Problem 1: Matrix of a projection

Let V be the plane $x+y+z=0$ in $\mathbf{R}^3$ through the origin, so V has an orthogonal basis $\{\mathbf{v},\mathbf{w}\}$ for $\mathbf{v}=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix}1\\1\\-2\end{bmatrix}$.

Let $L:\mathbf{R}^3\to\mathbf{R}^3$ be the function $L(\mathbf{x})=\mathbf{Proj}_V(\mathbf{x})$.

(a) Compute the 3×3 matrix A for L; the entries should be fractions with denominator 3. (Hint: what is the meaning of each column?)

(b) For $\mathbf{a}=\begin{bmatrix}1\\3\\5\end{bmatrix}$, compute $\mathbf{Proj}_V(\mathbf{a})$ in two ways: using the orthogonal basis $\{\mathbf{v},\mathbf{w}\}$ for V, and using the matrix-vector product against your answer in (a). (You should get the same answer both ways, a vector with integer entries.)

(c) The geometric definition of $\mathbf{Proj}_V$ gives that its output lies in V, on which $\mathbf{Proj}_V$ has no effect, so $\mathbf{Proj}_V\circ\mathbf{Proj}_V=\mathbf{Proj}_V$. Check that your answer A in (a) satisfies the corresponding matrix equality $A^2=A$. (Hint: if you write $A=(1/3)B$ for a matrix B with integer entries then the calculation will be cleaner.)

Problem 2: Matrix multiplication

(a) Compute the following matrix products.

$$\begin{bmatrix}1&0\\-2&1\end{bmatrix}\begin{bmatrix}1&9&11\\2&5&6\end{bmatrix}=$$

$$\begin{pmatrix}\text{for } \mathbf{v},\mathbf{w} \text{ two}\\ \text{vectors in } \mathbf{R}^n\end{pmatrix}\quad\begin{bmatrix}v_1&\cdots&v_n\end{bmatrix}\begin{bmatrix}w_1\\\vdots\\w_n\end{bmatrix}=$$

$$\begin{bmatrix}2&0\\0&10\end{bmatrix}\begin{bmatrix}1&9\\2&5\end{bmatrix}=$$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&9&11\\2&5&6\end{bmatrix}=$$

$$\begin{bmatrix}1&9\\2&5\end{bmatrix}\begin{bmatrix}2&0\\0&10\end{bmatrix}=$$

$$\begin{bmatrix}2&0&0\\0&-1&0\\0&0&3\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=$$

(b) Let $q(x,y,z)=x^2+2y^2-z^2-3xy+4xz+yz$. Find values of a, b, c, d, e, f that satisfy

$$\begin{bmatrix}x&y&z\end{bmatrix}\begin{bmatrix}a&d&e\\d&b&f\\e&f&c\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=q(x,y,z)$$

for every x,y,z. Strictly speaking, the left side multiplies out to be a 1×1 matrix and the equality means that the scalar $q(x,y,z)$ on the right side is the unique entry in that matrix. (Hint: multiply the left side fully, and compare coefficients on the two sides, such as for x^2, yz, etc.)

(c) (**Extra**) Is there a version of (b) for any $q(x,y,z)=Ax^2+By^2+Cz^2+Dxy+Exz+Fyz$ in general?

Problem 3: Some more matrix algebra

Consider the linear transformation $T:\mathbf{R}^3\to\mathbf{R}^2$ given by projecting a vector $\mathbf{v}\in\mathbf{R}^3$ onto its first two components (viewed as a 2-vector), then reflecting that projection across the line $x+y=0$ in $\mathbf{R}^2$, and finally adding to this the $45°$ clockwise rotation of the projection of $\mathbf{v}$ onto its last two components. Find the 2×3 matrix A that computes T.

