Problem 1: Recognizing Eigenvectors

For the following matrices A and nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, verify that the vectors are eigenvectors for A and find their corresponding eigenvalues.

(a) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

Solution:

(a) We calculate

$$A\mathbf{v}_1 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = 4\mathbf{v}_1,$$

so $\mathbf{v}_1$ is an eigenvector with eigenvalue 4. Furthermore

$$A\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = -2\mathbf{v}_2,$$

so $\mathbf{v}_2$ is an eigenvector with eigenvalue -2. Lastly

$$A\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1\mathbf{v}_3,$$

so $\mathbf{v}_3$ is an eigenvector with eigenvalue 1.

(a) We calculate

$$A\mathbf{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = 1\mathbf{v}_1,$$

so $\mathbf{v}_1$ is an eigenvector with eigenvalue 1. Furthermore

$$A\mathbf{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = -3\mathbf{v}_2,$$

so $\mathbf{v}_2$ is an eigenvector with eigenvalue -3. Lastly

$$A\mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix} = -3\mathbf{v}_3,$$

so $\mathbf{v}_3$ is an eigenvector with eigenvalue -3.

Problem 2: Geometric meaning of eigenvalues

Identify the eigenvalues of the following linear transformations $\mathbf{R}^3 \to \mathbf{R}^3$, and find *all* eigenvectors for each eigenvalue (expressed in terms of the given geometric data). The eigenvalues are explicit numbers, not depending on the given line or plane.

Hint: Think geometrically by looking for lines carried onto themselves (or crushed into $\{\mathbf{0}\}$: don't overlook the possibility of 0 as an eigenvalue!). In particular, if a line is not carried onto itself or crushed into the origin, it cannot provide any eigenvectors.

(a) The reflection across a plane $V \subset \mathbf{R}^3$ through the origin.

(b) The projection onto a plane $V \subset \mathbf{R}^3$ through the origin.

(c) The rotation by $90°$ around a line $L \subset \mathbf{R}^3$ through the origin.

(d) The rotation by $180°$ around the line $L \subset \mathbf{R}^3$ through the origin.

Solution:

(a) Each nonzero vector in V is carried onto itself, so they are all eigenvectors with eigenvalue 1. Nonzero vectors in its orthogonal complement are reflected exactly onto their negative, so they are eigenvector with eigenvalue -1. Visually we see that lines through the origin not contained in V nor perpendicular to V (in other words, the angle that the line makes with V is strictly between $0°$ and $90°$) are not carried into themselves under the reflection, so there are no other eigenvectors or eigenvalues than what we have found.

(b) Each nonzero vector in V is carried onto itself, so they are all eigenvectors to eigenvalue 1. Nonzero vectors in its orthogonal complement line are projected onto $\mathbf{0}$, so they are eigenvectors with eigenvalue 0. A line not perpendicular to V projects onto a line in V (rather than being crushed into the origin), so such a line cannot be the same as its projection unless it is already inside the plane. Hence, there are no eigenvectors outside the plane and its normal line, so there are no other eigenvectors or eigenvalues than what we have found

(c) The nonzero vectors on the line L will not be rotated at all, being kept where they are, so they are eigenvectors to eigenvalue 1. There are no other lines through the origin carried onto themselves (or crushed into the origin), so there are no other eigenvalues or eigenvectors.

(d) The nonzero vectors on the line L will not be rotated at all, so they are eigenvectors with eigenvalue 1 as in (d). Nonzero vectors in its orthogonal complement get rotated exactly to their negative, so they are eigenvectors to eigenvalue -1. If we visualize lines through the origin distinct from L and not in the plane perpendicular to L the we see that such lines are never carried into themselves under such a rotation, so there are no eigenvectors outside L and $L^{\perp}$. Hence, we have found all of the eigenvectors and eigenvalues.

Problem 3: Eigenvalues of 2×2 matrices

For each of the following 2×2 matrices, find all the eigenvalues and an eigenvector for each eigenvalue.

(a) $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$.

(c) $C = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$ for a general number $a \neq 0$. (Your answer may depend on a; for $a = 4$ it should recover the answer to (b).)

Solution:

(a) We calculate the characteristic polynomial as

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

So the eigenvalues of A are 2 and -1.

To find an eigenvector with eigenvalue 2, we write the vector equation $A\mathbf{x} = 2\mathbf{x}$ as a pair of scalar equations (by equating vector entries)

$$x_2 = 2x_1 \quad \text{and} \quad 2x_1 + x_2 = 2x_2;$$

these two equations are nonzero scalar multiples of each other (as we know must happen: two nonzero linear equations in two unknowns with no constant term have a nonzero simultaneous solution precisely when the equations are scalar multiples of each other). By inspecting either of these, an eigenvector is given by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; any nonzero scalar multiple also works.

To find an eigenvector with eigenvalue -1 we solve the equations

$$x_2 = -x_1 \quad \text{and} \quad 2x_1 + x_2 = -x_2.$$

By inspecting either of these equations, an eigenvector is given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; any nonzero scalar multiple also works (such as its negative).

(b) We calculate the characteristic polynomial as

$$\lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3).$$

So the eigenvalues of B are 5 and -3.

To find an eigenvector with eigenvalue 5, we solve the equations

$$x_1 + 4x_2 = 5x_1 \quad \text{and} \quad 4x_1 + x_2 = 5x_2,$$

which both are equivalent to $x_1 = x_2$. So an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (and any nonzero scalar multiple also works).

