Problem 1: Parametric form of a plane

Let P be the plane in $\mathbf{R}^3$ containing the points $(1,1,1)$, $(1,2,3)$, and $(3,2,1)$.

(a) Find a parametric representation of P. (Extra: can you write down many other parametrizations?)

(b) Use the dot product to find a normal vector to P. (Hint: Think about why it is the same as a vector perpendicular to two different "directions" within the plane, and then form some displacement vectors.)

(c) Find an equation for P of the form $ax + by + cz = d$ for some a, b, c, d in $\mathbf{R}$. (You can do this with or without (b).)

Solution: (a) The difference vectors connecting the point $(1,1,1)$ to the points $(1,2,3)$ and $(3,2,1)$ are $\begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$, respectively. Hence, one parametric representation of the plane P is

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\2\end{bmatrix} + t'\begin{bmatrix}2\\1\\0\end{bmatrix}, \quad t, t' \in \mathbf{R}.$$

To get other parametrizations of the plane, we can replace the point $\begin{bmatrix}1\\1\\1\end{bmatrix}$ in the above expression by any other point in the plane and then adjust the two difference vectors accordingly, or even leave the above difference vectors unchanged (which corresponds to simply using another reference point from which all motion in the plane is swept out as t and u vary). Or we could go back to the start and use one of the other two given vectors in place of $(1,1,1)$ as the " base point" whose displacement we compute from the others (and there are many other options that could be considered, as we will see later in the course).

(b) We want to find a nonzero vector $\begin{bmatrix}a\\b\\c\end{bmatrix}$ that is perpendicular to the difference vectors $\begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$ that point along different directions in the plane. This amounts to finding a nonzero solution to the system of equations

$$\begin{cases} a\cdot 0 + b\cdot 1 + c\cdot 2 = 0 \\ a\cdot 2 + b\cdot 1 + c\cdot 0 = 0 \end{cases}$$

The first equation means $b = -2c$. The second equation means $b = -2a$. Thus, the condition is $a = c = -\frac{1}{2}b$.

Clearly $a = 1, b = -2, c = 1$ is one such solution. Using the dot product, we can check that the vector $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ is indeed perpendicular to both $\begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$.

One may also notice that this system of equations has many solutions - indeed, for any nonzero t, we can take $a = t, b = -2t, c = t$. This reflects the fact that there are many vectors perpendicular to both $\begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$: if $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ is perpendicular to those two vectors, then so is every scalar multiple of $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ (the nonzero multiples being the interesting ones).

(c) Since $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a normal vector to P, it is perpendicular to every vector connecting any two points on P. In particular, for any point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ on P, the vector $\begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}$ connecting $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ must be perpendicular to $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. This perpendicularity is expressed by a dot product being 0:

$$1 \cdot (x-1) + (-2) \cdot (y-1) + 1 \cdot (z-1) = 0.$$

Thus, expanding out and combining terms, we obtain that

$$x - 2y + z = 0$$

is an equation for P.

Problem 2: Equation of a plane

(a) Consider the distinct points $A = (0,1,1)$, $B = (3,4,4)$, and $C = (1,-1,-4)$. Compute the nonzero displacement vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to confirm these are not scalar multiples of each other, so these three points lie in a unique common plane P. Find an equation for P of the form $ax + by + cz = d$.

(b) Find a *unit* vector (i.e., a vector of length 1) that is normal to the plane whose equation is $6x - 2y - 3z = 4$. Your answer should have entries that are fractions (no ugly square roots).

(c) Are the planes in (a) and (b) parallel to each other? How do you know?

Solution: (a) The displacement vectors are $\overrightarrow{AB} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$. These are clearly not scalar multiples of each other, so indeed there is a unique plane P through the three points. A normal vector to P is a vector perpendicular to the displacements, which is to say a normal vector $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ satisfies $\mathbf{n} \cdot \overrightarrow{AB} = 0 = \mathbf{n} \cdot \overrightarrow{AC}$, or equivalently

$$3n_1 + 3n_2 + 3n_3 = 0,$$
$$n_1 - 2n_2 - 5n_3 = 0.$$

We want to find some nonzero solution, so we can set $n_3 = 1$ and then solve the resulting pair of equations for n_1 and n_2, arriving at $n_1 = 1$ and $n_2 = -2$: the nonzero solution is $\mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, which we may take as our normal vector to the plane. (We could have instead set n_2 or n_1 to equal 1 and solved for the remaining two n_i's using the resulting two equations; each would yield a solution that is a nonzero scalar multiple of the $\mathbf{n}$ that we just found.)

The plane's equation takes the form $n_1(x-a) + n_2(y-b) + n_3(z-c) = 0$ for a normal vector $\mathbf{n}$ and a point (a,b,c) in the plane, so using $\mathbf{n}$ found above and the point A in the plane yields the equation

$$1(x-0) - 2(y-1) + 1(z-1) = 0,$$

which simplifies to $x - 2y + z = -1$.

