About this document:

The Math Department regularly seeks out feedback on the Math 50-series, and indeed the addition of "companion" materials is something that we have been working on. The end-of-chapter summary pages is something we added last year, and the Panopto video series on "How to read a mathematics text" was recently developed as part of an ongoing effort to develop such additional resources.

The creation of **additional practice homework exercises** *is also underway, and with this document we are sharing what has been produced. There are two file versions, one which contains statements of "practice HW exercises," and the other which contains problems together with solutions.*

Please note that this is a work-in-progress; the department has prioritized certain chapters over others during the pandemic when time has been short for everyone. The absence of problems for some chapters shouldn't be interpreted as indicating less value for the content of those chapters (more that the skills therein are covered enough by other chapters that they were a lower priority for their own "extra problems").

Feedback is always appreciated — please contact your instructor or administrative TA.

1. Vectors, vector addition, and scalar multiplication

2. Vector geometry in $\mathbb{R}^n$ and correlation coefficients

Practice 2.1 (Angle bisector)**.** In this exercise, we relate angle bisectors to vector geometry concepts: length, vector addition, dot product, and angles. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero n-vectors that are not scalar multiples of each other, so $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are visualized as the diagonals of the parallelogram P with edges along $\mathbf{v}$ and $\mathbf{w}$, as shown for $n=3$ in Figure 2.1 below.

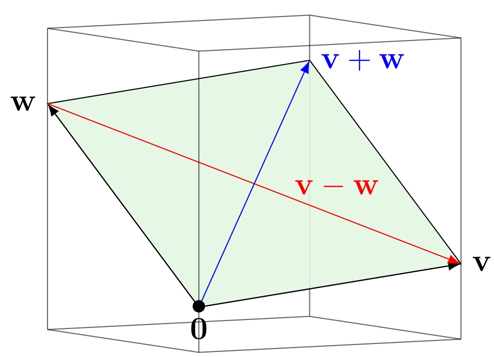


FIGURE 2.1. The parallelogram P with a vertex at $\mathbf{0}$ and edges along $\mathbf{v}$ and $\mathbf{w}$: its points are $s\mathbf{v}+t\mathbf{w}$ for $0 \leq s,t \leq 1$.

(a) Assume $\mathbf{v}$ and $\mathbf{w}$ have the same length (i.e., $\|\mathbf{v}\| = \|\mathbf{w}\|$), so P is a rhombus. Use algebra with dot products to show $\mathbf{v}+\mathbf{w}$ is orthogonal to $\mathbf{v}-\mathbf{w}$. In words: the diagonals of a rhombus are perpendicular to each other.

(b) Continue to assume $\mathbf{v}$ and $\mathbf{w}$ have the same length. Let θ_1 be the angle between $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}$, and let θ_2 be the angle between $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}$ (so $0° < \theta_1, \theta_2 < 180°$). Show $\theta_1 = \theta_2$. Hint: show $\cos\theta_1 = \cos\theta_2$.

In words: the diagonal $\mathbf{v} + \mathbf{w}$ through the vertex where the edges along $\mathbf{v}$ and $\mathbf{w}$ meet bisects the angle at that vertex (as it makes the same angle $\theta_1 = \theta_2$ with each of $\mathbf{v}$ and $\mathbf{w}$).

Solution:

(a) To check that two vectors are orthogonal, we calculate their dot product. Using the distributive law for dot products,

$$(\mathbf{v}+\mathbf{w}) \cdot (\mathbf{v}-\mathbf{w}) = (\mathbf{v}+\mathbf{w}) \cdot \mathbf{v} - (\mathbf{v}+\mathbf{w}) \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{w}.$$

Since the dot product is commutative, so $\mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} = 0$, the original dot product is equal to $\mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2$. We are assuming $\|\mathbf{v}\| = \|\mathbf{w}\|$, so this difference of squares vanishes as desired.

(b) To check the angles are equal, we just need to check their cosines are the same. We use the general formula for angles in terms of dot products and lengths of vectors:

$$\cos\theta_1 = \frac{(\mathbf{v}+\mathbf{w}) \cdot \mathbf{v}}{\|\mathbf{v}+\mathbf{w}\|\|\mathbf{v}\|}, \quad \cos\theta_2 = \frac{(\mathbf{v}+\mathbf{w}) \cdot \mathbf{w}}{\|\mathbf{v}+\mathbf{w}\|\|\mathbf{w}\|}.$$

To show these are equal, we note that the denominators are equal because $\|\mathbf{v}\| = \|\mathbf{w}\|$. Hence, it is enough to show that the numerators are equal:

$$(\mathbf{v}+\mathbf{w}) \cdot \mathbf{v} \stackrel{?}{=} (\mathbf{v}+\mathbf{w}) \cdot \mathbf{w}.$$

We check this by expanding each side via the distributive law: the left side is $\mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v}$ and the right side is $\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}$. The terms $\mathbf{w} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{w}$ are equal since the dot product is commutative, and the terms $\mathbf{v} \cdot \mathbf{v}$ and $\mathbf{w} \cdot \mathbf{w}$ are equal since these are respectively equal to $\|\mathbf{v}\|^2$ and $\|\mathbf{w}\|^2$ (which are the same since we are assuming $\mathbf{v}$ and $\mathbf{w}$ have the same length).

Practice 2.2 (Triangle incenter)**.** It is a remarkable fact of plane geometry that the angle bisectors of all three vertices of a triangle meet at a common point (called the *incenter* of the triangle). This exercise works out a specific example, using the parametric form of lines in $\mathbf{R}^2$. Concepts you'll learn later in the course ("linear transformation") allow one to establish the result for general triangles.

Consider the triangle in $\mathbf{R}^2$ with vertices $P = \mathbf{v} = \begin{bmatrix} 14 \\ 0 \end{bmatrix}$, $Q = \mathbf{w} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$, and $R = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(a) Use Exercise 2.1(b) to find a vector bisecting the angle $\angle PRQ$ at $R = \mathbf{0}$ between $\mathbf{v}$ and $\mathbf{w}$. Then, after scaling that vector to make the entries as simple as you can, find a parametric form for the angle bisector line through R. (Hint: $\mathbf{v}$ and $\mathbf{w}$ has different lengths, so to apply Exercise 2.1(b) consider the unit vectors $\mathbf{v}' = \mathbf{v}/\|\mathbf{v}\|$ and $\mathbf{w}' = \mathbf{w}/\|\mathbf{w}\|$.

(b) Similarly to part (a), find a parametric form for the line through P bisecting the angle $\angle RPQ$. (Hint: first work out the displacement vectors from each of R and Q to the vertex P of the angle, making sure to do subtraction in the correct order, to get vectors along those edges to use in roles analogous to $\mathbf{v}$ and $\mathbf{w}$ in part (a).) Make sure that your parametric form passes through P!

(c) Use the parametric forms of the bisector lines described in parts (a) and (b) to find the point where these lines meet. (This point has integer coordinates.)

(d) Adapting the procedure used in part (b), find a parametric form for the line through Q bisecting the angle $\angle PQR$ and check that it passes through the point that you found in (c) (so all three angle bisector lines pass through this common point).

Solution:

(a) First we make unit vectors along the directions of $\mathbf{v}$ and $\mathbf{w}$ by dividing by their lengths:

$$\mathbf{v}' = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}' = \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{1}{\sqrt{5^2+12^2}} \begin{bmatrix} 5 \\ 12 \end{bmatrix} = \begin{bmatrix} 5/13 \\ 12/13 \end{bmatrix}.$$

Now that we have two vectors of the same length along the directions of $\mathbf{v}$ and $\mathbf{w}$, so they still represent the same angle, Exercise 2.1 says that their vector sum bisects the angle:

$$\mathbf{v}' + \mathbf{w}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5/13 \\ 12/13 \end{bmatrix} = \begin{bmatrix} 18/13 \\ 12/13 \end{bmatrix}.$$

To make a cleaner description of the bisector line, we may scale this vector by any nonzero scalar. Let's multiply by 13 to remove the denominator, getting $\begin{bmatrix} 18 \\ 12 \end{bmatrix}$, and then remove the common factor of 6 by multiplying by $1/6$ to get $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Thus, the bisector line consists of points of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = R + t \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3t \\ 2t \end{bmatrix}$$

for a scalar t. (There are many other parametric forms, such as $\begin{bmatrix} 6t \\ 4t \end{bmatrix}$, $\begin{bmatrix} -3t \\ -2t \end{bmatrix}$, etc. This line also has the equation $y = (2/3)x$.)

(b) To work out the bisector, we will compute vectors along the displacements from each of R and Q to the vertex P, and then scale each by a positive scaling factor to get unit vectors (to attain the same length) and finally add those unit vectors (as we did in part (a)).

The displacement from R to P is $\mathbf{r} = \mathbf{0} - \mathbf{v} = \begin{bmatrix} -14 \\ 0 \end{bmatrix}$, and the displacement from Q to P is $\mathbf{q} = \mathbf{w} - \mathbf{v} = \begin{bmatrix} -9 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. To compute unit vectors in these directions, we divide by the lengths to get $\mathbf{r}' = \mathbf{r}/\|\mathbf{r}\| = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\mathbf{q}' = \mathbf{q}/\|\mathbf{q}\| = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ (for this latter one we can replace $\mathbf{q}$ with $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ since scaling $\mathbf{q}$ by the positive scalar $1/3$ has no effect on the associated unit vector).

Hence, an angle bisector is given by the vector sum

$$\mathbf{r}' + \mathbf{q}' = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} -8/5 \\ 4/5 \end{bmatrix}.$$

As in part (a), to compute a parametric form of the bisector line we may first multiply this by any nonzero scaling factor to clean up the vector entries, so we first multiply by 5 to get $\begin{bmatrix} -8 \\ 4 \end{bmatrix}$ and then remove the common factor of 4 by multiplying by $1/4$ to get $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Therefore, the angle bisector of $\angle RPQ$ is the line with parametric form,

$$\begin{bmatrix} x \\ y \end{bmatrix} = P + s\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14-2s \\ s \end{bmatrix}$$

for a scalar s.

(c) Setting the parametric equations of two lines to be equal (remember that t and s are parameters for different lines), we want to get the pair of simultaneous equations

$$\begin{cases} 3t &= 14 - 2s \\ 2t &= s. \end{cases}$$

Solving the system of equations yields $s = 4, t = 2$, and so the point of intersection is $(3t, 2t) = (6, 4)$ (or $(14 - 2s, s) = (6, 4)$).

(d) We follow the same steps as (b). The displacement from P to Q is $\mathbf{v} - \mathbf{w} = \begin{bmatrix} 9 \\ -12 \end{bmatrix} = 3\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and the displacement from R to Q is $\mathbf{0} - \mathbf{w} = -\begin{bmatrix} -5 \\ -12 \end{bmatrix}$. The respective unit vectors in these directions are $\mathbf{p}'' = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$ and $\mathbf{r}'' = \begin{bmatrix} -5/13 \\ -12/13 \end{bmatrix}$. An angle bisector at Q is given by the vector sum

$$\mathbf{p}'' + \mathbf{r}'' = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} + \begin{bmatrix} -5/13 \\ -12/13 \end{bmatrix} = \begin{bmatrix} 14/65 \\ -112/65 \end{bmatrix}.$$

To give a clean parametric form of the bisector line we may multiply this sum by any nonzero scalar, so we first multiply by 65 to get $\begin{bmatrix} 14 \\ -112 \end{bmatrix}$ and then (since $112 = 14 \times 8$) multiply by $1/14$ to get $\begin{bmatrix} 1 \\ -8 \end{bmatrix}$. Hence, the angle bisector line for $\angle RQP$ has the parametric form

$$\begin{bmatrix} x \\ y \end{bmatrix} = Q + r\begin{bmatrix} 1 \\ -8 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix} + r\begin{bmatrix} 1 \\ -8 \end{bmatrix} = \begin{bmatrix} 5+r \\ 12-8r \end{bmatrix}.$$

We seek an r for which $\begin{bmatrix} 5+r \\ 12-8r \end{bmatrix}$ is equal to $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ (the point found in (c) where the other two angle bisector lines meet). The value $r = 1$ works, so indeed all three angle bisector lines pass through this common point.

3. Planes in $\mathbb{R}^3$

Practice 3.1. Consider the three points

$$P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; Q = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; R = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

(a) Find a parametric form for the plane passing through these three points.

(b) Find an equational form $ax + by + cz = d$ for the plane through these three points .

(c) Find a parametric form for the plane defined by the equation $3x - y + 5z = 4$.

Solution:

(a) Let $\mathbf{u} = Q - P = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = R - P = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ be the displacements from P. The parametric representation is $P + t\mathbf{u} + t'\mathbf{v}$, which is to say

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t' \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + t' \\ 1 + t + 2t' \\ 1 + 2t \end{bmatrix}.$$

Alternative solutions. Using displacements from Q or R respectively gives the parametric forms

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} + t' \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 + t' \\ 2 - t + t' \\ 3 - 2t - 2t' \end{bmatrix}$$

and

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + t' \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 - t - t' \\ 3 - 2t - t' \\ 1 + 2t' \end{bmatrix}.$$

(b) We need a normal vector $\mathbf{n}$; i.e., a nonzero vector perpendicular to displacements from $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

This says $\mathbf{u} \cdot \mathbf{n} = 0$ and $\mathbf{v} \cdot \mathbf{n} = 0$, or more explicitly

$$0n_1 + 1n_2 + 2n_3 = 0 \quad \text{and} \quad 1n_1 + 2n_2 + 0n_3 = 0.$$

In other words, $n_2 + 2n_3 = 0$ and $n_1 + 2n_2 = 0$. If we choose $n_3 = 1$ then we get $n_2 = -2$ from the first equation and $n_1 = -2n_2 = 4$ from the second equation. Thus $\mathbf{n} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$ is such a vector, so the equational form $\mathbf{n} \cdot (\mathbf{x} - P) = 0$ says

$$\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = 0,$$

or equivalently

$$\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} = 0.$$

Working out the dot product, this says

$$4(x-1) - 2(y-1) + (z-1) = 0,$$

and bringing the constants to the right side turns this into

$$4x - 2y + z = 3$$

as an equation of the desired form.

(c) To get a parametric form from an equational form, use the equation to express one of the variables in terms of the others. The simplest such choice is $y = 3x + 5z - 4$, which yields the parametric form

$$\begin{bmatrix} x \\ 3x + 5z - 4 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}.$$

Alternative solutions. Solving for each of x and z in terms of the other two variables yields other parametric forms:

$$\begin{bmatrix} (y - 5z + 4)/3 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4/3 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \\ (-3x + y + 4)/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4/5 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ -3/5 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1/5 \end{bmatrix}.$$

Practice 3.2. Let P_1 be the plane in $\mathbf{R}^3$ described by the equation $x - 2y + z = 0$. The following parts of this exercise can be worked on independently of each other.

(a) Consider the plane P_2 with the parametric form $t\mathbf{v} + s\mathbf{w}$ for $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$. Which of the following is true? Justify your answer.

(i) $P_1 = P_2$; that is, P_2 is another description for the plane P_1.

(ii) P_1 and P_2 do not have any points in common (that is, they are parallel and distinct).

(iii) P_1 and P_2 are not the same plane, but they do meet each other (that is, P_1 and P_2 have at least one point in common).

(b) Let P_3 be the plane described by the parametric form $\begin{bmatrix} 1 + 3s \\ 3 + 2s + t \\ 1 + s + 2t \end{bmatrix}$. Which of the following is true?

Justify your answer.

(i) $P_1 = P_3$; that is, P_3 is another description for the plane P_1.

(ii) P_1 and P_3 do not have any points in common (that is, they are parallel and distinct).

(iii) P_1 and P_3 are not the same plane, but they do meet each other (that is, P_1 and P_3 have at least one point in common).

(c) Specify (with justification) a plane P_4 in $\mathbf{R}^3$ which meets P_1 exactly along the line

$$\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

You are free to specify P_4 *either* by an equation or by a parametric form, whichever is more convenient for you.

Solution:

(a) The correct option is (i).

Both P_1 and P_2 contain the origin, so option (ii) is not possible. We'll check that they are parallel by verifying they share a common normal direction, so since they also have a point in common they must then be the same plane (i.e., option (i) holds). The equational form of P_1 yields the normal vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and for the plane P_2 through $\mathbf{0}$ with displacements $\mathbf{v}$ and $\mathbf{w}$ we have

$$\mathbf{v} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = (2)(1) + (3)(-2) + (4)(1) = 2 - 6 + 4 = 0$$

$$\mathbf{w} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = (-3)(1) + (-1)(-2) + (1)(1) = -3 + 2 + 1 = 0$$

Thus, the planes P_1 and P_2 have a point in common and share a normal line, so they are the same plane in $\mathbf{R}^3$.

Alternate Solution. We could directly compute a normal vector for P_2; such a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ must satisfy the system

$$0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 2a + 3b + 4c, \qquad \text{and} \qquad 0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = -3a - b + c.$$

Letting $a = 1$ leads to the system $3b + 4c = -2$, $-b + c = 3$. The second of these implies that $c = 3 + b$, which substituted into the first equation yields $-2 = 3b + 4(3 + b) = 12 + 7b$, so

$b = -2$. Hence, $c = 3 + b = 3 - 2 = 1$, and we readily check that $(a, b, c) = (1, -2, 1)$ indeed satisfies both of the dot product equations, so it is a normal vector for P_2. This is also a normal vector to P_1 in view of its given equation, and since the planes also share a point (namely, the origin), they must be the same plane.

(b) The correct option is (ii).

We expand the parametric form for P_3 as follows:

$$\begin{aligned} \begin{bmatrix} 1+3s \\ 3+2s+t \\ 1+s+2t \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + s\,\mathbf{u}_1 + t\,\mathbf{u}_2, \end{aligned}$$

where we let $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Next we note that P_3 contains the point $(1, 3, 1)$ (by setting $s = t = 0$ above), and this point does not satisfy the equation for P_1: $1 - 2(3) + 1 = -4 \neq 0$. Thus, P_1 and P_3 cannot be the same plane, and option (i) is not possible. On the other hand, each of the displacements $\mathbf{u}_1$ and $\mathbf{u}_2$ along P_2 is perpendicular to the normal vector $\left[\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix}\right]$ arising from the given equation of P_1:

$$\begin{aligned} \mathbf{u}_1 \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} &= (3)(1) + (2)(-2) + (1)(1) = 3 - 4 + 1 = 0 \\ \mathbf{u}_2 \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} &= (0)(1) + (1)(-2) + (2)(1) = 0 - 2 + 2 = 0 \end{aligned}$$

Since the normal line to P_1 is normal to the two displacements $\mathbf{u}_1$ and $\mathbf{u}_2$ along distinct directions in P_3, this is a normal line to P_3. Hence, P_3 and P_1 are parallel, and we know they are distinct planes since we found a point of P_3 not in P_1, so option (iii) holds.

Alternate solution. We seek to find all points in P_3 that satisfy the equation of the plane P_1, by substituting the parametric expressions for the x-, y-, and z-coordinates of points on P_3 into the equation defining P_1:

$$\begin{aligned} x - 2y + z &= (1 + 3s) - 2(3 + 2s + t) + (1 + s + 2t) \\ &= (1 - 6 + 1) + (3s - 4s + s) + (-2t + 2t) = -4 + 0 + 0 \neq 0 \end{aligned}$$

This tells us that no point on P_3 satisfies the equation defining P_1. Thus, option (ii) must be true.

(c) Since we seek a plane P_4 which contains the given parametric line, it will be especially convenient to specify P_4 parametrically, in the following form:

$$\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} + s\,\mathbf{u} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $\mathbf{u}$ is a vector which is not a scalar multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$ (since otherwise P_4 would be a line, not a plane) and does not lie along the plane P_1 through $\mathbf{0}$ (since otherwise P_4 would be equal to P_1, rather than just meeting it along the desired line).

Many answers are possible; we just need to make sure that both conditions on $\mathbf{u}$ are satisfied. Perhaps the simplest way to specify a vector $\mathbf{u}$ not lying along P_1 is to choose a (nonzero) normal vector to P_1, such as $\mathbf{u} = (1, -2, 1)$. By inspection this is also not a scalar multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$, so we take P_4 to be

$$\begin{bmatrix}4\\1\\-2\end{bmatrix} + s\begin{bmatrix}1\\-2\\1\end{bmatrix} + t\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Remark. The condition that $\mathbf{u}$ not lie along P_1 is equivalent to $\mathbf{u}$ *not* being orthogonal to P_1's normal vector $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ (since otherwise $\mathbf{u}$ would lie along P_1, due to the equation defining P_1). So any $\mathbf{u}$ satisfying $\mathbf{u} \cdot \begin{bmatrix}1\\-2\\1\end{bmatrix} \neq 0$ which is also not a scalar multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$ will do the job.

Practice 3.3. Suppose P is the plane in $\mathbf{R}^3$ given by the equation $2x - 2y + z = 1$. Let A and C be the following points:

$$A = (3, -1, 2), \qquad C = (2, -9, -3).$$

(a) Find a nonzero vector $\mathbf{n}$ that is normal to P.

(b) Determine whether A and C lie on the same side, or on opposite sides, of the plane P.

(c) Find the point B on P for which the displacement vector $\overrightarrow{BA}$ is normal to the plane P.

(d) With B as in part (c), determine whether the angle between $\overrightarrow{BC}$ and $\overrightarrow{BA}$ is acute, right, or obtuse.

Solution:

(a) Using the coefficients of the variables in the equation that defines P, we may use $\mathbf{n} = \begin{bmatrix}2\\-2\\1\end{bmatrix}$ (any nonzero scalar multiple of this works just as well).

(b) We plug the coordinates of A and C into the left side of the equation of P, and see how the numerical output compares with the constant 1 on the right side of the equation of P.

At the point A, we have $2x - 2y + z = 2(3) - 2(-1) + (2) = 10 \neq 1$ (so A is not on P). At the point C, we have $2x - 2y + z = 2(2) - 2(-9) + (-3) = 19 \neq 1$ (so C is not on P either).

Since both 10 and 19 are greater than 1, A and C lie on the same side of P.

(c) We need $\overrightarrow{BA} = t\mathbf{n}$ for some scalar t. So if B has coordinates (x_0, y_0, z_0), then we want

$$\overrightarrow{BA} = \begin{bmatrix}x_0 - 3\\y_0 + 1\\z_0 - 2\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}2t\\-2t\\t\end{bmatrix} = t\mathbf{n}$$

for some scalar t. This says $x_0 = 3 + 2t$, $y_0 = -1 - 2t$, and $z_0 = 2 + t$ for some scalar t. Now since B lies in P, we also need $2x_0 - 2y_0 + z_0 = 1$; i.e., $1 = 2(3+2t) - 2(-1-2t) + (2+t) = 10 + 9t$. It follows that $t = -1$, so $B = (3 + 2t, -1 - 2t, 2 + t) = (1, 1, 1)$. (We also see that $\overrightarrow{BA} = t\mathbf{n} = -\mathbf{n}$.)

(d) This question is equivalent to asking whether the cosine of the angle θ between $\overrightarrow{BC}$ and $\overrightarrow{BA}$ is positive, zero, or negative (respectively); furthermore, since

$$\cos(\theta) = \frac{\overrightarrow{BC} \cdot \overrightarrow{BA}}{\|\overrightarrow{BC}\| \, \|\overrightarrow{BA}\|}$$

and the above denominator is always positive, we need only compute the numerator — i.e., the dot product of $\overrightarrow{BC}$ and $\overrightarrow{BA}$ — and find its sign. (Note that this formula makes sense only when vectors $\overrightarrow{BC}$ and $\overrightarrow{BA}$ are nonzero, which is the case here.)

We compute the displacement $\overrightarrow{BC} = \begin{bmatrix} 2-1 \\ -9-1 \\ -3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -10 \\ -4 \end{bmatrix}$, and we computed $\overrightarrow{BA} = -\mathbf{n} = \left[\begin{smallmatrix} -2 \\ 2 \\ 1 \end{smallmatrix}\right]$ at the end of the solution to part (c). Hence, the dot product is

$$\overrightarrow{BC} \cdot \overrightarrow{BA} = \begin{bmatrix} 1 \\ -10 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} = -2 - 20 + 4 = -18 < 0.$$

Thus, the cosine of the angle between the vectors is also negative, so the angle is obtuse.

4. Span, subspaces, and dimension

Practice 4.1. The parts of this exercise can be worked on independently of each other.

(a) Determine, with justification, the dimension (1, 2, or 3) of the linear subspace $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ of $\mathbf{R}^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

(b) Determine, with justification, the dimension (1, 2, or 3) of the linear subspace $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ of $\mathbf{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Solution:

(a) The vector $\mathbf{u}_3$ belongs to the span of $\mathbf{u}_1$ and $\mathbf{u}_2$; in fact, $\mathbf{u}_3 = \frac{1}{2}\mathbf{u}_2$ (or equivalently $\mathbf{u}_2 = 2\mathbf{u}_3$) by inspection. Hence dropping $\mathbf{u}_3$ from the spanning set does not change the linear subspace, so the dimension of the subspace is either 1 or 2.

On the other hand, neither of the nonzero $\mathbf{u}_1$ or $\mathbf{u}_2$ is a scalar multiple of each other (e.g., if $\mathbf{u}_2 = a\mathbf{u}_1$ for some a then equating each vector entry gives the simultaneous conditions $2 = a$, $0 = -a$, $4 = a$ that have no common solution); equivalently, neither is in the span of the other. Hence, there is no "redundancy" anymore, so the dimension is 2.

(b) None of the nonzero $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a scalar multiple of the other (one checks this by inspection, similarly to what was done in (a) with $\mathbf{u}_1$ and $\mathbf{u}_2$), so the dimension cannot be 1 and hence it is either 2 or 3. The value of the dimension depends on whether one of the three vectors is in the span of the other two (if not then there is "no redundancy" and the dimension is 3; otherwise the dimension is 2).

We'll analyze if $\mathbf{v}_1 = a\mathbf{v}_2 + b\mathbf{v}_3$ for some scalars a, b and find that there are such scalars (so the dimension is 2). If we instead were to try to write $\mathbf{v}_2$ or $\mathbf{v}_3$ as a linear combination of the other two, then the algebra would go similarly (yielding that $\mathbf{v}_2 = -\mathbf{v}_1 + 4\mathbf{v}_3$ and $\mathbf{v}_3 = (1/4)\mathbf{v}_1 + (1/4)\mathbf{v}_2$).

