Last time:

- orthogonality & orthogonal basis
- projection

Today:

- applications of projections
  - best fit lines
  - mathematical models
- multivariable functions
  - level sets
  - composition

## Problem 1: A best fit line

The collection of 5 data points $(-1, 6)$, $(0, 3)$, $(1, 0)$, $(2, -3)$, $(3, -4)$ lies close to a line of negative slope; see Figure 1. We are going to compute that line.

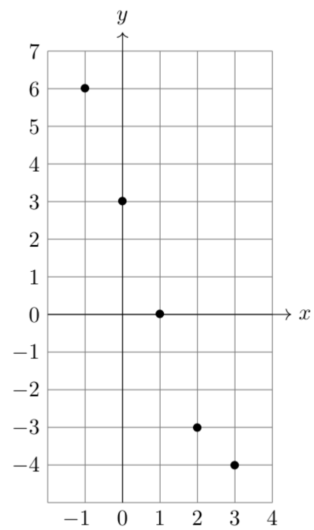


Figure 1: Five data points: $(-1, 6)$, $(0, 3)$, $(1, 0)$, $(2, -3)$, $(3, -4)$.

Suppose the line of best fit (in the least squares sense) is written as $y = mx + b$.

(a) Write down explicit 5-vectors $\mathbf{X}$ and $\mathbf{Y}$ so that for the 5-vector $\mathbf{1}$ whose entries are all equal to 1, the projection of $\mathbf{Y}$ into the plane $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ in $\mathbf{R}^5$ is $m\mathbf{X} + b\mathbf{1}$.

(b) Compute an orthogonal basis of $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ having the form $\{\mathbf{1}, \mathbf{v}\}$ for a 5-vector $\mathbf{v}$, and find scalars $t$ and $s$ so that $\mathbf{Proj}_V(\mathbf{Y}) = t\mathbf{v} + s\mathbf{1}$.

(c) By expressing $\mathbf{v}$ from (b) as a linear combination of $\mathbf{X}$ and $\mathbf{1}$, use your answer to (b) to find $m$ and $b$ so that the equation $y = mx + b$ gives the line of best fit. (As a safety check on your computations, you may want to plot your line on the above figure to see that it is a good fit for the data.)

## Problem 2: A linear mathematical model via closest vector and dot products

A researcher measures the basal metabolic rate[1], height, and weight for $100$ people and expresses the result as vectors:

$$\mathbf{B}, \mathbf{W}, \mathbf{H} \in \mathbf{R}^{100}$$

Here, the $i$th entry of $\mathbf{H}$ is the height of the $i$th person in inches, and similarly for $\mathbf{B}$ (basal metabolic rate in kilocalories per day) and $\mathbf{W}$ (weight in pounds).

The researcher would like to work out a linear formula to estimate the basal metabolic rate in terms of height and weight. In mathematical terms, she would like to find $a, b \in \mathbf{R}$ for which

$$a\mathbf{H} + b\mathbf{W} \text{ is as close to } \mathbf{B} \text{ as possible.}$$

(a) Suppose that the vectors were in $\mathbf{R}^3$ rather than $\mathbf{R}^{100}$. Draw a picture to explain why the $a, b$ we are looking for must satisfy

$$\mathbf{B} - (a\mathbf{H} + b\mathbf{W}) \text{ is perpendicular to } \mathbf{H}, \mathbf{W}.$$

(We know this is true in $\mathbf{R}^{100}$ by the Orthogonal Projection Theorem; the point is to understand it intuitively with a picture in $\mathbf{R}^3$.)

(b) Use the orthogonality as discussed in (a) (which must hold for 100-vectors) and dot products to write down a system of linear equations for $a, b$ (whose coefficients involve dot products among 100-vectors).

(c) The researcher computes that $\mathbf{H} \cdot \mathbf{H} = 1/2$, $\mathbf{W} \cdot \mathbf{W} = 3$ and $\mathbf{H} \cdot \mathbf{W} = 3/2$; also $\mathbf{B} \cdot \mathbf{W} = 3$ and $\mathbf{B} \cdot \mathbf{H} = 2$. Using the vanishing of dot products against $\mathbf{H}$ and $\mathbf{W}$ arising from (a), solve for $a$ and $b$. (In the real world, such dot products would usually be "ugly" numbers; we made them clean, as we do on exams, so the answer comes out cleanly without using a calculator.)

Observe that the solution did not require knowledge of the $100$-element vectors– just knowledge about their dot products! (Of course, to *compute* those dot products one has to know the 100-vectors, but the point is that the *only* way the knowledge of the 100-vectors is relevant is solely to compute those dot products.)

Review: Level sets

**Problem 3: Level sets of multivariable functions**

(a) Describe and sketch the level sets of $\ln(y - x^2)$ on the region where $y > x^2$, relating each level set to the parabola $y = x^2$.

(b) Describe and sketch the level sets of $\cos(x^2 + y^2)$.

(c) Express the surface graph of $f(x, y) = x^2 + y^2$ in $\mathbf{R}^3$ as a level set of a function $h(x, y, z)$.

(d) (Extra) By using polar coordinates, describe the part of the graph of $f(x, y) = x^2 + y^2$ from (c) that lies over a line in the $xy$-plane through the origin, and use that to sketch the actual surface graph. (Don't "cheat" by looking on a computer; the point is to learn for yourself how to use restriction over well-chosen lower-dimensional subspaces, such as lines through the origin in $\mathbf{R}^2$, to build up a mental model of what happens over the entire domain.)

**Problem 4: Computations with vector-valued functions**

For the functions $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ and $\mathbf{g} : \mathbf{R}^m \to \mathbf{R}^p$ below, compute $\mathbf{g} \circ \mathbf{f} : \mathbf{R}^n \to \mathbf{R}^p$ by working out its component functions; in each part also state the values of $n$, $m$, and $p$.

(a) $\mathbf{f}(x, y) = (e^x \cos(y), e^x \sin(y))$, $\mathbf{g}(v, w) = (v^2 - w^2, 2vw)$

(b) $\mathbf{f}(x, y) = (x^2 - y^2, 2xy)$, $\mathbf{g}(v, w) = (e^v \cos(w), e^v \sin(w))$

(c) $\mathbf{f}(t) = (1 - t^2, 2t, 1 + t^2)$, $\mathbf{g}(x, y, z) = x^2 + y^2 - z^2$

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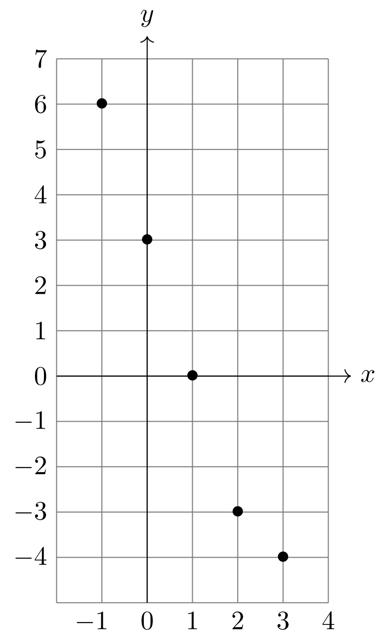


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[1] rate at which the body uses energy, measured in kilocalories per day, if the person is at rest