1. (2 points) Suppose the matrix A satisfies

$$A = ST$$

where

$$S = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

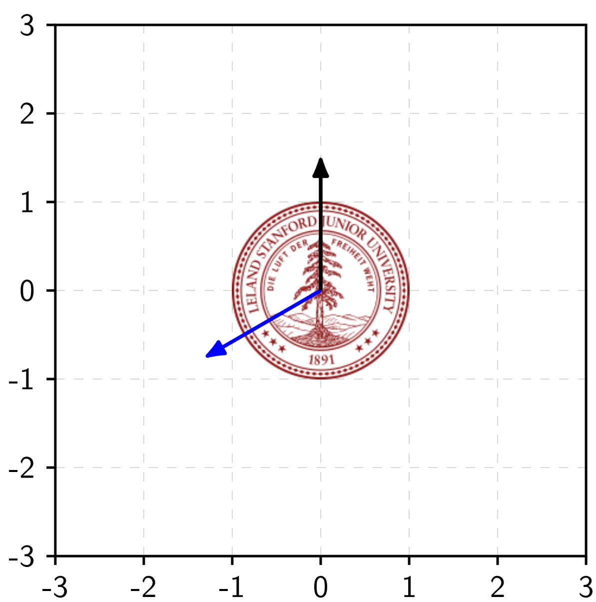
Observe that the columns of S are orthonormal.

What are the dimensions of $N(A)$ and $C(A)$ (i.e., the null space and column space of A, respectively)?

(i) 0 (ii) 1 (iii) 2 (iv) 3

2. (4 points) Suppose A is a 2×2 matrix with eigenvalues μ_1, μ_2; and that B is a *symmetric* 2×2 matrix with eigenvalues λ_1, λ_2.

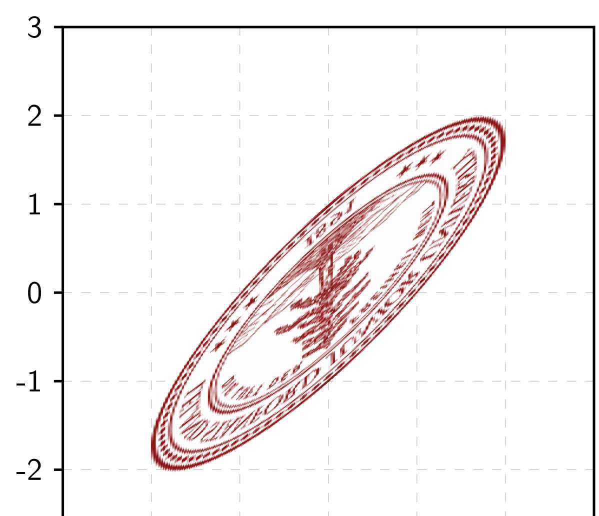
Consider also the following picture of the Stanford seal with two vectors (one black/vertical; one blue/non-vertical) superimposed, and suppose the additional three statements about these vectors, written alongside the figure:



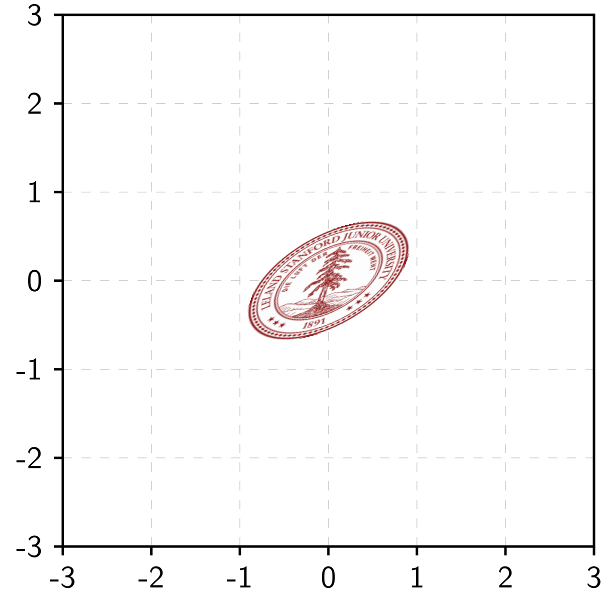
- The blue (non-vertical) vector shown above is an eigenvector of A with eigenvalue μ_1.
- The black (vertical) vector shown above is an eigenvector of A with eigenvalue μ_2.
- The blue (non-vertical) vector shown above is an eigenvector of B with eigenvalue λ_1.

