Solutions to Math 51 Quiz 3 Practice B

1. (10 points) Consider the function

$$f(x,y) = \ln\left(\frac{y}{x}\right).$$

Compute

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

and

$$x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2}.$$

$$\begin{aligned}
f_x &= \frac{x}{y}\left(-\frac{y}{x^2}\right) \\
&= -\frac{1}{x} \\
f_y &= \frac{x}{y}\left(\frac{1}{x}\right) \\
&= \frac{1}{y} \\
f_{xx} &= \frac{1}{x^2} \\
f_{yy} &= -\frac{1}{y^2} \\
x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= -\frac{x}{x} + \frac{y}{y} \\
&= 0 \\
x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} &= \frac{x^2}{x^2} + \frac{y^2}{y^2} \\
&= 2
\end{aligned}$$

2. (2 points) Below is a contour plot of a function $g(x,y)$ over the region of points (x,y) with $-3 \le x \le 1$ and $-2 \le y \le 2$, with the dashed line $y = 0$ drawn over it.

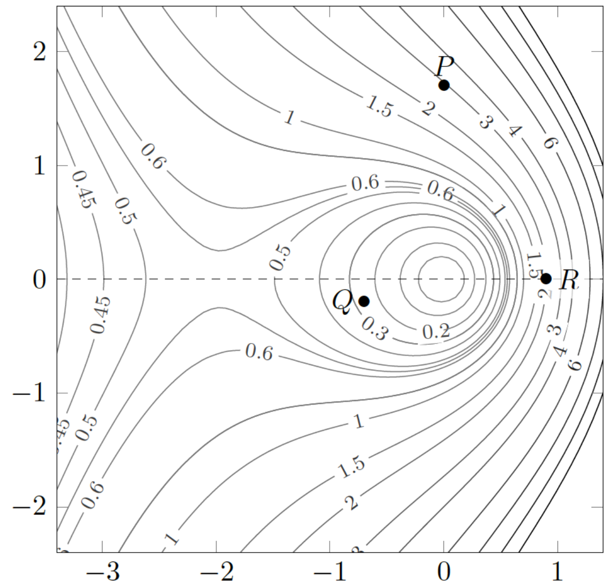


Figure 1: A contour plot for a function $g(x, y)$

For the points labeled P, Q, R, determine the if each of $g_x(P), g_y(Q), g_x(R)$, and $g_y(R)$ is positive, negative, or 0.

As we move through P horizontally from left to right, the numerical labels on the contour lines are increasing, so $g_x(P) > 0$.

As we move down through Q vertically the numerical labels on the contour lines are increasing, so $g_y(Q) < 0$.

As we move through R horizontally from left to right, the numerical labels on the contour lines are increasing, so $g_x(R) > 0$.

Finally, as we move up through R vertically the line of motion is tangent to the level curve through $R = (a, 0)$, and this indicates that $g_y(R) = 0$; more visually, just below and just above R the function values go down to $g(R) = 2$ and then back up again, so $g(a, y)$ has a local minimum at $y = 0$ and hence its derivative vanishes at $y = 0$. But this derivative is $g_y(a, 0) = g_y(R)$, so $g_y(R) = 0$.

3. (2 points) Which compositions of the function

$$f(x, y) = (x^2 + y^2, x^2 - y^2) \text{ and } g(x, y, z) = (xy, xz)$$

are possible?

a) $f \circ g$ is defined, but $g \circ f$ is not.

b) $g \circ f$ is defined, but $f \circ g$ is not.

c) Both $f \circ g$ and $g \circ f$ are defined.

d) Neither $f \circ g$ nor $g \circ f$ is defined.

In function notation, we have $f : \mathbf{R}^2 \to \mathbf{R}^2$ and $g : \mathbf{R}^3 \to \mathbf{R}^2$. As such, the output of f is a 2-vector, which cannot be an input to g. Therefore, $g \circ f$ is not defined.

On the other hand, since inputs to f and outputs of g are 2-vectors, the function $f \circ g : \mathbf{R}^3 \to \mathbf{R}^2$ is defined:

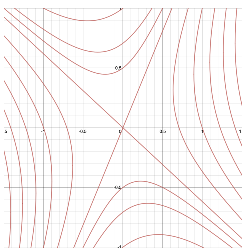
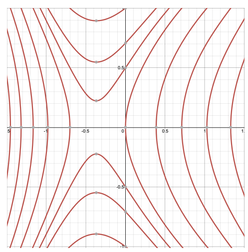
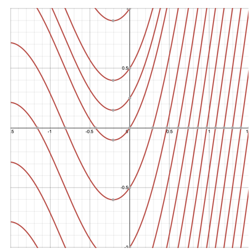
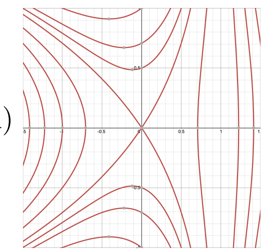
$$\begin{aligned}(f \circ g)(x, y, z) &= f(g(x, y, z)) \\ &= f(xy, xz) \\ &= (x^2(y^2 + z^2), x^2(y^2 - z^2)).\end{aligned}$$

4. (3 points) Consider a function $f(x, y)$ satisfying

$$\left|\frac{\partial f}{\partial x}(a, b)\right| \neq \left|\frac{\partial f}{\partial x}(-a, -b)\right|$$

for all $(a, b) \in \mathbf{R}^2$. Which contour plot is most likely to correspond to $f(x, y)$?

