Problem 1: Large powers of symmetric matrices (a 3×3 example)

Consider the matrix

$$M = \begin{bmatrix} 3/5 & 1/5 & 1/5 \\ 1/5 & 3/5 & 1/5 \\ 1/5 & 1/5 & 3/5 \end{bmatrix}$$

Since M is symmetric, the Spectral Theorem implies that there is an orthogonal basis for $\mathbf{R}^3$ consisting of eigenvectors for M. For this problem, assume that we are also given that the eigenvalues of M are $\lambda_1 = 1$ and $\lambda_2 = \dfrac{2}{5}$.

(a) Let V_1 be the λ_1-eigenspace. Verify that $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ lies in V_1, and explain why it spans that space.

(b) Find the λ_2-eigenspace V_2, and write it as the span of two orthogonal vectors $\mathbf{w}_2, \mathbf{w}_3$.

(c) Let $\mathbf{w}'_1, \mathbf{w}'_2, \mathbf{w}'_3$ be unit vectors obtained from $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. Let W be the matrix whose columns are $\mathbf{w}'_1, \mathbf{w}'_2, \mathbf{w}'_3$. Find the diagonal matrix D where $M = WDW^\top$.

(d) Using the fact that $(2/5)^{100} \approx 0$ to over thirty-five decimal places, calculate M^{100} explicitly.

Solution:

(a) We compute $M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a 1-eigenvector. The 1-eigenspace is the null space of $M - I_3 = \begin{bmatrix} -2/5 & 1/5 & 1/5 \\ 1/5 & -2/5 & 1/5 \\ 1/5 & 1/5 & -2/5 \end{bmatrix}$. By the Rank-Nullity Theorem, this is 1-dimensional if the rank is 2 (i.e., 2-dimensional column space). The first two columns of $M - \mathrm{I}_3$ are independent by inspection, and the sum of the columns is $\mathbf{0}$ (since $\mathbf{w}_1$ is in the null space), so the 3rd column is in the span of the other two.

Alternative solution: The λ_1-eigenspace of M is the null space of $M - I_3 = \begin{bmatrix} -2/5 & 1/5 & 1/5 \\ 1/5 & -2/5 & 1/5 \\ 1/5 & 1/5 & -2/5 \end{bmatrix}$, which is the collection of all vectors $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for which:

$$\begin{cases} -\frac{2}{5}x + \frac{1}{5}y + \frac{1}{5}z = 0 \\ \frac{1}{5}x - \frac{2}{5}y + \frac{1}{5}z = 0 \\ \frac{1}{5}x + \frac{1}{5}y - \frac{2}{5}z = 0 \end{cases} .$$

Subtracting the second line from the first yields $\dfrac{3}{5}x = \dfrac{3}{5}y$, that is, $x = y$. Substituting in any of the remaining equations gives $z = x = y$. So $\mathbf{x} = \begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $V_1 = \operatorname{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) We solve the system $(M - \lambda_2 I_3)\mathbf{x} = \mathbf{0}$, that is

$$\begin{cases} \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z = 0 \\ \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z = 0 \\ \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z = 0 \end{cases} ,$$

which is the same equation three times. We have $x = -(y+z)$, that is $\mathbf{x} = \begin{bmatrix} -(y+z) \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Let $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, then we get an orthogonal vector $\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_2} \mathbf{v}_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

(c) The matrix interpretation of the Spectral Theorem is that $M = WDW^\top$ where W has i-th column $\mathbf{w}'_i$ and D is diagonal with i-th entry the eigenvalue for $\mathbf{w}'_i$. Hence, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 2/5 \end{bmatrix}$.

