Solutions to Math 51 Quiz 3 Practice A

1. (10 points) The best line of fit through the 6 data points $(x_1, y_1), \ldots, (x_6, y_6)$

$$(4, -2),\ (2, -1),\ (2, 0),\ (1, 5),\ (-1, 7),\ (-2, 9)$$

is

$$y = mx + b.$$

Find m and b.

- We first write down explicit 6-vectors $\mathbf{X}$ and $\mathbf{Y}$ so that for the 6-vector $\mathbf{1}$ whose entries are all equal to 1, the projection of $\mathbf{Y}$ into the plane $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ in $\mathbf{R}^6$ is $m\mathbf{X} + b\mathbf{1}$. The vectors are

$$\mathbf{X} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 5 \\ 7 \\ 9 \end{bmatrix}.$$

(Note: entries can be rearranged, provided that both $\mathbf{X}$ and $\mathbf{Y}$ are rearranged in the same way.)

- We next compute an orthogonal basis of $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ having the form $\{\mathbf{1}, \hat{\mathbf{X}}\}$ for a 6-vector $\mathbf{v}$, and find scalars r and s so that $\mathbf{Proj}_V(\mathbf{Y}) = r\hat{\mathbf{X}} + s\mathbf{1}$. The vector $\hat{\mathbf{X}}$ can be taken to be $\mathbf{X} - \mathbf{Proj_1}(\mathbf{X}) = \mathbf{X} - \overline{x}\mathbf{1}$ with $\overline{x}$ equal to the average of the entries x_i in $\mathbf{X}$. This average is

$$\overline{x} = \frac{4 + 2 + 2 + 1 - 1 - 2}{6} = \frac{6}{6} = 1,$$

so

$$\hat{\mathbf{X}} = \mathbf{X} - \overline{x}\mathbf{1} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ -2 \\ -3 \end{bmatrix}.$$

The projection of $\mathbf{Y}$ into V is then given by

$$\frac{\mathbf{Y} \cdot \hat{\mathbf{X}}}{\hat{\mathbf{X}} \cdot \hat{\mathbf{X}}}\hat{\mathbf{X}} + \frac{\mathbf{Y} \cdot \mathbf{1}}{\mathbf{1} \cdot \mathbf{1}}\mathbf{1} = \left(\frac{-48}{24}\right)\hat{\mathbf{X}} + \left(\frac{18}{6}\right)\mathbf{1} = (-2)\hat{\mathbf{X}} + (3)\mathbf{1}.$$

Hence, $r = -2$ and $s = 3$.

- By expressing $\hat{\mathbf{X}}$ as a linear combination of $\mathbf{X}$ and $\mathbf{1}$, use your answer to (b) to find m and b so that the equation $y = mx + b$ gives the line of best fit. We have $\hat{\mathbf{X}} = \mathbf{X} - \mathbf{1}$, so

$$\begin{aligned} \mathbf{Proj}_V(\mathbf{Y}) = (-2)\mathbf{v} + (3)\mathbf{1} &= (-2)(\mathbf{X} - \mathbf{1}) + (3)\mathbf{1} \\ &= (-2)\mathbf{X} + (-(-2) + 3)\,\mathbf{1} \\ &= (-2)\mathbf{X} + (5)\mathbf{1}. \end{aligned}$$

Hence, the line of best fit is $y = -2x + 5$; so $m = -2$ and $b = 5$.

2. (2 points) Let

$$f(x, y) = (y^2 + 1)\cos(x) + x\cos(y) + x$$

Compute $\frac{\partial^2 f}{\partial x \partial y}$. At which points (a, b) is the value

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) > 0?$$

Select all that apply.

