Solutions to Math 51 Practice problems for Quiz 6

1. (3 points) Suppose that $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ satisfies

$$f(0,0) = (1,1), \quad f(1,1) = (0,0).$$

Let A be the derivative matrix of f at the point $(0,0)$; that is,

$$A = (Df)(0,0).$$

Furthermore, suppose we define the function $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ by the formula

$$h(\mathbf{x}) = A\mathbf{x}$$

for this *same* matrix A.

Given only this information about f and h, which of the following expressions must *always* be equal to A^2 ? Select all that apply.

(i) $(D(f \circ h))(0,0)$ (ii) $(D(f \circ f))(0,0)$ (iii) $(D(h \circ f))(0,0)$

(iv) $(D(f \circ h))(1,1)$ (v) $(D(h \circ h))(1,1)$

The correct choices are (i),(iii),(v).

By the Chain Rule, for any composition $p \circ q$ of two functions p and q, at a point $\mathbf{a}$ we have

$$(D(p \circ q))(\mathbf{a}) = (Dp)(q(\mathbf{a}))\ (Dq)(\mathbf{a})$$

Thus, being able to evaluate $(D(p \circ q))(\mathbf{a})$ requires knowing Dp at $q(\mathbf{a})$ and knowing Dq at $\mathbf{a}$; note that this in turn may also require knowing the value of $q(\mathbf{a})$, unless Dp is constant.

We only know the value of matrix Df at $(0,0)$, but we know matrix $Dh = A$ is constant. (Recall homework Exercise 13.9, where if g is a linear function represented by a matrix B, i.e. where $g(\mathbf{x}) = B\mathbf{x}$, then $(Dg)(\mathbf{x}) = B$. This can also be verified by a direct computation with a 2×2 matrix.)

The only value of h we know is $h(0,0) = (0,0)$ (because $A\mathbf{0} = \mathbf{0}$).

As a result, we have enough information to evaluate:

(i) $(D(f \circ h))(0,0) = (Df)(h(0,0))\ (Dh)(0,0) = (Df)(0,0)\ A = (A)(A) = A^2$,

(iii) $(D(h \circ f))(0,0) = (Dh)(f(0,0))\ (Df)(0,0) = (Dh)(1,1)\ (Df)(0,0) = (A)(A) = A^2$,

(v) $(D(h \circ h))(1,1) = (Dh)(h(1,1))\ (Dh)(1,1) = (A)(A) = A^2$.

but we can only proceed so far on the others:

(ii) $(D(f \circ f))(0,0) = (Df)(f(0,0))\ (Df)(0,0) = (Df)(1,1)\ (Df)(0,0) = (Df)(1,1)\ A$,

(iv) $(D(f \circ h))(1,1) = (Df)(h(1,1))\ (Dh)(1,1) = (Df)(h(1,1))\ A$

2. (4 points) Suppose A and B are invertible 51×51 matrices. Evaluate each of the following statements as either **true** (i.e., *always* true) or **false** (i.e., sometimes not true):

(a) If $A^8 = A$, then A must be equal to the identity matrix.

(b) The matrix $A^{-1}B^{-1}AB$ must be invertible.

(a) False. All we can conclude is that $A^7 = I$ (by multiplying both sides by A^{-1}), but this is not sufficient to know that $A = I$. Indeed, if we view A as the matrix of a linear transformation T_A, then it could be the case that T_A is a (non-identity) operation which when repeated (i.e., composed with itself) 7 times becomes the identity. For example, T_A could be counterclockwise rotation in the 2-dimensional subspace $\text{span}(\mathbf{e}_1, \mathbf{e}_2)$ (by an angle $\frac{2\pi}{7}$ or multiple of this); or alternatively T_A could be the linear transformation that sends the first seven standard basis vectors in a cycle (and leaves the others unchanged): $T_A(\mathbf{e}_1) = \mathbf{e}_2, T_A(\mathbf{e}_2) = \mathbf{e}_3, \ldots, T_A(\mathbf{e}_7) = \mathbf{e}_1$.

