About this document:

The Math Department regularly seeks out feedback on the Math 50-series, and indeed the addition of "companion" materials is something that we have been working on. The end-of-chapter summary pages is something we added last year, and the Panopto video series on "How to read a mathematics text" was recently developed as part of an ongoing effort to develop such additional resources.

The creation of **additional practice homework exercises** *is also underway, and with this document we are sharing what has been produced. There are two file versions, one which contains statements of "practice HW exercises," and the other which contains problems together with solutions.*

Please note that this is a work-in-progress; the department has prioritized certain chapters over others during the pandemic when time has been short for everyone. The absence of problems for some chapters shouldn't be interpreted as indicating less value for the content of those chapters (more that the skills therein are covered enough by other chapters that they were a lower priority for their own "extra problems").

Feedback is always appreciated — please contact your instructor or administrative TA.

1. Vectors, vector addition, and scalar multiplication

2. Vector geometry in $\mathbb{R}^n$ and correlation coefficients

Practice 2.1 (Angle bisector)**.** In this exercise, we relate angle bisectors to vector geometry concepts: length, vector addition, dot product, and angles. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero n-vectors that are not scalar multiples of each other, so $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are visualized as the diagonals of the parallelogram P with edges along $\mathbf{v}$ and $\mathbf{w}$, as shown for $n=3$ in Figure 2.1 below.

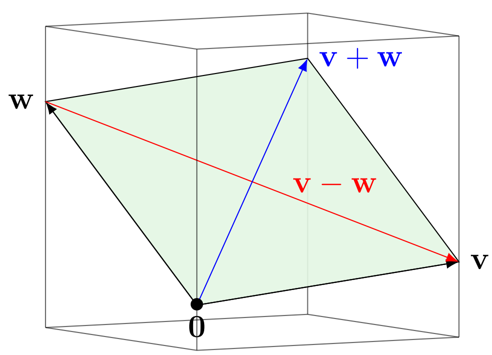


FIGURE 2.1. The parallelogram P with a vertex at $\mathbf{0}$ and edges along $\mathbf{v}$ and $\mathbf{w}$: its points are $s\mathbf{v}+t\mathbf{w}$ for $0 \leq s,t \leq 1$.

(a) Assume $\mathbf{v}$ and $\mathbf{w}$ have the same length (i.e., $\|\mathbf{v}\| = \|\mathbf{w}\|$), so P is a rhombus. Use algebra with dot products to show $\mathbf{v}+\mathbf{w}$ is orthogonal to $\mathbf{v}-\mathbf{w}$. In words: the diagonals of a rhombus are perpendicular to each other.

(b) Continue to assume $\mathbf{v}$ and $\mathbf{w}$ have the same length. Let θ_1 be the angle between $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}$, and let θ_2 be the angle between $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}$ (so $0^\circ < \theta_1, \theta_2 < 180^\circ$). Show $\theta_1 = \theta_2$. Hint: show $\cos\theta_1 = \cos\theta_2$.

In words: the diagonal $\mathbf{v}+\mathbf{w}$ through the vertex where the edges along $\mathbf{v}$ and $\mathbf{w}$ meet bisects the angle at that vertex (as it makes the same angle $\theta_1 = \theta_2$ with each of $\mathbf{v}$ and $\mathbf{w}$).

Practice 2.2 (Triangle incenter)**.** It is a remarkable fact of plane geometry that the angle bisectors of all three vertices of a triangle meet at a common point (called the *incenter* of the triangle). This exercise works out a specific example, using the parametric form of lines in $\mathbf{R}^2$. Concepts you'll learn later in the course ("linear transformation") allow one to establish the result for general triangles.

Consider the triangle in $\mathbf{R}^2$ with vertices $P = \mathbf{v} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}$, $Q = \mathbf{w} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$, and $R = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(a) Use Exercise 2.1(b) to find a vector bisecting the angle $\angle PRQ$ at $R = \mathbf{0}$ between $\mathbf{v}$ and $\mathbf{w}$. Then, after scaling that vector to make the entries as simple as you can, find a parametric form for the angle bisector line through R. (Hint: $\mathbf{v}$ and $\mathbf{w}$ has different lengths, so to apply Exercise 2.1(b) consider the unit vectors $\mathbf{v}' = \mathbf{v}/\|\mathbf{v}\|$ and $\mathbf{w}' = \mathbf{w}/\|\mathbf{w}\|$.

(b) Similarly to part (a), find a parametric form for the line through P bisecting the angle $\angle RPQ$. (Hint: first work out the displacement vectors from each of R and Q to the vertex P of the angle, making sure to do subtraction in the correct order, to get vectors along those edges to use in roles analogous to $\mathbf{v}$ and $\mathbf{w}$ in part (a).) Make sure that your parametric form passes through P!

(c) Use the parametric forms of the bisector lines described in parts (a) and (b) to find the point where these lines meet. (This point has integer coordinates.)

(d) Adapting the procedure used in part (b), find a parametric form for the line through Q bisecting the angle $\angle PQR$ and check that it passes through the point that you found in (c) (so all three angle bisector lines pass through this common point).

3. Planes in $\mathbf{R}^3$

Practice 3.1. Consider the three points

$$P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad R = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

(a) Find a parametric form for the plane passing through these three points.

(b) Find an equational form $ax + by + cz = d$ for the plane through these three points .

(c) Find a parametric form for the plane defined by the equation $3x - y + 5z = 4$.

Practice 3.2. Let P_1 be the plane in $\mathbf{R}^3$ described by the equation $x - 2y + z = 0$. The following parts of this exercise can be worked on independently of each other.

(a) Consider the plane P_2 with the parametric form $t\mathbf{v} + s\mathbf{w}$ for $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$. Which of the following is true? Justify your answer.

(i) $P_1 = P_2$; that is, P_2 is another description for the plane P_1.

(ii) P_1 and P_2 do not have any points in common (that is, they are parallel and distinct).

(iii) P_1 and P_2 are not the same plane, but they do meet each other (that is, P_1 and P_2 have at least one point in common).

(b) Let P_3 be the plane described by the parametric form $\begin{bmatrix} 1+3s \\ 3+2s+t \\ 1+s+2t \end{bmatrix}$. Which of the following is true? Justify your answer.

(i) $P_1 = P_3$; that is, P_3 is another description for the plane P_1.

(ii) P_1 and P_3 do not have any points in common (that is, they are parallel and distinct).

(iii) P_1 and P_3 are not the same plane, but they do meet each other (that is, P_1 and P_3 have at least one point in common).

(c) Specify (with justification) a plane P_4 in $\mathbf{R}^3$ which meets P_1 exactly along the line

$$\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

You are free to specify P_4 *either* by an equation or by a parametric form, whichever is more convenient for you.

Practice 3.3. Suppose P is the plane in $\mathbf{R}^3$ given by the equation $2x - 2y + z = 1$. Let A and C be the following points:

$$A = (3, -1, 2), \qquad C = (2, -9, -3).$$

(a) Find a nonzero vector $\mathbf{n}$ that is normal to P.

(b) Determine whether A and C lie on the same side, or on opposite sides, of the plane P.

(c) Find the point B on P for which the displacement vector $\overrightarrow{BA}$ is normal to the plane P.

(d) With B as in part (c), determine whether the angle between $\overrightarrow{BC}$ and $\overrightarrow{BA}$ is acute, right, or obtuse.

4. Span, subspaces, and dimension

Practice 4.1. The parts of this exercise can be worked on independently of each other.

