1. (a) The left side is $\begin{bmatrix} t \\ 2t \end{bmatrix} + \begin{bmatrix} 2t' \\ 3t' \end{bmatrix} = \begin{bmatrix} t+2t' \\ 2t+3t' \end{bmatrix}$, so the equality with $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ amounts to the simultaneous pair of conditions

$$\begin{aligned} t + 2t' &= 4 \\ 2t + 3t' &= 5. \end{aligned}$$

Solving this by elimination (e.g., double the first equation and subtract from the second to cancel t and get $-t' = 5 - 8 = -3$, so $t' = 3$) yields $t = -2$, $t' = 3$.

(b) The left side is $\begin{bmatrix} t \\ 2t \end{bmatrix} + \begin{bmatrix} 2t' \\ 3t' \end{bmatrix} = \begin{bmatrix} t+2t' \\ 2t+3t' \end{bmatrix}$, so the equality with $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ amounts to the simultaneous pair of conditions

$$\begin{aligned} t + 2t' &= -2 \\ 2t + 3t' &= -1. \end{aligned}$$

Solving this by elimination (e.g., double the first equation and subtract from the second to cancel t and get $-t' = -1 + 4 = 3$, so $t' = -3$) yields $t = 4$, $t' = -3$.

(c) Proceeding as in the first part yields the simultaneous pair of conditions

$$\begin{aligned} t + 2t' &= x \\ 2t + 3t' &= y. \end{aligned}$$

To solve for t (in terms of x and y) we must eliminate t', so we multiply the first equation by 3 and subtract from it twice the second to get $3t - 4t = 3x - 2y$, or $-t = 3x - 2y$, so $t = -3x + 2y$. Likewise, to solve for t' (in terms of x and y) we must eliminate t, so we multiply the first equation by 2 and subtract it from the second to get $3t' - 4t' = y - 2x$, or $-t' = y - 2x$, so $t' = 2x - y$.

Since $(t, t') = (-3x + 2y, 2x - y)$, for $x = 4$ and $y = 5$ it is $(-12 + 10, 8 - 5) = (-2, 3)$, exactly the answer to part (a).

Since $(t, t') = (-3x + 2y, 2x - y)$, for $x = -2$ and $y = -1$ it is $(6 - 2, -4 + 1) = (4, -3)$, exactly the answer to part (b).

◇

2. (a) $\mathbf{v} + \mathbf{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \mathbf{v} - \mathbf{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, 2\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, 3\mathbf{w} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \mathbf{v} + 2\mathbf{w} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \mathbf{v} + 3\mathbf{w} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$.

(b) $\mathbf{v} + \mathbf{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{v} - \mathbf{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, 2\mathbf{w} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, 3\mathbf{w} = \begin{bmatrix} -3 \\ -6 \end{bmatrix}, \mathbf{v} + 2\mathbf{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{v} + 3\mathbf{w} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$.