The conditions on a, b are

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3a+b \\ 2a+b \\ 5a+2b \end{bmatrix}.$$

This encodes the simultaneous conditions

$$1 = 3a + b, \;\; 2 = 2a + b, \;\; 3 = 5a + 2b.$$

Subtracting the second equation from the first to eliminate b yields $-1 = (3a+b) - (2a+b) = a$, which forces $a = -1$, so the three conditions become

$$1 = -3 + b, \;\; 2 = -2 + b, \;\; 3 = -5 + 2b,$$

which each have the same solution $b = 4$. Hence, $\mathbf{v}_1 = (-1)\mathbf{v}_2 + 4\mathbf{v}_3$ (as is then readily checked directly), so the dimension is 2.

Practice 4.2. Let $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -5 \end{bmatrix}$. Let V be the collection of all 4-vectors $\mathbf{v}$ that are orthogonal to *both* $\mathbf{w}_1$ and $\mathbf{w}_2$. Express V as the span of finitely many vectors.

Solution: A 4-vector $\mathbf{x}$ belongs to V precisely when $\mathbf{x} \cdot \mathbf{w}_1 = 0$ and $\mathbf{x} \cdot \mathbf{w}_2 = 0$. In terms of the entries of $\mathbf{x}$, these amount to the simultaneous equations

$$\begin{aligned} 2x_1 - x_3 - x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 - 5x_4 &= 0 \end{aligned}$$

The second equation involves x_2 and the first does not, so we solve the second for x_2 to eliminate it:

$$x_2 = -3x_1 - 2x_3 + 5x_4$$

We can solve the first equation for x_1 in terms of x_3 and x_4 as $x_1 = (1/2)x_3 + (1/2)x_4$, and plug that into our expression for x_2 to get that also in terms of x_3 and x_4:

$$x_2 = -3(x_3/2 + x_4/2) - 2x_3 + 5x_4 = -(7/2)x_3 + (7/2)x_4.$$

In this way we have solved for x_1 and x_2 in terms of x_3 and x_4 which may freely vary, and the process can be run in reverse to recover the original two equations.

Hence, V consists of exactly the 4-vectors of the form

$$\mathbf{x} = \begin{bmatrix} (1/2)x_3 + (1/2)x_4 \\ -(7/2)x_3 + (7/2)x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1/2 \\ -7/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1/2 \\ 7/2 \\ 0 \\ 1 \end{bmatrix}.$$

This expresses V as the span of $\mathbf{x}' = \begin{bmatrix} 1/2 \\ -7/2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}'' = \begin{bmatrix} 1/2 \\ 7/2 \\ 0 \\ 1 \end{bmatrix}$.

Remark. As with any linear subspace that is larger than the origin, there are *infinitely many* other valid spanning sets; the one we obtained is due to the choices we made when analyzing the system of equations describing the collection of vectors (e.g., to express x_1 and x_2 in terms of x_3 and x_4; we could have decided to express x_2 and x_3 in terms of x_1 and x_4, and so on).

5. Basis and orthogonality

6. Projections

Practice 6.1 (projection onto unit vectors)**.** The projection

$$\mathbf{Proj}_{\mathbf{w}}\,\mathbf{x} = \left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w}$$

is insensitive to replacing $\mathbf{w}$ with a nonzero scalar multiple $c\mathbf{w}$, and this can become a bit easier to work with when $\mathbf{w}$ is a unit vector. This exercise explores such use of unit vectors.

(a) Check that when $\mathbf{w}$ is a unit vector, we have the formula $\mathbf{Proj}_{\mathbf{w}}\,\mathbf{x} = (\mathbf{x}\cdot\mathbf{w})\mathbf{w}$. Express the length of this projection in terms of $\mathbf{x}\cdot\mathbf{w}$ (hint: if c is a scalar and $\mathbf{w}$ is a unit vector, what is the length of $c\mathbf{w}$ in terms of c?).

(b) For a numerical example, find the projection of the vector $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ onto the line spanned by $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, by directly applying the projection formula.

(c) Solve part (b) by instead computing the unit vector $\mathbf{w}' = \mathbf{w}/\|\mathbf{w}\|$ and then applying part (a) with that unit vector, using that the projection $\mathbf{Proj}_{\mathbf{w}}$ is the same as the projection $\mathbf{Proj}_{\mathbf{w}'}$. You should get the same answer as in (b).

Remark. The solution to (c) will demonstrate the pitfall of passing to the unit vector $\mathbf{w}'$ in place of $\mathbf{w}$: it usually requires working with square roots (which ultimately cancel out in the end, if the original vectors $\mathbf{x}$ and $\mathbf{w}$ have entries which are integers or rational numbers).

For theoretical considerations the intervention of such square roots doesn't matter, but if doing numerical work then it can be more pleasant not to get bogged down with the square roots. That is the reason that this course emphasizes the general formula for $\mathbf{Proj}_{\mathbf{w}}$ using dot products rather than requiring $\mathbf{w}$ to be a unit vector (as is done is many textbooks).

Solution:

(a) When $\mathbf{w}$ is a unit vector, $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 = 1$, so the projection formula becomes

$$\mathbf{Proj}_{\mathbf{w}}\,\mathbf{x} = \left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} = (\mathbf{x}\cdot\mathbf{w})\mathbf{w}$$

as desired. The length of this projection vector is $|\mathbf{x}\cdot\mathbf{w}|$ since the length of $c\mathbf{w}$ for a unit vector $\mathbf{w}$ and scalar c is $\|c\mathbf{w}\| = |c|\,\|\mathbf{w}\| = |c|$.

Remark. This formula has a concrete meaning in terms of direction and length: it says that the projection of $\mathbf{x}$ onto a unit vector $\mathbf{u}$ lies along the line through the origin passing through $\mathbf{u}$ with length equals the dot product $|\mathbf{x}\cdot\mathbf{u}|$, and it points in the direction of $\mathbf{u}$ (i.e., is a positive scalar multiple of $\mathbf{u}$) when $\mathbf{x}\cdot\mathbf{u} > 0$ and points in the direction of $-\mathbf{u}$ (i.e., is a positive scalar multiple of $-\mathbf{u}$) when $\mathbf{x}\cdot\mathbf{u} < 0$. The fact that, up to a sign, the length of the projection equals the dot product with the *unit vector* along that direction is very useful in a variety of applications of vector geometry (including Math 52 and many physics and engineering contexts).

(b) Calculating the dot products, $\mathbf{x}\cdot\mathbf{w} = 2+6+12 = 20$ and $\mathbf{w}\cdot\mathbf{w} = 1+4+9 = 14$. Directly applying the projection formula,

$$\mathbf{Proj}_{\mathbf{w}}\,\mathbf{x} = \left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{20}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{10}{7}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}10/7\\20/7\\30/7\end{bmatrix}.$$

(c) Since $\|\mathbf{w}\| = \sqrt{1+4+9} = \sqrt{14}$, we have $\mathbf{w}' = \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}$. Using the recipe from part (a) with this unit vector,

$$\mathbf{Proj}_{\mathbf{w}}(\mathbf{x}) = \mathbf{Proj}_{\mathbf{w}'}\,\mathbf{x} = (\mathbf{x}\cdot\mathbf{w}')\mathbf{w}' = \frac{20}{\sqrt{14}}\begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix} = \begin{bmatrix}20/14\\40/14\\60/14\end{bmatrix} = \begin{bmatrix}10/7\\20/7\\30/7\end{bmatrix}.$$

This is indeed the same answer as obtained in (b).

Practice 6.2. Let $\mathbf{u} = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$ be vectors in $\mathbf{R}^4$, and let $V = \mathrm{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ be their span.

(a) By calculating various dot products, verify that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ forms an orthogonal basis for V.

(b) Find the point on V that is closest to the point $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$. (The answer can be written as $1/3$ times a 4-vector whose entries are integers.)

Solution:

(a) To see if a spanning set of non-zero vectors form an orthogonal basis, we need to check that any pair among $\mathbf{u}, \mathbf{v}, \mathbf{w}$ has dot product equal to 0:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = (-3) \cdot (1) + (-1) \cdot 0 + 2 \cdot (1) + 1 \cdot (1) = -3 + 2 + 1 = 0,$$

$$\mathbf{u} \cdot \mathbf{w} = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix} = (-3) \cdot (-1) + (-1) \cdot (4) + 2 \cdot (0) + 1 \cdot (1) = 3 - 4 + 1 = 0,$$

$$\mathbf{v} \cdot \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot (-1) + 0 \cdot (4) + 1 \cdot (0) + 1 \cdot (1) = -1 + 1 = 0.$$

(b) The closest point on V to $\mathbf{x}$ is $\mathbf{Proj}_V(\mathbf{x})$. Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ forms an orthogonal basis of V, we have

$$\mathbf{Proj}_V(\mathbf{x}) = \mathbf{Proj}_{\mathbf{u}}(\mathbf{x}) + \mathbf{Proj}_{\mathbf{v}}(\mathbf{x}) + \mathbf{Proj}_{\mathbf{w}}(\mathbf{x}) = \left(\frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} + \left(\frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} + \left(\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}.$$

Next we calculate the scalar coefficients, which involve ratios among dot products of $\mathbf{x}$ with $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and dot products of each of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ with itself:

$$\mathbf{x} \cdot \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix} = -3 - 2 + 6 - 1 = 0, \qquad \mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix} = 9 + 1 + 4 + 1 = 15$$

so

$$\frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} = \frac{0}{15} = 0.$$

$$\mathbf{x}\cdot\mathbf{v} = \begin{bmatrix}1\\2\\3\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = 1+3-1=3, \qquad \mathbf{v}\cdot\mathbf{v} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = 1+1+1=3,$$

so

$$\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} = \frac{3}{3} = 1.$$

$$\mathbf{x}\cdot\mathbf{w} = \begin{bmatrix}1\\2\\3\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\4\\0\\1\end{bmatrix} = -1+8-1=6, \qquad \mathbf{w}\cdot\mathbf{w} = \begin{bmatrix}-1\\4\\0\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\4\\0\\1\end{bmatrix} = 1+16+1=18,$$

so

$$\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}} = \frac{6}{18} = \frac{1}{3}.$$

Finally, plugging all of this into the projection formula gives

$$\mathbf{Proj}_V(\mathbf{x}) = \mathbf{v} + \frac{1}{3}\mathbf{w} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-1\\4\\0\\1\end{bmatrix} = \begin{bmatrix}2/3\\4/3\\1\\4/3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\\4\\3\\4\end{bmatrix}.$$

Practice 6.3. Let $\mathcal{P}$ be the plane in $\mathbf{R}^3$ through $\mathbf{0}$ spanned by $\mathbf{v} = \begin{bmatrix}4\\0\\-1\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}0\\3\\1\end{bmatrix}$. In this exercise we will calculate the distance from the point $\mathbf{x} = \begin{bmatrix}7\\-1\\12\end{bmatrix}$ to $\mathcal{P}$, and find the point in $\mathcal{P}$ that's closest to $\mathbf{x}$.

(a) Find a (nonzero) normal vector $\mathbf{u}$ to the plane $\mathcal{P}$.

(b) Find the projection $\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$. (Your answer should be a vector whose entries are integers.)

(c) Using the general fact that $\mathbf{x} = \mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) + \mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$, explain why the length of $\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$ equals the distance from $\mathbf{x}$ to the plane $\mathcal{P}$, and compute this distance. (Your answer should be an integer.)

(d) Find the point on $\mathcal{P}$ that is closest to $\mathbf{x}$. (Hint: use the formula in part (c); it is *not* necessary to compute an orthogonal basis of $\mathcal{P}$!)

Solution:

(a) Writing the normal vector as $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we solve a system of linear equations

$$\begin{cases} 0 = \mathbf{u} \cdot \mathbf{v} = 4a - c \\ 0 = \mathbf{u} \cdot \mathbf{w} = 3b + c \end{cases}$$

so for any choice of c, we get $a = c/4$ and $b = -c/3$. To make the calculations below nicer, we will make an integer solution by choosing $c = 12$; i.e. $\mathbf{u} = \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix}$.

(b) We calculate some dot products, and then use the projection formula:

$$\mathbf{x} \cdot \mathbf{u} = \begin{bmatrix} 7 \\ -1 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix} = 21 + 4 + 144 = 169, \qquad \mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix} = 9 + 16 + 144 = 169,$$

so

$$\frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} = \frac{169}{169} = 1.$$

Hence,

$$\mathbf{Proj}_{\mathbf{u}}(\mathbf{x}) = \left(\frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} = \mathbf{u} = \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix}.$$

(c) The point on $\mathcal{P}$ closest to $\mathbf{x}$ is $\mathbf{Proj}_{\mathcal{P}}(\mathbf{x})$, so the distance from $\mathbf{x}$ to $\mathcal{P}$ is the length of the displacement vector $\mathbf{x} - \mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = \mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$ as desired. This length is $\sqrt{3^2 + (-4)^2 + 12^2} = 13$.

Note that to calculate the shortest distance (but not the closest point), we do not even need to calculate $\mathbf{Proj}_{\mathcal{P}}(\mathbf{x})$: knowing $\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$ (which is simpler to compute) and its length is enough.

(d) The closest point is

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = \mathbf{x} - \mathbf{Proj}_{\mathbf{u}}(\mathbf{x}) = \begin{bmatrix} 7 \\ -1 \\ 12 \end{bmatrix} - \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}.$$

(As sanity checks on this answer: by inspection it is indeed orthogonal to $\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$, and adding it to $\mathbf{Proj}_{\mathbf{u}}(\mathbf{x})$ yields $\mathbf{x}$. You can also check directly that $\mathbf{Proj}_{\mathcal{P}}(\mathbf{x})$ equals $\mathbf{v} + \mathbf{w}$, explicitly exhibiting it as a point in the plane $\mathcal{P}$.)

Practice 6.4. Let L be the line $y = \sqrt{3}\,x$ in $\mathbf{R}^2$ through the origin.

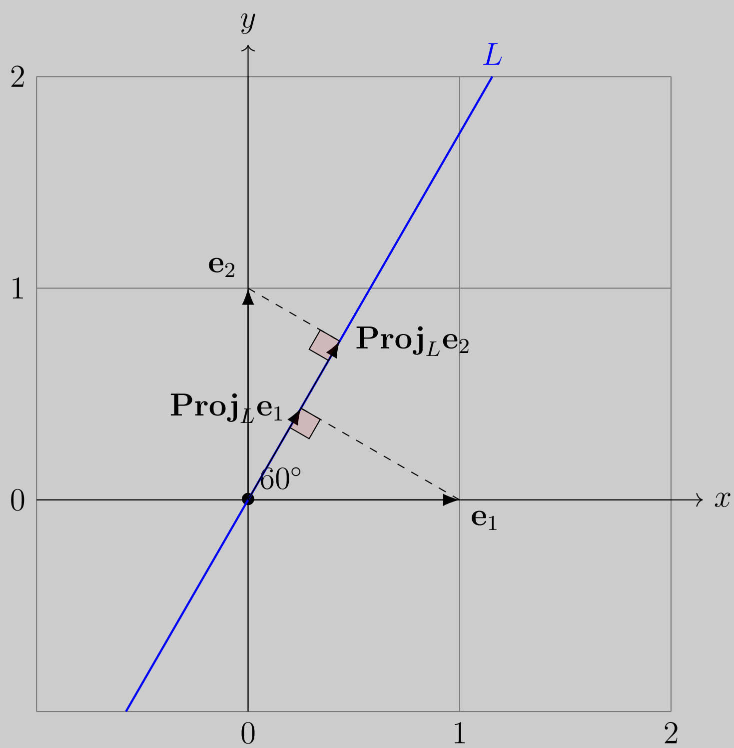
(a) Sketch the line L, the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and their projections $\mathbf{Proj}_L(\mathbf{e}_1)$ and $\mathbf{Proj}_L(\mathbf{e}_2)$ to L. With the help of your picture and your knowledge of trigonometry, fill in the following blanks: the angle between line L and x-axis is _____, so the length of $\mathbf{Proj}_L(\mathbf{e}_1)$ equals _____ and the length of $\mathbf{Proj}_L(\mathbf{e}_2)$ equals ______.

(b) Pick a nonzero vector $\mathbf{w}$ in L. Using the projection formula $\mathbf{Proj_w}(\mathbf{x}) = ((\mathbf{x}\cdot\mathbf{w})/(\mathbf{w}\cdot\mathbf{w}))\mathbf{w}$, calculate $\mathbf{Proj_w}(\mathbf{e}_1)$ and $\mathbf{Proj_w}(\mathbf{e}_2)$. Check that their lengths agree with your answers in (a).

(c) Using your answer in (b), calculate the projection of any point $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ onto the line L. (Your answer should be a 2-vector whose entries depend on x and y.) Check that for $\mathbf{v} = \mathbf{e}_1$ and $\mathbf{v} = \mathbf{e}_2$, this recovers what you found in (b).

Solution:

(a) Since the slope is $\sqrt{3} = (\sqrt{3}/2)/(1/2) = \tan(60°)$, the angle between L and the x-axis is 60°. Hence, the displacement vector $\mathbf{Proj}_L(\mathbf{e}_1)$ is the leg adjacent to the 60-degree angle in a 30-60-90 right triangle with hypotenuse $\mathbf{e}_1$ of length 1, and likewise the displacement vector $\mathbf{Proj}_L(\mathbf{e}_2)$ is the leg adjacent to the 30-degree angle in a 30-60-90 right triangle with hypotenuse $\mathbf{e}_2$ of length 1, as shown:



Thus, the projections in each case have length equal to the unit length of the hypotenuse times the cosine of the adjacent angle:

$$\|\mathbf{Proj}_L(\mathbf{e}_1)\| = 1\times\cos(60°) = 1/2, \quad \|\mathbf{Proj}_L(\mathbf{e}_2)\| = 1\times\cos(30°) = \sqrt{3}/2.$$

(b) By choosing $x = 1$ we get $y = \sqrt{3}$, so one choice is $\mathbf{w} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ (any nonzero scalar multiple is a valid choice too). Thus,

$$\mathbf{Proj}_L(\mathbf{e}_i) = \mathbf{Proj_w}(\mathbf{e}_i) = \left(\frac{\mathbf{e}_i\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} = \frac{\mathbf{e}_i\cdot\mathbf{w}}{4}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}.$$

Since $\mathbf{e}_1 \cdot \mathbf{w} = 1$ and $\mathbf{e}_2 \cdot \mathbf{w} = \sqrt{3}$, we have

$$\mathbf{Proj}_L(\mathbf{e}_1) = \frac{1}{4}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}, \quad \mathbf{Proj}_L(\mathbf{e}_2) = \frac{\sqrt{3}}{4}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/4 \\ 3/4 \end{bmatrix}.$$

Their respective lengths are $(1/4)\sqrt{1+(\sqrt{3})^2} = (1/4)(\sqrt{4}) = 1/2$ and $(\sqrt{3}/4)\sqrt{1+(\sqrt{3})^2} = \sqrt{3}/2$. These agree with the lengths found in (a).

(c) Using $\mathbf{w}$ from (b), we have

$$\mathbf{Proj}_L(\mathbf{v}) = \mathbf{Proj}_{\mathbf{w}}(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} = \frac{x+\sqrt{3}\,y}{4}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} (x+\sqrt{3}\,y)/4 \\ (\sqrt{3}\,x+3y)/4 \end{bmatrix}.$$

As the desired safety check, for $(x, y) = (1, 0)$ this becomes $\begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$ and for $(x, y) = (0, 1)$ this becomes $\begin{bmatrix} \sqrt{3}/4 \\ 3/4 \end{bmatrix}$, which are respectively what we computed for $\mathbf{Proj}_L(\mathbf{e}_1)$ and $\mathbf{Proj}_L(\mathbf{e}_2)$ in the solution to (b).

Practice 6.5. In this exercise we find the projection of any vector to coordinate axes and coordinate planes. The answers provide concrete instances of the general projection formula; you should think about their reasonableness.

(a) Find the projection of a 2-vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ to: (1) the x-axis, (2) the y-axis.

(b) Find the projection of a 3-vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to: (1) the xy-plane, (2) the yz-plane, (3) the xz-plane.

(c) Find the projection of a 4-vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ to: (1) the xy-plane (i.e., $z = 0, w = 0$), (2) the yw plane (i.e., $x = 0, z = 0$).

Solution:

(a) (1) The x-axis is spanned by $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We have $\mathbf{v} \cdot \mathbf{e}_1 = x, \mathbf{e}_1 \cdot \mathbf{e}_1 = 1$, so

$$\mathbf{Proj}_{\mathbf{e}_1}(\mathbf{v}) = (x/1)\mathbf{e}_1 = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

(2) The y-axis is spanned by $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We have $\mathbf{v} \cdot \mathbf{e}_2 = y, \mathbf{e}_2 \cdot \mathbf{e}_2 = 1$, so

$$\mathbf{Proj}_{\mathbf{e}_2}(\mathbf{v}) = (y/1)\mathbf{e}_2 = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

(b) We'll show the answers are: (1) $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, (2) $\begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$, (3) $\begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$. We have $\mathbf{v} \cdot \mathbf{e}_1 = x$, $\mathbf{v} \cdot \mathbf{e}_2 = y$, and $\mathbf{v} \cdot \mathbf{e}_3 = z$. Each $\mathbf{e}_i \cdot \mathbf{e}_i$ is equal to 1 (i.e., the $\mathbf{e}_i$'s are unit vectors), so the projection to each plane is obtained via the respective *orthogonal* bases for the planes: $\mathbf{e}_1, \mathbf{e}_2$ is an orthogonal basis for the xy-plane, $\mathbf{e}_2, \mathbf{e}_3$ is an orthogonal basis for the yz-plane, and $\mathbf{e}_1, \mathbf{e}_3$ is an orthogonal basis for the xz-plane.

Explicitly, the projection of $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to the xy-plane is

$$\left(\frac{\mathbf{v} \cdot \mathbf{e}_1}{\mathbf{e}_1 \cdot \mathbf{e}_1} \right) \mathbf{e}_1 + \left(\frac{\mathbf{v} \cdot \mathbf{e}_2}{\mathbf{e}_2 \cdot \mathbf{e}_2} \right) \mathbf{e}_2 = x\mathbf{e}_1 + y\mathbf{e}_2 = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix},$$

the projection to the yz-plane is similarly $y\mathbf{e}_2 + z\mathbf{e}_3 = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$, and the projection to the xz-plane is similarly $x\mathbf{e}_1 + z\mathbf{e}_3 = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$.

(c) We'll show the answers are: (1) $\begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix}$, (2) $\begin{bmatrix} 0 \\ y \\ 0 \\ w \end{bmatrix}$. The xy-plane in $\mathbf{R}^4$ consists of points of the form

$$\begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} = a\mathbf{e}_1 + b\mathbf{e}_2,$$

so this plane has orthogonal basis $\mathbf{e}_1, \mathbf{e}_2$. Likewise, the yw-plane consists of points of the form

$$\begin{bmatrix} 0 \\ a \\ 0 \\ b \end{bmatrix} = a\mathbf{e}_2 + b\mathbf{e}_4,$$

so this plane has orthogonal basis $\mathbf{e}_2, \mathbf{e}_4$.

Using the respective orthogonal bases, the projections to these planes are

$$\left(\frac{\mathbf{v}\cdot\mathbf{e}_1}{\mathbf{e}_1\cdot\mathbf{e}_1}\right)\mathbf{e}_1 + \left(\frac{\mathbf{v}\cdot\mathbf{e}_2}{\mathbf{e}_2\cdot\mathbf{e}_2}\right)\mathbf{e}_2 = x\mathbf{e}_1 + y\mathbf{e}_2 = \begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix}$$

and

$$\left(\frac{\mathbf{v}\cdot\mathbf{e}_2}{\mathbf{e}_2\cdot\mathbf{e}_2}\right)\mathbf{e}_2 + \left(\frac{\mathbf{v}\cdot\mathbf{e}_4}{\mathbf{e}_4\cdot\mathbf{e}_4}\right)\mathbf{e}_4 = y\mathbf{e}_2 + w\mathbf{e}_4 = \begin{bmatrix} 0 \\ y \\ 0 \\ w \end{bmatrix}.$$

7. Applications of projections in R^n: orthogonal bases of planes and linear regression

Practice 7.1. Let $\mathbf{u} = \begin{bmatrix} 11 \\ 0 \\ 15 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -5 \\ 4 \\ -3 \\ 0 \end{bmatrix}$, and consider the plane $\mathcal{P} = \mathrm{span}(\mathbf{u}, \mathbf{v})$. Let $\mathbf{x} = \begin{bmatrix} -3 \\ 8 \\ -1 \\ -3 \end{bmatrix}$.

(a) Find the vector $\mathbf{u}' = \mathbf{u} - t\mathbf{v}$ for a scalar t so that the pair of vectors $\{\mathbf{u}', \mathbf{v}\}$ is an orthogonal basis of $\mathcal{P}$. (The answer is a vector with integer entries, related to the history of Stanford University.)

(b) Using the Projection Formula, compute the scalars a and b fo which $\mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = a\mathbf{u}' + b\mathbf{v}$.

(c) Which linear combination of the original $\mathbf{u}$ and $\mathbf{v}$ is closest to the point $\mathbf{x}$? More precisely, using (a) and (b), find the scalars c and d for which $\mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = c\mathbf{u} + d\mathbf{v}$.