Finally, shown below are the outputs when these matrices are applied to the original picture of the Stanford seal:

Effect of applying the 2×2 matrix A:



Effect of applying the *symmetric* 2×2 matrix B:



Given all of this information, estimate: the eigenvalues μ_1, μ_2, of A, and the eigenvalues λ_1, λ_2 of B.

(i) -2 (ii) -1 (iii) -0.5 (iv) 0 (v) 0.5 (vi) 1 (vii) 2

3. (3 points) Suppose M is a symmetric 3×3 matrix with eigenvalues -2 and 1, and suppose additionally that:

- vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, unit-length eigenvectors for M with eigenvalue -2; and
- vector $\mathbf{c}$ is a unit-length eigenvector for M with eigenvalue 1.

If we define the vector $\mathbf{x}$ by

$$\mathbf{x} = M^{2021}(\mathbf{a} - \mathbf{c}),$$

then suppose

- the angle between $\mathbf{x}$ and $\mathbf{a}$ is A degrees; and
- the angle between $\mathbf{x}$ and $\mathbf{b}$ is B degrees; and
- the angle between $\mathbf{x}$ and $\mathbf{c}$ is C degrees.

What are the approximate values of A, B, C?

(i) 0 (ii) 45 (iii) 90 (iv) 135

(v) 180 (vi) not enough information to determine

4. (2 points) Suppose A and B are 2×2 symmetric matrices. In which of the following situations can we conclude that the collection of A's eigenvalues must be the same as the collection of B's eigenvalues? Select all that apply. (Note that each situation is to be considered separately.)

(i) A and B have the same determinant, and each has 5 as an eigenvalue.

(ii) A and B have the same trace, and each has $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as an eigenvector.

(iii) A and B have the same trace and determinant.

5. (3 points) Let A be a 3×3 matrix and let λ be an eigenvalue of A. Which of the following are possible dimensions of $C(A - \lambda I_3)$?
Select all that apply.

   (a) 0 (b) 1 (c) 2 (d) 3

6. (2 points) Recall from Exam 4 the Markov matrix

$$M = \begin{bmatrix} 3/4 & 1/6 & 0 \\ 1/4 & 2/3 & 1/2 \\ 0 & 1/6 & 1/2 \end{bmatrix}$$

describing the weekly dynamics of universities' opening and closing. M has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ with corresponding eigenvalues $1, 2/3, 1/4$.

$M^{2021}(\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3)$ is approximately

   (a) $\mathbf{v}_1$ (b) $2\mathbf{v}_2$ (c) $3\mathbf{v}_3$ (d) $\mathbf{0}$

7. (2 points) Let L be a lower triangular matrix, and $A = LL^\top$. Which of the following statements are always true? Select all that apply.

   (a) $LL^\top$ is an LU-decomposition of A.
   (b) $L^\top L$ is an LU-decomposition of A.
   (c) Solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L\mathbf{x} = \mathbf{b}$.
   (d) Solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L^\top\mathbf{x} = \mathbf{b}$.
   (e) If L is invertible, then solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L^\top\mathbf{x} = L^{-1}\mathbf{b}$.
   (f) Solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L\mathbf{x} = L^\top\mathbf{b}$.
   (g) Solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L^\top\mathbf{x} = L\mathbf{b}$.

