1. Whatever a, b, c would be, the condition $ax + by + cz$ must at least be satisfied by $\mathbf{v}$ and $\mathbf{w}$, giving the conditions

$$a + b + c = 0, \;\; 4a + 2b + c = 0.$$

Hence, $c = -a - b$ and $c = -4a - 2b$, so $-a - b = -4a - 2b$. Hence, $3a = -b$, so $b = -3a$ and $c = -a - b = -a - (-3a) = 2a$, so $(a, b, c) = (a, -3a, 2a)$. Setting $a = 1$, we're led to the triple

$$(a, b, c) = (1, -3, 2),$$

so let's check that this works. (The same will hold for any other $a \neq 0$, but $a = 1$ makes the calculations cleanest.)

We give two arguments, one geometric and one algebraic. Thinking geometrically, since $\mathbf{v}$ and $\mathbf{w}$ are nonzero and not scalar multiples of each other we know that their span is a plane through the origin, and we are just trying to check that $(1, -3, 2)$ is a normal vector to this plane. It suffices to check it is perpendicular to each of $\mathbf{v}$ and $\mathbf{w}$, and that is a direct calculation with dot products.

To argue algebraically, since $\mathbf{v}$ and $\mathbf{w}$ satisfy $x - 3y + 2z = 0$, so does any linear combination:

$$r\mathbf{v} + s\mathbf{w} = \begin{bmatrix} r \\ r \\ r \end{bmatrix} + \begin{bmatrix} 4s \\ 2s \\ s \end{bmatrix} = \begin{bmatrix} r + 4s \\ r + 2s \\ r + s \end{bmatrix}$$

because $(r + 4s) - 3(r + 2s) + 2(r + s) = r + 4s - 3r - 6s + 2r + 2s = 0$. It remains to show that if $x - 3y + 2z = 0$ then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r\mathbf{v} + s\mathbf{w}$ for some r, s to be found (in terms of x, y, z). This says

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r + 4s \\ r + 2s \\ r + s \end{bmatrix}$$

so we need exactly to find r, s so that

$$r + 4s = x, \;\; r + 2s = y, \;\; r + s = z.$$

Subtracting the third from the second gives $s = y - z$. Plugging this into the first two conditions gives

$$r = x - 4s = x - 4y + 4z, \;\; r = y - 2s = y - 2(y - z) = -y + 2z.$$

These final expressions are the *same* value for r because $x - 4y + 4z = -y + 2z$, or equivalently $x - 3y + 2z = 0$ (by hypothesis on (x, y, z)).

Now let's check that the values $r = -y + 2z$ and $s = y - z$ really work:

$$(-y + 2z) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (y - z) \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -y + 2z \\ -y + 2z \\ -y + 2z \end{bmatrix} + \begin{bmatrix} 4y - 4z \\ 2y - 2z \\ y - z \end{bmatrix} = \begin{bmatrix} 3y - 2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where the final equality uses the given relation $x - 3y + 2z = 0$. ◇

2. (a) We want to write $\mathbf{v}' = a\mathbf{v} + b\mathbf{w}$ and $\mathbf{w}' = c\mathbf{v} + d\mathbf{w}$ for some scalars a, b, c, d. The first of these says $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a + b \\ b \\ a - b \end{bmatrix}$, or equivalently is the system of 3 equations in 2 unknowns

$$3 = a + b, 1 = b, 1 = a - b.$$

The second says $b = 1$, and then both the first and third say $a = 2$, so $\mathbf{v}' = 2\mathbf{v} + \mathbf{w}$ (as is also readily checked directly).

Doing similarly for c, d yields $c = 1, d = -3$, so $\mathbf{w}' = \mathbf{v} - 3\mathbf{w}$ (as is also readily checked directly).

(b) If $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a nonzero vector perpendicular to the plane $\mathrm{span}(\mathbf{v}, \mathbf{w})$, we have that orthogonality to $\mathbf{v}$ and to $\mathbf{w}$ amounts to the conditions

$$a + c = 0, \quad a + b - c = 0.$$

Thus, $c = -a$ and $b = c - a = -2a$. Hence,

$$\mathbf{n} = \begin{bmatrix} a \\ -2a \\ -a \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

(with $a \neq 0$). Setting $a = 1$ for convenience, we obtain the normal vector $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$. Both $\mathbf{v}'$ and $\mathbf{w}'$ are perpendicular to this latter vector:

$$3 - 2 - 1 = 0, \quad -2 - 2(-3) + 4(-1) = -2 + 6 - 4 = 0.$$

◇

3. (a) Suppose $\mathbf{v}_1 = a\mathbf{v}_1 + b\mathbf{v}_2$. This is the system of 3 equations

$$1 = 2a + 3b, \quad -2 = -2b, \quad 3 = a + b.$$

The second forces $b = 1$, so the first and third equations become $1 = 2a + 3$ and $3 = a + 1$. These respectively say $a = -1$ and $a = 2$, which are incompatible, so no such (a, b) exist.