Solution:

(a) In order for $\mathbf{u}' = \mathbf{u} - t\mathbf{v}$ to be orthogonal to $\mathbf{v}$, we need to solve for t:

$$0 = (\mathbf{u} - t\mathbf{v})\cdot\mathbf{v} = \begin{bmatrix} 11+5t \\ -4t \\ 15+3t \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 4 \\ -3 \\ 0 \end{bmatrix} = -55 - 25t - 16t - 45 - 9t = -100 - 50t,$$

so $t = -2$. (This is saying $t = (\mathbf{u}\cdot\mathbf{v})/(\mathbf{v}\cdot\mathbf{v})$, so $t\mathbf{v}$ is exactly $\mathbf{Proj}_{\mathbf{v}}(\mathbf{u})$.) Therefore,

$$\mathbf{u}' = \mathbf{u} - (-2\mathbf{v}) = \begin{bmatrix} 1 \\ 8 \\ 9 \\ 1 \end{bmatrix}.$$

(b) Having the orthogonal basis $\{\mathbf{u}', \mathbf{v}\}$, we can calculate the projection in terms of dot products:

$$\mathbf{x}\cdot\mathbf{u}' = \begin{bmatrix} -3 \\ 8 \\ -1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 8 \\ 9 \\ 1 \end{bmatrix} = -3+64-9-3 = 49, \qquad \mathbf{u}'\cdot\mathbf{u}' = \begin{bmatrix} 1 \\ 8 \\ 9 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 8 \\ 9 \\ 1 \end{bmatrix} = 1+64+81+1 = 147,$$

so

$$a = \frac{\mathbf{x} \cdot \mathbf{u}'}{\mathbf{u}' \cdot \mathbf{u}'} = \frac{49}{147} = \frac{1}{3};$$

$$\mathbf{x} \cdot \mathbf{v} = \begin{bmatrix} -3 \\ 8 \\ -1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 4 \\ -3 \\ 0 \end{bmatrix} = 15 + 32 + 3 = 50, \qquad \mathbf{v} \cdot \mathbf{v} = \begin{bmatrix} -5 \\ 4 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 4 \\ -3 \\ 0 \end{bmatrix} = 25 + 16 + 9 = 50,$$

so

$$b = \frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = \frac{50}{50} = 1.$$

This says $\mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = (1/3)\mathbf{u}' + \mathbf{v}$.

(c) Since $\mathbf{u}' = \mathbf{u} + 2\mathbf{v}$, we can restate expressions in terms of $\mathbf{u}'$ and $\mathbf{v}$ to be in terms of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{x}) = \frac{1}{3}\mathbf{u}' + \mathbf{v} = \frac{1}{3}(\mathbf{u} + 2\mathbf{v}) + \mathbf{v} = \frac{1}{3}\mathbf{u} + \frac{5}{3}\mathbf{v},$$

so $c = 1/3, d = 5/3$.

Practice 7.2. Let V be a plane in $\mathbf{R}^4$ whose parametric form is

$$V = \left\{ \begin{bmatrix} s + 2t \\ 2s - 5t \\ s - 4t \\ 2s - 9t \end{bmatrix} : s, t \in \mathbf{R} \right\}.$$

We seek the scalars s and t for which the corresponding point in V is closest to $\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 0 \end{bmatrix}$. The following steps carry out this computation.

(a) Let $\mathbf{v}_1$ be the vector in V with $s = 1, t = 0$, and let $\mathbf{v}_2$ be the vector in V with $s = 0, t = 1$. Compute $\mathbf{v}_1$ and $\mathbf{v}_2$ explicitly, and then find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2'\}$ for V, where one of the basis vectors is $\mathbf{v}_1$.

(b) Find scalars a and b for which $\mathbf{Proj}_V(\mathbf{x}) = a\mathbf{v}_1 + b\mathbf{v}_2'$, and compute this projection explicitly as a 4-vector.

(c) Find scalars s and t for which $\mathbf{Proj}_V(\mathbf{x}) = s\mathbf{v}_1 + t\mathbf{v}_2$.

Solution:

(a) The parametric expression for a point in V yields $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -5 \\ -4 \\ -9 \end{bmatrix}$, so an orthogonal basis of V is given by $\{\mathbf{v}_1, \mathbf{v}_2'\}$ for $\mathbf{v}_2' = \mathbf{v}_2 - c\mathbf{v}_1$ with

$$c = \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} = \frac{-30}{10} = -3.$$

Hence, $\mathbf{v}_2' = \mathbf{v}_2 + 3\mathbf{v}_1 = \begin{bmatrix} 5 \\ 1 \\ -1 \\ -3 \end{bmatrix}$.

(b) By orthogonality,

$$a = \frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} = \frac{15}{10} = \frac{3}{2}, \quad b = \frac{\mathbf{x} \cdot \mathbf{v}_2'}{\mathbf{v}_2' \cdot \mathbf{v}_2'} = \frac{18}{36} = \frac{1}{2}.$$

Thus, explicitly

$$\mathbf{Proj}_V(\mathbf{x}) = \frac{3}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2' = \begin{bmatrix} 3/2 \\ 3 \\ 3/2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5/2 \\ 1/2 \\ -1/2 \\ -3/2 \end{bmatrix}.$$

(c) Using the answer from (b), we have

$$\mathbf{Proj}_V(\mathbf{x}) = \frac{3}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2' = \frac{3}{2}\mathbf{v}_1 + \frac{1}{2}(\mathbf{v}_2 + 3\mathbf{v}_1) = 3\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2.$$

Hence, $s = 3$ and $t = 1/2$.

Practice 7.3. Consider the 5 data points $(x_1, y_1), \cdots, (x_5, y_5)$ given by

$$(-2,-1), (0,-2), (1,-3), (2,-4), (4,-5).$$

We wish to find a line of best fit of the form $y = mx + b$, minimizing the sum of squares of $y_i - (mx_i + b)$. The task of finding m and b for the best fit line in $\mathbf{R}^2$ amounts to finding the linear combination $m\mathbf{X} + b\mathbf{1} \in \mathbf{R}^5$ closest to $\mathbf{Y}$, where (as usual) $\mathbf{1}$ denotes the 5-vector with all entries equal to 1.

(a) Explicitly write down the 5-vectors $\mathbf{X}$ and $\mathbf{Y}$.

(b) The 5-vectors $\mathbf{X}$ and $\mathbf{1}$ span a plane P in $\mathbf{R}^5$. Find an orthogonal basis of this plane having the form $\{\mathbf{w}, \mathbf{1}\}$, and then express $\mathbf{Proj}_P(\mathbf{Y})$ as a linear combination of $\mathbf{w}$ and $\mathbf{1}$ (i.e., fill in the blanks: $\mathbf{Proj}_P(\mathbf{Y}) = \_\_\_\_\mathbf{w} + \_\_\_\_\mathbf{1}$).

(c) Fill in the blanks: $\mathbf{Proj}_P(\mathbf{Y}) = \_\_\_\_\mathbf{X} + \_\_\_\_\mathbf{1}$. (Hint: in part (b) you should have written $\mathbf{w}$ as a linear combination of $\mathbf{X}$ and $\mathbf{1}$.) Use these coefficients to write down the line $y = mx + b$ of best fit for the 5 given data points.

Solution:

(a) From the data points, we have $\mathbf{X} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \\ -5 \end{bmatrix}$.

(b) The average $\overline{x}$ of the entries in $\mathbf{X}$ is 1, so

$$\mathbf{w} = \mathbf{X} - \mathbf{Proj}_{\mathbf{1}}(\mathbf{X}) = \mathbf{X} - \overline{x}\,\mathbf{1} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

(The vector $\mathbf{w}$ is denoted as $\widehat{\mathbf{X}}$ in the discussion of line of best fit.)

To compute $\mathbf{Proj}_P(\mathbf{Y})$, we need to compute the projections of $\mathbf{Y}$ onto the orthogonal directions of $\mathbf{w}$ and $\mathbf{1}$. These latter projections are computed using some dot products: $\mathbf{Y} \cdot \mathbf{w} = -14$ and $\mathbf{w} \cdot \mathbf{w} = 20$, so $(\mathbf{w} \cdot \mathbf{Y})/(\mathbf{w} \cdot \mathbf{w}) = -7/10$, and also

$$\frac{\mathbf{1} \cdot \mathbf{Y}}{\mathbf{1} \cdot \mathbf{1}} = \frac{-15}{5} = -3.$$

Thus, $\mathbf{Proj}_P(\mathbf{Y}) = -(7/10)\mathbf{w} - (3)\mathbf{1}$.

(c) From part (b) we have $\mathbf{w} = \mathbf{X} - \mathbf{1}$, so the expression at the end of (b) yields

$$\mathbf{Proj}_P(\mathbf{Y}) = -(7/10)\mathbf{w} - (3)\mathbf{1} = -(7/10)(\mathbf{X} - \mathbf{1}) - (3)\mathbf{1} = -(7/10)\mathbf{X} - (23/10)\mathbf{1}.$$

Hence, the line of best fit is $y = -(7/10)x - (23/10)$.

Practice 7.4. Consider the 5 data points $(1, 7)$, $(2, 5)$, $(4, -1)$, $(5, -5)$, $(8, -6)$. Using linear algebra in $\mathbf{R}^5$, compute the equation $y = mx + b$ of the line of best fit for these points. (The values you find for m and b should be integers.)

Solution: We introduce the data vectors

$$\mathbf{X} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 7 \\ 5 \\ -1 \\ -5 \\ -6 \end{bmatrix}$$

in $\mathbf{R}^5$. Since the sum of the entries in $\mathbf{X}$ is 20, the average of the entries in $\mathbf{X}$ is $\overline{x} = 20/5 = 4$. Hence, the projection of $\mathbf{X}$ onto the 5-vector $\mathbf{1}$ (whose entries are all equal to 1) is $(4)\mathbf{1}$. It follows that the plane $V = \mathrm{span}(\mathbf{1}, \mathbf{X})$ has an orthogonal basis of the form $\{\mathbf{1}, \widehat{\mathbf{X}}\}$ for the 5-vector

$$\widehat{\mathbf{X}} = \mathbf{X} - (4)\mathbf{1} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 4 \end{bmatrix}.$$

The line of best fit is $mx + b$ where $\mathbf{Proj}_V(\mathbf{X}) = m\mathbf{X} + b\mathbf{1}$, so we seek to compute this projection as a linear combination of $\mathbf{X}$ and $\mathbf{1}$ (to read off the values of m and b as the coefficients). The projection is first computed readily in terms of the orthogonal basis $\{\mathbf{1}, \widehat{\mathbf{X}}\}$:

$$\mathbf{Proj}_V(\mathbf{Y}) = \left(\frac{\mathbf{Y}\cdot\mathbf{1}}{\mathbf{1}\cdot\mathbf{1}}\right)\mathbf{1} + \left(\frac{\mathbf{Y}\cdot\widehat{\mathbf{X}}}{\widehat{\mathbf{X}}\cdot\widehat{\mathbf{X}}}\right)\widehat{\mathbf{X}} = \left(\frac{0}{5}\right)\mathbf{1} + \frac{-60}{30}\widehat{\mathbf{X}} = -2\widehat{\mathbf{X}}.$$

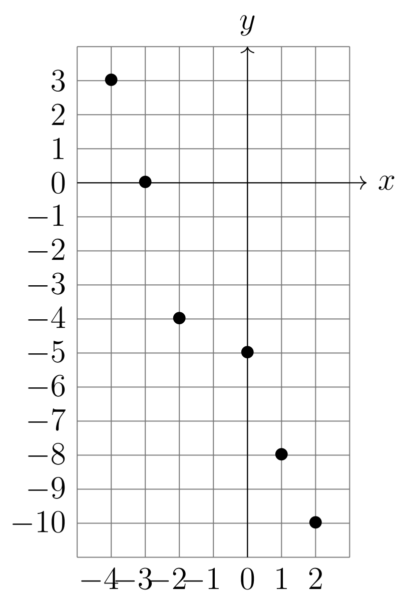
Substituting into this final expression the formula $\widehat{\mathbf{X}} = \mathbf{X} - (4)\mathbf{1}$, the projection is equal to

$$-2(\mathbf{X} - (4)\mathbf{1}) = -2\mathbf{X} + (8)\mathbf{1}.$$

Hence, the equation of the best fit line is $y = -2x + 8$.

Practice 7.5. Consider the 6 data points $(x_1, y_1), \ldots, (x_6, y_6)$ given as follows:

$$(-4, 3),\ (-3, 0),\ (-2, -4),\ (0, -5),\ (1, -8),\ (2, -10).$$



Suppose the line of best fit (in the least-squares sense) is written as $y = mx + b$.

(a) Based on the picture, is the correlation coefficient r closest to $-1, -1/2, 0, 1/2$, or 1?

(b) Write down explicit 6-vectors $\mathbf{X}$ and $\mathbf{Y}$ so that for the 6-vector $\mathbf{1}$ whose entries are all equal to 1, the projection of $\mathbf{Y}$ into the plane $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ in $\mathbf{R}^6$ is $m\mathbf{X} + b\mathbf{1}$. (You are just being asked to write down such $\mathbf{X}$ and $\mathbf{Y}$, nothing more.)

(c) Find a 6-vector $\mathbf{v}$ so that $\{\mathbf{1}, \mathbf{v}\}$ is an orthogonal basis for $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$.

(d) Compute the projection $\mathbf{Proj}_V(\mathbf{Y})$ and express your answer as a linear combination $s\mathbf{v} + t\mathbf{1}$ for scalars s and t that you should determine. (The values of s and t are integers.)

(e) Use your answer to (d) to express $\mathbf{Proj}_V(\mathbf{Y})$ as a linear combination of $\mathbf{X}$ and $\mathbf{1}$, and obtain from this the values of m and b.

Solution:

(a) The value of r should be closest to -1 because the data points appear to align very closely to a line (so the absolute value of r is close to 1) with negative slope (so r is negative).

(b) From the given data points, the 6-vectors are

$$\mathbf{X} = \begin{bmatrix} -4 \\ -3 \\ -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ -5 \\ -8 \\ -10 \end{bmatrix}.$$

(Note: entries can be rearranged, provided that both $\mathbf{X}$ and $\mathbf{Y}$ are rearranged in the same way.)

(c) The vector $\mathbf{v}$ can be taken to be $\mathbf{X} - \mathbf{Proj}_{\mathbf{1}}(\mathbf{X}) = \mathbf{X} - \overline{x}\,\mathbf{1}$ with $\overline{x}$ equal to the average of the entries in $\mathbf{X}$. This average is

$$\frac{-4-3-2+0+1+2}{6} = \frac{-6}{6} = -1,$$

so

$$\mathbf{v} = \mathbf{X} - (-1)\mathbf{1} = \mathbf{X} + \mathbf{1} = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

(The vector $\mathbf{v}$ is denoted as $\widehat{\mathbf{X}}$ in the discussion of line of best fit.)

(d) Since $\{\mathbf{1}, \mathbf{v}\}$ is an orthogonal basis for V, the projection of $\mathbf{Y}$ into V is given by

$$\left(\frac{\mathbf{Y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} + \left(\frac{\mathbf{Y}\cdot\mathbf{1}}{\mathbf{1}\cdot\mathbf{1}}\right)\mathbf{1} = \left(\frac{-56}{28}\right)\mathbf{v} + \left(\frac{-24}{6}\right)\mathbf{1} = (-2)\mathbf{v} + (-4)\mathbf{1}.$$

Hence, $s = -2$ and $t = -4$.

(e) From part (c) we have $\mathbf{v} = \mathbf{X} + \mathbf{1}$, so the result of part (d) gives

$$\begin{aligned} \mathbf{Proj}_V(\mathbf{Y}) = (-2)\mathbf{v} + (-4)\mathbf{1} &= (-2)(\mathbf{X} + \mathbf{1}) + (-4)\mathbf{1} \\ &= (-2)\mathbf{X} + ((-2) + (-4))\,\mathbf{1} \\ &= (-2)\mathbf{X} + (-6)\mathbf{1}. \end{aligned}$$

Since we also know that $\mathbf{Proj}_V(\mathbf{Y}) = m\mathbf{X} + b\mathbf{1}$, we read off the values $m = -2$ and $b = -6$ (i.e., the line of best fit has equation $y = -2x - 6$).

8. Multivariable functions, level sets, and contour plots

Practice 8.1. For the two functions

$$g(x, y) = (x^2,\ xy) \quad \text{and} \quad h(u, v) = (u + v,\ u - v,\ \sin(uv)),$$

which of the following compositions make sense? If a composition makes sense, explicitly compute its output as an expression in terms of its input variables.

(a) $g \circ h$.

(b) $h \circ g$

Solution:

(a) The function h has input in $\mathbf{R}^2$ and output in $\mathbf{R}^3$ while the function g has input in $\mathbf{R}^2$ and output in $\mathbf{R}^2$. Thus, the composition $g \circ h$ does not make sense because the output of h belong to $\mathbf{R}^3$ while the input for g belongs to $\mathbf{R}^2$ (so it doesn't make sense to evaluate g on an output of h).

(b) Since $g : \mathbf{R}^2 \to \mathbf{R}^2$ and $h : \mathbf{R}^2 \to \mathbf{R}^3$, the composition $h \circ g$ makes sense because the output of g lies in $\mathbf{R}^2$ and we can evaluate h on any input from $\mathbf{R}^2$.

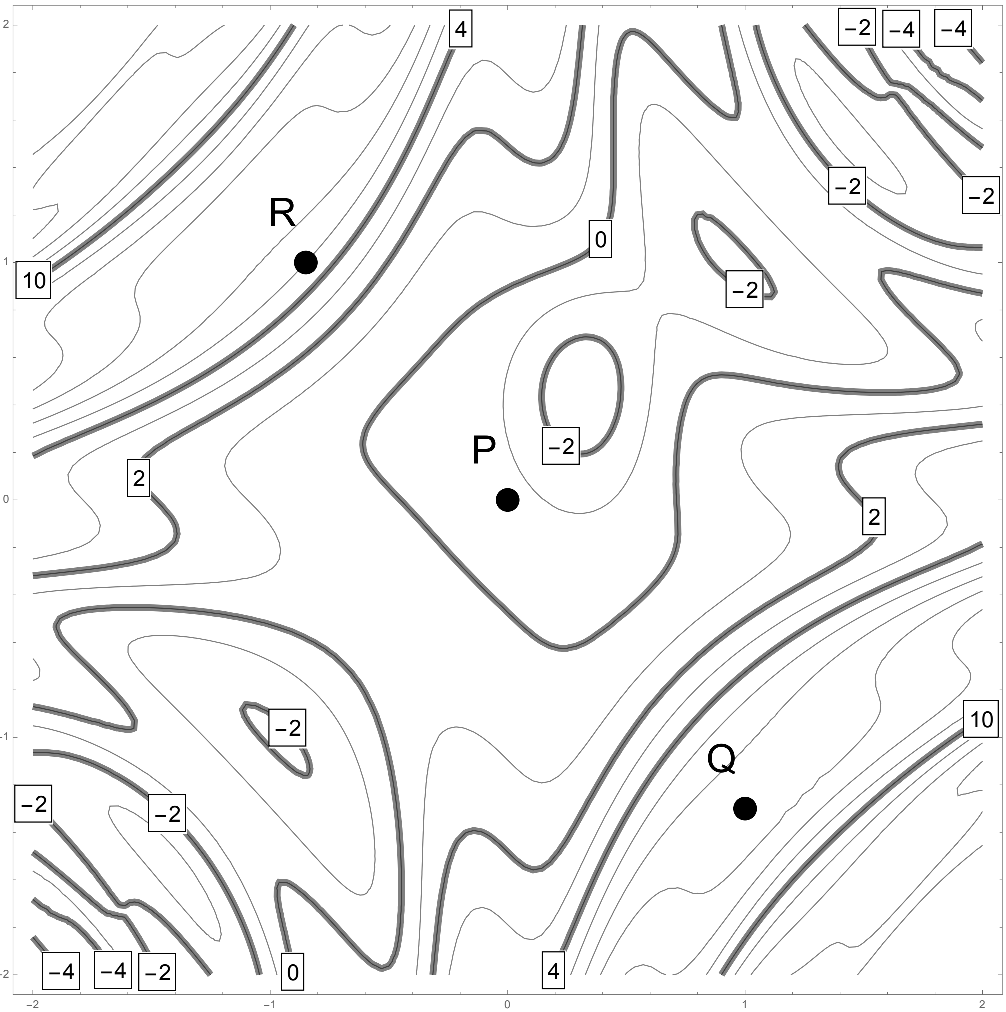
To explicitly calculate $h \circ g$, we use the definitions of g and h:

$$(h \circ g)(x, y) = h(g(x, y)) = h(x^2, xy) = (x^2 + xy, x^2 - xy, \sin(x^3 y)).$$

9. Partial derivatives and contour plots

Practice 9.1. Below is a collection of level sets of a function $f\colon \mathbf{R}^2 \to \mathbf{R}$. Each **thick** level set has its value labeled inside a $\boxed{\text{box}}$ — and in general every level set (thick or thin; labeled or not) corresponds to an *integer* value. (As usual, x is horizontal and y is vertical; the length scales in the x- and y-directions are equal.)

The points **P**, **Q**, and **R** in the xy-plane are each marked with a dot (•).



For each part below, choose one answer.

(a) $\dfrac{\partial f}{\partial x}$ at **Q** is: NEGATIVE ZERO POSITIVE

(b) $\dfrac{\partial f}{\partial y}$ at **Q** is: NEGATIVE ZERO POSITIVE

(c) $\dfrac{\partial f}{\partial x}$ at **R** is: NEGATIVE ZERO POSITIVE

(d) $\dfrac{\partial f}{\partial y}$ at **R** is: NEGATIVE ZERO POSITIVE

(e) Which partial derivative is largest, in *numerical value*? $f_x(\mathbf{P})$ $f_x(\mathbf{Q})$ $f_x(\mathbf{R})$

(f) Which partial derivative is larger, in *absolute value*? $|f_x(\mathbf{Q})|$ $|f_x(\mathbf{R})|$

Solution:

(a) Answer: positive.

Moving a small distance in the positive x ("east") direction from **Q** changes the value of f from $f(\mathbf{Q})$ (i.e., some value between 6 and 7) to a larger value (i.e., towards value 7); thus, f increases.

(b) Answer: negative.

Moving a small distance in the positive y ("north") direction from **Q** changes the value of f from $f(\mathbf{Q})$ (some value between 6 and 7) to a smaller value (i.e., towards value 6); thus, f decreases.

(c) Answer: negative.

Moving a small distance in the positive x ("east") direction from **R** changes the value of f from 5 (precisely) to some value between 4 and 5 (i.e., towards value 4); thus, f decreases.

(d) Answer: positive.

Moving a small distance in the positive y ("north") direction from **R** changes the value of f from 5 (precisely) to some value between 5 and 6 (i.e., towards value 6); thus, f increases.

(e) Answer: $f_x(\mathbf{Q})$.

We have seen that $f_x(\mathbf{R})$ is negative and $f_x(\mathbf{Q})$ is positive; so $f_x(\mathbf{R})$ cannot be the largest numerical value among the choices. Moreover, $f_x(\mathbf{P})$ also appears to be negative: moving a small amount in the positive x ("east") direction from **P** changes the value of f from $f(\mathbf{P})$ (i.e., some value between -1 and 0) to a smaller value (i.e., towards value -1); that is, f decreases. Thus, the positive number $f_x(\mathbf{Q})$ must be the largest among these three choices.

(f) Answer: $f_x(\mathbf{R})$.

Consecutive level curves are horizontally spaced more closely together near **R** than they are near **Q**; so the absolute value of the horizontal rate of change at **R** is greater than that at **Q**.

Practice 9.2. Consider the function $f(x, y) = y\cos(x) + x\sin(y)$.

(a) Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(b) Compute the "mixed" second partial derivative $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ in two different ways, verifying that these two ways give the same formula.

(c) Compute $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$.

Solution:

(a)

$$\frac{\partial f}{\partial x} = -y\sin(x) + \sin(y), \qquad \frac{\partial f}{\partial y} = \cos(x) + x\cos(y)$$

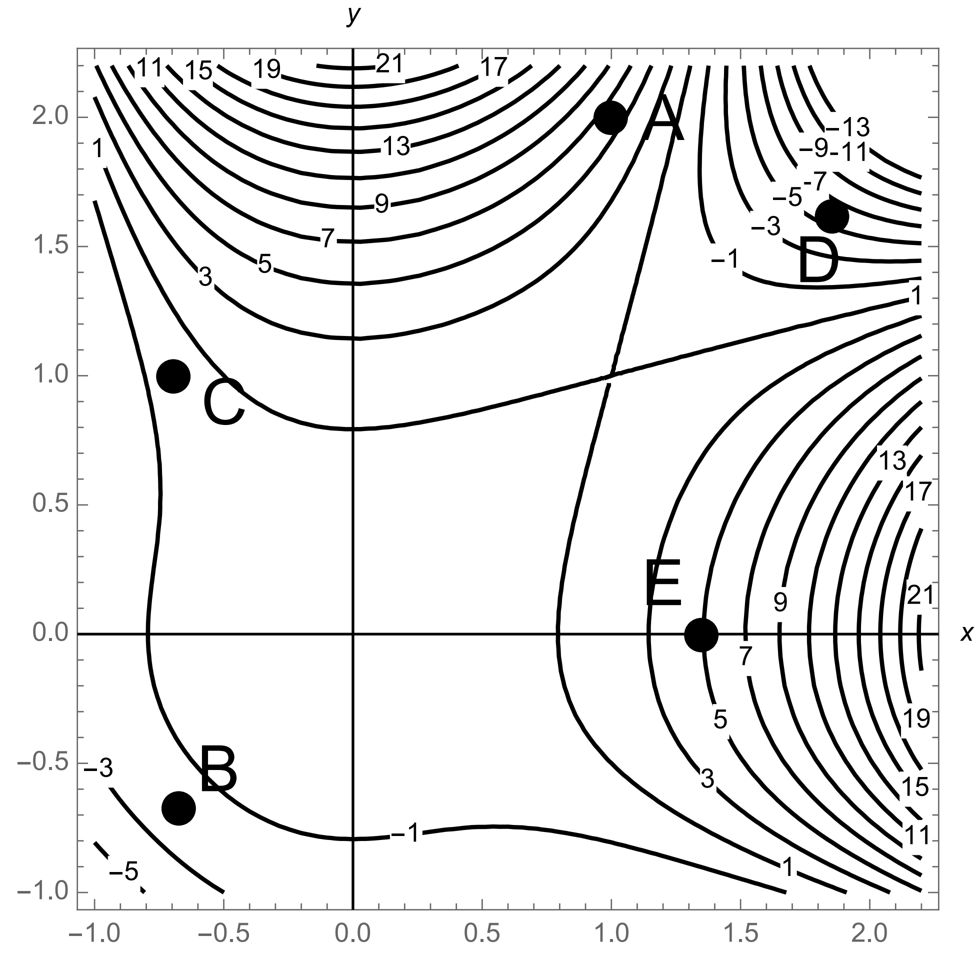
(b)

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\Big(\cos(x) + x\cos(y)\Big) = -\sin(x) + \cos(y)$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\Big(-y\sin(x) + \sin(y)\Big) = -\sin(x) + \cos(y)$$

(c)

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\Big(-y\sin(x) + \sin(y)\Big) = -y\cos(x)$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\Big(\cos(x) + x\cos(y)\Big) = -x\sin(y)$$

Practice 9.3. Below is a contour plot of a function $g(x, y)$ over the square of points (x, y) with $-1 \leq x \leq 2$ and $-1 \leq y \leq 2$. Five points A, B, C, D, E are marked on the plot.



