Problem 1: Determining linear independence

For each of the following collections of vectors, determine if it is linearly independent (think in terms of expressing a vector as a linear combination of others or studying if "$\sum c_j \mathbf{v}_j = \mathbf{0}$" can happen with some nonzero c_j, not by using Gram–Schmidt), and give a basis of their span in each case.

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix}$.

(b) $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

(c) $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}$.

(d) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Solution:

(a) They satisfy the linear relation $\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0}$. To find such a relation, note that all of these three vectors are nonzero and are not scalar multiples of each other. Hence, if any one of them were to be a linear combination of the others then all coefficients in the linear combination would have to be nonzero and so each of them would be a linear combination of the others. Thus, we lose nothing by just focusing on trying to write $\mathbf{u}$ in terms of $\mathbf{v}$ and $\mathbf{w}$ (i.e., if any linear dependence relation exists among the three then this approach must succeed). Trying to write $\mathbf{u} = x\mathbf{v} + y\mathbf{w}$ is a system of 3 equations in 2 unknowns, and solving two of those equations gives one solution that is also checked to satisfy the third equation, leading to the expression $2\mathbf{v} - \mathbf{w}$ for $\mathbf{u}$, hence the linear relation mentioned at the start.

A basis for the span is given by any two of the three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ since none of them are scalar multiples of another and each can be written as a linear combination of the other two (as can be seen from their linear relation $\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0}$).

(b) These are linearly independent. If we express the condition $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ in terms of scalar equations we obtain $\begin{cases} c_2 + 2c_3 = 0 \\ c_1 - c_3 = 0 \\ c_1 + 5c_2 + c_3 = 0 \end{cases}$ from where we immediately see $c_3 = c_1$ and $c_2 = -2c_1$. Substituting these into the last equation gives $c_1 - 10c_1 + c_1 = 0$ so $c_1 = 0$ and hence $c_2 = c_3 = 0$. This confirms linear independence. (Alternatively, since the vectors are all nonzero and none is a scalar multiple of another, if there is any linear dependence relation among the vectors then all coefficients must be nonzero and so we could write each of them as a linear combination of the other two. Hence, it is equivalent to show that the situation $\mathbf{u} = x\mathbf{v} + y\mathbf{w}$ is impossible: this is a system of 3 equations in x and y, and one can check that it has no simultaneous solution.)

Since these vectors are linearly independent, they are a basis of their span.

(c) This is three vectors in $\mathbf{R}^2$, so they cannot be linearly independent since $\dim \mathbf{R}^2 = 2$. But the first two vectors are nonzero and neither is a scalar multiple of the other, so their span is 2-dimensional. Being inside $\mathbf{R}^2$, they span $\mathbf{R}^2$ and so are a basis of that span. In particular, the third vector must be a linear combination of the first two. (The specific linear combination is found by solving the pair of equations $11 = a + 3b$ and $6 = -2a + b$, which is $(a, b) = (-1, 4)$.)

In fact, any two of the three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ will form a basis for the span (which is $\mathbf{R}^2$) since all are nonzero and none of them is a scalar multiple of any of the others.

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(d) These are linearly independent. If we express $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} + c_4\mathbf{x} = \mathbf{0}$ in terms of 4 scalar equations via each vector entry, we get $\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 + c_2 + c_4 = 0 \\ c_1 + c_3 + c_4 = 0 \\ c_2 + c_3 + c_4 = 0 \end{cases}$

Comparing the equations pairwise, we conclude that $c_1 = c_2 = c_3 = c_4$ (e.g., the difference of the first and third gives $c_2 = c_4$, the difference of the first and second gives $c_3 = c_4$, and the difference of the third and fourth gives $c_1 = c_2$, so all c_i's are equal to each other). Now substituting this into any of the equations gives that all c_i's vanish.

Consequently, these four vectors form a basis for their span, which must be $\mathbf{R}^4$ for dimension reasons ($\mathbf{R}^n$ has only itself as an n-dimensional subspace).

Problem 2: Computing the Gram–Schmidt process

Run the Gram–Schmidt process on the following collection of vectors, and obtain an orthogonal basis for its span.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -4 \\ 5 \\ -2 \\ -3 \end{bmatrix}.$$

Using the outcome of your calculations, also compute the dimension of the span and if you encounter any $\mathbf{w}_i$ equal to $\mathbf{0}$ then use this to produce a linear dependence relation among the $\mathbf{v}_j$'s (in which case, and as a safety check, confirm such a linear dependence relation by direct computation once you have found one).

