Problem 1: Matrix of a projection

Let V be the plane $x+y+z=0$ in $\mathbf{R}^3$ through the origin, so V has an orthogonal basis $\{\mathbf{v},\mathbf{w}\}$ for $\mathbf{v}=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix}1\\1\\-2\end{bmatrix}$.
Let $L:\mathbf{R}^3\to\mathbf{R}^3$ be the function $L(\mathbf{x})=\mathbf{Proj}_V(\mathbf{x})$.

(a) Compute the 3×3 matrix A for L; the entries should be fractions with denominator 3. (Hint: what is the meaning of each column?)

(b) For $\mathbf{a}=\begin{bmatrix}1\\3\\5\end{bmatrix}$, compute $\mathbf{Proj}_V(\mathbf{a})$ in two ways: using the orthogonal basis $\{\mathbf{v},\mathbf{w}\}$ for V, and using the matrix-vector product against your answer in (a). (You should get the same answer both ways, a vector with integer entries.)

(c) The geometric definition of $\mathbf{Proj}_V$ gives that its output lies in V, on which $\mathbf{Proj}_V$ has no effect, so $\mathbf{Proj}_V\circ\mathbf{Proj}_V=\mathbf{Proj}_V$. Check that your answer A in (a) satisfies the corresponding matrix equality $A^2=A$. (Hint: if you write $A=(1/3)B$ for a matrix B with integer entries then the calculation will be cleaner.)

Solution:

(a) The jth column is $L(\mathbf{e}_j)$, and in general for any $\mathbf{x}\in\mathbf{R}^3$ we have

$$\mathbf{Proj}_V(\mathbf{x})=\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}+\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}.$$

Setting $\mathbf{x}=\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ respectively, we obtain

$$\mathbf{Proj}_V(\mathbf{e}_1)=\frac{1}{2}\mathbf{v}+\frac{1}{6}\mathbf{w}=\begin{bmatrix}1/2+1/6\\-1/2+1/6\\-2/6\end{bmatrix}=\begin{bmatrix}2/3\\-1/3\\-1/3\end{bmatrix},$$

$$\mathbf{Proj}_V(\mathbf{e}_2)=\frac{-1}{2}\mathbf{v}+\frac{1}{6}\mathbf{w}=\begin{bmatrix}-1/2+1/6\\1/2+1/6\\-2/6\end{bmatrix}=\begin{bmatrix}-1/3\\2/3\\-1/3\end{bmatrix},$$

$$\mathbf{Proj}_V(\mathbf{e}_3)=0\mathbf{v}+\frac{-2}{6}\mathbf{w}=\begin{bmatrix}-1/3\\-1/3\\2/3\end{bmatrix}.$$

Assembling these together,

$$A=\begin{bmatrix}2/3&-1/3&-1/3\\-1/3&2/3&-1/3\\-1/3&-1/3&2/3\end{bmatrix}.$$

(b) The general formula gives the projection as

$$\frac{\mathbf{a}\cdot\mathbf{v}}{2}\mathbf{v}+\frac{\mathbf{a}\cdot\mathbf{w}}{6}\mathbf{w}=-\mathbf{v}-\mathbf{w}=\begin{bmatrix}-2\\0\\2\end{bmatrix}.$$

The matrix-vector product is

$$A\mathbf{a}=\begin{bmatrix}2/3&-1/3&-1/3\\-1/3&2/3&-1/3\\-1/3&-1/3&2/3\end{bmatrix}\begin{bmatrix}1\\3\\5\end{bmatrix}=\begin{bmatrix}2/3-1-5/3\\-1/3+2-5/3\\-1/3-1+10/3\end{bmatrix}=\begin{bmatrix}-3/3-1\\2-6/3\\9/3-1\end{bmatrix}=\begin{bmatrix}-2\\0\\2\end{bmatrix},$$

the same as the output of the direct calculation.

