Problem 1: Orthogonality and projections

(a) In the span of $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $\begin{bmatrix}1\\-2\\3\\-4\end{bmatrix}$ find a non-zero vector $\mathbf{v}$ orthogonal to $\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$.

(b) Here is a geometric analogue to the algebra in (a): for a plane P through the origin in $\mathbf{R}^3$ and a nonzero 3-vector $\mathbf{w}$ not orthogonal to P, why should there always be nonzero vectors in P orthogonal to $\mathbf{w}$? (Hint: visualize the plane W through $\mathbf{0}$ with normal vector $\mathbf{w}$, and think about how it meets the plane P).

(c) Find a nonzero vector $\mathbf{u} \in \mathbf{R}^3$ for which the projections of $\mathbf{v} = \begin{bmatrix}3\\0\\4\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}2\\2\\1\end{bmatrix}$ onto $\mathbf{u}$ are equal. (Recall that the projection of $\mathbf{x}$ onto a nonzero vector $\mathbf{u}$ is given by the formula $\left(\frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$.) There are many answers. Informally, the condition says that $\mathbf{v}$ and $\mathbf{w}$ make the same "shadow" onto the line spanned by $\mathbf{u}$.

Solution:

(a) We seek a vector of the form

$$\mathbf{v} = c_1\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + c_2\begin{bmatrix}1\\-2\\3\\-4\end{bmatrix} = \begin{bmatrix}c_1+c_2\\2c_1-2c_2\\3c_1+3c_2\\4c_1-4c_2\end{bmatrix}$$

satisfying $\mathbf{v}\cdot\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = 0$, which says $c_1+c_2+2c_1-2c_2+3c_1+3c_2-4c_1+4c_2 = 0$, or equivalently $2c_1+6c_2 = 0$.

We can now choose $c_1 = 3, c_2 = -1$ to obtain $\mathbf{v} = 3\begin{bmatrix}1\\2\\3\\4\end{bmatrix} - \begin{bmatrix}1\\-2\\3\\-4\end{bmatrix} = \begin{bmatrix}2\\8\\6\\16\end{bmatrix}$. (Any nonzero scalar multiple of this works just as well.)

(b) The visualization is that P and W are two planes through the origin that are *distinct* (since $\mathbf{w}$ is on the normal line to the second of these planes, but not the first). By drawing a picture we see that any two such planes meet exactly along a line through the origin. Anything nonzero in that line does the job: it lies in P and in W, so it is orthogonal to $\mathbf{w}$.

(c) From the projection formula, we seek a nonzero vector $\mathbf{u}$ for which $\mathbf{v}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{u}$, or equivalently $(\mathbf{v}-\mathbf{w})\cdot\mathbf{u} = 0$. Writing $\mathbf{u} = \begin{bmatrix}x\\y\\z\end{bmatrix}$, since $\mathbf{v}-\mathbf{w} = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ this says

$$0 = (\mathbf{v}-\mathbf{w})\cdot\mathbf{u} = x - 2y + 3z.$$

We can solve for any one among x, y, z in terms of the other two, so we can choose two of coordinates to be whatever we want that isn't both 0 and then uniquely fill in the third coordinate to get a nonzero $\mathbf{u}$ as desired.

For example, one nonzero triple (x, y, z) that works is $x = -1, y = 1, z = 1$, which is to say $\mathbf{u} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$. (There are *many* nonzero solutions: the condition of being orthogonal to $\mathbf{v}-\mathbf{w}$ defines an entire plane through the origin, and the $\mathbf{u}$ we found is but one nonzero point in that plane.)

Problem 2: An orthogonal basis

Let V be the set of vectors $\mathbf{v} \in \mathbf{R}^3$ satisfying $\mathbf{v} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ (this says that both of these explicit 3-vectors have the same projection onto $\mathbf{v}$, or in other words make the same "shadow" onto the line spanned by $\mathbf{v}$).

(a) Express V as the collection of 3-vectors orthogonal to a single nonzero 3-vector.

(b) By fiddling with orthogonality equations, build an orthogonal basis of V. There are many possible answers.

(c) Use your answer to (b) to give an orthonormal basis for V.