(b) Suppose $\mathbf{v}_2 = a\mathbf{v}_1 + b\mathbf{v}_3$. This is the system of 3 equations

$$2 = a + 3b, \quad 0 = -2a - 2b, \quad 1 = 3a + b.$$

The second equation forces $b = -a$, and plugging this into the other two equations yields $2 = a - 3a = -2a$ and $1 = 3a - a = 2a$. These latter two equations have incompatible solutions ($a = -1$ and $a = 1/2$), so no such (a, b) exists.

Next, suppose $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$. This is the system of 3 equations

$$3 = a + 2b, \quad -2 = -2a, \quad 1 = 3a + b.$$

The second equation forces $a = 1$, and plugging this into the other two equations yields $3 = 1 + 2b$ and $1 = 3 + b$, whose respective solutions are $b = 1$ and $b = -2$, which are incompatible. Hence, once again, no such (a, b) exists.

(c) By (a) and (b), it we remove any $\mathbf{v}_i$ then the span of what remains is smaller than V (since $\mathbf{v}_i$ is not in the span of what remains). Hence, by Theorem 4.2.5 , we have $\dim V = 3$.

◇

4. (a) To show that the span or 1 or 2 nonzero 3-vectors has a nonzero orthogonal vector, it suffices to show that any single nonzero 3-vector $\mathbf{v}$ or pair of nonzero 3-vectors $\mathbf{v}, \mathbf{w}$ has a nonzero orthogonal vector $\mathbf{n}$ (since then anything in their span is also orthogonal to $\mathbf{n}$, since being orthogonal to $\mathbf{n}$ is preserved under passage to linear combinations, due to the good behavior of dot products: $(c\mathbf{v}) \cdot \mathbf{n} = c(\mathbf{v} \cdot \mathbf{n})$ and $(a\mathbf{v} + b\mathbf{w}) \cdot \mathbf{n} = a(\mathbf{v} \cdot \mathbf{n}) + b(\mathbf{w} \cdot \mathbf{n})$).

Arguing geometrically: we can see that any line has lots of perpendicular lines in $\mathbf{R}^3$, and likewise for any plane.

Arguing algebraically: if $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is nonzero then an orthogonal vector is any $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying $ax + by + cz = 0$. At least one of x, y, z is nonzero. If $z \neq 0$ then we can use $b = 1$, let a be whatever we like (such as 0, $\sqrt{2}$, or whatever), and $c = (-ax - y)/z$. If instead $x \neq 0$ or $y \neq 0$ then we can do similarly. That shows algebraically that the span of a nonzero vector has a nonzero orthogonal vector. Likewise, for the span of 2 vectors we can also argue algebraically that there is a nonzero orthogonal vector, since the span is either a line (in which case we have already settled that) or it is a plane (in which case our method of parametrizing planes through the origin yields a nonzero normal vector).

(b) The conditions on $\mathbf{n}$ amount to the simultaneous conditions

$$a - 2b + 3c = 0, \;\; 2a + c = 0, \;\; 3a - 2b + c = 0.$$

The second condition gives that $c = -2a$, so plugging this into the first and third conditions yields

$$0 = a - 2b + 3(-2a) = -5a - 2b, \;\; 0 = 3a - 2b - 2a = a - 2b,$$

so $a = (2/5)b$ and $a = 2b$. These latter two conditions force $a, b = 0$, so also $c = 0$.

(c) If the dimension of the span V of the three given nonzero 3-vectors isn't equal to 3 then it must be 1 or 2. But then by (a) there would be a *nonzero* 3-vector $\mathbf{n}$ orthogonal to V, and by (b) no such $\mathbf{n}$ exists.

