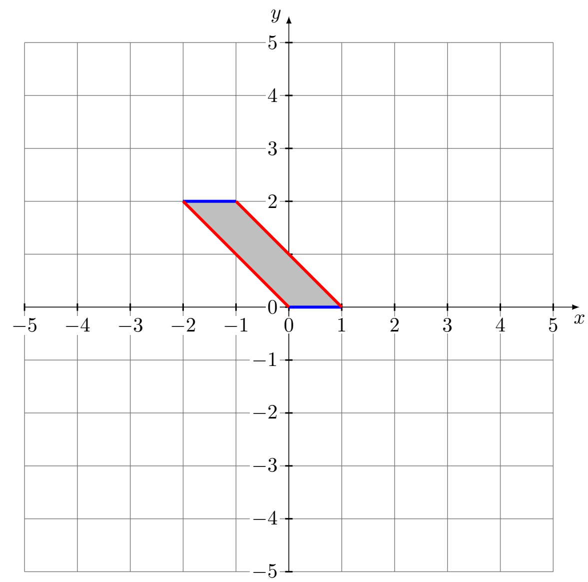
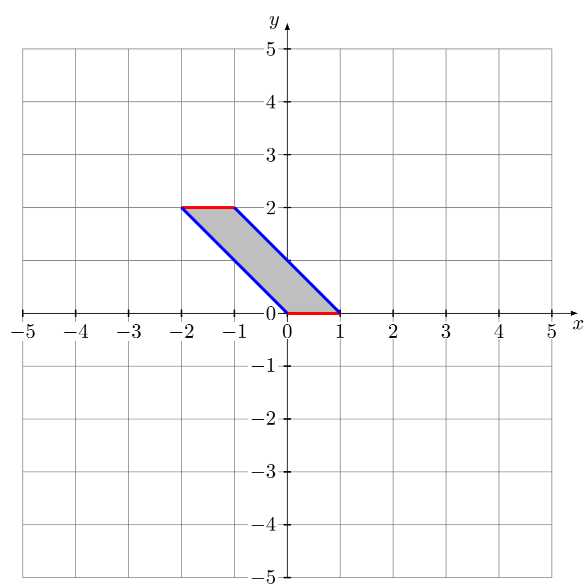
Solutions to Math 51 Quiz 5

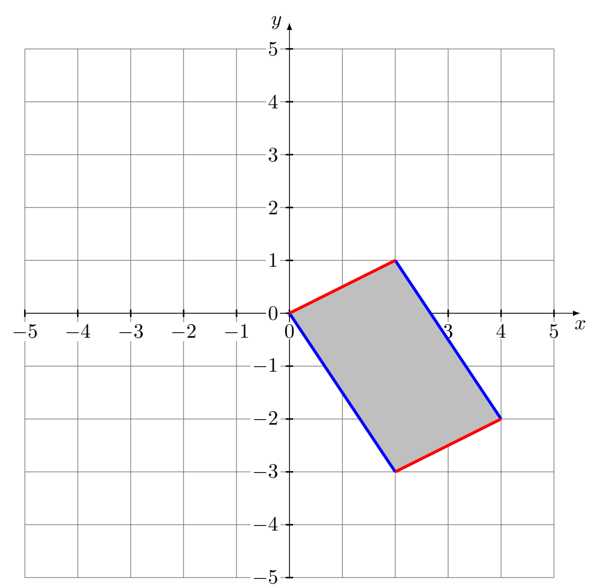
1. (10 points) Let P be the parallelogram with corners at $\mathbf{0}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, and $\mathbf{u}+\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. P is shown in the diagram below.



$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ are linear functions, and $\mathbf{f}(P)$ and $\mathbf{g}(P)$ are shown in the diagrams below. The blue edges get mapped to blue edges, and the red edges get mapped to the red edges.



$\mathbf{f}(P)$



$\mathbf{g}(P)$

Draw $\mathbf{f} \circ \mathbf{g}(P)$ and give the coordinates of the four vertices.

By inspection, we see that

$$\mathbf{f}(\mathbf{u}) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \qquad \mathbf{f}(\mathbf{v}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{g}(\mathbf{u}) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \qquad \mathbf{g}(\mathbf{v}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Noting that $\mathbf{e}_2 = \mathbf{u} + \frac{1}{2}\mathbf{v}$, we get

$$\mathbf{f}(\mathbf{e}_2) = \mathbf{f}(\mathbf{u}) + \frac{1}{2}\mathbf{f}(\mathbf{v}) = \begin{bmatrix} -\frac{3}{2} \\ 2 \end{bmatrix}$$

$$\mathbf{g}(\mathbf{e}_2) = \mathbf{g}(\mathbf{u}) + \frac{1}{2}\mathbf{g}(\mathbf{v}) = \begin{bmatrix} 3 \\ -\frac{5}{2} \end{bmatrix}.$$

Thus, the matrices associated with $\mathbf{f}$ and $\mathbf{g}$ are

$$T_{\mathbf{f}} = \begin{bmatrix} -2 & -\frac{3}{2} \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad T_{\mathbf{g}} = \begin{bmatrix} 2 & 3 \\ -3 & -\frac{5}{2} \end{bmatrix},$$

