Solutions to Math 51 Quiz 2 Practice B

1. (10 points) Find the (shortest) distance between the two parallel planes

$$2x - y + 2z = 3$$

and

$$2x - y + 2z = 12.$$

Since the two planes are parallel, the distance between them is the same as the distance from a point P on $2x - y + 2z = 3$ to the other plane $2x - y + 2z = 12$. $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ is normal to both planes. We choose $P = (0, -3, 0)$ in the plane $2x - y + 2z = 3$ and choose $Q = (6, 0, 0)$ in the plane $2x - y + 2z = 12$. To find the distance between the two planes, we project the displacement vector $\mathbf{v} = \overrightarrow{PQ} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$ onto the normal vector $\mathbf{n}$ to obtain $\mathbf{Proj_n\, v}$, and the length of this projection vector is the distance between the two planes.

$$\mathbf{Proj_n\, v} = \frac{\mathbf{v} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} = \frac{9}{9} \mathbf{n} = \mathbf{n}.$$

$\|\mathbf{n}\| = 3$, so the distance between the two planes is 3.
Note that if you projected $\mathbf{v}$ onto plane Π given by $2x - y + 2z = 12$ instead, you would need an orthogonal basis for $2x - y + 2z = 12$. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ are two vectors normal to $\mathbf{n}$,

$$\mathbf{u}_3 = \mathbf{u}_2 - \mathbf{Proj}_{\mathbf{u}_1}(\mathbf{u}_2) = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{\mathbf{u}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 = \frac{1}{5} \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix}.$$

$$\mathbf{Proj}_\Pi\, \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{v} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} \mathbf{u}_3 = \frac{12}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{18}{5} \frac{25}{45} \frac{1}{5} \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}.$$

The distance from P to Π is

$$|\mathbf{v} - \mathbf{Proj}_\Pi\, \mathbf{v}| = \left\| \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\| = \|\mathbf{n}\| = 3$$

2. (2 points) **True or False:** Let V and W be linear subspaces of $\mathbb{R}^{51}$ and consider the set $U = \{\mathbf{v} + \mathbf{w} : \mathbf{v} \in V, \mathbf{w} \in W\}$. Then, U is a linear subspace of $\mathbb{R}^{51}$.

Since V is a linear subspace of $\mathbb{R}^{51}$, there is a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for V. Similarly, there is a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_l\}$ for W. Then, $U = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_l)$, and so, U is indeed a linear subspace of $\mathbb{R}^{51}$; the statement is **TRUE**.

3. (2 points) **True or False:** Let $\mathbf{u}$, $\mathbf{v}$ be fixed vectors in $\mathbf{R}^n$ such that $W = \text{span}(\mathbf{u}, \mathbf{v})$ is a linear subspace of dimension 2. Suppose $\mathbf{x}$ is a vector in $\mathbf{R}^n$. Then it is always the case that

$$\|\mathbf{Proj}_W(\mathbf{x})\| \geq \|\mathbf{Proj}_\mathbf{u}(\mathbf{x})\|.$$

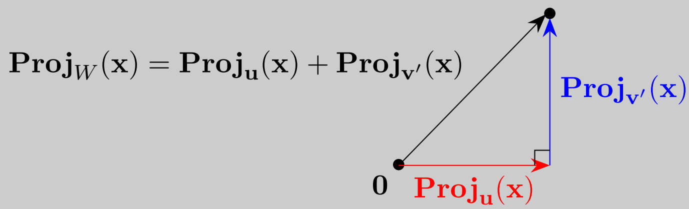
True. If $\mathbf{u}$, $\mathbf{v}'$ form an *orthogonal* basis for W (e.g., if $\mathbf{v}' = \mathbf{v} - \mathbf{Proj}_{\mathbf{u}}(\mathbf{v})$), then by the Orthogonal Projection Theorem,

$$\mathbf{Proj}_W(\mathbf{x}) = \mathbf{Proj}_{\mathbf{u}}(\mathbf{x}) + \mathbf{Proj}_{\mathbf{v}'}(\mathbf{x}).$$

But the two vectors $\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$ and $\mathbf{Proj}_{\mathbf{v}'}(\mathbf{x})$ are *orthogonal*, because they are scalar multiples of the orthogonal basis vectors $\mathbf{u}$ and $\mathbf{v}'$ respectively. As a result,

$$\|\mathbf{Proj}_W(\mathbf{x})\|^2 = \|\mathbf{Proj}_{\mathbf{u}}(\mathbf{x}) + \mathbf{Proj}_{\mathbf{v}'}(\mathbf{x})\|^2 = \|\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})\|^2 + \|\mathbf{Proj}_{\mathbf{v}'}(\mathbf{x})\|^2$$

by the Pythagorean Theorem[1]:



Now since

$$\|\mathbf{Proj}_W(\mathbf{x})\|^2 = \|\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})\|^2 + \|\mathbf{Proj}_{\mathbf{v}'}(\mathbf{x})\|^2,$$

and since any squared-magnitude $\|\mathbf{Proj}_{\mathbf{v}'}(\mathbf{x})\|^2 \geq 0$, we have

$$\|\mathbf{Proj}_W(\mathbf{x})\|^2 \geq \|\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})\|^2,$$

so $\|\mathbf{Proj}_W(\mathbf{x})\| \geq \|\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})\|$.