Determine from this contour plot which point clearly satisfies each of the following conditions. No justification needed.

(i) g is approximately 0

(ii) $\dfrac{\partial g}{\partial y} = 0$

(iii) $\dfrac{\partial g}{\partial x} < 0$ AND $\dfrac{\partial g}{\partial y} > 0$

(iv) $\dfrac{\partial g}{\partial x} < 0$ AND $\dfrac{\partial g}{\partial y} < 0$

Solution:

(i) g is approximately 0: C since C is in the narrow space between the -1 and 1 contour lines.

(ii) $\frac{\partial g}{\partial y} = 0$: E since E is the only point where the contour line has vertical tangent. This means the instantaneous rate of change in the y-axis direction is 0, since one remains on the same level curve if one moves a little bit further up or down.

(iii) The correct answer is A, by reasoning as follows:

First condition: Among the five labeled points, only A and D have the property $\dfrac{\partial g}{\partial x} < 0$, since the function value on the contour decreases as one moves to the right.

Second condition: Among these two candidate points, only at point A is $\dfrac{\partial g}{\partial y} > 0$, as the function value on the contour increases as one moves upward.

(iv) The correct answer is D, by reasoning as follows:

First condition: As noted in (iii), only A and D among the five labeled points have $\dfrac{\partial g}{\partial x} < 0$.

Second condition: Among these two candidate points, only at point D is $\dfrac{\partial g}{\partial y} < 0$, as the function value on the contour decreases as one moves to the right.

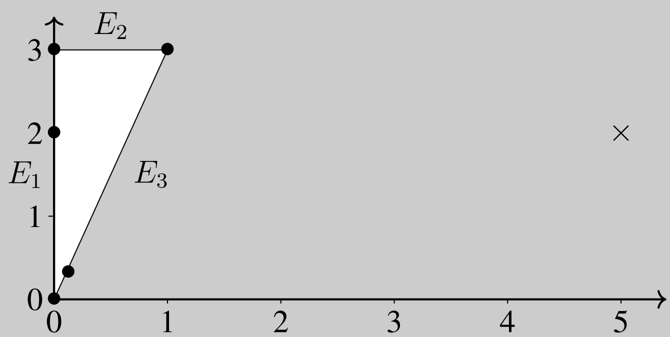
10. Maxima, minima, and critical points

Practice 10.1. Let T be the region in the xy-plane given by the triangle — *together with its interior* — whose vertices are $(0, 0)$, $(0, 3)$, and $(1, 3)$. Draw an approximate picture of T (it need not be to scale), and determine the maximum and minimum values of the function

$$f(x, y) = (x - 5)^2 - (y - 2)^2$$

on this region, and the point(s) at which each is attained.

Solution:



We must build a list of candidates for extrema, consisting either of critical points in the interior of T, or potential extrema on the boundary of T. This boundary consists of three segments:

- vertical edge E_1, line segment $x = 0$ with $0 \leq y \leq 3$
- horizontal edge E_2, line segment $y = 3$ with $0 \leq x \leq 1$
- diagonal edge E_3, line $y = 3x$ between $(0, 0)$ and $(1, 3)$

First, to find candidates on the interior we seek where ∇f vanishes on the interior. This gradient is $\begin{bmatrix} 2(x-5) \\ -2(y-2) \end{bmatrix}$, which equals $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ precisely when $(x, y) = (5, 2)$. However, as indicated in the picture, that critical point marked as "$\times$" in the plane lies outside the region T, so it is *not* a candidate for extremum.

We now look for potential extrema on the edge E_1. This consists of certain points $(0, y)$ (with $0 \leq y \leq 3$), at which the value of f is $f(0, y) = 25 - (y - 2)^2$ of y. This single-variable function has derivative $-2(y - 2)$, so its only critical point is $y = 2$, which is in the permitted range. So $(0, 2)$ is a candidate. The endpoints (i.e., vertices) $(0, 0)$ and $(0, 3)$ are always candidates too.

Next, we look for potential extrema on the edge E_2. This consists of certain points $(x, 3)$ (with $0 \leq x \leq 1$), at which the value of f is $f(x, 3) = (x - 5)^2 - 1$. This single-variable function has derivative $2(x - 5)$, so its only the critical point is $x = 5$, which does not lie in the permitted range $0 \leq x \leq 1$. Hence, that does not lead to a candidate extremum. The endpoints (i.e., vertices) $(0, 3)$ and $(1, 3)$ are always candidates too.

Finally, we look for potential extrema on the edge E_3. This consists of certain points $(x, 3x)$ (with $0 \leq x \leq 1$), at which the value of f is $f(x, 3x) = (x - 5)^2 - (3x - 2)^2 = -8x^2 + 2x + 21$. This single-variable function has derivative $-16x + 2$, so its only critical point is $x = 1/8$, which is in the permitted range. So $(1/8, 3/8)$ is a candidate. The endpoints (i.e., vertices) $(0,0)$ and $(1,3)$ are always candidates too.

Altogether we have five candidates (see picture): the three vertices, and $(0,2)$ and $(1/8, 3/8)$. We evaluate:

$$\begin{aligned} f(0,0) &= 25 - 4 = 21 \\ f(0,2) &= 25 \qquad\qquad \leftarrow \text{MAX} \\ f(0,3) &= 25 - 1 = 24 \\ f(1,3) &= 16 - 1 = 15 \quad \leftarrow \text{MIN} \\ f\left(\frac{1}{8}, \frac{3}{8}\right) &= -8\left(\frac{1}{8}\right)^2 + 2\left(\frac{1}{8}\right) + 21 = -\frac{1}{8} + \frac{1}{4} + 21 = 21\frac{1}{8} \end{aligned}$$

Thus, the maximum value 25 is attained at $(0,2)$, and the minimum value 15 is attained at $(1,3)$.

Practice 10.2. Let D be the region

$$D = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2 : x \geq y^2 + 1 \text{ and } x \leq 5 \right\}.$$

Define $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(x,y) = xe^x(y^2+1)$.

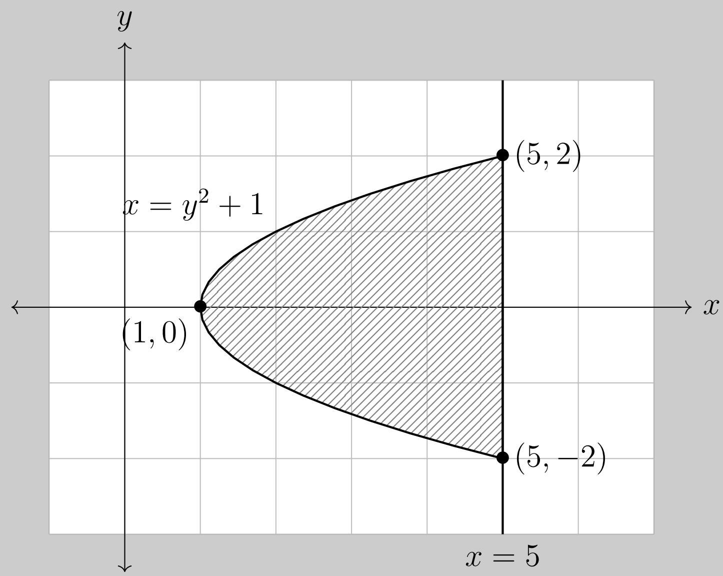
(a) Draw the region D. (It doesn't have to be to scale; just do your best to draw an approximately correct shape.)

(b) Find all the critical points of f in $\mathbf{R}^2$. (There is only one, but your answer must justify this.)

(c) Find the minimum and maximum of f on the region D and all points at which they are attained. (You may take for granted that these extrema exist. It may be helpful to recall that $e > 1$.)

Solution:

(a) The equation $x = y^2 + 1$ defines a sideways parabola, so we care about the region inside the parabola which lies to the left of the line $x = 5$. This is shown in the picture below.



(b) We have

$$\frac{\partial f}{\partial x}(x,y) = e^x(x+1)(y^2+1) \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = 2xye^x.$$

If (x,y) is a critical point, then both of these quantities are zero. The equation

$$e^x(x+1)(y^2+1) = 0$$

tells us $x = -1$ because neither e^x nor y^2+1 can ever be zero. Thus every critical point is of the form $(-1,y)$. We also know $2xye^x = 0$, and if $x = -1$ then we must have $y = 0$. Therefore, $(-1,0)$ is the only critical point.

(c) Observe that our only critical point $(-1,0)$ is not in D because it is not a solution of $x \geq y^2+1$. Therefore, the extrema lie on the boundary.

First, we consider the portion of the boundary where $x = y^2+1$: points of the form (y^2+1, y) with $-2 \leq y \leq 2$. Along this boundary component, we have $f(y^2+1, y) = (y^2+1)^2 e^{y^2+1}$ (all we've done is plug $x = y^2+1$ into the formula for f). Thus, we must find the extrema of that function of y on the interval $-2 \leq y \leq 2$. The derivative of the one-variable function $h(y) = (y^2+1)^2 e^{y^2+1}$ is

$$\begin{aligned} h'(y) = 2(y^2+1)2ye^{y^2+1} + (y^2+1)^3 e^{y^2+1}(2y) &= 2y(y^2+1)^2(2+(y^2+1))e^{y^2+1} \\ &= 2y(y^2+1)^2(y^2+3)e^{y^2+1}. \end{aligned}$$

The factors y^2+1, y^2+3, e^{y^2+1} never vanish, so the only critical point is at $y = 0$, corresponding to the point $(1,0)$ (the "vertex" of the parabola). There are also, as always, the endpoints $y = \pm 2$; these correspond to the points $(5,2)$ and $(5,-2)$.

Now we consider the other boundary piece, where $x = 5$ and $-2 \leq y \leq 2$. Along this boundary component, the value of f at $(5,y)$ is $f(5,y) = 5e^5(y^2+1)$. This has derivative $5e^5(2y) = 10e^5 y$, so its only critical point is $y = 0$, which now corresponds to $(5,0)$. We should also include its endpoints (corresponding to $y = \pm 2$), but those are the points $(5,-2)$ and $(5,2)$ already on our list from the parabolic boundary curve.

Finally, we evaluate f at all four candidate points:

$$f(1,0) = e, \qquad f(5,-2) = f(5,2) = 25e^5, \qquad f(5,0) = 5e^5.$$

Therefore, the minimum for f on D is e at the point $(1,0)$, and the maximum is $25e^5$ at the points $(5,\pm 2)$ (here we have used the fact $e > 1$).

Practice 10.3. Find all the critical points of the following functions of two variables:

(a) $f(x,y) = x^2 + xy + y^2 - 2x - y$

(b) $f(x,y) = x^3y^2(6 - x - y)$, where $x > 0, y > 0$.

(c) $f(x,y) = \dfrac{1 + x - y}{\sqrt{1 + x^2 + y^2}}$.

Solution:

(a) We have $\dfrac{\partial f}{\partial x} = 2x + y - 2$, $\dfrac{\partial f}{\partial y} = x + 2y - 1$.

Setting these equal to 0 is two equations in two unknowns, which we solve to get $(1,0)$. Therefore the only critical point of f is $(1,0)$.

(b) We expand out the product to get $f(x,y) = 6x^3y^2 - x^4y^2 - x^3y^3$, and then compute the partial derivatives: $\dfrac{\partial f}{\partial x} = 18x^2y^2 - 4x^3y^2 - 3x^2y^3 = x^2y^2(18 - 4x - 3y)$. Since we're working in the region $x, y > 0$, the vanishing of this partial derivative says $18 - 4x - 3y = 0$.

Likewise $\dfrac{\partial f}{\partial y} = 12x^3y - 2x^4y - 3x^3y^2 = x^3y(12 - 2x - 3y)$, so the vanishing of this on the region $x, y > 0$ says $12 - 2x - 3y = 0$.

So the critical points are where the simultaneous equations $4x + 3y = 18$ and $2x + 3y = 12$. This is solved to get $(3,2)$, so the only critical point of f is $(3,2)$.

(c) Using the (single variable) quotient rule and chain rule to calculate the partial derivatives gives

$$\frac{\partial f}{\partial x} = \frac{\sqrt{1 + x^2 + y^2} - (1 + x - y)\dfrac{2x}{2\sqrt{1 + x^2 + y^2}}}{1 + x^2 + y^2} = \frac{2(1 + x^2 + y^2) - (1 + x - y)2x}{2(1 + x^2 + y^2)^{3/2}}$$
$$= \frac{(1 + x^2 + y^2) - (x + x^2 - xy)}{(1 + x^2 + y^2)^{3/2}}$$
$$= \frac{1 - x + y^2 + xy}{(1 + x^2 + y^2)^{3/2}}$$

(we combined terms in the numerator for the second equality, and cancelled the factors of 2 for the third equality) and

$$\frac{\partial f}{\partial y} = \frac{-\sqrt{1+x^2+y^2} - (1+x-y)\dfrac{2y}{2\sqrt{1+x^2+y^2}}}{1+x^2+y^2} \quad = \quad \frac{-2(1+x^2+y^2) - (1+x-y)2y}{2(1+x^2+y^2)^{3/2}}$$

$$= \frac{(-1+x^2+y^2) - (y+xy-y^2)}{(1+x^2+y^2)^{3/2}}$$

$$= \frac{-1-y-x^2-xy}{(1+x^2+y^2)^{3/2}}.$$

The critical points are where both of these vanish, which amounts to the simultaneous conditions

$$1-x+y^2+xy=0, \;\; 1+y+x^2+xy=0.$$

We see $1+xy$ in each, so these say

$$1+xy = x-y^2, \;\; 1+xy = -y-x^2.$$

Equating the right sides says $x-y^2 = -y-x^2$, or equivalently

$$x+y = y^2-x^2 = (y-x)(y+x).$$

Hence, either $y+x=0$ or (cancelling $y+x \neq 0$ from both sides) $1 = y-x$. That is, $y=-x$ or $y=x+1$. Going back to the two simultaneous vanishing conditions and plugging in these two possibilities for y, for $y=-x$ we have

$$0 = 1-x+(-x)^2+x(-x) = 1-x+x^2-x^2 = 1-x, \;\; 0 = 1-x+x^2+x(-x) = 1-x$$

(in other words, $x=1$) and for $y=x+1$ we have

$$0 = 1-x+(1+x)^2+x(1+x) = 1-x+1+2x+x^2+x+x^2 = 2+2x+2x^2$$

and

$$0 = 1+(1+x)+x^2+x(1+x) = 2+x+x^2+x+x^2 = 2+2x+2x^2$$

(in other words, $x^2+x+1=0$).

Putting it all together, either $y=-x$ with $x=1$ or $y=1+x$ with $x^2+x+1=0$. The first of these yields $(1,-1)$ as a critical point. The second yields nothing because x^2+x+1 has no real roots (its discriminant is $1-4=-3<0$). In conclusion, the only critical point of function f is $(1,-1)$.

Practice 10.4. Find all the critical points of the following functions of three variables:

(a) $f(x,y,z) = x^2+y^2+z^2-xy+x-2z$.

(b) $f(x,y,z) = x + \dfrac{y^2}{4x} + \dfrac{z^2}{y} + \dfrac{2}{z}$, where $x>0, y>0, z>0$.

Note: in (b), the fact that x, y, z are all positive simplifies calculations.

Solution:

(a) To find the critical points, we calculate the partial derivatives, and set them to 0.
$\dfrac{\partial f}{\partial x} = 2x - y + 1$, $\dfrac{\partial f}{\partial y} = 2y - x$, $\dfrac{\partial f}{\partial z} = 2z - 2 = 2(z-1)$.
The vanishing of the third says $z = 1$. The vanishing of the second says $x = 2y$; plugging that in to the first equation gives $2x - y + 1 = 3y + 1$, so its vanishing forces $y = -1/3$ and hence $x = 2y = -2/3$. Therefore the only critical point of the function f is $(-2/3, -1/3, 1)$.

(b) We compute the partial derivatives
$\dfrac{\partial f}{\partial x} = 1 - \dfrac{y^2}{4x^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{y}{2x} - \dfrac{z^2}{y^2}$, $\dfrac{\partial f}{\partial z} = \dfrac{2z}{y} - \dfrac{2}{z^2}$.
The vanishing of the first says $4x^2 = y^2$. The vanishing of the second says $y/2x = z^2/y^2$, or equivalently (after cross-multiplying) $y^3 = 2xz^2$. The vanishing of the third says $2z/y = 2/z^2$, or equivalently (after cross-multiplying and cancelling 2 from both sides) $z^3 = y$.

Since $x, y > 0$, we can pass to (positive) square roots of both sides of the equation $4x^2 = y^2$ to get $2x = y$. But $y = z^3$, so $x = (1/2)y = (1/2)z^3$. This expresses x and y in terms of z, so the condition $y^3 = 2xz^2$ says $z^9 = 2((1/2)z^3)z^2 = z^5$. But $z > 0$, so we can cancel z^5 from both sides to get $z^4 = 1$. Since $z > 0$, this last condition says $z = 1$, so then $y = z^3 = 1$ and $x = y/2 = 1/2$.

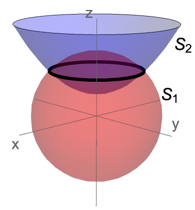
Hence, there is exactly one critical point: $(1/2, 1, 1)$.

11. Gradients, local approximations, and gradient descent

Practice 11.1. Consider two surfaces S_1 and S_2 defined by

$$S_1 : \ x^2 + y^2 + z^2 = 2 \ \text{ and } \ S_2 : \ 2z = x^2 + y^2 + c.$$

There is exactly one scalar c for which S_1 and S_2 cross each other orthogonally, i.e. the normal lines to the respective tangent planes to S_1 and S_2 are perpendicular at all points where these surfaces meet. Determine the value of c.



Solution: By definition, S_1 is the 2-level surface of $F_1(x, y, z) = x^2 + y^2 + z^2$ and S_2 is the c-level surface of $F_2(x, y, z) = 2z - x^2 - y^2$. Thus, normal directions to the surfaces at a common point are given by the respective gradient vectors:

$$\mathbf{n}_1 = \nabla F_1 = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}, \qquad \mathbf{n}_2 = \nabla F_2 = \begin{bmatrix} -2x \\ -2y \\ 2 \end{bmatrix}.$$

For S_1 and S_2 to intersect orthogonally says

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 0, \quad \text{so} \quad -4x^2 - 4y^2 + 4z = 0.$$

In other words, the orthogonality condition says

$$z = x^2 + y^2.$$

But remember that this is happening at points (x, y, z) on both surfaces, so such points satisfy the defining equations for S_1 and S_2. On S_1 we have $x^2 + y^2 + z^2 = 2$, so plugging in $z = x^2 + y^2$ yields

$$2 = x^2 + y^2 + z^2 = z + z^2, \quad \text{so} \quad 0 = z^2 + z - 2 = (z + 2)(z - 1).$$

Hence, $z = 1$ or $z = -2$. But $z = -2$ is not possible since $z = x^2 + y^2 \geq 0$, so $z = 1$.

Now look at the equation for S_2. This says $2z = x^2 + y^2 + c = z + c$ (since $x^2 + y^2 = z$ for the points under consideration), so $c = 2z - z = z$. But $z = 1$, so $c = 1$.

Practice 11.2. Suppose $f(x, y, z) = ye^{(z^2 - x^2)} - y^3 z^2$, and consider the surface $\mathcal{S}$ in $\mathbf{R}^3$ given by the equation $f(x, y, z) = 6$. Notice that this surface contains the point $\mathbf{a} = (1, -2, 1)$.

(a) Find an equation for the tangent plane to $\mathcal{S}$ at the point $\mathbf{a}$.

(b) There is only one point on $\mathcal{S}$ near $\mathbf{a}$ with $y = -2.2$ and $z = 0.9$ (you do not have to prove this). Use linear approximations to estimate the x-coordinate of this point. Simplify your answer as much as possible, but if you wish you may give your estimate to one decimal place (i.e., to the nearest tenth).

(c) Use linear approximations to estimate

$$(-1.9)e^{\left((0.8)^2 - (1.3)^2\right)} - (-1.9)^3(0.8)^2$$

Simplify your answer as much as possible. (*Hint:* this value is $f(1.3, -1.9, 0.8)$.)

Solution:

(a) We first compute the gradient of f at the point $\mathbf{a} = (1, -2, 1)$.

$$\nabla f(x,y,z) = \begin{bmatrix} -2xye^{z^2-x^2} \\ e^{z^2-x^2} - 3y^2z^2 \\ 2yze^{z^2-x^2} - 2y^3z \end{bmatrix} \implies \nabla f(1,-2,1) = \begin{bmatrix} (-2)(-2)e^{(1^2-1^2)} \\ e^{(1^2-1^2)} - 3(-2)^2(1^2) \\ (2)(-2)e^{(1^2-1^2)} - 2(-2)^3(1) \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 1-12 \\ -4-2(-8) \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ 12 \end{bmatrix}$$

Since the gradient of f is perpendicular to the level sets of f, the vector $\mathbf{n} = \nabla f(\mathbf{a}) = \left[\begin{smallmatrix} 4 \\ -11 \\ 12 \end{smallmatrix}\right]$ is perpendicular to $\mathcal{S}$ at $\mathbf{a} = (1,-2,1)$. Therefore the tangent plane P is the plane perpendicular to $\mathbf{n}$ and passing through $\mathbf{a}$, so its equation is

$$\begin{aligned} 4(x-1) - 11(y-(-2)) + 12(z-1) = 0 &\iff 4(x-1) - 11(y+2) + 12(z-1) = 0 \\ &\iff 4x - 11y + 12z = 38. \end{aligned}$$

(b) At points (x,y,z) near $\mathbf{a}$, we have the approximation

$$f(x,y,z) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 6 + 4(x-1) - 11(y+2) + 12(z-1)$$

Since $\mathbf{a} = (1,-2,1)$, we are looking for x near 1 that satisfies $f(x,-2.2,0.9) = 6$. By the approximation above we may estimate x as follows:

$$\begin{aligned} 6 \approx 6 + 4(x-1) - 11(-2.2+2) + 12(0.9-1) &\iff 0 \approx 4(x-1) - 11(-0.2) + 12(-0.1) \\ &\iff 0 \approx 4(x-1) + 2.2 - 1.2 \\ &\iff 0 \approx 4x - 3 \\ &\iff x \approx 3/4 \end{aligned}$$

(or 0.8, to the nearest tenth).

(c) We are asked to approximate f at $(1.3, -1.9, 0.8)$. This is very close to the point $\mathbf{a} = (1, -2, 1)$, namely it is $\mathbf{a} + \mathbf{h}$ for the small vector $\mathbf{h} = \begin{bmatrix} 0.3 \\ 0.1 \\ -0.2 \end{bmatrix}$. The local approximation of f near the point $\mathbf{a}$ is given by

$$\begin{aligned} f(\mathbf{a}+\mathbf{h}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} &= 6 + \begin{bmatrix} 4 \\ -11 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 0.3 \\ 0.1 \\ -0.2 \end{bmatrix} \\ &= 6 + (1.2 - 1.1 - 2.4) \\ &= 6 - 2.3 \\ &= 3.7. \end{aligned}$$

Practice 11.3. Define $g : \mathbf{R}^3 \to \mathbf{R}$ by $g(x, y, z) = x\cos(y) + xz^2$.

(a) Compute the gradient of g.

(b) Compute the equation for the tangent plane to the level set $\{(x, y, z) : g(x, y, z) = 2\}$ at the point $\mathbf{a} = (1, 0, 1)$.

(c) Compute the best linear approximation to $g(x, y, z)$ at the point $\mathbf{a} = (1, 0, 1)$, written in the form $ax + by + cz + d$ for scalars a, b, c, d (some of which may be 0). (Hint: your answer to part (b) may be useful.)

Solution:

(a) We have

$$\begin{aligned} \frac{\partial g}{\partial x}(x, y, z) &= \cos(y) + z^2 \\ \frac{\partial g}{\partial y}(x, y, z) &= -x\sin(y) \\ \frac{\partial g}{\partial z}(x, y, z) &= 2xz. \end{aligned}$$

Therefore, the gradient is

$$\nabla g(x, y, z) = \begin{bmatrix} \cos(y) + z^2 \\ -x\sin(y) \\ 2xz \end{bmatrix}.$$

(b) The tangent plane at $\mathbf{a}$ has normal vector $(\nabla g)(\mathbf{a})$ and passes through the point $\mathbf{a}$, so its equation is

$$((\nabla g)(\mathbf{a})) \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \mathbf{a} \right) = 0.$$

We compute

$$(\nabla g)(\mathbf{a}) = \begin{bmatrix} \cos(0) + 1^2 \\ -1\sin(0) \\ 2 \cdot 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

Thus,

$$((\nabla g)(\mathbf{a})) \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \mathbf{a} \right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y \\ z-1 \end{bmatrix}$$
$$= 2(x-1) + 0(y) + 2(z-1)$$
$$= 2x + 2z - 4.$$

Therefore, the equation for the tangent plane is $2x + 2z - 4 = 0$, or $x + z = 2$. Any nonzero scalar multiple of this is also correct.

(c) First, we compute
$$g(\mathbf{a}) = 1\cos(0) + 1 \cdot 1^2 = 2.$$
Using the tangent plane equation computed in part (b), we see that the best linear approximation for $g(x, y, z)$ is

$$g(\mathbf{a}) + ((\nabla g)(\mathbf{a})) \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \mathbf{a} \right) = 2 + (2x + 2z - 4) = 2x + 2z - 2.$$

12. Constrained optimization via Lagrange multipliers

Practice 12.1. Let R be the region of the plane $x + y + z = 12$ where $x, y, z > 0$, as shown in Figure 12.1. Note that we are *not* considering points on the boundary of this region. It is a fact (which you may accept) that the function $f(x, y, z) = xy^2z^3$ on R attains a maximal value at one point in R. Find that point.