(a) Determine, with justification, the dimension (1, 2, or 3) of the linear subspace $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ of $\mathbf{R}^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

(b) Determine, with justification, the dimension (1, 2, or 3) of the linear subspace $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ of $\mathbf{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Practice 4.2. Let $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -5 \end{bmatrix}$. Let V be the collection of all 4-vectors $\mathbf{v}$ that are orthogonal to *both* $\mathbf{w}_1$ and $\mathbf{w}_2$. Express V as the span of finitely many vectors.

5. Basis and orthogonality

6. Projections

Practice 6.1 (projection onto unit vectors)**.** The projection

$$\mathbf{Proj}_{\mathbf{w}}\,\mathbf{x} = \left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w}$$

is insensitive to replacing $\mathbf{w}$ with a nonzero scalar multiple $c\mathbf{w}$, and this can become a bit easier to work with when $\mathbf{w}$ is a unit vector. This exercise explores such use of unit vectors.

(a) Check that when $\mathbf{w}$ is a unit vector, we have the formula $\mathbf{Proj}_{\mathbf{w}}\,\mathbf{x} = (\mathbf{x}\cdot\mathbf{w})\mathbf{w}$. Express the length of this projection in terms of $\mathbf{x}\cdot\mathbf{w}$ (hint: if c is a scalar and $\mathbf{w}$ is a unit vector, what is the length of $c\mathbf{w}$ in terms of c?).

(b) For a numerical example, find the projection of the vector $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ onto the line spanned by $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, by directly applying the projection formula.

(c) Solve part (b) by instead computing the unit vector $\mathbf{w}' = \mathbf{w}/\|\mathbf{w}\|$ and then applying part (a) with that unit vector, using that the projection $\mathbf{Proj}_{\mathbf{w}}$ is the same as the projection $\mathbf{Proj}_{\mathbf{w}'}$ You should get the same answer as in (b).

Remark. The solution to (c) will demonstrate the pitfall of passing to the unit vector $\mathbf{w}'$ in place of $\mathbf{w}$: it usually requires working with square roots (which ultimately cancel out in the end, if the original vectors $\mathbf{x}$ and $\mathbf{w}$ have entries which are integers or rational numbers).

For theoretical considerations the intervention of such square roots doesn't matter, but if doing numerical work then it can be more pleasant not to get bogged down with the square roots. That is the reason that this course emphasizes the general formula for $\mathbf{Proj}_{\mathbf{w}}$ using dot products rather than requiring $\mathbf{w}$ to be a unit vector (as is done is many textbooks).

Practice 6.2. Let $\mathbf{u} = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$ be vectors in $\mathbf{R}^4$, and let $V = \operatorname{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ be their span.

(a) By calculating various dot products, verify that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ forms an orthogonal basis for V.

(b) Find the point on V that is closest to the point $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$. (The answer can be written as $1/3$ times a 4-vector whose entries are integers.)

Practice 6.3. Let $\mathcal{P}$ be the plane in $\mathbf{R}^3$ through $\mathbf{0}$ spanned by $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$. In this exercise we will calculate the distance from the point $\mathbf{x} = \begin{bmatrix} 7 \\ -1 \\ 12 \end{bmatrix}$ to $\mathcal{P}$, and find the point in $\mathcal{P}$ that's closest to $\mathbf{x}$.

(a) Find a (nonzero) normal vector $\mathbf{u}$ to the plane $\mathcal{P}$.

(b) Find the projection $\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$. (Your answer should be a vector whose entries are integers.)

(c) Using the general fact that $\mathbf{x} = \mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) + \mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$, explain why the length of $\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$ equals the distance from $\mathbf{x}$ to the plane $\mathcal{P}$, and compute this distance. (Your answer should be an integer.)

(d) Find the point on $\mathcal{P}$ that is closest to $\mathbf{x}$. (Hint: use the formula in part (c); it is *not* necessary to compute an orthogonal basis of $\mathcal{P}$!)

Practice 6.4. Let L be the line $y = \sqrt{3}\,x$ in $\mathbf{R}^2$ through the origin.

(a) Sketch the line L, the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and their projections $\mathbf{Proj}_L(\mathbf{e}_1)$ and $\mathbf{Proj}_L(\mathbf{e}_2)$ to L. With the help of your picture and your knowledge of trigonometry, fill in the following blanks: the angle between line L and x-axis is _____, so the length of $\mathbf{Proj}_L(\mathbf{e}_1)$ equals _____ and the length of $\mathbf{Proj}_L(\mathbf{e}_2)$ equals _______.

(b) Pick a nonzero vector $\mathbf{w}$ in L. Using the projection formula $\mathbf{Proj}_{\mathbf{w}}(\mathbf{x}) = ((\mathbf{x} \cdot \mathbf{w})/(\mathbf{w} \cdot \mathbf{w}))\mathbf{w}$, calculate $\mathbf{Proj}_{\mathbf{w}}(\mathbf{e}_1)$ and $\mathbf{Proj}_{\mathbf{w}}(\mathbf{e}_2)$. Check that their lengths agree with your answers in (a).

(c) Using your answer in (b), calculate the projection of any point $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ onto the line L. (Your answer should be a 2-vector whose entries depend on x and y.) Check that for $\mathbf{v} = \mathbf{e}_1$ and $\mathbf{v} = \mathbf{e}_2$, this recovers what you found in (b).

Practice 6.5. In this exercise we find the projection of any vector to coordinate axes and coordinate planes. The answers provide concrete instances of the general projection formula; you should think about their reasonableness.

(a) Find the projection of a 2-vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ to: (1) the x-axis, (2) the y-axis.

(b) Find the projection of a 3-vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to: (1) the xy-plane, (2) the yz-plane, (3) the xz-plane.

(c) Find the projection of a 4-vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ to: (1) the xy-plane (i.e., $z = 0, w = 0$), (2) the yw plane (i.e., $x = 0, z = 0$).

7. Applications of projections in R^n: orthogonal bases of planes and linear regression

Practice 7.1. Let $\mathbf{u} = \begin{bmatrix} 11 \\ 0 \\ 15 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -5 \\ 4 \\ -3 \\ 0 \end{bmatrix}$, and consider the plane $\mathcal{P} = \mathrm{span}(\mathbf{u}, \mathbf{v})$. Let $\mathbf{x} = \begin{bmatrix} -3 \\ 8 \\ -1 \\ -3 \end{bmatrix}$.

(a) Find the vector $\mathbf{u}' = \mathbf{u} - t\mathbf{v}$ for a scalar t so that the pair of vectors $\{\mathbf{u}', \mathbf{v}\}$ is an orthogonal basis of $\mathcal{P}$. (The answer is a vector with integer entries, related to the history of Stanford University.)

(b) Using the Projection Formula, compute the scalars a and b fo which $\mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = a\mathbf{u}' + b\mathbf{v}$.

(c) Which linear combination of the original $\mathbf{u}$ and $\mathbf{v}$ is closest to the point $\mathbf{x}$? More precisely, using (a) and (b), find the scalars c and d for which $\mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = c\mathbf{u} + d\mathbf{v}$.

Practice 7.2. Let V be a plane in $\mathbf{R}^4$ whose parametric form is

$$V = \left\{ \begin{bmatrix} s + 2t \\ 2s - 5t \\ s - 4t \\ 2s - 9t \end{bmatrix} : s, t \in \mathbf{R} \right\}.$$

We seek the scalars s and t for which the corresponding point in V is closest to $\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 0 \end{bmatrix}$. The following steps carry out this computation.

(a) Let $\mathbf{v}_1$ be the vector in V with $s = 1, t = 0$, and let $\mathbf{v}_2$ be the vector in V with $s = 0, t = 1$. Compute $\mathbf{v}_1$ and $\mathbf{v}_2$ explicitly, and then find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2'\}$ for V, where one of the basis vectors is $\mathbf{v}_1$.

