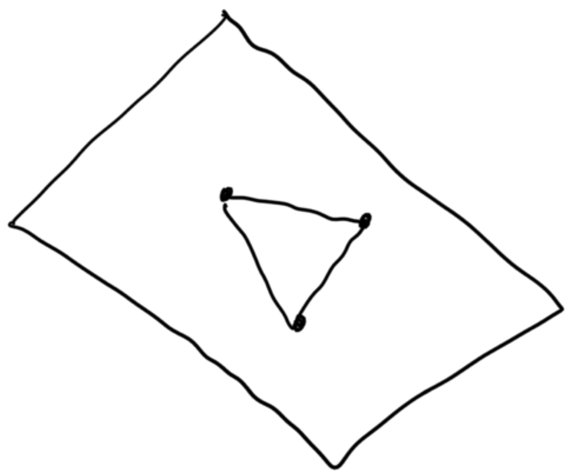
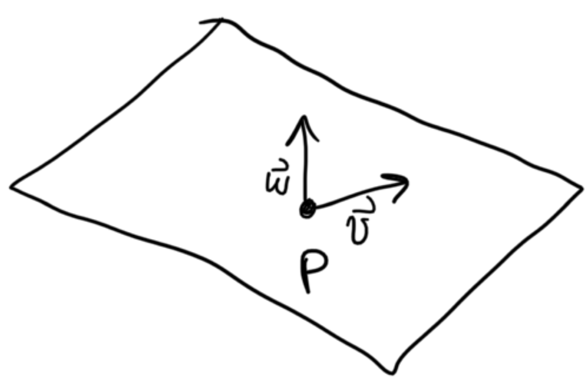
# Review: Planes in $\mathbb{R}^3$

Different ways of describing a plane:

- with 3 points (not all on the same line)

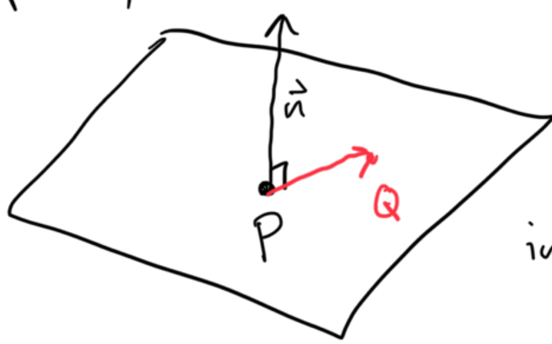


- with one point, two directions




Equation: $P + t\vec{w} + t'\vec{v}$ (parametric form)

• with one point, a normal vector




($\vec{n}$ is perpen to all vectors in the plane.)

Equation: $ax+by+cz=d$, where $\vec{n}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$,

$P=(p_1,p_2,p_3)$ where $d=ap_1+bp_2+cp_3$.

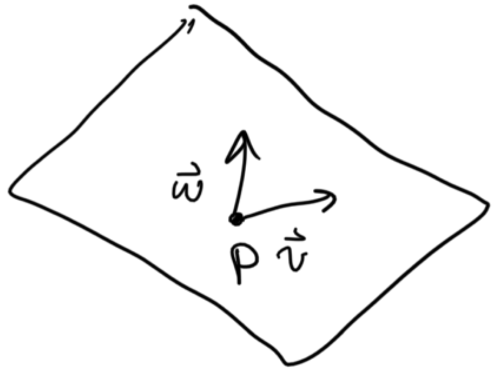
Why? $\vec{n}\cdot(Q-P)=0$ for any $Q=(x,y,z)$ in the plane. So $\begin{bmatrix} a \\ b \\ c \end{bmatrix}\cdot\begin{bmatrix} x-p_1 \\ y-p_2 \\ z-p_3 \end{bmatrix}=a(x-p_1)+b(y-p_2)+c(z-p_3)=0$

$\rightsquigarrow ax+by+cz=\underbrace{ap_1+bp_2+cp_3}_{=d}$

## Problem 1

Let $P$ be the plane in $\mathbb{R}^3$ containing $(1,1,1)$, $(1,2,3)$, and $(3,2,1)$

(a) Find a parametric representation of P.




