Last time

- gradients
- gradient is perpendicular to tangent line/plane of level set

Today:

- optimization using Lagrange multipliers
- linear functions and matrices
- the derivative matrix

Review: Lagrange multipliers

Problem 1: Constrained optimization

In what follows, you may accept that $f(x, y) = xy$ attains maximal and minimal values on the curve $x^2 - xy + y^2 = 3$.

(a) Use the method of Lagrange multipliers to find these extreme values and the point(s) where they are attained.

(b) The quadratic formula allows you to solve for y in terms of x on the curve: $y(x) = (x \pm \sqrt{x^2 - 4(x^2 - 3)})/2 = (x \pm \sqrt{12 - 3x^2})/2$ (with $|x| \leq 2$ so that the square root makes sense). Hence, we could instead try to find the extreme values for $f(x, y(x)) = x \cdot y(x) = (x^2 \pm x\sqrt{12 - 3x^2})/2$ for $-2 \leq x \leq 2$ via single-variable calculus. Is that more or less appetizing than the method in (a)?

Problem 2: Optimization review (what technique(s) would you use?)

(a) Given the function $f(x, y) = x + y$, find the maximum and minimum values of f on the domain

$$D_1 = \{(x, y) \colon 0 \leq y \leq x^2 \text{ and } -1 \leq x \leq 1\}.$$

(b) Find the maximum and minimum values of $G(x, y) = 3x^2 + 4xy$ on the region

$$D_2 = \{(x, y)\colon y \geq 0 \text{ and } x^2 + y^2 \leq 9\}.$$

(When doing this, one part of the boundary will be a mess via single-variable calculus, so employ Lagrange multipliers there with the boundary curve as a constraint condition. You may encounter the expression $2x^2 - 3xy - 2y^2$, in which case it will be useful to then observe that this factors as $(2x + y)(x - 2y)$.)

(c) (**Extra**) Let C be the curve in $\mathbf{R}^2$ defined by the equation

$$y^2 = x^3 - 4x^2 + 5x$$

Determine all points on C at minimal distance to $(5/2, 0)$.

<u>Review</u>: Linear functions

Problem 3: Identifying linear functions

In each case below, is $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^3$ linear? If it is, find the matrix representing it. If not, explain why not.

(a) $\mathbf{f}(x_1, x_2) = (x_1, x_2^2, 2x_1 + x_2)$

(b) $\mathbf{f}(x_1, x_2) = (1, x_2, 2x_1 + x_2)$

(c) $\mathbf{f}(x_1, x_2) = (0, x_2, 2x_1 + x_2)$

(d) $\mathbf{f}(x_1, x_2) = (0, x_1 x_2, 2x_1 + x_2)$

(e) $\mathbf{f}(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2, ex_1 + fx_2)$

Problem 4: Derivative matrix and numerical linear approximation

Consider the function $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$f(x, y) = (x^3 y^2, 4x + y^3 + xy).$$

(a) Compute the derivative matrix $(Df)(x, y)$, and then use it to give the linear approximation to f at $(1, 1)$.

(b) Use your answer to (a) to estimate the 2-vector $f(0.8, 1.1)$, and then compare it with an exact calculation using a calculator. Is it a good approximation?

(c) Give the linear approximation to f at $(2, -2)$ and use it to estimate the 2-vector $f(2.1, -1.9)$ and then compare this to the exact 2-vector using a calculator. Is the approximation good or bad?

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