(b) Find scalars a and b for which $\mathbf{Proj}_V(\mathbf{x}) = a\mathbf{v}_1 + b\mathbf{v}_2'$, and compute this projection explicitly as a 4-vector.

(c) Find scalars s and t for which $\mathbf{Proj}_V(\mathbf{x}) = s\mathbf{v}_1 + t\mathbf{v}_2$.

Practice 7.3. Consider the 5 data points $(x_1, y_1), \cdots, (x_5, y_5)$ given by

$$(-2,-1), (0,-2), (1,-3), (2,-4), (4,-5).$$

We wish to find a line of best fit of the form $y = mx + b$, minimizing the sum of squares of $y_i - (mx_i + b)$. The task of finding m and b for the best fit line in $\mathbf{R}^2$ amounts to finding the linear combination $m\mathbf{X} + b\mathbf{1} \in \mathbf{R}^5$ closest to $\mathbf{Y}$, where (as usual) $\mathbf{1}$ denotes the 5-vector with all entries equal to 1.

(a) Explicitly write down the 5-vectors $\mathbf{X}$ and $\mathbf{Y}$.

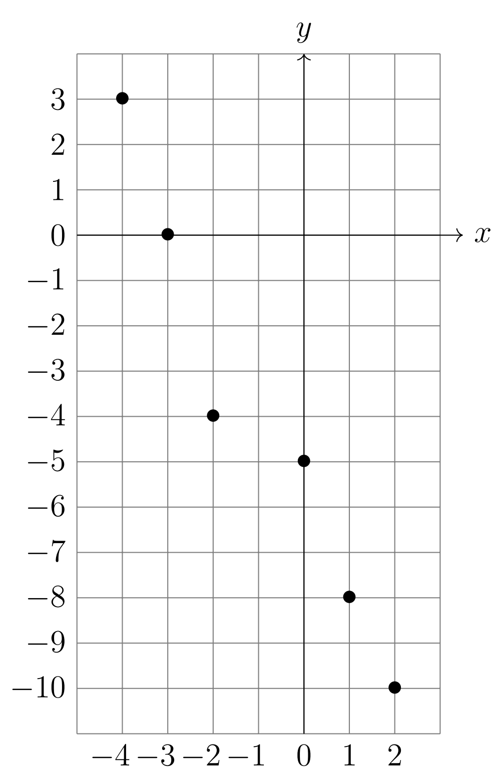
(b) The 5-vectors $\mathbf{X}$ and $\mathbf{1}$ span a plane P in $\mathbf{R}^5$. Find an orthogonal basis of this plane having the form $\{\mathbf{w}, \mathbf{1}\}$, and then express $\mathbf{Proj}_P(\mathbf{Y})$ as a linear combination of $\mathbf{w}$ and $\mathbf{1}$ (i.e., fill in the blanks: $\mathbf{Proj}_P(\mathbf{Y}) = \underline{\quad\quad}\mathbf{w} + \underline{\quad\quad}\mathbf{1}$).

(c) Fill in the blanks: $\mathbf{Proj}_P(\mathbf{Y}) = \underline{\quad\quad}\mathbf{X} + \underline{\quad\quad}\mathbf{1}$. (Hint: in part (b) you should have written $\mathbf{w}$ as a linear combination of $\mathbf{X}$ and $\mathbf{1}$.) Use these coefficients to write down the line $y = mx + b$ of best fit for the 5 given data points.

Practice 7.4. Consider the 5 data points $(1,7)$, $(2,5)$, $(4,-1)$, $(5,-5)$, $(8,-6)$. Using linear algebra in $\mathbf{R}^5$, compute the equation $y = mx + b$ of the line of best fit for these points. (The values you find for m and b should be integers.)

Practice 7.5. Consider the 6 data points $(x_1, y_1), \ldots, (x_6, y_6)$ given as follows:

$$(-4,3),\ (-3,0),\ (-2,-4),\ (0,-5),\ (1,-8),\ (2,-10).$$



Suppose the line of best fit (in the least-squares sense) is written as $y = mx + b$.

(a) Based on the picture, is the correlation coefficient r closest to $-1, -1/2, 0, 1/2$, or 1?

(b) Write down explicit 6-vectors $\mathbf{X}$ and $\mathbf{Y}$ so that for the 6-vector $\mathbf{1}$ whose entries are all equal to 1, the projection of $\mathbf{Y}$ into the plane $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ in $\mathbf{R}^6$ is $m\mathbf{X} + b\mathbf{1}$. (You are just being asked to write down such $\mathbf{X}$ and $\mathbf{Y}$, nothing more.)

(c) Find a 6-vector $\mathbf{v}$ so that $\{\mathbf{1}, \mathbf{v}\}$ is an orthogonal basis for $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$.

(d) Compute the projection $\mathbf{Proj}_V(\mathbf{Y})$ and express your answer as a linear combination $s\mathbf{v} + t\mathbf{1}$ for scalars s and t that you should determine. (The values of s and t are integers.)

(e) Use your answer to (d) to express $\mathbf{Proj}_V(\mathbf{Y})$ as a linear combination of $\mathbf{X}$ and $\mathbf{1}$, and obtain from this the values of m and b.

8. Multivariable functions, level sets, and contour plots

Practice 8.1. For the two functions

$$g(x, y) = (x^2,\ xy) \quad \text{and} \quad h(u, v) = (u + v,\ u - v,\ \sin(uv)),$$

which of the following compositions make sense? If a composition makes sense, explicitly compute its output as an expression in terms of its input variables.

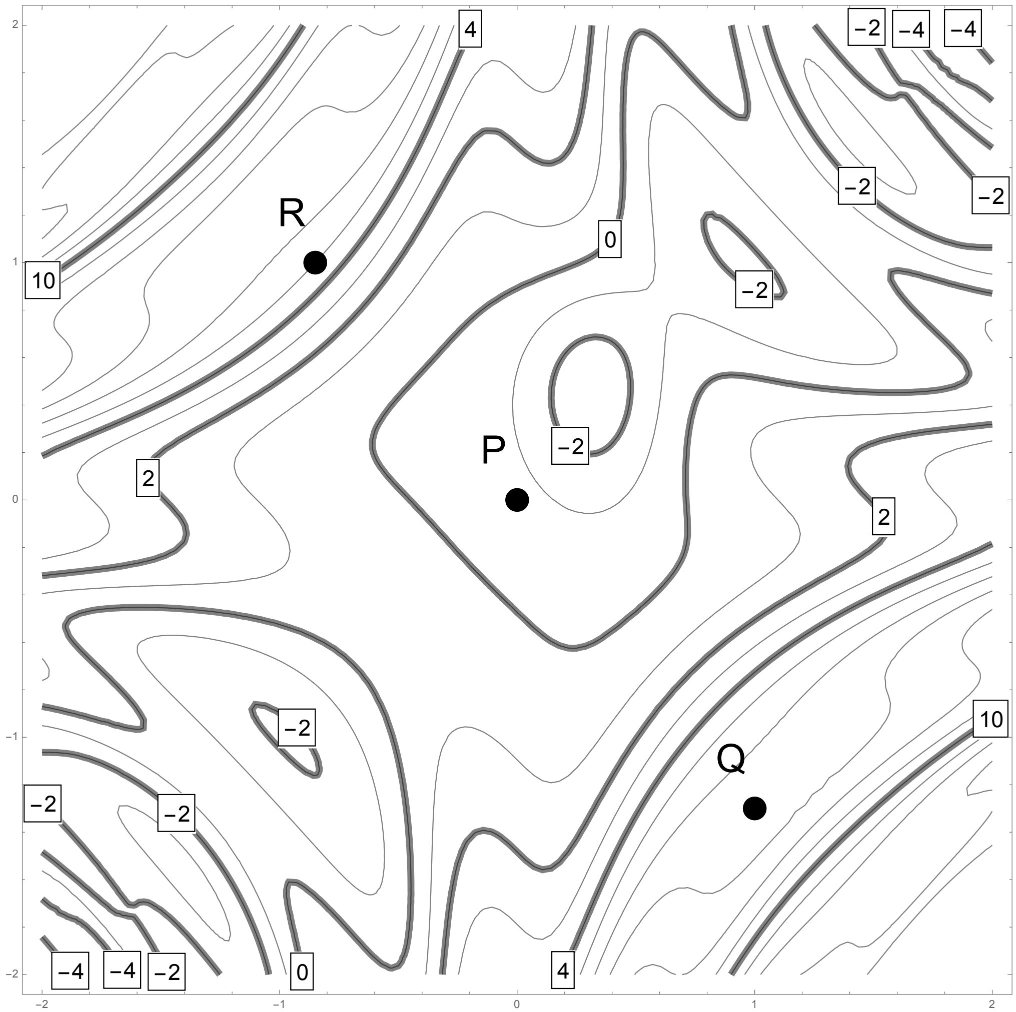
(a) $g \circ h$.

