Problem 1: Visualizing vectors and convex combinations in $\mathbb{R}^2$

Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

(a) Compute $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$. Draw $\mathbf{a}, \mathbf{b}$ and $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$ in a coordinate plane, and describe geometrically where the sum lies relative to $\mathbf{a}$ and $\mathbf{b}$. Do you expect such a geometric relationship is true for any 2-vectors $\mathbf{a}, \mathbf{b}$? How about for 3-vectors?

(b) Compute $\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$ and $\frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$, and plot these. Do you notice a pattern that should hold for any 2-vectors $\mathbf{a}$ and $\mathbf{b}$?

(c) Compute $\frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c}$, plot this point and draw segments joining it to each of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, and describe geometrically where this lies relative to $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(d) Find a nonzero vector that is perpendicular to $\mathbf{a}$. (Hint: draw a picture.)

Solution:

(a) The vector $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has both entries halfway between the corresponding entries for $\mathbf{a}$ and $\mathbf{b}$, and when plotted it is the midpoint of the segment joining the tips of $\mathbf{a}$ and $\mathbf{b}$, as shown in Figure 1. This midpoint relation works for any $\mathbf{a}$ and $\mathbf{b}$ since it can be analyzed using each vector entry separately, in which case it becomes the fact that the average of two numbers lies halfway between them. This argument applies equally well to 3-vectors.

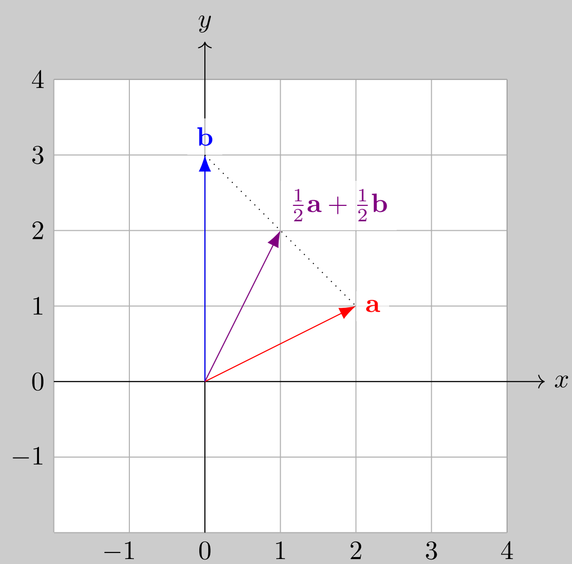

Figure 1: The vectors $\mathbf{a}$, $\mathbf{b}$ and $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.

(b) We compute $\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b} = \begin{bmatrix} 2/3 \\ 7/3 \end{bmatrix}$ and $\frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$, which when plotted are each on the segment joining $\mathbf{a}$ and $\mathbf{b}$, respectively 1/3 and 3/4 of the way from the $\mathbf{a}$ endpoint (as in part (a), we can think about this entry by entry), as shown in Figure 2. By the same reasoning, for any convex combination $t\mathbf{a} + (1-t)\mathbf{b}$ with $0 \le t \le 1$ we get a point on the segment joining $\mathbf{a}$ and $\mathbf{b}$ whose proportion of distance along the segment from the tip of $\mathbf{a}$ is $1-t$.

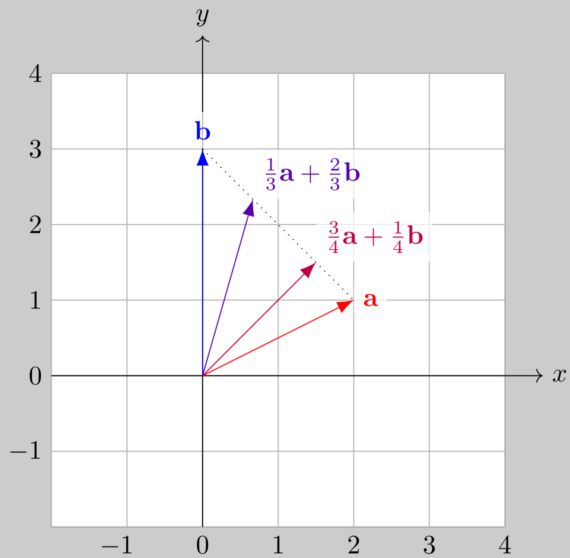

Figure 2: The vectors $\mathbf{a}$, $\mathbf{b}$, $\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}$ and $\frac{3}{4}\mathbf{a}+\frac{1}{4}\mathbf{b}$.

