1. (2 points) Determine whether the following statement is **true** (i.e., *always* true) or **false** (i.e., sometimes not true):

   Every orthogonal collection of seven nonzero vectors in $\mathbf{R}^{13}$ is linearly independent.

2. (3 points) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are unit vectors in $\mathbf{R}^{691}$, with

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \qquad \mathbf{v}_1 \cdot \mathbf{v}_3 = \frac{5}{13}, \qquad \mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{12}{13}.$$

   It is a fact that when we apply the Gram-Schmidt process to $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$, we obtain the vectors $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ in sequence, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are nonzero and $\mathbf{w}_3 = \mathbf{0}$.

   Given this information, which of the following is a valid linear dependence relation among the $\mathbf{v}_i$'s ?

   (i) $5\mathbf{v}_1 + 12\mathbf{v}_2 - 13\mathbf{v}_3 = \mathbf{0}$ (ii) $5\mathbf{v}_1 - 12\mathbf{v}_2 + 13\mathbf{v}_3 = \mathbf{0}$

   (iii) $5\mathbf{v}_1 + 12\mathbf{v}_2 + 13\mathbf{v}_3 = \mathbf{0}$ (iv) $5\mathbf{v}_1 - 12\mathbf{v}_2 - 13\mathbf{v}_3 = \mathbf{0}$

3. (3 points) Suppose the four 51-vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are all nonzero, and that

$$2\mathbf{v}_1 + 4\mathbf{v}_3 + 2\mathbf{v}_4 = \mathbf{0}.$$

   Which of the following are possible numbers of nonzero vectors that lie in an orthogonal basis for $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$, obtained via the Gram-Schmidt process?

   [Select all options that are possible with the information given.]

   (i) zero (ii) one (iii) two (iv) three (v) four

4. (3 points) Consider the following 3×3 matrix:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & a \\ -1/\sqrt{2} & 1/\sqrt{3} & b \\ 0 & -1/\sqrt{3} & c \end{bmatrix}$$

   where $a, b, c \in \mathbf{R}$. How many 3×3 orthogonal matrices exist that have the above form?

   (i) none (ii) one (iii) two

   (iv) more than two, but finitely many (v) infinitely many

5. (4 points) Suppose that $\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in \mathbf{R}^{2021}$ are linearly independent. Which of the following sets are linearly independent?

   (a) $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}$

   (b) $\{\mathbf{u}+\mathbf{v}+\mathbf{w}, \mathbf{u}-\mathbf{v}-2\mathbf{w}, \mathbf{u}+3\mathbf{v}+4\mathbf{w}\}$

   (c) $\{\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}\}$

   (d) $\{\mathbf{u}+\mathbf{v}+\mathbf{w}, \mathbf{u}-\mathbf{v}-\mathbf{w}, \mathbf{u}+\mathbf{v}-\mathbf{w}, \mathbf{u}-\mathbf{v}+\mathbf{w}\}$.

6. (4 points) Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -4 \\ -5 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 2 \\ -6 \\ -6 \\ 6 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 0 \\ -3 \\ -6 \\ -9 \end{bmatrix}$$

and $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. What is the dimension of V?
Hint: You can use the Gram-Schmidt process.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

7. (2 points) **True or False:**
If the Gram-Schmidt process is applied to

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 17 \\ 19 \\ 23 \end{bmatrix}$$

to yield $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ sequentially, then $\mathbf{w}_4 = \mathbf{0}$.
Hint: Do not use brute force to actually carry out the Gram-Schmidt process.

8. (3 points) Let A and B be two invertible 3×3 matrices, and P be the matrix for projection onto a plane in $\mathbf{R}^3$ through the origin. Which of the following matrices must **always** be invertible?

(a) $A + B$. (b) $AB^\top$. (c) AP.

9. (2 points) **True or False:**
If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set in $\mathbf{R}^n$, then $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$ is also a linearly independent set in $\mathbf{R}^n$.

10. (4 points) If we apply the Gram-Schmidt process to the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix},$$

Which of the following expressions is used to compute $\mathbf{w}_4$?
Note that $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

(a) $\begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{12}{18}\begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix} - \frac{17}{27}\begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{7}{15}\begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$

(c) $\frac{5}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{12}{18}\begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix} + \frac{17}{27}\begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{7}{15}\begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{17}{27}\begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

11. (3 points) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ be a set of n-vectors. Suppose that the Gram-Schmidt process applies to this list yields, in order, the vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ where

$$\mathbf{w}_3 = \mathbf{w}_4 = \mathbf{0}$$

and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_5$ are all non-zero. Then a basis for $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5)$ is given by:

Select all that apply.