- Pick one point to be P, say $P=(1,1,1)$
- $\vec{v}=\begin{bmatrix}1\\2\\3\end{bmatrix}-\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\2\end{bmatrix}$ (difference between another point and P)
- $\vec{w}=\begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\1\\0\end{bmatrix}$ (difference between the other point and P)

$$\rightsquigarrow \boxed{P+t\vec{v}+t'\vec{w}=\begin{bmatrix}1\\1\\1\end{bmatrix}+t\begin{bmatrix}0\\1\\2\end{bmatrix}+t'\begin{bmatrix}2\\1\\0\end{bmatrix}}$$

(b) Use the dot product to find a normal vector to P.

normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ must be perpendicular

to $\vec{v}$ and $\vec{w}$.

$$\vec{n} \cdot \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = b + 2c = 0 \rightsquigarrow b = -2c$$

$$\vec{n} \cdot \vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2a + b = 0 \rightsquigarrow b = -2a$$

e.g. let $a = c = 1$, then $b = -2$

$$\boxed{\vec{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

(c) Find an equation for P of the form $ax + by + cz = d$ for $a, b, c, d$ in $\mathbb{R}$.

Method 1: $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\rightsquigarrow x-2y+z=d$ use any point in the plane to fin d

e.g. $(x,y,z)=(1,1,1)$

$\rightsquigarrow 1-2\cdot 1+1=d$

$d=0$

So $\boxed{x-2y+z=0}$

Method 2: Plug in all 3 points to $ax+by+cz=d$

$$\left.\begin{aligned}(1,1,1) &\rightsquigarrow a+b+c=d \\ (1,2,3) &\rightsquigarrow a+2b+3c=d \\ (3,2,1) &\rightsquigarrow 3a+2b+c=d\end{aligned}\right\}$$

Solve system of equations (not a unique solution)

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$\boxed{\text{Problem 2}}$

(a) Consider the distinct points $A=(0,1,1)$, $B=(3,4,4)$, $C=(1,-1,-4)$. Compute the displacement vectors $\vec{AB}$ and $\vec{AC}$ to confirm these are not scalar multiples of one another, and find an equation of the form $ax+by+cz=d$ for the plane they lie in.

$$\vec{AB} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$$

Every multiple of $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ looks like

$$\begin{bmatrix} 3t \\ 3t \\ 3t \end{bmatrix} \neq \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$$

$\rightsquigarrow$ So these points define a plane

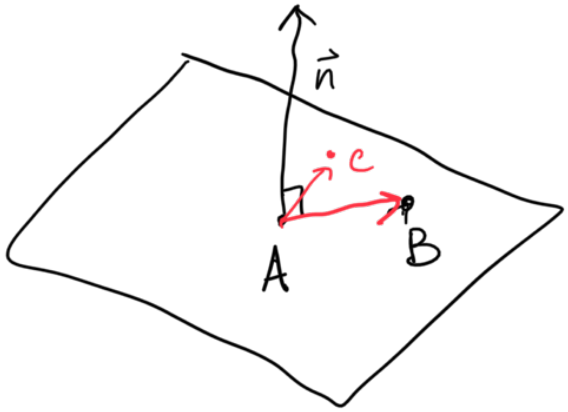
Normal vector is perpendicular to both $\vec{AB}$ and $\vec{AC}$

$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\vec{n} \cdot \vec{AB} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3a+3b+3c = 0$$

$$\vec{n}\cdot AC = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = a - 2b - 5c = 0 \,\Big)$$

$\rightsquigarrow \underline{a = 2b + 5c}$




So $3(2b+5c)+3b+3c=0$

$9b+18c=0$

$b+2c=0$

$\underline{b=-2c}$

Pick $c=1 \rightsquigarrow b=-2 \rightsquigarrow a=1$   so $\vec{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\rightsquigarrow \quad x - 2y + z = d$

Plug in a point, e.g. $(x,y,z) = A = (0,1,1)$

$\hookrightarrow \quad -2 + 1 = d$

$d = -1$

$$\boxed{x - 2y + z = -1}$$

(b) Find a $\boxed{\text{unit}}$ (length 1) vector that is normal to the plane whose equation is $6x-2y-3z=4$.