(c) The vector $\dfrac{1}{3}\mathbf{a}+\dfrac{1}{3}\mathbf{b}+\dfrac{1}{3}\mathbf{c}$ is equal to $\begin{bmatrix}1/3\\1\end{bmatrix}$, and when plotted it lies inside the triangle whose vertices are the tips of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. It is equally balanced from each of the corners as shown in Figure 3 since the coefficient $1/3$ on each vector make each entry of the sum be the average of the corresponding entries. Informally, this vector average is the "center of mass," or balancing point, of the triangle.

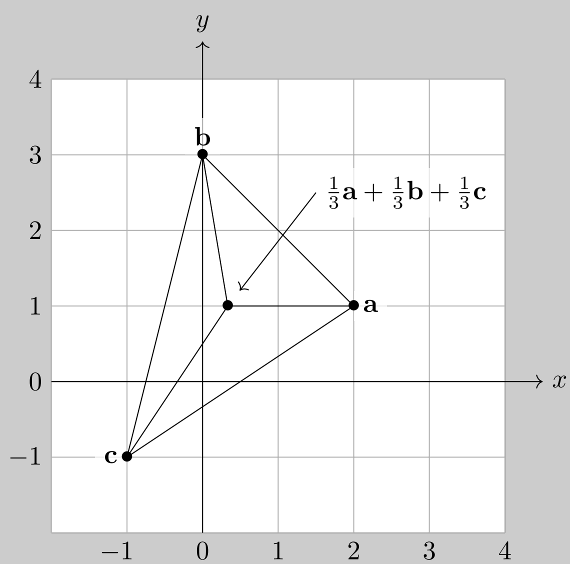

Figure 3: The point $\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}+\frac{1}{3}\mathbf{c}$ is the "center of mass" of the triangle formed by vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(d) By staring at a grid with $\mathbf{a}$ marked, we see that 90 degrees to the left is $\begin{bmatrix}-1\\2\end{bmatrix}$, so that works (as does any nonzero scalar multiple of it).

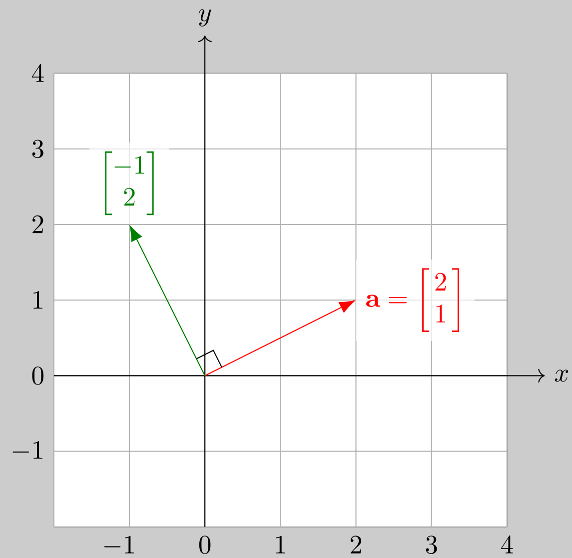

Figure 4: A vector that is perpendicular to $\mathbf{a}$.

Problem 2: Linear combinations

(a) Express $\begin{bmatrix}5\\4\end{bmatrix}$ as a linear combination of $\mathbf{a} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}0\\1\end{bmatrix}$.

(b) Express $\begin{bmatrix}5\\4\end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}1\\2\end{bmatrix}$. (This amounts to solving 2 equations in 2 unknowns.)

(c) Write a general 2-vector $\begin{bmatrix}x\\y\end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}1\\2\end{bmatrix}$. The coefficients of the linear combination will depend on x and y. Make sure that for $x = 5$ and $y = 4$ it agrees with your answer to (b)!

