Review

The <u>dot product</u> of n-vectors $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$,

$\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ is the scalar

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n.$$

<u>Careful</u>: Don't forget that $\vec{v} \cdot \vec{w}$ is a real number, <u>not</u> a vector.

If $\vec{v}$ and $\vec{w}$ are nonzero, then the angle θ between $\vec{v}$ and $\vec{w}$ is $0° \le \theta \le 180°$ such that
$(0 \le \theta \le \pi)$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}.$$

<u>Note</u>: $\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$, so

$$\|\vec{v}\|^2 = v_1^2 + \cdots + v_n^2 = \vec{v} \cdot \vec{v}$$

$\vec{v}$ and $\vec{w}$ are <u>perpendicular</u> when $\theta = 90° \; (\pi/2)$, or, equivalently, if $\vec{v} \cdot \vec{w} = 0$.

Problem 1

(a) Using that perpendicularity is governed by dot products being equal to 0, find a nonzero vector in $\mathbb{R}^3$ that is perpendicular to $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Then find another that is not a scalar multiple.

$\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is perpendicular to $\vec{v}$ if

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 2x - y + z = 0$$

Choose answers by inspection, e.g.

If $x = 0$, then $z = y$. So $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ works.

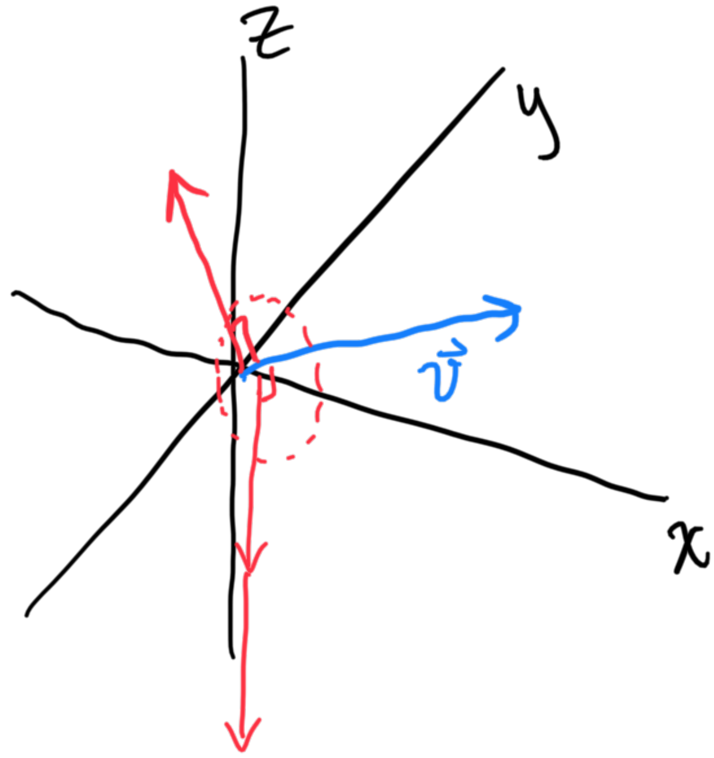
If $x = y = 1$, then $z = y - 2x = -1$, so $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ works.

(b) Find an equation in x, y, z that characterizes when $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. What does this collection of vectors look like:

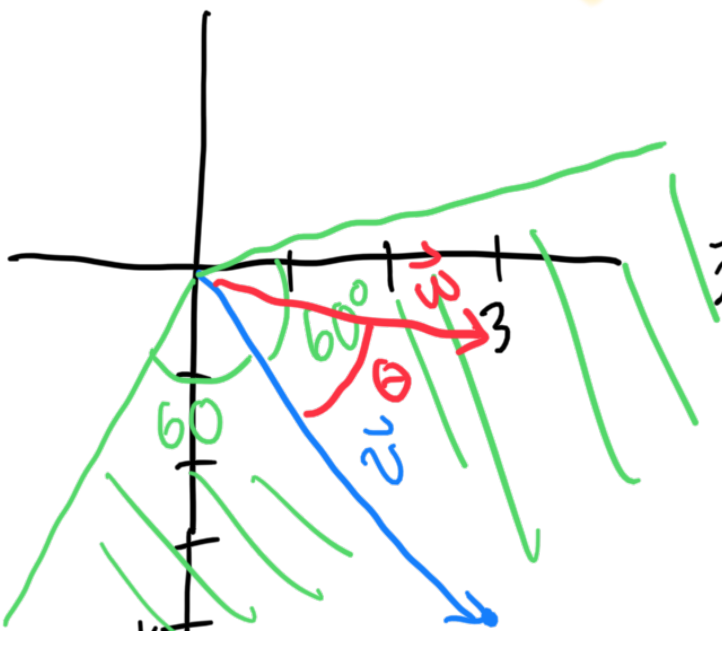
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 2x - y + z = 0$$

This equation determines a plane in $\mathbb{R}^3$.