Solution:

(a) Some vector algebra simplifies the condition for belonging to V: the given condition

$$\mathbf{v} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

says exactly that $\mathbf{v} \cdot \left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = 0$, which is to say

$$\mathbf{v} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0.$$

That is, V is the collection of 3-vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) The vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ belongs to V precisely when

$$x + y + z = 0,$$

which is the equation defining a plane through the origin (i.e., its dimension is 2) having normal vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. So to find an orthogonal basis for V, we need to find two nonzero vectors $\mathbf{v}_1, \mathbf{v}_2$ perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* perpendicular to each other. For $\mathbf{v}_1$, we can pick any nonzero vector in the given plane; that is, any solution to the defining equation other than $(0, 0, 0)$. One such vector is $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (and again, there are many other choices). Having chosen $\mathbf{v}_1$, the condition on $\mathbf{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is that it is a solution to the pair of equations

$$\begin{aligned} x + y + z &= 0 \\ y - z &= 0 \end{aligned}$$

other than the zero-vector solution $(0,0,0)$. We have $y = z$, and $x = -y - z$, so $x = -z - z = -2z$. One solution is given by choosing $z = 1$, so then $y = 1$ and $x = -2$. That is, we can use $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ (and any nonzero scalar multiple of this $\mathbf{v}_2$ works just as well, given the choice we made for $\mathbf{v}_1$). In other words, the pair of vectors

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is an orthogonal basis for the plane V.

(c) Now dividing by lengths gives an orthonormal basis:

$$\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

Problem 3: Subspaces defined by orthogonality, orthogonal bases, and shortest distances in R^3

(a) For each linear subspace V_i in $\mathbf{R}^3$ given below, exhibit the set

$$V_i' = \{\mathbf{x} \in \mathbf{R}^3 \mid \mathbf{x} \text{ is orthogonal to every vector in } V_i\}$$

as the span of a finite collection of vectors (so, as a linear subspace), and give a basis for V_i'.

(i) $V_1 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$

(ii) V_2 is the set of solutions in $\mathbf{R}^3$ to the pair of equations $\begin{cases} x_1 + 2x_2 + 3x_3 = 0, \\ 4x_1 + 5x_2 + 6x_3 = 0. \end{cases}$ (*Hint:* relate this to V_1 and think geometrically.)

(b) For each of the two V_i's given above, compute an orthogonal basis for it and *set up* how you'd find the distance from the point $\begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$ to V_i (i.e. the minimal distance from $\begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$ to a point in V_i) using such a basis. Finally, compute each distance. (*Hint for computation:* first treat the case of V_2. For the case of the plane V_1, use projections to compute an orthogonal basis and to give an expression for a vector whose length is the distance you want. It gets cumbersome to carry out that distance calculation by hand, so instead compute the distance to V_1 by relating it to the distance to V_2. Try drawing a picture of an orthogonal line and plane to get an idea.)

Solution:

(a)(i) A vector $\mathbf{x}$ in V_1' must be orthogonal to both $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and if it is perpendicular to those two vectors then it is perpendicular to everything in their span:

$$\mathbf{x} \cdot \left(a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) = a\left(\mathbf{x} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) + b\left(\mathbf{x} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) = 0a + 0b = 0.$$

So V_1' consists of exactly those $\mathbf{x}$ that are orthogonal to these two vectors, which is expressed by satisfying the pair of equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0, \\ 4x_1 + 5x_2 + 6x_3 = 0. \end{cases}$$

Since the two given 3-vectors spanning V_1 are nonzero and not scalar multiples of each other, V_1 is a plane through the origin in $\mathbf{R}^3$. Since the vectors perpendicular to V_1 make up its normal line, the solutions to this system constitute a line through the origin: the multiples of a single nonzero solution.

To make a nonzero solution, we try setting one of the x_i's to be 1 and solve for the others in the resulting pair of equations in the two remaining variables. Setting $x_3 = 1$, for example, we get the equations $x_1 + 2x_2 = -3$ and $4x_1 + 5x_2 = -6$, for which the solution is found to be $(1, -2)$. Hence, V_1' is the span of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, so that single vector is a basis for V_1'.

(a)(ii) By inspection, V_2 is defined by exactly the orthogonality equations found in (i) that defined V_1', so $V_2 = V_1'$ is the normal line to the plane V_1 through the origin. By visualization in $\mathbf{R}^3$, the vectors perpendicular to the normal line to a plane through the origin is that very same plane. Hence, $V_2' = V_1$. (Later in the course we will discuss the general fact that for any linear subspace V of any $\mathbf{R}^n$, those vectors perpendicular to V' are precisely those in V. In the present case we can "see" this for $V = V_1$ inside $\mathbf{R}^3$.) Thus, a basis for V_2' is the same thing as a basis for V_1, such as the pair of nonzero vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ whose span defined V_1 (and which aren't scalar multiples of each other).