$\diamond$

5. (a) The nonzero vector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, so these two constitute an orthogonal basis of $\mathbf{R}^2$. Both of these have length equal to 5. Dividing each by its length gives an orthonormal basis:

$$\left\{ \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} \right\}.$$

(b) We need to find a nonzero vector perpendicular to $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, and then another nonzero vector that is perpendicular to both of those. A nonzero vector perpendicular to $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ is any nonzero $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $2x + 2y + z = 0$. There are lots of such options; let's work with $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. To find a nonzero vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ perpendicular to both of these amounts to finding a solution $(a, b, c) \neq (0, 0, 0)$ to the system of equations

$$\begin{aligned} 2a + 2b + c &= 0 \\ -a + b &= 0 \end{aligned}$$

In other words, we have $b = a$ and $c = -2a - 2b = -4a$. The cleanest choice is $a = 1$ (though there is nothing actually special about this), giving $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$. This yields the orthogonal basis

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} \right\}.$$

Dividing each by its length gives an orthonormal basis

$$\left\{ \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{18} \\ 1/\sqrt{18} \\ -4/\sqrt{18} \end{bmatrix} \right\}$$

(and $\sqrt{18} = 3\sqrt{2}$, but that doesn't matter).

(c) The answer is "no" because the two given vectors are not perpendicular: their dot product is 1.

$\diamond$

6. Let's begin with the vector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (there is nothing special about this choice; there are many possible answers). For $\mathbf{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, orthogonality to $\mathbf{v}_1$ says $x + y + z = 0$, so we want $z = -x - y$ with $x, y \neq 0$. For example, we can

use $x = 1$, $y = 2$ and $z = -3$ (many other options are possible). That is, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$. Finally, for $\mathbf{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ the orthogonality to $\mathbf{v}_1$ and $\mathbf{v}_2$ says

$$a + b + c = 0, \quad a + 2b - 3c = 0$$

with $a, b, c \neq 0$. The first condition says $c = -a - b$, and plugging this into the second yields $0 = a + 2b - 3c = a + 2b - 3(-a - b) = 4a + 5b$, so $b = -(4/5)a$. Hence, $c = -a - b = -a$, so $\mathbf{v}_3 = \begin{bmatrix} a \\ -(4/5)a \\ -a \end{bmatrix}$ with $a \neq 0$. The cleanest choice is with $a = 5$, giving $\mathbf{v}_3 = \begin{bmatrix} 5 \\ -4 \\ -1 \end{bmatrix}$.

Since these explicit 3-vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are nonzero and pairwise orthogonal, they constitute an orthogonal basis of $\mathbf{R}^3$ $\diamond$

7. (a) We can use the equations to solve for 2 of the variables in terms of the others. For instance, $x_2 = -x_1$ and $x_4 = -x_1 - x_3$, so the vectors in W are exactly those of the form

$$\begin{bmatrix} x_1 \\ -x_1 \\ x_3 \\ -x_1 - x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ -x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

This exhibits W as precisely the span of 2 vectors.

(b) Let's use $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$. For $\mathbf{w}_2 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, there are 3 conditions: membership in W (i.e., orthogonality to $\mathbf{v}$ and $\mathbf{v}'$) amounts to 2 equations on a, b, c, d and then orthogonality to $\mathbf{w}_1$ is a third equation. In the solution to (a) we already worked out the effect of the first two equations, namely that $\mathbf{w}_2$ must have the form

$$\mathbf{w}_2 = \begin{bmatrix} a \\ -a \\ c \\ -a - c \end{bmatrix},$$

so then orthogonality of this to $\mathbf{w}_1$ is the condition

$$0 = \mathbf{w}_1 \cdot \mathbf{w}_2 = a - (-a) - (-a - c) = 3a + c,$$

which is to say $c = -3a$. Hence,

$$\mathbf{w}_2 = \begin{bmatrix} a \\ -a \\ -3a \\ -a + 3a \end{bmatrix} = \begin{bmatrix} a \\ -a \\ -3a \\ 2a \end{bmatrix}$$

with $a \neq 0$. The cleanest choice is $a = 1$ (though any nonzero a works just as well), yielding

$$\mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \\ -3 \\ 2 \end{bmatrix}.$$

As a safety-check, one can compute directly that the nonzero $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal to each other and belong to W.