Visualize:



(c) (Extra) What does the collection of nonzero vectors $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix}$ making an angle of at most $60°$ against $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ look like? Describe this region with a pair of conditions $ax^2 + bxy + cy^2 \geq 0$ and $y \leq (3/4)x$ (away from origin).



$0 \leq \theta \leq 60°$

$\frac{1}{2} = \cos 60° \leq \cos\theta \leq 1$

ignore

$\frac{1}{2} \leq \cos\theta$

$$\boxed{\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{3x-4y}{5\sqrt{x^2+y^2}}}$$

$$\vec{v}\cdot\vec{w} = \begin{bmatrix}3\\-4\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix} = 3x-4y$$

$$\|\vec{v}\| = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\|\vec{w}\| = \sqrt{x^2+y^2}$$

$$\frac{1}{2} \le \frac{3x-4y}{5\sqrt{x^2+y^2}}$$

$$\rightsquigarrow\ 5\sqrt{x^2+y^2} \le 2(3x-4y)$$

Square both sides and remember $0<3x-4y$

$$25(x^2+y^2) \le 4(3x-4y)^2$$

$$25x^2+25y^2 \le 4(9x^2-24xy+16y^2)$$

$$= 36x^2-96xy+64y^2$$

$$0 \leq 11x^2 - 96xy + 39y^2$$

$$4y \leq 3x$$

$$y \leq \frac{3}{4}x$$

Problem 2

(a) For $\vec{a} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$ show that

$$\vec{a} \cdot (\vec{b} - \vec{c}) = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}.$$

$$\vec{b} - \vec{c} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ -1 \end{bmatrix}$$

$$\vec{a} \cdot (\vec{b} - \vec{c}) = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 9 \\ -1 \end{bmatrix} = 4(-5) + (-2)9 + 3(-1)$$

$$= -20 - 18 - 3$$

$$= -41$$

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} = 4 - 10 - 6 = -12$$

$$\vec{a} \cdot \vec{c} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix} = 24 + 8 - 3 = 29$$

$$\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = -12 - 29 = -41$$

(b) Give an example of 2-vectors $\vec{a}, \vec{b}, \vec{c}$ for which

$(\vec{a}\cdot\vec{b})\vec{c} \neq (\vec{a}\cdot\vec{c})\vec{b}$

Remember: $\vec{a}\cdot\vec{b}$ and $\vec{a}\cdot\vec{c}$ are scalars!

One possibility: if $\vec{a}\cdot\vec{b}=0$ ($\vec{a}$ & $\vec{b}$ perpend.)
but $\vec{a}\cdot\vec{c}\neq 0$

e.g. $\vec{a}=\begin{bmatrix}1\\0\end{bmatrix}, \vec{b}=\begin{bmatrix}0\\1\end{bmatrix}, \vec{c}=\begin{bmatrix}1\\1\end{bmatrix}$

Then $\vec{a}\cdot\vec{b}=\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix}=0$

$\vec{a}\cdot\vec{c}=\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}=1$

So $(\vec{a}\cdot\vec{b})\vec{c}=0\cdot\vec{c}=\begin{bmatrix}0\\0\end{bmatrix}$

$(\vec{a}\cdot\vec{c})\vec{b}=1\cdot\vec{b}=\begin{bmatrix}0\\1\end{bmatrix}$

(c) (Extra) Explain in terms of variables why

$\vec{v}\cdot(\vec{w}_1+\vec{w}_2)=\vec{v}\cdot\vec{w}_1+\vec{v}\cdot\vec{w}_2$ for $\vec{v}=\begin{bmatrix}x\\y\\z\end{bmatrix}, \vec{w}_1=\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}, \vec{w}_2=\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}$.

Why does it follow that

$(\vec{v}_1+\vec{v}_2)\cdot(\vec{w}_1+\vec{w}_2)=\vec{v}_1\cdot\vec{w}_1+\vec{v}_2\cdot\vec{w}_1+\vec{v}_1\cdot\vec{w}_2+\vec{v}_2\cdot w_2$?

Does this work for n-vectors for any n?

