Problem 1: Large powers of symmetric matrices (a 3×3 example)

Consider the matrix

$$M = \begin{bmatrix} 3/5 & 1/5 & 1/5 \\ 1/5 & 3/5 & 1/5 \\ 1/5 & 1/5 & 3/5 \end{bmatrix}$$

Since M is symmetric, the Spectral Theorem implies that there is an orthogonal basis for $\mathbf{R}^3$ consisting of eigenvectors for M. For this problem, assume that we are also given that the eigenvalues of M are $\lambda_1 = 1$ and $\lambda_2 = \dfrac{2}{5}$.

(a) Let V_1 be the λ_1-eigenspace. Verify that $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ lies in V_1, and explain why it spans that space.

(b) Find the λ_2-eigenspace V_2, and write it as the span of two orthogonal vectors $\mathbf{w}_2, \mathbf{w}_3$.

(c) Let $\mathbf{w}_1', \mathbf{w}_2', \mathbf{w}_3'$ be unit vectors obtained from $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. Let W be the matrix whose columns are $\mathbf{w}_1', \mathbf{w}_2', \mathbf{w}_3'$. Find the diagonal matrix D where $M = WDW^\top$.

(d) Using the fact that $(2/5)^{100} \approx 0$ to over thirty-five decimal places, calculate M^{100} explicitly.

Problem 2: Large powers of symmetric matrices (a 2×2 example)

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, a symmetric matrix.

(a) Compute the eigenvalues $\lambda_1 > \lambda_2$ of A and find eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ for λ_1, λ_2, respectively. Check that $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal.

(b) Write $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combinations of the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(c) Use your expressions from part (b) to give an exact expression for A^{100}. (Hint: note that the first column of A^{100} is equal to $A^{100}\mathbf{e}_1$, and similarly for the second column. Use (b) to compute $A^{100}\mathbf{e}_i$.)

(d) Using the (very accurate!) approximation $(\lambda_2/\lambda_1)^{100} \approx 0$, give a much simpler approximate expression for A^{100}.

Problem 3: Calculating multiple derivatives

Consider the function

$$f(x, y) = e^{x \sin y}.$$

(a) Find all first and second partial derivatives of f.

(b) Find the gradient vector and Hessian matrix of f at $(1, 0)$.

(c) Find the quadratic approximation to $f(1 + h, k)$ for h and k near 0.

Problem 4: Level sets of quadratic forms

Consider the quadratic form $Q(x, y) = x^2 + 6xy + y^2$.

(a) Find the symmetric 2×2 matrix A for which $\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = Q(x, y)$.

(b) Find the eigenvalues λ_1 and λ_2 of A, and find unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for these respective eigenvalues.

(c) We can use the eigenvalues to express Q when its input is written in a basis of *unit* eigenvectors of A :

$$Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = \lambda_1 x'^2 + \lambda_2 y'^2.$$

Use this to sketch the level curves $Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = \pm 8$ as well as $Q(x'\mathbf{v}_1 + y'\mathbf{v}_2) = 0$ in an $x'y'$-coordinate plane, indicating where each crosses a coordinate axis.

(d) Explain why rotating the "standard basis" onto the orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ carries $x'\mathbf{e}_1 + y'\mathbf{e}_2$ onto $x'\mathbf{v}_1 + y'\mathbf{v}_2$, and carries the x', y' coordinate axes onto the "eigenlines" of A. Sketch the resulting rotation of the picture in (c); why is it the level sets $Q(x, y) = -8, 0, 8$? (This gives a *general technique* to draw level sets of any $Ax^2 + Bxy + Cy^2$.)