As a safety check on your work, make sure at each step that each $\mathbf{w}_i$ is orthogonal to the previous $\mathbf{w}_j$'s. (The $\mathbf{v}_i$'s have been designed so that you only have to work with integers throughout, and in particular the $\mathbf{w}_i$'s have integer entries. If you find yourself at any step grappling with things like $-5/3$ or $11/4$ and so on, you have made a mistake.)

Solution: We start with

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

The second vector is given by

$$\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{w}_1} \mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 = \mathbf{v}_2 - \frac{-6}{6}\mathbf{w}_1 = \mathbf{v}_2 + \mathbf{w}_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

which (as a safety check) is directly checked to be orthogonal to $\mathbf{w}_1$. The third vector is

$$\begin{aligned} \mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_1} \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_2} \mathbf{v}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \mathbf{w}_2 \\ &= \mathbf{v}_3 - \frac{6}{6}\mathbf{w}_1 - \frac{12}{12}\mathbf{w}_2 \\ &= \mathbf{v}_3 - \mathbf{w}_1 - \mathbf{w}_2 \\ &= \begin{bmatrix} -2 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

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This means that $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In fact, the vanishing of $\mathbf{w}_3$ gives that $\mathbf{v}_3 = \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{w}_1) = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_1) = 2\mathbf{v}_1 + \mathbf{v}_2$ (as can also be confirmed directly once we have discovered it).

For the last step, we disregard the vanishing $\mathbf{w}_3$ and compute

$$\begin{aligned}
\mathbf{w}_4 = \mathbf{v}_4 - \mathbf{Proj}_{\mathbf{w}_1}\mathbf{v}_4 - \mathbf{Proj}_{\mathbf{w}_2}\mathbf{v}_4 = \mathbf{v}_4 - \frac{\mathbf{v}_4\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_4\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\mathbf{w}_2 &= \mathbf{v}_4 - \frac{-12}{6}\mathbf{w}_1 - \frac{12}{12}\mathbf{w}_2 \\
&= \mathbf{v}_4 + 2\mathbf{w}_1 - \mathbf{w}_2 \\
&= \begin{bmatrix} -4 \\ 5 \\ -2 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 4 \\ -1 \\ 0 \end{bmatrix}.
\end{aligned}$$

This can be directly checked to be orthogonal to $\mathbf{w}_1$ and $\mathbf{w}_2$ (as a safety check).

So the nonzero $\mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_4$ are an orthogonal basis for the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$, and hence this span is therefore 3-dimensional.

Problem 3: Determining independence with vector algebra (Extra)

If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbf{R}^{1000}$ are linearly independent, show that $\mathbf{u}+\mathbf{v}, 2\mathbf{u}+\mathbf{v}+\mathbf{w}, -\mathbf{u}+\mathbf{v}+\mathbf{w}$ are linearly independent. (Hint: don't think in terms of vector entries! Think in terms of the formulation of linear independence as: "$\sum c_i\mathbf{v}_i = \mathbf{0}$ implies all c_i vanish".)

Solution: Assuming $c_1(\mathbf{u}+\mathbf{v}) + c_2(2\mathbf{u}+\mathbf{v}+\mathbf{w}) + c_3(-\mathbf{u}+\mathbf{v}+\mathbf{w}) = \mathbf{0}$, we want to show that the c_i's all vanish. Let's collect common terms to rewrite this as

$$(c_1 + 2c_2 - c_3)\mathbf{u} + (c_1 + c_2 + c_3)\mathbf{v} + (c_2 + c_3)\mathbf{w} = \mathbf{0}.$$

But $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, so such vanishing is equivalent to the vanishing of all three coefficients. In other words, this amounts to the system of three simulataneous equations

$$c_1 + 2c_2 - c_3 = 0, \;\; c_1 + c_2 + c_3 = 0, \;\; c_2 + c_3 = 0.$$

The last equation says $c_3 = -c_2$, so substituting this into each of the first two equations turns those into the conditions

$$c_1 + 2c_2 + c_2 = 0, \;\; c_1 + c_2 - c_2 = 0,$$

or equivalently

$$c_1 + 3c_2 = 0, \;\; c_1 = 0.$$

The second of these says $c_1 = 0$, so then the first of these says $c_2 = 0$, so also $c_3 = -c_2 = 0$. Hence, all c_i's must vanish, which establishes the desired linear independence.

(This particular example was designed so that one could attack the system of 3 equations in 3 unknowns by bare hands. For a more complicated situation this wouldn't have been feasible. We will see soon how to systematically attack the problem of solving large linear systems by using results in matrix algebra.)

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