Last time

- · gradients
- · gradient is perpendicular to tangent line/plane of level set

Today:

- · Optimization using Lagrange multipliers
- · linear functions and matrices
- · the derivative motrix

Review: Lagrange multipliers

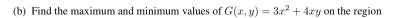


(b) The quadratic formula allows you to solve for y in terms of x on the curve: $y(x)=(x\pm\sqrt{x^2-4(x^2-3)})/2=(x\pm\sqrt{12-3x^2})/2$ (with $|x|\leq 2$ so that the square root makes sense). Hence, we could instead try to find the extreme values for $f(x,y(x))=x\cdot y(x)=(x^2\pm x\sqrt{12-3x^2})/2$ for $-2\leq x\leq 2$ via single-variable calculus. Is that more or less appetizing than the method in (a)?

Problem 2: Optimization review (what technique(s) would you use?)

(a) Given the function f(x,y) = x + y, find the maximum and minimum values of f on the domain

$$D_1 = \{(x, y) \colon 0 \le y \le x^2 \text{ and } -1 \le x \le 1\}.$$



$$D_2 = \{(x,y) \colon y \ge 0 \text{ and } x^2 + y^2 \le 9\}.$$

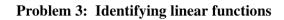
(When doing this, one part of the boundary will be a mess via single-variable calculus, so employ Lagrange multipliers there with the boundary curve as a constraint condition. You may encounter the expression $2x^2-3xy-2y^2$, in which case it will be useful to then observe that this factors as (2x+y)(x-2y).)

(c) (Extra) Let C be the curve in ${\bf R}^2$ defined by the equation

$$y^2 = x^3 - 4x^2 + 5x$$

Determine all points on C at minimal distance to (5/2,0).

Review: Linear functions



In each case below, is $\mathbf{f}: \mathbf{R}^2 \to \mathbf{R}^3$ linear? If it is, find the matrix representing it. If not, explain why not.

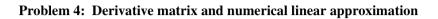
(a)
$$\mathbf{f}(x_1, x_2) = (x_1, x_2^2, 2x_1 + x_2)$$

(b)
$$\mathbf{f}(x_1, x_2) = (1, x_2, 2x_1 + x_2)$$

(c)
$$\mathbf{f}(x_1, x_2) = (0, x_2, 2x_1 + x_2)$$

(d)
$$\mathbf{f}(x_1, x_2) = (0, x_1 x_2, 2x_1 + x_2)$$

(e)
$$\mathbf{f}(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2, ex_1 + fx_2)$$



Consider the function $f: \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$f(x,y) = (x^3y^2, 4x + y^3 + xy).$$

(a) Compute the derivative matrix (Df)(x,y), and then use it to give the linear approximation to f at (1,1).





Problem 1: Constrained optimization

In what follows, you may accept that f(x,y) = xy attains maximal and minimal values on the curve $x^2 - xy + y^2 = 3$.

- (a) Use the method of Lagrange multipliers to find these extreme values and the point(s) where they are attained.
- (b) The quadratic formula allows you to solve for y in terms of x on the curve: $y(x) = (x \pm \sqrt{x^2 4(x^2 3)})/2 = (x \pm \sqrt{12 3x^2})/2$ (with $|x| \le 2$ so that the square root makes sense). Hence, we could instead try to find the extreme values for $f(x, y(x)) = x \cdot y(x) = (x^2 \pm x\sqrt{12 3x^2})/2$ for $-2 \le x \le 2$ via single-variable calculus. Is that more or less appetizing than the method in (a)?

Problem 2: Optimization review (what technique(s) would you use?)

(a) Given the function f(x,y) = x + y, find the maximum and minimum values of f on the domain

$$D_1 = \{(x, y) \colon 0 \le y \le x^2 \text{ and } -1 \le x \le 1\}.$$

(b) Find the maximum and minimum values of $G(x,y) = 3x^2 + 4xy$ on the region

$$D_2 = \{(x, y) \colon y \ge 0 \text{ and } x^2 + y^2 \le 9\}.$$

(When doing this, one part of the boundary will be a mess via single-variable calculus, so employ Lagrange multipliers there with the boundary curve as a constraint condition. You may encounter the expression $2x^2 - 3xy - 2y^2$, in which case it will be useful to then observe that this factors as (2x + y)(x - 2y).)

(c) (Extra) Let C be the curve in \mathbb{R}^2 defined by the equation

$$y^2 = x^3 - 4x^2 + 5x$$

Determine all points on C at minimal distance to (5/2, 0).

Problem 3: Identifying linear functions

In each case below, is $\mathbf{f}: \mathbf{R}^2 \to \mathbf{R}^3$ linear? If it is, find the matrix representing it. If not, explain why not.

(a)
$$\mathbf{f}(x_1, x_2) = (x_1, x_2^2, 2x_1 + x_2)$$

(b)
$$\mathbf{f}(x_1, x_2) = (1, x_2, 2x_1 + x_2)$$

(c)
$$\mathbf{f}(x_1, x_2) = (0, x_2, 2x_1 + x_2)$$

(d)
$$\mathbf{f}(x_1, x_2) = (0, x_1x_2, 2x_1 + x_2)$$

(e)
$$\mathbf{f}(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2, ex_1 + fx_2)$$

Problem 4: Derivative matrix and numerical linear approximation

Consider the function $f: \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$f(x,y) = (x^3y^2, 4x + y^3 + xy).$$

- (a) Compute the derivative matrix (Df)(x,y), and then use it to give the linear approximation to f at (1,1).
- (b) Use your answer to (a) to estimate the 2-vector f(0.8, 1.1), and then compare it with an exact calculation using a calculator. Is it a good approximation?
- (c) Give the linear approximation to f at (2, -2) and use it to estimate the 2-vector f(2.1, -1.9) and then compare this to the exact 2-vector using a calculator. Is the approximation good or bad?