## Problem 1: Matrix of a quadratic form

(a) Consider the quadratic form $q(x, y, z) = x^2 + 2y^2 - z^2 + 4xy + 6xz - 2yz$. Find the symmetric matrix $A$ so that $q(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(b) Consider the quadratic form $q(w, x, y, z) = 2x^2 + y^2 + 3xy + 4yw - zw$ (note that $w$ is the first coordinate, $x$ is the second coordinate, etc.). Find the symmetric matrix $A$ so that $q(w, x, y, z) = \begin{bmatrix} w & x & y & z \end{bmatrix} A \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$.

## Problem 2: Linear system solution set, in parametric form

Let $L$ be the line determined by the overlap of two planes in $\mathbf{R}^3$ (not passing through the origin):

$$x + y + z = 4 \quad \text{and} \quad 2x + 3y + z = 9.$$

(a) Find a point on $L$ whose $z$-coordinate is 5.

(b) Find a point on $L$ whose $z$-coordinate is 6.

(c) Find general formulas for $x$ and $y$ in terms of $z$ for all points $(x, y, z)$ in the line $L$; this should recover your answers to (a) and (b) upon plugging in $z = 5$ and $z = 6$ respectively. (Hint: think about $z$ as a "constant" and $x$ and $y$ as "variables" to guide your algebraic work in the style of what you did for (a) and (b).)

## Problem 3: Recognizing sets as null spaces (or not)

Often in linear algebra one builds a collection of $n$-vectors satisfying a variety of conditions, and it can be useful to know if the collection is the null space of an $m \times n$ matrix for some $m$. (in effect: is the collection describable as the simultaneous solution set for a system of $m$ linear equations in $n$ unknowns?)

(a) Explain why the subset $S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbf{R}^3 : v_1 + v_2 = v_3 + 1 \right\} \subset \mathbf{R}^3$ cannot be the null space of a matrix $A$ (such an $A$ would have to be $m \times 3$ for some $m$). Hint: try to find some general property of null spaces that $S$ violates.

(b) Find a $2 \times 3$ matrix $A$ for $N(A)$ is equal to the set $S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbf{R}^3 : v_1 + v_2 - v_3 = 0,\ v_2 + v_3 = 0 \right\} \subset \mathbf{R}^3$.

 

## Problem 4: Basis and dimension

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) Write the third column of $A$ as a linear combination of the first two columns. Use this to find a basis for $C(A)$.

(b) Find a basis for $C(A)$ that contains $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$.

(c) (**Extra**) How can having a basis for the column space $C(M)$ of an $m \times n$ matrix $M$ help in figuring out for a given $m$-vector $\mathbf{b}$ if $M\mathbf{x} = \mathbf{b}$ has a solution? (Hint: think about $\mathbf{Proj}_{C(M)}$.)

## Problem 5: Column spaces and an overdetermined linear system

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix}$, and let the 5-vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ denote the columns of $A$ (from left to right). For a 5-vector $\mathbf{b}$, the vector equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} \in \mathbf{R}^3$ encodes a system of 5 scalar linear equations in 3 unknowns (the entries of $\mathbf{x}$), so it is overdetermined (more equations than unknowns) and hence the rule of thumb is that only for rather special $\mathbf{b}$ should a solution exist. This problem works out a description of those special $\mathbf{b}$ for this specific $A$ by thinking in terms of column spaces.

(a) If a vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \in \mathbf{R}^5$ belongs to the column space of $A$ (i.e., $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ for some scalars $x_1, x_2, x_3$, or equivalently $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3$) then by turning this vector condition into a collection of 5 simultaneous scalar conditions explain why $b_3 - b_1 - b_2 = 0$, $b_4 + b_1 - b_2 = 0$, and $b_5 - 2b_2 + b_1 = 0$. (Hint: use that each $b_i$ can be expressed in a specific way in terms of the $x_j$'s).

(b) The three equations in the $b_i$'s at the end of (a) can be rewritten to say $b_3 = b_1 + b_2$, $b_4 = -b_1 + b_2$, and $b_5 = -b_1 + 2b_2$, so collectively they say exactly that $\mathbf{b}$ is a 5-vector of the special form $\begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \\ -b_1 + b_2 \\ -b_1 + 2b_2 \end{bmatrix}$ with $b_1, b_2 \in \mathbf{R}$. Check explicitly that any $\mathbf{b}$ of this form actually is in $C(A)$. (Hint: for $\mathbf{b}$ of this special form in terms of $b_1$ and $b_2$, adapt the type of argument in Problem 2(c) to find an $\mathbf{x} \in \mathbf{R}^3$ with $x_3 = 0$ for which the first two entries of $A\mathbf{x}$ are $b_1, b_2$ respectively, and then use the assumed three equations on the $b_i$'s to check that actually $A\mathbf{x} = \mathbf{b}$.). This says that three conditions on $\mathbf{b}$ at the end of (a) are not merely a consequence of membership in $C(A)$ but even exactly characterize it.

(c) Give an explicit 5-vector $\mathbf{b}$ for which the linear system "$A\mathbf{x} = \mathbf{b}$" does *not* have a solution. (There are many answers.)

 

## Problem 6: Normal matrices (Extra)

An $n \times n$ matrix $A$ is called *normal* if $A$ commutes with its transpose $A^\top$ (i.e., $AA^\top = A^\top A$). For example, every symmetric matrix (and in particular every diagonal matrix) is normal, since it is even equal to its own transpose.

(a) Check that every matrix of the form $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (which is not symmetric when $b \neq 0$) is normal, and that $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is normal. (When computing $MM^\top$ and $M^\top M$ you only need to compute the part on or above the diagonal, since the product is symmetric; saving work in this way assumes one doesn't make a miscalculation.)

(b) Explain why any orthogonal $n \times n$ matrix is normal, and that if $M$ is normal then so are $M^2$ and $M^3$. (How about $M^r$ for general $r \geq 1$?)

(c) In (a) we give a type of non-symmetric normal $2 \times 2$ matrix (with $b \neq 0$). Show by explicit example that for a symmetric (hence normal) $2 \times 2$ matrix $M$ and normal $2 \times 2$ matrix $M'$ of the type in (a), the product $A = MM'$ can fail to be normal. (Nearly anything you try for $M$ and $M'$ should work, as long as you avoid too many matrix entries equal to 0.)

In connection with (c), it is a fact that if $M$ and $M'$ are normal $n \times n$ matrices that *commute* (i.e., $MM' = M'M$) then the product $MM'$ is normal. However, this has no short explanation; it requires a big generalization of an upcoming important result called the Spectral Theorem.