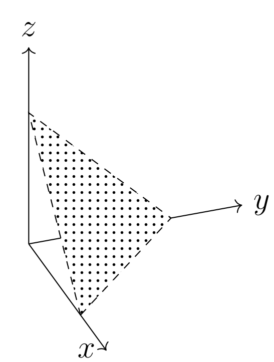


FIGURE 12.1. The region $x + y + z = 12$ with $x, y, z > 0$

Solution: Letting $g(x, y, z) = x + y + z$, the point P of interest is a local maximum for f under the constraint $g = 12$. The gradient of g is equal to the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ everywhere, and

$$(\nabla f)(x, y, z) = \begin{bmatrix} y^2 z^3 \\ 2zyz^3 \\ 3xy^2 z^2 \end{bmatrix}.$$

By the theorem on Lagrange multipliers, there is a scalar λ for which $(\nabla f)(P) = \lambda(\nabla g)(P)$. Writing $P = (a, b, c)$, this says

$$\begin{bmatrix} b^2 c^3 \\ 2abc^3 \\ 3ab^2 c^2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}.$$

Hence,

$$b^2 c^3 = \lambda, \;\; 2abc^3 = \lambda, \;\; 3ab^2 c^2 = \lambda.$$

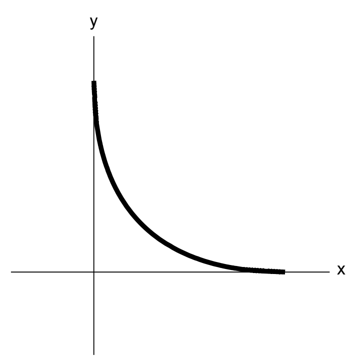
In particular, all three terms on the left are equal to each other, so we have

$$b^2 c^3 = 2abc^3 = 3ab^2 c^2.$$

Since we are in the region where $a, b, c > 0$, we can divide by any of these. Thus, dividing the equality $b^2 c^3 = 2abc^3$ by the common factor bc^3 yields $b = 2a$. Likewise, dividing the equality $b^2 c^3 = 3ab^2 c^2$ by the common factor $b^2 c^2$ yields $c = 3a$. We have now expressed b and c in terms of a.

Going back to the constraint $g = 12$, we have $12 = g(a, b, c) = a + b + c = 6a$, so $a = 2$ and hence $b = 4$ and $c = 6$. Thus, the point P is $(2, 4, 6)$.

Practice 12.2. Find the global extrema of the function $f(x, y) = 2x + 3y$ on the curve $g(x, y) = \sqrt{x} + \sqrt{y} = 5$ where $x, y \geq 0$; this curve is sketched below.



Solution: We want to optimize the function $f(x, y) = 2x + 3y$ subject to the constraint $g(x, y) = \sqrt{x} + \sqrt{y} = 5$. We set up the Lagrange system

$$\begin{cases} \nabla f & = \lambda \nabla g \\ g(x, y) & = 5. \end{cases}$$

and will solve it to find candidates on the curve (along with the endpoints).

Working out the partial derivatives of f and g, and then equating entries in 2-vectors, we get

$$\begin{aligned}2 &= \lambda\frac{1}{2\sqrt{x}}\\ 3 &= \lambda\frac{1}{2\sqrt{y}}\\ 5 &= \sqrt{x}+\sqrt{y}\end{aligned}$$

By the first two equations,

$$\lambda = 4\sqrt{x}, \quad \lambda = 6\sqrt{y}, \quad \text{so} \quad 2\sqrt{x} = 3\sqrt{y}.$$

Squaring $x, y \geq 0$ thereby yields $4x = 9y$, which is to say $y = (4/9)x$. Thus, from the third equation in the Lagrange system we have

$$5 = \sqrt{x}+\sqrt{y} = \sqrt{x} + \frac{2}{3}\sqrt{x} = \frac{5}{3}\sqrt{x}, \quad \text{so} \quad 3 = \sqrt{x} \quad \text{and hence} \quad x = 9.$$

It follows that $y = \frac{4}{9}x = 4$.

Thus, the candidates for the extrema at $(9, 4)$ and the endpoints on the coordinate axes: $(0, 25)$ and $(25, 0)$. We evaluate $f(x, y) = 2x + 3y$ at each of these points. First,

$$f(9, 4) = 2 \cdot 9 + 3 \cdot 4 = 30.$$

Next, $f(0, 25) = 75$ and $f(25, 0) = 50$. Hence, the maximum value is 75 and the minimum value is 30.

Practice 12.3. Find all points $P = (a, b, c)$ on the surface

$$y^2 - 9xz = 9$$

that are closest to the origin (equivalently: points on this surface that minimize $x^2 + y^2 + z^2$).

Solution: Let $f(x, y, z) = x^2+y^2+z^2$ be the squared distance from the origin; let $g(x, y, z) = y^2-9xz$. We are trying to locate minimum points of f subject to the constraint $g = 9$. We can use the method of Lagrange multipliers to identify candidate locations for constrained extrema.

Case 1: Gradient of constraint vanishes. Since $(\nabla g)(x, y, z) = \begin{bmatrix} -9z \\ 2y \\ -9x \end{bmatrix}$, we have $\nabla g = \mathbf{0}$ only if $(x, y, z) = (0, 0, 0)$. But $g(0, 0, 0) = 0 \neq 9$, so this point does not lie on the surface; thus, this case produces no candidate points.

Case 2: Constraint has non-vanishing gradient. In this case we must have

$$\nabla f = \lambda \nabla g \quad \text{for some } \lambda.$$

Computing ∇f, we find this means $\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} -9z \\ 2y \\ -9z \end{bmatrix}$; together with the constraint we get the system

$$\begin{aligned} 2x &= -9\lambda z \\ 2y &= 2\lambda y \\ 2z &= -9\lambda x \\ y^2 - 9xz &= 9 \end{aligned}$$

When solving any of the above equations for λ (so that it can be eliminated), we must make sure not to be dividing by zero. For example, we can only solve the equation $2y = 2\lambda y$ to say $\lambda = 1$ if $y \neq 0$; and then the case $y = 0$ needs to be checked separately because it may lead to valid solutions. (We could alternatively say that the equation $2y = 2\lambda y$ *factors* as $2y(1-\lambda) = 0$; here again we see that *either* $y = 0$ *or* $\lambda = 1$.)

First we treat the possibility $y = 0$, so the system of remaining equations is

$$2x = -9\lambda z, \qquad 2z = -9\lambda x, \qquad -9xz = 9.$$

Notice that within this sub-case, the last equation implies that $xz = -1$, so in particular neither x nor z can be zero. Thus, we can now divide: $-\lambda = \frac{2x}{9z} = \frac{2z}{9x}$. Cross-multipliying, we find that $18x^2 = 18z^2$, which implies that $x = \pm z$. But $-1 = xz$. The case $x = z$ is then ruled out (it yields $-1 = x^2$, an impossibility), and the case $x = -z$ yields $-1 = xz = -x^2$, so $x = \pm 1$ and then $z = -x = \mp 1$. Hence, this sub-case with $y = 0$ yields the candidate points $(1, 0, -1)$ and $(-1, 0, 1)$ (also $\lambda = 2/9$, but this won't be used).

Next we treat the possibility $y \neq 0$, so (as we have seen) $\lambda = 1$. The first and third equations in the Lagrange system then say $2x = -9z$ and $2z = -9x$. These last two simultaneous equations force $x = z = 0$. Thus, the constraint equation $y^2 - 9xz = 9$ becomes $y^2 = 9$, which implies $y = \pm 3$. Hence, the two points $(x, y, z) = (0, \pm 3, 0)$ are candidates for extrema (also $\lambda = 1$, but we don't need that).

We've obtained four candidates: $(0, 3, 0)$, $(0, -3, 0)$, $(1, 0, -1)$, $(-1, 0, 1)$. Our task was finding the points closest to the origin, so by checking all four points we get $(1, 0, -1)$ and $(-1, 0, 1)$.

Practice 12.4. Let $f(x, y) = 12x^2 - 4xy + 3y^2$.

(a) In what *unit-vector* direction should one move from the point $(0, 1)$ in order to decrease the value of f as rapidly as possible? (You may leave your answer unsimplified, but the vector given as your final answer should be in terms of numbers only, not symbols.)

(b) Determine the maximum and minimum values of f on the curve C that is given by the equation $4x^2 + y^2 = 8$, indicating all points where each extremum is attained.

Solution:

(a) We should move along $-\nabla f(0,1)$, but with unit length. Since $\nabla f = \left[\begin{smallmatrix} 24x-4y \\ -4x+6y \end{smallmatrix}\right]$, at $(0,1)$ this is $\left[\begin{smallmatrix} -4 \\ 6 \end{smallmatrix}\right]$. The unit-vector direction we need is $-\frac{1}{\sqrt{4^2+6^2}}\left[\begin{smallmatrix} -4 \\ 6 \end{smallmatrix}\right]$, or $\frac{1}{\sqrt{52}}\left[\begin{smallmatrix} 4 \\ -6 \end{smallmatrix}\right] = \frac{1}{\sqrt{13}}\left[\begin{smallmatrix} 2 \\ -3 \end{smallmatrix}\right]$ (no need to carry out this final simplification).

(b) Let $g(x,y) = 4x^2 + y^2$, so that the "constraint curve" C is given by the equation $g(x,y) = 8$. Now $\nabla g = \left[\begin{smallmatrix} 8x \\ 2y \end{smallmatrix}\right]$, which equals $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ only for $(x,y) = (0,0)$. That does *not* lie on the constraint curve.

Thus, the method of Lagrange multipliers can be applied to locate points that are candidates for extrema; we must solve $\nabla f = \lambda \nabla g$ on C. Computing $\nabla f = \left[\begin{smallmatrix} 24x-4y \\ -4x+6y \end{smallmatrix}\right]$, we set up the equation system

$$\begin{aligned} 24x - 4y &= \lambda(8x) \\ -4x + 6y &= \lambda(2y) \\ 4x^2 + y^2 &= 8 \end{aligned}$$

To equation expressions for λ from the first two equations we want to divide by $8x$ and $2y$, so let's first dispose of the case when one of those vanishes (so then we can avoid division by zero).

If $x = 0$ then the first equation implies $y = 0$. Separately, if $y = 0$ then the second equation implies $x = 0$. But if $x = y = 0$, we've already determined that the constraint equation is not satisfied. Thus, x and y are both nonzero, so

$$\frac{24x - 4y}{8x} = \lambda = \frac{-4x + 6y}{2y}.$$

Cross-multiplying, we find $48xy - 8y^2 = -32x^2 + 48xy$, so $y^2 = 4x^2$. But we have the constraint $8 = 4x^2 + y^2 = 8x^2$, so $x = \pm 1$ and $y^2 = 4x^2 = 4$, so $y = \pm 2$. This gives four candidate points:

$$(1,2), (-1,2), (1,-2), (-1,-2).$$

The values of f at each of these candidate points are as follows:

$$\begin{aligned} f(1,2) &= 12 - 8 + 12 = 16 \\ f(1,-2) &= 12 + 8 + 12 = 32 \\ f(-1,2) &= 12 + 8 + 12 = 32 \\ f(-1,-2) &= 12 - 8 + 12 = 16 \end{aligned}$$

Thus, the minimum value of 16 is attained at both $(1,2)$ and $(-1,-2)$; and the maximum value of 32 is attained at both $(1,-2)$ and $(-1,2)$.

Practice 12.5. Use Lagrange multipliers to find the extrema of the function $f(x,y,z) = xy + z$ on the sphere of radius 3 defined by the equation $x^2 + y^2 + z^2 = 9$, and the points at which these extrema are attained. (You may take for granted that $f(x,y,z)$ has extrema on this region. The extreme values of f are both integers.)

Solution: We define $g : \mathbf{R}^3 \to \mathbf{R}$ by $g(x, y, z) = x^2 + y^2 + z^2$. Our goal is to optimize f subject to the constraint $g(x, y, z) = 9$. We know by the theorem of Lagrange multipliers that the extrema must be points on the level set of g at level 9 which are solutions of $(\nabla g)(x, y, z) = 0$ or $(\nabla f)(x, y, z) = \lambda(\nabla g)(x, y, z)$ for some $\lambda \in \mathbf{R}$. Therefore, we begin by computing the gradients:

$$(\nabla f)(x, y, z) = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix} \qquad \text{and} \qquad (\nabla g)(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}.$$

The only solution to the equation $\nabla g(x, y, z) = \mathbf{0}$ is the point $(0, 0, 0)$, which does not fit the constraint of being on the sphere $g = 9$.

Therefore, we are left with the solutions of $\nabla f = \lambda(\nabla g)$, which are points (x, y, z) solving the system

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ 1 = 2\lambda z \end{cases}$$

Note that the last equation forces $z \neq 0$. We wish to equate expressions for λ from the first two equations, but first we address when the quantities $2xy$ or $2y$ by which we want to divide may be 0; i.e., the special cases $x = 0$ or $y = 0$.

First, consider the case $y = 0$. Then the second equation $x = 2\lambda y$ tells us $x = 0$. The only points of the form $(0, 0, z)$ which satisfy the constraint $g = 9$ are $(0, 0, -3)$ and $(0, 0, 3)$. For each of these points, we can find some λ which solves the system of equations (namely $\lambda = 1/(2z) = \pm 1/6$ via the third equation in the Lagrange system), so these points are potential extrema.

Now assume $y \neq 0$, so we can cancel y in the first equation of the Lagrange system to get $1 = 2\lambda x$. Thus, we get three expressions for λ, and we equate them:

$$\frac{y}{2x} = \frac{x}{2y} = \frac{1}{2z}.$$

Cross-multiplying the first two yields $y^2 = x^2$, so $y = \pm x$, and hence $y/x = x/y = \pm 1$, so $z = \pm 1$. The constraint then gives

$$9 = g(x, y, z) = g(x, \pm x, \pm 1) = 2x^2 + 1,$$

so $2x^2 = 8$, and hence $x = \pm 2$. Then $y = \pm 2$ (unrelated sign), and $z = \pm 1$ with the same sign as y/x/

Putting it all together, we have 6 candidate points:

$$\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Finally, we evaluate the function $f(x, y, z) = xy + z$ at these points:

$$f\left(\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}\right) = -3, \quad f\left(\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\right) = 3$$

$$f\left(\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}\right) = -5, \quad f\left(\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right) = 5.$$

Thus the global minimum of f is -5 (attained at $(-2,2,-1)$ and $(2,-2,-1)$), and the global maximum is 5 (attained at $(-2,-2,1)$ and $(2,2,1)$).

13. Linear functions, matrices, and the derivative matrix

Practice 13.1. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear function for which

$$f(1,0) = (2,3), \qquad f(1,1) = (1,9).$$

Find the matrix A for which $f(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbf{R}^2$.

Solution:

Since f is a linear function from $\mathbf{R}^2$ to $\mathbf{R}^2$, A will be a 2x2 matrix. In particular,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for some numbers $a, b, c, d \in \mathbf{R}$ to be determined. Using the given conditions

$$f(1,0) = (2,3), \qquad f(1,1) = (1,9)$$

and the fact that $f(\mathbf{x}) = A\,\mathbf{x}$ for any $\mathbf{x} \in \mathbf{R}^2$, we deduce that

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = f(1,0) = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 9 \end{bmatrix} = f(1,1) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}.$$

Equating corresponding entries in these 2-vectors gives $a = 2$, $c = 3$, $a + b = 1$, and $c + d = 9$. Thus, $b = -1$ and $d = 6$, so

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}.$$

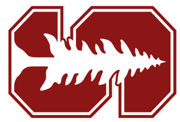
Practice 13.2. Consider the effect a linear transformation on the Stanford emblem.

For each of the following matrices M, identify which picture shows the output when M is applied to the Stanford emblem above.

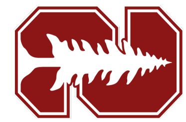
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} : \underline{\qquad} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} : \underline{\qquad} \qquad \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} : \underline{\qquad}$$

$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$: ______ $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$: ______ $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$: ______

(1)


(2)


(3)


(4)


(5)


(6)


(7)


(8)


Solution: The first column of each matrix M is $M\mathbf{e}_1$, which is the effect on $\mathbf{e}_1$ under the corresponding linear transformation. The second column of M is likewise the effect on $\mathbf{e}_2$ under the corresponding linear transformation. (In each description below, we describe the effect on the $\mathbf{e}_j$'s in words, and the conclusion we draw is sometimes aided by making your own picture of the effect on the standard basis that fits the verbal description.)

For $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, $\mathbf{e}_1$ stays put but $\mathbf{e}_2$ changes sign. The effect is an upside-down flip of the emblem, namely refection with respect to the x-axis. So it is (3).

For $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$, both $\mathbf{e}_1$ and $\mathbf{e}_2$ change signs. This is achieved by a $180°$ rotation. So it is (8).

For $\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$, $\mathbf{e}_1$ expands two-fold but $\mathbf{e}_2$ changes sign. The effect is an upside down flip of the emblem (reflection with respect to the x-axis) with two-fold expansion horizontally. So it is (5).

For $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $\mathbf{e}_1$ becomes $-\mathbf{e}_2$ and $\mathbf{e}_2$ becomes $\mathbf{e}_1$. This is achieved by a $90°$ clockwise rotation. So it is (2).

For $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$, $\mathbf{e}_1$ becomes $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_2$ becomes $-\mathbf{e}_1 + \mathbf{e}_2$. This is achieved by a $45°$ counter-clockwise rotation along with $\sqrt{2}$-fold expansion in every direction. So it is (6).

For $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, $\mathbf{e}_1$ stays put but $\mathbf{e}_2$ becomes $\mathbf{e}_1 + \mathbf{e}_2$. This corresponds a shearing transformation along the horizontal direction. So it is (7).

Remark: The remaining two pictures, (1) and (4), do not correspond to the effect on the Stanford emblem by any of the 6 given matrices. (1) corresponds to reflection in the y-axis given by matrix

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$; (4) is (1) followed by a $90°$ clockwise rotation, so it corresponds to matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We mention this in case anyone wonders about the matrices for these two cases.

14. Linear transformations and matrix multiplication

Practice 14.1. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear function that

- first rotates a given vector counterclockwise by $45°$ about the origin,
- then projects the result onto the line $y = x$.

Find the matrix B for which $T(\mathbf{x}) = B\mathbf{x}$ for any $\mathbf{x} \in \mathbf{R}^2$.

Solution:

The matrix of the given rotation is

$$C = \begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

On the other hand, to find the matrix of the given projection, note that it must be again a 2x2 matrix, and that its columns are obtained by projecting the basis vectors $(1, 0)$ and $(0, 1)$ onto the line $y = x$. Since the vector $(1, 1)$ is a basis of this line, we find that the projections of the basis vectors $(1, 0)$ and $(0, 1)$ onto the line are

$$\mathrm{Proj}_{(1,1)}(1, 0) = \frac{(1, 1) \cdot (1, 0)}{(1, 1) \cdot (1, 1)}(1, 1) = \frac{1}{2}(1, 1)$$

.

$$\mathrm{Proj}_{(1,1)}(0, 1) = \frac{(1, 1) \cdot (0, 1)}{(1, 1) \cdot (1, 1)}(1, 1) = \frac{1}{2}(1, 1).$$

Hence, the matrix of the projection is

$$D = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Finally, the matrix of the composition T is the product of the previous two matrices in the right order, which is

$$B = DC = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Practice 14.2. Suppose $S\colon \mathbf{R}^2 \to \mathbf{R}^2$ and $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ are the following linear functions:

- T rotates a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **clockwise** by 45 degrees and then multiplies the x_1-coordinate by 2. (For example, $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.)

- S scales a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by the scalar -1 and then multiplies the x_1-coordinate by $\frac{1}{2}$.

(*Note:* in the parts below, you do *not* need to prove that S, T, or any composition involving them, is a linear function; this may be taken as known.)

(a) Since T is a linear function, it is represented by a matrix. Find this matrix for T.

(b) Similarly, let M be the matrix for the linear function $S \circ T$. Compute M, simplifying as much as possible.

(c) Determine M^8; simplify your answer as much as possible. (*Hint:* interpret the linear function $S \circ T$ geometrically, for example by plotting the effect of $S \circ T$ on $\mathbf{e}_1$ and $\mathbf{e}_2$; what happens if you keep applying it successively 2, 3, 4, etc., times?)

Solution:

(a) We give two possible solutions; it's useful to think of each as a safety check against the other.

Solution #1: For $i = 1, 2$, the i-th column of the matrix for T is the value of $T(\mathbf{e}_i)$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $\mathbf{R}^2$. We have

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{rotate clockwise by } 45^\circ} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \xrightarrow{\text{horizontally double}} \begin{bmatrix} \sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \text{ (as given); and}$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{rotate clockwise by } 45^\circ} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \xrightarrow{\text{horizontally double}} \begin{bmatrix} \sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

That is, the results $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ of our operation on the standard basis vectors are:

$$T(\mathbf{e}_1) = \begin{bmatrix} \sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \qquad T(\mathbf{e}_2) = \begin{bmatrix} \sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Therefore, the matrix we're looking for is

$$\begin{bmatrix} | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Solution #2: The operation T is a composition of a rotation and a horizontal doubling; these individual functions have matrices

$$R = \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

respectively. The matrix of a composition of linear functions is the product of the matrices of each function; in the case of T, the order of muliplication must be DR, because $T(\mathbf{x})$ rotates first, then horizontally doubles. (To check this, note that $DR\mathbf{x} = D(R\mathbf{x})$ is the diagonal matrix applied to the rotated vector $R\mathbf{x}$.) Thus, the matrix of T is

$$DR = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(b) We give two possible solutions; it's useful to think of each as a safety check against the other.

Solution #1: For $i = 1, 2$, the i-th column of M is the value of $(S \circ T)(\mathbf{e}_i) = S(T(\mathbf{e}_i))$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $\mathbf{R}^2$. Applying S to each $T(\mathbf{e}_i)$, we find

$$T(\mathbf{e}_1) = \begin{bmatrix} \sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \xrightarrow{\text{scale by } -1} \begin{bmatrix} -\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \xrightarrow{\text{horizontally halve}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{; and}$$
$$T(\mathbf{e}_2) = \begin{bmatrix} \sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \xrightarrow{\text{scale by } -1} \begin{bmatrix} -\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \xrightarrow{\text{horizontally halve}} \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix},$$

That is, the values $S(T(\mathbf{e}_1))$ and $S(T(\mathbf{e}_2))$ are:

$$S(T(\mathbf{e}_1)) = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \qquad S(T(\mathbf{e}_2)) = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

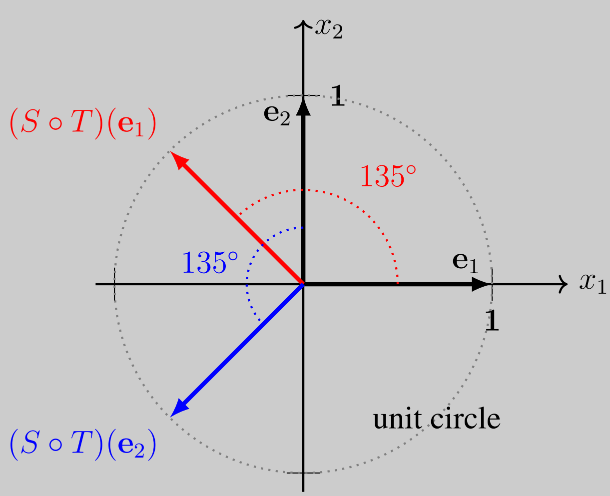
Therefore, the matrix M for $S \circ T$ is

$$M = \begin{bmatrix} | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Solution #2: The matrix of S is the diagonal matrix $\begin{bmatrix} -1/2 & 0 \\ 0 & -1 \end{bmatrix}$, since the overall result of applying S to any vector is to multiply its x_1-component by $-1/2$ and multiply the x_2 component by -1. Thus, the matrix M of $S \circ T$ is the product of the matrix for S with the matrix for T:

$$M = \begin{bmatrix} -1/2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(c) The operation $S \circ T$ is rotation by $90^\circ + 45^\circ = 135^\circ$ (or $3\pi/4$ radians) counterclockwise: this can be seen geometrically by using the two columns of M to plot the effect of $S \circ T$'s on $\mathbf{e}_1$ and $\mathbf{e}_2$, as in the figure:



Alternatively, we could verify this algebraically by observing that M is the matrix

$$M = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \cos(135^\circ) & -\sin(135^\circ) \\ \sin(135^\circ) & \cos(135^\circ) \end{bmatrix}$$

It follows that $S \circ T$, applied 8 times in succession, rotates every vector in $\mathbf{R}^2$ by 1080 $(= 135 \times 8)$ degrees, which is $3 \times 360^\circ$, or 6π $(= (3\pi/4) \times 8)$ radians. This amounts to three full rotations. Hence $S \circ T$, applied 8 times in succession, is the identity function; its matrix M^8 must then be given by the identity matrix:

$$M^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

15. Matrix algebra

16. Applications of matrix algebra: population dynamics, PageRank, and gambling

Practice 16.1. The number of supporters for each candidate in a certain three-person electoral race has been rather variable, due to daily changes in voter opinion.

- Among the voters that support candidate A today, 50% will still support candidate A tomorrow, and 30% will support candidate B tomorrow, and 20% will support candidate C tomorrow.
- Among the voters that support candidate B today, 20% will support candidate A tomorrow, and 80% will still support candidate B tomorrow.
- Among the voters that support candidate C today, 30% will support candidate A tomorrow, and 30% will support candidate B tomorrow, and 40% will still support candidate C tomorrow.

(a) Write down the Markov matrix M for this process of changing candidate support; that is, if the entries of the vector $\mathbf{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$ are today's numbers of supporters for each candidate, then find the matrix M for which the entries of the product $M\mathbf{x}$ are *tomorrow's* numbers of supporters (again, for candidates in the order A, B, C). Write a formula in terms of x_A, x_B, x_C for the number of supporters of A tomorrow.

(b) What proportion of supporters of candidate B will support candidate C precisely two days later? Show your reasoning, and simplify your answer as much as possible.

