Solutions to Math 51 Quiz 8

1. (10 points) Let A be a 3×3 matrix, and let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the columns of A. Let $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be the output of the Gram-Schmidt process applied to the columns of A, and let $A = QR$ be a QR-decomposition of A, where the columns of Q are obtained by normalizing the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. Suppose, in doing the Gram-Schmidt process, you find the following relations:

$$\begin{aligned}\mathbf{w}_1 &= \mathbf{v}_1\\ \mathbf{w}_2 &= \mathbf{v}_2 + \mathbf{w}_1\\ \mathbf{w}_3 &= \mathbf{v}_3 - \frac{1}{3}\mathbf{w}_1 - 2\mathbf{w}_2.\end{aligned}$$

Suppose furthermore that $\|\mathbf{w}_1\| = 9$, $\|\mathbf{w}_2\| = \sqrt{5}$, and $\|\mathbf{w}_3\| = 9\sqrt{5}$.

We are interested in solving $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = \begin{bmatrix}12\\-4\\1\end{bmatrix}$.

(a) (3 points) Find the matrix R in the above QR-decomposition of A.

(b) (4 points) Suppose we also know that

$$\mathbf{w}_1 = \begin{bmatrix}8\\4\\1\end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix}1\\-2\\0\end{bmatrix}, \qquad \mathbf{w}_3 = \begin{bmatrix}2\\1\\-20\end{bmatrix}.$$

Recalling that we want to solve $A\mathbf{x} = \mathbf{b}$, find a vector $\mathbf{y}$ for which this equation reduces to solving the equation $R\mathbf{x} = \mathbf{y}$. The components of $\mathbf{y}$ may contain square roots, but they should not contain any fractions.

Hint. You may want to find Q.

(c) (3 points) Using the vector $\mathbf{y}$ obtained in part (b), solve the equation $R\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. The components of $\mathbf{x}$ should all be integers.

(a) Solving for the $\mathbf{v}_i$'s from the above relations, we get

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{w}_1 &&= 9\mathbf{w}_1'\\ \mathbf{v}_2 &= -\mathbf{w}_1 + \mathbf{w}_2 &&= -9\mathbf{w}_1' + \sqrt{5}\mathbf{w}_2'\\ \mathbf{v}_3 &= \frac{1}{3}\mathbf{w}_1 + 2\mathbf{w}_2 + \mathbf{w}_3 &&= 3\mathbf{w}_1' + 2\sqrt{5}\mathbf{w}_2' + 9\sqrt{5}\mathbf{w}_3'\end{aligned}$$

where $\mathbf{w}_i' = \frac{1}{\|\mathbf{w}_i\|}\mathbf{w}_i$ for $i = 1, 2, 3$. Thus, the matrix R is

$$R = \begin{bmatrix}9 & -9 & 3\\ 0 & \sqrt{5} & 2\sqrt{5}\\ 0 & 0 & 9\sqrt{5}\end{bmatrix}.$$

(b) We know that

$$Q = \begin{bmatrix}8/9 & 1/\sqrt{5} & 2/9\sqrt{5}\\ 4/9 & -2/\sqrt{5} & 1/9\sqrt{5}\\ 1/9 & 0 & -20/9\sqrt{5}\end{bmatrix}.$$

Since $\mathbf{b} = A\mathbf{x} = Q(R\mathbf{x})$, the vector $\mathbf{y}$ we are looking for is

$$\mathbf{y} = Q^{-1}\mathbf{b} = Q^{\intercal}\mathbf{b} = \begin{bmatrix}8/9 & 4/9 & 1/9\\ 1/\sqrt{5} & -2/\sqrt{5} & 0\\ 2/9\sqrt{5} & 1/9\sqrt{5} & -20/9\sqrt{5}\end{bmatrix}\begin{bmatrix}12\\-4\\1\end{bmatrix} = \begin{bmatrix}9\\4\sqrt{5}\\0\end{bmatrix}.$$

(c) The system $R\mathbf{x} = \mathbf{y}$, for $\mathbf{x} = (x, y, z)$, is

$$\begin{aligned} 9x - 9y + 3z &= 9 \\ \sqrt{5}y + 2\sqrt{5}z &= 4\sqrt{5} \\ 9\sqrt{5}z &= 0 \end{aligned}$$

Using back-substitution, we see that $z = 0$, $y = 4$, and $x = 5$. Thus, $\mathbf{x} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$.

