- Last time
  · linearity & matrices
  · the derivative matrix

# Today

- ·matrices for projections and more complicated linear transformations
- · matrix "algebra" (addition & multiplication, plugaina matrices into polynomials)

#### **Problem 1: Matrix of a projection**

Let V be the plane x+y+z=0 in  $\mathbf{R}^3$  through the origin, so V has an orthogonal basis  $\{\mathbf{v},\mathbf{w}\}$  for  $\mathbf{v}=\begin{bmatrix}1\\-1\\0\end{bmatrix}$  and  $\mathbf{w}=\begin{bmatrix}1\\1\\-2\end{bmatrix}$ . Let  $L: \mathbf{R}^3 \to \mathbf{R}^3$  be the function  $L(\mathbf{x}) = \mathbf{Proj}_V(\mathbf{x})$ .

(a) Compute the  $3 \times 3$  matrix A for L; the entries should be fractions with denominator 3. (Hint: what is the meaning of each column?)

(b) For  $\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ , compute  $\mathbf{Proj}_V(\mathbf{a})$  in two ways: using the orthogonal basis  $\{\mathbf{v}, \mathbf{w}\}$  for V, and using the matrix-vector product against your answer in (a). (You should get the same answer both ways, a vector with integer entries.)

(c) The geometric definition of  $\mathbf{Proj}_V$  gives that its output lies in V, on which  $\mathbf{Proj}_V$  has no effect, so  $\mathbf{Proj}_V \circ \mathbf{Proj}_V = \mathbf{Proj}_V$ . Check that your answer A in (a) satisfies the corresponding matrix equality  $A^2 = A$ . (Hint: if you write A = (1/3)B for a matrix B with integer entries then the calculation will be cleaner.)

## **Problem 2: Matrix multiplication**

(a) Compute the following matrix products.

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 9 & 11 \\ 2 & 5 & 6 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} =$$

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$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

(b) Let  $q(x, y, z) = x^2 + 2y^2 - z^2 - 3xy + 4xz + yz$ . Find values of a, b, c, d, e, f that satisfy

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = q(x, y, z)$$

for every x,y,z. Strictly speaking, the left side multiplies out to be a  $1 \times 1$  matrix and the equality means that the scalar q(x,y,z) on the right side is the unique entry in that matrix. (Hint: multiply the left side fully, and compare coefficients on the two sides, such as for  $x^2$ , yz, etc.)

(c) (Extra) Is there a version of (b) for any  $q(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz$  in general?

#### Problem 3: Some more matrix algebra

Consider the linear transformation  $T: \mathbf{R}^3 \to \mathbf{R}^2$  given by projecting a vector  $\mathbf{v} \in \mathbf{R}^3$  onto its first two components (viewed as a 2-vector), then reflecting that projection across the line x+y=0 in  $\mathbf{R}^2$ , and finally adding to this the  $45^\circ$  clockwise rotation of the projection of  $\mathbf{v}$  onto its last two components. Find the  $2\times 3$  matrix A that computes T.

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