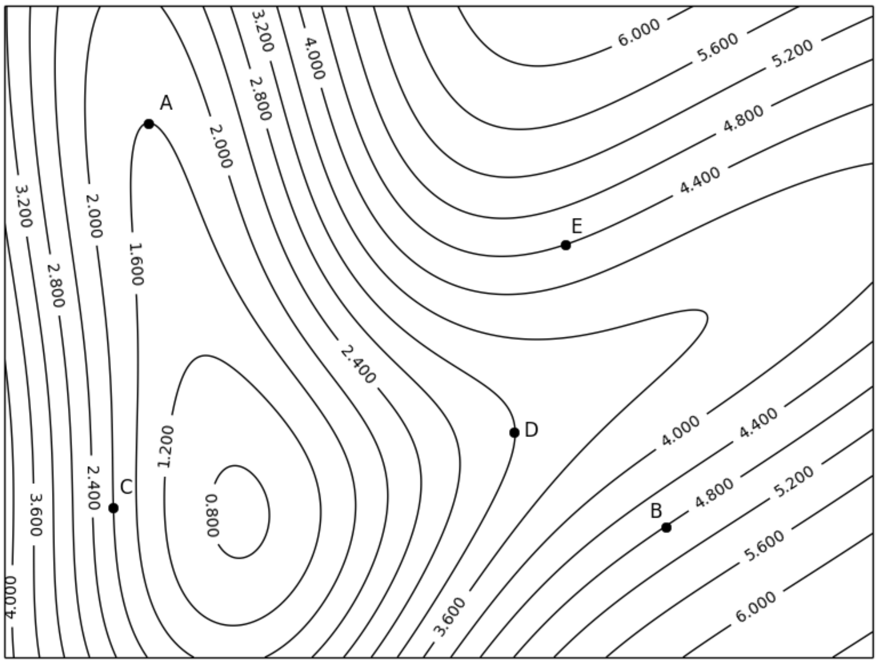
Problem 1: Visually interpreting derivatives

Below is a collection of level sets of a function $f\colon \mathbf{R}^2 \to \mathbf{R}$. (As usual, x is horizontal and y is vertical, and the length scales in the x- and y-directions are equal.)



(a) (Choose one) $\dfrac{\partial f}{\partial y}$ at **A** is: NEGATIVE ZERO $\boxed{\text{POSITIVE}}$

(b) (Choose one) $\dfrac{\partial f}{\partial y}$ at **B** is: $\boxed{\text{NEGATIVE}}$ ZERO POSITIVE

(c) (Choose one) $\dfrac{\partial f}{\partial y}$ at **C** is: NEGATIVE $\boxed{\text{ZERO}}$ POSITIVE

(d) (Choose one) $\dfrac{\partial f}{\partial x}$ at **D** is: NEGATIVE ZERO $\boxed{\text{POSITIVE}}$

(e) Which partial derivative is larger, in *absolute value*? $|f_y(\mathbf{A})|$ $\boxed{|f_y(\mathbf{B})|}$

(f) Which partial derivative is larger, in *absolute value*? $|f_x(\mathbf{E})|$ $\boxed{|f_y(\mathbf{E})|}$

(g) At what point(s) (not necessarily labeled) in the region depicted does f reasonably seem to have a local minimum? a local maximum? What can you say about the value taken by f at each of these points?

A picture to help you visualize contour plots.
This is not part of the problem.

Solution:

(a) Answer: positive.

Moving a small distance in the positive y ("north") direction from **A** changes the value of f from 1.6 (precisely) to some value in the range between 1.6 and 2.0; thus, f increases.

(b) Answer: negative.

Moving a small distance in the positive y ("north") direction from **B** changes the value of f from 4.8 (precisely) to some value in the range between 4.4 and 4.8; thus, f decreases.

(c) Answer: zero.

Moving a small distance in the positive y ("north") direction from **C** is simply moving along (i.e., tangent to) the level set of f at level 2.0; thus, f is (instantaneously) unchanging.

(d) Answer: positive.

Moving a small distance in the positive x ("east") direction from **D** changes the value of f from 3.2 (precisely) to some value in the range between 3.2 and 3.6; thus, f increases.

(e) Answer: $|f_y(\mathbf{B})|$.

Consecutive level curves are vertically spaced much closer together near **B** than near **A**; so the (absolute value of the) vertical rate of change at **B** is greater than at **A**.

