Solutions to Math 51 Quiz 2

1. (10 points) Consider the plane $\mathcal{P}$ given by $x + y - z = 0$ with an orthogonal basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$.

Let $\mathbf{v} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$.

(a) (3 points) Compute $\mathbf{Proj}_{\mathcal{P}}(\mathbf{v})$.

(b) (7 points) Find a 3-vector $\mathbf{w}$ for which $\mathbf{Proj}_{\mathcal{P}}(\mathbf{w}) = \mathbf{Proj}_{\mathcal{P}}(\mathbf{v})$ and the first component of $\mathbf{w}$ is 51.

Labeling the orthogonal basis as $\{\mathbf{u}_1, \mathbf{u}_2\}$, we can compute

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{v}) = \mathbf{Proj}_{\mathbf{u}_1}(\mathbf{v}) + \mathbf{Proj}_{\mathbf{u}_2}(\mathbf{v}) = \frac{2}{2}\mathbf{u}_1 + \frac{-12}{6}\mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}.$$

Now, since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}\}$, where $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a vector normal to $\mathcal{P}$, forms an orthogonal basis for $\mathbb{R}^3$, we can write

$$\mathbf{w} = \alpha \mathbf{u}_1 + \beta \mathbf{u}_2 + \gamma \mathbf{n}$$

for any 3-vector $\mathbf{w}$. However, if we project $\mathbf{w}$ onto $\mathcal{P}$ with orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$, we can see that

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{w}) = \alpha \mathbf{u}_1 + \beta \mathbf{u}_2.$$

Since we want $\mathbf{Proj}_{\mathcal{P}}(\mathbf{w}) = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$ (from part (a)), we can conclude that

$$\mathbf{w} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \gamma - 1 \\ \gamma + 4 \\ -\gamma + 3 \end{bmatrix}.$$

Setting the first component equal to 51, we get that $\gamma = 52$, and the desired vector is $\mathbf{w} = \begin{bmatrix} 51 \\ 56 \\ -49 \end{bmatrix}$.

2. (2 points) **True or False:** Suppose V is a 21-dimensional linear subspace of $\mathbb{R}^{51}$. Then, for every $\mathbf{u} \in \mathbb{R}^{51}$,

$$\mathbf{Proj}_V(\mathbf{Proj}_V(\mathbf{u})) = \mathbf{Proj}_V(\mathbf{u}).$$

TRUE. Since $\mathbf{Proj}_V(\mathbf{u}) \in V$ for every 51-vector $\mathbf{u}$, it follows that

$$\mathbf{Proj}_V(\mathbf{Proj}_V(\mathbf{u})) = \mathbf{Proj}_V(\mathbf{u}).$$

3. (2 points) **True or False:** Let V be a linear subspace of $\mathbb{R}^4$ consisting of points (w, x, y, z) satisfying $-2w + 3x + 2y + 4z = 0$, i.e. $V = \{(w, x, y, z) : -2w + 3x + 2y + 4z = 0\}$. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 3 \\ 0 \end{bmatrix}$, and suppose $\mathbf{v}_3 \in V$ is a non-zero vector that is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Then, every vector $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

TRUE. First, we note that $\mathbf{v}_1, \mathbf{v}_2 \in V$ (they both satisfy the equation for V). Now, since $\mathbf{v}_3$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, it follows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set and since all three vectors are in V and V is 3-dimensional, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is actually an orthogonal basis for V. Therefore, every vector $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

4. (3 points) Suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^8$, with $\|\mathbf{v}\| = 2, \|\mathbf{w}\| = 4$, and $\mathbf{v} \cdot \mathbf{w} = -1$. Suppose $\mathbf{w}' = \mathbf{w} + \alpha\mathbf{v}$ for some number α. Which of the following values of α will make $\{\mathbf{v}, \mathbf{w}'\}$ an orthogonal basis for $\text{span}(\mathbf{v}, \mathbf{w})$?

a) $-\frac{1}{2}$ b) 0 c) $\frac{1}{4}$ d) $-\frac{1}{2}$ e) 1 f) There is no such α.

Since we want $\mathbf{v}$ and $\mathbf{w}'$, we can look at the dot product:

$$\mathbf{v} \cdot \mathbf{w}' = \mathbf{v} \cdot (\mathbf{w} + \alpha\mathbf{v}) = \mathbf{v} \cdot \mathbf{w} + \alpha\mathbf{v} \cdot \mathbf{v} = -1 + 4\alpha.$$

Hence, if $\alpha = \frac{1}{4}$, $\{\mathbf{v}, \mathbf{w}\}$ is an orthogonal basis for $\text{span}(\mathbf{v}, \mathbf{w})$.

5. (3 points) Consider a plane $\mathcal{P}$ in $\mathbb{R}^3$ defined by the equation $x + 12y + 123z = 0$. Let $\mathbf{v}$ be a fixed non-zero 3-vector in $\mathcal{P}$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{51}$ are 51 distinct 3-vectors for which

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{u}_i) = \mathbf{v}$$

for all $i = 1, 2, \ldots, 51$. What is the dimension of $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{51})$?

a) 1 b) 2 c) 3 d) 51 e) Not enough info

Define $\mathbf{n} = \begin{bmatrix} 1 \\ 12 \\ 123 \end{bmatrix}$, a vector normal to $\mathcal{P}$. Then, we know that (via the reasoning in question 1 or Practice Quiz A question 5) if $\mathbf{Proj}_{\mathcal{P}}(\mathbf{w}) = \mathbf{v}$ for some $\mathbf{w}$, then

$$\mathbf{w} = \mathbf{v} + t\mathbf{n}$$

for some scalar t. Hence, for all $i = 1, 2, \ldots, 51$, there are distinct scalars $\alpha_1, \alpha_2, \ldots, \alpha_{51}$ for which

$$\mathbf{u}_1 = \mathbf{v} + \alpha_1\mathbf{n}, \qquad \mathbf{u}_2 = \mathbf{v} + \alpha_2\mathbf{n}, \qquad \ldots \qquad \mathbf{u}_{51} = \mathbf{v} + \alpha_{51}\mathbf{n}.$$

Therefore, it follows that $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{51}) = \text{span}(\mathbf{v}, \mathbf{n})$ which is 2-dimensional.