(d) We have seen that $M^k = WD^kW^{-1} = WD^kW^\top$ for any integer $k > 0$. So for $k = 100$, we have $M^{100} = WD^{100}W^\top$. But D is diagonal, so D^{100} is obtained by raising diagonal entries to the 100-th power. Since $(2/5)^{100} \approx 0$, we have $D^{100} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore,

$$
\begin{aligned}
M^{100} &\approx \begin{bmatrix} | & | & | \\ \mathbf{w}'_1 & \mathbf{w}'_2 & \mathbf{w}'_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} - & \mathbf{w}'^\top_1 & - \\ - & \mathbf{w}'^\top_2 & - \\ - & \mathbf{w}'^\top_3 & - \end{bmatrix} \\
&= \begin{bmatrix} | & | & | \\ \mathbf{w}'_1 & \mathbf{0} & \mathbf{0} \\ | & | & | \end{bmatrix} \begin{bmatrix} - & \mathbf{w}'^\top_1 & - \\ - & \mathbf{w}'^\top_2 & - \\ - & \mathbf{w}'^\top_3 & - \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ - & \mathbf{w}'^\top_2 & - \\ - & \mathbf{w}'^\top_3 & - \end{bmatrix} \\
&= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.
\end{aligned}
$$

Problem 2: Large powers of symmetric matrices (a 2×2 example)

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, a symmetric matrix.

(a) Compute the eigenvalues $\lambda_1 > \lambda_2$ of A and find eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ for λ_1, λ_2, respectively. Check that $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal.

(b) Write $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combinations of the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(c) Use your expressions from part (b) to give an exact expression for A^{100}. (Hint: note that the first column of A^{100} is equal to $A^{100}\mathbf{e}_1$, and similarly for the second column. Use (b) to compute $A^{100}\mathbf{e}_i$.)

(d) Using the (very accurate!) approximation $(\lambda_2/\lambda_1)^{100} \approx 0$, give a much simpler approximate expression for A^{100}.

Solution:

(a) The characteristic polynomial of A is $P_A(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$. So the eigenvalues of A are $1, 3$. The eigenvectors for $\lambda_1 = 3$ are those $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ for which $A\mathbf{x} - 3\mathbf{x} = 0$. That is, $\begin{bmatrix} 2x - y \\ -x + 2y \end{bmatrix} - \begin{bmatrix} 3x \\ 3y \end{bmatrix} = \mathbf{0}$. This amounts to the equation $-x - y = 0$, i.e. $x = -y$. So an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda_2 = 1$, the same setup yields the equations $\begin{bmatrix} x - y \\ -x + y \end{bmatrix} = \mathbf{0}$, which is to say $x = y$, so $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector. We directly compute $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, as claimed.

(b) We can either see "by inspection" that $\mathbf{e}_1 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$ and $\mathbf{e}_2 = \frac{1}{2}(-\mathbf{v}_1 + \mathbf{v}_2)$, or set up the problems $\mathbf{e}_1 = a\mathbf{v}_1 + b\mathbf{v}_2$ and $\mathbf{e}_2 = a'\mathbf{v}_1 + b'\mathbf{v}_2$ are systems of 2 equations in 2 unknowns that one can solve via high school algebra.

(c) We apply the linearity of A^{100} to compute $A^{100}\mathbf{e}_1$ and $A^{100}\mathbf{e}_2$:

$$A^{100}\mathbf{e}_1 = \frac{1}{2}(A^{100}\mathbf{v}_1 + A^{100}\mathbf{v}_2) = \frac{1}{2}(3^{100}\mathbf{v}_1 + 1^{100}\mathbf{v}_2) = \frac{1}{2}\left(\begin{bmatrix} 3^{100} + 1 \\ -3^{100} + 1 \end{bmatrix}\right)$$

and

$$A^{100}\mathbf{e}_2 = \frac{1}{2}(-A^{100}\mathbf{v}_1 + A^{100}\mathbf{v}_2) = \frac{1}{2}(-3^{100}\mathbf{v}_1 + \mathbf{v}_2) = \frac{1}{2}\left(\begin{bmatrix} -3^{100} + 1 \\ 3^{100} + 1 \end{bmatrix}\right).$$

Therefore,

$$A^{100} = \frac{1}{2}\begin{bmatrix} 1 + 3^{100} & 1 - 3^{100} \\ 1 - 3^{100} & 1 + 3^{100} \end{bmatrix}.$$

(d) If we divide the expression from (c) through by 3^{100}, we obtain

$$\frac{A^{100}}{3^{100}} = \frac{1}{2}\begin{bmatrix} 3^{-100} + 1 & 3^{-100} - 1 \\ 3^{-100} - 1 & 3^{-100} + 1 \end{bmatrix}.$$

But $3^{-100} \approx 0$, so this gives us the approximate equality

$$A^{100} \approx \frac{3^{100}}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Problem 3: Calculating multiple derivatives

Consider the function

$$f(x, y) = e^{x \sin y}.$$

(a) Find all first and second partial derivatives of f.

