Problem 1: Unconstrained local extrema via Hessian

For each of the following functions $\mathbf{R}^2 \to \mathbf{R}$, use the gradient to find all critical points and characterize each critical point (i.e., local maximum, local minimum, saddle point, or otherwise) by computing the Hessian in general and analyzing it at each critical point.

(a) $x^4y^4 - 2x^2 - 2y^2$

(b) $-3x^2 + 2xy - (3/2)y^2 + y^3$

Solution:

(a) The gradient is $\begin{bmatrix} 4x^3y^4 - 4x \\ 4x^4y^3 - 4y \end{bmatrix}$, so this vanishes when both partial derivatives vanish. This is the simultaneous vanishing of

$$4x^3y^4 - 4x = 4x(x^2y^4 - 1), \;\; 4x^4y^3 - 4y = 4y(x^4y^2 - 1).$$

If $x = 0$ then the second vanishing condition forces $y = 0$, and likewise if $y = 0$ then the first vanishing condition forces $x = 0$. Hence, $(0, 0)$ is a critical point and at any other critical point both coordinates are nonzero, so $x^2y^4 - 1 = 0$ and $x^4y^2 - 1 = 0$, which is to say

$$x^2y^4 = 1 = x^4y^2.$$

Dividing the left side by the right side gives that $y^2/x^2 = 1$, so $y/x = \pm 1$, i.e. $y = \pm x$. Plugging this equality into either of the conditions gives $x^6 = 1$ (notice that the $\pm$ sign cancels out since we have even powers). This gives $x = \pm 1$ (here the $\pm$ doesn't correspond to the $\pm$ in $y = \pm x$, i.e. for both cases $y = x$ and $y = -x$ we can have $x = 1$ or $x = -1$), so for $x = 1$ we can have $y = \pm 1$, and similarly for $x = -1$, and hence we get four more critical points: namely, $(x, y) = \pm(1, 1)$ and $(x, y) = \pm(1, -1)$.

To summarize, there are five critical points: $(0, 0)$, $\pm(1, 1)$, and $\pm(1, -1)$.

The Hessian is $\begin{bmatrix} 12x^2y^4 - 4 & 16x^3y^3 \\ 16x^3y^3 & 12x^4y^2 - 4 \end{bmatrix}$, and we now compute this at each critical point. The Hessian at (0,0) is $\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$, which is diagonal with negative diagonal entries, so it is negative-definite and hence $(0, 0)$ is a local maximum.

The Hessian at $\pm(1, 1)$ is $\begin{bmatrix} 8 & 16 \\ 16 & 8 \end{bmatrix}$ (note in each term that the $\pm$ cancels since there are an even number of combined x's and y's in each term), and the Hessian at $\pm(1, -1)$ is $\begin{bmatrix} 8 & -16 \\ -16 & 8 \end{bmatrix}$. The determinants of the two matrices are $64 - (\pm 16)^2 < 0$, so the remaining critical points are all saddle points.

(b) The gradient is $\begin{bmatrix} -6x + 2y \\ 2x - 3y + 3y^2 \end{bmatrix}$, so this vanishes when $-6x + 2y = 0$ and $2x - 3y + 3y^2 = 0$. The first of these says $y = 3x$, and plugging this into the second gives

$$0 = 2x - 9x + 27x^2 = -7x + 27x^2 = x(27x - 7),$$

so $x = 0$ or $x = 7/27$. Since $y = 3x$, we obtain two critical points: $(0, 0)$ and $(7/27, 7/9)$.

Based on the determination of the gradient, the Hessian is

$$(\mathrm{H}f)(x, y) = \begin{bmatrix} -6 & 2 \\ 2 & -3 + 6y \end{bmatrix}.$$

We need to analyze this at each critical point. At $(0, 0)$ this is $\begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}$ which we encountered on an earlier worksheet: its characteristic polynomial is $\lambda^2 + 9\lambda + 14 = (\lambda + 2)(\lambda + 7)$, so its eigenvalues are $-2, -7$ that are negative. Thus, $(\mathrm{H}f)(0, 0)$ is negative-definite, so the origin is a local maximum for f.

At the other critical point $(7/27, 7/9)$, the Hessian is

$$\begin{bmatrix} -6 & 2 \\ 2 & -3+14/3 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 2 & 5/3 \end{bmatrix}.$$

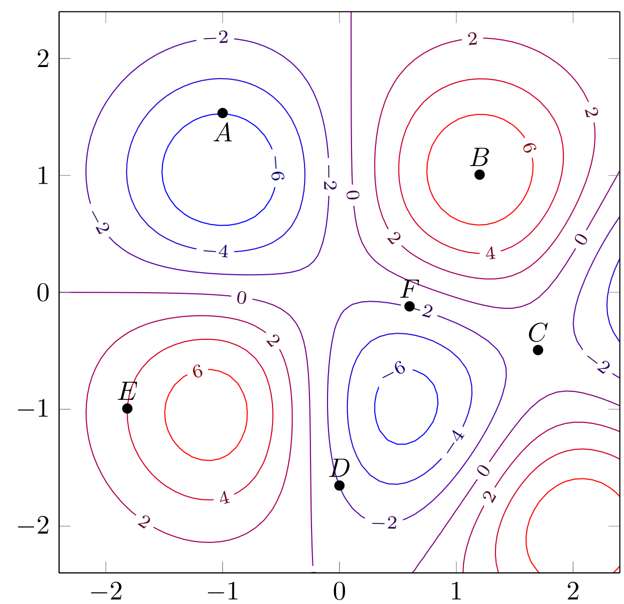
The determinant is $-10-4=-14<0$, so its eigenvalues have opposite signs and thus it is indefinite. This critical point is therefore a saddle point.

Problem 2: Visually interpreting critical points

Consider the given contour plot for a function $f\colon \mathbf{R}^2 \to \mathbf{R}$.

(a) Assuming B is a critical point, is the Hessian matrix of f at B: (i) positive-definite, (ii) negative-definite, or (iii) indefinite? (Assume it is one of these.)

(b) Assuming C is a critical point, is the Hessian matrix of f at C: (i) positive-definite, (ii) negative-definite, or (iii) indefinite? (Assume it is one of these.)



Solution:

(a) The contour plot indicates B is a local maximum for f, so the Hessian at B is negative definite.

(b) The contour plot indicates C is a saddle point (hyperbolic level sets). Hence, its Hessian is indefinite.

Problem 3: Using Hessian eigenvalues to characterize critical points

Consider a critical point $\mathbf{a}$ of $f : \mathbf{R}^n \to \mathbf{R}$ whose Hessian $(\mathrm{H}f)(\mathbf{a})$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ for some orthogonal basis (as we are guaranteed always happens, by the Spectral Theorem). For each of the following possibilities for the list of eigenvalues, is the behavior of f at $\mathbf{a}$ a local maximum, local minimum, or saddle point? (It is one of these in each case below.)

(a) eigenvalues 43, 5, 1

(b) eigenvalues 5 , -3, -7

(c) eigenvalues 1, 0, -1

(d) eigenvalues 1, 1, 1, 1

(e) eigenvalues -1, -5

Solution:

(a) Positive-definite Hessian, so local minimum.

(b) Indefinite Hessian, so saddle point.

(c) Indefinite Hessian (even though one eigenvalue is 0), so saddle point.

(d) Positive-definite Hessian (even though “repeated” eigenvalues), so local minimum.

(e) Negative-definite Hessian, so local maximum.