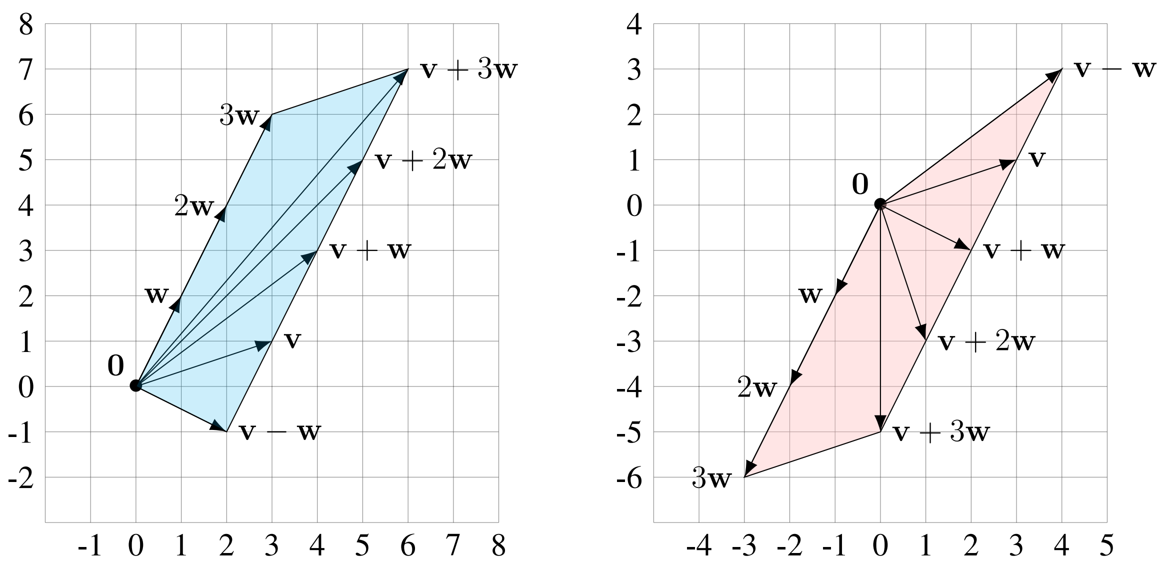

Figure 1: The solutions to part (a) (left) and part (b) (right).

$\diamond$

3. (a) If a course is worth u units then its grade is weighted by $u/20$ (the fraction of units from that course out of all 20 units taken), so the vector of GPA's is

$$\frac{5}{20}\mathbf{v}_1 + \frac{5}{20}\mathbf{v}_2 + \frac{4}{20}\mathbf{v}_3 + \frac{3}{20}\mathbf{v}_4 + \frac{3}{20}\mathbf{v}_5.$$

With common denominator 20, the numerators are exactly the unit-values. The sum of the numerators is 20, which is how we defined the denominator, so the sum of the fractions is the numerator-sum divided by 20, which is 20/20 = 1. Thus, this is a convex combination. In this way of describing it, the only role of 20 is as the sum of the unit values. Hence, we get convexity by the same reasoning, regardless of the total unit sum.

$\diamond$

4. (a) Consider the parallelogram formed by two 2-vectors $\mathbf{v}$ and $\mathbf{w}$ as shown in Figure 2. We will apply the Law of Cosines to the two triangles shown in Figure 2.

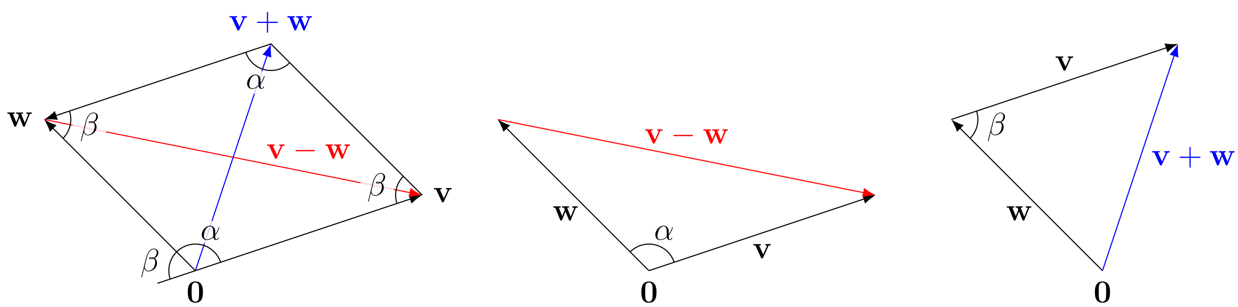

Figure 2: The figure used in the geometric proof of the parallelogram law when $n = 2$, using the Law of Cosines.