(b) Find the gradient vector and Hessian matrix of f at $(1, 0)$.

(c) Find the quadratic approximation to $f(1 + h, k)$ for h and k near 0.

Solution:

(a) We calculate

$$\frac{\partial f}{\partial x}(x, y) = (\sin y)e^{x \sin y}$$

and

$$\frac{\partial f}{\partial y}(x, y) = x(\cos y)e^{x \sin y}.$$

From these we compute

$$\frac{\partial^2 f}{\partial x^2}(x, y) = (\sin^2 y)e^{x \sin y}$$

and

$$\frac{\partial^2 f}{\partial y^2}(x, y) = x^2(\cos^2 y)e^{x \sin y} - x(\sin y)e^{x \sin y}$$

and

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y) = (\cos y)e^{x\sin y} + x(\cos y)(\sin y)e^{x\sin y}.$$

(b) From this we get $(\nabla f)(1,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and

$$(\mathrm{H}f)(1,0) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(c) The quadratic approximation is

$$f(1+h,k) \approx f(1,0) + (\nabla f)(1,0)\cdot \begin{bmatrix} h \\ k \end{bmatrix} + \frac{1}{2}\begin{bmatrix} h & k \end{bmatrix}((\mathrm{H}f)(1,0))\begin{bmatrix} h \\ k \end{bmatrix} = 1 + k + \frac{1}{2}(2hk + k^2) = 1 + k + hk + \frac{1}{2}k^2.$$

Problem 4: Level sets of quadratic forms

Consider the quadratic form $Q(x,y) = x^2 + 6xy + y^2$.

(a) Find the symmetric 2×2 matrix A for which $\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = Q(x,y)$.

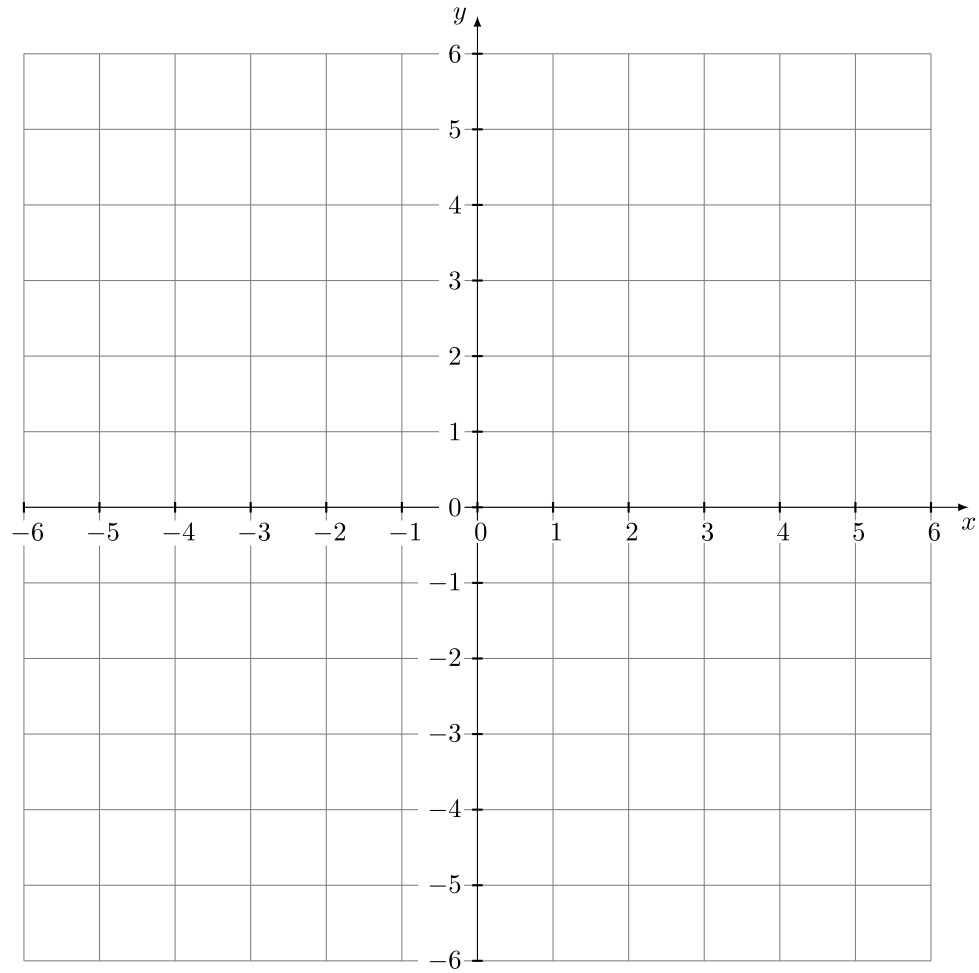
(b) Find the eigenvalues λ_1 and λ_2 of A, and find unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for these respective eigenvalues.