(f) Answer: $|f_y(\mathbf{E})|$.

The *vertical* spacing of consecutive level curves near **E** is closer together than the *horizontal* spacing here; so the (absolute value of the) vertical rate of change is greater than the (absolute value of the) horizontal rate of change.

(g) We expect that f has a local extremum at some point *inside* the loop, labeled "0.800," that is located to the right of point **C**. Judging from the decreasing label-values near this point, we conclude this extremum is a local minimum for f, with f-value less than 0.8 (but greater than 0.4, since otherwise there would be an additional loop depicted in the diagram). No other local extrema are apparent in the region of $\mathbf{R}^2$ shown.

Problem 2: Partial derivative practice

Compute the first and second partial derivatives in general, verifying equality of mixed partials directly, and evaluate the first partials at the indicated point $\mathbf{a}$.

(a) $g(x_1, x_2) = \sin(x_1x_2 - x_1 + x_2)$, $\mathbf{a} = (\sqrt{\pi}, \sqrt{\pi})$.

(b) $h(x, y) = e^x(x - y)^2$, $\mathbf{a} = (0, 1)$.

Solution:

(a) $g_{x_1} = (x_2 - 1)\cos(x_1x_2 - x_1 + x_2)$ and $g_{x_2} = (x_1 + 1)\cos(x_1x_2 - x_1 + x_2)$, so

$$g_{x_1}(\sqrt{\pi}, \sqrt{\pi}) = 1 - \sqrt{\pi}, \;\; g_{x_2}(\sqrt{\pi}, \sqrt{\pi}) = -1 - \sqrt{\pi}.$$

Also, $g_{x_1x_1} = -(x_2 - 1)^2 \sin(x_1x_2 - x_1 + x_2)$, $g_{x_2x_2} = -(x_1 + 1)^2 \sin(x_1x_2 - x_1 + x_2)$, and (in both ways) $g_{x_1x_2} = -\cos(x_1x_2 - x_1 + x_2) - (x_2 - 1)(x_1 + 1)\sin(x_1x_2 - x_1 + x_2)$.

(b) $h_x = e^x(x - y)^2 + 2e^x(x - y)$ and $h_y = -2e^x(x - y)$, so

$$h_x(0, 1) = -1, \;\; h_y(0, 1) = 2.$$

Also, $h_{xx} = 2e^x(x-y) + e^x(x-y)^2 + 2e^x + 2e^x(x-y)$, $h_{yy} = 2e^x$, and (in both ways) $h_{xy} = -2e^x(x-y) - 2e^x$.

Problem 3: Finding candidates for local extrema

For each of the following functions $f : \mathbf{R}^2 \to \mathbf{R}$, find all critical points.

(a) $x_1^2 + 4x_1x_2 + 5x_2^2 - 4x_1 + 2x_2$.

(b) $x^4y^4 - 2x^2 - 2y^2$.

(c) $\cos(\pi(x^2 + y^2))$.

(d) $x_1^3 - 3x_1x_2^2 + 3x_2^2$.

Solution:

(a) We have $f_{x_1} = 2x_1 + 4x_2 - 4$ and $f_{x_2} = 4x_1 + 10x_2 + 2$. These simultaneously vanish only at $(12, -5)$

(b) We have $f_x = 4x^3y^4 - 4x$ and $f_y = 4x^4y^3 - 4y$. These simultaneously vanish only at $(0, 0)$, $(\pm 1, \pm 1)$

(c) We have $f_x = -2x\pi \sin(\pi(x^2 + y^2))$ and $f_y = -2y\pi \sin(\pi(x^2 + y^2))$. These simultaneously vanish exactly when $x^2 + y^2$ is an integer since $\sin(\pi t) = 0$ precisely for t an integer, as we see by thinking about the meaning of such vanishing in terms of the unit circle and angles. (The case $x = y = 0$ is treated separately.)

(d) We have $f_{x_1} = 3x_1^2 - 3x_2^2$ and $f_{x_2} = -6x_1x_2 + 6x_2$. These simultaneously vanish only at $(0, 0)$, $(1, \pm 1)$

Problem 4: Computing extrema on a region I

Find the global extreme values of $f(x, y) = 2x^2 + y^2 + 5y$ on the disk of points (x, y) satisfying $x^2 + y^2 \leq 16$.

