

Last time

- linear independence (& bases)
- Gram-Schmidt (algorithm for obtaining an orthogonal basis)

Today

- quadratic forms & symmetric matrices
- column and null spaces
- more solving systems of equations with matrices

Problem 1: Matrix of a quadratic form

(a) Consider the quadratic form $q(x, y, z) = x^2 + 2y^2 - z^2 + 4xy + 6xz - 2yz$. Find the symmetric matrix A so that

$$q(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

- (b) Consider the quadratic form $q(w, x, y, z) = 2x^2 + y^2 + 3xy + 4yw - zw$ (note that w is the first coordinate, x is the second coordinate, etc.). Find the symmetric matrix A so that $q(w, x, y, z) = \begin{bmatrix} w & x & y & z \end{bmatrix} A \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$.

Problem 2: Linear system solution set, in parametric form

Let L be the line determined by the overlap of two planes in \mathbf{R}^3 (not passing through the origin):

$$x + y + z = 4 \quad \text{and} \quad 2x + 3y + z = 9.$$

- (a) Find a point on L whose z -coordinate is 5.

(b) Find a point on L whose z -coordinate is 6.

- (c) Find general formulas for x and y in terms of z for all points (x, y, z) in the line L ; this should recover your answers to (a) and (b) upon plugging in $z = 5$ and $z = 6$ respectively. (Hint: think about z as a “constant” and x and y as “variables” to guide your algebraic work in the style of what you did for (a) and (b).)

Problem 3: Recognizing sets as null spaces (or not)

Often in linear algebra one builds a collection of n -vectors satisfying a variety of conditions, and it can be useful to know if the collection is the null space of an $m \times n$ matrix for some m . (in effect: is the collection describable as the simultaneous solution set for a system of m linear equations in n unknowns?)

- (a) Explain why the subset $S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbf{R}^3 : v_1 + v_2 = v_3 + 1 \right\} \subset \mathbf{R}^3$ cannot be the null space of a matrix A (such an A would have to be $m \times 3$ for some m). Hint: try to find some general property of null spaces that S violates.

(b) Find a 2×3 matrix A for $N(A)$ is equal to the set $S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbf{R}^3 : v_1 + v_2 - v_3 = 0, v_2 + v_3 = 0 \right\} \subset \mathbf{R}^3$.

Problem 4: Basis and dimension

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}.$$

- (a) Write the third column of A as a linear combination of the first two columns. Use this to find a basis for $C(A)$.

- (b) Find a basis for $C(A)$ that contains $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$.

- (c) **(Extra)** How can having a basis for the column space $C(M)$ of an $m \times n$ matrix M help in figuring out for a given m -vector \mathbf{b} if $M\mathbf{x} = \mathbf{b}$ has a solution? (Hint: think about $\mathbf{Proj}_{C(M)}$.)

Problem 5: Column spaces and an overdetermined linear system

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix}$, and let the 5-vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ denote the columns of A (from left to right). For a 5-vector \mathbf{b} , the

vector equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} \in \mathbf{R}^3$ encodes a system of 5 scalar linear equations in 3 unknowns (the entries of \mathbf{x}), so it is overdetermined (more equations than unknowns) and hence the rule of thumb is that only for rather special \mathbf{b} should a solution exist. This problem works out a description of those special \mathbf{b} for this specific A by thinking in terms of column spaces.

- (a) If a vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \in \mathbf{R}^5$ belongs to the column space of A (i.e., $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ for some scalars

x_1, x_2, x_3 , or equivalently $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3$) then by turning this vector condition into a collection of 5 simultaneous scalar conditions explain why $b_3 - b_1 - b_2 = 0$, $b_4 + b_1 - b_2 = 0$, and $b_5 - 2b_2 + b_1 = 0$. (Hint: use that each b_i can be expressed in a specific way in terms of the x_j 's).

(b) The three equations in the b_i 's at the end of (a) can be rewritten to say $b_3 = b_1 + b_2$, $b_4 = -b_1 + b_2$, and $b_5 = -b_1 + 2b_2$,

so collectively they say exactly that \mathbf{b} is a 5-vector of the special form
$$\begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \\ -b_1 + b_2 \\ -b_1 + 2b_2 \end{bmatrix} \text{ with } b_1, b_2 \in \mathbf{R}. \text{ Check explicitly}$$

that any \mathbf{b} of this form actually is in $C(A)$. (Hint: for \mathbf{b} of this special form in terms of b_1 and b_2 , adapt the type of argument in Problem 2(c) to find an $\mathbf{x} \in \mathbf{R}^3$ with $x_3 = 0$ for which the first two entries of $A\mathbf{x}$ are b_1, b_2 respectively, and then use the assumed three equations on the b_i 's to check that actually $A\mathbf{x} = \mathbf{b}$.) This says that three conditions on \mathbf{b} at the end of (a) are not merely a consequence of membership in $C(A)$ but even exactly characterize it.

(c) Give an explicit 5-vector \mathbf{b} for which the linear system " $A\mathbf{x} = \mathbf{b}$ " does *not* have a solution. (There are many answers.)

Problem 6: Normal matrices (Extra)

An $n \times n$ matrix A is called *normal* if A commutes with its transpose A^\top (i.e., $AA^\top = A^\top A$). For example, every symmetric matrix (and in particular every diagonal matrix) is normal, since it is even equal to its own transpose.

- (a) Check that every matrix of the form $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (which is not symmetric when $b \neq 0$) is normal, and that $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is normal. (When computing MM^\top and $M^\top M$ you only need to compute the part on or above the diagonal, since the product is symmetric; saving work in this way assumes one doesn't make a miscalculation.)

- (b) Explain why any orthogonal $n \times n$ matrix is normal, and that if M is normal then so are M^2 and M^3 . (How about M^r for general $r \geq 1$?)

- (c) In (a) we give a type of non-symmetric normal 2×2 matrix (with $b \neq 0$). Show by explicit example that for a symmetric (hence normal) 2×2 matrix M and normal 2×2 matrix M' of the type in (a), the product $A = MM'$ can fail to be normal. (Nearly anything you try for M and M' should work, as long as you avoid too many matrix entries equal to 0.)

In connection with (c), it is a fact that if M and M' are normal $n \times n$ matrices that *commute* (i.e., $MM' = M'M$) then the product MM' is normal. However, this has no short explanation; it requires a big generalization of an upcoming important result called the Spectral Theorem.

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