(c) We can use the eigenvalues to express Q when its input is written in a basis of *unit* eigenvectors of A :

$$Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = \lambda_1 x'^2 + \lambda_2 y'^2.$$

Use this to sketch the level curves $Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = \pm 8$ as well as $Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = 0$ in an $x'y'$-coordinate plane, indicating where each crosses a coordinate axis.

(d) Explain why rotating the "standard basis" onto the orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ carries $x'\mathbf{e}_1 + y'\mathbf{e}_2$ onto $x'\mathbf{v}_1 + y'\mathbf{v}_2$, and carries the x', y' coordinate axes onto the "eigenlines" of A. Sketch the resulting rotation of the picture in (c); why is it the level sets $Q(x,y) = -8, 0, 8$? (This gives a *general technique* to draw level sets of any $Ax^2 + Bxy + Cy^2$.)



Solution:

(a) The matrix corresponding to this quadratic form is $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(b) This has eigenvalue $\lambda_1 = 4$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and eigenvalue $\lambda_2 = -2$ with eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The corresponding unit eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

(c) This means in the basis of unit eigenvectors, the quadratic form becomes $4x'^2 - 2y'^2$. Hence, $Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = \pm 8$ is the condition $4x'^2 - 2y'^2 = \pm 8$, which is the same as $2x'^2 - y'^2 = \pm 4$. This is a pair of hyperbolas, meeting the x'-axis ($y' = 0$) at $\pm\sqrt{2} \approx \pm 1.414$ and the y'-axis ($x' = 0$) at ± 2. The level set $4x'^2 - 2y'^2 = 0$ is the pair of lines $y' = \pm\sqrt{2}\,x'$ that are the green asymptotes to these hyperbolas in Figure **??** (with hyperbolas in red and blue).

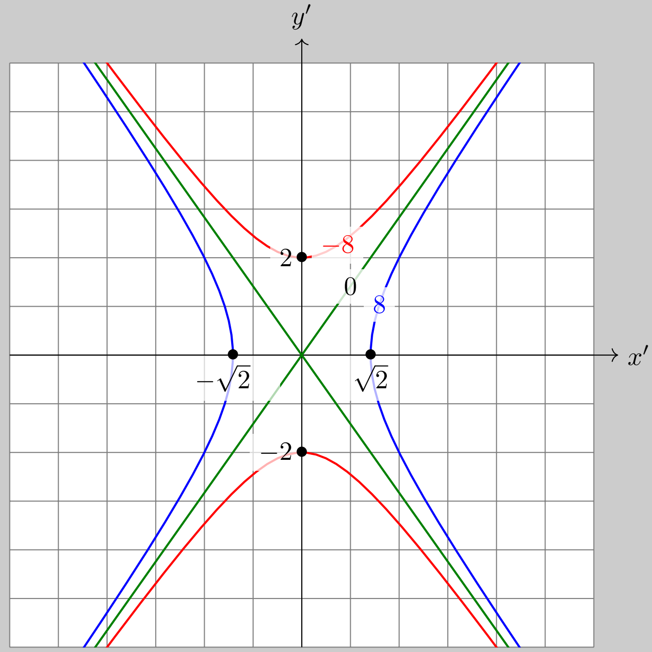


Figure 1: The level set $Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = -8$ in red crosses the y'-axis at ± 2, the level set $Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = 8$ in blue crosses the x'-axis at $\pm\sqrt{2} \approx \pm 1.414$, and their asymptotes are the green level set $Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = 0$.