Applying the Law of Cosines to the first triangle (with the long edge labeled by $\mathbf{v} - \mathbf{w}$) yields

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\alpha. \tag{$*$}$$

Applying the Law of Cosines to the second triangle yields

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\beta. \tag{$**$}$$

Adding equations ($*$) and ($**$) yields

$$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|(\cos\alpha + \cos\beta).$$

Since $\alpha + \beta = \pi$, as shown in the first drawing in Figure 2, we have $\cos\alpha = \cos(\pi - \beta) = -\cos\beta$ (an angle between 0 and π and its supplement mark off points on the unit circle with opposite x-coordinates). Therefore $\cos\alpha + \cos\beta = 0$, so we obtain the equality

$$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2,$$

as desired.

(b) Writing $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, we have

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}, \quad \mathbf{v} - \mathbf{w} = \begin{bmatrix} v_1 - w_1 \\ \vdots \\ v_n - w_n \end{bmatrix},$$

so

$$\begin{aligned}
\|\mathbf{v}+\mathbf{w}\|^2+\|\mathbf{v}-\mathbf{w}\|^2 &= \sum_{i=1}^{n}(v_i+w_i)^2+\sum_{i=1}^{n}(v_i-w_i)^2 \\
&= \sum_{i=1}^{n}(v_i^2+2v_iw_i+w_i^2)+\sum_{i=1}^{n}(v_i^2-2v_iw_i+w_i^2) \\
&= \sum_{i=1}^{n}(v_i^2+2v_iw_i+w_i^2+v_i^2-2v_iw_i+w_i^2) \\
&= \sum_{i=1}^{n}(2v_i^2+2w_i^2) \\
&= 2\sum_{i=1}^{n}v_i^2+2\sum_{i=1}^{n}w_i^2 \\
&= 2\|\mathbf{v}\|^2+2\|\mathbf{w}\|^2
\end{aligned}$$

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5. We calculate each using three ingredients: properties of the dot product, that $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$ for vectors $\mathbf{v}$, and the assumption that $\mathbf{a},\mathbf{b},\mathbf{c}$ are all unit vectors (so each satisfies $\mathbf{v}\cdot\mathbf{v}=1$).
   (a) $\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} = 1+0+0+1=2.$
   (b) Similarly, we have $\|\mathbf{b}-\mathbf{c}\|^2 = \mathbf{b}\cdot\mathbf{b}-2(\mathbf{b}\cdot\mathbf{c})+\mathbf{c}\cdot\mathbf{c} = 1-\frac{2}{5}+1=\frac{8}{5}$.
   (c) Using (a), we have

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}+\mathbf{b}\|^2+2(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}+\|\mathbf{c}\|^2 = 2+2(\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c})+1 = 2+2\left(\frac{1}{2}+\frac{1}{5}\right)+1=\frac{22}{5}$$

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6. (a) The vectors of interest are nonzero $\begin{bmatrix}x\\y\\z\end{bmatrix}$ with $x+2y+2z=0$. One way to make such are by setting $y=0$ or by setting $x=0$. These are $\begin{bmatrix}-2z\\0\\z\end{bmatrix}=z\begin{bmatrix}-2\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\-z\\z\end{bmatrix}=z\begin{bmatrix}0\\-1\\1\end{bmatrix}$, so two such vectors not scalar multiples of each other are $\begin{bmatrix}2\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\-1\\1\end{bmatrix}$ (there are many other answers). The collection of vectors orthogonal to $\begin{bmatrix}1\\2\\2\end{bmatrix}$ is a plane.
   (b) These are the unit vectors in the plane found in (a), so this is a circle. One way to get these is to pick anything nonzero in the plane found in (a) and divide by its length to get a unit vector. For example:

$$\frac{1}{\sqrt{5}}\begin{bmatrix}2\\0\\-1\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

   are two such (you only need to give one such unit vector; many answers are possible).

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7. (a) We can use $\mathbf{v}=(1,-1,1,-1,\ldots,-1)$ and $\mathbf{w}=(1,1,\ldots,1)$. There are many other answers, such as $\mathbf{v}=(1,1,\ldots,1,999)$ and $\mathbf{w}=(1,1,\ldots,1,-1)$.
   (b) We can use $\mathbf{v}=(1,1,\ldots,1,998)$ and $\mathbf{w}=(1,1,\ldots,1,-1)$.

