Solutions to Math 51 Quiz 1

1. (10 points) Consider the plane $\mathcal{P}$ give by the parametric form

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + t' \begin{bmatrix} -2 \\ 1 \\ \alpha \end{bmatrix}.$$

   1. (4 points) If the point $(-3, 4, -11)$ is on $\mathcal{P}$, what is the value of α?
   2. (6 points) Give an equation for $\mathcal{P}$.

Since $(-3, 4, -11)$ is on $\mathcal{P}$, there are t and t' for which

$$\begin{bmatrix} -3 \\ 4 \\ -11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + t' \begin{bmatrix} -2 \\ 1 \\ \alpha \end{bmatrix},$$

from which we get the following system of linear equations:

$$\begin{aligned} -3 &= 1 - 2t' \\ 4 &= 2 + 3t + t' \\ -11 &= -1 + t + \alpha t' \end{aligned}$$

From the first two equations, we get $t = 0$ and $t' = 2$, and so, from the third equation we get $\alpha = -5$.

We can try to find a normal vector $\mathbf{n}$ for $\mathcal{P}$; this $\mathbf{n}$ has to be perpendicular to both $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$.

If $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we get

$$\begin{aligned} 3b + c &= 0 \\ -2a + b - 5c &= 0 \end{aligned}$$

from which we can obtain $c = -3b$ and $a = 8b$. Therefore, $\mathbf{n} = \begin{bmatrix} 8 \\ 1 \\ -3 \end{bmatrix}$ is normal to $\mathcal{P}$. Setting up the equation $8x + y - 3z = d$, we get $d = 13$ by plugging in the point $(1, 2, -1)$ (or $(-3, 4, -11)$). Hence, one such equation for $\mathcal{P}$ is

$$8x + y - 3z = 13.$$

2. (2 points) **True or False:** Consider the line given by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\mathbf{v}.$$

There is a 3-vector $\mathbf{v}$ for which the above line lies on the plane $2x - 3y - z = 0$.

FALSE. Since $2(1) - 3(1) = (0) = -1 \neq 0$, the point $(1, 1, 0)$ does not lie on the given plane. Hence, it is impossible for the line to lie on the plane.

3. (2 points) **True or False:** Suppose the set of 51 points

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_{51}, y_{51})\}$$

has correlation coefficient 1. It is possible to add a point (x_{52}, y_{52}) for which the set of 52 points

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_{52}, y_{52})\}$$

has correlation coefficient -1.

FALSE. Since the correlation of the first set is 1, the 51 points all lie on a single line with *positive* slope. Hence, it is impossible for these 51 points to be on a single line with *negative* slope, no matter what the 52nd point is.

4. (3 points) Let $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. How many unit vectors $\mathbf{u}$ are there for which the angle between $\mathbf{u}$ and $\mathbf{v}$ is $60°$?

a) 0 b) 1 c) 2 d) 4 e) infinitely many

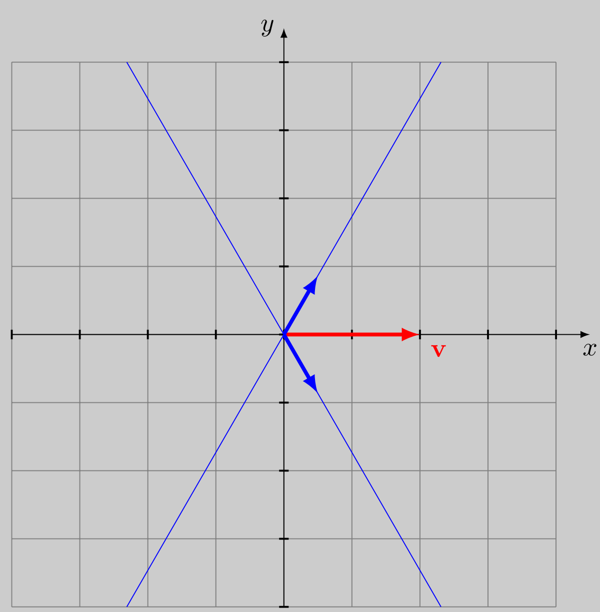
Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$. Then,

- $a^2 + b^2 = 1$
- $\cos(60°) = \dfrac{2a}{2} = a$, and so, $a = \dfrac{1}{2}$.

Thus, there are two possible values for b, $\dfrac{\sqrt{3}}{2}$ and $-\dfrac{\sqrt{3}}{2}$. Therefore, there are **TWO** such vectors,

$$\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

The fact that there are two such vectors can also be easily realized by graphing on the xy-plane:



5. (3 points) Which of the following is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$?

a) $\begin{bmatrix} 51 \\ 0 \\ 21 \end{bmatrix}$ b) $\begin{bmatrix} 51 \\ -21 \\ -42 \end{bmatrix}$ c) $\begin{bmatrix} 51 \\ -21 \\ 42 \end{bmatrix}$ d) $\begin{bmatrix} 51 \\ -42 \\ 21 \end{bmatrix}$ e) $\begin{bmatrix} 51 \\ -42 \\ -21 \end{bmatrix}$

A linear combination of the two given vectors is of the form

$$\alpha \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta \\ \alpha \\ -2\alpha \end{bmatrix};$$

note that the third component has to be -2 times the second component. The only choice that satisfies this condition is **c**. In particular,

$$\begin{bmatrix} 51 \\ -21 \\ 42 \end{bmatrix} = -21 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + 36 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$