Last time:

- quadratic forms & symmetric matrices
- column and null spaces
- solving systems of linear equations

Today:

- LU decomposition
- QR decomposition

Goal of decompositions: write any matrix as a product of other matrices that are easier to work with (e.g. easily invertible)

Problem 1: LU-decomposition

Let $A = \begin{bmatrix} 12 & 9 & 3 \\ -4 & 1 & 7 \\ 4 & 3 & 2 \end{bmatrix}$ and $L = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Verify that $LU = A$, so this is an LU-decomposition of A.

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Let $\mathbf{b} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$. Find all solutions to $L\mathbf{y} = \mathbf{b}$. (You should get that $\mathbf{y}_0 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is the only solution.)

(c) Find all solutions to $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b}$ as in (b). (Hint: This means solving $LU\mathbf{x} = \mathbf{b}$, which is the same as $U\mathbf{x} = \mathbf{y}_0$. Why?)

$$U = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2: QR-decomposition

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 5 \\ 1 & 5 & 3 \end{bmatrix}$, and define $\mathbf{v}_i$ to be the ith column of A.

(a) Apply the Gram–Schmidt process to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The output vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ that you obtain should all be nonzero, and as a check on your work make sure that they are pairwise orthogonal.

(b) Examine your calculations from (a) to express each $\mathbf{v}_i$ as a linear combination of the orthogonal basis of $\mathbf{w}_j$'s. (This should be found from the work already done in (a); do *not* directly compute the projections of $\mathbf{v}_i$ onto each $\mathbf{w}_j$, as that would be defeating the point of the work in (a).) Then compute the unit vectors $\mathbf{w}'_j = \mathbf{w}_j/\|\mathbf{w}_j\|$ and express $\mathbf{v}_i$ as a linear combination of the $\mathbf{w}'_j$'s.

(c) Use (b) to find a decomposition $A = QR$ where Q is an orthogonal matrix and R is an upper triangular matrix. Check your answer is correct by computing the product QR of the Q and R that you find.

(d) Use (c) to find A^{-1} as an explicit 3×3 matrix (with entries that are fractions with denominator that is a factor of 10, no $\sqrt{5}$ anywhere), and check that its product against A on the left or the right is equal to I_3; it is fine to compute just one of those products.

Hint: when computing R^{-1}, you may find it convenient to first extract $\sqrt{5}$ as a factor from every entry of R (i.e., write $R = \sqrt{5}R'$ for an upper triangular matrix R', so $R^{-1} = (1/\sqrt{5}){R'}^{-1}$; it is easier to find ${R'}^{-1}$.)

Problem 1: LU-decomposition

Let $A = \begin{bmatrix} 12 & 9 & 3 \\ -4 & 1 & 7 \\ 4 & 3 & 2 \end{bmatrix}$ and $L = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Verify that $LU = A$, so this is an LU-decomposition of A.

(b) Let $\mathbf{b} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$. Find all solutions to $L\mathbf{y} = \mathbf{b}$. (You should get that $\mathbf{y}_0 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is the only solution.)

(c) Find all solutions to $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b}$ as in (b). (Hint: This means solving $LU\mathbf{x} = \mathbf{b}$, which is the same as $U\mathbf{x} = \mathbf{y}_0$. Why?)

Problem 2: QR-decomposition

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 5 \\ 1 & 5 & 3 \end{bmatrix}$, and define $\mathbf{v}_i$ to be the ith column of A.

(a) Apply the Gram–Schmidt process to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The output vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ that you obtain should all be nonzero, and as a check on your work make sure that they are pairwise orthogonal.

(b) Examine your calculations from (a) to express each $\mathbf{v}_i$ as a linear combination of the orthogonal basis of $\mathbf{w}_j$'s. (This should be found from the work already done in (a); do *not* directly compute the projections of $\mathbf{v}_i$ onto each $\mathbf{w}_j$, as that would be defeating the point of the work in (a).) Then compute the unit vectors $\mathbf{w}'_j = \mathbf{w}_j/\|\mathbf{w}_j\|$ and express $\mathbf{v}_i$ as a linear combination of the $\mathbf{w}'_j$'s.

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