2. (2 points) **True or False:** Suppose that A and B are 2×2 matrices with the same characteristic polynomial, i.e. $P_A(\lambda) = P_B(\lambda)$. Then, $A = B$.

This statement is **FALSE**. Consider the two matrices we saw in Lecture 26, Example 7: $A = \begin{bmatrix} 16 & 6 \\ 6 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 16 & -6 \\ -6 & 0 \end{bmatrix}$. Both matrices have the same characteristic polynomial $P_A(\lambda) = P_B(\lambda) = \lambda^2 - 16\lambda - 36$; however, $A \neq B$.

3. (2 points) **True of False:** Suppose A and B are $n \times n$ matrices with B invertible. Suppose that $\mathbf{v}$ is a nonzero n-vector in $N(A + 51B^{-1})$. Then, $\mathbf{v}$ is an eigenvector of BA with eigenvalue -51.

This statement is **TRUE**. Since $\mathbf{v} \in N(A + 51B^{-1})$, we have $(A + 51B^{-1})\mathbf{v} = \mathbf{0}$, and so, $A\mathbf{v} = -51B^{-1}\mathbf{v}$. Multiplying by B on the left, we get

$$BA\mathbf{v} = -51\mathbf{v},$$

and so, $\mathbf{v}$ is an eigenvector of BA with eigenvalue -51.

4. (3 points) Let A be a 4×4 symmetric matrix. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of A with eigenvalues -1 and 5, respectively. Let $\mathbf{u} = 2\mathbf{v} - 3\mathbf{w}$. Which of the following must be true about $\mathbf{u}$?

(a) $\mathbf{u}$ is an eigenvector of A with eigenvalue $-(17)^2$

(b) $\mathbf{u}$ is an eigenvector of A with eigenvalue -17.

(c) $\mathbf{u}$ is an eigenvector of A with eigenvalue $-\sqrt{17}$.

(d) $\mathbf{u}$ is an eigenvector of A with eigenvalue $\sqrt{17}$.

(e) $\mathbf{u}$ is an eigenvector of A with eigenvalue 17.

(f) $\mathbf{u}$ is an eigenvector of A with eigenvalue $(17)^2$.

(g) We need to know $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ in order to determine the eigenvalue associated with $\mathbf{u}$.

(h) $\mathbf{u}$ cannot be an eigenvector of A.

By the hypothesis, we have $A\mathbf{v} = -\mathbf{v}$ and $A\mathbf{w} = 5\mathbf{w}$. Suppose that $\mathbf{u}$ is an eigenvector of A with eigenvalue λ. Then,

$$A\mathbf{u} = A(2\mathbf{v} - 3\mathbf{w}) = -2\mathbf{v} - 15\mathbf{w}.$$

However, we also have

$$A\mathbf{u} = \lambda\mathbf{u} = \lambda(2\mathbf{v} - 3\mathbf{w}) = 2\lambda\mathbf{v} - 3\lambda\mathbf{w}.$$

Thus, we have $-2\mathbf{v} - 15\mathbf{w} = 2\lambda\mathbf{v} - 3\lambda\mathbf{w}$, and so,

$$(3\lambda - 15)\mathbf{w} = (2\lambda + 2)\mathbf{v},$$

which is impossible since $\mathbf{v}$ and $\mathbf{w}$ are orthogonal (due to the Spectral theorem). Hence, $\mathbf{u}$ cannot be an eigenvector of A.

5. (3 points) Note that

$$\begin{bmatrix} -1/2 & -9/2 \\ -9/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Compute $\begin{bmatrix} -1/2 & -9/2 \\ -9/2 & -1/2 \end{bmatrix}^{5151}$.

Note that $Q = \begin{bmatrix} -1/2 & -9/2 \\ -9/2 & -1/2 \end{bmatrix}$ is an orthogonal matrix, $D = \begin{bmatrix} -5 & 0 \\ 0 & 4 \end{bmatrix}$ is a diagonal matrix, and $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = Q^{\intercal} = Q^{-1}$. Hence,

$$\begin{aligned} \begin{bmatrix} -1/2 & -9/2 \\ -9/2 & -1/2 \end{bmatrix}^{5151} &= \left(QDQ^{-1}\right)^{5151} \\ &= QD^{5151}Q^{-1} \\ &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} (-5)^{5151} & 0 \\ 0 & 4^{5151} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -5^{5151} + 4^{5151} & -5^{5151} - 4^{5151} \\ -5^{5151} - 4^{5151} & -5^{5151} + 4^{5151} \end{bmatrix} \end{aligned}$$