8. (3 points) The symmetric matrix A has the property that

$$A\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -14 \\ -7 \\ 7 \end{bmatrix}.$$

Which of the following vectors $\mathbf{v}$ satisfies $q_A(\mathbf{v}) = \mathbf{v}^T A\mathbf{v} = 0$?

   (a) $\begin{bmatrix} 0 \\ 3 \\ 9 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

   (b) $\begin{bmatrix} -27 \\ -12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 14\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

   (c) $\begin{bmatrix} 12 \\ 27 \\ 57 \end{bmatrix} = 14\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

   (d) $\begin{bmatrix} -36 \\ -9 \\ 45 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 21\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

9. (2 points) Suppose a 3×3 invertible matrix A has QR-decomposition given by $A = QR$. Suppose A_1 is obtained from A by scaling its second column by a factor of c, while the other two columns are the same as the corresponding columns of A. What can you say about the QR decomposition of A_1?

   (a) $A_1 = Q_1 R_1$ where $Q_1 = Q$ and R_1 is obtained from R by scaling its second column by a factor of c, while the other two columns of R_1 are the same as the corresponding columns of R.

   (b) $A_1 = Q_1 R_1$ where $Q_1 = Q$ and R_1 is obtained from R by scaling its second row by a factor of c, while the other two rows of R_1 are the same as the corresponding rows of R.

   (c) $A_1 = Q_1 R_1$ where $R_1 = R$ and Q_1 is obtained from Q by scaling its second column by a factor of c, while the other two columns of Q_1 are the same as the corresponding columns of Q.

   (d) $A_1 = Q_1 R_1$ where $R_1 = R$ and Q_1 is obtained from Q by scaling its second row by a factor of c, while the other two rows of Q_1 are the same as the corresponding rows of Q.

10. (3 points) Suppose A is a 2×2 matrix with eigenvalues 2 and 3. Let

$$B = A^2 - 5A + 6I_2.$$

Which of the following statements are correct? Select all that apply.

   (a) $\mathbf{R}^2$ has a basis consisting of eigenvectors of A.

   (b) If $\mathbf{v}$ is an eigenvector for A, then $\mathbf{v}$ is a also an eigenvector for B.

   (c) If $\mathbf{v}$ is an eigenvector for B, then $\mathbf{v}$ is a also an eigenvector for A.

   (d) $B = 0$, i.e. B is a 2×2 matrix all of whose entries are 0.

   (e) $B \neq 0$, i.e. B is not a 2×2 matrix all of whose entries are 0.

11. (3 points) A population of 60 goats moves between three meadows, labeled A, B, and C. Their daily movement is modeled by a symmetric Markov matrix M (whose rows and columns correspond to meadows A, B, and C, in that order).

Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Assume that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of M with eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{5}{9}$, and $\lambda_3 = \frac{1}{3}$, respectively.

If k is large enough that $\left(\frac{5}{9}\right)^k \approx 0$ and $\left(\frac{1}{3}\right)^k \approx 0$, how many goats are there in meadow B after k days? (Your answer should be an integer.)

12. (4 points) Suppose A is a symmetric matrix with eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and corresponding eigenvalues $1, 2, 3$. Match each of the following inputs, (a)–(d), to its output (among (1)–(6); some inputs may match to the same output).

   (a) $(\mathbf{v}_2 \cdot \mathbf{v}_3)\mathbf{v}_1$

   (b) $(A\mathbf{v}_2 \cdot A\mathbf{v}_3)\mathbf{v}_1$

   (c) $\mathbf{Proj}_{\mathbf{v}_2} A(\mathbf{v}_2 + \mathbf{v}_3)$

   (d) $A\left(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{Proj}_{\mathbf{v}_2}(\mathbf{v}_3)\right)$

   (1) $\mathbf{0}$ (2) $6\mathbf{v}_1$ (3) $2\mathbf{v}_2$ (4) $2\mathbf{v}_1$ (5) $\mathbf{v}_1 + 2\mathbf{v}_2$ (6) $\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$