(c) Suppose $M^{20} \approx \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.6 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$. Over the long term, what proportion of voters will support candidate C? Justify your answer.

Solution:

(a)

$$M = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}$$

Thus, tomorrow's number of supporters for candidate A will be $(0.5)x_A + (0.2)x_B + (0.3)x_C$ (which can also be seen due to how many A supporters tomorrow come from each type of supporter today, and that gives the first row of M).

(b) Suppose today's vector of candidate support totals is $\mathbf{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$, as in part (a). Then in precisely two days, the number of supporters for candidate C is the third entry of the vector $M^2\mathbf{x}$. Such an entry has the form $m_{31}x_A + m_{32}x_B + m_{33}x_C$, where m_{ij} is the (i, j)-entry of M^2. Thus, m_{32} is the proportion of B's supporters that have become C's supporters two days later.

Computing the $(3, 2)$ entry of M^2, we get $(0.2)(0.2) + (0)(0.8) + (0.4)(0) = 0.04$, or 4%.

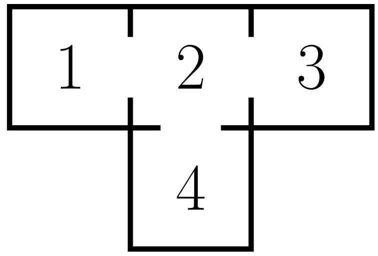
(c) No matter the initial candidate support totals x_A, x_B, x_C, the number of supporters for candidate C after 20 days will be the third entry of the vector

$$\begin{aligned} M^{20} \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} &\approx x_A \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix} + x_B \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix} + x_C \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix} \quad \text{(column combinations)} \\ &= (x_A + x_B + x_C) \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix} \quad \text{(column vectors equal).} \end{aligned}$$

So after 20 days, the *proportion* of supporters for candidate C is about $\dfrac{(0.1)(x_A + x_B + x_C)}{x_A + x_B + x_C}$, or 10%. But this calculation has ended up independent of the individual entries in $\mathbf{x}$ — it only depends on the *total* of the three entries, which is the total number of voters (something that doesn't change from one day to the next by the mechanism of this problem). What we have used most crucially is that all columns of M^{20} are *essentially the same*.

So after $k > 20$ days, the vector of support totals will be $M^k\mathbf{x} = M^{20}(M^{k-20}\mathbf{x}) = M^{20}\mathbf{z} \approx (z_A + z_B + z_C) \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix}$. But for the vector $\mathbf{z} = M^{k-20}\mathbf{x}$, we know its entries sum to $x_A + x_B + x_C$ just like $\mathbf{x}$. Thus, the third entry of $M^k\mathbf{x}$ is *always* about $(0.1)(x_A + x_B + x_C)$, for all $k \geq 20$; and the proportion of supporters for candidate C will be about 10% in the long term.

Practice 16.2. Remy the rat runs through the maze pictured below. For every 15-second interval, Remy stays put and then exits the room for one chosen at random from adjacent rooms (so when in Rooms 1, 3, or 4, next Remy enters Room 2).



(a) After n steps, let

$$\mathbf{p}_n = \begin{bmatrix} \text{probability(equivalently, chances) that Remy is in Room } 1 \\ \text{probability(equivalently, chances) that Remy is in Room } 2 \\ \text{probability(equivalently, chances) that Remy is in Room } 3 \\ \text{probability(equivalently, chances) that Remy is in Room } 4 \end{bmatrix}.$$

Suppose Remy starts in Room 1. What are $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$?

(b) Write the matrix M so that $\mathbf{p}_{n+1} = M\mathbf{p}_n$. (This is handled similarly to the reasoning with the Gambler's Ruin discussion in the course text.)

(c) Compute M^2.

(d) It turns out that $M^3 = M$ – no need to verify this. Deduce what M^{2019} and M^{2020} must be.

(e) Remy's friend Linguini puts a block of delicious cheddar in Room 4. Once Remy reaches the cheese, he stops running and remains in Room 4. Write the matrix N for which $N\mathbf{p}_n = \mathbf{p}_{n+1}$ in this new scenario.

(f) Explain informally why N^{100} is very close to

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Solution:

(a) If Remy starts in Room 1 then $\mathbf{p}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. From Room 1, Remy has to go to Room 2, so $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. From Room 2, Remy can go to Rooms $1, 3$, or 4 with equal likelihood, so $\mathbf{p}_2 = \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix}$

(b) The i-th column of the matrix describes Remy's choices when he starts from Room i. Given the room arrangement, if Remy starts in any room but Room 2, then he has no choice and has to go to Room 2. If he starts from Room 2, he'll pick any other room with probability $1/3$. Hence:

$$M = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \end{bmatrix}.$$

(c)

$$M^2 = MM = \begin{bmatrix} 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{bmatrix}.$$

(d) $M^4 = MM^3 = MM = M^2$, and $M^5 = M^3 = M$. We see that even powers of M are all equal to one another, and odd powers of M are all equal to one another. So $M^{2019} = M$, $M^{2020} = M^2$.

(e)

$$N = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 1 \end{bmatrix}.$$

(f) The ith column of $N^{100}\mathbf{p}_0$ consists of the probabilities that, starting from Room i, Remy ends up in each of the rooms after the 100th step. As soon as Remy reaches Room 4, Remy never leaves. In contrast, from any other room the next step will put Remy in some other room. So the odds that Remy is in Room 4 after 100 steps should be extremely close to 1, no matter where the initial position was (it would otherwise require that every time in Room 2 always avoids going into Room 4, and note that every time in Rooms 1 or 3 is followed by Room 2).

Practice 16.3. A certain city has two soccer teams, FC and United, that compete for the allegiances of its citizens. Each citizen of the city is either an FC supporter, a United supporter, or a non-watcher (doesn't watch soccer at all). Suppose the following:

- Every year, 20% of FC supporters will give up and become non-watchers for the next year; the remaining 80% will stay with their team.
- Every year, 20% of United supporters will give up and become non-watchers for the next year; the remaining 80% will stay with their team.
- Every year, 10% of non-watchers will decide to become FC supporters for the next year, and 20% of non-watchers will become United supporters the next year; the remaining 70% will continue to be non-watchers the next year.

(You may also ignore births, deaths, and migration; so there is no other mechanism for the number of each type of citizen to change.)

(a) Write down the Markov matrix M for this process of changing soccer allegiances; that is, if the entries of the vector $\mathbf{x} = \begin{bmatrix} x_F \\ x_U \\ x_N \end{bmatrix}$ are this year's numbers of each type of citizen (FC, United, and non-watcher, respectively), then find the matrix M for which the entries of the product $M\mathbf{x}$ are *next year's* numbers of citizen types (again, in the order FC, United, non-watcher).

(b) What proportion of current FC supporters will be non-watchers precisely two years later? Show your reasoning, and simplify your answer as much as possible.

(c) Suppose $M^{20} \approx \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}$. Over the *long term*, what is the proportion of citizens that are United supporters? Justify your answer.

Solution:

(a)

$$M = \begin{bmatrix} 0.8 & 0 & 0.1 \\ 0 & 0.8 & 0.2 \\ 0.2 & 0.2 & 0.7 \end{bmatrix}$$

(For example, next year's number of supporters for FC will be $(0.8)x_F + 0x_U + (0.1)x_N$, which gives the first row of M; and so on.)

(b) Suppose this year's vector of support totals is $\mathbf{x} = \left[\begin{smallmatrix} x_F \\ x_U \\ x_N \end{smallmatrix}\right]$, as in part (a). Then in precisely two years, the number of non-watchers is the third entry of the vector $M^2\mathbf{x}$. Such an entry has the form $m_{31}x_F + m_{32}x_U + m_{33}x_N$, where m_{ij} is the (i, j)-entry of M^2. Thus, m_{31} is the proportion of FC supporters that become non-watchers two years later.

Computing the $(3, 1)$ entry of M^2, we get $(0.2)(0.8) + (0.2)(0) + (0.7)(0.2) = 0.30$, or 30%.

(c) No matter the initial totals x_F, x_U, x_N, the number of United supporters after 20 years will be the second entry of the vector

$$\begin{aligned} M^{20} \begin{bmatrix} x_F \\ x_U \\ x_N \end{bmatrix} &\approx x_F \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} + x_U \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} + x_N \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} \qquad \text{(column combinations)} \\ &= (x_F + x_U + x_N) \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} \qquad \text{(column vectors equal).} \end{aligned}$$

So after 20 days, the *proportion* of United supporters is about $\frac{(0.4)(x_F+x_U+x_N)}{x_F+x_U+x_N}$, or 40%. But this calculation has ended up independent of the individual entries in $\mathbf{x}$ — it only depends on the *total* of the three entries, which is the total number of citizens (something that doesn't change from one year to the next by the mechanism of this problem). Here we have used crucially that all columns of M^{20} are *essentially the same*.

So after $k > 20$ years, the vector of support totals will be $M^k\mathbf{x} = M^{20}(M^{k-20}\mathbf{x}) = M^{20}\mathbf{z} \approx (z_F + z_U + z_N)\left[\begin{smallmatrix} 0.2 \\ 0.4 \\ 0.4 \end{smallmatrix}\right]$, where the only thing we are using about $\mathbf{z} = M^{k-20}\mathbf{x}$ is that it is also a vector whose entries sum to $x_F + x_U + x_N$, just like $\mathbf{x}$. Thus, the third entry of $M^k\mathbf{x}$ is *always* about $(0.4)(x_F + x_U + x_N)$, for all $k \geq 20$; and the proportion of United supporters will be about 40% in the long term.

17. Multivariable Chain Rule

Practice 17.1. Let $h : \mathbf{R} \to \mathbf{R}$ be a single-variable function.

(a) For $g : \mathbf{R}^2 \to \mathbf{R}$ and $F(x, y) = h(g(x, y))$, use the matrix form of the Chain Rule to show

$$\frac{\partial F}{\partial x} = h'(g(x,y))\frac{\partial g}{\partial x}, \quad \frac{\partial F}{\partial y} = h'(g(x,y))\frac{\partial g}{\partial y}.$$

(b) Show that the function $f(x, y) = yh(x^2 - y^2)$ satisfies

$$\frac{1}{x}\frac{\partial f}{\partial x} + \frac{1}{y}\frac{\partial f}{\partial y} = \frac{f(x,y)}{y^2}$$

(away from the coordinate axes, so $x \neq 0$ and $y \neq 0$). Part (a) is helpful for this.

(c) Show that the function $f(x, y) = xy + xh(y/x)$ (away from the y-axis, so $x \neq 0$) satisfies

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = xy + f(x, y).$$

Again, part (a) is helpful for this.

Solution:

(a) By the Chain Rule,

$$(DF)(x, y) = (Dh)(g(x, y))(Dg)(x, y) = \begin{bmatrix} h'(g(x,y)) \end{bmatrix} \begin{bmatrix} g_x & g_y \end{bmatrix} = \begin{bmatrix} h'(g(x,y))g_x & h'(g(x,y))g_y \end{bmatrix}.$$

But $(DF)(x, y)$ is the 1×2 matrix given by $\begin{bmatrix} F_x & F_y \end{bmatrix}$, so comparing matrix entries on the two sides gives exactly the desired equalities.

(b) Using part (a), we have

$$\frac{\partial f}{\partial x} = y\frac{\partial}{\partial x}(h(x^2 - y^2)) = yh'(x^2 - y^2)(2x) = 2xyh'(x^2 - y^2)$$

and again using part (a) along with the product rule we have

$$\frac{\partial f}{\partial y} = h(x^2-y^2)+y\frac{\partial}{\partial y}(h(x^2-y^2)) = h(x^2-y^2)+yh'(x^2-y^2)(-2y) = h(x^2-y^2)-2y^2h'(x^2-y^2).$$

Hence,

$$\frac{1}{x}\frac{\partial f}{\partial x} + \frac{1}{y}\frac{\partial f}{\partial y} = 2yh'(x^2 - y^2) + \frac{h(x^2 - y^2)}{y} - 2yh'(x^2 - y^2) = \frac{h(x^2 - y^2)}{y} = \frac{f(x, y)}{y^2},$$

the final equality obtained by multiplying and dividing by y.

(c) Using part (a) and the product rule (now for the x-partial), we have

$$\frac{\partial f}{\partial x} = y+h(y/x)+x\frac{\partial}{\partial x}(h(y/x)) = y+h(y/x)+xh'(y/x)(-y/x^2) = y+h(y/x)-yh'(y/x)/x$$

and

$$\frac{\partial f}{\partial y} = x + x\frac{\partial}{\partial y}(h(y/x)) = x + xh'(y/x)(1/x) = x + h'(y/x).$$

Hence,

$$\begin{aligned}
x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x(y + h(y/x) - yh'(y/x)/x) + y(x + h'(y/x)) \\
&= xy + xh(y/x) - yh'(y/x) + xy + yh'(y/x) \\
&= 2xy + xh(y/x) \\
&= xy + f(x, y).
\end{aligned}$$

Practice 17.2. For this problem, suppose $f\colon \mathbf{R} \to \mathbf{R}$ is a single-variable function.

(a) Let $F(x, y) = f(x^2 + y^2)$. Use the matrix-based form of the Chain Rule to find functions $h_1(x, y)$ and $h_2(x, y)$ having nothing to do with f so that

$$\frac{\partial F}{\partial x} = h_1(x, y)f'(x^2 + y^2),$$
$$\frac{\partial F}{\partial y} = h_2(x, y)f'(x^2 + y^2).$$

(b) Show that $F(x, y) = f(x^2 + y^2)$ satisfies $y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y} = 0$. (*Hint:* part (a) might be helpful.)

(c) Show that $G(x, y) = e^x f\big(y - \frac{1}{2}x^2\big)$ satisfies $\frac{\partial G}{\partial x} + x\frac{\partial G}{\partial y} = G$.

Solution:

(a) Let g be the function $g\colon \mathbf{R}^2 \to \mathbf{R}$ given by $g(x, y) = x^2 + y^2$; thus, we have that $F = f \circ g$. By the Chain Rule,

$$\begin{aligned}
(DF)(x, y) = (Df)(g(x, y))(Dg)(x, y) &= \begin{bmatrix} f'(g(x, y)) \end{bmatrix} \begin{bmatrix} g_x & g_y \end{bmatrix} \\
&= \begin{bmatrix} f'(x^2 + y^2) \end{bmatrix} \begin{bmatrix} 2x & 2y \end{bmatrix} \\
&= \begin{bmatrix} 2xf'(x^2 + y^2) & 2yf'(x^2 + y^2) \end{bmatrix}
\end{aligned}$$

But $(DF)(x, y)$ is the 1×2 matrix given by $\begin{bmatrix} F_x & F_y \end{bmatrix}$, so comparing matrix entries on the two sides gives formulas of the form specified in the problem statement, provided that $h_1(x, y) = 2x$ and $h_2(x, y) = 2y$.

(b) Using part (a), we have

$$\begin{aligned}\frac{\partial F}{\partial x} &= 2xf'(x^2+y^2)\\ \frac{\partial F}{\partial y} &= 2yf'(x^2+y^2)\end{aligned}$$

Hence,

$$y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y} = 2xyf'(x^2+y^2) - 2xyf'(x^2+y^2) = 0$$

(c) A Chain Rule calculation similar to (a) (using inner function $g(x,y) = y - \frac{1}{2}x^2$) yields that

$$\begin{aligned}\frac{\partial}{\partial x}\Big(f\big(y-\tfrac{1}{2}x^2\big)\Big) &= -xf'\left(y-\tfrac{1}{2}x^2\right)\\ \frac{\partial}{\partial y}\Big(f\big(y-\tfrac{1}{2}x^2\big)\Big) &= f'\big(y-\tfrac{1}{2}x^2\big)\end{aligned}$$

Hence if $G(x,y) = e^x f\big(y-\frac{1}{2}x^2\big)$, then

$$\begin{aligned}\frac{\partial G}{\partial x} &= \left(\frac{\partial}{\partial x}\left(e^x\right)\right) f\big(y-\tfrac{1}{2}x^2\big) + e^x\frac{\partial}{\partial x}\Big(f\big(y-\tfrac{1}{2}x^2\big)\Big)\\ &= e^x f\big(y-\tfrac{1}{2}x^2\big) - xe^x f'\left(y-\tfrac{1}{2}x^2\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial G}{\partial y} &= \left(\frac{\partial}{\partial y}\left(e^x\right)\right) f\big(y-\tfrac{1}{2}x^2\big) + e^x\frac{\partial}{\partial y}\Big(f\big(y-\tfrac{1}{2}x^2\big)\Big)\\ &= 0 + e^x\frac{\partial}{\partial y}\Big(f\big(y-\tfrac{1}{2}x^2\big)\Big)\\ &= e^x f'\big(y-\tfrac{1}{2}x^2\big)\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial G}{\partial x} + x\frac{\partial G}{\partial y} &= e^x f\big(y-\tfrac{1}{2}x^2\big) - xe^x f'\left(y-\tfrac{1}{2}x^2\right) + xe^x f'\big(y-\tfrac{1}{2}x^2\big)\\ &= e^x f\big(y-\tfrac{1}{2}x^2\big) = G(x,y).\end{aligned}$$

Practice 17.3. Let $F\colon \mathbf{R}^2 \to \mathbf{R}^3$ be given by $F(x,y) = \begin{bmatrix} xy^2 \\ x^2+y^2 \\ x^2/y \end{bmatrix}$, and suppose $G\colon \mathbf{R}^2 \to \mathbf{R}^2$ satisfies

$$G(2,1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad G(2.01,1) = \begin{bmatrix} 2.02 \\ 1.99 \end{bmatrix}, \quad \text{and} \quad G(2,1.01) = \begin{bmatrix} 2.03 \\ 2.01 \end{bmatrix}.$$

(a) Compute $(DF)(x,y)$.

(b) Estimate the 2×2 derivative matrix $(DG)(2,1)$; show all reasoning, and simplify your answer as much as possible. (Hint: try to relate the first column of $(DG)(2,1)$ to $G(2,1)$ and $G(2.01,1)$.)

(c) Use your answer to (b) to compute $(D(F\circ G))(2,1)$, and *use this matrix* to estimate $(F\circ G)(2.2, 0.9)$. Simplify your answers as much as possible. (*Note:* if you did not find an answer to (b), in its place you may use the (incorrect) matrix $(DG)(2,1) = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$.)

Solution:

(a) $DF = \begin{bmatrix} y^2 & 2xy \\ 2x & 2y \\ 2x/y & -x^2/y^2 \end{bmatrix}$

(b) Using $G(\mathbf{a}+\mathbf{h}) - G(\mathbf{a}) \approx (DG)(\mathbf{a})\mathbf{h}$ at the point $\mathbf{a} = (2,1)$, and successsively $\mathbf{h} = \frac{1}{100}\mathbf{e}_1$ and $\mathbf{h} = \frac{1}{100}\mathbf{e}_2$, we find that the given infomation implies

$$(DG)(\mathbf{a})\left(\frac{\mathbf{e}_1}{100}\right) \approx \begin{bmatrix} 0.02 \\ -0.01 \end{bmatrix} \Longleftrightarrow (DG)(\mathbf{a})\mathbf{e}_1 \approx \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$(DG)(\mathbf{a})\left(\frac{\mathbf{e}_2}{100}\right) \approx \begin{bmatrix} 0.03 \\ 0.01 \end{bmatrix} \Longleftrightarrow (DG)(\mathbf{a})\mathbf{e}_2 \approx \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

So the first column of $DG(\mathbf{a})$ is approximately $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and the second column of $DG(a)$ is approximately $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, i.e., $DG(\mathbf{a}) \approx \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$.

(c) Since $G(2,1) = (2,2)$, we have

$$(D(F\circ G))(2,1) = (DF)(2,2)\,(DG)(2,1) = \begin{bmatrix} 4 & 8 \\ 4 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 20 \\ 4 & 16 \\ 5 & 5 \end{bmatrix}.$$

Since $(2.2, 0.9) = (2,1) + (0.2, -0.1)$,

$$\begin{aligned}
(F\circ G)(2.2, 0.9) &\approx (F\circ G)(2,1) + (D(F\circ G)(2,1)) \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix} \\
&= F(2,2) + \begin{bmatrix} 0 & 20 \\ 4 & 16 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix} \\
&= \begin{bmatrix} 8 \\ 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 0.8 - 1.6 \\ 1 - 0.5 \end{bmatrix} = \begin{bmatrix} 6 \\ 7.2 \\ 2.5 \end{bmatrix}
\end{aligned}$$

Practice 17.4. Let

$$f(x,y) = (-x^2 + xy, x^2 + y^3), \qquad A(v,w) = (v+w, v).$$

We define

$$F(x,y) = (f\circ A\circ f)(x,y).$$

Compute the derivative matrix $(DF)(1,1)$ and use this matrix to estimate $F(1.1, 1.1)$.

Solution: We first compute

$$f(1,1) = (0,2), \qquad (A \circ f)(1,1) = A(0,2) = (2,0)$$

$$F(1,1) = (f \circ A \circ f)(1,1) = f(2,0) = (-4,4)$$

$$(Df)(x,y) = \begin{bmatrix} -2x+y & x \\ 2x & 3y^2 \end{bmatrix}, \qquad (DA)(v,w) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

By repeated application of the Chain Rule, we have

$$\begin{aligned}
(DF)(1,1) &= (D(f \circ A \circ f))(1,1) \\
&= (Df((A \circ f))(1,1))\,(D(A \circ f))(1,1) \\
&= (Df)((A \circ f)(1,1))\,(DA)(f(1,1))\,(Df)(1,1) \\
&= (Df)(2,0) \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (Df)(1,1) \\
&= \begin{bmatrix} -4 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -2 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -6 & -14 \\ 4 & 16 \end{bmatrix}.
\end{aligned}$$

Thus,

$$F(1.1,1.1) = F(1,1) + ((DF)(1,1)) \begin{bmatrix} 1.1-1 \\ 1.1-1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} + \begin{bmatrix} -6 & -14 \\ 4 & 16 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix}.$$

18. Matrix inverses and multivariable Newton's method for zeros

19. Linear independence and the Gram-Schmidt process

Practice 19.1. Consider the matrix $A = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 4 & 2 \\ -1 & -2 & 1 \end{bmatrix}$, whose columns from left to right are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Carry out the Gram-Schmidt process for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, in the manner in which it is presented in Chapter 19 of the course textbook, to construct an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of $\mathbf{R}^3$.

As a safety check on your work, verify directly that the $\mathbf{w}_j$'s you compute are perpendicular to each other. (The entries in the $\mathbf{w}_j$'s are all integers.)

Solution: . We have: $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$,

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \mathbf{v}_2 - \frac{6}{2}\mathbf{w}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \mathbf{v}_3 - \frac{1}{2}\mathbf{w}_1 - \frac{-3}{6}\mathbf{w}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

The dot products among these are:

$$\mathbf{w}_1 \cdot \mathbf{w}_2 = 0 + 1 - 1 = 0, \quad \mathbf{w}_1 \cdot \mathbf{w}_3 = 0 + 2 - 2 = 0, \quad \mathbf{w}_2 \cdot \mathbf{w}_3 = -4 + 2 + 2 = 0.$$

20. Matrix transpose, quadratic forms, and orthogonal matrices

21. Linear systems, column space, and null space

Practice 21.1. Let S be the collection of all vectors in $\mathbf{R}^4$ satisfying:

$$4x_1 + 2x_2 - 4x_3 + x_4 = 0, \quad \text{and} \quad x_3 + x_4 = 0.$$

(a) Express S as the null space $N(A)$ for some matrix A.

(b) Express S as the column space $C(B)$ for some matrix B.

Solution:

(a) Since S is expressed as set of solutions with right side equal to zero, the coefficients of those two equations can be the rows of a 4×2 matrix that does the job:

$$A = \begin{bmatrix} 4 & 2 & -4 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(b) (There are many possible correct B's.) To express S as a column space (i.e., the span of some (column) vectors), first we need to solve the linear system. The second equation gives $x_3 = -x_4$, which plugged into the first equation gives:

$$4x_1 = -2x_2 + 4x_3 - x_4 = -2x_2 - 5x_4$$

Hence the solution is of the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -(1/2)x_2 - (5/4)x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5/4 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Thus S is spanned by $\begin{bmatrix}-1/2\\1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}-5/4\\0\\-1\\1\end{bmatrix}$. If you don't like fractions, you may change those vectors into

$$\mathbf{v} = \begin{bmatrix}-1\\2\\0\\0\end{bmatrix}, \mathbf{w} = \begin{bmatrix}-5\\0\\-4\\4\end{bmatrix}.$$

These two 4-vectors are the columns of one possible B (for which $S = C(B)$): $B = \begin{bmatrix}-1 & -5\\2 & 0\\0 & -4\\0 & 4\end{bmatrix}$.

Practice 21.2. The following is a series of short questions about null spaces. The parts are independent of each other.

(a) For the matrix $A = \begin{bmatrix}1 & 3 & 5\\2 & 4 & 6\end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix}1\\-2\\1\end{bmatrix}$, verify if $\mathbf{v} \in N(A)$.

(b) Give a parametric equation of a line that is contained in $N(\begin{bmatrix}2 & -3 & 5\end{bmatrix})$.

(c) Let V be the collection of all vectors in $\mathbf{R}^3$ perpendicular to $\mathbf{v} = \begin{bmatrix}8\\-5\\6\end{bmatrix}$. Write V as $N(A)$ for some matrix A.

(d) Give an example of matrix A for which $\mathbf{R}^3 = N(A)$.

(e) Find the dimension of $N(A)$ where A is the diagonal matrix $\begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$.

Solution:

(a) We need to check if $A\mathbf{v} = \mathbf{0}$:

$$A\mathbf{v} = \begin{bmatrix}1\cdot 1 + 3\cdot(-2) + 5\cdot 1\\2\cdot 1 + 4\cdot(-2) + 6\cdot 1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

so indeed $\mathbf{v}$ is in $N(A)$.

(b) The null space in question consists of vectors with coordinates x_1, x_2, x_3 satisfying $2x_1 - 3x_2 + 5x_3 = 0$. Knowing any two of the coordinates, we can solve for the third one. For example, if $x_1 = 1, x_2 = -1$, then the equation becomes $2 - 3\cdot(-1) + 5x_3 = 0$, so $x_3 = -1$. Therefore the vector $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is in the null space, and so are its scalar multiples: the line $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \end{bmatrix}$ is a line contained in the null space. (There are many other correct answers, since the null space is a plane in $\mathbf{R}^3$ through the origin, so it contains infinitely many lines through the origin.)