(c) We have $A = (1/3)B$ for the matrix

$$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

so $A^2 = (1/9)B^2$. By direct matrix multiplication we have

$$B^2 = \begin{bmatrix} 4+1+1 & -2-2+1 & -2+1-2 \\ -2-2+1 & 1+4+1 & 1-2-2 \\ -2+1-2 & 1-2-2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix},$$

and by factoring out the factor of 3 we see this is $3B$. Hence, dividing by 9 gives that $(1/9)B^2$ is equal to $(1/9)(3B) = (1/3)B = A$, as desired.

Problem 2: Matrix multiplication

(a) Compute the following matrix products.

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 9 & 11 \\ 2 & 5 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 9 & 11 \\ 0 & -13 & -16 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ 2 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 18 \\ 20 & 50 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 9 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 90 \\ 4 & 50 \end{bmatrix}}$$

$$\left(\begin{matrix}\text{for } \mathbf{v}, \mathbf{w} \text{ two} \\ \text{vectors in } \mathbf{R}^n\end{matrix}\right) \quad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \boxed{\begin{bmatrix} \mathbf{v} \cdot \mathbf{w} \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 9 & 11 \\ 2 & 5 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 5 & 6 \\ 1 & 9 & 11 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \boxed{\begin{bmatrix} 2a & 2b & 2c \\ -d & -e & -f \\ 3g & 3h & 3i \end{bmatrix}}$$

(b) Let $q(x, y, z) = x^2 + 2y^2 - z^2 - 3xy + 4xz + yz$. Find values of a, b, c, d, e, f that satisfy

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = q(x, y, z)$$

for every x, y, z. Strictly speaking, the left side multiplies out to be a 1×1 matrix and the equality means that the scalar $q(x, y, z)$ on the right side is the unique entry in that matrix. (Hint: multiply the left side fully, and compare coefficients on the two sides, such as for x^2, yz, etc.)

(c) (**Extra**) Is there a version of (b) for any $q(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz$ in general?

Solution:

(a) See boxed answers above.

(b) Carrying out the matrix multiplication of the firsr two matrices on the left, we get

$$\begin{bmatrix} ax + dy + ez & dx + by + fz & ex + fy + cz \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and this is the 1×1 matrix whose entry is

$$(ax + dy + ez)x + (dx + by + fz)y + (ex + fy + cz)z = ax^2 + dyx + ezx + dxy + by^2 + fzy + exz + fyz + cz^2.$$

But on the right side each of xy, xz, and yz appears twice with the same coefficient, so combining terms yields

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz.$$

Equating this to the given $q(x, y, z)$ and comparing coefficients yields $a = 1$, $b = 2$, $c = -1$, $d = -3/2$, $e = 2$, $f = 1/2$.

(c) $a = A$, $b = B$, $c = C$, $d = D/2$, $e = E/2$, $f = F/2$.

Problem 3: Some more matrix algebra

Consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by projecting a vector $\mathbf{v} \in \mathbf{R}^3$ onto its first two components (viewed as a 2-vector), then reflecting that projection across the line $x + y = 0$ in $\mathbf{R}^2$, and finally adding to this the 45° clockwise rotation of the projection of $\mathbf{v}$ onto its last two components. Find the 2×3 matrix A that computes T.

Solution: We first find the 2×3 matrix B representing the composition of projection on the first two components followed by the reflection. We will do this by using that the ith column of B is the effect of the composition on $\mathbf{e}_i$. The line $x+y=0$ is the diagonal going from the upper left to the lower right corner, so the composition sends the 3-vector $\mathbf{e}_1$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and sends $\mathbf{e}_2$ to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Finally, $\mathbf{e}_3$ is killed by the projection. Hence, the matrix is

$$B = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

(You could also have arrived at this by computing matrices for the projection and the reflection and multiplying them in the correct order.)

Now we calculate the matrix C given by composing projection onto the last two components and the rotation. We'll do this by multiplying matrices. The projection matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the rotation matrix is $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, so

$$C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Altogether, the matrix A is given as

$$A = B + C = \begin{bmatrix} 0 & -1 + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

