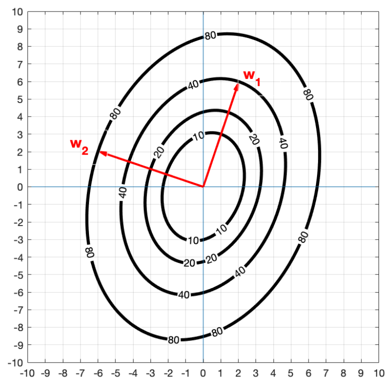
1. (3 points) Each picture below is a contour plot of $q_M(x, y)$ for some symmetric matrix M; the contour labels indicate the values of $q_M(x, y)$. In each case, two eigenvectors $\mathbf{w}_1$ and $\mathbf{w}_2$ of M are also sketched, where

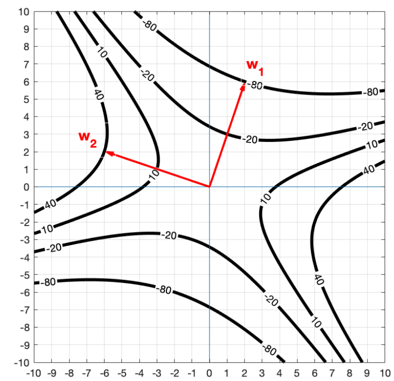
$$\|\mathbf{w}_1\| = \|\mathbf{w}_2\| = 2\sqrt{10}, \qquad M\mathbf{w}_1 = \lambda_1 \mathbf{w}_1, \qquad M\mathbf{w}_2 = \lambda_2 \mathbf{w}_2.$$

Determine $a = \frac{\lambda_2}{\lambda_1}$ for contour plot A, and $b = \frac{\lambda_2}{\lambda_1}$ for contour plot B.

(A)



(B)



Note that
For (A): $q_M(\mathbf{w}_1) = 40$ and $q_M(\mathbf{w}_2) = 80$.
For (B): $q_M(\mathbf{w}_1) = -80$ and $q_M(\mathbf{w}_2) = 40$.

(a) 0 (b) 1 (c) -1 (d) 2 (e) -2

(f) 3 (g) -3 (h) 4 (i) -4 (j) 1/2

(k) $-1/2$ (l) 1/3 (m) $-1/3$ (n) 1/4 (o) $-1/4$

2. (3 points) For which values b does the function

$$f(x, y) = 3x^2 - 2bxy + 12y^2$$

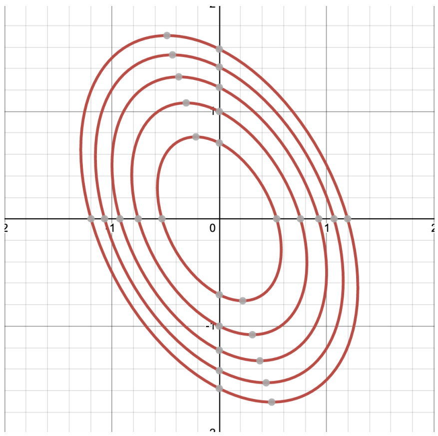
attain its global minimum value at *exactly one* point in $\mathbf{R}^2$?

(a) $b < 0$ (b) $b > 0$ (c) $|b| > 6$ (d) $-6 < b < 6$

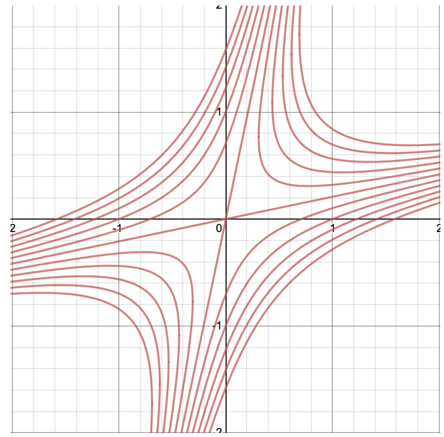
3. (4 points) Below are contour plots of **quadratic approximations at the origin** of four functions; each of the four functions has a critical point at $(0, 0)$. For each of the given Hessian matrices at the origin, select the picture representing the associated contour plot.

(a) $(\mathrm{Hf})(0,0) = \begin{bmatrix} 14 & 4 \\ 4 & 8 \end{bmatrix}$ (b) $(\mathrm{Hg})(0,0) = \begin{bmatrix} 4 & 10 \\ 10 & 4 \end{bmatrix}$

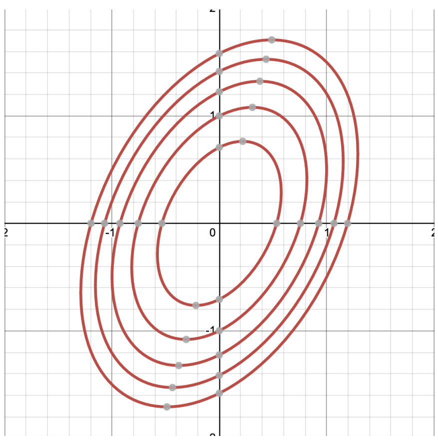
(1)