(b) **Distance to V_2 (setup and computation):** we have seen that V_2 is a line spanned by $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ (which by itself constitutes an orthogonal basis for V_2). The distance from a point $\mathbf{x}$ to a 1-dimensional subspace L of $\mathbf{R}^n$ is the length of the difference vector vector $\mathbf{x} - \mathbf{Proj}_L(\mathbf{x})$, since $\mathbf{Proj}_L(\mathbf{x})$ is the point in L closest to $\mathbf{x}$. Thus, the distance from $\mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$ to V_2 is the length of the difference vector

$$\begin{aligned} \mathbf{x} - \mathbf{Proj}_{V_2}(\mathbf{x}) = \mathbf{x} - \mathbf{Proj}_{\mathbf{u}}(\mathbf{x}) &= \mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \\ &= \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} - \frac{7}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 - 7/6 \\ 7/3 \\ -7/6 \end{bmatrix} \\ &= \frac{7}{6}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}, \end{aligned}$$

whose length is $(7/6)\sqrt{25+4+1} = 7\sqrt{30}/6$. (This answer can be written in various other ways, such as $7\sqrt{5/6}$.)

Distance to V_1 (setup using orthogonal basis): The distance from a point $\mathbf{x}$ to the plane V_1 is the length of the difference $\mathbf{x} - \mathbf{Proj}_{V_1}(\mathbf{x})$ between $\mathbf{x}$ and the unique point $\mathbf{Proj}_{V_1}(\mathbf{x})$ on V_1 closest to $\mathbf{x}$. This latter projection can be computed using an orthogonal basis of V_1, which can be computed from a basis of V_1 (such as the two vectors that span V_1 in its definition). More specifically, if we define $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ then an orthogonal basis of V_1 is given by $\mathbf{v}$ and

$$\mathbf{w}' = \mathbf{w} - \mathbf{Proj}_{\mathbf{v}}(\mathbf{w}) = \mathbf{w} - \frac{\mathbf{w}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \mathbf{w} - \frac{32}{14}\mathbf{v} = \begin{bmatrix} 24/14 \\ 6/14 \\ -12/14 \end{bmatrix} = \frac{6}{14}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} = \frac{3}{7}\mathbf{w}''$$

for $\mathbf{w}'' = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$. Hence, for any $\mathbf{x}$ (such as $\begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$) we can then compute the difference

$$\mathbf{x} - \mathbf{Proj}_{V_1}(\mathbf{x}) = \mathbf{x} - \mathbf{Proj}_{\mathbf{v}}(\mathbf{x}) - \mathbf{Proj}_{\mathbf{w}''}(\mathbf{x})$$

and its length is the desired minimal distance. (The final vector and its length are written below, at the bottom of the following simpler alternate solution.)

Distance to V_1 (simpler computation using $V_2 = V_1'$): Since the computation we've set up above can get rather cumbersome to do by hand (though a computer can do it rapidly), here is a nice alternative that takes advantage of knowing that $V_1' = V_2$.

Observe that by the Orthogonal Projection Theorem applied to $\mathbf{x}$ and V_1, $\mathbf{x}$ can be expressed uniquely as a sum

$$\mathbf{x} = \mathbf{y} + \mathbf{y}'$$

where $\mathbf{y}$ lies in V_1 and $\mathbf{y}'$ lies in $V_1' = V_2$; and furthermore $\mathbf{y} = \mathbf{Proj}_{V_1}(\mathbf{x})$. But also observe that by the same theorem as applied to $\mathbf{x}$ and V_2, $\mathbf{x}$ can be expressed uniquely as a sum of vectors

$$\mathbf{x} = \mathbf{z} + \mathbf{z}'$$

where $\mathbf{z}$ lies in V_2 and $\mathbf{z}'$ lies in $V_2' = V_1$; and furthermore $\mathbf{z} = \mathbf{Proj}_{V_2}(\mathbf{x})$.

Now, the uniqueness condition of the Orthogonal Projection Theorem means that these *two ways* of expressing $\mathbf{x}$ as a sum of vectors from each of V_1 and V_2 is actually *the same way*! We finally conclude that

$$\mathbf{x} = \mathbf{Proj}_{V_1}(\mathbf{x}) + \mathbf{Proj}_{V_2}(\mathbf{x}), \text{ crucially using the facts that } V_1' = V_2 \text{ and } V_2' = V_1.$$

(Since V_1 is a plane in $\mathbf{R}^3$ and V_2 is its normal line, you can draw a picture to confirm this!) Thus, the vector whose length is the distance we need is simply

$$\mathbf{x} - \mathbf{Proj}_{V_1}(\mathbf{x}) = \mathbf{Proj}_{V_2}(\mathbf{x}) = \mathbf{Proj}_{\mathbf{u}}(\mathbf{x}).$$

For $\mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$, this projection was computed in our work on the distance to V_2: it is $(7/6)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. This has length $(7/6)\sqrt{6} = 7/\sqrt{6}$, so this is the distance from $\mathbf{x}$ to V_1.

