Last time:

- powers of symmetric matrices
- Hessian matrix and quadratic approximation of functions
- level sets of quadratic forms

Today:

- using the Hessian to analyze critical points
- critical points & contour plots 2.0

Problem 1: Unconstrained local extrema via Hessian

For each of the following functions $\mathbf{R}^2 \to \mathbf{R}$, use the gradient to find all critical points and characterize each critical point (i.e., local maximum, local minimum, saddle point, or otherwise) by computing the Hessian in general and analyzing it at each critical point.

(a) $x^4y^4 - 2x^2 - 2y^2$

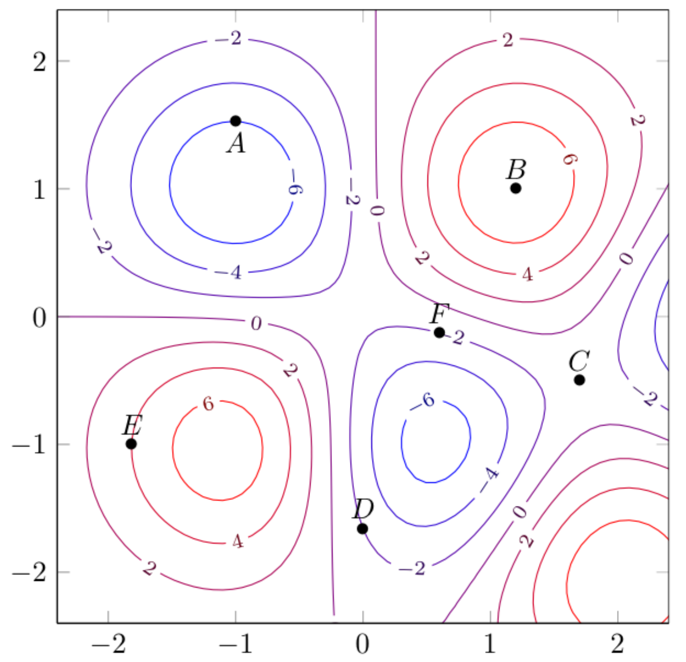
(b) $-3x^2 + 2xy - (3/2)y^2 + y^3$

Problem 2: Visually interpreting critical points

Consider the given contour plot for a function $f \colon \mathbf{R}^2 \to \mathbf{R}$.

(a) Assuming B is a critical point, is the Hessian matrix of f at B: (i) positive-definite, (ii) negative-definite, or (iii) indefinite? (Assume it is one of these.)

(b) Assuming C is a critical point, is the Hessian matrix of f at C: (i) positive-definite, (ii) negative-definite, or (iii) indefinite? (Assume it is one of these.)



Problem 3: Using Hessian eigenvalues to characterize critical points

Consider a critical point $\mathbf{a}$ of $f : \mathbf{R}^n \to \mathbf{R}$ whose Hessian $(\mathrm{H}f)(\mathbf{a})$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ for some orthogonal basis (as we are guaranteed always happens, by the Spectral Theorem). For each of the following possibilities for the list of eigenvalues, is the behavior of f at $\mathbf{a}$ a local maximum, local minimum, or saddle point? (It is one of these in each case below.)

(a) eigenvalues 43, 5, 1

(b) eigenvalues 5 , -3, -7

(c) eigenvalues 1, 0, -1

(d) eigenvalues 1, 1, 1, 1

(e) eigenvalues -1, -5

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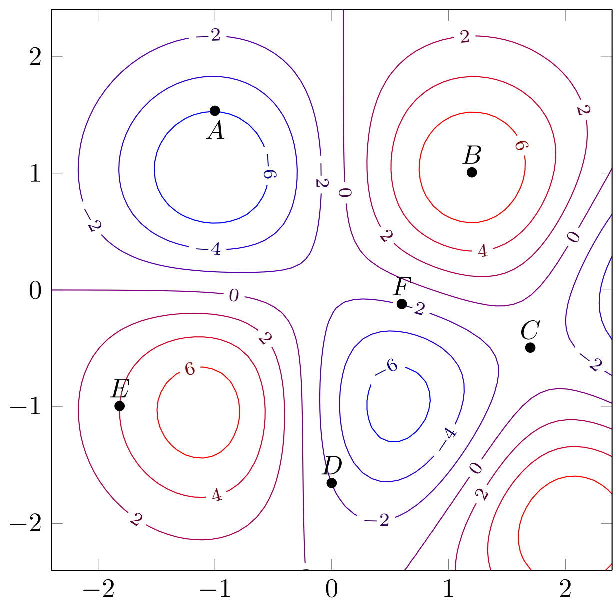

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