(b) $h \circ g$

9. Partial derivatives and contour plots

Practice 9.1. Below is a collection of level sets of a function $f \colon \mathbf{R}^2 \to \mathbf{R}$. Each **thick** level set has its value labeled inside a $\boxed{\text{box}}$ — and in general every level set (thick or thin; labeled or not) corresponds to an *integer* value. (As usual, x is horizontal and y is vertical; the length scales in the x- and y-directions are equal.)

The points **P**, **Q**, and **R** in the xy-plane are each marked with a dot (•).



For each part below, choose one answer.

(a) $\dfrac{\partial f}{\partial x}$ at **Q** is: NEGATIVE ZERO POSITIVE

(b) $\dfrac{\partial f}{\partial y}$ at **Q** is: NEGATIVE ZERO POSITIVE

(c) $\dfrac{\partial f}{\partial x}$ at **R** is: NEGATIVE ZERO POSITIVE

(d) $\dfrac{\partial f}{\partial y}$ at **R** is: NEGATIVE ZERO POSITIVE

(e) Which partial derivative is largest, in *numerical value*? $f_x(\mathbf{P})$ $f_x(\mathbf{Q})$ $f_x(\mathbf{R})$

(f) Which partial derivative is larger, in *absolute value*? $|f_x(\mathbf{Q})|$ $|f_x(\mathbf{R})|$

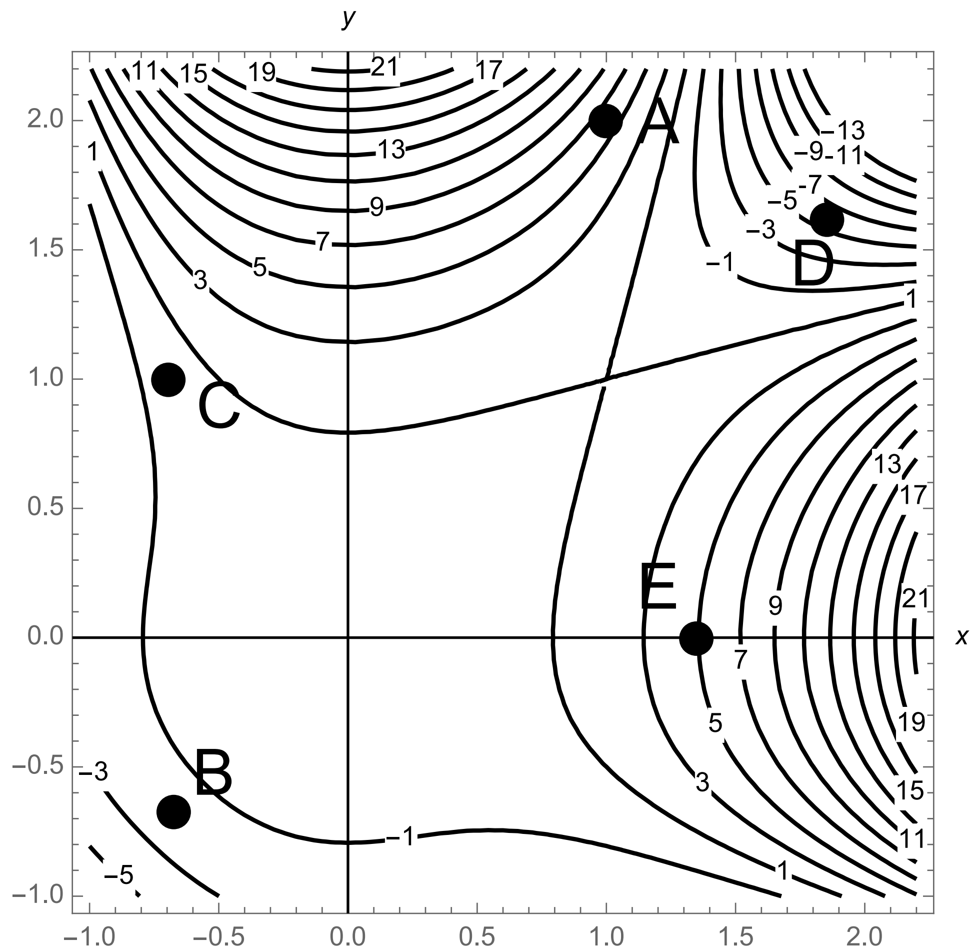
Practice 9.2. Consider the function $f(x, y) = y\cos(x) + x\sin(y)$.

(a) Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(b) Compute the "mixed" second partial derivative $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ in two different ways, verifying that these two ways give the same formula.

(c) Compute $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$.

Practice 9.3. Below is a contour plot of a function $g(x, y)$ over the square of points (x, y) with $-1 \leq x \leq 2$ and $-1 \leq y \leq 2$. Five points A, B, C, D, E are marked on the plot.



Determine from this contour plot which point clearly satisfies each of the following conditions. No justification needed.

(i) g is approximately 0

(ii) $\frac{\partial g}{\partial y} = 0$

(iii) $\frac{\partial g}{\partial x} < 0$ AND $\frac{\partial g}{\partial y} > 0$

(iv) $\frac{\partial g}{\partial x} < 0$ AND $\frac{\partial g}{\partial y} < 0$

10. Maxima, minima, and critical points

Practice 10.1. Let T be the region in the xy-plane given by the triangle — *together with its interior* — whose vertices are $(0,0)$, $(0,3)$, and $(1,3)$. Draw an approximate picture of T (it need not be to scale), and determine the maximum and minimum values of the function

$$f(x,y) = (x-5)^2 - (y-2)^2$$

on this region, and the point(s) at which each is attained.

Practice 10.2. Let D be the region

$$D = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2 : x \geq y^2 + 1 \text{ and } x \leq 5 \right\}.$$

Define $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(x,y) = xe^x(y^2+1)$.

(a) Draw the region D. (It doesn't have to be to scale; just do your best to draw an approximately correct shape.)

(b) Find all the critical points of f in $\mathbf{R}^2$. (There is only one, but your answer must justify this.)

(c) Find the minimum and maximum of f on the region D and all points at which they are attained. (You may take for granted that these extrema exist. It may be helpful to recall that $e > 1$.)

Practice 10.3. Find all the critical points of the following functions of two variables:

(a) $f(x,y) = x^2 + xy + y^2 - 2x - y$

(b) $f(x,y) = x^3y^2(6-x-y)$, where $x > 0, y > 0$.