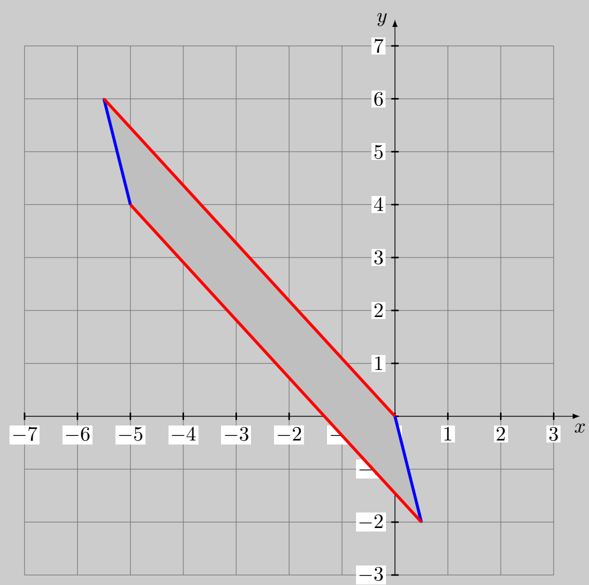
respectively. Therefore, the matrix associated with $\mathbf{f} \circ \mathbf{g}$ is

$$T_{\mathbf{f}\circ\mathbf{g}} = T_{\mathbf{f}} T_{\mathbf{g}} = \begin{bmatrix} -2 & -\frac{3}{2} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{9}{4} \\ -2 & 1 \end{bmatrix}.$$

Hence,

$$(\mathbf{f} \circ \mathbf{g})(\mathbf{u}) = \begin{bmatrix} \frac{1}{2} \\ -2 \end{bmatrix} \qquad \text{and} \qquad (\mathbf{f} \circ \mathbf{g})(\mathbf{v}) = \begin{bmatrix} -\frac{11}{2} \\ 6 \end{bmatrix}$$

The picture of $(\mathbf{f} \circ \mathbf{g})(P)$ is shown below:



The four vertices are $(0, 0)$, $(0.5, -2)$, $(-5.5, 6)$, and $(-5, 4)$.

2. (2 points) **True or False:** Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ be two functions which are affine but not linear. Then, it is possible that $\mathbf{f} \circ \mathbf{g}$ is a linear function.

This statement is **TRUE**. Take $\mathbf{f}(x, y) = (x + 1, y)$ and $\mathbf{g}(x, y) = (x - 1, y)$; $\mathbf{f}$ translates to the right by 1, and $\mathbf{g}$ translates to the left by 1. Clearly, both $\mathbf{f}$ and $\mathbf{g}$ are affine, but not linear. However, the composition

$$(\mathbf{f} \circ \mathbf{g})(x, y) = \mathbf{f}(x - 1, y) = (x, y)$$

is linear.

3. (2 points) **True or False:** Let D be a diagonal 2×2 matrix. Then, for all 2×2 matrices A, we have $AD = DA$.

This statement is **FALSE**. Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then,

$$AD = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = DA.$$

4. (3 points) Suppose $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -4 \\ -1 & 0 & 0 & 1 \\ -1 & 3 & 1 & 3 \end{bmatrix}$ and M is a 4×4 matrix whose second row is $\begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix}$.
Which of the following statements *must* be true?

(a) The second row of AM must be $\begin{bmatrix} -4 & 9 & 3 & 10 \end{bmatrix}$.

(b) The second row of MA must be $\begin{bmatrix} -4 & 9 & 3 & 10 \end{bmatrix}$.

(c) The second column of AM must be $\begin{bmatrix} -1 \\ -12 \\ 3 \\ 10 \end{bmatrix}$.

(d) The second column of MA must be $\begin{bmatrix} -1 \\ -12 \\ 3 \\ 10 \end{bmatrix}$.

If we think intuitively about how matrix multiplication works, since we only know the second row of M, the only thing we can figure out is the second row of MA.

More concretely, if we take $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & 0 & 1 & 3 \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$, then

$$AM = \begin{bmatrix} m_{11} - m_{31} & m_{12} - m_{32} & m_{13} - m_{33} & m_{14} - m_{34} \\ -4m_{41} & -4m_{42} & 2 - 4m_{43} & 6 - 4m_{44} \\ -m_{11} + m_{41} & -m_{12} + m_{42} & -m_{13} + m_{43} & -m_{14} + m_{44} \\ -m_{11} + m_{31} + 3m_{41} & -m_{12} + m_{32} + 3m_{42} & -m_{13} + 3 + m_{33} & -m_{14} + 9 + m_{34} + 3m_{44} \end{bmatrix}$$

$$MA = \begin{bmatrix} m_{11} - m_{13} - m_{14} & 3m_{12} + 3m_{14} & -m_{11} + m_{14} & -4m_{12} + m_{13} + 3m_{14} \\ -4 & 9 & 3 & 10 \\ m_{31} - m_{33} - m_{34} & 3m_{32} + 3m_{34} & -m_{31} + m_{34} & -4m_{32} + m_{33} + 3m_{34} \\ m_{41} - m_{43} - m_{44} & 3m_{42} + 3m_{44} & -m_{41} + m_{44} & -4m_{42} + m_{43} + 3m_{44} \end{bmatrix}$$

5. (3 points) Suppose that A is a 3×1 matrix and B is a 1×3 matrix. If

$$AB = \begin{bmatrix} 6 & -4 & 12 \\ -3 & 2 & -6 \\ \frac{51}{2} & -17 & \xi \end{bmatrix},$$

what is the value of ξ?

(a) No such ξ exists.

(b) 51

(c) $\dfrac{289}{2}$

(d) $\dfrac{867}{2}$

(e) There are infinitely many possible values for ξ.

Similar to what we saw in Lecture 15, Example 8, if we take $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$, we get

$$AB = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}.$$

This is a "rank 1 matrix" where the rows are scalar multiples of each other and the columns are scalar multiples of each other as well. So inspecting the given matrix, we see that the first two rows are scalar multiples of each other and the first two columns are scalar multiples of each other as well. In order for the third row to be a scalar multiple of the first two rows and the third column to be a scalar multiple of the first two columns, the only possible value is $\xi = 51$.