Solutions to Math 51 Practice problems for Quiz 5

1. (3 points) For the following pairs of matrices, which of them satisfy $AB = BA$? Select all that apply.

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) none of these

Only $\boxed{\text{(c)}}$ is such a pair:

(a) BA is not defined, since B is 2×3 and A is 2×2.

(b) $AB = \begin{bmatrix} -2 & -8 \\ 0 & 3 \end{bmatrix}$, but $BA = \begin{bmatrix} 3 & 0 \\ -7 & -2 \end{bmatrix}$.

(c) $AB = \begin{bmatrix} 8 & -2 \\ 4 & 2 \end{bmatrix} = BA$.

(d) $AB = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, but $BA = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2. (1 point) **True or False:** If

$$A = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -1 & -2 \\ 0 & 15 \end{bmatrix},$$

then $AB = BA$.

This is False, since AB is a 2×2 matrix while BA is a 3×3 matrix, so they are certainly not equal.

3. (4 points) Consider the effect of a linear transformation on the following image of the bear flag without the black background.

For each of the following matrices M_i, identify which picture shows the output when M_i is applied to the original image, i.e. the bear flag without the black background.

Note that we are not specifying where the images are with respect to the origin, as this problem can be solved without this information.

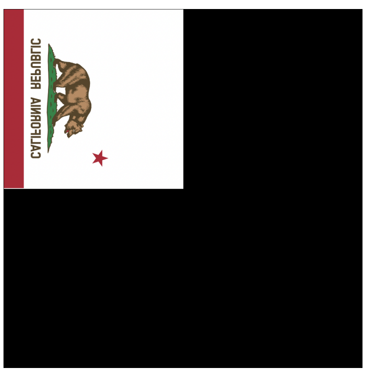
$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

(A)


(B)


(C)


(D)


(E)


(F)


Note that the first column of each matrix M, $M\mathbf{e}_1$, is the effect on $\mathbf{e}_1$ under the corresponding linear transformation; the second column of M is the effect on $\mathbf{e}_2$ under the corresponding linear transformation.

For $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{e}_1$ stays put, but $\mathbf{e_2}$ changes sign. The effect is an upside down flip of the image, namely refection with respect to the x-axis. So it is E.

For $M_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\mathbf{e}_1$ goes to $\mathbf{e}_1 + \mathbf{e}_2$, and $\mathbf{e}_2$ stays put. This corresponds to a upward shearing linear transformation. So it is C.

For $M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{e}_1$ goes to $\mathbf{e}_2$ and $\mathbf{e}_2$ goes to $\mathbf{e}_1$. The roles of the corrediate axes are swapped, this corresponds to a reflection across the line$y = x$. So it is D.

For $M_4 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{e}_2$ expands two fold, but $\mathbf{e}_1$ changes sign. The effect is an right-left flip of the image (reflection with respect to the y-axis) with two-fold expansion vertically. So it is A.

Note that image B corresponds to a 180° rotation, so the corresponding matrix must be $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; image F corresponds to a composition of M_1 (image E) followed by a 90° clockwise rotation, so the corresponding matrix is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Alternative, the image F also corresponds to a composition of M_3 (image D) followed by a 180° rotation, so the corresponding matrix is again

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4. (3 points) $f : \mathbf{R}^2 \to \mathbf{R}^3$ is given by

$$f(x, y) = \begin{bmatrix} x^2 y \\ x - y + xy \\ 3y - x^3 \end{bmatrix}$$

Which of the following represents the linear approximation to f at the point $(1, 2)$?

(a)

$$f(x, y) \approx \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 & -3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

(b)

$$f(x, y) \approx \begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 3 & 0 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x-2 \\ y-1 \end{bmatrix}$$

(c)

$$f(x, y) \approx \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 3 & 0 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

(d)

$$f(x, y) \approx \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 3 & 0 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

First compute

$$f(1, 2) = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

The component functions of $f(x, y)$ are $f_1(x, y) = x^3 y$, $f_2(x, y) = x - y + xy$, and $f_3(x, y) = 3y - x^4$. Plugging these in to the formula for the derivative matrix gives

$$\mathrm{D}f = \begin{bmatrix} 2xy & x^2 \\ 1 + y & -1 + x \\ -3x^x & 3 \end{bmatrix}$$

So in particular,

$$(\mathrm{D}f)(1, 2) = \begin{bmatrix} 4 & 1 \\ 3 & 0 \\ -3 & 3 \end{bmatrix}.$$

Plugging this in to the formula for linear approximation at $(x, y) = (1, 2)$, we see that

$$f(x, y) \approx \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 3 & 0 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix},$$

i.e., the answer is (c).