(c) $f(x,y) = \dfrac{1+x-y}{\sqrt{1+x^2+y^2}}$.

Practice 10.4. Find all the critical points of the following functions of three variables:

(a) $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$.

(b) $f(x,y,z) = x + \dfrac{y^2}{4x} + \dfrac{z^2}{y} + \dfrac{2}{z}$, where $x > 0, y > 0, z > 0$.

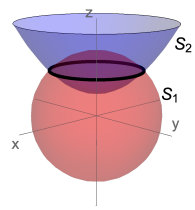
Note: in (b), the fact that x, y, z are all positive simplifies calculations.

11. Gradients, local approximations, and gradient descent

Practice 11.1. Consider two surfaces S_1 and S_2 defined by

$$S_1 : \ x^2 + y^2 + z^2 = 2 \ \text{ and } \ S_2 : \ 2z = x^2 + y^2 + c.$$

There is exactly one scalar c for which S_1 and S_2 cross each other orthogonally, i.e. the normal lines to the respective tangent planes to S_1 and S_2 are perpendicular at all points where these surfaces meet. Determine the value of c.



Practice 11.2. Suppose $f(x, y, z) = ye^{(z^2 - x^2)} - y^3 z^2$, and consider the surface $\mathcal{S}$ in $\mathbf{R}^3$ given by the equation $f(x, y, z) = 6$. Notice that this surface contains the point $\mathbf{a} = (1, -2, 1)$.

(a) Find an equation for the tangent plane to $\mathcal{S}$ at the point $\mathbf{a}$.

(b) There is only one point on $\mathcal{S}$ near $\mathbf{a}$ with $y = -2.2$ and $z = 0.9$ (you do not have to prove this). Use linear approximations to estimate the x-coordinate of this point. Simplify your answer as much as possible, but if you wish you may give your estimate to one decimal place (i.e., to the nearest tenth).

(c) Use linear approximations to estimate

$$(-1.9)e^{\left((0.8)^2 - (1.3)^2\right)} - (-1.9)^3(0.8)^2$$

Simplify your answer as much as possible. (*Hint:* this value is $f(1.3, -1.9, 0.8)$.)

Practice 11.3. Define $g : \mathbf{R}^3 \to \mathbf{R}$ by $g(x, y, z) = x\cos(y) + xz^2$.

(a) Compute the gradient of g.

(b) Compute the equation for the tangent plane to the level set $\{(x, y, z) : g(x, y, z) = 2\}$ at the point $\mathbf{a} = (1, 0, 1)$.

(c) Compute the best linear approximation to $g(x, y, z)$ at the point $\mathbf{a} = (1, 0, 1)$, written in the form $ax + by + cz + d$ for scalars a, b, c, d (some of which may be 0). (Hint: your answer to part (b) may be useful.)

12. Constrained optimization via Lagrange multipliers

Practice 12.1. Let R be the region of the plane $x+y+z=12$ where $x,y,z>0$, as shown in Figure 12.1. Note that we are *not* considering points on the boundary of this region. It is a fact (which you may accept) that the function $f(x,y,z)=xy^2z^3$ on R attains a maximal value at one point in R. Find that point.

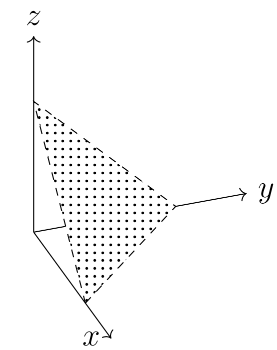
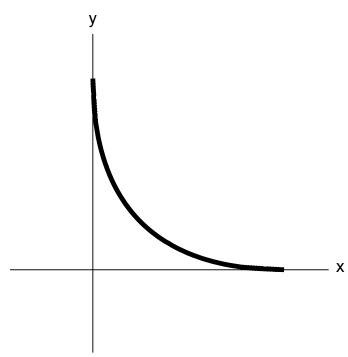


FIGURE 12.1. The region $x+y+z=12$ with $x,y,z>0$

Practice 12.2. Find the global extrema of the function $f(x,y)=2x+3y$ on the curve $g(x,y)=\sqrt{x}+\sqrt{y}=5$ where $x,y\geq 0$; this curve is sketched below.



Practice 12.3. Find all points $P=(a,b,c)$ on the surface

$$y^2-9xz=9$$

that are closest to the origin (equivalently: points on this surface that minimize $x^2+y^2+z^2$).

Practice 12.4. Let $f(x,y)=12x^2-4xy+3y^2$.

(a) In what *unit-vector* direction should one move from the point $(0,1)$ in order to decrease the value of f as rapidly as possible? (You may leave your answer unsimplified, but the vector given as your final answer should be in terms of numbers only, not symbols.)

(b) Determine the maximum and minimum values of f on the curve C that is given by the equation $4x^2+y^2=8$, indicating all points where each extremum is attained.

Practice 12.5. Use Lagrange multipliers to find the extrema of the function $f(x,y,z)=xy+z$ on the sphere of radius 3 defined by the equation $x^2+y^2+z^2=9$, and the points at which these extrema are attained. (You may take for granted that $f(x,y,z)$ has extrema on this region. The extreme values of f are both integers.)

13. Linear functions, matrices, and the derivative matrix

Practice 13.1. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear function for which

$$f(1,0) = (2,3), \qquad f(1,1) = (1,9).$$

Find the matrix A for which $f(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbf{R}^2$.

Practice 13.2. Consider the effect a linear transformation on the Stanford emblem.

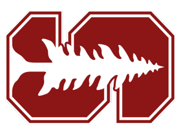
For each of the following matrices M, identify which picture shows the output when M is applied to the Stanford emblem above.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} : \underline{\qquad} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} : \underline{\qquad} \qquad \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} : \underline{\qquad}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : \underline{\qquad} \qquad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} : \underline{\qquad} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} : \underline{\qquad}$$

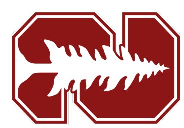
(1)

(2)



(3)

(4)



(5)

(6)

(7)

(8)

14. Linear transformations and matrix multiplication

Practice 14.1. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear function that

- first rotates a given vector counterclockwise by $45°$ about the origin,
- then projects the result onto the line $y = x$.

Find the matrix B for which $T(\mathbf{x}) = B\mathbf{x}$ for any $\mathbf{x} \in \mathbf{R}^2$.

Practice 14.2. Suppose $S\colon \mathbf{R}^2 \to \mathbf{R}^2$ and $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ are the following linear functions:

- T rotates a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **clockwise** by 45 degrees and then multiplies the x_1-coordinate by 2. (For example, $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.)
- S scales a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by the scalar -1 and then multiplies the x_1-coordinate by $\dfrac{1}{2}$.

(*Note:* in the parts below, you do *not* need to prove that S, T, or any composition involving them, is a linear function; this may be taken as known.)

(a) Since T is a linear function, it is represented by a matrix. Find this matrix for T.

(b) Similarly, let M be the matrix for the linear function $S \circ T$. Compute M, simplifying as much as possible.

(c) Determine M^8; simplify your answer as much as possible. (*Hint:* interpret the linear function $S \circ T$ geometrically, for example by plotting the effect of $S \circ T$ on $\mathbf{e}_1$ and $\mathbf{e}_2$; what happens if you keep applying it successively 2, 3, 4, etc., times?)

15. Matrix algebra

16. Applications of matrix algebra: population dynamics, PageRank, and gambling

Practice 16.1. The number of supporters for each candidate in a certain three-person electoral race has been rather variable, due to daily changes in voter opinion.

