Solutions to Math 51 Quiz 4

1. (10 points) Find the points on $x^2 - xy + y^2 = 3$ that are closest and farthest from the point $(4, -4)$. Make sure to fully justify your answer.

We are trying to minimize $\sqrt{(x-4)^2 + (y+4)^2}$ subject to $x^2 - xy + y^2 = 3$. As we saw in Lecture 12, it is much simpler to minimize $f(x, y) = (x-4)^2 + (y+4)^2$. Note that

$$\nabla f = \begin{bmatrix} 2x - 8 \\ 2y + 8 \end{bmatrix} \quad \text{and} \quad \nabla g = \begin{bmatrix} 2x - y \\ -x + 2y \end{bmatrix}.$$

First, we check whether ∇g vanishes – it does vanish at $(0, 0)$, which is not on the constraint curve.

Next, we check the Lagrange multiplier equation:

$$\begin{bmatrix} 2x - 8 \\ 2y + 8 \end{bmatrix} = \lambda \begin{bmatrix} 2x - y \\ -x + 2y \end{bmatrix} \quad \Rightarrow \quad \begin{cases} 2x - 8 = \lambda(2x - y) \\ 2y + 8 = \lambda(-x + 2y) \end{cases}$$

Adding the two equations, we get

$$2x + 2y = \lambda(x + y),$$

and so, $(x + y)(\lambda - 2) = 0$. Thus, either $x + y = 0$ or $\lambda - 2 = 0$.

- $x + y = 0$: Substituting $y = -x$ into the constraint curve, we see that $3x^2 = 3$, and so, $x = \pm 1$. Thus, there are two points of interest, $(-1, 1)$ and $(1, -1)$.
- $\lambda - 2 = 0$: If $\lambda = 2$, the two equations from the Lagrange multiplier equation both reduce to $y = x + 4$. Plugging this into the constraint curve, we get $x^2 - x(x+4) + (x+4)^2 = x^2 + 4x + 16 = 3$, which has no real solutions.

We see that $f(-1, 1) = 50$ and $f(1, -1) = 18$. Therefore, $(-1, 1)$ is the farthest point ($5\sqrt{2}$ units away) and $(1, -1)$ is the closest point ($3\sqrt{2}$ units away) from $(4, -4)$.

2. (2 points) **True or False:** Consider the sphere S defined by $x^2 + y^2 + z^2 = 51$, and let $\mathbf{v}$ be a fixed nonzero vector in $\mathbb{R}^3$. Then, there are exactly two points on S for which the tangent plane at the points are perpendicular to $\mathbf{v}$.

This statement is **TRUE**. The sphere is the level set of $f(x, y, z) = x^2 + y^2 + z^2$ at the level 51. If the tangent plane at the point (a, b, c) is perpendicular to $\mathbf{v}$, then

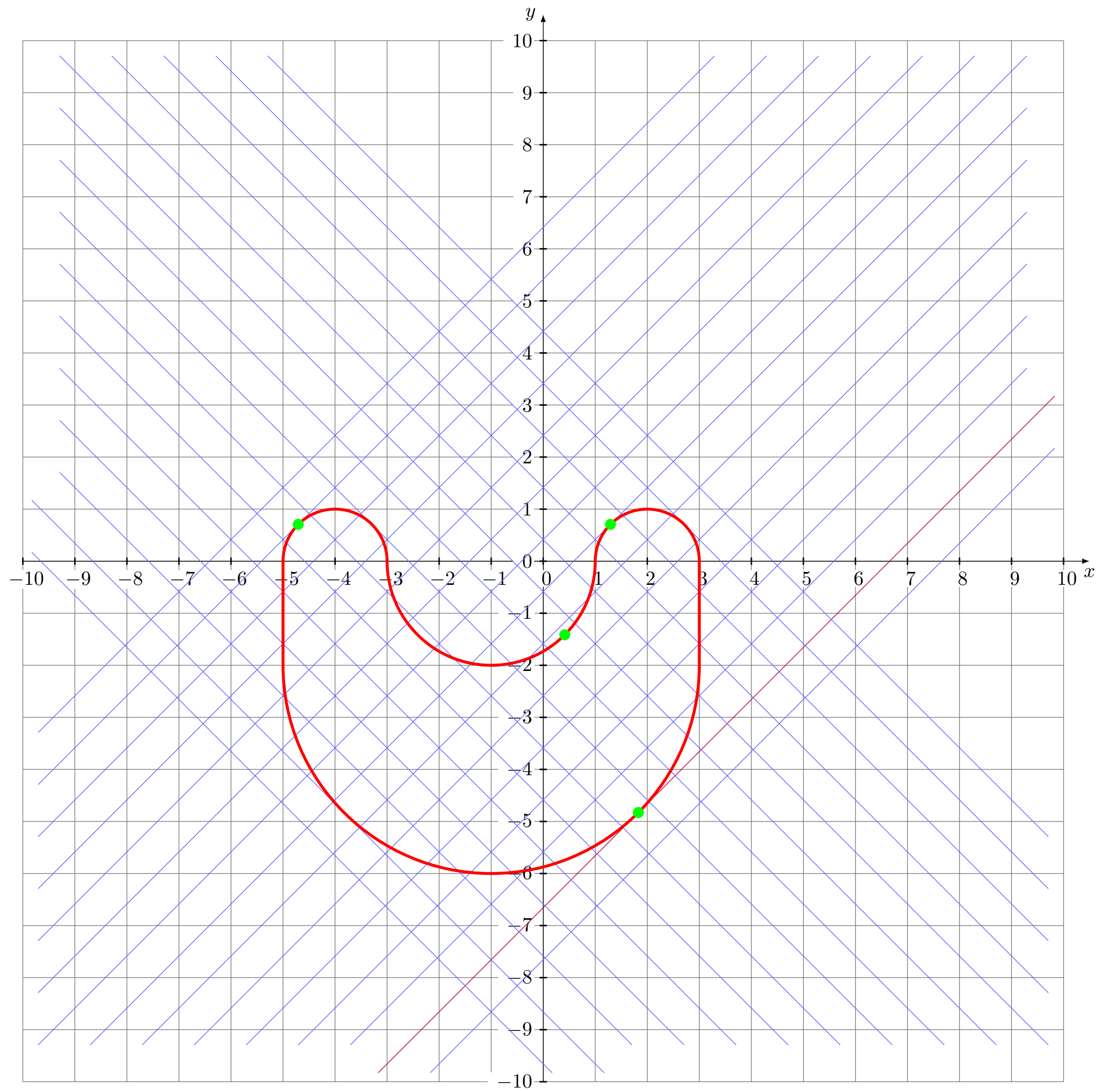
$$\nabla f(a, b, c) = \begin{bmatrix} 2a \\ 2b \\ 2c \end{bmatrix} = \alpha \mathbf{v}.$$