(d) (**Extra**) Draw a picture for each of (a) and (b). Then draw all points $n\mathbf{v}+m\mathbf{w}$ for integers n and m with $-2 \le n, m \le 3$, and draw lines through these parallel to each of $\mathbf{v}$ and $\mathbf{w}$, which should yield a tiling of the plane with copies of the parallelogram P whose corners are at the tips of the vectors $\mathbf{0}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v}+\mathbf{w}$.

Interpret geometrically (without any calculations) the meaning of the answer to (b) in terms of these parallelograms, and do the same for the result in (c) that such a linear combination always exists. (Hint: for (c), mark $(6, 5)$ and $(3, 4)$ and compare where they lie among the parallelograms with the general formula in (c) for these two cases.)

Solution:

(a) If we write $\begin{bmatrix}5\\4\end{bmatrix} = a\mathbf{a} + b\mathbf{b} = \begin{bmatrix}a\\b\end{bmatrix}$ then we see that $a = 5$ and $b = 4$ works: $\begin{bmatrix}5\\4\end{bmatrix} = 5\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix}$. This one could also have been done by inspection, thinking about the meaning of coordinates of a point.

(b) If we write $\begin{bmatrix}5\\4\end{bmatrix} = a\mathbf{v} + b\mathbf{w} = \begin{bmatrix}2a+b\\a+2b\end{bmatrix}$ then the conditions are $2a + b = 5$ and $a + 2b = 4$, which we solve to get $a = 2$ and $b = 1$: $\begin{bmatrix}5\\4\end{bmatrix} = 2\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$ (which can be directly checked as a safety measure).

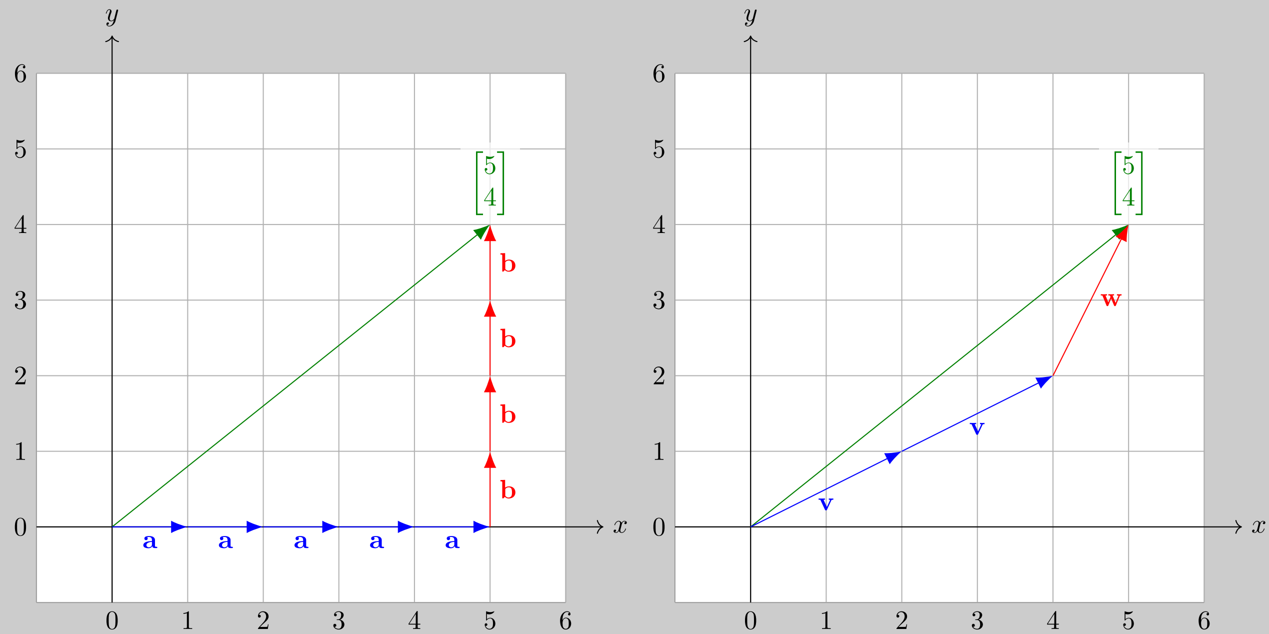

Figure 5: Drawing of solution for (a) (left) and solution for (b) (right).

