# Solutions to Math 51 Quiz 7

1. (10 points) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 0 & 1 & -4 \\ 2 & 8 & -18 \end{bmatrix}$. In this problem, we are interested in finding a basis for $C(A)^\perp$.

   (a) Find the matrix $B$ satisfying the property that $\mathbf{x} \cdot (A\mathbf{y}) = (B\mathbf{x}) \cdot \mathbf{y}$ for all $\mathbf{x} \in \mathbb{R}^4$ and $\mathbf{y} \in \mathbb{R}^3$, and explain why asking if $\mathbf{x} \in C(A)^\perp$ is the same as asking if $\mathbf{x} \in N(B)$ (i.e. show that $C(A)^\perp = N(B)$).

   (b) Let $B$ be the matrix you found in part (a). Apply the Gram-Schmidt process to the columns of $B$. You must show all your work.

   (c) Let $B$ be the matrix you found in part (a). Find a basis for $N(B)$ (and hence a basis for $C(A)^\perp$).

   (d) Consider the following statement.
   "Let $S_1 = \{(w, x, y, z) \in \mathbb{R}^4 : aw + bx + cy + dz = 0\}$ and $S_2 = \{(w, x, y, z) \in \mathbb{R}^4 : ew + fx + gy + hz = 0\}$. Then, a vector $\mathbf{b} \in \mathbb{R}^4$ is equal to $A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^3$ precisely when $\mathbf{b}$ lies in both $S_1$ and $S_2$."
   Find numbers $a, b, c, d, e, f, g, h$ which makes the above statement true, and justify your answer.

(a) The desired matrix is $B = A^T$; explicitly $B = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 1 & 8 \\ -1 & -3 & -4 & -18 \end{bmatrix}$. Note that if $\mathbf{x} \in N(B)$, then that precisely means that $B\mathbf{x} = \mathbf{0}$; however this also means that $\mathbf{x} \cdot (A\mathbf{y}) = (B\mathbf{x}) \cdot \mathbf{y} = \mathbf{0} \cdot \mathbf{y} = 0$ for any $\mathbf{y} \in \mathbb{R}^3$. This would mean that $\mathbf{x}$ is orthogonal to any vector of the form $A\mathbf{y}$, and hence it belongs in the orthogonal complement of $C(A)$, i.e. $\mathbf{x} \in C(A)^\perp$. Conversely, if $\mathbf{x} \in C(A)^\perp$, then $(B\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathbb{R}^3$; this is enough to guarantee that $B\mathbf{x} = \mathbf{0}$ (e.g. by noting that by choosing $\mathbf{y} = \mathbf{e}_i$, we obtain that the $i$th component of $B\mathbf{x}$ is zero, and hence all of the components of $B\mathbf{x}$ must be zero). By definition, this would mean that $\mathbf{x} \in N(B)$. Thus, we see that $\mathbf{x}$ is in $C(A)^\perp$ precisely when it is in $N(B)$.

(b) The collection of vectors in question is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 2 \\ 8 \\ -18 \end{bmatrix}$.

The first vector $\mathbf{w}_1$ obtained in the Gram-Schmidt process is just $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. To find the second vector $\mathbf{w}_2$, we note by inspection that $\mathbf{v}_2 = 3\mathbf{v}_1 = 3\mathbf{w}_1$, i.e. $\mathbf{v}_2$ lies in the span of $\mathbf{w}_1$, so $\text{Proj}_{\mathbf{w}_1}\mathbf{v}_2 = \mathbf{v}_2$, and hence $\mathbf{w}_2 = \mathbf{v}_2 - \text{Proj}_{\mathbf{w}_1}\mathbf{v}_2 = \mathbf{v}_2 - \mathbf{v}_2 = \mathbf{0}$ (alternatively, we can explicitly compute

$$\mathbf{w}_2 = \mathbf{v}_2 - \text{Proj}_{\mathbf{w}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1(3) + 2(6) - 1(-3)}{1(1) + 2(2) - 1(-1)} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} - \frac{18}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \mathbf{0},$$

as desired.) To find $\mathbf{w}_3$, we need compute $\mathbf{w}_3 = \mathbf{v}_3 - \text{Proj}_{\mathbf{w}_1}\mathbf{v}_3$ (there is no projection onto $\mathbf{w}_2$ since $\mathbf{w}_2 = \mathbf{0}$). We thus have