5. (4 points) Let P be the parallelogram with corners at $\mathbf{0}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\mathbf{v}+\mathbf{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, as shown in Figure ??:

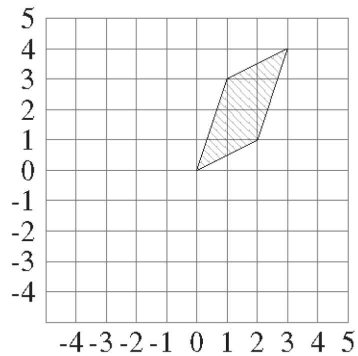
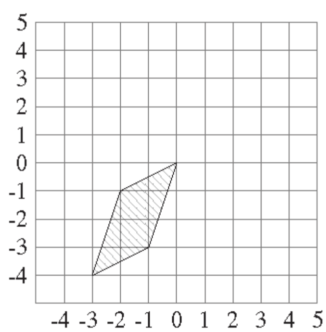


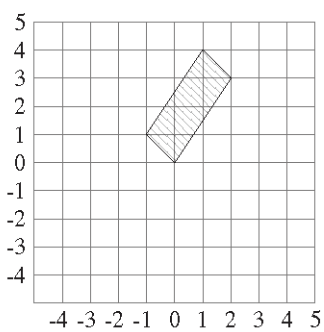
Figure 1: The parallelogram P

For each of the following four matrices M, determine which of the six pictures below shows the output when the corresponding matrix is applied to all points of P.

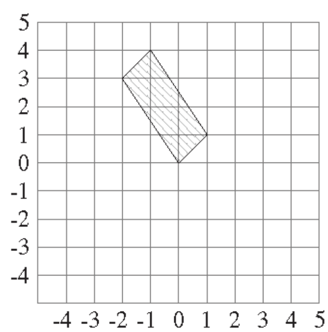
$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ____ $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ ____ $C = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$ ____ $D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ____



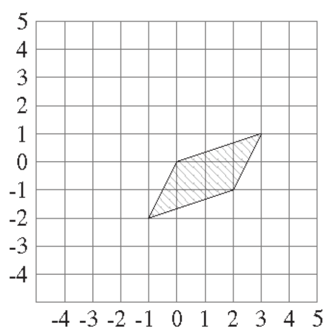
(1)



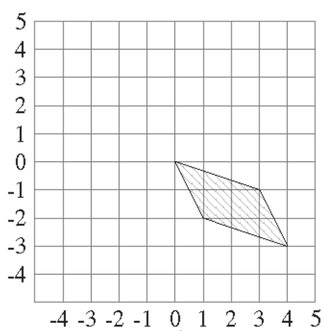
(2)



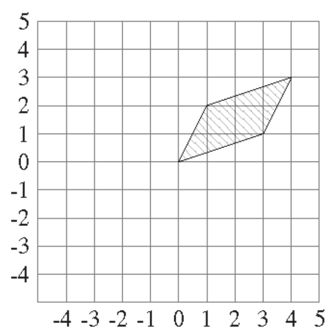
(3)



(4)



(5)



(6)

The parallelogram P consists of points of the form $a\mathbf{v} + b\mathbf{w}$ with $0 \le a, b \le 1$, so for each matrix M the output on points of P consists of points of the form

$$M(a\mathbf{v} + b\mathbf{w}) = a(M\mathbf{v}) + b(M\mathbf{w})$$

with $0 \le a, b \le 1$. In other words, it is the parallelogram with corners at $\mathbf{0}$, $M\mathbf{v}$, $M\mathbf{w}$, and the vector sum $M\mathbf{v} + M\mathbf{w}$. In particular, two of the edges are the segments joining $\mathbf{0}$ to $M\mathbf{v}$ and to $M\mathbf{w}$. So for each of the 6 matrices given, we apply them to $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and then look for the picture which has those as its edges emanating from the origin. We can try to compute $M\mathbf{v}$ and $M\mathbf{w}$ either algebraically or geometrically.

For $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, this corresponds to a 90° clockwise rotation, it carries $\mathbf{v}$ to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and carries $\mathbf{w}$ to $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$, so it is (5).

For $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, this is a shearing linear transformation, it carries $\mathbf{v}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and carries $\mathbf{w}$ to $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, so it is (3).

For $C = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$, it carries $\mathbf{v}$ to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and carries $\mathbf{w}$ to $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$, so it is (4).

For $D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, this negates everything, it is a 180° rotation, so both edges from the origin are reflected across the origin, yielding (1). More algebraically, this matrix carries $\mathbf{v}$ to $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ and carries $\mathbf{w}$ to $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$, which indeed is (1).

As for the unmatched diagrams, (2) is (3) flipped (reflected) across the the y-axis, so it corresponds to the matrix product

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

As an algebraic check, this matrix carries $\mathbf{v}$ to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and carries $\mathbf{w}$ to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, which is indeed (2).

Finally, (6) is the original image flipped (reflected) across the line $y = x$, with corresponds to matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which swaps the roles of the coordinate axes. As an algebraic check, this matrix carries $\mathbf{v}$ to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and carries $\mathbf{w}$ to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, which indeed is (6).