(d) For n-vectors $\vec{w}_1$ and $\vec{w}_2$, verify that

$\|\vec{w}_1+\vec{w}_2\|^2=\|\vec{w}_1\|^2+2(\vec{w}_1\cdot\vec{w}_2)+\|\vec{w}_2\|^2$

by using the relation $\|\vec{w}\|^2=\vec{w}\cdot\vec{w}$ and general properties of dot products as stated in (c).

$$\|\vec{w}_1+\vec{w}_2\|^2 = (\vec{w}_1+\vec{w}_2)\cdot(\vec{w}_1+\vec{w}_2)$$

$$\overset{\text{by part(c)}}{=} \vec{w}_1\cdot\vec{w}_1+\vec{w}_2\cdot\vec{w}_1+\vec{w}_1\cdot\vec{w}_2+\vec{w}_2\cdot\vec{w}_2$$

$$\boxed{\vec{w}_1\cdot\vec{w}_2=\vec{w}_2\cdot\vec{w}_1}$$

$$= \|\vec{w}_1\|^2+\vec{w}_2\cdot\vec{w}_1+\vec{w}_1\cdot\vec{w}_2+\|\vec{w}_2\|^2$$

$$= \|\vec{w}_1\|^2+2(\vec{w}_1\cdot\vec{w}_2)+\|\vec{w}_2\|^2.$$

Problem 3 Correlation coefficients

Given n points $(x_1,y_1),\ldots,(x_n,y_n)$ in $\mathbb{R}^2$ (satisfying some assumptions)

let $\vec{X}=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{Y}=\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

The <u>correlation coefficient</u> is

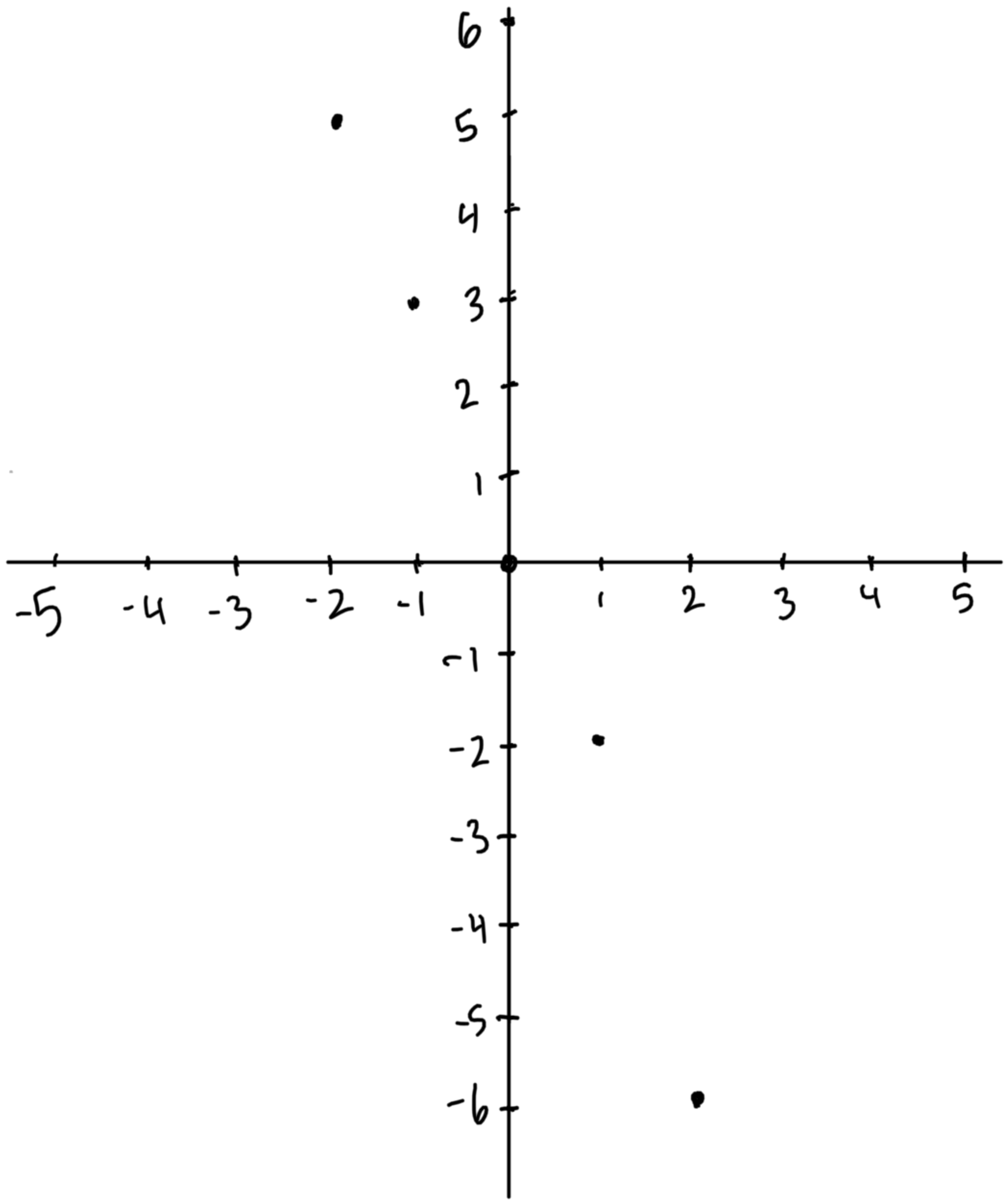
$$r=\frac{\vec{X}\cdot\vec{Y}}{\|\vec{X}\|\|\vec{Y}\|}$$ = cosine of the angle between $\vec{X}$ and $\vec{Y}$.

