Problem 1: Checking for Linearity

For the following functions F, analyze their interaction with vector addition and scalar multiplication to determine if they are linear or not. If not linear, give an explicit pair of vectors $\mathbf{v}, \mathbf{w}$ for which $F(\mathbf{v} + \mathbf{w}) \neq F(\mathbf{v}) + F(\mathbf{w})$ or an explicit vector $\mathbf{v}$ and scalar c for which $F(c\mathbf{v}) \neq c\,F(\mathbf{v})$.

(a) $f(x) = \begin{bmatrix} 2x \\ 2x+3 \end{bmatrix}$

(b) $g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + y.$

(c) $h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \\ x^2 \end{bmatrix}$

(d) $k\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \\ 0 \end{bmatrix}$

Problem 2: Composition of linear maps and matrix multiplication

(a) Let $R : \mathbf{R}^2 \to \mathbf{R}^2$ be the operation that carries any vector $\mathbf{v}$ to its reflection across the y-axis. Explain in words or with a picture why $R\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -a \\ b \end{bmatrix}$ for any a, b, so R is linear. Then calculate the 2×2 matrix A for which $R(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v} \in \mathbf{R}^2$.

(b) Find the matrix corresponding to the linear transformation that first rotates a vector in $\mathbf{R}^2$ by an angle of α counterclockwise and then reflects across the y-axis. Then find the matrix corresponding to the opposite order of these operations (it is a different matrix when $\alpha \neq 0, \pi$, which is to say when $\sin\alpha \neq 0$).

(c) Draw a picture to illustrate visually (so without any appeal to matrices) why the composition of the two operations in (b) (rotation and reflection) depends on the order in which they are carried out when $\alpha \neq 0, \pi$.

Problem 3: Algebra and geometry with hyperbolas

This problem studies hyperbolas, using some algebra and thinking (more instructive than a computer) to determine the basic geometry of a hyperbola from an equation. This will be useful later when relating contour plots to the multivariable second derivative test. You do *not* need a calculator for this exercise; human brainpower is sufficient!

(a) For $A, B, C > 0$, the equations $Ax^2 - By^2 = \pm C$ (one for each sign) are hyperbolas with asymptotes $y = \pm\sqrt{A/B}\,x$. For the pair of hyperbolas $H_\pm$ in each of (i) and (iii) below (treating each $\pm$ case separately), or else in each of (ii) and (iv) below (treating each $\pm$ case separately), compute:

- which of the coordinate axes each crosses (the x-axis consists of points $(u, 0)$, the y-axis consists of points $(0, v)$),
- the slopes $\pm m$ of the asymptotes,
- which coordinate axis is "nearer" to the asymptotes (note: if the slope c of a line $y = cx$ satisfies $|c| < 1$ then the line is "closer" to the x-axis, whereas if $|c| > 1$ then the line is "closer" to the y-axis; draw the cases $c = \pm 2, \pm 1/2$ to see why this is reasonable).

(i) $x^2 - 6y^2 = \pm 10$, (ii) $3x^2 - 5y^2 = \pm 13$, (iii) $7x^2 - 2y^2 = \pm 18$, (iv) $-5x^2 + y^2 = \pm 21$

(b) Use the information found in (a) to approximately draw each pair of hyperbolas (both signs) on a common coordinate grid (one grid for each pair), indicating the approximate axis crossings and the asymptotes drawn "closer" to the appropriate coordinate axis (a qualitatively correct picture is sufficient).

(c) Based on your work, for a general hyperbola $Ax^2 - By^2 = C$ with $C \neq 0$ and both A and B with the same sign, how does the coordinate axis (x or y) to which the asymptotes are "closer" related to how $|A|$ compares to $|B|$ (in terms of which is bigger and which is smaller)?