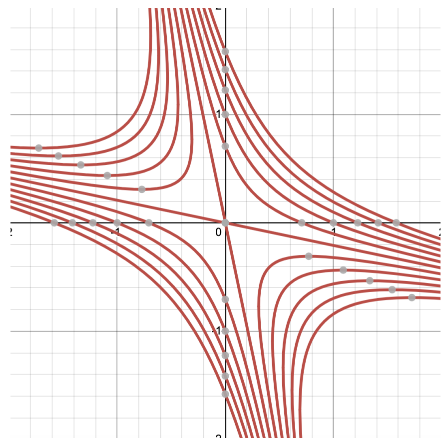
(2)



(3)



(4)



4. (2 points) A researcher collected 100 data points

$$(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_{100}, y_{100})$$

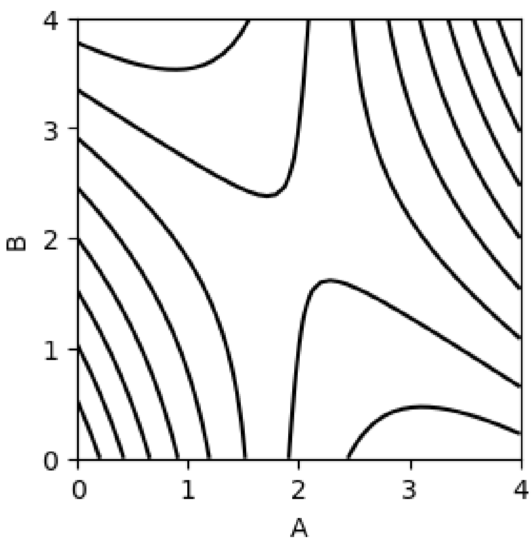
and attempted to find the best-fit line $y = A + Bx$; i.e., she wanted to find A and B that minimizes the total squared-error function

$$f(A, B) = (y_1 - A - Bx_1)^2 + (y_2 - A - Bx_2)^2 + \cdots + (y_{100} - A - Bx_{100})^2.$$

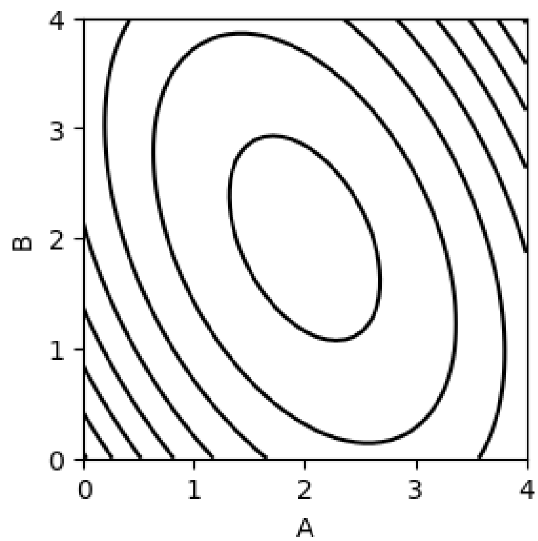
Note that $f(A, B)$ is a quadratic function of two variables A, B.

Which of the following are possible contour plots of f (with horizontal A-axis and vertical B-axis)? Select all that apply.

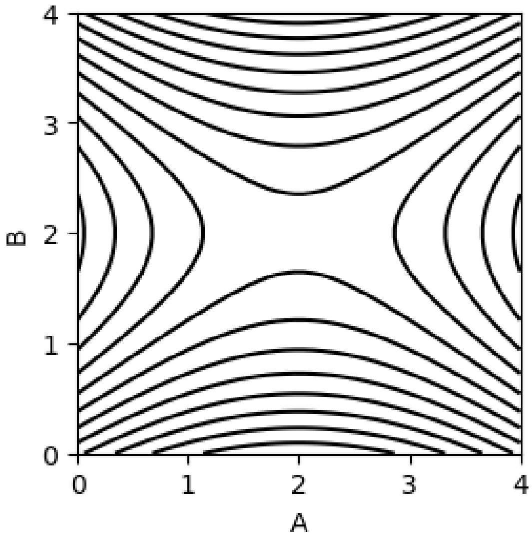
(i)