(c) All vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ perpendicular to $\begin{bmatrix} 8 \\ -5 \\ 6 \end{bmatrix}$ satisfies the linear equation

$$8x_1 - 5x_2 + 6x_3 = 0,$$

so V is exactly the null space $N(\begin{bmatrix} 8 & -5 & 6 \end{bmatrix})$; i.e. the matrix A can be taken to be the 1×3 matrix $\begin{bmatrix} 8 & -5 & 6 \end{bmatrix}$.

(d) The matrix needs to be $n \times 3$ for its input to consist of 3-vectors. The requirement is that $A\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \mathbf{R}^3$. But the columns are $A\mathbf{e}_j$ for the standard basis of $\mathbf{R}^3$, so the vanishing of the columns says A is an n matrix with all entries equal to 0 (so there are many answers, depending on which $n \geq 1$ you pick).

(e) For a direct solution, first we write down the linear system,

$$\begin{cases} 2x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + 1x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

The first equation says $x_1 = 0$, the second equation says $x_2 = 0$, and the third equation doesn't tell us anything. So all vectors in $N(A)$ are of the form $\begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$, which is a 1 dimensional subspace spanned by $\mathbf{e}_3$. (This corresponds to the fact that the 3rd diagonal entry of A is 0.)

Alternative solution: the matrix A has rank 2 (2 independent columns), so by rank-nullity theorem, the dimension of $N(A)$ is $3 - 2 = 1$.

Practice 21.3. The three parts of this exercise are practice with null spaces; they are not related to each other and can be worked on independently.

(a) For an $m \times n$ matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbf{R}^m$, explain why $N(A) = \{\mathbf{0}\}$ when the $\mathbf{a}_j$'s are linearly independent and why $N(A)$ contains a nonzero vector when the $\mathbf{a}_j$'s are linearly dependent.

(b) Let A be an $n \times n$ matrix, and suppose $\mathbf{v} \in \mathbf{R}^n$ is a vector for which $A\mathbf{v} = \lambda\mathbf{v}$, where λ is a scalar. Explain why if $\lambda = 0$ then $\mathbf{v} \in N(A)$, and why if $\lambda \neq 0$ then $\mathbf{v} \in C(A)$.

(c) For every linear subspace V of $\mathbf{R}^8$ and the 8×8 matrix A for $\mathbf{Proj}_V : \mathbf{R}^8 \to \mathbf{R}^8$, explain why $C(A) = V$ and $N(A) = V^\perp$. (There is no need to compute or describe entries in A.)

Solution:

(a) Suppose the $\mathbf{a}_j$'s are linearly independent. For any $\mathbf{x} \in \mathbf{R}^n$, if we write $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then we have $A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$, so if $A\mathbf{x} = \mathbf{0}$ then $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$. By linear independence, this forces the coefficients $x_1, \ldots, x_n$ to all vanish, so $\mathbf{x} = \mathbf{0}$. This says $N(A) = \{\mathbf{0}\}$ as desired.

Next, suppose the $\mathbf{a}_j$'s are linearly dependent. This means that there are scalars $c_1, \ldots, c_n$ not all 0 for which $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$. Thus, the n-vector $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is nonzero yet $A\mathbf{c} = \sum_{j=1}^{n} c_j\mathbf{a}_j = \mathbf{0}$, so $\mathbf{c}$ is a nonzero vector belonging to $N(A)$. This says $N(A)$ contains a nonzero vector, as desired.

(b) If $\lambda = 0$ then $A\mathbf{v} = \lambda\mathbf{v} = 0\mathbf{v} = \mathbf{0}$, so $\mathbf{v} \in N(A)$. Suppose $\lambda \neq 0$, so we can multiply by $1/\lambda$ on both sides of "$A\mathbf{v} = \lambda\mathbf{v}$" to get $(1/\lambda)A\mathbf{v} = \mathbf{v}$. The left side is equal to $A((1/\lambda)\mathbf{v})$, so we conclude that $\mathbf{v}$ can be written as $A\mathbf{x}$ for $\mathbf{x} = (1/\lambda)\mathbf{v}$. But every vector of the form $A\mathbf{x}$ is a linear combination of the columns and hence belongs to the column space of A, so $\mathbf{v} \in C(A)$.

(c) The column space $C(A)$ is the span of the columns $\mathbf{a}_1, \ldots, \mathbf{a}_8$ of A, and that consists of the vectors of the form $A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_8\mathbf{a}_8$. But by definition of A we have $A\mathbf{x} = \mathbf{Proj}_V(\mathbf{x})$, so $A\mathbf{x}$ always belongs to V. This says $C(A)$ lies inside V. But it exhausts V because for any $\mathbf{v} \in V$ we claim $A\mathbf{v} = \mathbf{v}$, or in other words $\mathbf{Proj}_V(\mathbf{v}) = \mathbf{v}$. In general $\mathbf{Proj}_V(\mathbf{x})$ is the point in V closest to $\mathbf{x}$, and if $\mathbf{x} \in V$ (such as $\mathbf{x} = \mathbf{v} \in V$ above) then it is its own closest point in V; i.e. $\mathbf{Proj}_V(\mathbf{x}) = \mathbf{x}$ as claimed.

Another of the characterizations of $\mathbf{Proj}_V(\mathbf{x})$ is that it is the unique point $\mathbf{v} \in V$ for which $\mathbf{x} - \mathbf{v}$ is orthogonal to everything in V; i.e., $\mathbf{x} - \mathbf{v} \in V^\perp$. Hence, $\mathbf{Proj}_V(\mathbf{x}) = \mathbf{0}$ precisely when $\mathbf{x} = \mathbf{x} - \mathbf{0} \in V^\perp$. But the condition "$\mathbf{Proj}_V(\mathbf{x}) = \mathbf{0}$" says $A\mathbf{x} = \mathbf{0}$, so we can equivalently say $V^\perp = N(A)$.

Practice 21.4. For each of the following statements, choose either TRUE (meaning, "always true") or FALSE (meaning, "not always true"). No justification required, although it helps if you can explain why a statement is true, or give examples of when a statement is false.

(a) For nonzero matrices A and B for which the product AB makes sense, the column space $C(AB)$ is equal to $C(A)$.

(b) Some 5×7 matrix A satisfies $\dim N(A) = 3$ and $\dim C(A) = 2$.

(c) If A is an $n \times n$ matrix with $\dim(C(A)) = n$, then A is invertible.

Solution:

(a) Not always true:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$C(A) = \mathbf{R}^2$ and $C(AB) = \text{span}(\mathbf{e}_1)$.

(b) Never true: By the Rank-Nullity Theorem, $\text{dim}C(A) + \text{dim}N(A)$ is equal to the number of columns of A, which is 7. But the given information yields that the sum is 5, so this is impossible.

(c) Always true: we have $\dim(N(A)) = n - \dim(C(A)) = n - n = 0$ (by the Rank-Nullity Theorem), so the null space of A is $\{\mathbf{0}\}$. We have learned in class that a square matrix with vanishing null space is invertible.

Practice 21.5. (a) Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 5 & 8 \\ 1 & 2 & 1 & 1 \\ 2 & 4 & 4 & 6 \end{bmatrix}.$$

Label the column vectors of A as $\begin{bmatrix} | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ | & | & | & | \end{bmatrix}$. Note that

$$\mathbf{a_2} = 2\mathbf{a_1}, \qquad \mathbf{a_4} = 2\mathbf{a_3} - \mathbf{a_1}.$$

Find a 4×2 matrix B whose columns are not scalar multiples of each other and for which $AB = 0$, i.e. the 4×2 zero matrix. *There are many valid answers.*

(b) Explain why any vector in $C(B)$ must also be in $N(A)$.

Solution:

(a) Matrix-vector products can be expressed as column linear combinations as follows:

$$\begin{bmatrix} | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4.$$

Since

$$2\mathbf{a_1} - \mathbf{a_2} = \mathbf{0}, \qquad -\mathbf{a_1} + 2\mathbf{a_3} - \mathbf{a_4} = \mathbf{0},$$

we have

$$\begin{bmatrix} | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}, \qquad \begin{bmatrix} | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = \mathbf{0}.$$

so

$$\begin{bmatrix} | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 0 \\ 0 & 2 \\ 0 & -1 \end{bmatrix} = 0$$

as a 4×2 matrix. So this explicit 4×2 matrix can be taken as B. (There are many other correct answers.)

(b) Any vector in the column space $C(B)$ is of the form $\mathbf{v} = B\,\mathbf{x}$. To check if it is in the null space $N(A)$, multiply the matrix A with the vector: $A\,\mathbf{v} = A\,(B\,\mathbf{x}) = (AB)\mathbf{x}$, and this is indeed the zero vector because $AB = 0$.

22. Matrix decompositions: QR-decomposition and LU-decomposition

Practice 22.1. The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 4 & -4 \end{bmatrix}$ is equal to LU with $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) For $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, use the given LU-decomposition of A to solve $A\mathbf{x} = \mathbf{b}$ via repeated back-substitutions, and then check directly that your solution really is a solution. (All entries in the solution vector $\mathbf{x}$ are integers, and likewise for the output of back-substitutions using L.)

(b) Use the given LU-decomposition to compute A^{-1} (its entries are integers, with upper-left entry equal to 20), and check that what you obtain really is an inverse to A by multiplying it against A in some order (you do not need to compute the matrix product in both orders; it is recommended to check your calculations of U^{-1} and L^{-1} really work before computing A^{-1}).

Solution:

(a) We need to solve $LU\mathbf{x} = \mathbf{b}$, so first we solve $L\mathbf{y} = \mathbf{b}$ and then solve $U\mathbf{x} = \mathbf{y}$. The lower triangular system $L\mathbf{y} = \mathbf{b}$ says

$$y_1 = 1, \;\; 2y_1 + y_2 = 3 \;\; y_1 - 3y_2 + y_3 = -2.$$

The first equation gives the value of y_1, then the second gives $y_2 = 3 - 2y_1 = 3 - 2 = 1$, and the third then gives $y_3 = -2 - y_1 + 3y_2 = -2 - 1 + 3 = 0$.

Next, we solve the upper triangular system $U\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ by back-substitution. The system of equations is

$$x_1 + x_2 + x_3 = 1, \;\; -x_2 + 2x_3 = 1, \;\; x_3 = 0.$$

The last equation gives $x_3 = 0$, and then the second equation gives $x_2 = -1$. Then the first equation says $x_1 = 1 - x_2 - x_3 = 1 + 1 = 2$. Thus, the solution vector is $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

To see this works, we plug it in:

$$1(2) + 1(-1) + 1(0) = 2 - 1 = 1, \;\; 2(2) + 1(-1) + 4(0) = 4 - 1 = 3,$$

and

$$1(2) + 4(-1) - 4(0) = 2 - 4 = -2,$$

as desired.

(b) Since $A = LU$ with L and U each invertible (due to having no 0 in their diagonals), we have $A^{-1} = U^{-1}L^{-1}$. To calculate U^{-1} and L^{-1}, we set them up as

$$U^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & -1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a' & 1 & 0 \\ b' & c' & 1 \end{bmatrix}$$

and need to solve for a, b, c and a', b', c'.

For U^{-1}, we use the requirement

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = UU^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & b \\ 0 & -1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a-1 & b+c+1 \\ 0 & 1 & -c+2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Looking just above the diagonal, we have $a = 1$ and $c = 2$. Then looking in the upper-right, we get $0 = b + c + 1 = b + 3$, so $b = -3$.

Next, for L^{-1} we use the requirement

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = LL^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a' & 1 & 0 \\ b' & c' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2+a' & 1 & 0 \\ 1-3a'+b' & -3+c' & 1 \end{bmatrix}.$$

Looking just below the diagonal, we have $a' = -2$ and $c' = 3$. Going into the lower-left corner, $0 = 1 - 3a' + b' = 1 + 6 + b' = 7 + b'$, so $b' = -7$.

Having computed U^{-1} and L^{-1}, we multiply them (in the correct order!) to obtain that the inverse $A^{-1} = U^{-1}L^{-1}$ is equal to

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -7 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1-2+21 & 1-9 & -3 \\ 2-14 & -1+6 & 2 \\ -7 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 20 & -8 & -3 \\ -12 & 5 & 2 \\ -7 & 3 & 1 \end{bmatrix}.$$

To check this works, we compute the matrix product

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 4 & -4 \end{bmatrix} \begin{bmatrix} 20 & -8 & -3 \\ -12 & 5 & 2 \\ -7 & 3 & 1 \end{bmatrix} &= \begin{bmatrix} 20-12-7 & -8+5+3 & -3+2+1 \\ 40-12-28 & -16+5+12 & -6+2+4 \\ 20-48+28 & -8+20-12 & -3+8-4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

as desired.

Practice 22.2. (a) Consider the following matrix A and its column vectors

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 2 & 3 & 5 \\ 2 & 4 & 4 \\ 0 & 2 & -2 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 9 \\ 5 \\ 4 \\ -2 \end{bmatrix}.$$

If you perform Gram-Schmidt on $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, you get the following:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \mathbf{v}_2 - 2\mathbf{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \mathbf{v}_3 - 3\mathbf{w}_1 - \mathbf{w}_2 = \begin{bmatrix} 4 \\ 0 \\ -2 \\ -4 \end{bmatrix}.$$

Use this information to find the QR decomposition of A as $A = QR$. *The entries in R should all be integers.*

(b) The matrix $B = \begin{bmatrix} 3/5 & 3/5 & -1/5 \\ 0 & 1 & 1 \\ 4/5 & 4/5 & 7/5 \end{bmatrix}$ has QR decomposition

$$B = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Use this decomposition to compute B^{-1}.

Solution:

(a) We can express $\mathbf{v}_i$'s in terms of the $\mathbf{w}_j$'s as follows:

$$\mathbf{v}_1 = \mathbf{w}_1, \qquad \mathbf{v}_2 = 2\mathbf{w}_1 + \mathbf{w}_2, \qquad \mathbf{v}_3 = 3\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3,$$

$$\|\mathbf{w}_1\| = \|\mathbf{w}_2\| = \sqrt{1^2 + 2^2 + 2^2} = 3, \qquad \|\mathbf{w}_3\| = \sqrt{4^2 + (-2)^2 + 4^2} = 6.$$

Set

$$\mathbf{w}_1' = \mathbf{w}_1/3, \qquad \mathbf{w}_2' = \mathbf{w}_2/3, \qquad \mathbf{w}_3' = \mathbf{w}_3/6.$$

$$\mathbf{v}_1 = 3\mathbf{w}_1', \qquad \mathbf{v}_2 = 6\mathbf{w}_1' + 3\mathbf{w}_2', \qquad \mathbf{v}_3 = 9\mathbf{w}_1' + 3\mathbf{w}_2' + 6\mathbf{w}_3',$$

$$\begin{aligned} A &= \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & | \\ 3\mathbf{w}_1' & 6\mathbf{w}_1' + 3\mathbf{w}_1' & 9\mathbf{w}_1' + 3\mathbf{w}_2' + 6\mathbf{w}_3' \\ | & | & | \end{bmatrix} \\ &= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 0 \\ 2/3 & 0 & -1/3 \\ 0 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 3 \\ 0 & 0 & 6 \end{bmatrix}. \end{aligned}$$

(b) Since R has 1's along the diagonal, R^{-1} must have 1's along the diagonal. Solving for a, b, c in

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = RR^{-1} = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&a&b\\0&1&c\\0&0&1\end{bmatrix} = \begin{bmatrix}1&a+1&b+c+1\\0&1&c+1\\0&0&1\end{bmatrix},$$

we find that

$$a = c = -1, \qquad b = 0.$$

$$B^{-1} = R^{-1}Q^{-1} = R^{-1}Q^{\top} = \begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}3/5&0&4/5\\0&1&0\\-4/5&0&3/5\end{bmatrix} = \begin{bmatrix}3/5&-1&4/5\\4/5&1&-3/5\\-4/5&0&3/5\end{bmatrix}.$$

Practice 22.3. Let $A = \begin{bmatrix}4&-7&-2\\2&-4&-1\\0&2&0\end{bmatrix}$ and $U = \begin{bmatrix}4&-7&-2\\0&-\frac{1}{2}&0\\0&0&0\end{bmatrix}$.

(a) It is a fact, which you do not have to check, that

$$\begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix}\begin{bmatrix}1&0&0\\-\frac{1}{2}&1&0\\0&0&1\end{bmatrix}A = U.$$

Find an LU-decomposition of A; that is, find a lower-triangular matrix L satisfying $A = LU$.

(b) Let $\mathbf{b} = \begin{bmatrix}1\\2\\-6\end{bmatrix}$. Using repeated back-substitutions, find all solutions to the system $A\mathbf{x} = \mathbf{b}$, and then make a direct check that your answer really satisfies the equation.

Solution:

(a) The two matrices involved in the product with A and U are lower triangular, so their product is a lower triangular matrix L_1. Assuming we can find an inverse to L_1 (which would also be lower triangular), we'll be able to convert the above statement "$L_1A = U$" into "$A = LU$," specifically for $L = L_1^{-1}$. We'll begin by computing L_1:

$$L_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix}\begin{bmatrix}1&0&0\\-\frac{1}{2}&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\-\frac{1}{2}&1&0\\-2&4&1\end{bmatrix}$$

Then L_1^{-1} is a lower triangular matrix whose diagonal entries are all $1/1 = 1$ (the reciprocals of the diagonal entries of L_1), and whose off-diagonal entries can be found as follows:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\-\frac{1}{2}&1&0\\-2&4&1\end{bmatrix}\begin{bmatrix}1&0&0\\a&1&0\\b&c&1\end{bmatrix} = \begin{bmatrix}1&0&0\\-\frac{1}{2}+a&1&0\\-2+4a+b&4+c&1\end{bmatrix} \implies \left\{\begin{matrix}-\frac{1}{2}+a=0\\4+c=0\\-2+4a+b=0\end{matrix}\right\}$$

The first two equations imply $a = \frac{1}{2}$ and $c = -4$; then plugging these into the third equation leads to $b = 0$. Thus, $A = LU$ as follows:

$$\begin{bmatrix} 4 & -7 & -2 \\ 2 & -4 & -1 \\ 0 & 2 & 0 \end{bmatrix} = A = LU = L_1^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 4 & -7 & -2 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(As a safety check, we can compute the right-hand product directly to verify it equals A.)

(b) We first solve $L\mathbf{y} = \mathbf{b}$, and then $U\mathbf{x} = \mathbf{y}$. The lower triangular system of equations is

$$y_1 = 1, \quad \frac{1}{2}y_1 + y_2 = 2, \quad -4y_2 + y_3 = -6.$$

This gives $y_1 = 1$, then $\frac{1}{2} + y_2 = 2$, so $y_2 = \frac{3}{2}$, and finally $-4\left(\frac{3}{2}\right) + y_3 = -6$, so $y_3 = 0$.

Next, the upper triangular system is

$$4x_1 - 7x_2 - 2x_3 = 1, \quad -\frac{1}{2}x_2 = \frac{3}{2}, \quad 0 = 0,$$

so going backwards gives $x_2 = -3$, so $4x_1 - 7(-3) - 2x_3 = 1$, yielding $4x_1 = 2x_3 - 20$, so $x_1 = \frac{1}{2}x_3 - 5$ (where x_3 can be anything).

Hence, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 - 5 \\ -3 \\ x_3 \end{bmatrix}$ (notice the solutions form a line). This really works, since for any x_3,

$$\begin{aligned} 4\left(\tfrac{1}{2}x_3 - 5\right) - 7(-3) - 2x_3 &= -20 + 21 = 1, \\ 2\left(\tfrac{1}{2}x_3 - 5\right) - 4(-3) - x_3 &= -10 + 12 = 2, \\ 2(-3) &= -6. \end{aligned}$$

Practice 22.4. Consider the matrix $A = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 4 & 2 \\ -1 & -2 & 1 \end{bmatrix}$, whose columns from left to right are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(a) (This part is the same as Practice 19.1.) Carry out the Gram-Schmidt process for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, in the manner in which it is presented in Chapter 19 of the course textbook, to construct an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of $\mathbf{R}^3$.

(b) Using your work in the previous part, express each $\mathbf{v}_j$ as a linear combination of the $\mathbf{w}_i$'s, and use that to compute the QR-decomposition of A.

(c) Let $\mathbf{b} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$. Using the method explained Chapter 22, apply the QR-decomposition of A to solve the linear system $A\mathbf{x} = \mathbf{b}$, showing all of your steps.

Solution:

(a) $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \mathbf{v}_2 - \frac{6}{2}\mathbf{w}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \mathbf{v}_3 - \frac{1}{2}\mathbf{w}_1 - \frac{-3}{6}\mathbf{w}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

(b) $\mathbf{v}_1 = \mathbf{w}_1$

$\mathbf{v}_2 = 3\mathbf{w}_1 + \mathbf{w}_2$

$\mathbf{v}_3 = (1/2)\mathbf{w}_1 - (1/2)\mathbf{w}_2 + \mathbf{w}_3$

$$Q = \begin{bmatrix} 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 3\sqrt{2} & (1/2)\sqrt{2} \\ 0 & \sqrt{6} & (-1/2)\sqrt{6} \\ 0 & 0 & \sqrt{12} \end{bmatrix}$$

(c) To solve: $R\mathbf{x} = Q^\top \mathbf{b} = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -24/\sqrt{6} \\ 12/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ -4\sqrt{6} \\ 4\sqrt{3} \end{bmatrix}$

$x_3\sqrt{12} = 4\sqrt{3} \implies x_3 = 2$

$x_2\sqrt{6} - (1/2)x_3\sqrt{6} = -4\sqrt{6} \implies x_2\sqrt{6} = -3\sqrt{6} \implies x_2 = -3$

$x_1\sqrt{2} + 3x_2\sqrt{2} + (1/2)x_3\sqrt{2} = 0 \implies x_1\sqrt{2} - 9\sqrt{2} + \sqrt{2} = 0 \implies x_1 = 8$

Verify: $A\mathbf{x} = A\begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix} = 8\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - 3\begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$

23. Eigenvalues and eigenvectors

Practice 23.1. For each eigenvalue λ of $A = \begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ 7 & 5 & 3 \end{bmatrix}$, compute a basis for the nonzero linear subspace $N(A - \lambda \mathrm{I}_3)$ in $\mathbf{R}^3$, and as a check on your work verify directly that each vector in that basis is an eigenvector for A with eigenvalue λ.

Solution: The eigenvalues for a lower triangular matrix are its diagonal entries. So this matrix has as its eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$. We need to compute a basis for each of the null spaces $N(A - 3\mathrm{I}_3)$ and $N(A - \mathrm{I}_3)$.

For the first of these, we have

$$A - 3\mathrm{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 6 & -2 & 0 \\ 7 & 5 & 0 \end{bmatrix},$$

so a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in its null space precisely when it satisfies the equations

$$0 = 0, \quad 6x_1 - 2x_2 = 0, \quad 7x_1 + 5x_2 = 0.$$

The first equation gives nothing, and the second and third equations are two equations in two unknowns for which the equations are not scalar multiples of each other and hence has as its only solution $(0, 0)$ (as can be checked directly: the second equation says $x_2 = 3x_1$, and plugging this into the third equation forces x_1 and x_2 to vanish). Thus, altogether this null space consists of vectors of the form

$$\begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, by inspection we see that a basis of this null space is given by the vector

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

direct matrix-vector multiplication gives

$$A\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = 3\mathbf{v},$$

as desired.

Turning to the null space of $A - \mathrm{I}_3$, we have

$$A - \mathrm{I}_3 = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 0 & 0 \\ 7 & 5 & 2 \end{bmatrix}.$$

Hence, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ lies in the null space of this precisely when it satisfies the equations

$$2x_1 = 0, \quad 6x_1 = 0, \quad 7x_1 + 5x_2 + 2x_3 = 0.$$

The first two both say $x_1 = 0$, and then plugging this into the third gives $5x_2 + 2x_3 = 0$, so $x_3 = -(5/2)x_2$. Hence,

$$\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ -(5/2)x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -5/2 \end{bmatrix},$$

so a basis for this null space is given by the vector $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -5/2 \end{bmatrix}$, or any nonzero scalar multiple of that. Direct matrix-vector multiplication establishes the eigenvector condition:

$$A\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 5 - 15/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5/2 \end{bmatrix} = \mathbf{w},$$

as desired.

Practice 23.2. Suppose $A = \begin{bmatrix} 1 & 6 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) For each eigenvalue λ of A, compute a basis for the nonzero linear subspace $N(A - \lambda \mathrm{I}_3)$ in $\mathbf{R}^3$, and as a check on your work verify directly that each vector in that basis is an eigenvector for A with eigenvalue λ.

(b) What is the *second column* of A^{10}? Give your answer as a 3-vector containing numbers; though you are permitted to leave simple exponential expressions (e.g., 17^5, etc.) unevaluated. (*Hint:* write $\mathbf{e}_2$ as a combination of eigenvectors you found, and consider how to use matrix-vector products.)

Solution:

(a) As we have seen in the course, the eigenvalues for an upper triangular matrix (as well as lower triangular) are its diagonal entries. So this matrix has as its eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -2$. We need to compute a basis for each of the null spaces $N(A - \mathrm{I}_3)$ and $N(A + 2\mathrm{I}_3)$.

For the first of these, we have

$$A - \mathrm{I}_3 = \begin{bmatrix} 0 & 6 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

so a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in its null space precisely when it satisfies the equations

$$6x_2 + 3x_3 = 0, \quad -3x_2 + 2x_3 = 0, \quad 0 = 0.$$

The third equation gives nothing, and the first and second equations are two homogeneous equations in two unknowns for which the equations are not scalar multiples of each other and hence has as its only solution $(0, 0)$ (as can be checked directly: the second equation says $x_3 = -2x_2$, and plugging this into the third equation forces x_2 and x_3 to vanish). Thus, altogether this null space consists of vectors of the form

$$\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, by inspection we see that a basis of this null space is given by the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

direct matrix-vector multiplication gives

$$A\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{v},$$

as desired.

