Solutions to Math 51 Quiz 1 Practice B

1. (10 points) Given three points $\mathbf{a} = (6, 3)$, $\mathbf{b} = (4, 5)$, $\mathbf{c} = (10, 9)$, and a linear combination $\mathbf{p} = \frac{1}{5}\mathbf{a} + \frac{1}{5}\mathbf{b} + \frac{3}{5}\mathbf{c}$.

   The line through $\mathbf{p}$ and $\mathbf{c}$ intersects the line segment between $\mathbf{a}$ and $\mathbf{b}$ in a single point, which we call $\mathbf{X} = (x, y)$.

   Compute the values of x and y. Your answers should be non-negative integers.

   [Hint: Write $\mathbf{p}$ as a linear combination of $\mathbf{c}$ with another point that is located on the line segment between $\mathbf{a}$ and $\mathbf{b}$.]

Notice that since $\frac{1}{5} + \frac{1}{5} + \frac{3}{5} = 1$, the point $\mathbf{p}$ is a convex linear combination of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$; so it lies within the triangle defined by these three points. (So indeed, a line drawn from one vertex of this triangle, through $\mathbf{p}$, will intersect the opposite side of the triangle somwhere between the other two vertices.)

In order to follow the hint, we should recognize that "a point that is located on the line segment between $\mathbf{a}$ and $\mathbf{b}$" is any point that can be expressed as a convex linear combination of $\mathbf{a}$ and $\mathbf{b}$; this requires nonnegative scalar coefficients that add to 1. In order to do this for $\mathbf{p}$, rewrite the coefficients of $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{p}$ as scalings of numbers that do add to 1:

$$\begin{aligned}\frac{1}{5}\mathbf{a} + \frac{1}{5}\mathbf{b} &= \frac{1}{5}(\mathbf{a} + \mathbf{b}) \\ &= \frac{2}{5}\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right)\end{aligned}$$

That is,

$$\mathbf{p} = \frac{1}{5}\mathbf{a} + \frac{1}{5}\mathbf{b} + \frac{3}{5}\mathbf{c} = \frac{2}{5}\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right) + \frac{3}{5}\mathbf{c}$$

In this expression, notice that we are now describing $\mathbf{p}$ as a convex linear combination of $\mathbf{c}$ and $\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$, the latter of which is a point on the line segment between $\mathbf{a}$ and $\mathbf{b}$. This is precisely describing the situation in the problem: the point $\mathbf{X}$ is the point $\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$, and $\mathbf{p}$ lies on the segment between $\mathbf{X}$ and $\mathbf{c}$!

We compute that $(x, y) = \mathbf{X} = \left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right) = \boxed{(5, 4)}$.

2. (2 points) **True or False:** For two non-zero vectors $\mathbf{v}$ and $\mathbf{w}$, $\|\mathbf{v} + 2\mathbf{w}\|$ is always greater than $\|\mathbf{v}\|$.

False (i.e., not always true): If the angle between $\mathbf{v}$ and $2\mathbf{w}$ is obtuse, then $\|\mathbf{v}+2\mathbf{w}\|$ can be less than $\|\mathbf{v}\|$. For example, set

$$\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix}, \qquad \mathbf{v} + 2\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \|\mathbf{v} + 2\mathbf{w}\| = 1 < \|\mathbf{v}\|.$$

3. (2 points) **True or False:** the line through the two points $(1, 0, 1)$ and $(3, 1, 2)$ is parallel to the plane $x - y - z = 4$.

True (i.e., "always true"): Note that the line through the two points $(1, 0, 1)$ and $(0, 1, 2)$ has direction $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$. And the plane $x - y - z = 4$ has normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Since $\mathbf{n} \cdot \mathbf{v} = 0$, the line must be perpendicular to the normal of the plane. Hence the line must be parallel to the plane.

4. (3 points) The planes $2x + y - 2z = 2$ and $x - y + 2z = 1$ intersect in a line L with parametric form:

(i) $\begin{bmatrix}1\\2\\1\end{bmatrix} + t\begin{bmatrix}0\\2\\1\end{bmatrix}$ (ii) $\begin{bmatrix}1\\2\\1\end{bmatrix} + t\begin{bmatrix}1\\1\\0\end{bmatrix}$ (iii) $\begin{bmatrix}1\\2\\1\end{bmatrix} + t_1\begin{bmatrix}2\\1\\-2\end{bmatrix} + t_2\begin{bmatrix}1\\-1\\2\end{bmatrix}$

The line L must have direction $\mathbf{v} = \begin{bmatrix}a\\b\\c\end{bmatrix}$ perpendicular to the two normals to the planes $\mathbf{n}_1 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix}1\\-1\\2\end{bmatrix}$. Solving simultaneously

$$\begin{aligned}\mathbf{v}\cdot\mathbf{n}_1 &= 2a + b - 2c = 0\\ \mathbf{v}\cdot\mathbf{n}_2 &= a - b + 2c = 0\end{aligned}$$

we find $b = a + 2c$ from the second equation, and substitute this into the first equation, we have $a = 0$. So $\mathbf{v} = c\begin{bmatrix}0\\2\\1\end{bmatrix}$. The point $(1, 0, 0)$ and $(1, 2, 1)$ lies on both planes since they satisfy both plane equations. So **??** is an equation for line L.

5. (3 points) Let $\mathbf{v}$ be a fixed nonzero vector in $\mathbf{R}^3$, and $d \in \mathbf{R}$ a scalar. Geometrically, the collection of vectors $\mathbf{w} = \begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbf{R}^3$ satisfying the condition

$$\mathbf{v}\cdot\mathbf{w} = d$$

is a

a) line. b) plane. c) $\mathbf{R}^3$. d) might take different shapes depending on what $\mathbf{v}$ and $\mathbf{w}$ are.

Let $\mathbf{v} = \begin{bmatrix}a\\b\\c\end{bmatrix}$, then

$$\mathbf{v}\cdot\mathbf{w} = ax + by + cz = d$$

gives an equation of a plane.