Solutions to Math 51 Quiz 6

1. (10 points) Let $\mathbf{f}(x,y) = (x^2y^3 - xy + x + y + 1, x^3y^2 + xy^5 + 2y + 1)$, and let $\mathbf{g} = \mathbf{f} \circ \mathbf{f}$.

   (a) (4 points) Compute $D\mathbf{g}(0,0)$.

   (b) (2 points) Let $\mathbf{a} : \mathbb{R} \to \mathbb{R}^2$ and suppose that $\mathbf{a}(0) = (0,0)$ and $\mathbf{a}'(0) = (1,-1)$. Let $\mathbf{b} = \mathbf{g} \circ \mathbf{a}$. What is $\mathbf{b}'(0)$?

   (c) (4 points) Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$, and suppose that $\mathbf{r}(0) = (0,0)$. Let $\mathbf{s} = \mathbf{g} \circ \mathbf{r}$, and suppose that $\mathbf{s}'(0) = \begin{bmatrix} 10 \\ 32 \end{bmatrix}$. What is $\mathbf{r}'(0)$?

(a) Note that $D\mathbf{f}(x,y) = \begin{bmatrix} 2xy^3 - y + 1 & 3x^2y^2 - x + 1 \\ 3x^2y^2 + y^5 & 2x^3y + 5xy^4 + 2 \end{bmatrix}$. Hence, by the Chain Rule,

$$D\mathbf{g}(0,0) = D\mathbf{f}(\mathbf{f}(0,0))D\mathbf{f}(0,0) = D\mathbf{f}(1,1)D\mathbf{f}(0,0) = \begin{bmatrix} 2 & 3 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 4 & 22 \end{bmatrix}.$$

(b) Again, by the Chain Rule,

$$\mathbf{b}'(0) = D\mathbf{b}(0) = D\mathbf{g}(\mathbf{a}(0))D\mathbf{a}(0) = D\mathbf{g}(0,0)\mathbf{a}'(0) = \begin{bmatrix} 2 & 8 \\ 4 & 22 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -18 \end{bmatrix}.$$

(c) We know by the Chain Rule,

$$\mathbf{s}'(0) = D\mathbf{s}(0) = D\mathbf{g}(\mathbf{r}(0))D\mathbf{r}(0) = D\mathbf{g}(0,0)\mathbf{r}'(0) = \begin{bmatrix} 2 & 8 \\ 4 & 22 \end{bmatrix} \mathbf{r}'(0).$$

The inverse of $D\mathbf{g}(0,0)$ is, since the determinant is 12,

$$(D\mathbf{g}(0,0))^{-1} = \frac{1}{12} \begin{bmatrix} 22 & -8 \\ -4 & 2 \end{bmatrix}.$$

Thus,

$$\mathbf{r}'(0) = (D\mathbf{g}(0,0))^{-1}\mathbf{s}'(0) = \frac{1}{12} \begin{bmatrix} 22 & -8 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 32 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

2. (2 points) **True or False:** Suppose that A, B, and C are $n \times n$ matrices and that A is invertible. Furthermore, suppose that $AB = CA$. Then, $B = C$.

This statement is **FALSE**. See Example 18.4.1 in the textbook for a counterexample. More generally, all we can deduce from $AB = CA$, with A being invertible, is that $B = A^{-1}CA$ and $C = ABA^{-1}$. The fact that the expression on the right-hand sides don't automatically cancel to C and B, respectively, is crucial for some more advanced results about matrices to hold (e.g. "diagonalization.")

3. (2 points) **True or False:** Let A be a 2×2 Markov matrix. Then, for large values of n,

$$A^n \approx \begin{bmatrix} \alpha & \alpha \\ 1-\alpha & 1-\alpha \end{bmatrix}$$

for some scalar $0 \leq \alpha \leq 1$. Note that that matrix above is a Markov matrix as well.

This statement is **FALSE**. We saw a counterexample in Lecture 16, at the top of page 2. Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is a Markov matrix. However, the odd powers are $A^{2k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and the even powers are $A^{2k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The higher powers of A do not converge, no matter how large the powers get.

4. (3 points) A Ghanaian zoologist observes the migratory pattern of caracals that stay within two savannahs A and B. He observes the following migratory patterns:

Every winter,

- 10% of the caracals in savannah A migrate to savannah B; and
- 25% of the caracals in savannah B migrate to savannah A.

Every summer,

- 20% of the caracals in savannah A migrate to savannah B; and
- 15% of the caracals in savannah B migrate to savannah A.

We are assuming that no new caracals are born and no caracal dies. Let $\mathbf{c} = \begin{bmatrix} c_A \\ c_B \end{bmatrix}$ be the caracal vector, where c_A is the number of caracals in savannah A in fall 2051, and c_B is the number of caracals in savannah B in fall 2051. Suppose that

$$A_1 = \begin{bmatrix} 0.10 & 0.75 \\ 0.90 & 0.25 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.90 & 0.25 \\ 0.10 & 0.75 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0.75 & 0.10 \\ 0.25 & 0.90 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0.25 & 0.90 \\ 0.75 & 0.10 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.15 & 0.80 \\ 0.85 & 0.20 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.85 & 0.20 \\ 0.15 & 0.80 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0.80 & 0.15 \\ 0.20 & 0.85 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0.20 & 0.85 \\ 0.80 & 0.15 \end{bmatrix}$$

Use the matrices above and $\mathbf{c}$ to describe the vector with the number of caracals in each savannah in spring 2054.

Upon inspection, the Markov matrices for the winter migration and the summer migration are A_2 and B_3, respectively. From fall 2051 to spring 2054, it goes through a winter, a summer, a winter, a summer, then a winter. Hence, the quantity $A_2 B_3 A_2 B_3 A_2 \mathbf{c}$ represents the vector with the numbers of caracals in each savannah in spring 2054.

5. (3 points) Let $\mathbf{f}(x, y) = (x^2 + (x-1)e^{y-2}, y^3 + xy^2)$. What is the result after applying one iteration of Newton's method with respect to $\mathbf{f}$, starting at $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

We see that $D\mathbf{f}(x, y) = \begin{bmatrix} 2x + e^{y-2} & (x-1)e^{y-2} \\ y^2 & 3y^2 + 2xy \end{bmatrix}$, and so, $D\mathbf{f}(1, 2) = \begin{bmatrix} 3 & 0 \\ 4 & 16 \end{bmatrix}$. Since $(D\mathbf{f}(1, 2))^{-1} = \frac{1}{48}\begin{bmatrix} 16 & 0 \\ -4 & 3 \end{bmatrix}$, it follows that the result after one iteration of Newton's method is (since $\mathbf{f}(1, 2) = (1, 12)$)

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{48}\begin{bmatrix} 16 & 0 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \end{bmatrix}.$$