(c) If we write $\begin{bmatrix}x\\y\end{bmatrix} = a\mathbf{v} + b\mathbf{w} = \begin{bmatrix}2a+b\\a+2b\end{bmatrix}$ then the conditions are $2a+b=x$ and $a+2b=y$. We can then apply the method from high school algebra to isolate each of a and b on its own (e.g., double the first equation and subtract the second to get rid of b), eventually obtaining $a=(2x-y)/3$ and $b=(2y-x)/3$, so $\begin{bmatrix}x\\y\end{bmatrix} = \frac{2x-y}{3}\begin{bmatrix}2\\1\end{bmatrix} + \frac{2y-x}{3}\begin{bmatrix}1\\2\end{bmatrix}$ (which can be directly checked, as a safety measure). For $x=5$ and $y=4$ this indeed agrees with the answer to (b), and in Figure 6 we illustrate the "general case".

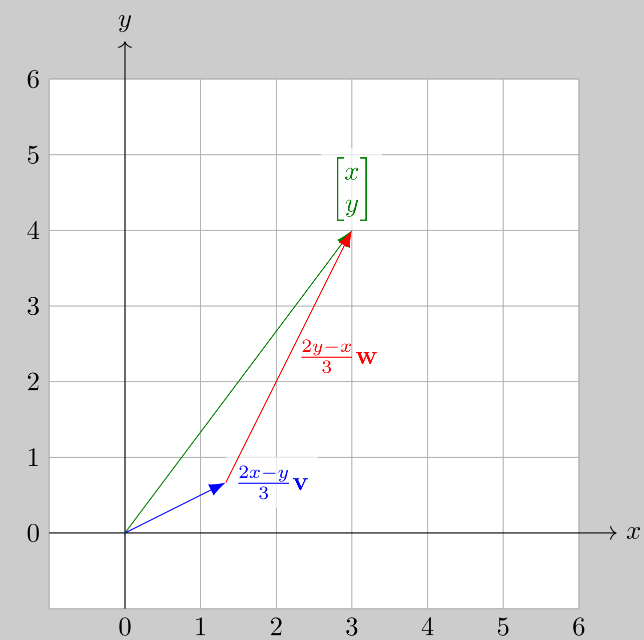

Figure 6: Drawing of solution to (c).

(d) The drawings for (a), (b), (c) are shown in Figures 5 and 6. In each case, the parallelogram P as in the hint can be translated around to exactly tile the plane, with the corners corresponding to the linear combinations $n\mathbf{v} + m\mathbf{w}$ where n and m are *integers* (counting how many steps up/down or left/right we moved the parallelogram), as shown in Figure 7.