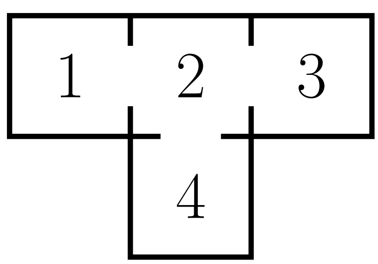
- Among the voters that support candidate A today, 50% will still support candidate A tomorrow, and 30% will support candidate B tomorrow, and 20% will support candidate C tomorrow.
- Among the voters that support candidate B today, 20% will support candidate A tomorrow, and 80% will still support candidate B tomorrow.
- Among the voters that support candidate C today, 30% will support candidate A tomorrow, and 30% will support candidate B tomorrow, and 40% will still support candidate C tomorrow.

(a) Write down the Markov matrix M for this process of changing candidate support; that is, if the entries of the vector $\mathbf{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$ are today's numbers of supporters for each candidate, then find the matrix M for which the entries of the product $M\mathbf{x}$ are *tomorrow's* numbers of supporters (again, for candidates in the order A, B, C). Write a formula in terms of x_A, x_B, x_C for the number of supporters of A tomorrow.

(b) What proportion of supporters of candidate B will support candidate C precisely two days later? Show your reasoning, and simplify your answer as much as possible.

(c) Suppose $M^{20} \approx \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.6 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$. Over the long term, what proportion of voters will support candidate C? Justify your answer.

Practice 16.2. Remy the rat runs through the maze pictured below. For every 15-second interval, Remy stays put and then exits the room for one chosen at random from adjacent rooms (so when in Rooms 1, 3, or 4, next Remy enters Room 2).



(a) After n steps, let

$$\mathbf{p}_n = \begin{bmatrix} \text{probability(equivalently, chances) that Remy is in Room } 1 \\ \text{probability(equivalently, chances) that Remy is in Room } 2 \\ \text{probability(equivalently, chances) that Remy is in Room } 3 \\ \text{probability(equivalently, chances) that Remy is in Room } 4 \end{bmatrix}.$$

Suppose Remy starts in Room 1. What are $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$?

(b) Write the matrix M so that $\mathbf{p}_{n+1} = M\mathbf{p}_n$. (This is handled similarly to the reasoning with the Gambler's Ruin discussion in the course text.)

(c) Compute M^2.

(d) It turns out that $M^3 = M$ – no need to verify this. Deduce what M^{2019} and M^{2020} must be.

(e) Remy's friend Linguini puts a block of delicious cheddar in Room 4. Once Remy reaches the cheese, he stops running and remains in Room 4. Write the matrix N for which $N\mathbf{p}_n = \mathbf{p}_{n+1}$ in this new scenario.

(f) Explain informally why N^{100} is very close to

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Practice 16.3. A certain city has two soccer teams, FC and United, that compete for the allegiances of its citizens. Each citizen of the city is either an FC supporter, a United supporter, or a non-watcher (doesn't watch soccer at all). Suppose the following:

- Every year, 20% of FC supporters will give up and become non-watchers for the next year; the remaining 80% will stay with their team.
- Every year, 20% of United supporters will give up and become non-watchers for the next year; the remaining 80% will stay with their team.
- Every year, 10% of non-watchers will decide to become FC supporters for the next year, and 20% of non-watchers will become United supporters the next year; the remaining 70% will continue to be non-watchers the next year.

(You may also ignore births, deaths, and migration; so there is no other mechanism for the number of each type of citizen to change.)

(a) Write down the Markov matrix M for this process of changing soccer allegiances; that is, if the entries of the vector $\mathbf{x} = \begin{bmatrix} x_F \\ x_U \\ x_N \end{bmatrix}$ are this year's numbers of each type of citizen (FC, United, and non-watcher, respectively), then find the matrix M for which the entries of the product $M\mathbf{x}$ are *next year's* numbers of citizen types (again, in the order FC, United, non-watcher).

(b) What proportion of current FC supporters will be non-watchers precisely two years later? Show your reasoning, and simplify your answer as much as possible.

(c) Suppose $M^{20} \approx \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}$. Over the *long term*, what is the proportion of citizens that are United supporters? Justify your answer.

17. Multivariable Chain Rule

Practice 17.1. Let $h : \mathbf{R} \to \mathbf{R}$ be a single-variable function.

(a) For $g : \mathbf{R}^2 \to \mathbf{R}$ and $F(x,y) = h(g(x,y))$, use the matrix form of the Chain Rule to show

$$\frac{\partial F}{\partial x} = h'(g(x,y))\frac{\partial g}{\partial x}, \quad \frac{\partial F}{\partial y} = h'(g(x,y))\frac{\partial g}{\partial y}.$$

(b) Show that the function $f(x,y) = yh(x^2 - y^2)$ satisfies

$$\frac{1}{x}\frac{\partial f}{\partial x} + \frac{1}{y}\frac{\partial f}{\partial y} = \frac{f(x,y)}{y^2}$$

(away from the coordinate axes, so $x \neq 0$ and $y \neq 0$). Part (a) is helpful for this.

(c) Show that the function $f(x,y) = xy + xh(y/x)$ (away from the y-axis, so $x \neq 0$) satisfies

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = xy + f(x,y).$$

Again, part (a) is helpful for this.

Practice 17.2. For this problem, suppose $f\colon \mathbf{R} \to \mathbf{R}$ is a single-variable function.

(a) Let $F(x,y) = f(x^2 + y^2)$. Use the matrix-based form of the Chain Rule to find functions $h_1(x,y)$ and $h_2(x,y)$ having nothing to do with f so that

$$\frac{\partial F}{\partial x} = h_1(x,y) f'(x^2 + y^2),$$
$$\frac{\partial F}{\partial y} = h_2(x,y) f'(x^2 + y^2).$$

(b) Show that $F(x,y) = f(x^2 + y^2)$ satisfies $y\dfrac{\partial F}{\partial x} - x\dfrac{\partial F}{\partial y} = 0$. (*Hint:* part (a) might be helpful.)

(c) Show that $G(x,y) = e^x f\big(y - \frac{1}{2}x^2\big)$ satisfies $\dfrac{\partial G}{\partial x} + x\dfrac{\partial G}{\partial y} = G$.

Practice 17.3. Let $F\colon \mathbf{R}^2 \to \mathbf{R}^3$ be given by $F(x,y) = \begin{bmatrix} xy^2 \\ x^2 + y^2 \\ x^2/y \end{bmatrix}$, and suppose $G\colon \mathbf{R}^2 \to \mathbf{R}^2$ satisfies

$$G(2,1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad G(2.01,1) = \begin{bmatrix} 2.02 \\ 1.99 \end{bmatrix}, \quad \text{and} \quad G(2,1.01) = \begin{bmatrix} 2.03 \\ 2.01 \end{bmatrix}.$$

(a) Compute $(DF)(x,y)$.

(b) Estimate the 2×2 derivative matrix $(DG)(2,1)$; show all reasoning, and simplify your answer as much as possible. (Hint: try to relate the first column of $(DG)(2,1)$ to $G(2,1)$ and $G(2.01,1)$.)

(c) Use your answer to (b) to compute $(D(F \circ G))(2,1)$, and *use this matrix* to estimate $(F \circ G)(2.2, 0.9)$. Simplify your answers as much as possible. (*Note:* if you did not find an answer to (b), in its place you may use the (incorrect) matrix $(DG)(2,1) = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$.)

Practice 17.4. Let

$$f(x,y) = (-x^2 + xy, x^2 + y^3), \qquad A(v,w) = (v+w, v).$$

We define

$$F(x,y) = (f \circ A \circ f)(x,y).$$

Compute the derivative matrix $(DF)(1,1)$ and use this matrix to estimate $F(1.1, 1.1)$.