In particular, $-1\le r\le 1$. If the points are close to being on a line, then r will be close to -1 or 1; if not, r will be close to zero.

Consider the collection of data points:

$(-2,5), (-1,3), (0,0), (1,-2), (2,-6).$

(a) Plot the points and see if they look close to a line.



(b) Compute the correlation coefficient exactly.
Using a calculator, approximate it to 3 decimal digits to see if its nearness to ± 1 fits well with the visual quality of fit of the line to the data plot in (a).

$(-2,5), (-1,3), (0,0), (1,-2), (2,-6)$.

$$\vec{x} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ -2 \\ -6 \end{bmatrix}$$

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 3 \\ 0 \\ -2 \\ -6 \end{bmatrix} = -10 - 3 + 0 - 2 - 12$$

$$= -27$$

$$\|\vec{x}\| = \sqrt{(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2}$$
$$= \sqrt{4 + 1 + 1 + 4}$$
$$= \sqrt{10}$$

$$\|\vec{y}\| = \sqrt{5^2 + 3^2 + 0^2 + (-2)^2 + (-6)^2}$$
$$= \sqrt{25 + 9 + 4 + 36}$$
$$= \sqrt{74}$$

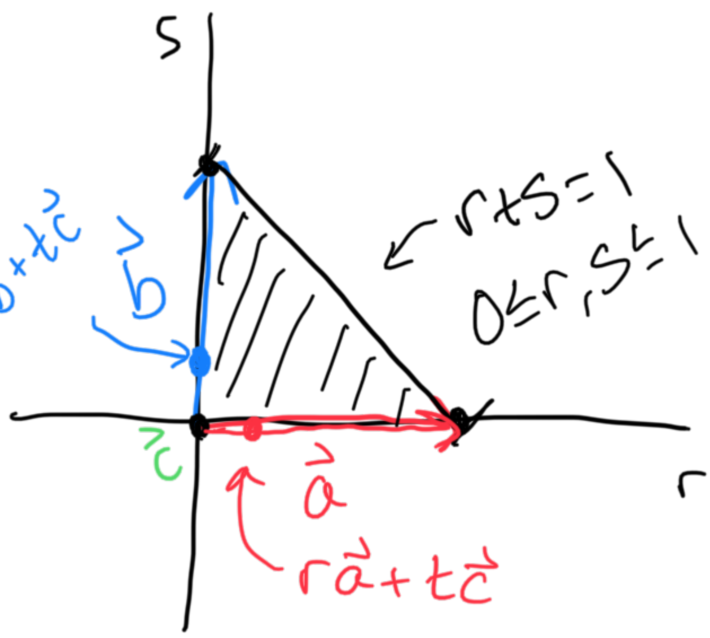
$$r = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \frac{-27}{\sqrt{10}\sqrt{74}} \approx -0.992$$

very close to -1!

Fits with picture from pt (a) b/c the points were close to a line with negative slope.

Problem 4 (Extra)

(a) For the 2-vectors $\vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b} = \begin{bmatrix}0\\1\end{bmatrix}$, $\vec{c} = \begin{bmatrix}0\\0\end{bmatrix}$, describe the set of all possible vectors $r\vec{a} + s\vec{b} + t\vec{c}$, where $r+s+t=1$ with $0 \le r,s,t \le 1$. Which points correspond to $t=0$? $s=0$? $r=0$?



If $t=0$,

$$r\vec{a} + s\vec{b} = \begin{bmatrix}r\\s\end{bmatrix}$$

$$r\begin{bmatrix}1\\0\end{bmatrix} + s\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}r\\0\end{bmatrix} + \begin{bmatrix}0\\s\end{bmatrix}$$

where $r+s=1$ and $0 \le r,s \le 1$

If $s=0$, $r\vec{a} + t\vec{c} = r\vec{a} = \begin{bmatrix}r\\0\end{bmatrix}$ $\quad r+t=1$

$0 \le r \le 1$

If $r=0$, $s\vec{b} + t\vec{c} = s\vec{b} = \begin{bmatrix}0\\s\end{bmatrix}$ $\quad 0 \le s \le 1$

(b) Try the same thing using the 3-vectors $\vec{a} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{b} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{c} = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

(c) Can you explain why your description in (a) applies to any three 2-vectors $\vec{a}, \vec{b}, \vec{c}$ not on a common line?

(d) Is there a version for a triple of 3-vectors not all on a common line? Why does it work?