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8. Part (a) is the case $\mathbf{v}\cdot\mathbf{w} = 0$ (i.e., orthogonality), part (b) is the case $\mathbf{v}\cdot\mathbf{w} > 0$ (i.e., acute angle), and part (c) is the case $\mathbf{v}\cdot\mathbf{w} < 0$ (i.e., obtuse angle). We can give such vectors in the xy-plane, for example (though many other options are possible), such as $\mathbf{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ for (a), $\mathbf{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ for (b), and $\mathbf{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}-1\\-1\\0\end{bmatrix}$ for (c). ◇

9. (a) The perpendicularity of $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ implies

$$0 = (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2,$$

so $\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2$ and hence $\|\mathbf{v}\| = \|\mathbf{w}\|$; i.e., $\mathbf{v}$ and $\mathbf{w}$ have the same length.

(b) We can run the calculation in the solution to (a) in reverse. More specifically, since $\|\mathbf{v}\| = \|\mathbf{w}\|$ we have $\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2$ and so

$$(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2 = 0.$$

This says that $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are perpendicular,

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10. By definition

$$r = \frac{\mathbf{X}\cdot\mathbf{Y}}{\|\mathbf{X}\|\|\mathbf{Y}\|}.$$

(b) Here $\mathbf{X} = \begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix}-5\\3\\1\\-3\\4\end{bmatrix}$ so we get

$$r = \frac{\mathbf{X}\cdot\mathbf{Y}}{\|\mathbf{X}\|\|\mathbf{Y}\|} = \frac{12}{\sqrt{10}\sqrt{60}} = \frac{\sqrt{6}}{5} \approx 0.49.$$

This is a small positive correlation (a bit less than 0.5) and in the picture that the points can be crudely approximated by a line with positive slope.

(c) Here $\mathbf{X} = \begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix}6\\2\\-1\\-2\\-5\end{bmatrix}$ so we get

$$r = \frac{\mathbf{X}\cdot\mathbf{Y}}{\|\mathbf{X}\|\|\mathbf{Y}\|} = \frac{-26}{\sqrt{10}\sqrt{70}} = \frac{-13}{5\sqrt{7}} \approx -0.983.$$

This is a negative number a bit larger than -1, which is the minimal possible value. The points almost lie on a line with negative slope.

These examples illustrate the general fact that correlation measures how well the set of points can be fit to a line. A correlation close to 1 means the data is close to a line with positive slope, and correlation close to -1 means that the data is close to a line with negative slope (when the correlation is exactly 1 or exactly -1 then the data is exactly on such a line).

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11. (a) To find a point in common on these lines, we search for t_1, t_2 for which $\begin{bmatrix}-4\\2+2t_1\\7+3t_1\end{bmatrix} = \begin{bmatrix}1+5t_2\\0\\5+t_2\end{bmatrix}$. From the second coordinate, we can see that $t_1 = -1$. From the first coordinate, we can see that $t_2 = -1$. Plugging these values in, we observe the third coordinates are equal. In particular, $(-4, 0, 4)$ is a point on both lines.

(b) The parametric form is given by this point and nonzero directions along the *directions* of the two respective lines, such as $(0,2,3)$ and $(5,0,1)$. Putting everything together, a parametric form for $\mathcal{P}$ is that it consists of points of the form

$$\begin{bmatrix} -4+5t' \\ 2t \\ 4+3t+t' \end{bmatrix}$$

for scalars t and t'.

(c) A nonzero normal vector $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to the plane must satisfy $2b+3c=0$ and $5a+c=0$. So $a=-c/5$ and $b=-3c/2$. Setting $c=10$, we get $\mathbf{n} = \begin{bmatrix} -2 \\ -15 \\ 10 \end{bmatrix}$. Since $(-4,0,4)$ is a point on the plane, an equation form is given by $-2(x+4)-15y+10(z-4)=0$, or equivalently $-2x-15y+10z=48$..