Turning to the null space of $A + 2\mathrm{I}_3$, we have

$$A + 2\mathrm{I}_3 = \begin{bmatrix} 3 & 6 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Hence, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ lies in the null space of this precisely when it satisfies the equations

$$3x_1 + 6x_2 + 3x_3 = 0, \;\; 2x_3 = 0, \;\; 3x_3 = 0.$$

The last two equations both say $x_3 = 0$, and plugging this into the first turns it into the condition $3x_1 + 6x_2 = 0$, which says $x_1 = -2x_2$. Hence,

$$\mathbf{x} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},$$

so a basis for this null space is given by the vector $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, or any nonzero scalar multiple of that. Direct matrix-vector multiplication establishes the eigenvector condition:

$$A\mathbf{w} = \begin{bmatrix} -2+6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = -2\mathbf{w}.$$

(b) As works for any matrix, the second column of A^{10} is given by $A^{10}\mathbf{e}_2$. To find this vector without extensive computations, we need to write $\mathbf{e}_2$ as a linear combination of the eigenvectors $\mathbf{v}$ and $\mathbf{w}$, since it is quite straightforward to compute $A^{10}\mathbf{v}$ or $A^{10}\mathbf{w}$: indeed, anytime $M\mathbf{u} = \lambda\mathbf{u}$ for some square matrix M and scalar λ, then

$$M^{10}\mathbf{u} = M^9(M\mathbf{u}) = M^9(\lambda\mathbf{u}) = \lambda M^9\mathbf{u},$$

so

$$M^{10}\mathbf{u} = \lambda M^9\mathbf{u} = \lambda^2 M^8\mathbf{u} = \cdots = \lambda^9 M\mathbf{u} = \lambda^{10}\mathbf{u}.$$

To write $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as a combination of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, we could note that between $\mathbf{v}$ and $\mathbf{w}$, only $\mathbf{w}$ has a nonzero second entry; this allows us to find that

$$\mathbf{e}_2 - \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\mathbf{v}, \quad \text{so} \quad \mathbf{e}_2 = 2\mathbf{v} + \mathbf{w}.$$

(Alternatively we could set up and solve a system $\mathbf{e}_2 = a\mathbf{v} + b\mathbf{w}$ to find that $a = 2$ and $b = 1$.)

Finally, we find that

$$\begin{aligned} A^{10}\mathbf{e}_2 &= A^{10}(2\mathbf{v} + \mathbf{w}) \\ &= 2A^{10}\mathbf{v} + A^{10}\mathbf{w} \\ &= 2\mathbf{v} + (-2)^{10}\mathbf{w} \\ &= 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-2)^{10}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 - 2^{11} \\ 2^{10} \\ 0 \end{bmatrix} \end{aligned}$$

Practice 23.3. Let A be the 3×3 matrix corresponding to the linear transformation that reflects about the plane $2x + 3y + 6z = 0$ (you do not need to write down A, or explain why it is indeed a linear transformation). List the eigenvalues of A, and the corresponding eigenspaces. *Hint:* Under the reflection, which vectors are reversed, and which vectors are not changed?

Solution: In the direction normal to the plane, i.e. along the normal vector $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, the matrix A reverses the vector, so -1 is an eigenvalue of A, with eigenspace $\mathrm{span}\left(\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}\right)$.

Reflection of any vector on the plane is the same vector, so the whole plane $2x + 3y + 6z = 0$ is the eigenspace corresponding to eigenvalue 1. You can express this plane as a span, for example $\mathrm{span}\left(\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\right)$.

24. Applications of eigenvalues: Spectral Theorem, quadratic forms, and matrix powers

Practice 24.1. Let

$$M = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

be a Markov matrix. In this exercise we will study M in the context of Spectral Theorem.

(a) Briefly indicate why there is an orthogonal basis of eigenvectors for the matrix M.

(b) The eigenvalues of M are $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$. Verify that $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the λ_1-eigenspace, and explain why it spans that space. Hint: you can either use the rank-nullity theorem, or directly find the λ_1-eigenspace by solving a system of linear equations.

(c) Find the λ_2-eigenspace V_2 and write it as the span of two orthogonal vectors $\mathbf{w}_2, \mathbf{w}_3$.

(d) Let $\mathbf{w}'_1, \mathbf{w}'_2, \mathbf{w}'_3$ be unit vectors obtained from $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ by dividing each by its length. Let W be the matrix whose columns are $\mathbf{w}'_1, \mathbf{w}'_2, \mathbf{w}'_3$. Find the diagonal matrix D where $M = WDW^\top$.

(e) Using the fact that $(1/2)^{100} \approx 0$ to 30 decimal places, calculate M^{100} explicitly.

Solution:

(a) The matrix M is symmetric, therefore it has an orthogonal basis of eigenvectors by the Spectral Theorem.

(b) We compute $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a 1-eigenvector. The 1-eigenspace is the null space of $M - I_3 = \begin{bmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{bmatrix}$. By the Rank-Nullity Theorem, this is 1-dimensional if the rank is 2 (i.e., 2-dimensional column space). The first two columns of $M - \mathrm{I}_3$ are independent by inspection, and the sum of the columns is $\mathbf{0}$ (since $\mathbf{w}_1$ is in the null space), so the 3rd column is in the span of the other two.

Alternative solution: The λ_1-eigenspace of M is the null space of $M - I_3 = \begin{bmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{bmatrix}$, which is the collection of all vectors $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for which:

$$\begin{cases} -x + \frac{1}{2}y + \frac{1}{2}z = 0 \\ \frac{1}{2}x - y + \frac{1}{2}z = 0 \\ \frac{1}{2}x + \frac{1}{2}y - z = 0 \end{cases} .$$

Subtracting the second line from the first yields $\frac{3}{2}x = \frac{3}{2}y$; that is, $x = y$. Substituting in any of the remaining equations gives $z = x = y$. So $\mathbf{x} = \begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $V_1 = \operatorname{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) We solve the system $(M - \lambda_2 I_3)\mathbf{x} = \mathbf{0}$, that is

$$\begin{cases} \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z = 0 \\ \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z = 0 \\ \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z = 0 \end{cases},$$

which is the same equation three times. We have $x = -(y+z)$, that is $\mathbf{x} = \begin{bmatrix} -(y+z) \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Let $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, then we get an orthogonal vector $\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_2} \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

(d) The matrix interpretation of the Spectral Theorem is that $M = WDW^\top$ where W has i-th column $\mathbf{w}'_i$ and D is diagonal with i-th entry the eigenvalue for $\mathbf{w}'_i$. Hence, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$.

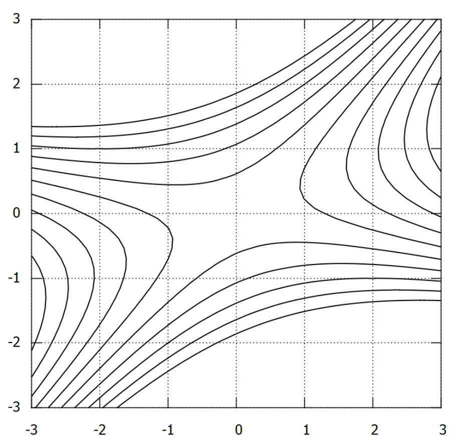
(e) We have seen that $M^k = WD^kW^{-1} = WD^kW^\top$ for any integer $k > 0$. So for $k = 100$, we have $M^{100} = WD^{100}W^\top$. But D is diagonal, so D^{100} is obtained by raising diagonal entries to the 100-th power. Since $(1/2)^{100} \approx 0$, we have $D^{100} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore,

$$\begin{aligned} M^{100} &\approx \begin{bmatrix} | & | & | \\ \mathbf{w}_1' & \mathbf{w}_2' & \mathbf{w}_3' \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} - & \mathbf{w}_1'^{\top} & - \\ - & \mathbf{w}_2'^{\top} & - \\ - & \mathbf{w}_3'^{\top} & - \end{bmatrix} \\ &\approx \begin{bmatrix} | & | & | \\ \mathbf{w}_1' & \mathbf{0} & \mathbf{0} \\ | & | & | \end{bmatrix} \begin{bmatrix} - & \mathbf{w}_1'^{\top} & - \\ - & \mathbf{w}_2'^{\top} & - \\ - & \mathbf{w}_3'^{\top} & - \end{bmatrix} \\ &\approx \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ - & \mathbf{w}_2'^{\top} & - \\ - & \mathbf{w}_3'^{\top} & - \end{bmatrix} \\ &\approx \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}. \end{aligned}$$

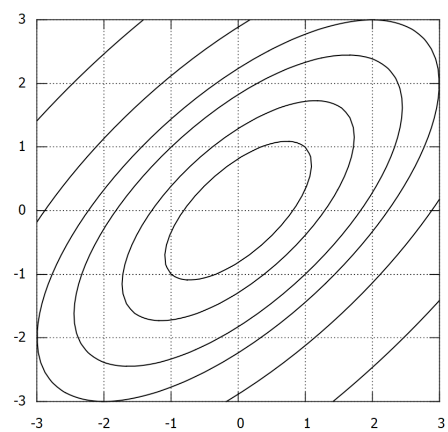
Practice 24.2. (a) Find the eigenvalues and a corresponding eigenvector for each for the 2×2 matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

(b) Find the eigenvalues and a corresponding eigenvector for each for the 2×2 matrix $B = \begin{bmatrix} 3 & 6 \\ 6 & -13 \end{bmatrix}$.

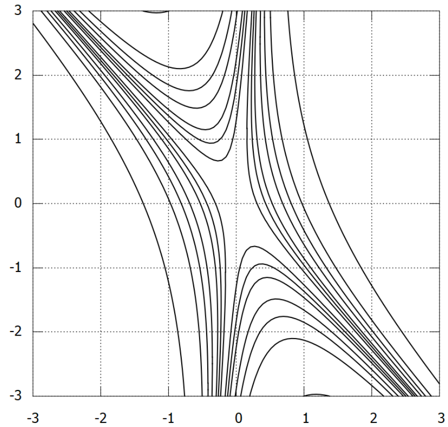
(c) Which of the following 4 pictured contour plots represents (i) the quadratic form $q_A(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$; (ii) the quadratic form $q_B(\mathbf{x}) = \mathbf{x}^\top B\mathbf{x}$? Justify your answer using part (a) and (b).



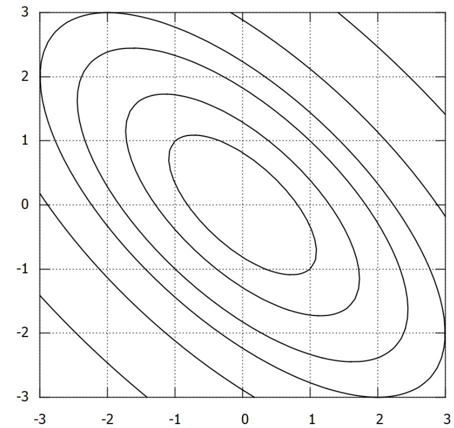
(1)



(2)



(3)



(4)

Solution:

(a) The eigenvalues of the matrix A are the roots of the polynomial:

$$P_A(\lambda) = \lambda^2 - \operatorname{tr}(A) + \det A = \lambda^2 - 6\lambda + 5.$$

The two roots are $\lambda_1 = 1$, $\lambda_2 = 5$.

The eigenvectors associated to the value $\lambda_1 = 1$ are the vectors in the null space of $A - I_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. By inspection, $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector.

Similarly, the eigenvectors for $\lambda_2 = 5$ are in the null space of $A - 5I_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$, so $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

(b) The eigenvalues of the matrix B are the roots of the polynomial:

$$P_B(\lambda) = \lambda^2 - \operatorname{tr}(B) + \det B = \lambda^2 + 10\lambda - 75 = (\lambda + 15)(\lambda - 5).$$

The two roots (which can also be found using the quadratic formula since $10^2 - 4(-75) = 100 + 300 = 400 = 20^2$) are $\lambda_1 = -15$, $\lambda_2 = 5$. The eigenvectors associated to the value $\lambda_1 = -15$ are the vectors in the null space of $B + 15I_2$, that is $\begin{bmatrix} 18 & 6 \\ 6 & 2 \end{bmatrix}$. By inspection, $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector.

Similarly, the eigenvectors for λ_2 are in the null space of $B - 5I_2 = \begin{bmatrix} -2 & 6 \\ 6 & -18 \end{bmatrix}$, so $\mathbf{w}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector.

(c) Both eigenvalues of A (namely 1 and 5) are positive, therefore the quadratic form q_A is positive definite, and its level sets are ellipses. These ellipses have axes of symmetry given by the eigenvectors of A, namely $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The ellipses are longer along the line spanned by the first vector since its corresponding eigenvalue is smaller ($1 < 5$). This is picture (4).

The eigenvalues of B (namely -15 and 5) are of opposite signs, therefore the quadratic form q_B is indefinite, and its level sets are hyperbolas. These hyperbolas have axes of symmetry given by the eigenvectors of B, namely $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Comparing the absolute value of the eigenvalues ($15 > 5$), the hyperbolas should be more "squeezed" in the direction of the first eigenvector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. This is picture (1).

25. The Hessian and quadratic approximation

26. Grand finale: application of the Hessian to local extrema, and bon voyage

Practice 26.1. Let $f(x,y) = x^3 + 3xy^2 - 15x - 12y$.

(a) Compute f_x and f_y, and verify that the points $P = (1,2)$, $Q = (2,1)$, $R = (-1,-2)$, and $S = (-2,-1)$ are critical points. (These are the only critical points, but we are not asking you to justify that.)

(b) Compute the Hessian $(\mathrm{H}f)(x,y)$, and evaluate this at each of the 4 critical points considered in part (a). Use this to give the quadratic approximation to f near $P = (1,2)$ (i.e., the quadratic approximation to $f(1+h, 2+k)$ for h, k near 0) and to determine for each of the other three critical points Q, R, S if it is a local maximum, local minimum, or a saddle point.

Solution:

(a) The partial derivatives are $f_x = 3x^2+3y^2-15 = 3(x^2+y^2-5)$ and $f_y = 6xy-12 = 6(xy-2)$. At all 4 points of interest, one of x^2 or y^2 is equal to 1 and the other is equal to 4, so $x^2+y^2 = 5$ at all of the points and hence f_x vanishes at all of these points. By inspection, at all 4 points we have $xy = 2$ and so f_y vanishes at all of these points as well.

(b) From the determination of f_x and f_y in part (a), we have

$$f_{xx} = 6x, \quad f_{xy} = 6y, \quad f_{yy} = 6x.$$

Hence,

$$(\mathrm{H}f)(x,y) = \begin{bmatrix} 6x & 6y \\ 6y & 6x \end{bmatrix},$$

so

$$(\mathrm{H}f)(P) = \begin{bmatrix} 6 & 12 \\ 12 & 6 \end{bmatrix}, \quad (\mathrm{H}f)(Q) = \begin{bmatrix} 12 & 6 \\ 6 & 12 \end{bmatrix},$$

$$(\mathrm{H}f)(R) = \begin{bmatrix} -6 & -12 \\ -12 & -6 \end{bmatrix}, \quad (\mathrm{H}f)(S) = \begin{bmatrix} -12 & -6 \\ -6 & -12 \end{bmatrix}.$$

Thus, the quadratic approximation at $P = (1,2)$ is

$$\begin{aligned} f(1+h, 2+k) &\approx f(1,2) + \frac{1}{2} \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} 6 & 12 \\ 12 & 6 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} \\ &= (1 + 12 - 15 - 24) + \frac{1}{2}(6h^2 + 24hk + 6k^2) \\ &= -26 + 3h^2 + 12hk + 3k^2. \end{aligned}$$

At R the determinant is $36 - 144 < 0$, so the eigenvalues have opposite signs and hence this a saddle point. At Q and S the determinant is $144 - 36 > 0$, so the eigenvalues have the same sign, which must therefore coincide with the sign of the trace. At Q the trace is $24 > 0$ and at S the trace is $-24 < 0$, so the eigenvalues of the Hessian at Q are positive and the eigenvalues of the Hessian at S are negative. Hence, Q is a local minimum and S is a local maximum.

Practice 26.2. Let

$$f(x, y) = 3xe^y - 3e^y - x^3.$$

(a) This function has exactly one critical point. Find it, and determine its nature (local maximum, local minimum, or saddle point).

(b) Approximate $f(0.1, 0.2)$ to two decimal places using a quadratic approximation of f at a nearby point.

Solution:

(a) Set

$$\nabla f(x, y) = \begin{bmatrix} 3e^y - 3x^2 \\ e^y(3x - 3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so to find critical points, we have to solve simultaneously

$$3e^y - 3x^2 = 0, \tag{26.0.1}$$
$$e^y(3x - 3) = 0. \tag{26.0.2}$$

Since e^y is never 0, it follows from the second equation that $x = 1$ must hold, so the first equation shows that $e^y = 1$. Hence, $y = 0$. Thus, the only critical point is $(1, 0)$.

We have

$$(\mathrm{H}f)(x, y) = \begin{bmatrix} -6x & 3e^y \\ 3e^y & e^y(3x - 3) \end{bmatrix}, \qquad \text{so} \quad (\mathrm{H}f)(1, 0) = \begin{bmatrix} -6 & 3 \\ 3 & 0 \end{bmatrix}$$

The determinant of $(\mathrm{H}f)(1, 0)$ is negative, so $(\mathrm{H}f)(1, 0)$ has a positive and a negative eigenvalue and hence $(1, 0)$ is a saddle point.

(b)

$$\begin{aligned} f(0.1, 0.2) &\approx f(0, 0) + (\nabla f)(0, 0) \cdot \begin{bmatrix} 0.1 - 0 \\ 0.2 - 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0.1 & 0.2 \end{bmatrix} (\mathrm{H}f)(0, 0) \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \\ &= -3 + \begin{bmatrix} 3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \\ &= -3 - 0.3 + \frac{1}{2}(6(0.1)(0.2) - 3(0.2)(0.2)) \\ &= -3.3 + \frac{1}{2}(0.12 - 0.12) \\ &= -3.30 \end{aligned}$$

Practice 26.3. Let f be the function

$$f(x,y) = x^2y - \frac{3}{2}x^2 + \frac{1}{2}y^3 + 6x - \frac{35}{2}y$$

You can accept (or check if you wish) that $f(-2,3) = -45$.

(a) Verify that $(-2,3)$ is a critical point of f, and compute the symmetric Hessian matrix $(\mathrm{H}f)(-2,3)$ and the quadratic approximation for f at $(-2,3)$ (i.e., the quadratic approximation to $f(-2+h,3+k)$ for $h,k \approx 0$). (The entries of $(\mathrm{H}f)(-2,3)$ are integers.)

(b) For the Hessian matrix $H = (\mathrm{H}f)(-2,3)$ that you found in part (a), compute its eigenvalues and use this to determine if $(-2,3)$ is a local maximum, local minimum, or saddle point. Use the eigenvalues and eigenvectors of H to sketch what the contour plot of f looks like near $(-2,3)$. (The eigenvalues of H are integers.)

(*Note:* it only matters to sketch approximate ellipses or hyperbolas aligned with the appropriate perpendicular lines through the critical point; precise asymptotic directions for a hyperbola, and precisely measured ratio of major and minor axes of an ellipse, don't matter as long as the "longer" or "wider" axis direction is apparent on your picture. There is also no need to label the contours with numerical values.)

Solution:

(a) We compute symbolically that

$$(\nabla f)(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} = \begin{bmatrix} 2xy - 3x + 6 \\ x^2 + (3/2)y^2 - 35/2 \end{bmatrix}$$

and

$$(\mathrm{H}f)(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{bmatrix} = \begin{bmatrix} 2y-3 & 2x \\ 2x & 3y \end{bmatrix}.$$

Thus, at $(-2,3)$, the gradient is $(\nabla f)(-2,3) = \begin{bmatrix} -12+6+6 \\ 4+(27/2)-(35/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which confirms that $(-2,3)$ is a critical point of f. The Hessian there is

$$H = (\mathrm{H}f)(-2,3) = \begin{bmatrix} 3 & -4 \\ -4 & 9 \end{bmatrix}.$$

Now since the quadratic form associated to this Hessian matrix H is $q_H(h,k) = 3h^2 - 8hk + 9k^2$, it follows that the quadratic approximation for f near $(-2,3)$ is:

$$\begin{aligned} f(-2+h,3+k) &\approx f(-2,3) + (\nabla f)(-2,3) \cdot \begin{bmatrix} h \\ k \end{bmatrix} + \frac{1}{2}\begin{bmatrix} h & k \end{bmatrix} (\mathrm{H}f)(-2,3) \begin{bmatrix} h \\ k \end{bmatrix} \\ &= -45 + 0 + \tfrac{1}{2}q_H(h,k) \\ &= -45 + \frac{1}{2}(3h^2 - 8hk + 9k^2) \end{aligned}$$

(b) The Hessian matrix $H = (\mathrm{H}f)(-2,3) = \begin{bmatrix} 3 & -4 \\ -4 & 9 \end{bmatrix}$ has trace 12 and determinant $27-16=11$, so it has characteristic polynomial

$$\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1),$$

which has as its roots $\lambda_1 = 11$ and $\lambda_2 = 1$ (which can also be found by the quadratic formula, if you didn't notice how it factors). These are integers as promised and have the same positive sign, so q_H is positive-definite and hence $(-2,3)$ is a local minimum.

To sketch the contour plot, we need to work out perpendicular eigenvectors $\mathbf{w}_1$ and $\mathbf{w}_2$ for λ_1 and λ_2 respectively so as to write the quadratic form q_H associated to the Hessian H in a more convenient reference frame. The lines for the eigenvalues are the null spaces of $H - (11)\mathrm{I}_2$ and $H - \mathrm{I}_2$. We compute these matrices to be

$$H - (11)\mathrm{I}_2 = \begin{bmatrix} -8 & -4 \\ -4 & -2 \end{bmatrix}, \qquad H - \mathrm{I}_2 = \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}.$$

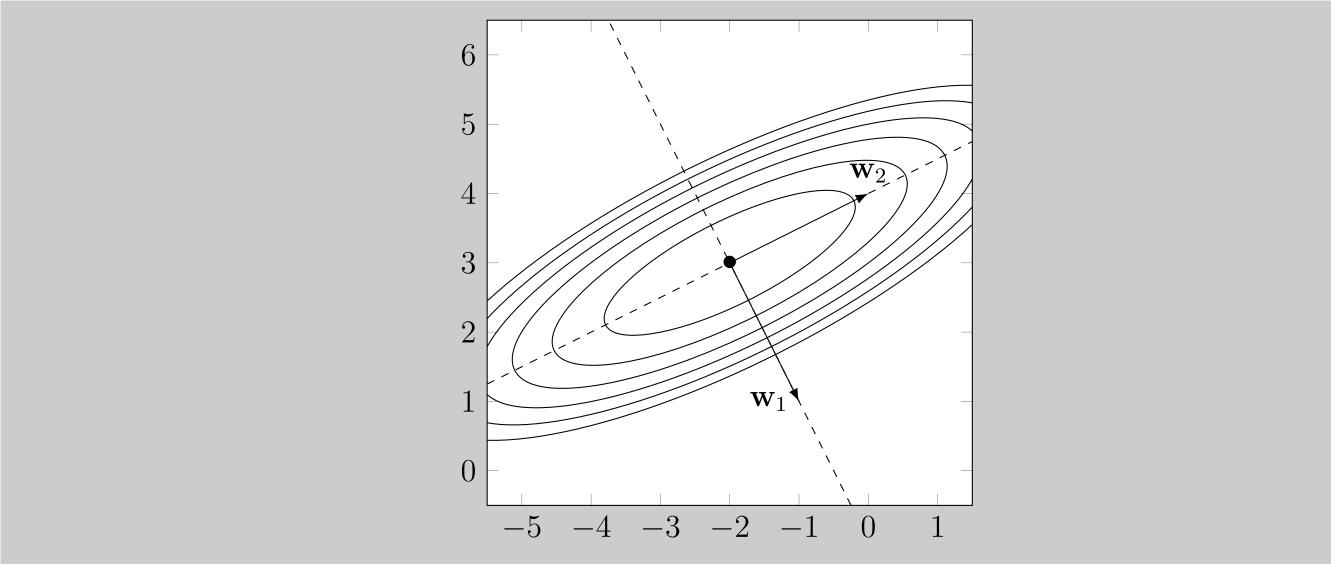
The first has null space corresponding to the pair of equations $-8x - 4y = 0$ and $-4x - 2y = 0$, which are scalar multiples of each other (as they must be for a line of an eigenvalue): this is the line $y = -2x$, so it is the span of the vector $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (or any nonzero scalar multiple of this). The second of these has null space corresponding to the pair of equations $2x - 4y = 0$ and $-4x + 8y = 0$, which are likewise scalar multiples of each other (as they must be for a line of an eigenvalue): this is the line $y = \frac{1}{2}x$, so it is the span of the vector $\mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (or any nonzero scalar multiple of this).

Therefore, in terms of the *orthonormal* basis $\mathbf{u}_1 = \dfrac{\mathbf{w}_1}{\|\mathbf{w}_1\|}$, $\mathbf{u}_2 = \dfrac{\mathbf{w}_2}{\|\mathbf{w}_2\|}$, we know

$$q_H(t_1\mathbf{u}_1 + t_2\mathbf{u}_2) = \lambda_1 t_1^2 + \lambda_2 t_2^2 = 11t_1^2 + t_2^2.$$

This has level curves that are ellipses centered at $(-2,3)$ with symmetry lines through that point along the directions of $\mathbf{w}_1$ and $\mathbf{w}_2$ and stretched out more along the $\mathbf{w}_2$-line than along the $\mathbf{w}_1$-line (since $11t_1^2 + t_2^2 = c$ crosses the t_1-axis at $\pm\sqrt{c/11}$ and the t_2-axis at $\pm\sqrt{c}$, so the ratio of the length along the $\mathbf{w}_2$-line to the length along the $\mathbf{w}_1$-line is $\sqrt{c}/\sqrt{c/11} = \sqrt{11} > 1$).

The approximate contour plot of f via its quadratic approximation at $(-2,3)$ is shown below:


6
5
4
3
2
1
0
$\mathbf{w}_2$
$\mathbf{w}_1$
−5
−4
−3
−2
−1
0
1