Problem 1: Orthogonality and projections

(a) In the span of $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $\begin{bmatrix}1\\-2\\3\\-4\end{bmatrix}$ find a non-zero vector $\mathbf{v}$ orthogonal to $\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$.

(b) Here is a geometric analogue to the algebra in (a): for a plane P through the origin in $\mathbf{R}^3$ and a nonzero 3-vector $\mathbf{w}$ not orthogonal to P, why should there always be nonzero vectors in P orthogonal to $\mathbf{w}$? (Hint: visualize the plane W through $\mathbf{0}$ with normal vector $\mathbf{w}$, and think about how it meets the plane P).

(c) Find a nonzero vector $\mathbf{u} \in \mathbf{R}^3$ for which the projections of $\mathbf{v} = \begin{bmatrix}3\\0\\4\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}2\\2\\1\end{bmatrix}$ onto $\mathbf{u}$ are equal. (Recall that the projection of $\mathbf{x}$ onto a nonzero vector $\mathbf{u}$ is given by the formula $\left(\frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$.) There are many answers. Informally, the condition says that $\mathbf{v}$ and $\mathbf{w}$ make the same "shadow" onto the line spanned by $\mathbf{u}$.

Problem 2: An orthogonal basis

Let V be the set of vectors $\mathbf{v} \in \mathbf{R}^3$ satisfying $\mathbf{v} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix}2\\3\\4\end{bmatrix}$ (this says that both of these explicit 3-vectors have the same projection onto $\mathbf{v}$, or in other words make the same "shadow" onto the line spanned by $\mathbf{v}$).

(a) Express V as the collection of 3-vectors orthogonal to a single nonzero 3-vector.

(b) By fiddling with orthogonality equations, build an orthogonal basis of V. There are many possible answers.

(c) Use your answer to (b) to give an orthonormal basis for V.

Problem 3: Subspaces defined by orthogonality, orthogonal bases, and shortest distances in $\mathbb{R}^3$

(a) For each linear subspace V_i in $\mathbf{R}^3$ given below, exhibit the set

$$V_i' = \{\mathbf{x} \in \mathbf{R}^3 \mid \mathbf{x} \text{ is orthogonal to every vector in } V_i\}$$

as the span of a finite collection of vectors (so, as a linear subspace), and give a basis for V_i'.

(i) $V_1 = \operatorname{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right)$

(ii) V_2 is the set of solutions in $\mathbf{R}^3$ to the pair of equations $\begin{cases} x_1 + 2x_2 + 3x_3 = 0, \\ 4x_1 + 5x_2 + 6x_3 = 0. \end{cases}$ (*Hint:* relate this to V_1 and think geometrically.)

(b) For each of the two V_i's given above, compute an orthogonal basis for it and *set up* how you'd find the distance from the point $\begin{bmatrix}7\\0\\0\end{bmatrix}$ to V_i (i.e. the minimal distance from $\begin{bmatrix}7\\0\\0\end{bmatrix}$ to a point in V_i) using such a basis. Finally, compute each distance.

(*Hint for computation:* first treat the case of V_2. For the case of the plane V_1, use projections to compute an orthogonal basis and to give an expression for a vector whose length is the distance you want. It gets cumbersome to carry out that distance calculation by hand, so instead compute the distance to V_1 by relating it to the distance to V_2. Try drawing a picture of an orthogonal line and plane to get an idea.)

Problem 4: Building another orthogonal vector (Extra)

If $\{\mathbf{v}, \mathbf{w}\}$ is a pair of nonzero orthogonal vectors in $\mathbf{R}^3$ then we can always enlarge it to an orthogonal basis $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ of $\mathbf{R}^3$ by taking $\mathbf{u}$ to be a nonzero normal vector to the plane $\mathrm{span}(\mathbf{v}, \mathbf{w})$. If $n > 3$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ are mutually orthogonal nonzero vectors in $\mathbf{R}^n$ then can we always find a nonzero $\mathbf{v}_n$ orthogonal to those (so $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis of $\mathbf{R}^n$)?