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12. (a) A normal vector to $\mathcal{P}$ is $\mathbf{n} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{n}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a normal vector to the xy-plane. Hence, the angle θ between the two satisfies

$$\cos(\theta) = \frac{|\mathbf{n}\cdot\mathbf{n}'|}{\|\mathbf{n}\|\|\mathbf{n}'\|} = \frac{|1|}{\sqrt{9}\sqrt{1}} = \frac{1}{3}.$$

(b) The conditions on $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are orthogonality to difference vectors between one of the 3 points and the other two. That is, $\mathbf{n}$ is orthogonal to

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix},$$

so the conditions are $-2a+3c=0$ and $-2a-b=0$. This says $b=-2a$ and $c=2a/3$. This yields the normal vector $\mathbf{n} = \begin{bmatrix} a \\ -2a \\ 2a/3 \end{bmatrix} = a\begin{bmatrix} 1 \\ -2 \\ 2/3 \end{bmatrix}$ for whatever nonzero a we like.

Setting $a=3$ to remove the denominator (for convenience, certainly not necessary) gives $\mathbf{n} = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$ and again $\mathbf{n}'$ to be $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we have $\mathbf{n}\cdot\mathbf{n}' = 2$, so

$$\cos(\theta) = \frac{|\mathbf{n}\cdot\mathbf{n}'|}{\|\mathbf{n}\|\,\|\mathbf{n}'\|} = \frac{2}{\sqrt{9+36+4}\cdot 1} = \frac{2}{7}.$$

(c) The plane $\mathcal{P}$ has normal vector $\mathbf{n_1} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$ and the plane $\mathcal{Q}$ has normal vector $\mathbf{n_2} = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$. These (nonzero) normal vectors are not scalar multiples of each other, so the two planes are not parallel. Let $\mathbf{v}$ be a displacement vector between distinct points in the overlap line L, so $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is perpendicular to both normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ (as the line L is in each plane):

$$\mathbf{v}\cdot\mathbf{n}_1 = 0 \text{ and } \mathbf{v}\cdot\mathbf{n}_2 = 0,$$

which is to say

$$2x-2y-z=0 \text{ and } 3x-6y+2z=0.$$

There are many non-zero solutions to these two simultaneous equations; we seek just one such to get a $\mathbf{v}$ that we can then use to obtain a parametric form for L.
Letting $z = 1$, we arrive at the pair of equations

$$2x - 2y = 1 \text{ and } 3x - 6y = -2$$

that has the solution $(5/3, 7/6)$, so we obtain a direction vector $\begin{bmatrix} 5/3 \\ 7/6 \\ 1 \end{bmatrix}$. Since any nonzero multiple of $\mathbf{v}$ is also a direction vector along L, for a cleaner final answer let's multiply by 6 to get rid of the denominator and thereby work with $\mathbf{v} = \begin{bmatrix} 10 \\ 7 \\ 6 \end{bmatrix}$. The line L then has the parametric form $\mathbf{x}_0 + t\mathbf{v}$ where $\mathbf{x}_0$ is any point of L, which is to say a common point on the planes $\mathcal{P}$ and $\mathcal{Q}$. We next find such an $\mathbf{x}_0$.
These planes have as their respective equations

$$2x - 2y - z = 3 \text{ and } 3x - 6y + 2z = 7,$$

for which we seek a common solution to get a choice of $\mathbf{x}_0$. For instance, setting z to be 0 gives the pair of equations $2x - 2y = 3$ and $3x - 6y = 7$ whose common solution is $(2/3, -5/6)$, so we obtain the parametric form

$$\begin{bmatrix} 2/3 \\ -5/6 \\ 0 \end{bmatrix} + t\mathbf{v} = \begin{bmatrix} 2/3 + 10t \\ -5/6 + 7t \\ 6t \end{bmatrix}$$

(many others are possible).

$\diamond$