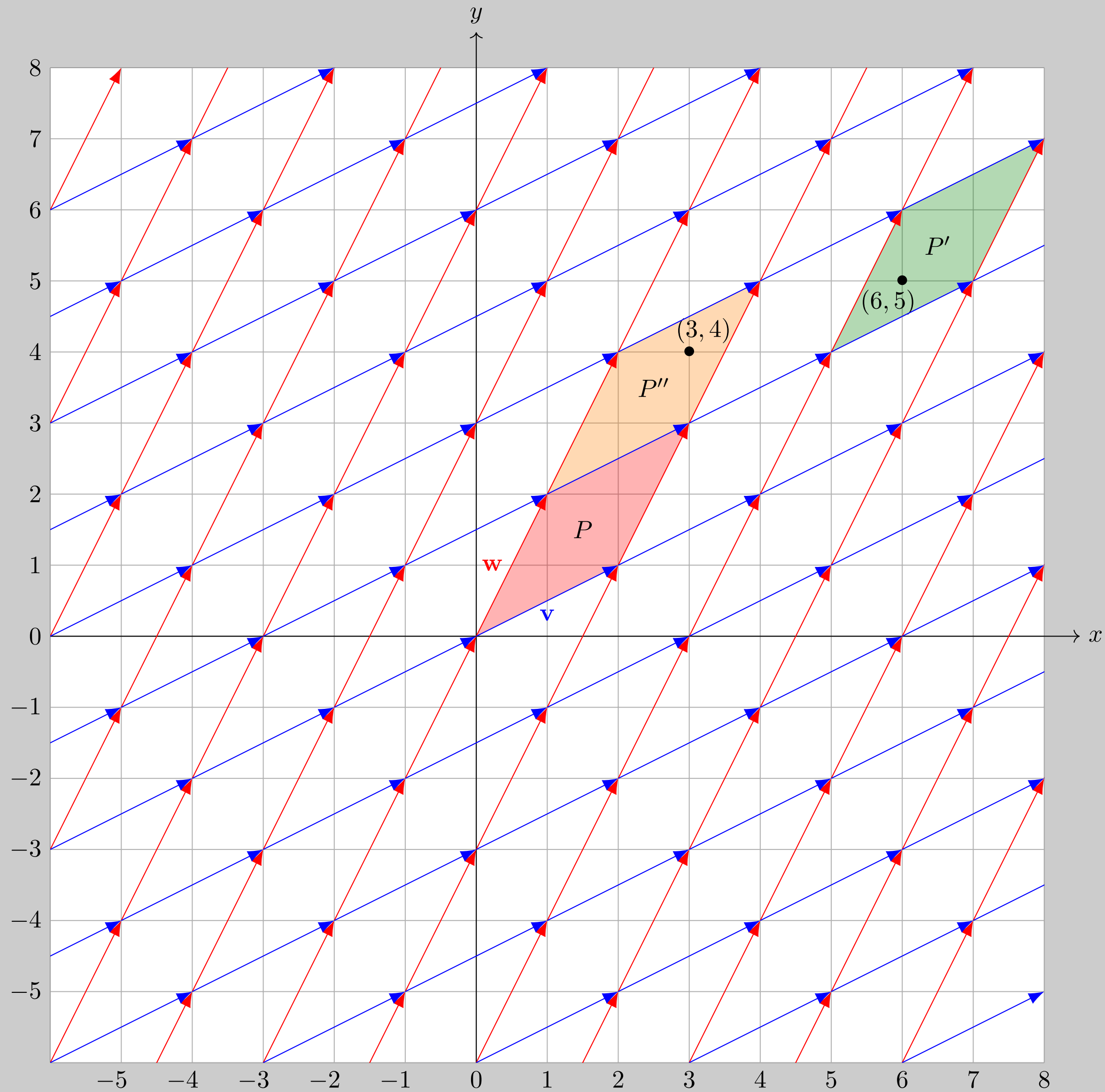

Figure 7: Translated copies of P tile the plane

The points $(6,5)$ and $(3,4)$ are marked in Figure 7. The general formula in (c) yields in these two cases the expressions

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \frac{7}{3}\mathbf{v} + \frac{4}{3}\mathbf{w}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{2}{3}\mathbf{v} + \frac{5}{3}\mathbf{w},$$

and in Figure 7 these lie in the parallelograms as shown that are respectively translated by P as follows. The point $(6,5)$ is in the green parallelogram P' obtained from P by moving 2 steps in the direction of $\mathbf{v}$ and 1 step in the direction of $\mathbf{w}$, with $(6,5)$ lying inside P' an amount 1/3 of the way inside in the directions of each of $\mathbf{v}$ and $\mathbf{w}$, encoding that $7/3 = 2 + 1/3$ and $5/3 = 1 + 1/3$. The point $(3,4)$ is in the orange parallelogram P'' obtained from P by moving 0 steps in the direction of $\mathbf{v}$ and 1 step in the direction of $\mathbf{w}$, with $(3,4)$ lying inside P' an amount 2/3 of the way inside in the directions of each of $\mathbf{v}$ and $\mathbf{w}$, encoding that $2/3 = 0 + 1/3$ and $5/3 = 1 + 2/3$.