(b) True. From Chapter 18, any product (or inverse) of invertible matrices is invertible. So AB is invertible, and thus $B^{-1}AB$ is invertible, and finally $A^{-1}B^{-1}AB$ is invertible.

3. (4 points) A Peruvian lives near a mountain range and notices goats moving between mountain A and mountain B every year. He observes the following migration pattern, for certain constant values α, β:

- α% of the goats on mountain A in one year migrate to mountain B in the next year; and
- β% of the goats on mountain B in one year migrate to mountain A in the next year.

We are assuming that no new goats are born and no goat dies.

Let $\mathbf{g}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$ be the goat vector, where a_n is the number of goats on mountain A after n years and b_n is the number of goats on mountain B after n years.

Let M be the Markov matrix for which $\mathbf{g}_{n+1} = M\mathbf{g}_n$, and suppose that, for certain constant values γ, δ, the matrix M takes the form:

$$M = \begin{bmatrix} \gamma & 0.4 \\ \delta & 0.6 \end{bmatrix}$$

If we learn that for large values of n,

$$\mathbf{g}_n = \begin{bmatrix} 2000 \\ 1000 \end{bmatrix},$$

then what are the values of each of α and β ?

(Note that these will be percentages; i.e., numbers between 0 and 100.)

(i) 20 (ii) 40 (iii) 60

(iv) 80 (v) not able to be determined from the given information

The observations about migration imply that mountain A's goat population in year $n+1$ is a mix of two populations: those that migrated from mountain B over the previous year (there are $(\beta/100)b_n$ of these), plus those that remained on mountain A from the previous year, or equivalently those from mountain A that did *not* migrate to mountain B (there are $a_n - (\alpha/100)a_n = (1 - \alpha/100)a_n$) of these). A similar statement can be made about mountain B's population in year $n+1$.

Thus, we obtain the following equations determining the mountains' populations in year $n+1$ in terms of those in year n:

$$\begin{aligned} a_{n+1} &= \left(1 - \frac{\alpha}{100}\right) a_n + \left(\frac{\beta}{100}\right) b_n \\ b_{n+1} &= \left(\frac{\alpha}{100}\right) a_n + \left(1 - \frac{\beta}{100}\right) b_n \end{aligned}$$

In matrix and vector terms, since $\mathbf{g}_n = \begin{bmatrix} 2000 \\ 1000 \end{bmatrix}$, and so forth, this system may be expressed as

$$\mathbf{g}_{n+1} = \begin{bmatrix} 1 - \dfrac{\alpha}{100} & \dfrac{\beta}{100} \\ \dfrac{\alpha}{100} & 1 - \dfrac{\beta}{100} \end{bmatrix} \mathbf{g}_n$$

Comparing this matrix with the form of M given, we see that $\boxed{\beta = 40}$.

To find α, we need to additionally use the fact given about large values of n (and therefore of $n+1$), which implies that

$$\begin{aligned} \begin{bmatrix} 2000 \\ 1000 \end{bmatrix} &= \begin{bmatrix} 1 - \dfrac{\alpha}{100} & 0.4 \\ \dfrac{\alpha}{100} & 0.6 \end{bmatrix} \begin{bmatrix} 2000 \\ 1000 \end{bmatrix} \\ &= \begin{bmatrix} \left(1 - \dfrac{\alpha}{100}\right)(2000) + (0.4)(1000) \\ \left(\dfrac{\alpha}{100}\right)(2000) + (0.6)(1000) \end{bmatrix} \\ &= \begin{bmatrix} 2400 - 20\alpha \\ 600 + 20\alpha \end{bmatrix} \end{aligned}$$

Equating components, we find that $20\alpha = 400$, so $\boxed{\alpha = 20}$.

4. (3 points) Suppose

$$f : \mathbf{R}^2 \to \mathbf{R}, \quad (\nabla f)(0,0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$g : \mathbf{R}^2 \to \mathbf{R}^2, \quad g(x,y) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Which of the following is $(\nabla(f \circ g))(0,0)$?