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \mathbf{v}_3 - \frac{1(0) + 2(1) - 1(-4)}{6}\mathbf{w}_1 = \mathbf{v}_3 - (1)\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix}.$$

Finally, to compute $\mathbf{w}_4$, we compute

$$\begin{aligned}
\mathbf{w}_4 &= \mathbf{v}_4 - \mathrm{Proj}_{\mathbf{w}_1}\mathbf{v}_4 - \mathrm{Proj}_{\mathbf{w}_3}\mathbf{v}_4 \\
&= \mathbf{v}_4 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_4}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{w}_3 \cdot \mathbf{v}_4}{\mathbf{w}_3 \cdot \mathbf{w}_3}\mathbf{w}_3 \\
&= \mathbf{v}_4 - \frac{1(2)+2(8)-1(-18)}{6}\mathbf{w}_1 - \frac{-1(2)-1(8)-3(-18)}{(-1)(-1)+(-1)(-1)+(-3)(-3)}\mathbf{w}_3 \\
&= \mathbf{v}_4 - \frac{36}{6}\mathbf{w}_1 - \frac{44}{11}\mathbf{w}_3 \\
&= \mathbf{v}_4 - 6\mathbf{w}_1 - 4\mathbf{w}_3 \\
&= \begin{bmatrix} 2 \\ 8 \\ -18 \end{bmatrix} - 6\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 4\begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2-6(1)-4(-1) \\ 8-6(2)-4(-1) \\ -18-6(-1)-4(-3) \end{bmatrix} = \mathbf{0}.
\end{aligned}$$

Thus, the output of Gram-Schmidt is $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{w}_2 = \mathbf{0}$, $\mathbf{w}_3 = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix}$, and $\mathbf{w}_4 = \mathbf{0}$.

(c) Since $B$ is a $3 \times 4$ matrix, the null space $N(B)$ consists of vectors $\mathbf{x} \in \mathbb{R}^4$ such that $B\mathbf{x} = \mathbf{0}$. Recall that if $\mathbf{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$, then $B\mathbf{x}$ is the linear combination $w\mathbf{v}_1 + x\mathbf{v}_2 + y\mathbf{v}_3 + z\mathbf{v}_4$; hence we're looking for linear dependence relations among $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Note that our work in part (b) can help find linear dependence relations, since some of the $\mathbf{w}_i$ were zero, and the $\mathbf{w}_i$ are obtained by linear combinations of the $\mathbf{v}_i$.

Specifically, we found that $\mathbf{w}_2 = \mathbf{0}$; however algebraically $\mathbf{w}_2$ was obtained from $\mathbf{v}_2 - 3\mathbf{w}_1 = \mathbf{v}_2 - 3\mathbf{v}_1$. Hence, we have $-3\mathbf{v}_1 + \mathbf{v}_2 = 0$. This implies that $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(B)$. Similarly, we found that $\mathbf{w}_4 = \mathbf{0}$, with $\mathbf{w}_4$ also equaling $\mathbf{v}_4 - 6\mathbf{w}_1 - 4\mathbf{w}_3$. Since $\mathbf{w}_1 = \mathbf{v}_1$, and $\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{w}_1 = \mathbf{v}_3 - \mathbf{v}_1$, it follows that

$$\mathbf{0} = \mathbf{w}_4 = \mathbf{v}_4 - 6\mathbf{w}_1 - 4\mathbf{w}_3 = \mathbf{v}_4 - 6\mathbf{v}_1 - 4(\mathbf{v}_3 - \mathbf{v}_1) = -2\mathbf{v}_1 - 4\mathbf{v}_3 + \mathbf{v}_4.$$