Problem 4: Building another orthogonal vector (Extra)

If $\{\mathbf{v}, \mathbf{w}\}$ is a pair of nonzero orthogonal vectors in $\mathbf{R}^3$ then we can always enlarge it to an orthogonal basis $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ of $\mathbf{R}^3$ by taking $\mathbf{u}$ to be a nonzero normal vector to the plane $\mathrm{span}(\mathbf{v}, \mathbf{w})$. If $n > 3$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ are mutually orthogonal nonzero vectors in $\mathbf{R}^n$ then can we always find a nonzero $\mathbf{v}_n$ orthogonal to those (so $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis of $\mathbf{R}^n$)?

Solution: It feels like this should always be possible, though it may be very challenging to figure out how to actually do it at the present point of the course. First, we observe that there *are* n-vectors outside $V = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$; i.e., V does not exhaust $\mathbf{R}^n$. Indeed, V has dimension $n-1$ (as any orthogonal collection of nonzero vectors is a basis for its span) whereas $\dim \mathbf{R}^n = n$, so definitely $V \neq \mathbf{R}^n$. Pick a vector $\mathbf{x} \in \mathbf{R}^n$ outside V. Any vector

$$\mathbf{x}' = \mathbf{x} - c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - \cdots - c_{n-1}\mathbf{v}_{n-1}$$

obtained from $\mathbf{x}$ by adjusting it by a linear combinations of the $\mathbf{v}_j$'s is also outside V: by inspection $\mathbf{x}' = \mathbf{x} - \mathbf{v}$ for $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1} \in V$, so if we were to have $\mathbf{x}' \in V$ then also $\mathbf{x} = \mathbf{x}' + \mathbf{v}$ would belong to V, contrary to how $\mathbf{x}$ was chosen. Hence, for any scalars $c_1, \ldots, c_{n-1}$ the resulting vector $\mathbf{x}'$ is not in V (in particular, $\mathbf{x}'$ is nonzero!). We will find such $c_1, \ldots, c_{n-1}$ making $\mathbf{x}'$ orthogonal to $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$; such an $\mathbf{x}'$ can be taken as $\mathbf{v}_n$.

We want to make each $\mathbf{x}' \cdot \mathbf{v}_j$ vanish (for $1 \leq j \leq n-1$). To ease the notation, let's first consider $\mathbf{x}' \cdot \mathbf{v}_1$. Since $\mathbf{v}_1 \cdot \mathbf{v}_i = 0$ for all $i > 1$, we have

$$\mathbf{x}' \cdot \mathbf{v}_1 = \mathbf{x} \cdot \mathbf{v}_1 - c_1(\mathbf{v}_1 \cdot \mathbf{v}_1) - c_2(\mathbf{v}_2 \cdot \mathbf{v}_1) - \cdots - c_{n-1}(\mathbf{v}_{n-1} \cdot \mathbf{v}_1) = \mathbf{x} \cdot \mathbf{v}_1 - c_1(\mathbf{v}_1 \cdot \mathbf{v}_1).$$

To make this vanish, there is exactly one choice for c_1 that does the job:

$$c_1 = \frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}$$

(the division makes sense since $\mathbf{v}_1 \cdot \mathbf{v}_1 = \|\mathbf{v}_1\|^2 > 0$ since $\mathbf{v}_1$ is nonzero). By exactly the same type of calculation, using that each $\mathbf{v}_j$ is orthogonal to the rest and is nonzero, for each $1 \leq j \leq n-1$ there is exactly one value of c_j which makes $\mathbf{x}'$ orthogonal to $\mathbf{v}_j$:

$$c_j = \frac{\mathbf{x} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}.$$

Thus, we are led to define

$$\mathbf{v}_n = \mathbf{x} - \sum_{j=1}^{n-1} \left(\frac{\mathbf{x} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j} \right) \mathbf{v}_j;$$

the preceding work shows that $\mathbf{v}_n$ is orthogonal to $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ and moreover $\mathbf{v}_n \notin V$, so $\mathbf{v}_n$ is nonzero. This does the job.