Note that the contour plots below all have uniform increments in f-values: the gaps between f-values for successive level curves are the same.

a)  b)  c)  d) 

The answer is (c). The function described by graph (c) appears to satisfy $|f_x(a, b)| = |f_x(-a, -b)|$ for all $(a, b) \in \mathbb{R}^2$. The functions described by graphs (b) and (d) appear to satisfy $|f_x(0, a)| = |f_x(0, -a)|$. Note that (a) is the only plot where

$$\left|\frac{\partial f}{\partial x}(a, b)\right| = \left|\frac{\partial f}{\partial x}(-a, -b)\right|$$

since the functions described by graphs (b) and (d) have, for example, $|f_x(-1, 0)| > |f_x(1, 0)|$ while the function described by the graph (C) has $|f_x(1, 0)| > f_x(-1, 0)|$.

5. (3 points) The line of best fit for a collection of data points $(x_1, y_1), \ldots, (x_{100}, y_{100})$ is

$$y = -4x + 30.$$

Suppose the x-coordinates and the y-coordinates have the same mean, i.e. $\bar{x} = \bar{y}$. What is $\bar{x}$?

a) 0 b) 6 c) 7.5 d) 30 e) -10

Let $\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_{100} \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_{100} \end{bmatrix}$. To compute the line of best fit, we first compute an orthogonal basis for span$(\mathbf{X}, \mathbf{1})$, by defining

$$\begin{aligned}\widehat{\mathbf{X}} &= \mathbf{X} - \mathrm{Proj}_{\mathbf{1}}(\mathbf{X}) \\ &= \mathbf{X} - \left(\frac{x_1 + \cdots + x_{100}}{100}\right)\mathbf{1} \\ &= \mathbf{X} - \bar{x}\mathbf{1}.\end{aligned}$$

The projection of $\mathbf{Y}$ onto $\mathrm{span}(\mathbf{X}, \mathbf{1})$ is

$$\begin{aligned}
\mathrm{Proj}_{\mathrm{span}(\mathbf{X},\mathbf{1})}\mathbf{Y} &= \mathrm{Proj}_{\widehat{\mathbf{X}}}(\mathbf{Y}) + \mathrm{Proj}_{\mathbf{1}}(\mathbf{Y}) \\
&= \left(\frac{\mathbf{Y}\cdot\widehat{\mathbf{X}}}{\widehat{\mathbf{X}}\cdot\widehat{\mathbf{X}}}\right)\widehat{\mathbf{X}} + \bar{y}\mathbf{1} \\
&= \left(\frac{\mathbf{Y}\cdot\widehat{\mathbf{X}}}{\widehat{\mathbf{X}}\cdot\widehat{\mathbf{X}}}\right)(\mathbf{X} - \bar{x}\mathbf{1}) + \bar{y}\mathbf{1} \\
&= \left(\frac{\mathbf{Y}\cdot\widehat{\mathbf{X}}}{\widehat{\mathbf{X}}\cdot\widehat{\mathbf{X}}}\right)\mathbf{X} + \left(\bar{x} - \left(\frac{\mathbf{Y}\cdot\widehat{\mathbf{X}}}{\widehat{\mathbf{X}}\cdot\widehat{\mathbf{X}}}\right)\bar{x}\right)\mathbf{1}.
\end{aligned}$$

The coefficients of $\mathbf{X}$ and of $\mathbf{1}$ are the coefficients of the line of best fit. Hence we must have $\left(\frac{\mathbf{Y}\cdot\widehat{\mathbf{X}}}{\widehat{\mathbf{X}}\cdot\widehat{\mathbf{X}}}\right) = -4$, and so $\bar{x} + 4\bar{x} = 30$. This implies $\bar{x} = 6$.
Alternatively, note that in the line of best fit, we have

$$m = \frac{\mathbf{Y}\cdot\widehat{\mathbf{X}}}{\widehat{\mathbf{X}}\cdot\widehat{\mathbf{X}}}, \quad b = \bar{y} - m\bar{x}$$

$$y = mx + b = mx + (\bar{y} - m\bar{x}).$$

If $\bar{y} = \bar{x}$, $m = -4$ and $b = 30$, then

$$b = \bar{x} + 4\bar{x} = 5\bar{x} = 30, \quad \bar{x} = 6.$$