(a) $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$

We apply the chain rule here. For this, we need to compute the derivative matrices Df and Dg.

First, recall that the gradient of a function is the transpose of its derivative matrix. (In other words, ∇f is the same as Df except written vertically). Thus, we know that

$$(Df)(0,0) = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

As for Dg, a direct computation shows that

$$(Dg)(x,y) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

regardless of x and y. You may also recall homework Exercise 13.9, where if g is a linear function represented by a matrix B, i.e. where $g(\mathbf{x}) = B\mathbf{x}$, then $(Dg)(\mathbf{x}) = B$.

Thus, since $g(0,0) = (0,0)$, applying the chain rule gives

$$(D(f \circ g))(0,0) = (Df)(0,0) \cdot (Dg)(0,0) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \end{bmatrix}.$$

Again, since the gradient of $f \circ g$ is just the transpose of $D(f \circ g)$, we conclude that

$$(\nabla(f \circ g))(0,0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Remark: It might be tempting to take the matrix-vector multiplication

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

and take the right hand side to be the answer. This is not correct: here the chain rule is written in the wrong order.

5. (4 points) Match the appropriate Markov matrix with the description about annual population movement between the cities. In the matrix, the rows/columns correspond to A,B,C,D in that order.

$$M_1 = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0.3 & 0.8 & 0 \\ 0 & 0.7 & 0.2 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.7 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0 & 0 \\ 0 & 0.4 & 0.9 & 0 \\ 0 & 0.3 & 0.1 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0.6 & 0.6 & 0.2 & 0 \\ 0.2 & 0.1 & 0.4 & 0 \\ 0.2 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0.1 & 0 & 0 & 0.9 \\ 0 & 0.1 & 0.9 & 0 \\ 0.9 & 0 & 0.1 & 0 \\ 0 & 0.9 & 0 & 0.1 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0.4 & 0.2 & 0.4 & 0.7 \\ 0.1 & 0.5 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0.1 & 0.2 & 0.1 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0.4 & 0.8 & 0 & 0 \\ 0.6 & 0.2 & 0 & 0 \\ 0 & 0 & 0.9 & 0.7 \\ 0 & 0 & 0.1 & 0.3 \end{bmatrix}.$$

(a) People move between A and B, people move between C and D, but there is no movement from [A or B] to [C or D] or vice versa.

(b) After many years, almost everyone will end up in D.

(c) From any city, people move to any other city eventually, but it sometimes requires multiple years to move between two particular cities.

(d) People move between A and D, people move between B and C, but there is no movement from [A or D] to [B or C] or vice versa.

(a) This description matches M_6, because transition rates between A and B are positive, transition rates between C and D are positive, but all other transition rates are zero.

(b) This description matches M_2, because in all other cities, some people either move directly into D, or some people move into a city where they might next move into D. However, no one ever leaves D, so eventually almost everyone will end up there.

(c) This description matches M_4. Only in M_4 and M_5 can anyone in one city eventually visit any other city; in M_5, anyone in one city might move to any other city immediately. But in M_4, there is a loop of movement from A to C to B to D back to A, so this is the only city that might require multiple years to move between some pair of cities.

(d) This description matches M_1, because transition rates between A and D are positive, transition rates between B and C are positive, but all other transition rates are zero.

6. (3 points) Consider the matrices

$$M = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & -1 & -2 & 0 \\ 6 & 0 & 0 & 3 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

It is a fact that

$$M_1 = L_1 M R_1, \quad M_2 = L_2 M R_2, \quad M_3 = L_3 M R_3$$

where each of L_1, R_1, L_2, R_2, L_3 and R_3 is a certain one of the following matrices:

(a) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (h) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Identify which matrix each of L_1, R_1, L_2, R_2, L_3 and R_3 must be.
Hint: For each i, one of L_i and R_i is an identity matrix; the other one manipulates either the columns or rows of M.