4. (3 points) Consider the three vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$$

and the three planes

$$P_1 = \text{span}(\mathbf{v}_2, \mathbf{v}_3), \;\; P_2 = \text{span}(\mathbf{v}_1, \mathbf{v}_3), \;\; P_3 = \text{span}(\mathbf{v}_1, \mathbf{v}_2).$$

Note that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \;\; \mathbf{v}_1 \cdot \mathbf{v}_3 = 0.$$

What is $\text{span}\left(\mathbf{Proj}_{P_1}(\mathbf{v}_1), \mathbf{Proj}_{P_2}(\mathbf{v}_2), \mathbf{Proj}_{P_3}(\mathbf{v}_3)\right)$?

a) The plane P_1. b) The plane P_2. c) The plane P_3. d) All of $\mathbf{R}^3$.

Note that $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$ and $\mathbf{v}_3$, and hence to P_1, so $\mathbf{Proj}_{P_1}(\mathbf{v}_1) = \mathbf{0}$.
$\{\mathbf{v}_1, \mathbf{v}_3\}$ is an orthogonal basis for P_2,

$$\mathbf{Proj}_{P_2}(\mathbf{v}_2) = \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 = -\frac{12}{18}\mathbf{v}_3 = -\frac{2}{3}\mathbf{v}_3$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for P_3,

$$\mathbf{Proj}_{P_3}(\mathbf{v}_3) = \frac{\mathbf{v}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = -\frac{12}{12}\mathbf{v}_2 = -\mathbf{v}_2$$

Hence,

$$\operatorname{span}\left(\mathbf{Proj}_{P_1}(\mathbf{v}_1), \mathbf{Proj}_{P_2}(\mathbf{v}_2), \mathbf{Proj}_{P_3}(\mathbf{v}_3)\right) = \operatorname{span}\left(\mathbf{0}, -\frac{2}{3}\mathbf{v}_3, -\mathbf{v}_2\right) = \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3) = P_1.$$

$$\operatorname{span}\left(\mathbf{v}_1 - \mathbf{Proj}_{P_1}(\mathbf{v}_1), \mathbf{Proj}_{P_2}(\mathbf{v}_2), \mathbf{Proj}_{P_3}(\mathbf{v}_3)\right) = \operatorname{span}\left(\mathbf{v}_1, -\frac{2}{3}\mathbf{v}_3, -\mathbf{v}_2\right) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathbf{R}^3.$$

$$\operatorname{span}\left(\mathbf{Proj}_{P_1}(\mathbf{v}_1), \mathbf{v}_2 - \mathbf{Proj}_{P_2}(\mathbf{v}_2), \mathbf{Proj}_{P_3}(\mathbf{v}_3)\right) = \operatorname{span}\left(\mathbf{0}, \mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3, -\mathbf{v}_2\right) = \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3) = P_1.$$

$$\operatorname{span}\left(\mathbf{Proj}_{P_1}(\mathbf{v}_1), \mathbf{Proj}_{P_2}(\mathbf{v}_2), \mathbf{v}_3 - \mathbf{Proj}_{P_3}(\mathbf{v}_3)\right) = \operatorname{span}\left(\mathbf{0}, -\frac{2}{3}\mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_2\right) = \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3) = P_1.$$

5. (3 points) For each of the following sets V, if it is a linear subspace, determine its dimension $\dim(V)$; if it is not a linear subspace, write 0.

(A) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2 : x^2 = 4y^2 \right\}$

(B) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3 : x + y = z + 1 \right\}$

(C) $V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbf{R}^4 : x + w = y + z \right\}$

(D) $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbf{R}^5 : x_1 + 2x_2 + 3x_3 = 0 \text{ and } x_1 + x_2 + x_3 = x_4 + x_5 \right\}$

(A) Note that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \in V$, but $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ is not in V since $4^2 \neq 4 \cdot 0^2$. V cannot be a linear subspace by Prop. 4.1.11 in the textbook.

(B) Note that $\mathbf{0}$ is not in V. Since any linear subspace must contain $\mathbf{0}$, V cannot be a linear subspace if it doesn't contain $\mathbf{0}$.

(C) V consists of vectors of the form

$$\begin{bmatrix} x \\ y \\ z \\ -x + y + z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for V, so V is 3-dimensional.

(D) $x_1 + 2x_2 + 3x_3 = 0$ implies $x_1 = -2x_2 - 3x_3$; plug this into $x_1 + x_2 + x_3 = x_4 + x_5$, we have $-x_2 - 2x_3 = x_4 + x_5$, so $x_5 = -x_2 - 2x_3 - x_4$. V consists of vectors of the form

$$\begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \\ x_4 \\ -x_2 - 2x_3 - x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ forms a basis for V, so V is 3-dimensional.