Review

The <u>dimension</u> of a span/linear subspace is the smallest number of vectors needed to span it.

$\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a <u>basis</u> for $\mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k)$ if $\dim(\mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k)) = k$.

In general: if one of the $\vec{v}_i$ can be written as a linear combination of the other $\vec{v}_i$'s

(e.g. $\vec{v}_1 = c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$)

then that vector is "redundant" and can be thrown out without changing the span.

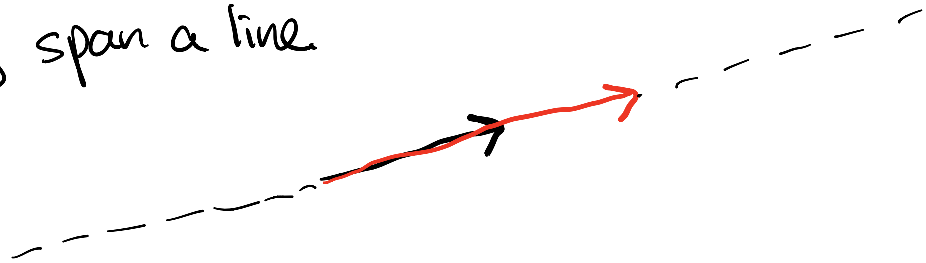
When none of the $\vec{v}_i$ can be written as a lin. comb. of the others, then you have a basis.

Problem 1: Determining the nature of a span

For each collection of 3-vectors, determine whether its span is a point, a line, a plane, or all of $\mathbf{R}^3$. Give a basis of the span in each case. (Keep in mind that if a vector in the collection is a linear combination of others then it can be dropped without affecting the span.)

(a) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}$

If one is a scalar multiple of the other, then they span a line.



If not, they span a plane.

Check: $\begin{bmatrix}1\\2\\1\end{bmatrix} \overset{?}{=} c\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}2c\\c\\2c\end{bmatrix}$

$$\rightsquigarrow \begin{cases} 2c=1 \leftarrow c=\frac{1}{2} \\ c=2 \leftarrow \\ 2c=1 \end{cases}$$

No solution $\Rightarrow$ they span a plane, which is 2 dimensional.

so $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right\}$ is a basis for $\mathrm{span}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right)$

(b) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}$

Check for redundancies:

$$\begin{bmatrix}-1\\1\\-1\end{bmatrix} \overset{?}{=} c_1\begin{bmatrix}1\\2\\1\end{bmatrix} + c_2\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}c_1+2c_2\\2c_1+c_2\\c_1+2c_2\end{bmatrix}$$

$$\begin{cases} c_1+2c_2=-1 \\ 2c_1+c_2=1 \\ c_1+2c_2=-1 \end{cases} \quad \begin{aligned} &\rightsquigarrow c_1=-1-2c_2 \\ &\rightsquigarrow 2(-1-2c_2)+c_2=1 \end{aligned}$$

$$-2-3c_2=1$$

$$-3c_2=3$$

$$c_2=-1$$

$$c_1=-1-2(-1)=1$$

$$\begin{bmatrix}-1\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}2\\1\\2\end{bmatrix}$$

So can throw out $\begin{bmatrix}-1\\1\\-1\end{bmatrix}$:

should again check for redundancies but already did in part (a)

$$\text{span}\left(\begin{bmatrix}-1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right)$$

by part (a), this is a plane with basis $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right\}$

(c) $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}2\\1\\2\end{bmatrix}$, $\begin{bmatrix}-1\\1\\0\end{bmatrix}$

Check for redundancy:

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \stackrel{?}{=} c_1\begin{bmatrix}2\\1\\2\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}2c_1-c_2\\c_1+c_2\\2c_1\end{bmatrix}$$

$$\begin{cases} 2c_1 - c_2 = 1 & \rightsquigarrow\ 2\cdot\frac{3}{2} - \frac{1}{2} = 3 - \frac{1}{2} \neq 1 \\ c_1 + c_2 = 2 & \rightsquigarrow\ c_2 = 2 - c_1 = 2 - \frac{3}{2} = \frac{1}{2} \\ 2c_1 = 3 & \rightsquigarrow\ c_1 = \frac{3}{2} \end{cases}$$

No solution exists, so no redundancy.

$\Rightarrow$ the span is $\mathbb{R}^3$, and

$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$ is a basis.

Problem 2: More recognizing and describing linear subspaces

Which of the following subsets S of $\mathbf{R}^3$ are linear subspaces? If a set S is a linear subspace, exhibit it as a span. If it is not a linear subspace, describe it geometrically and explain why it is not a linear subspace.

need not be a basis for this problem

(a) The set S_1 of points (x, y, z) in $\mathbf{R}^3$ with both $z = x + 2y$ and $z = 5x$.

$z = x + 2y$ is a plane

$z = 5x$ is a plane

Plug in $(0,0,0)$: $0=0+2\cdot 0$ ✓

$0=5\cdot 0$ ✓

Both contain $(0,0,0)$, so they are not parallel

$\Rightarrow$ they intersect in a line containing $(0,0,0)$, which is a span.

So S_1 is a span.