(d) The rotation R of $\{\mathbf{e}_1, \mathbf{e}_2\}$ onto $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a $45°$ counterclockwise rotation. Applying this to Figure **??** yields Figure **??**. We have to explain why this yields the level sets $Q(x, y) = -8, 0, 8$ (e.g., the blue hyperbola below is $Q(x, y) = 8$).

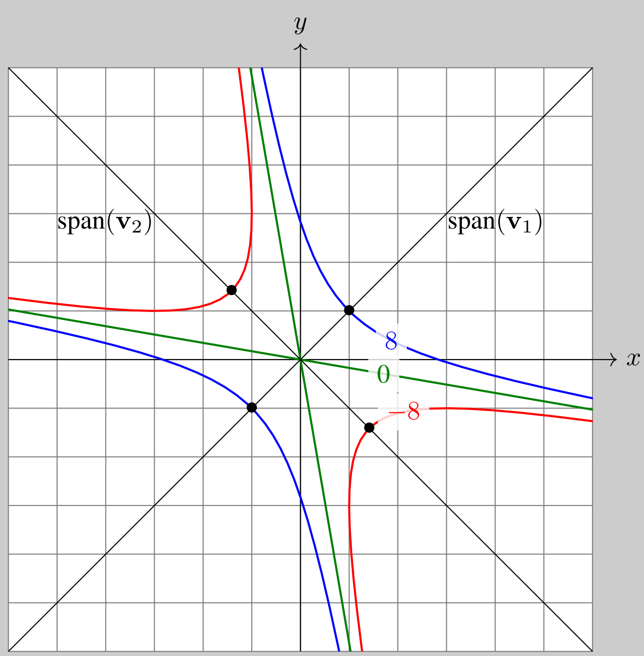


Figure 2: The level sets $Q(x, y) = -8$ (red), $Q(x, y) = 8$ (blue), $Q(x, y) = 0$ (green) and "eigenlines" $y = \pm x$ crossing the hyperbolas at the black dots. This is a rotation of Figure **??** to put its coordinate axes on the "eigenlines".

The rotation R is *linear*, so it carries $x'\mathbf{e}_1 + y'\mathbf{e}_2$ to $x'\mathbf{v}_1 + y'\mathbf{v}_2$. Setting $y' = 0$ and letting x' vary, this says that the x'-axis is carried onto the line through $\mathbf{v}_1$, and by setting $x' = 0$ and letting y' vary this says that the y'-axis is carried onto the line through $\mathbf{v}_2$ (this behavior on the lines is also seen visually, without needing to do such algebra).

To figure out when a point P lies in the level set $Q = c$ (which is what we want to do to make the sketches for $c = -8, 0, 8$), first write $P = x'\mathbf{v}_1 + y'\mathbf{v}_2$ as in the hint, so the picture in (c) shows for which (x', y') we have $Q(P) = c$. But $P = R(x\mathbf{e}_1 + y'\mathbf{e}_2)$, so the blue hyperbola in (c) consists of those (x', y') for which applying R takes $x'\mathbf{e}_1 + y'\mathbf{e}_2$ into the level set $Q = 8$, and similarly with the red hyperbola for $Q = -8$ and the green asymptotes for $Q = 0$.

This is saying that R applied to the blue hyperbola in (c) is the level set $Q = 8$, and similarly for the red hyperbola with $Q = -8$ and the green asymptotes with $Q = 0$. In other words, this says that R applied to the picture in (c) yields the desired level sets for Q, and this effect of R on the picture in (c) is exactly the rotation we have drawn in Figure **??**.