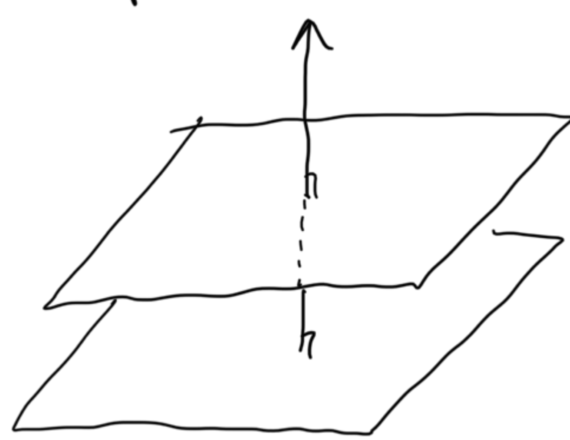
normal vector $\begin{bmatrix} 6 \end{bmatrix}$

$$\begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$\left\|\begin{bmatrix} 6 \\ -2 \\ -3 \end{bmatrix}\right\| = \sqrt{6^2+(-2)^2+(-3)^2} = \sqrt{36+4+9}$$
$$= \sqrt{49}$$
$$= 7$$

So $\frac{1}{7}\begin{bmatrix} 6 \\ -2 \\ -3 \end{bmatrix} = \boxed{\begin{bmatrix} 6/7 \\ -2/7 \\ -3/7 \end{bmatrix}}$ is a unit vector normal to the plane

(c) Are the planes in (a) and (b) parallel? Why?




planes are parallel when their normal vectors lie on the same line

i.e. when they are scalar multiples of one another

$\begin{bmatrix} 1 \end{bmatrix}$ $\begin{bmatrix} 6 \end{bmatrix}$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \neq c \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$ so these planes are not parallel

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## Review: Spans and subspaces

The span of n-vectors $\vec{v}_1, \ldots, \vec{v}_k$ is

$$\text{span}(\vec{v}_1, \ldots, \vec{v}_k) = \{\text{linear combinations } c_1\vec{v}_1 + \ldots + c_k\vec{v}_k\}$$

$\{\ \}$ "set of"

Note: If $c_1 = \ldots = c_k = 0$, get $\vec{0}$ so $\vec{0}$ is always in the span.

A linear subspace of $\mathbb{R}^n$ is the span of any (finite) collection of n-vectors, i.e.

$$\text{span}(\vec{v}_1, \ldots, \vec{v}_k), \ \vec{v}_i \in \mathbb{R}^n$$

"is an element of"

~ Every linear subspace contains $0$!

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## Problem 3

For each of the following subsets of $\mathbb{R}^2$ or $\mathbb{R}$ write down a collection of finitely many vectors whose span is that set, or explain why there is no such collection.

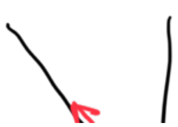
(a) The line $x+y=1$


$y=1-x$

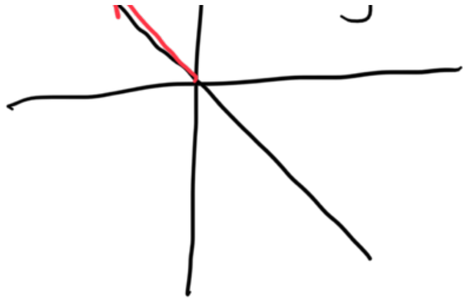
Does not include $(0,0)$!

So cannot be a span.

(b) The line $x+y=0$



$y=-x$ Lines through $(0,0)$



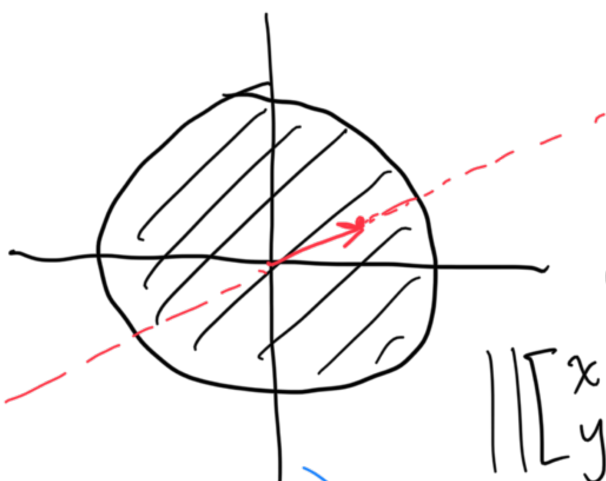
are <u>always</u> spans.

$$\{x+y=0\} = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

Check: $\mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \left\{ c\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{array}{c} \begin{bmatrix} -c \\ c \end{bmatrix} \end{array} \begin{array}{l} x \\ y \end{array} \right\}$

$= \{x+y=0\}$

(c) The unit disc $x^2+y^2 \leq 1$



<u>Not</u> a span.

