since the columns are $A\mathbf{e}_1$ and $A\mathbf{e}_2$, at least one of which is nonzero by assumption. This rules out the case $\dim N(A) = 2$, so $\dim N(A) \leq 1$. The 1-dimensional case is when it is a line through the origin, and the 0-dimensional case is when it consists of just the origin.

(b) If some column vanishes then the other column is nonzero and $C(A)$ is the line spanned by that nonzero column. Likewise, $N(A)$ is the line corresponding to the coordinate axis for the vanishing column. So the case when some column vanishes checks out fine, and now we may and do suppose both columns are nonzero. In particular, the columns are linearly dependent precisely when each column is a scalar multiple of the other.

Since A has a nonzero column, the column space is a line exactly when such a column spans the column space (any nonzero vector in a line through the origin spans that line). In such cases the other column (which could be $\mathbf{0}$) also belongs to the same line and so is a scalar multiple of the initial choice of nonzero column. In other words, if $C(A)$ is a line then some column is a scalar multiple of the other, and so the columns are linearly dependent. On the other hand, when the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent then we have $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{0}$ with some $c_i \neq 0$, so dividing by the nonzero coefficient expresses one column as a scalar multiple of the other. This forces the latter column to span the entire $C(A)$, but $C(A)$ is nonzero and so (being spanned by one vector) it must be a line.

Next, we turn our attention to $N(A)$. This is a line exactly when it is nonzero (since its dimension is 0 or 1), and to say a nonzero vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ lies in $N(A)$ is to say $A\mathbf{x} = \mathbf{0}$. But

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$$

where $\mathbf{a}_i$ is the ith column of A, so the vanishing of $A\mathbf{x}$ for a nonzero $\mathbf{x}$ is exactly the vanishing of a linear combination $x_1\mathbf{a}_1 + x_2\mathbf{a}_2$ with $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \mathbf{0}$, which is to say $x_1 \neq 0$ or $x_2 \neq 0$. But this in turn expresses precisely linear dependence of the columns (i.e., the vanishing of some linear combination using at least one nonzero coefficient).

(c) First suppose $C(A) = \mathbf{R}^2$, so $C(A)$ is not a line; i.e., $\dim C(A) \neq 1$. By (b) it follows that $\dim N(A) \neq 1$. By (a), $N(A)$ is either a line through the origin or just the origin itself, so the only option we have is that $N(A)$ is equal to the origin. Next, suppose $N(A)$ is equal to the origin, so $\dim N(A) \neq 1$. By (b), it follows that $\dim C(A) \neq 1$. But by (a) the only possibilities for $\dim C(A)$ are 1 and 2, so $\dim C(A) = 2$. The only 2-dimensional subspace of $\mathbf{R}^2$ is itself, so $C(A) = \mathbf{R}^2$ as desired.

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12. (a) Because both systems are overdetermined, they have more equations than variables. Thus, the combined system also does (i.e. it is overdetermined). So we expect it not to have any solutions.

(b) We cannot necessarily say that the combined system is overdetermined or underdetermined (it depends on *how* over/underdetermined the two original systems are). But we can expect the first system not to have any solutions; therefore, also expect the combined system not to have any solutions.

(c) Both systems have the same number of equations as variables. Thus, the combined system has twice as many equations as variables. That is, it is overdetermined. We therefore expect it not to have any solutions.

(d) We've combined the two systems into one that has twice as many variables as each of the original linear systems and the same number of equations as each of the original linear systems. Since the two original linear systems are underdetermined, so is the combined system. So we expect it to have infinitely many solutions.

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