(b) From the equation $\begin{bmatrix} 6 \\ -2 \\ -3 \end{bmatrix}$ is a normal vector, but this vector has length $\sqrt{6^2 + (-2)^2 + (-3)^2} = \sqrt{49} = 7$. To find a normal vector of unit length, scale by $1/7$ to obtain $\begin{bmatrix} 6/7 \\ -2/7 \\ -3/7 \end{bmatrix}$. (Its negative also works.)

(c) Two planes in $\mathbf{R}^3$ are parallel precisely when their normal vectors are scalar multiples of one another. (Why?) The normal vectors have been computed and are not scalar multiples of each other in this case, so the planes are not parallel.

Problem 3: What sets can be linear subspaces, and what cannot?

For each of the following subsets of $\mathbf{R}^2$ or $\mathbf{R}^3$, write down a collection of finitely many vectors whose span is that set or explain why there is no such collection.

(a) The line $x + y = 1$ (b) The line $x + y = 0$ (c) The unit disk $x^2 + y^2 \leq 1$ (d) $\{\mathbf{0}\}$ (e) The plane $x + y + z = 0$

Solution:

(a) Since $\mathbf{0}$ is not on that line ($0 + 0 \neq 1$), it cannot be a span.

(b) This is a line through the origin. The vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is on the line $(1 + (-1) = 0)$ and hence this vector spans the line. Any nonzero multiple of it will also span the line.

(c) Any non-zero vector $\mathbf{v}$ in the unit disk has a multiple $a \cdot \mathbf{v}$ (where a might have to be very large) that is not in the unit disk. Hence the unit disk cannot be the span of anything, since for anything in a span every scalar multiple is also in the span $(c(a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k) = (ca)\mathbf{v}_1 + \cdots + (ca)\mathbf{v}_k)$.

(d) This is the span of $\{\mathbf{0}\}$.

(e) These are $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $x + y + z = 0$, so $z = -x - y$, so they are the vectors $\begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. This exhibits it as the span of two vectors, namely $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Problem 4: More recognizing and describing linear subspaces

Which of the following subsets S of $\mathbf{R}^3$ are linear subspaces? If a set S is a linear subspace, exhibit it as a span. If it is not a linear subspace, describe it geometrically and explain why it is not a linear subspace.

(a) The set S_1 of points (x, y, z) in $\mathbf{R}^3$ with both $z = x + 2y$ and $z = 5x$.

(b) The set S_2 of points (x, y, z) in $\mathbf{R}^3$ with either $z = x + 2y$ or $z = 5x$.

(c) The set S_3 of points (x, y, z) in $\mathbf{R}^3$ of the form $t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t'\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ for some scalars t and t' (which are allowed to be anything, depending on the point (x, y, z)).

Solution: Points in S_1 are those satisfying $z = 5x$ and $5x = x + 2y$, where the second equation says $4x = 2y$ or equivalently $y = 2x$. Hence, these are points of the form $\begin{bmatrix} x \\ 2x \\ 5x \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, so this is the span of the vector $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$. Geometrically, this is the overlap of two planes through the origin (meeting in a line through the origin).

The subset S_2 not a subspace. Informally, it is the collection of points on either of two planes through the origin, and this cannot be a span: any span is "closed" under forming linear combinations, but if we add a point in one plane to a point in the other plane then the vector sum (in terms of the parallelogram law) is generally outside both planes (apart from the special case when we begin with vectors on the line from (a) along which the planes meet).

Finally, S_3 is obtained by shifting in space by a point $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ some plane through the origin. But this point by which we shift is in the initial plane through the origin:

$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

In other words, we can rewrite the expressions giving S_3 as vectors of the form

$$t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t' \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = (t+1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (t'+1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

with arbitrary t and t' (so equivalently, arbitrary $t+1$ and $t'+1$). Hence, this is a span of two vectors in $\mathbf{R}^3$ that aren't scalar multiples of each other, so it is a plane through the origin. In particular, we have exhibited it as a linear subspace.

Problem 5: Visualizing a span

For each collection of vectors in $\mathbf{R}^2$, sketch its span: is it a point, a line, or all of $\mathbf{R}^2$?

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For each collection of vectors in $\mathbf{R}^3$ sketch its span: is it a point, a line, a plane, or all of $\mathbf{R}^3$?

(f) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (g) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (h) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (i) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution:

(a) This is the x-axis.

(b) The span is the entirety of $\mathbf{R}^2$ since $x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

(c) This again is the x-axis.

(d) Since $\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ this is the x-axis.

(e) This is just the origin.

(f) This is the xy-plane.

(g) This is the entirety of $\mathbf{R}^3$, by an analogue of (b): $x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(h) Since $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so this third vector contributes nothing new to the span, we again get the xy-plane. This is an instance of the observation that for any vectors $\mathbf{v}, \mathbf{w}$ and scalars a, b, c, we have that

$$a\mathbf{v} + b\mathbf{w} + c(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + b\mathbf{w} + c\mathbf{v} + c\mathbf{w} = (a+c)\mathbf{v} + (b+c)\mathbf{w}$$

belongs to the span of $\mathbf{v}$ and $\mathbf{w}$.

(i) This is just the x-axis (including the origin in the collection of vectors contributes nothing to the span).