$\diamond$

8. (a) We calculate $\mathbf{b}_1 \cdot \mathbf{b}_1 = 30$, $\mathbf{b}_2 \cdot \mathbf{b}_2 = 5$, $\mathbf{b}_1 \cdot \mathbf{v} = 30$, and $\mathbf{b}_2 \cdot \mathbf{v} = 20$, so

$$\mathbf{v}' = \mathbf{b}_1 + 4\mathbf{b}_2 = \begin{bmatrix} 9 \\ 5 \\ -2 \end{bmatrix}.$$

(b) By inspection $\mathbf{v}' \neq \mathbf{v}$. The reason $\mathbf{v}$ cannot lie in the plane $\mathcal{P}$ is because if it were in this plane, so it is in the span of the orthogonal collection of nonzero vectors $\{\mathbf{b}_1, \mathbf{b}_2\}$, then by Theorem 5.3.6 it would be equal to the expression defining $\mathbf{v}'$. But we observed that $\mathbf{v}' \neq \mathbf{v}$, so this is not possible.

$\diamond$

9. (a) By computation, $\mathbf{v} \cdot \mathbf{w}' = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} = 12 + 0 - 12 = 0$. Also, $a\mathbf{v} + b\mathbf{w} = a\mathbf{v} + b(\mathbf{w}' - \mathbf{v}) = a\mathbf{v} + b\mathbf{w}' - b\mathbf{v} = (a-b)\mathbf{v} + b\mathbf{w}'$, so the vectors $\mathbf{v}, \mathbf{w}' \in \mathcal{P}$ so span $\mathcal{P}$. Hence, they constitute an orthogonal basis of $\mathcal{P}$.

(b) The shortest distance from $\mathbf{x}$ to $\mathcal{P}$ is the length of the vector between $\mathbf{x}$ and the closest point on $\mathcal{P}$ to $\mathbf{x}$. But the latter is also known as $\mathbf{Proj}_{\mathcal{P}}(\mathbf{x})$. So the length we are looking for is $\|\mathbf{x} - \mathbf{Proj}_{\mathcal{P}}(\mathbf{x})\|$. Using our orthonormal basis $\{\mathbf{v}, \mathbf{w}'\}$ for $\mathcal{P}$, we have

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = \mathbf{Proj}_{\mathbf{v}}(\mathbf{x}) + \mathbf{Proj}_{\mathbf{w}'}(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} + \frac{\mathbf{x} \cdot \mathbf{w}'}{\mathbf{w}' \cdot \mathbf{w}'}\mathbf{w}'$$

with

$$\mathbf{x} \cdot \mathbf{v} = -40 + 15 = -25, \quad \mathbf{v} \cdot \mathbf{v} = 25, \quad \mathbf{x} \cdot \mathbf{w}' = -30 + 24 - 20 = -26, \quad \mathbf{w}' \cdot \mathbf{w}' = 26,$$

so

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = \frac{-25}{25}\mathbf{v} + \frac{-26}{26}\mathbf{w}' = -\mathbf{v} - \mathbf{w}' = -\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, $\mathbf{x} - \mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = \begin{bmatrix} -10 - (-7) \\ -24 - 1 \\ 5 - 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -25 \\ 4 \end{bmatrix}$, whose length is $\sqrt{9 + 625 + 16} = \sqrt{650}$.

$\diamond$

10. (a) For the first assertion, by Theorem 5.2.2 , it's enough to show that the four given vectors are pairwise orthogonal. Because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthonormal basis for V, they are pairwise orthogonal. So we just need to check that $\begin{bmatrix} 2 \\ -4 \\ 1 \\ 2 \end{bmatrix}$ is orthogonal to each of these three. In fact, $\begin{bmatrix} 2 \\ -4 \\ 1 \\ 2 \end{bmatrix}$ is perpendicular to *any* vector in V: if $\mathbf{u} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \in V$ then the equation defining V is exactly the condition that $\mathbf{u} \cdot \begin{bmatrix} 2 \\ -4 \\ 1 \\ 2 \end{bmatrix} = 0$. By Theorem 6.2.1 ,

$$\mathbf{Proj}_V(\mathbf{u}) = \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{u}) + \mathbf{Proj}_{\mathbf{v}_2}(\mathbf{u}) + \mathbf{Proj}_{\mathbf{v}_3}(\mathbf{u}).$$

On the other hand,

$$\mathbf{u} = \mathbf{Proj}_{\mathbf{R}_4}(\mathbf{u}) = \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{u}) + \mathbf{Proj}_{\mathbf{v}_2}(\mathbf{u}) + \mathbf{Proj}_{\mathbf{v}_3}(\mathbf{u}) + \mathbf{Proj}_{\mathbf{v}_4}(\mathbf{u}).$$