- $(a, b) = (0, 0)$
- $(a, b) = (0, -\pi/2)$
- $(a, b) = (-\pi/2, \pi)$
- $(a, b) = (\pi/2, \pi/2)$
- None of the these four

We have

$$\frac{\partial f}{\partial x} = -(y^2 + 1)\sin(x) + \cos(y) + 1,$$

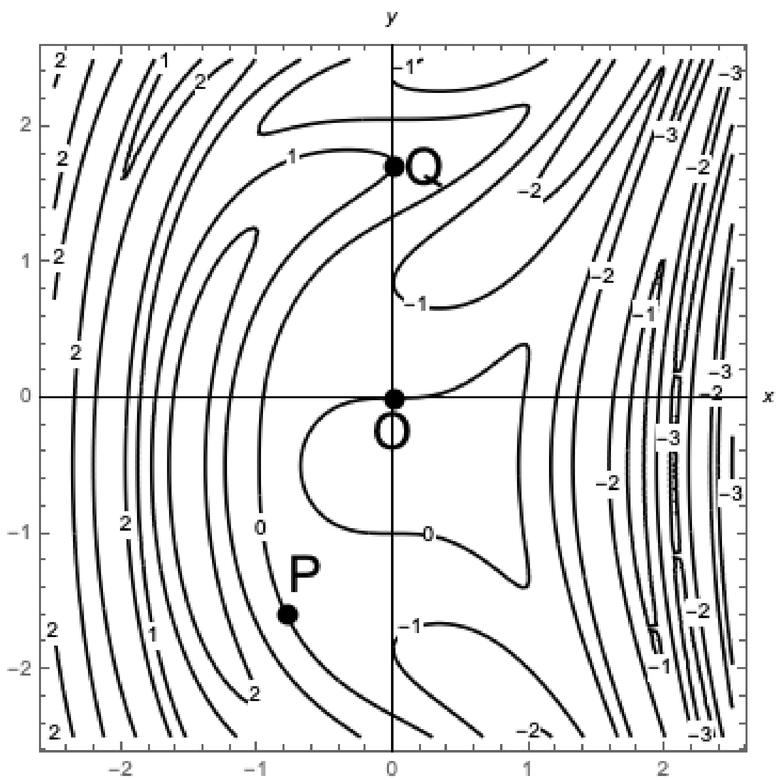
so

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ &= \frac{\partial}{\partial y}\big(-(y^2 + 1)\sin(x) + \cos(y) + 1\big) \\ &= -2y\sin(x) - \sin(y) \end{aligned}$$

Among the points given, this quantity is greater than zero at $(a, b) = (0, -\pi/2)$ and $(a, b) = (-\pi/2, \pi)$:

- At $(a, b) = (0, 0)$, $\frac{\partial^2 f}{\partial x \partial y}(a, b) = 0 - \sin(0) = 0$.
- At $(a, b) = (0, -\pi/2)$, $\frac{\partial^2 f}{\partial x \partial y}(a, b) = -2(-\pi/2)\sin(0) - \sin(-\pi/2) = 0 + 1 > 0$.
- At $(a, b) = (-\pi/2, \pi)$, $\frac{\partial^2 f}{\partial x \partial y}(a, b) = -2(\pi)\sin(-\pi/2) - \sin(\pi) = 2\pi - 0 > 0$.
- At $(a, b) = (\pi/2, \pi/2)$, $\frac{\partial^2 f}{\partial x \partial y}(a, b) = -2(\pi/2)\sin(\pi/2) - \sin(\pi/2) = -\pi - 1 < 0$.

3. (2 points) Below is a contour plot of a function $g(x, y)$ over the region of points (x, y) where $-2.5 \leq x \leq 2.5$ and $-2.5 \leq y \leq 2.5$.



For the points labeled O, P, Q, determine whether the following is negative, positive or 0.

a) $\frac{\partial g}{\partial x}(O) = g_x(O)$ b) $\frac{\partial g}{\partial y}(O) = g_y(O)$ c) $\frac{\partial g}{\partial x}(P) = g_x(P)$ d) $\frac{\partial g}{\partial y}(Q) = g_y(Q)$

As we move through O horizontally from left to right the line of motion is tangent to the level curve through O, this indicates that $g_x(O) = 0$. Likewise, as we move up through O vertically the numerical labels on the contour lines are decreasing, so $g_y(O) < 0$.

As we move through P horizontally from left to right, the numerical labels on the contour lines are decreasing, so $g_x(P) < 0$.

Finally, as we move up through Q vertically the line of motion is tangent to the level curve through Q, and this indicates that $g_y(Q) = 0$.

