Solutions to Math 51 Quiz 3 Practice B

1. (10 points) Consider the data set of 51 points

$$\{(x_1, y_1), \ldots, (x_{51}, y_{51})\},$$

and suppose that its line of best fit is $y = -3x + 4$. Now, consider the following data set of 51 points

$$\{(2x_1, -y_1 + 1), \ldots, (2x_{51}, -y_{51} + 1)\}.$$

What is the line of best fit for the new data set? Make sure to justify your answer fully.

If we take $\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_{51} \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_{51} \end{bmatrix}$, we know that

$$\mathbf{Proj}_{\mathrm{span}(\mathbf{1},\mathbf{X})}(\mathbf{Y}) = -3\mathbf{X} + 4\mathbf{1}.$$

The data vectors for the new vectors are $2\mathbf{X}$ and $-\mathbf{Y} + \mathbf{1}$. In order to find the line of best fit for the new data set, say $y = mx + b$, we need to look for m and b for which

$$\mathbf{Proj}_{\mathrm{span}(\mathbf{1},2\mathbf{X})}(-\mathbf{Y} + \mathbf{1}) = m(2\mathbf{X}) + b\mathbf{1}.$$

Now, we can compute

$$\begin{aligned} \mathbf{Proj}_{\mathrm{span}(\mathbf{1},2\mathbf{X})}(-\mathbf{Y} + \mathbf{1}) &= \mathbf{Proj}_{\mathrm{span}(\mathbf{1},\mathbf{X})}(-\mathbf{Y} + \mathbf{1}) \\ &= -\mathbf{Proj}_{\mathrm{span}(\mathbf{1},\mathbf{X})}(\mathbf{Y}) + \mathbf{Proj}_{\mathrm{span}(\mathbf{1},\mathbf{X})}(\mathbf{1}) \\ &= -(-3\mathbf{X} + 4\mathbf{1}) + \mathbf{1} \\ &= 3\mathbf{X} - 3\mathbf{1} \\ &= \frac{3}{2}(2\mathbf{X}) - 3\mathbf{1}, \end{aligned}$$

and so, the line of best fit for the new data set is $\boxed{y = \frac{3}{2}x - 3}$.

2. (2 points) **True or False:** Let $f(x, y, z) = x^3y^2 - \sin\left(e^{yz} - e^{x^2} - x\right) + \cos(xyz)$ and $g(x, y, z) = 3\left(x^3y^2 - \sin\left(e^{yz} - e^{x^2} - x\right) + \cos(xyz)\right) + 2$. Then the level set of f at the level 1 is the same set as the level set of g at the level 1.

The statement is **FALSE**. Note that $g(x, y, z) = 3f(x, y, z) + 2$, so on the level set of f at the level 1, i.e. for (x, y, z) where $f(x, y, z) = 1$, we have $g(x, y, z) = 3(1) + 2$, so in fact the level set of f at the level 1 coincides with the level set of g at the level 5.

3. (2 points) **True or False:** There exists a continuous function $f(x, y)$ for which

$$f_x(x, y) = f_y(x, y) = e^{x^2+y^2}.$$

The statement is **FALSE**. We can compute

$$f_{xy}(x, y) = 2ye^{x^2+y^2}$$
$$f_{yx}(x, y) = 2xe^{x^2+y^2}$$

Since f is continuous, f_{xy} and f_{yx} must be equal by Clairaut-Schwartz; however, they are not. Hence, no such f can exist.

4. (3 points) Let $f(x, y) = x^2y + x$ and $g(x, y) = (x - 3y, 2x + 4y)$. Which of the following is true?

   (a) The composition $g \circ f$ makes sense, and we can write $g \circ f : \mathbb{R}^2 \to \mathbb{R}$.

   (b) We can write $f : \mathbb{R} \to \mathbb{R}^2$.

   (c) The point $(-2, 2)$ is on the level set of f at the level 0.

   (d) The point $(2, -1)$ is on the level set of $f \circ g$ at the level 5.

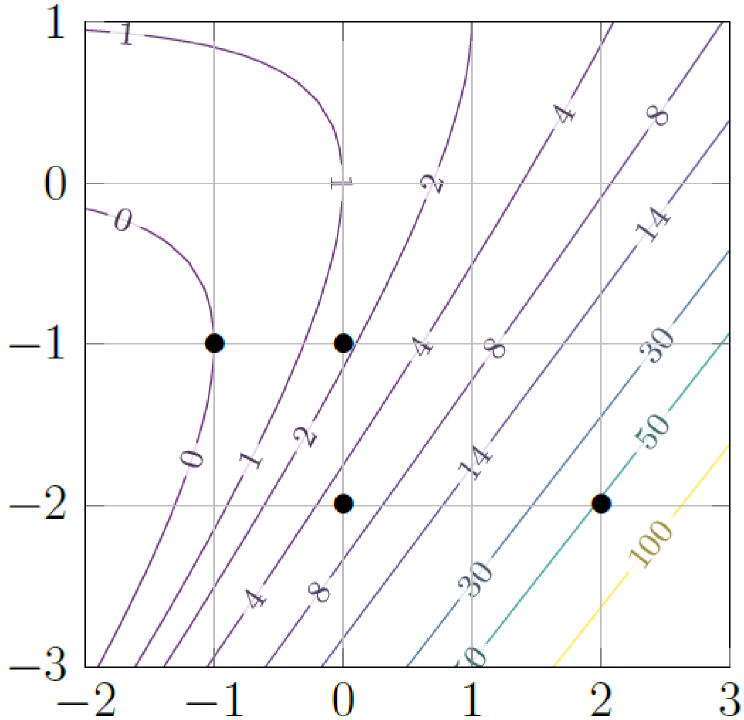
(a) Since $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g \circ f$ does not make sense.

(b) $f : \mathbb{R}^2 \to \mathbb{R}$, not $\mathbb{R} \to \mathbb{R}^2$.

(c) Since $f(-2, 2) = 6$, $(-2, 2)$ is not on the level set of f at the level 0.

(d) Since $f \circ g(2, -1) = f(5, 0) = 5$, $(2, -1)$ is on the level set of $f \circ g$ at the level 5.

5. (3 points) Suppose a function $f(x, y)$ has the contour plot below.



Which of the following quantities is the greatest?

a) $f_x(-1,-1)$ b) $f_y(-1,-1)$ c) $f_x(0,-1)$ d) $f_x(0,-2)$ e) $f_y(0,-2)$ f) $f_y(2,-2)$

First, we note that $f_x(-1,-1)$, $f_x(0,-1)$, and $f_x(0,-2)$ are positive, while $f_y(-1,-1)$ is zero and both $f_y(0,-2)$ and $f_y(2,-2)$ are negative. Comparing $f_x(-1,-1)$, $f_x(0,-1)$, and $f_x(0,-2)$, we can see that $f_x(0,-2)$ is the greatest.