Subtracting these two,

$$\mathbf{Proj}_V(\mathbf{u}) = \mathbf{u} - \mathbf{Proj}_{\mathbf{v}_4}(\mathbf{u}).$$

(b) Using the formula from (a),

$$\mathbf{Proj}_V(\mathbf{u}) = \mathbf{u} - \left(\frac{2w - 4x + y + 2z}{4 + 16 + 1 + 4}\right)\mathbf{v}_4 = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} - \frac{2w - 4x + y + 2z}{25}\begin{bmatrix} 2 \\ -4 \\ 1 \\ 2 \end{bmatrix}.$$

(c) We plug into the formula from (b) for vector $\mathbf{u} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 4 \end{bmatrix}$, which gives us

$$\mathbf{Proj}_V(\mathbf{u}) = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 4 \end{bmatrix} - \frac{2 + 8 + 2 + 8}{25}\begin{bmatrix} 2 \\ -4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 6/5 \\ 6/5 \\ 12/5 \end{bmatrix}.$$

$\diamond$

11. (a) To show that necessarily $c_1 = 0$ and $c_2 = 0$, we have to rule out the case $c_1 \neq 0$ and the case $c_2 \neq 0$. If $c_1 \neq 0$ then we can divide by c_1 (or equivalently, multiply by $1/c_1$) to get $\mathbf{v}_1 + (c_2/c_1)\mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_1 = -(c_2/c_1)\mathbf{v}_2$. But this is impossible since $\mathbf{v}_1$ isn't a scalar multiple of $\mathbf{v}_2$.

    Likewise, if $c_2 \neq 0$ then we can divide by c_2 (or equivalently, multiply by $1/c_2$) to get $(c_1/c_2)\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_2 = -(c_1/c_2)\mathbf{v}_1$. But this is impossible since $\mathbf{v}_2$ isn't a scalar multiple of $\mathbf{v}_1$.

    (b) If $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ for some scalar coefficients then by subtracting the right side from the left side we get

$$\mathbf{0} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 - b_1\mathbf{v}_1 - b_2\mathbf{v}_2 = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2.$$

    By (a), this can only happen when both coefficients vanish, which is to say $a_1 - b_1 = 0$ and $a_2 - b_2 = 0$, which is exactly the same as saying $a_1 = b_1$ and $a_2 = b_2$.

$\diamond$

12. (a) We expand out the right side and then subtract the left side:

$$(a + 3t)\mathbf{v}_1 + (b - 5t)\mathbf{v}_2 = a\mathbf{v}_1 + 3t\mathbf{v}_1 + b\mathbf{v}_2 - 5t\mathbf{v}_2 = (a\mathbf{v}_1 + b\mathbf{v}_2) + t(3\mathbf{v}_1 - 5\mathbf{v}_2),$$

    so subtracting the left side from this leaves us with $t(3\mathbf{v}_1 - 5\mathbf{v}_2)$. But

$$3\mathbf{v}_1 - 5\mathbf{v}_2 = 3((5/3)\mathbf{v}_2) - 5\mathbf{v}_2 = 5\mathbf{v}_2 - 5\mathbf{v}_2 = \mathbf{0},$$

    so the t-part vanishes.

    (b) Again we expand out the right side and then subtract the left side:

$$\begin{aligned}(a - 3t)\mathbf{w}_1 + (b + 8t)\mathbf{w}_2 + (c + 4t)\mathbf{w}_3 &= a\mathbf{w}_1 - 3t\mathbf{w}_1 + b\mathbf{w}_2 + 8t\mathbf{w}_2 + c\mathbf{w}_3 + 4t\mathbf{w}_3 \\ &= (a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3) + t(-3\mathbf{w}_1 + 8\mathbf{w}_2 + 4\mathbf{w}_3),\end{aligned}$$

    so subtracting the left side from this leaves us with $t(-3\mathbf{w}_1 + 8\mathbf{w}_2 + 4\mathbf{w}_3)$. But

$$-3\mathbf{w}_1 + 8\mathbf{w}_2 + 4\mathbf{w}_3 = -3\mathbf{w}_1 + 8\mathbf{w}_2 + 4((3/4)\mathbf{w}_1 - 2\mathbf{w}_2) = -3\mathbf{w}_1 + 8\mathbf{w}_2 + 3\mathbf{w}_1 - 8\mathbf{w}_2 = \mathbf{0},$$

    so the t-part vanishes.

$\diamond$