6. (4 points) Suppose f is a linear function where

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Then $f(\mathbf{x}) = A\mathbf{x}$ where the matrix A must be

a) $\begin{bmatrix} 3 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$

f) $\begin{bmatrix} 3 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix}$

Note that the input of f is a 3-vector, and the output of f has 2 components, so the matrix A representing f must be 2×3. The first column $\mathbf{c}_1$ of A is given by $A\mathbf{e}_1 = f(\mathbf{e}_1)$, the second column $\mathbf{c}_2$ of A is given by $A\mathbf{e}_2 = f(\mathbf{e}_2)$, and the third column $\mathbf{c}_3$ of A is given by $A\mathbf{e}_3 = f(\mathbf{e}_3)$.

Since f is linear,

$$\mathbf{c}_1 = f(\mathbf{e}_1) = f\left(\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}\right) = f\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\3\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}.$$

$$\mathbf{c}_3 = f(\mathbf{e}_2) = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\mathbf{c}_3 = f(\mathbf{e}_3) = f\left(\begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}\right) = f\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) - f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}.$$

Hence,

$$A = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 1\end{bmatrix}$$

7. (3 points) Suppose f is a linear function where

$$f\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}y-z\\y-x\end{bmatrix}$$

Then $f(\mathbf{x}) = A\mathbf{x}$ where the matrix A must be _____________.

f takes a vector in $\mathbf{R}^3$ and outputs a vector in $\mathbf{R}^2$, so A must be 2×3. The three columns of A are the values $f(\mathbf{e}_1), f(\mathbf{e}_2), f(\mathbf{e}_3)$; so we find that

$$A = \begin{bmatrix}0 & 1 & -1\\-1 & 1 & 0\end{bmatrix}$$

8. (3 points) Suppose the linear transformation $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ first rotates a vector $\begin{bmatrix}x\\y\end{bmatrix}$ counterclockwise by 45 degrees, *then* stretches it by a factor of 2 in the y-direction. Then $F\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = M\begin{bmatrix}x\\y\end{bmatrix}$, where the matrix M must be

a) $\begin{bmatrix}\frac{\sqrt{2}}{2} & \sqrt{2}\\-\frac{\sqrt{2}}{2} & \sqrt{2}\end{bmatrix}$ b) $\begin{bmatrix}\frac{\sqrt{2}}{2} & -\sqrt{2}\\\frac{\sqrt{2}}{2} & \sqrt{2}\end{bmatrix}$ c) $\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\-\sqrt{2} & \sqrt{2}\end{bmatrix}$ d) $\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\\sqrt{2} & \sqrt{2}\end{bmatrix}$

We give two possible solutions; it's useful to think of each as a safety check against the other.

Solution #1: For $i = 1, 2$, the i-th column of the matrix for F is the value of $F(\mathbf{e}_i)$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$

is the standard basis of $\mathbf{R}^2$. We have

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{rotate CCW by } 45^\circ} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \xrightarrow{\text{vertically double}} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix} \text{ and}$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{rotate CCW by } 45^\circ} \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \xrightarrow{\text{vertically double}} \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2} \end{bmatrix},$$

That is, the results $F(\mathbf{e}_1)$ and $F(\mathbf{e}_2)$ of our operation on the standard basis vectors are:

$$F(\mathbf{e}_1) = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix}, \qquad F(\mathbf{e}_2) = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2} \end{bmatrix}.$$

Therefore, the matrix M we're looking for is

$$M = \begin{bmatrix} | & | \\ F(\mathbf{e}_1) & F(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}.$$

Solution #2: The operation F is a composition of a rotation and a vertical doubling; these individual functions have matrices

$$R = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

respectively. The matrix of a composition of linear functions is the product of the matrices of each function; in the case of F, the order of muliplication must be DR, because $F(\mathbf{x})$ rotates first, then horizontally doubles. (To check this, note that $DR\mathbf{x} = D(R\mathbf{x})$ is the diagonal matrix applied to the rotated vector $R\mathbf{x}$.) Thus, the matrix of F is

$$M = DR = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

So the answer is $\boxed{\text{(iv)}}$.

Remark. The most common error on this problem was mistaking the function F, and its matrix computation, with a similar function T that first stretches vertically by a factor of 2, and *then* rotates counterclockwise by 45 degrees. The matrix A of this function T is also the product of the matrices R and D in Solution #2 above, but in the opposite order: $A = RD$. (To see this, note that $RD\mathbf{x} = R(D\mathbf{x})$ is the rotation matrix applied to the vertically-doubled vector $D\mathbf{x}$.) That is, the matrix of such a function T is

$$A = RD = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2} \end{bmatrix}$$

which is not equal to M. (For finding the matrix A of such a function T, one could alternatively arrive at A via a technique analogous to Solution #1.)