(a) $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$

(b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$

(c) $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{w}_5\}$

(d) $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_5\}$

(e) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$

12. (2 points) Let $A : \mathbf{R}^{51} \to \mathbf{R}^{51}$ be a linear transformation. Suppose that the null space of A is a plane. Fix $\mathbf{b} \in \mathbf{R}^{51}$. What could the set $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$ be?

(a) the empty set: there are no solutions

(b) a line

(c) a plane

(d) all of $\mathbf{R}^{51}$

(e) none of the other choices.

13. (3 points) Suppose that A is an **invertible** 3×3 matrix such that

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

What is the solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$?

Note that $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ *are orthogonal to each other.*

(a) $\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

(e) There is not enough information to determine this.

14. (2 points) **True or False:**
If two $m \times n$ matrices A and B have the same column space and same null space, i.e.

$$C(A) = C(B), \;\; N(A) = N(B),$$

then $A = B$ always.

15. (2 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ -1 & 3 \end{bmatrix}.$$

Suppose that we append two columns to the right of second column of the matrix A, obtaining a 3×4 matrix A'. Which of the following are the possible value(s) for the dimension of $N(A')$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

16. (3 points) Let M be a 5×4 matrix and $\mathbf{b} \in \mathbf{R}^5$ a vector. Assume that there is at least one solution $\mathbf{x} \in \mathbf{R}^4$ to the equation $M\mathbf{x} = \mathbf{b}$.

If A is an invertible 5×5 matrix, does $M\mathbf{y} = A\mathbf{b}$ have a solution $\mathbf{y} \in \mathbf{R}^4$?

(a) Yes: No matter what A is, there is some $\mathbf{y} \in \mathbf{R}^4$ so that $M\mathbf{y} = A\mathbf{b}$.
(b) Maybe: Depending on what A is, there may or may not be some $\mathbf{y} \in \mathbf{R}^4$ so that $M\mathbf{y} = A\mathbf{b}$.
(c) No: No matter what A is, there is no $\mathbf{y} \in \mathbf{R}^4$ so that $M\mathbf{y} = A\mathbf{b}$.

17. (3 points) Suppose that A is an **invertible** 3×3 matrix such that

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

What is the solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$?

Note that $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ *are orthogonal to each other.*

(a) $\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ (e) There is not enough information to determine this.

18. (3 points) Let A be an $m \times n$ matrix where $m > n$. Let B be the $m \times (m+n)$ matrix as follows:

$$B = \begin{bmatrix} A & I_m \end{bmatrix},$$

where the $m \times m$ identity matrix I_m is appended to the right of A. Which of the following statements are true? Select all that apply.

(a) $N(A) = N(B)$.
(b) $N(B)$ is n-dimensional.
(c) $N(B) = \{\mathbf{0}\}$.
(d) $C(A) = C(B)$.

19. (3 points) Suppose A is a 3×3 matrix, and that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent collection of vectors in $\mathbf{R}^3$ for which

- $\mathbf{u}$, $\mathbf{v}$ lie in the null space of A; and
- $\mathbf{w}$ lies in the column space of A.

Which of the following systems of equations must have at least one solution? Select all that apply.

(i) $A\mathbf{x} = \mathbf{0}$ (ii) $A\mathbf{x} = 2\mathbf{u} + 3\mathbf{v}$ (iii) $A\mathbf{x} = 3\mathbf{w}$ (iv) $A\mathbf{x} = \mathbf{u} + 3\mathbf{v} + 2\mathbf{w}$

20. (3 points) Suppose that A is a 3×3 matrix such that:

$$A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \qquad A\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix}, \qquad A\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

What statement describes the set of solutions to $A\mathbf{x} = \begin{bmatrix}4\\3\\1\end{bmatrix}$?

Note that $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ *and* $\begin{bmatrix}2\\1\\3\end{bmatrix}$ *are orthogonal to each other.*

(i) There is not enough information to choose a single answer here.
(ii) There are no solutions (i.e., empty set).
(iii) There is a unique solution (i.e., set consisting of exactly one point in $\mathbf{R}^3$)
(iv) There are infinitely many solutions, and graphically the solution set forms a line in $\mathbf{R}^3$.
(v) There are infinitely many solutions, and graphically the solution set forms a plane in $\mathbf{R}^3$.