If $(x,y)$ is in the unit disc, then

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sqrt{x^2+y^2} \leq 1$$

(if $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$) $\left\| c\begin{bmatrix} x \\ y \end{bmatrix} \right\| = c\sqrt{x^2+y^2}$

c can be arbitrarily large, so that

this length is $>1$

$\Rightarrow$ this cannot be a span

In general, if $S$ is a span, and $\vec{v} \in S$, then $c\vec{v} \in S$ for any $c \in \mathbb{R}$. So since we found a vector in the unit disc whose scalar multiple is not in the disc, the disc cannot be a span.

(d) $\{\vec{0}\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \operatorname{span}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$

$$\operatorname{span}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \left\{ c \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

(e) The plane $x+y+z=0$.

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$\boxed{\text{Problem 4}}$

Which of the following subsets $S$ of $\mathbb{R}^3$ are linear subspaces? If $S$ is a linear subspace, write it as a span. If not, describe it geometrically and explain why not.

why not.

(a) The set $S_1$ of points $(x, y, z)$ in $\mathbb{R}^3$ with both $z = x + 2y$ and $z = 5x$.

$z = x + 2y$ (or $x + 2y - z = 0$) is a plane

$z = 5x$ (or $5x - z = 0$) is a plane

Both contain $(0,0,0)$, so they are not parallel $\Rightarrow$ they intersect in a line.

So $S_1$ is a line containing $(0,0,0)$, which is a span.

Find this intersection:

$$z = 5x$$

$$\rightsquigarrow 5x = x + 2y$$

$$4x = 2y$$

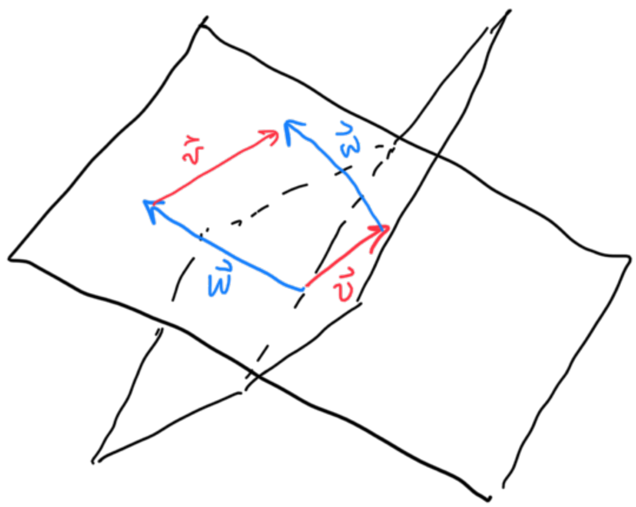
$$2x = y$$

Choose $x = 1 \rightsquigarrow y = 2 \rightsquigarrow z = 5$

So $S_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$

$\left(\begin{bmatrix} 5 \end{bmatrix}\right)$

(b) The set $S_2$ of points $(x,y,z)$ in $\mathbb{R}^3$ with either $z = x + 2y$ or $z = 5x$.




$S_2$

$\vec{v} + \vec{w}$ may not be in either plane

So <u>not</u> a span.

$S = \text{span}(\vec{v}_1, \ldots, \vec{v}_k)$

If $\vec{v}, \vec{w} \in S$, then $\vec{v} + \vec{w} \in S$, and $c\vec{v} \in S$

E.g. $\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ satisfies $z = 5x$, $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ satisfies $z = x + 2y$

but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ does not satisfy either

5 3 8 equation.

(c) The set $S_3$ of points $(x, y, z)$ in $\mathbb{R}^3$ of the form

$$t\begin{bmatrix}1\\2\\3\end{bmatrix} + t'\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}3\\3\\3\end{bmatrix} \text{ for some scalars } t$$

and $t'$.

$\boxed{\text{Problem 5}}$

For each collection of vectors in $\mathbb{R}^2$, sketch its span. Is it a point, line, or all of $\mathbb{R}^2$?

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For each collection of vectors in $\mathbb{R}^3$, sketch its span. Is it a point, line, plane, or all of $\mathbb{R}^3$

(f) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(h) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(i) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$