Solution: By computing the partial derivatives f_x and f_y to see where both vanish at the same point, the function f has one critical point on the interior of the disk: $(x, y) = (0, -5/2)$. On the boundary $x^2 + y^2 = 16$ the function can be expressed entirely in terms of y as

$$2(16 - y^2) + y^2 + 5y, \; -4 \leq y \leq 4$$

which by single-variable calculus has a critical point when its derivative $-4y + 2y + 5$ vanishes, which is to say $y = 5/2$ (this is in the open interval $(-4, 4)$), in which case $x = \pm\sqrt{39}/2$ due to being on the boundary circle $x^2 + y^2 = 16$. Of course the interval's endpoints $y = \pm 4$ also must be considered, corresponding to $x = 0$.

So overall we have five candidates for where extrema are attained: $(0, -5/2)$, $(0, \pm 4)$, $(\pm\sqrt{39}/2, 5/2)$. Evaluating the function and seeing which values are largest and smallest, we get a global maximum $f(\pm\sqrt{39}/2, 5/2) = 153/4 = 38.25$ and global minimum $f(0, -5/2) = -25/4 = -6.25$. (The values of f at $(0, \pm 4)$ are $16 \pm 20 = -4, 36$.)

Problem 5: Computing extrema on a region II

Find the global extreme values of $f(x, y) = x^4y^4 - 2x^2 - 2y^2$ on the region of points (x, y) that lies on or inside the triangle with vertices $(-2, -2)$, $(-2, 2)$, $(2, -2)$. (Sketch this triangle first, to get oriented.)

Solution: The function f has five critical points $(0, 0)$, $(\pm 1, \pm 1)$ (seen by computing the partial derivatives f_x and f_y and determining they simultaneously vanish, which requires a bit of algebra). All but $(1, 1)$ lie in the region (though only $(-1, -1)$ is on the interior). Next, we extremize on the boundary, by substitution on each boundary segment:

(i) On the segment from $(-2, -2)$ to $(-2, 2)$ that is the part of the line $x = -2$ with $-2 \leq y \leq 2$, we substitute $x = -2$ to set up the task:

$$\text{extremize } 16y^4 - 8 - 2y^2 \text{ when } -2 \leq y \leq 2.$$

The derivative is $64y^3 - 4y$, which vanishes for $y = 0, \pm 1/4$. Accounting for the endpoints of this range of y-values, we get candidates $(-2, 0)$, $(-2, \pm 1/4)$, $(-2, -2)$, $(-2, 2)$ on this boundary segment at which an extreme value could be attained.

(ii) On the segment from $(-2, -2)$ to $(2, -2)$, which is the part of the line $y = -2$ with $-2 \leq x \leq 2$, we substitute $y = -2$ to set up the task:

$$\text{extremize } 16x^4 - 2x^2 - 8 \text{ when } -2 \leq x \leq 2$$

The derivative is $64x^3 - 4x$, which vanishes for $x = 0, \pm 1/4$. Accounting for endpoints of this range of x-values, we get the candidates $(0, -2)$, $(\pm 1/4, -2)$, $(-2, -2)$, $(2, -2)$ on this boundary segment at which an extreme value could be attained.

(iii) On the line segment from $(-2, 2)$ to $(2, -2)$, which is the part of the line $y = -x$ with $-2 \leq x \leq 2$, we substitute $y = -x$ to set up the task:

$$\text{extremize } x^8 - 4x^2 \text{ when } -2 \leq x \leq 2$$

The derivative is $8x^7 - 8x$, which vanishes for $x = 0, \pm 1$. Accounting for the endpoints of this range of x-values, we get the candidates of $(0, 0)$, $(1, -1)$, $(-1, 1)$, $(-2, 2)$, $(2, -2)$ on this boundary segment at which an extreme value could be attained.

Comparing values of the function of interest $f(x, y) = x^4y^4 - 2x^2 - 2y^2$ at all candidate points we have obtained, both the interior critical point $(-1, -1)$ and the various boundary points, we find a global minimum value $f\left(\pm\frac{1}{4}, -2\right) = f\left(-2, \pm\frac{1}{4}\right) = -\frac{2055}{256} \approx -8.06$ and a global maximum value $f(2, -2) = f(-2, 2) = f(-2, -2) = 240$.