Last time:

- QR & LU decompositions

Today:

- eigenvalues & eigenvectors
- "definiteness" of quadratic forms/symmetric matrices

Problem 1: Recognizing Eigenvectors

For the following matrices A and nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, verify that the vectors are eigenvectors for A and find their corresponding eigenvalues.

(a) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

Problem 2: Geometric meaning of eigenvalues

Identify the eigenvalues of the following linear transformations $\mathbf{R}^3 \to \mathbf{R}^3$, and find *all* eigenvectors for each eigenvalue (expressed in terms of the given geometric data). The eigenvalues are explicit numbers, not depending on the given line or plane.

Hint: Think geometrically by looking for lines carried onto themselves (or crushed into $\{\mathbf{0}\}$: don't overlook the possibility of 0 as an eigenvalue!). In particular, if a line is not carried onto itself or crushed into the origin, it cannot provide any eigenvectors.

(a) The reflection across a plane $V \subset \mathbf{R}^3$ through the origin.

(b) The projection onto a plane $V \subset \mathbf{R}^3$ through the origin.

(c) The rotation by 90° around a line $L \subset \mathbf{R}^3$ through the origin.

(d) The rotation by 180° around the line $L \subset \mathbf{R}^3$ through the origin.

Problem 3: Eigenvalues of 2×2 matrices

For each of the following 2×2 matrices, find all the eigenvalues and an eigenvector for each eigenvalue.

(a) $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$.

(c) $C = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$ for a general number $a \neq 0$. (Your answer may depend on a; for $a = 4$ it should recover the answer to (b).)

Problem 4: Additional practice with eigenvalues and eigenvectors (triangular examples)

For each eigenvalue λ of the given matrix A, compute a basis for the nonzero linear subspace $N(A - \lambda \mathrm{I}_3)$ in $\mathbf{R}^3$ (the "λ-eigenspace"), and as a check on your work verify directly that each vector in that basis is an eigenvector for A with eigenvalue λ.

(a) $A = \begin{bmatrix} 1 & 6 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 3 & 0 \\ 6 & -2 & 2 \end{bmatrix}$.

Problem 5: Quadratic forms and definiteness I

(a) For each of the following 2×2 symmetric matrices M, compute the quadratic form $q_M(x, y) = \begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix}$:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 17 & 4 \\ 4 & 2 \end{bmatrix}, D = \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}.$$

(b) For each M in (a), use its characteristic polynomial to find its eigenvalues (they are all integers in these cases), and from that determine if $q_M(x, y)$ is positive-definite, negative-definite, or indefinite.

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 17 & 4 \\ 4 & 2 \end{bmatrix}, D = \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}.$$

Problem 6: Quadratic forms and definiteness II

For each of the following symmetric 3×3 matrices M and given nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, carry out two tasks:

(i) Compute the associated quadratic form $q_M(x, y, z)$, and verify that $\mathbf{v}_i$'s are pairwise orthogonal and eigenvectors, determining the eigenvalue for each.

(ii) Use your answer to (i) to write down the quadratic form $q(u, v, w) = q_M(u\mathbf{v}_1' + v\mathbf{v}_2' + w\mathbf{v}_3')$ when everything is described in terms of the basis of orthonormal eigenvectors $\mathbf{v}_i' = \mathbf{v}_i/\|\mathbf{v}_i\|$, from which you should determine if q_M is positive-definite, negative-definite, indefinite, positive-semidefinite (but not positive-definite), or negative-semidefinite (but not negative-definite). You *do not* need to compute the lengths $\|\mathbf{v}_i\|$.

(a) $\begin{bmatrix} 5 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

(b) $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$

(c) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

(d) $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

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For each eigenvalue λ of the given matrix A, compute a basis for the nonzero linear subspace $N(A - \lambda I_3)$ in $\mathbf{R}^3$ (the "λ-eigenspace"), and as a check on your work verify directly that each vector in that basis is an eigenvector for A with eigenvalue λ.

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(b) $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

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