1. (3 points) Suppose that $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ satisfies

$$f(0,0) = (1,1), \quad f(1,1) = (0,0).$$

Let A be the derivative matrix of f at the point $(0,0)$; that is,

$$A = (Df)(0,0).$$

Furthermore, suppose we define the function $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ by the formula

$$h(\mathbf{x}) = A\mathbf{x}$$

for this *same* matrix A.

Given only this information about f and h, which of the following expressions must *always* be equal to A^2 ? Select all that apply.

(i) $(D(f \circ h))(0,0)$ (ii) $(D(f \circ f))(0,0)$ (iii) $(D(h \circ f))(0,0)$

(iv) $(D(f \circ h))(1,1)$ (v) $(D(h \circ h))(1,1)$

2. (4 points) Suppose A and B are invertible 51×51 matrices. Evaluate each of the following statements as either **true** (i.e., *always* true) or **false** (i.e., sometimes not true):

(a) If $A^8 = A$, then A must be equal to the identity matrix.

(b) The matrix $A^{-1}B^{-1}AB$ must be invertible.

3. (4 points) A Peruvian lives near a mountain range and notices goats moving between mountain A and mountain B every year. He observes the following migration pattern, for certain constant values α, β:

- $\alpha\%$ of the goats on mountain A in one year migrate to mountain B in the next year; and
- $\beta\%$ of the goats on mountain B in one year migrate to mountain A in the next year.

We are assuming that no new goats are born and no goat dies.

Let $\mathbf{g}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$ be the goat vector, where a_n is the number of goats on mountain A after n years and b_n is the number of goats on mountain B after n years.

Let M be the Markov matrix for which $\mathbf{g}_{n+1} = M\mathbf{g}_n$, and suppose that, for certain constant values γ, δ, the matrix M takes the form:

$$M = \begin{bmatrix} \gamma & 0.4 \\ \delta & 0.6 \end{bmatrix}$$

If we learn that for large values of n,

$$\mathbf{g}_n = \begin{bmatrix} 2000 \\ 1000 \end{bmatrix},$$

then what are the values of each of α and β ?

(Note that these will be percentages; i.e., numbers between 0 and 100.)

(i) 20 (ii) 40 (iii) 60

(iv) 80 (v) not able to be determined from the given information

4. (3 points) Suppose

$$f : \mathbf{R}^2 \to \mathbf{R}, \quad (\nabla f)(0,0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$g : \mathbf{R}^2 \to \mathbf{R}^2, \quad g(x,y) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Which of the following is $(\nabla(f \circ g))(0,0)$?

(a) $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$

5. (4 points) Match the appropriate Markov matrix with the description about annual population movement between the cities. In the matrix, the rows/columns correspond to A,B,C,D in that order.

$$M_1 = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0.3 & 0.8 & 0 \\ 0 & 0.7 & 0.2 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.7 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0 & 0 \\ 0 & 0.4 & 0.9 & 0 \\ 0 & 0.3 & 0.1 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0.6 & 0.6 & 0.2 & 0 \\ 0.2 & 0.1 & 0.4 & 0 \\ 0.2 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0.1 & 0 & 0 & 0.9 \\ 0 & 0.1 & 0.9 & 0 \\ 0.9 & 0 & 0.1 & 0 \\ 0 & 0.9 & 0 & 0.1 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0.4 & 0.2 & 0.4 & 0.7 \\ 0.1 & 0.5 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0.1 & 0.2 & 0.1 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0.4 & 0.8 & 0 & 0 \\ 0.6 & 0.2 & 0 & 0 \\ 0 & 0 & 0.9 & 0.7 \\ 0 & 0 & 0.1 & 0.3 \end{bmatrix}.$$

(a) People move between A and B, people move between C and D, but there is no movement from [A or B] to [C or D] or vice versa.

(b) After many years, almost everyone will end up in D.

(c) From any city, people move to any other city eventually, but it sometimes requires multiple years to move between two particular cities.

(d) People move between A and D, people move between B and C, but there is no movement from [A or D] to [B or C] or vice versa.

6. (3 points) Consider the matrices

$$M = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & -1 & -2 & 0 \\ 6 & 0 & 0 & 3 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

It is a fact that

$$M_1 = L_1 M R_1, \quad M_2 = L_2 M R_2, \quad M_3 = L_3 M R_3$$

where each of L_1, R_1, L_2, R_2, L_3 and R_3 is a certain one of the following matrices:

(a) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (h) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Identify which matrix each of L_1, R_1, L_2, R_2, L_3 and R_3 must be.
Hint: For each i, one of L_i and R_i is an identity matrix; the other one manipulates either the columns or rows of M.

7. (3 points) Let

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

Which of the following are always true?
Select all that apply.

(a) $AB = BA$ (b) $AC = CA$ (c) $BC = CB$

8. (3 points) **True or False:** Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f(x,y) = (x^3 + y^3 + x, x^3 - y^3 - y)$. $(Df)(x,y)$ is invertible for every point $(x,y) \in \mathbf{R}^2$.

9. (1 point) **True or False:** The matrix $\begin{bmatrix} 0 & 1 & 8 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 1 & 1 & 8 \\ 0 & 2 & 2 & 2 \end{bmatrix}$ is invertible.

10. (4 points) Stanford University's President is trying to decide whether Stanford should be fully open, partially open, or remote-only in the spring. The data collected from college campuses around the country so far this academic year seems to indicate that from one week to the next:

- 1/4 of the fully open universities were partially open the following week, the rest remain fully open;
- 1/6 of the partially open universities were fully open the following week, 1/6 of the partially open universities were remote-only the following week, and the rest of the partially open universities remain partially open;
- 1/2 of the remote-only universities were partially open the following week, the rest remain remote-only.

Let M be the Markov matrix describing the weekly dynamics of universities' opening and closing, where the rows/columns correspond to fully open, partially open, and remote-only in that order. It turns out that

$$M^{15} \approx \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 1/2 \\ 1/6 & 1/6 & 1/6 \end{bmatrix}.$$

With what probability will a fully open university be remote-only after 2 weeks?
With what probability will a university be fully open after 15 weeks?