18. Matrix inverses and multivariable Newton's method for zeros

19. Linear independence and the Gram-Schmidt process

Practice 19.1. Consider the matrix $A = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 4 & 2 \\ -1 & -2 & 1 \end{bmatrix}$, whose columns from left to right are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Carry out the Gram-Schmidt process for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, in the manner in which it is presented in Chapter 19 of the course textbook, to construct an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of $\mathbf{R}^3$.

As a safety check on your work, verify directly that the $\mathbf{w}_j$'s you compute are perpendicular to each other. (The entries in the $\mathbf{w}_j$'s are all integers.)

20. Matrix transpose, quadratic forms, and orthogonal matrices

21. Linear systems, column space, and null space

Practice 21.1. Let S be the collection of all vectors in $\mathbf{R}^4$ satisfying:

$$4x_1 + 2x_2 - 4x_3 + x_4 = 0, \quad \text{and} \quad x_3 + x_4 = 0.$$

(a) Express S as the null space $N(A)$ for some matrix A.

(b) Express S as the column space $C(B)$ for some matrix B.

Practice 21.2. The following is a series of short questions about null spaces. The parts are independent of each other.

(a) For the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, verify if $\mathbf{v} \in N(A)$.

(b) Give a parametric equation of a line that is contained in $N(\begin{bmatrix} 2 & -3 & 5 \end{bmatrix})$.

(c) Let V be the collection of all vectors in $\mathbf{R}^3$ perpendicular to $\mathbf{v} = \begin{bmatrix} 8 \\ -5 \\ 6 \end{bmatrix}$. Write V as $N(A)$ for some matrix A.

(d) Give an example of matrix A for which $\mathbf{R}^3 = N(A)$.

(e) Find the dimension of $N(A)$ where A is the diagonal matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Practice 21.3. The three parts of this exercise are practice with null spaces; they are not related to each other and can be worked on independently.

(a) For an $m \times n$ matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbf{R}^m$, explain why $N(A) = \{\mathbf{0}\}$ when the $\mathbf{a}_j$'s are linearly independent and why $N(A)$ contains a nonzero vector when the $\mathbf{a}_j$'s are linearly dependent.

(b) Let A be an $n \times n$ matrix, and suppose $\mathbf{v} \in \mathbf{R}^n$ is a vector for which $A\mathbf{v} = \lambda\mathbf{v}$, where λ is a scalar. Explain why if $\lambda = 0$ then $\mathbf{v} \in N(A)$, and why if $\lambda \neq 0$ then $\mathbf{v} \in C(A)$.

(c) For every linear subspace V of $\mathbf{R}^8$ and the 8×8 matrix A for $\mathbf{Proj}_V : \mathbf{R}^8 \to \mathbf{R}^8$, explain why $C(A) = V$ and $N(A) = V^\perp$. (There is no need to compute or describe entries in A.)

Practice 21.4. For each of the following statements, choose either TRUE (meaning, "always true") or FALSE (meaning, "not always true"). No justification required, although it helps if you can explain why a statement is true, or give examples of when a statement is false.

(a) For nonzero matrices A and B for which the product AB makes sense, the column space $C(AB)$ is equal to $C(A)$.

(b) Some 5×7 matrix A satisfies $\dim N(A) = 3$ and $\dim C(A) = 2$.

(c) If A is an $n \times n$ matrix with $\dim(C(A)) = n$, then A is invertible.

Practice 21.5. (a) Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 5 & 8 \\ 1 & 2 & 1 & 1 \\ 2 & 4 & 4 & 6 \end{bmatrix}.$$

Label the column vectors of A as $\begin{bmatrix} | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ | & | & | & | \end{bmatrix}$. Note that

$$\mathbf{a_2} = 2\mathbf{a_1}, \qquad \mathbf{a_4} = 2\mathbf{a_3} - \mathbf{a_1}.$$

Find a 4×2 matrix B whose columns are not scalar multiples of each other and for which $AB = 0$, i.e. the 4×2 zero matrix. *There are many valid answers.*

(b) Explain why any vector in $C(B)$ must also be in $N(A)$.

22. Matrix decompositions: QR-decomposition and LU-decomposition

Practice 22.1. The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 4 & -4 \end{bmatrix}$ is equal to LU with $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) For $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, use the given LU-decomposition of A to solve $A\mathbf{x} = \mathbf{b}$ via repeated back-substitutions, and then check directly that your solution really is a solution. (All entries in the solution vector $\mathbf{x}$ are integers, and likewise for the output of back-substitutions using L.)

(b) Use the given LU-decomposition to compute A^{-1} (its entries are integers, with upper-left entry equal to 20), and check that what you obtain really is an inverse to A by multiplying it against A in some order (you do not need to compute the matrix product in both orders; it is recommended to check your calculations of U^{-1} and L^{-1} really work before computing A^{-1}).

Practice 22.2. (a) Consider the following matrix A and its column vectors

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 2 & 3 & 5 \\ 2 & 4 & 4 \\ 0 & 2 & -2 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 9 \\ 5 \\ 4 \\ -2 \end{bmatrix}.$$

If you perform Gram-Schmidt on $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, you get the following:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \mathbf{v}_2 - 2\mathbf{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \mathbf{v}_3 - 3\mathbf{w}_1 - \mathbf{w}_2 = \begin{bmatrix} 4 \\ 0 \\ -2 \\ -4 \end{bmatrix}.$$

Use this information to find the QR decomposition of A as $A = QR$. *The entries in R should all be integers.*

(b) The matrix $B = \begin{bmatrix} 3/5 & 3/5 & -1/5 \\ 0 & 1 & 1 \\ 4/5 & 4/5 & 7/5 \end{bmatrix}$ has QR decomposition

$$B = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Use this decomposition to compute B^{-1}.

Practice 22.3. Let $A = \begin{bmatrix} 4 & -7 & -2 \\ 2 & -4 & -1 \\ 0 & 2 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 4 & -7 & -2 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) It is a fact, which you do not have to check, that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = U.$$

Find an LU-decomposition of A; that is, find a lower-triangular matrix L satisfying $A = LU$.

(b) Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$. Using repeated back-substitutions, find all solutions to the system $A\mathbf{x} = \mathbf{b}$, and then make a direct check that your answer really satisfies the equation.

Practice 22.4. Consider the matrix $A = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 4 & 2 \\ -1 & -2 & 1 \end{bmatrix}$, whose columns from left to right are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(a) (This part is the same as Practice 19.1.) Carry out the Gram-Schmidt process for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, in the manner in which it is presented in Chapter 19 of the course textbook, to construct an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of $\mathbf{R}^3$.

(b) Using your work in the previous part, express each $\mathbf{v}_j$ as a linear combination of the $\mathbf{w}_i$'s, and use that to compute the QR-decomposition of A.

(c) Let $\mathbf{b} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$. Using the method explained Chapter 22, apply the QR-decomposition of A to solve the linear system $A\mathbf{x} = \mathbf{b}$, showing all of your steps.

23. Eigenvalues and eigenvectors

Practice 23.1. For each eigenvalue λ of $A = \begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ 7 & 5 & 3 \end{bmatrix}$, compute a basis for the nonzero linear subspace $N(A - \lambda \mathrm{I}_3)$ in $\mathbf{R}^3$, and as a check on your work verify directly that each vector in that basis is an eigenvector for A with eigenvalue λ.

Practice 23.2. Suppose $A = \begin{bmatrix} 1 & 6 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) For each eigenvalue λ of A, compute a basis for the nonzero linear subspace $N(A - \lambda \mathrm{I}_3)$ in $\mathbf{R}^3$, and as a check on your work verify directly that each vector in that basis is an eigenvector for A with eigenvalue λ.