This implies that $\begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \in N(B)$. We have thus found two (linearly independent) vectors in $N(B)$; this forms a basis for $N(B)$ as long as we can show that $\dim(N(B)) = 2$. In general, the number of vectors in the output of the Gram-Schmidt process which equal $\mathbf{0}$ will equal the dimension of the null space of the corresponding matrix; in this case we can also see this using the Rank-Nullity Theorem, which says that $C(B) + N(B) = 4$ since $B$ has 4 columns. The column space $C(B)$ is at least 2-dimensional, since the first and third columns are linearly independent; hence $\dim(N(B))$ is at most 2, and thus equal to 2 since we already found 2 linearly independent vectors in $N(B)$. Hence, a basis for $N(B)$ is given by $\boxed{\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}}$.

(d) Note that a vector $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ lying in both $S_1$ and $S_2$ means precisely that it is orthogonal to the two vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $\begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$. Furthermore, note that if a vector $\mathbf{b}$ is orthogonal to a basis of a linear subspace $V$, then it is orthogonal to the entire subspace, i.e. $\mathbf{b} \in V^{\perp}$, and furthermore if we can write $V$ as an orthogonal complement $V = W^{\perp}$, then $\mathbf{b} \in (W^{\perp})^{\perp} = W$; conversely $\mathbf{x}$ being in $W$ means it must be orthogonal to a basis of $V = W^{\perp}$. Thus, if we can take these two vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $\begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$ to span $C(A)^{\perp}$, then $\mathbf{b}$ being orthogonal to these two vectors is precisely the condition of being in $C(A)$, i.e. equal to $A\mathbf{x}$ for some $\mathbf{x}$. Thus, we can take $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $\begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$ to be the two vectors obtained in the basis of $N(B)$ (i.e. $C(A)^{\perp}$) found in part (c), i.e. $\boxed{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$ and $\boxed{\begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}}$. (Other answers are definitely possible.)

2. (2 points) **True or False:** Suppose $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are nonzero vectors in $\mathbb{R}^n$ with $\mathbf{v}_2 \perp \mathbf{v}_3$. Let $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be the output of the Gram-Schmidt process applied to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Then, if $\mathbf{w}_3 = \mathbf{v}_3 - \alpha\mathbf{v}_1 - \beta\mathbf{v}_2$, $\beta$ must equal 0.

This statement is **FALSE**. If $\mathbf{w}_3 = \mathbf{v}_3 - \alpha\mathbf{v}_1 - \beta\mathbf{v}_2$, then

$$\mathbf{Proj}_{V_2}(\mathbf{v}_3) = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2.$$

However, there is no reason for $\mathbf{Proj}_{V_2}(\mathbf{v}_3)$ to be a scalar multiple of $\mathbf{v}_1$. We can construct a counterexample this way – the intuition is to create a plane that is slanted with respect to $\mathbf{v}_3$ (i.e. $\mathbf{v}_3$ is not normal to the plane); the projection of $\mathbf{v}_3$ onto the plane will be perpendicular to $\mathbf{v}_2$, but if we take $\mathbf{v}_1$ to be a vector that is not perpendiular to $\mathbf{v}_2$, then the projection has to have a nonzero $\mathbf{v}_2$-component.

Take $\mathbf{v}_2 = \mathbf{e}_2$, $\mathbf{v}_3 = \mathbf{e}_3$, and take $\mathbf{v}_1 = (1, 1, 1)$. Note that $\mathbf{v}_2$ and $\mathbf{v}_3$ are perpendicular. An orthogonal basis for $V_2 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ is $\{(0, 1, 0), (1, 0, 1)\}$. Hence,

$$\mathbf{Proj}_{V_2}(\mathbf{e}_3) = \mathbf{Proj}_{(1,0,1)}(\mathbf{e}_3) + \mathbf{Proj}_{\mathbf{e}_2}(\mathbf{e}_3) = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2,$$

and so, $\beta$ does not have to equal zero.