4. (3 points) Define the functions $f(x, y)$ and $g(x, y)$ on $\mathbf{R}^2$ by

$$f(x, y) = e^x(x\cos(y) - y\sin(y)), \qquad g(x, y) = e^x(y\cos(y) + x\sin(y)).$$

Compute $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial g}{\partial x}$, and $\dfrac{\partial g}{\partial y}$. Answer each of the following two questions:

1. Which among a, b, c, d below equals $\dfrac{\partial f}{\partial x}$?
2. Which among a, b, c, d below equals $\dfrac{\partial f}{\partial y}$?

a) $\dfrac{\partial g}{\partial x}$ b) $\dfrac{\partial g}{\partial y}$ c) $-\dfrac{\partial g}{\partial x}$ d) $-\dfrac{\partial g}{\partial y}$

Compute $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial g}{\partial y} = g_y$, and confirm that these are equal. By the product rule in terms of x (regarding y as "constant") we have

$$f_x = e^x(x\cos(y) - y\sin(y)) + e^x\cos(y) = e^x((x+1)\cos(y) - y\sin(y)),$$

and by treating x as "constant" we have

$$g_y = e^x(\cos(y) - y\sin(y) + x\cos(y)) = e^x((x+1)\cos(y) - y\sin(y)).$$

By inspection, these are the same.
Compute $\frac{\partial f}{\partial y} = f_y$ and $-g_x = -\frac{\partial g}{\partial x}$, and confirm that these are equal. Using the product rule in terms of y now (treating x as "constant") we have

$$f_y = e^x(-x\sin(y) - \sin(y) - y\cos(y)) = -e^x((x+1)\sin(y) + y\cos(y)),$$

and by treating y as "constant" we have

$$g_x = e^x(y\cos(y) + x\sin(y)) + e^x\sin(y) = e^x((x+1)\sin(y) + y\cos(y)).$$

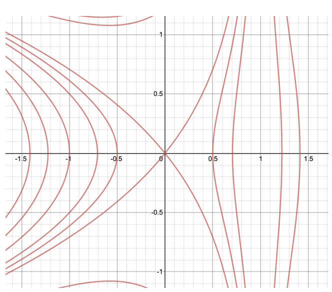
By inspection we see that $f_y = -g_x$.

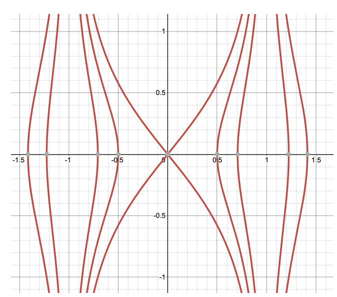
5. (3 points) Consider a function $f(x, y)$ satisfying

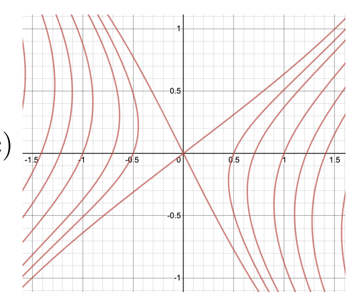
$$\left|\frac{\partial f}{\partial y}(a, b)\right| = \left|\frac{\partial f}{\partial y}(a, -b)\right| = \left|\frac{\partial f}{\partial y}(-a, b)\right|$$

for all $(a, b) \in \mathbf{R}^2$. Which contour plot is most likely to correspond to $f(x, y)$?

Note that the contour plots below all have uniform increments in f-values: the gaps between f-values for successive level curves are the same.

a) 

b) 

c) 

The answer is (B). The the level curves exhibited in graph (B) have symmetry with respect to x-axis and y-axis, so

$$|f_y(a, b)| = |f_y(a, -b)| = |f_y(-a, b)| = |f_y(-a, -b)|$$

for all $(a, b) \in \mathbb{R}^2$.
In (A), there is symmetry in the x-axis but not the y-axis, so $|f_y(a, b)| \neq |f_y(-a, b)|$; in (C), there is no symmetry in the x-axis or the y-axis, so $|f_y(a, b)| \neq |f_y(a, -b)|$ and $|f_y(-a, b)| \neq |f_y(a, b)|$.