- M_1 is the same as M, except M_1's second column is the sum of column 1 and column 2 of M. So
$$M_1 = I_3 M \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
- M_2 is obtained from M by having M's three rows scaled by factors of $2, -1, 3$ respectively. So
$$M_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} M I_4.$$
- M_3 shares the same second and third rows as M, but the first row of M_3 is the sum of rows 1, 2, and 3 of M. So
$$M_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M I_4.$$

7. (3 points) Let

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

Which of the following are always true?
Select all that apply.

(a) $AB = BA$ (b) $AC = CA$ (c) $BC = CB$

$$AB = BA = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & a & ac \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad CA = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$BC = CB = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

8. (3 points) **True or False:** Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f(x, y) = (x^3 + y^3 + x, x^3 - y^3 - y)$. $(Df)(x, y)$ is invertible for every point $(x, y) \in \mathbf{R}^2$.

Always true: *Solution*: This is true. First we compute

$$(Df)(x,y) = \begin{bmatrix} 3x^2 + 1 & 3y^2 \\ 3x^2 & -3y^2 - 1 \end{bmatrix}.$$

To check whether this is invertible, we compute its determinant:

$$\det((Df)(x,y)) = (3x^2 + 1)(-3y^2 - 1) - (3x^2)(3y^2) = -18x^2y^2 - 3x^2 - 3y^2 - 1.$$

This expression is ≤ -1 for all x, y (since $x^2, y^2 \geq 0$), in particular never vanishes, implying that Df is everywhere invertible.

9. (1 point) **True or False:** The matrix $\begin{bmatrix} 0 & 1 & 8 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 1 & 1 & 8 \\ 0 & 2 & 2 & 2 \end{bmatrix}$ is invertible.

False. If $A = \begin{bmatrix} 0 & 1 & 8 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 1 & 1 & 8 \\ 0 & 2 & 2 & 2 \end{bmatrix}$ were invertible, then there would be a matrix B where $BA = I_4$. The (1,1)-entry of BA would have to be 1. But the (1,1) entry of BA would be the first row of B dotted with the first columns of A. The first column of A is the zero vector, so its product with any vector would be 0, not 1, contradicting the fact that $BA = I_4$.

10. (4 points) Stanford University's President is trying to decide whether Stanford should be fully open, partially open, or remote-only in the spring. The data collected from college campuses around the country so far this academic year seems to indicate that from one week to the next:

- 1/4 of the fully open universities were partially open the following week, the rest remain fully open;
- 1/6 of the partially open universities were fully open the following week, 1/6 of the partially open universities were remote-only the following week, and the rest of the partially open universities remain partially open;
- 1/2 of the remote-only universities were partially open the following week, the rest remain remote-only.

Let M be the Markov matrix describing the weekly dynamics of universities' opening and closing, where the rows/columns correspond to fully open, partially open, and remote-only in that order. It turns out that

$$M^{15} \approx \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 1/2 \\ 1/6 & 1/6 & 1/6 \end{bmatrix}.$$

With what probability will a fully open university be remote-only after 2 weeks?
With what probability will a university be fully open after 15 weeks?

Let M be the Markov matrix, we have

$$M = \begin{bmatrix} 3/4 & 1/6 & 0 \\ 1/4 & 2/3 & 1/2 \\ 0 & 1/6 & 1/2 \end{bmatrix}$$

The probability a fully open university becomes remote-only after 2 weeks is determined by the (3,1) entry of M^2 which is $0 \cdot 3/4 + 1/6 \cdot 1/4 + 1/2 \cdot 0 = 1/24$.

Since all the columns of M^{15} are the same, $M^{15}\mathbf{e}_i = \begin{bmatrix} 1/3 \\ 1/2 \\ 1/6 \end{bmatrix}$ for $i = 1, 2, 3$. Regardless of whether a university is initially fully open, partially open, or remote-only, it will be fully open with probability 1/3 after 15 weeks.