Find the intersection:

$$z=5x$$

$$\hookrightarrow 5x=x+2y$$

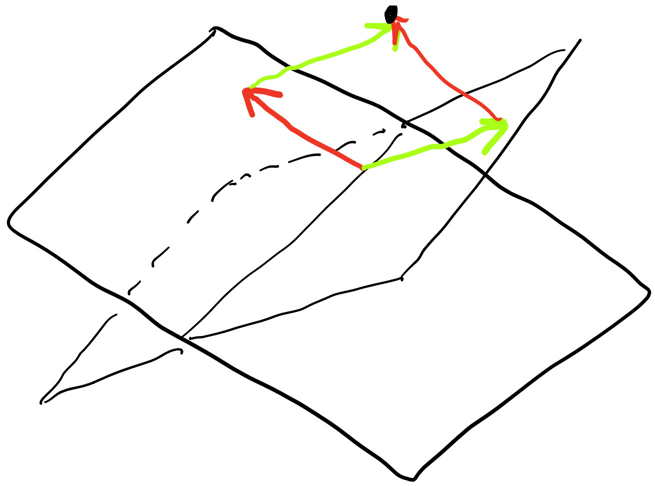
$$4x=2y$$

$$2x=y$$

Choose $x=1$. $\rightsquigarrow$ $y=2$ and $z=5\cdot 1=5$

So $\text{span}\left(\begin{bmatrix}1\\2\\5\end{bmatrix}\right)=S_1$

(b) The set S_2 of points (x, y, z) in $\mathbf{R}^3$ with either $z = x + 2y$ or $z = 5x$.



Not a span because sums of vectors in S_2 may not be in S_2.

E.g. $z = x + 2y$

$x = y = 1 \rightsquigarrow z = 3$ $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \in S_2$

$z = 5x$

$x = 1, y = 0 \rightsquigarrow z = 5$ $\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \in S_2$

$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} \end{array} \begin{array}{c} x \\ y \\ z \end{array}$ $8 \overset{?}{\neq} 2 + 2 \cdot 1 = 4$

$8 \neq 5 \cdot 2$

$\begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$ is not in S_2. $\Rightarrow$ S_2 not a span

(c) The set S_3 of points (x, y, z) in $\mathbf{R}^3$ of the form $t\begin{bmatrix}1\\2\\3\end{bmatrix} + t'\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}3\\3\\3\end{bmatrix}$ for some scalars t and t' (which are allowed to be anything, depending on the point (x, y, z)).

This is the parametric form of a plane.

If it contains $(0,0,0)$, then it's a span.

Check: $\begin{bmatrix}0\\0\\0\end{bmatrix} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + t'\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}t+2t'+3\\2t+t'+3\\3t+3\end{bmatrix}$

$3t+3=0 \rightsquigarrow 3t=-3$
$t=-1$

$2t+t'+3=0 \rightsquigarrow t'=-3-2t$
$=-3-2(-1)$
$=-1$

$t+2t'+3=0 \rightsquigarrow (-1)+2\cdot(-1)+3=0$
$-1-2+3=0 \checkmark$

$\Rightarrow \begin{bmatrix}0\\0\\0\end{bmatrix}$ is in the plane $\Rightarrow S_3$ is a span.

$S_3 = \mathrm{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right)$.

Problem 3: Multiple descriptions as a span (Extra)

Let $\mathbf{v}, \mathbf{w}$ be two vectors in $\mathbf{R}^{12}$. Show that $\operatorname{span}(\mathbf{v}, \mathbf{w}) = \operatorname{span}(\mathbf{v}+\mathbf{w}, \mathbf{v}-\mathbf{w})$. (Hint: You can show that two sets S and T are equal in two steps: everything belong to S also belong to T, and everything belonging to T also belongs to S.)

Problem 4: Linear subspaces and orthogonality (computations)

looking for 2 vectors

Let V be the set of vectors in $\mathbf{R}^4$ orthogonal to both $\begin{bmatrix}1\\0\\4\\2\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$. Find a pair of vectors that span V, so it is a linear subspace.

$$\begin{bmatrix}x\\y\\z\\w\end{bmatrix} \in V \quad \text{if} \quad \begin{bmatrix}x\\y\\z\\w\end{bmatrix} \cdot \begin{bmatrix}1\\0\\4\\2\end{bmatrix} = x+4z+2w = 0$$

$$\rightsquigarrow x = -4z-2w$$

$$\underline{\text{and}} \quad \begin{bmatrix}x\\y\\z\\w\end{bmatrix} \cdot \begin{bmatrix}0\\1\\1\\1\end{bmatrix} = y+z+w = 0$$

$$\rightsquigarrow y = -z-w$$

$$\begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}-4z-2w\\-z-w\\z\\w\end{bmatrix} = z\begin{bmatrix}-4\\-1\\1\\0\end{bmatrix} + w\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$$

$$\Rightarrow V = \operatorname{span}\left(\begin{bmatrix}-4\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}\right)$$

Problem 5: An orthogonal basis

Let V be the set of vectors $\mathbf{v} \in \mathbf{R}^3$ satisfying $\mathbf{v} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ (this says that both of these explicit 3-vectors have the same projection onto $\mathbf{v}$, or in other words make the same "shadow" onto the line spanned by $\mathbf{v}$).

(a) Express V as the collection of 3-vectors orthogonal to a single nonzero 3-vector.

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

$$x + 2y + 3z = 2x + 3y + 4z$$

$$0 = x + y + z$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \right\}$$

(b) By fiddling with orthogonality equations, build an orthogonal basis of V. There are many possible answers.

(c) Use your answer to (b) to give an orthonormal basis for V.