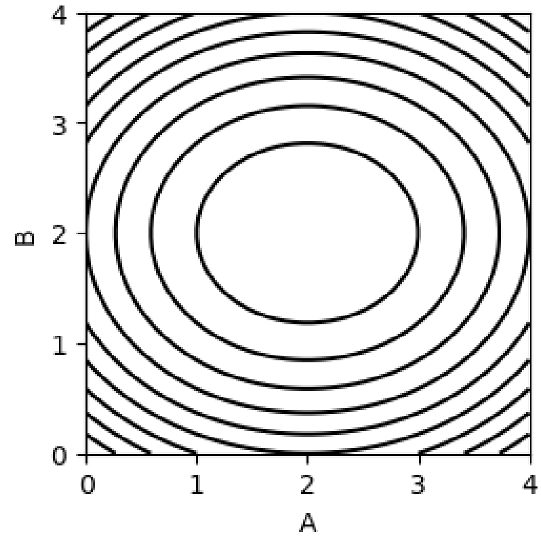
(ii)



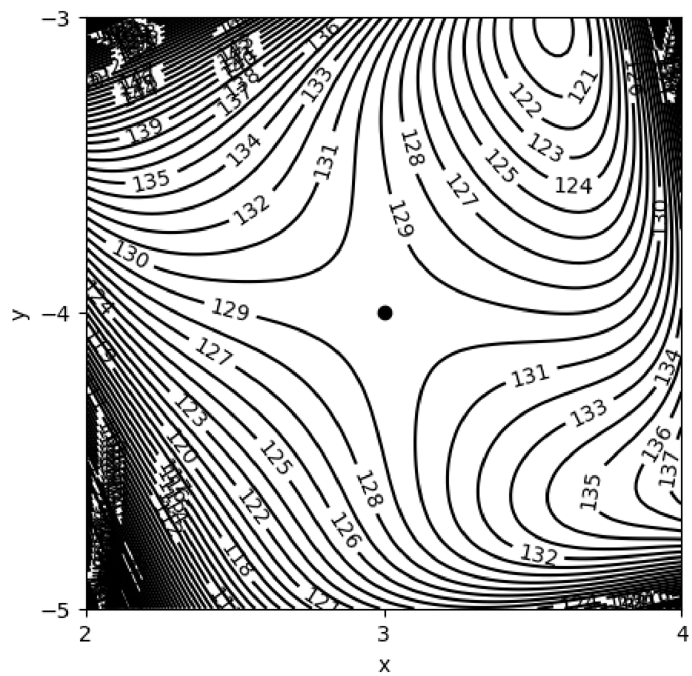
(iii)



(iv)



5. (4 points) Below is a contour plot of $f(x, y)$ around a critical point $(3, -4)$:



At this critical point, the Hessian matrix $(\mathrm{Hf})(3, -4)$ has one positive eigenvalue, λ, and one negative eigenvalue, μ.

(a) Which of the following is approximately equal to an eigenvector of $(\mathrm{Hf})(3, -4)$ with positive eigenvalue (i.e., with eigenvalue λ)?

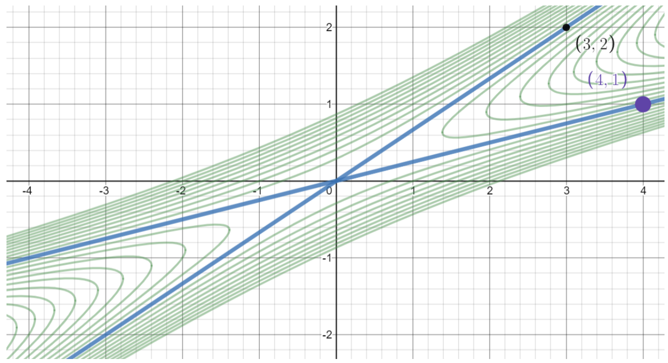
(b) Which of the following is approximately equal to an eigenvector of $(\mathrm{Hf})(3, -4)$ with negative eigenvalue (i.e., with eigenvalue μ)?

(i) $(1, 1)$ (ii) $(1, 0)$ (iii) $(-1, 1)$ (iv) $(0, 1)$

6. (2 points) **True or False:** Suppose $f(x, y)$ has the property that $\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = 1$ for all (x, y). Then f has no local maxima.

7. (2 points) **True or False:** Suppose $f(x, y)$ has the property that $Hf(x, y)$ is positive definite for any point (x, y), and let $g(x, y) = e^{f(x,y)}$. Then $Hg(x, y)$ is also positive definite for any point (x, y).

8. (2 points) **True or False:** Consider the following partial contour plot of some quadratic form:



The blue contour corresponds to the level set at the level 0.

Then: $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ are eigenvectors of the matrix associated to this quadratic form.

9. (3 points) For which value of a is the line $y = mx$ an eigenline for the Hessian matrix of the function $f(x, y) = y^2 - axy$ at $(0, 0)$?

   (a) 0
   (b) $m/2$
   (c) m
   (d) $2m$
   (e) m^2
   (f) $2m^2 - 1$
   (g) $\frac{1}{m}$
   (h) $\frac{2m}{1-m^2}$
   (i) $\frac{m}{1+m^2}$

10. (10 points) Find and classify all critical points of $f(x, y) = (x + y - 2)^2 + (x - y)^3 - (x - y)^2$.