There are exactly two points on S that satisfy this: $\sqrt{51}\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ and $-\sqrt{51}\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

3. (2 points) **True or False:** Suppose you are trying to find a local minimum of some function $f : \mathbb{R}^2 \to \mathbb{R}$ and you run gradient descent from some point (a, b) with $t = -0.25$. Suppose that gradient descent converges to a point (α, β). Then, f has a local minimum at (α, β).

This statement is **FALSE**. We saw a counterexample in Example 9 from Lecture 11 – if we start gradient descent at $(1, 0)$ for $f(x, y) = x^2 - y^2$, we saw that it converged to $(0, 0)$, which is a saddle point.

4. (3 points) The graph of $g(x, y) = 51$ is drawn in the xy-coordinate system below for some function $g : \mathbb{R}^2 \to \mathbb{R}$. Consider the function $f(x, y) = x - y$. We are concerned with the maximum value of $f(x, y)$ on the curve $g(x, y) = 51$. For your convenience, some lines of slopes ± 1 are provided.



The maximum value of $f(x, y)$ on the curve $g(x, y) = 51$ is between which two consecutive integers?

We are trying to maximize $c = x - y$; this is equivalent to $y = x - c$, which has y-intercept $-c$. Thus, we are trying to find the line with slope 1 passing through the red curve ($g(x, y) = 51$) that has the *lowest* y-intercept (maximizing c is equivalent to minimizing $-c$, the y-intercept). This line is shown in purple in the diagram above; we can see that $-c$ is between -7 and -6. Thus, the maximum is between 6 and 7.

We can also take a Lagrange multiplier approach. Since $\nabla f = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we are looking for points on $g(x, y) = 51$, the red curve, which have a scalar multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. In other words, we are looking for points on the red curve that have tangent lines perpendicular to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. There are four such points, marked in green on the diagram above, and by inspection, $f(x, y) = x - y$ achieves its maximum at the bottom right green point, and the value is between 6 and 7.

5. (3 points) Suppose $f(x,y)$ attains its minimum value on the set $g(x,y)=0$ at the point (a,b), and that $\nabla g(a,b)\neq \mathbf{0}$. Furthermore, suppose the curve $y=x^3+3x+1$ passes through (a,b) and is tangent to $g(x,y)=0$ at (a,b). Which of the following could be the gradient of f at (a,b)?

a) $\begin{bmatrix}1\\0\end{bmatrix}$ b) $\begin{bmatrix}0\\1\end{bmatrix}$ c) $\begin{bmatrix}-1\\1\end{bmatrix}$ d) $\begin{bmatrix}-1\\-1\end{bmatrix}$ e) $\begin{bmatrix}5\\-1\end{bmatrix}$ f) $\begin{bmatrix}5\\1\end{bmatrix}$

$\nabla f(a,b)$ must be perpedicular to the tangent line to $y=x^3+3x+1$ at (a,b). $y=x^3+3x+1$ is the level curve of $h(x,y)=y-x^3-3x-1$ at the level 0. Note that

$$\nabla h(a,b)=\begin{bmatrix}-3a^2-3\\1\end{bmatrix}.$$

The only choice that can be a scalar multiple of $\nabla h(a,b)$ is $\begin{bmatrix}5\\-1\end{bmatrix}$.