(b) What is the *second column* of A^{10}? Give your answer as a 3-vector containing numbers; though you are permitted to leave simple exponential expressions (e.g., 17^5, etc.) unevaluated. (*Hint:* write $\mathbf{e}_2$ as a combination of eigenvectors you found, and consider how to use matrix-vector products.)

Practice 23.3. Let A be the 3×3 matrix corresponding to the linear transformation that reflects about the plane $2x + 3y + 6z = 0$ (you do not need to write down A, or explain why it is indeed a linear transformation). List the eigenvalues of A, and the corresponding eigenspaces. *Hint:* Under the reflection, which vectors are reversed, and which vectors are not changed?

24. Applications of eigenvalues: Spectral Theorem, quadratic forms, and matrix powers

Practice 24.1. Let

$$M = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

be a Markov matrix. In this exercise we will study M in the context of Spectral Theorem.

(a) Briefly indicate why there is an orthogonal basis of eigenvectors for the matrix M.

(b) The eigenvalues of M are $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$. Verify that $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the λ_1-eigenspace, and explain why it spans that space. Hint: you can either use the rank-nullity theorem, or directly find the λ_1-eigenspace by solving a system of linear equations.

(c) Find the λ_2-eigenspace V_2 and write it as the span of two orthogonal vectors $\mathbf{w}_2, \mathbf{w}_3$.

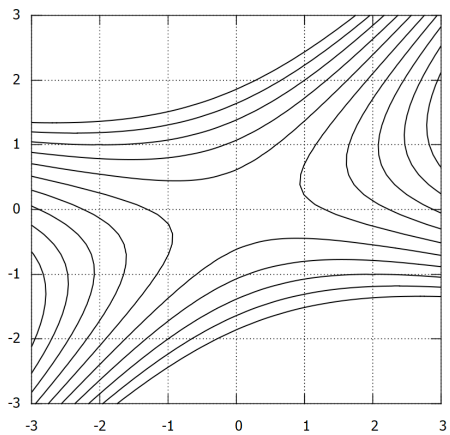
(d) Let $\mathbf{w}_1', \mathbf{w}_2', \mathbf{w}_3'$ be unit vectors obtained from $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ by dividing each by its length. Let W be the matrix whose columns are $\mathbf{w}_1', \mathbf{w}_2', \mathbf{w}_3'$. Find the diagonal matrix D where $M = WDW^\top$.

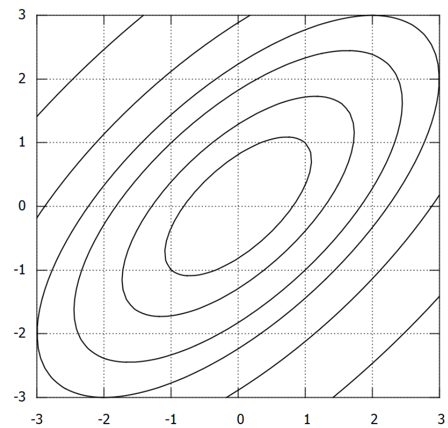
(e) Using the fact that $(1/2)^{100} \approx 0$ to 30 decimal places, calculate M^{100} explicitly.

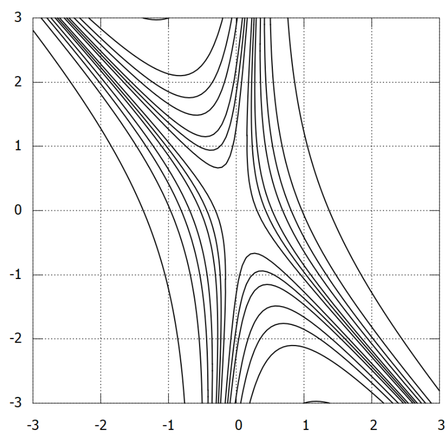
Practice 24.2. (a) Find the eigenvalues and a corresponding eigenvector for each for the 2×2 matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

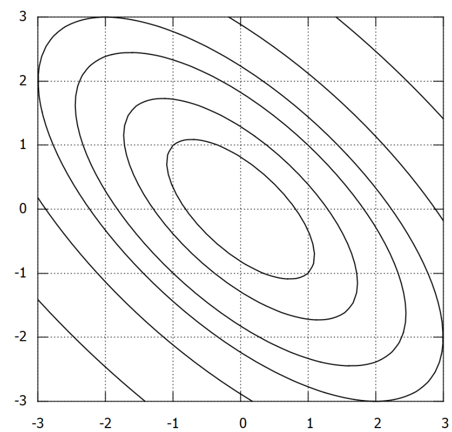
(b) Find the eigenvalues and a corresponding eigenvector for each for the 2×2 matrix $B = \begin{bmatrix} 3 & 6 \\ 6 & -13 \end{bmatrix}$.

(c) Which of the following 4 pictured contour plots represents (i) the quadratic form $q_A(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$; (ii) the quadratic form $q_B(\mathbf{x}) = \mathbf{x}^\top B\mathbf{x}$? Justify your answer using part (a) and (b).

(1) 

(2) 

(3) 

(4) 

25. The Hessian and quadratic approximation

26. Grand finale: application of the Hessian to local extrema, and bon voyage

Practice 26.1. Let $f(x, y) = x^3 + 3xy^2 - 15x - 12y$.

(a) Compute f_x and f_y, and verify that the points $P = (1, 2)$, $Q = (2, 1)$, $R = (-1, -2)$, and $S = (-2, -1)$ are critical points. (These are the only critical points, but we are not asking you to justify that.)

(b) Compute the Hessian $(\mathrm{H}f)(x, y)$, and evaluate this at each of the 4 critical points considered in part (a). Use this to give the quadratic approximation to f near $P = (1, 2)$ (i.e., the quadratic approximation to $f(1+h, 2+k)$ for h, k near 0) and to determine for each of the other three critical points Q, R, S if it is a local maximum, local minimum, or a saddle point.

Practice 26.2. Let

$$f(x, y) = 3xe^y - 3e^y - x^3.$$

(a) This function has exactly one critical point. Find it, and determine its nature (local maximum, local minimum, or saddle point).

(b) Approximate $f(0.1, 0.2)$ to two decimal places using a quadratic approximation of f at a nearby point.

Practice 26.3. Let f be the function

$$f(x, y) = x^2y - \frac{3}{2}x^2 + \frac{1}{2}y^3 + 6x - \frac{35}{2}y$$

You can accept (or check if you wish) that $f(-2, 3) = -45$.

(a) Verify that $(-2, 3)$ is a critical point of f, and compute the symmetric Hessian matrix $(\mathrm{H}f)(-2, 3)$ and the quadratic approximation for f at $(-2, 3)$ (i.e., the quadratic approximation to $f(-2+h, 3+k)$ for $h, k \approx 0$). (The entries of $(\mathrm{H}f)(-2, 3)$ are integers.)

(b) For the Hessian matrix $H = (\mathrm{H}f)(-2, 3)$ that you found in part (a), compute its eigenvalues and use this to determine if $(-2, 3)$ is a local maximum, local minimum, or saddle point. Use the eigenvalues and eigenvectors of H to sketch what the contour plot of f looks like near $(-2, 3)$. (The eigenvalues of H are integers.)

(*Note:* it only matters to sketch approximate ellipses or hyperbolas aligned with the appropriate perpendicular lines through the critical point; precise asymptotic directions for a hyperbola, and precisely measured ratio of major and minor axes of an ellipse, don't matter as long as the "longer" or "wider" axis direction is apparent on your picture. There is also no need to label the contours with numerical values.)