3. (2 points) **True or False:** It is possible to find a $2 \times 2$ matrix $A$ for which $N(A) = C(A)$.

If $N(A) = C(A)$ were possible, then it must be true that $\dim N(A) = \dim C(A) = 1$. The fact that $\dim C(A) = 1$ implies that the two columns must be scalar multiples of each other. So, $A$ must be of the form

$$A = \begin{bmatrix} a & \alpha a \\ b & \alpha b \end{bmatrix}.$$

Every vector in $C(A)$ is a scalar multiple of $\begin{bmatrix} a \\ b \end{bmatrix}$. Hence, we need to check if it is possible for $\beta \begin{bmatrix} a \\ b \end{bmatrix}$ to be in $N(A)$ for all scalars $\beta$. In other words,

$$\begin{bmatrix} a & \alpha a \\ b & \alpha b \end{bmatrix} \begin{bmatrix} \beta a \\ \beta b \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which would force $\beta a^2 + \alpha\beta ab = 0$ and $\beta ab + \alpha\beta b^2 = 0$. Note that these factor into

$$\beta a(a + \alpha b) = 0 \qquad \text{and} \qquad \beta b(a + \alpha b) = 0.$$

Since both $a$ and $b$ cannot be simultaneously zero (this would make $\dim C(A) = 0$), this forces $a + \alpha b = 0$. Picking $a = 2$ and $b = -4$ (any choice of nonzero $a$ and $b$ works), we see that $\alpha = \frac{1}{2}$, and so

$$A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}.$$

It is easily verified that $C(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$ and $N(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$, and so, $C(A) = N(A)$.

*Remark.* $A$ was featured in PCRQ 20 (Chapter 21) questions 2 and 3, where we saw that $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$ was in both the null space and the column space.

4. (3 points) Suppose $A$ and $B$ are symmetric $n \times n$ matrices and that $C = AB - BA$. Which of the following statements *must* be true?

   (a) $C = 0$

   (b) $C^\intercal = C$

   (c) $C^\intercal = -C$

   (d) None of the above statements must be true; it depends on $A$ and $B$.

We see that

$$C^\intercal = (AB - BA)^\intercal = (AB)^\intercal - (BA)^\intercal = B^\intercal A^\intercal - A^\intercal B^\intercal = BA - AB = -(AB - BA) = -C.$$

5. (3 points) Consider the following system of linear equations

$$\begin{aligned} x + 2y + 3z &= 3 \\ 2x + 4y + 7z &= 6 \\ 3x + 6y + \alpha z &= 9 \end{aligned}$$

Which of the following statements is true about the number of solutions to the system?

(a) The system always has a unique solution.

(b) The system could have a unique solution, depending on the choice of $\alpha$.

(c) The system has no solution.

(d) There are some choices of $\alpha$ that cause the system to have no solutions, and the other choices of $\alpha$ that cause the system to have infinitely many solutions.

(e) The system always has an infinite number of solutions, regardless of the choice of $\alpha$.

The linear system can be represented by

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & \alpha \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}}_{\mathbf{b}}.$$

Writing the columns of $A$ as $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, it is very easy to see that $\mathbf{b} \in C(A)$ by noticing $\mathbf{b} = 3\mathbf{v}_1$, $\mathbf{b} = \frac{3}{2}\mathbf{v}_2$, $\mathbf{b} = \mathbf{v}_1 + \mathbf{v}_2$, etc. Hence, the system has at least one solution.

Now, since $\mathbf{v}_2 = 2\mathbf{v}_1$, we know that $\dim C(A)$ has to be strictly less than 3 (i.e. at least one of the three column vectors is redundant). Thus, by Rank-Nullity, we see that $N(A) \geq 1$, and so, there is a nonzero vector in the null space. Hence, we are guaranteed the system has infinitely many solutions regardless of the value of $\alpha$.