Problem 1: Distributing money

Five people sit around a circular table, each with some amount of money in their wallets. Simultaneously, each person divides their money into three equal thirds and gives one third to the person on the left, one third to the person on the right, and keeps the remaining third for themselves.

Write down a 5×5 matrix A that describes this operation. In other words, label the people as $\#1$, $\#2$, etc. going around the table in some way (beginning with some choice of "first" person), so person $\#5$ is sitting next to person $\#1$. Then A should take as input a 5-vector $\mathbf{x}$ whose ith entry x_i is the amount of money the ith person has *before* doing this, and $A\mathbf{x}$ should be the 5-vector whose ith entry is the amount of money the ith person has *after* the operation. (The answer is a Markov matrix!)

Solution: If the "input" to the above operation is the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$, then the "output" should be $\begin{bmatrix} \frac{1}{3}x_5 + \frac{1}{3}x_1 + \frac{1}{3}x_2 \\ \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \\ \frac{1}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 \\ \frac{1}{3}x_3 + \frac{1}{3}x_4 + \frac{1}{3}x_5 \\ \frac{1}{3}x_4 + \frac{1}{3}x_5 + \frac{1}{3}x_1 \end{bmatrix}$. In terms of matrix-vector products, we may write this output vector as the product

$$\begin{bmatrix} \frac{1}{3}x_5 + \frac{1}{3}x_1 + \frac{1}{3}x_2 \\ \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \\ \frac{1}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 \\ \frac{1}{3}x_3 + \frac{1}{3}x_4 + \frac{1}{3}x_5 \\ \frac{1}{3}x_4 + \frac{1}{3}x_5 + \frac{1}{3}x_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Thus, the desired matrix is $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

Problem 2: Migration

Assume there are 3 cities A, B, and C. Assume that in every given year

- 80% of the residents of A stay in A, while 10% move to B and 10% move to C.
- 70% of the residents of B stay in B, while 10% move to A and 20% move to C.
- 60% of the residents of C stay in C, while 10% move to A and 30% move to B.

(We disregard births and deaths, and people moving to or from other locations.)

(a) Write down the 3×3 Markov matrix for this process (where we use a "population vector" in $\mathbf{R}^3$ whose first entry is the population of A, second entry is the population of B, and third entry is the population of C). Why must its columns add up to 1, even if the given percentages for movement among the cities were changed?

(b) Assume that initially each of the cities has 10,000 inhabitants. How many inhabitants does each city have after 1 year?

(c) How many inhabitants does city B have after 2 years?

(d) Now assume that of the 80% of residents who stay in city A, 5% die every year (so 75% remain). Write down the matrix for this new process. Do its columns still add up to 1? If not, where does your argument for column sums in (a) break down in this new setting?

Solution:

(a) The Markov matrix is
$$M = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.2 & 0.6 \end{bmatrix}.$$
The sum of its second column entries accounts for all former residents of city B, so this must add up to $100\% = 1$ even if the percentages for moving from city B were to change. The same reasoning applies to the first and third columns (using cities A and C respectively).

(b) We can calculate this population vector as
$$M \begin{bmatrix} 10000 \\ 10000 \\ 10000 \end{bmatrix} = \begin{bmatrix} 10000 \\ 11000 \\ 9000 \end{bmatrix}.$$

(c) We can calculate this population vector as
$$M \begin{bmatrix} 10000 \\ 11000 \\ 9000 \end{bmatrix},$$
but we only need the second entry, which is 11400.

(d) The new Markov matrix would be
$$M' = \begin{bmatrix} 0.75 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.2 & 0.6 \end{bmatrix}.$$
The first column does not add up to 1 any more because not all citizens of A will survive to the next year: the argument in (a) for column sums being 100% assumes that everyone in a given year contributes to the movement among cities next year (so no new people are added to or removed from the process); this doesn't work when people may exit the process (such as through death).

Problem 3: Fibonacci Numbers (Extra)

The *Fibonacci numbers* are the sequence of numbers $a_1, a_2, a_3, \ldots$ obtained given by starting with initial values $a_1 = 1$ and $a_2 = 1$ with each successive term being the sum of the two preceding terms; i.e., $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$.

(a) Write down the first 8 Fibonacci numbers.

(b) Fnd a (non-Markov!) 2×2 matrix M for which $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = M \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ for every $n \geq 1$.

(c) Obtain an expression for a_{n+4} in terms of a_{n+1} and a_n in two ways: first do it by directly feeding the defining formula for the sequence (each term is the sum of the two preceding terms) into itself a few times, and then do it by computing M^3. Check that you get the same expression via each method. (The second method is the more useful approach when a_{n+4} is replaced with a_{n+k} for k much much bigger than 4.)

Using techniques from later in the course, one can study powers of M to find a slick explicit formula for a_n in terms of a_1 and a_2 (which are just numbers) for any choice of those two initial values.

Solution:

(a) The Fibonacci numbers are given by 1,1,2,3,5,8,13,21,$\ldots$.

(b) We have $a_{n+2} = a_{n+1} + a_n$, so $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} a_{n+1} + a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$. Hence, the matrix M is

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

(c) From the definition using the preceding two terms,

$$a_{n+4} = a_{n+3} + a_{n+2} = (a_{n+2} + a_{n+1}) + a_{n+2} = 2a_{n+2} + a_{n+1} = 2(a_{n+1} + a_n) + a_{n+1} = 3a_{n+1} + 2a_n.$$

Alternatively, we calculate $M^3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$. Since we have

$$\begin{bmatrix} a_{n+4} \\ a_{n+3} \end{bmatrix} = M^3 \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix},$$

we read off $a_{n+4} = 3a_{n+1} + 2a_n$.