Problem 1: Determining the nature of a span

For each collection of 3-vectors, determine whether its span is a point, a line, a plane, or all of $\mathbf{R}^3$. Give a basis of the span in each case. (Keep in mind that if a vector in the collection is a linear combination of others then it can be dropped without affecting the span.)

(a) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}$ (b) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}$ (c) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}$

Problem 2: More recognizing and describing linear subspaces

Which of the following subsets S of $\mathbf{R}^3$ are linear subspaces? If a set S is a linear subspace, exhibit it as a span. If it is not a linear subspace, describe it geometrically and explain why it is not a linear subspace.

(a) The set S_1 of points (x, y, z) in $\mathbf{R}^3$ with both $z = x + 2y$ and $z = 5x$.

(b) The set S_2 of points (x, y, z) in $\mathbf{R}^3$ with either $z = x + 2y$ or $z = 5x$.

(c) The set S_3 of points (x, y, z) in $\mathbf{R}^3$ of the form $t\begin{bmatrix}1\\2\\3\end{bmatrix} + t'\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}3\\3\\3\end{bmatrix}$ for some scalars t and t' (which are allowed to be anything, depending on the point (x, y, z)).

Problem 3: Multiple descriptions as a span (Extra)

Let $\mathbf{v}, \mathbf{w}$ be two vectors in $\mathbf{R}^{12}$. Show that $\mathrm{span}(\mathbf{v}, \mathbf{w}) = \mathrm{span}(\mathbf{v} + \mathbf{w}, \mathbf{v} - \mathbf{w})$. (Hint: You can show that two sets S and T are equal in two steps: everything belong to S also belong to T, and everything belonging to T also belongs to S.)

Problem 4: Linear subspaces and orthogonality (computations)

Let V be the set of vectors in $\mathbf{R}^4$ orthogonal to both $\begin{bmatrix}1\\0\\4\\2\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$. Find a pair of vectors that span V, so it is a linear subspace.

Problem 5: An orthogonal basis

Let V be the set of vectors $\mathbf{v} \in \mathbf{R}^3$ satisfying $\mathbf{v} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix}2\\3\\4\end{bmatrix}$ (this says that both of these explicit 3-vectors have the same projection onto $\mathbf{v}$, or in other words make the same "shadow" onto the line spanned by $\mathbf{v}$).

(a) Express V as the collection of 3-vectors orthogonal to a single nonzero 3-vector.

(b) By fiddling with orthogonality equations, build an orthogonal basis of V. There are many possible answers.

(c) Use your answer to (b) to give an orthonormal basis for V.