In general, each point $\mathbf{x}$ in the plane lies in *some* parallelogram $\mathcal{P}$ as in Figure 7, with its position inside $\mathcal{P}$ nailing down the specific coefficients a and b for which $a\mathbf{v} + b\mathbf{w} = \mathbf{x}$: the “integer parts” of a and b record the lower-left corner of $\mathcal{P}$, and their “fractional parts” record what fraction of each side direction into $\mathcal{P}$ the point $\mathbf{x}$ lies relative to the lower-left corner (as we worked out for $\mathbf{x} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ above).

Problem 3: Length and distance

(a) Compute the distance between $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$. (The answer is an integer.)

(b) Compute the distance between $\begin{bmatrix} 4 \\ -1 \\ 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -6 \\ 1 \\ -3 \end{bmatrix}$. (The answer is an integer.)

(c) If a nonzero vector $\mathbf{v}$ lies at an angle $30°$ counterclockwise from the positive x-axis, what is the unit vector in the same direction as $\mathbf{v}$? (Draw a picture to get an idea.) What if $30°$ is replaced with a general angle θ?

Solution:

(a) The difference vector is $\begin{bmatrix} 12 \\ -5 \end{bmatrix}$ (up to a sign), whose length is $\sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$.

(b) The difference vector is $\begin{bmatrix} -3 \\ 5 \\ -1 \\ 1 \end{bmatrix}$ (up to a sign), whose length is $\sqrt{9 + 25 + 1 + 1} = \sqrt{36} = 6$.

(c) If we write $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ then it lies on the line through the origin making an angle of $30°$, so the unit vector of interest is where that line crosses the unit circle in the first quadrant. This is $(\cos(30°), \sin(30°)) = (\sqrt{3}/2, 1/2)$. By the same method, for a general angle the unit vector is $(\cos\theta, \sin\theta)$.

Problem 4: Vector operations with data

Suppose there are three students in Math 51 with the following components for their course grades:
Student 1: 81/100 on homework, 83/100 on midterm A, 70/100 on midterm B, 75/100 on the final.
Student 2: 73/100 on homework, 75/100 on midterm A, 74/100 on midterm B, 88/100 on the final.
Student 3: 90/100 on homework, 95/100 on midterm A, 88/100 on midterm B, 92/100 on the final.

(a) Write down vectors $\mathbf{v}_{\mathrm{HW}}, \mathbf{v}_{\mathrm{A}}, \mathbf{v}_{\mathrm{B}}, \mathbf{v}_{\mathrm{Final}}$ (all in $\mathbf{R}^3$) representing respectively the grades as percentages on homework, midterm A, midterm B, and the final exam (e.g., for a score of $83/100$, the vector entry should be 83 rather than .83).

(b) Give a general formula as a linear combination of those four vectors in $\mathbf{R}^3$ for a 3-vector $\mathbf{v}_{\mathrm{CG}}$ whose entries are the course grades of the three students in order from student 1 to student 3, assuming the breakdown of the total grade for the course is 16% homework, 36% final, 24% each midterm, and then compute it (you may use a calculator).

Solution:

(a) We have

$$\mathbf{v}_{\mathrm{HW}} = \begin{bmatrix} 81 \\ 73 \\ 90 \end{bmatrix}, \mathbf{v}_{\mathrm{A}} = \begin{bmatrix} 83 \\ 75 \\ 95 \end{bmatrix}, \mathbf{v}_{\mathrm{B}} = \begin{bmatrix} 70 \\ 74 \\ 88 \end{bmatrix}, \mathbf{v}_{\mathrm{Final}} = \begin{bmatrix} 75 \\ 88 \\ 92 \end{bmatrix}.$$

(b) We have

$$\mathbf{v}_{\mathrm{CG}} = \frac{16}{100}\mathbf{v}_{\mathrm{HW}} + \frac{24}{100}\mathbf{v}_{\mathrm{A}} + \frac{24}{100}\mathbf{v}_{\mathrm{B}} + \frac{36}{100}\mathbf{v}_{\mathrm{Final}}$$

since in each individual vector entry (first, second, or third) this expresses precisely the uniform rule for computing the course grade. Using a calculator, we get $\mathbf{v}_{\mathrm{CG}} = \begin{bmatrix} 76.68 \\ 79.12 \\ 91.44 \end{bmatrix}$.