To find the eigenvectors with eigenvalue -3, we solve the equations

$$x_1 + 4x_2 = -3x_1 \quad \text{and} \quad 4x_1 + x_2 = -3x_2,$$

which both are equivalent to $x_1 = -x_2$. So an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (and any nonzero scalar multiple also works, such as its negative).

(c) We calculate the characteristic polynomial as

$$\lambda^2 - 2\lambda + (1 - a^2) = \lambda^2 - 2\lambda + ((1-a)(1+a)) = (\lambda - (1+a))(\lambda - (1-a)).$$

So the eigenvalues of C are $1 + a$ and $1 - a$ (*not* $a - 1$; watch out for sign errors).

To find an eigenvector with eigenvalue $1 + a$, we solve the equations

$$x_1 + ax_2 = (1+a)x_1, \;\; ax_1 + x_2 = (1+a)x_2,$$

which each say $ax_2 = ax_1$, or equivalently (since $a \neq 0$) $x_2 = x_1$. So an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (which curiously doesn't depend on a; this is a coincidence for this particular class of matrices).

To find an eigenvector with eigenvalue $1 - a$, we solve the equations

$$x_1 + ax_2 = (1-a)x_1, \;\; ax_1 + x_2 = (1-a)x_2,$$

which each say $ax_2 = -ax_1$, or equivalently (since $a \neq 0$) $x_2 = -x_1$. So an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (which curiously again doesn't depend on a; this is a coincidence for this particular class of matrices).

Problem 4: Additional practice with eigenvalues and eigenvectors (triangular examples)

For each eigenvalue λ of the given matrix A, compute a basis for the nonzero linear subspace $N(A-\lambda \mathrm{I}_3)$ in $\mathbf{R}^3$ (the "λ-eigenspace"), and as a check on your work verify directly that each vector in that basis is an eigenvector for A with eigenvalue λ.

(a) $A = \begin{bmatrix} 1 & 6 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 3 & 0 \\ 6 & -2 & 2 \end{bmatrix}$.

Solution:

(a) As we have seen in the textbook, the eigenvalues for an upper triangular matrix (as well as lower triangular) are its diagonal entries. So this matrix has as its eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -2$. We need to compute a basis for each of the null spaces $N(A - \mathrm{I}_3)$ and $N(A + 2\mathrm{I}_3)$.

For the first of these, we have

$$A - \mathrm{I}_3 = \begin{bmatrix} 0 & 6 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

so a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in its null space precisely when it satisfies the equations

$$6x_2 + 3x_3 = 0, \;\; -3x_2 + 2x_3 = 0, \;\; 0 = 0.$$

The third equation gives nothing, and the first and second equations are two homogeneous equations in two unknowns for which the equations are not scalar multiples of each other and hence has as its only solution $(0,0)$ (as can be checked directly: the second equation says $x_3 = -2x_2$, and plugging this into the third equation forces x_2 and x_3 to vanish). Thus, altogether this null space consists of vectors of the form

$$\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, by inspection we see that a basis of this null space is given by the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

direct matrix-vector multiplication gives

$$A\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{v},$$

as desired.

Turning to the null space of $A + 2\mathrm{I}_3$, we have

$$A + 2\mathrm{I}_3 = \begin{bmatrix} 3 & 6 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Hence, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ lies in the null space of this precisely when it satisfies the equations

$$3x_1 + 6x_2 + 3x_3 = 0, \;\; 2x_3 = 0, \;\; 3x_3 = 0.$$

The last two equations both say $x_3 = 0$, and plugging this into the first turns it into the condition $3x_1 + 6x_2 = 0$, which says $x_1 = -2x_2$. Hence,

$$\mathbf{x} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},$$

so a basis for this null space is given by the vector $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, or any nonzero scalar multiple of that. Direct matrix-vector multiplication establishes the eigenvector condition:

$$A\mathbf{w} = \begin{bmatrix} -2+6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = -2\mathbf{w}.$$

(b) Since this matrix is lower-triangular, we may read the diagonal entries of A (similarly as in (a)) to conclude that its eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. We need to compute a basis for each of the null spaces $N(A - 2\mathrm{I}_3)$ and $N(A - 3\mathrm{I}_3)$.

For the first of these, we have

$$A - 2\mathrm{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -2 & 0 \end{bmatrix},$$

so a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in its null space precisely when it satisfies the equations

$$0 = 0, \;\; -3x_1 + x_2 = 0, \;\; 6x_1 - 2x_2 = 0.$$

The first equation gives nothing, and the second and third equations both say $x_2 = 3x_1$. Thus, the null space consists of vectors of the form

$$\begin{bmatrix} x_1 \\ 3x_1 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where x_1 and x_3 can each be any scalar. Hence, we see that a basis of this null space is given by the linearly independent vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Direct matrix-vector multiplication gives

$$A\mathbf{v} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} = 2\mathbf{v}, \qquad A\mathbf{v}' = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2\mathbf{v}',$$

as desired.

Turning to the null space of $A - 3\mathrm{I}_3$, we have

$$A - 3\mathrm{I}_3 = \begin{bmatrix} -1 & 0 & 0 \\ -3 & 0 & 0 \\ 6 & -2 & -1 \end{bmatrix}.$$

Hence, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ lies in the null space of this precisely when it satisfies the equations

$$-x_1 = 0, \;\; -3x_1 = 0, \;\; 6x_1 - 2x_2 - x_3 = 0.$$

The first two equations both say $x_1 = 0$, and plugging this into the last turns it into the condition $-2x_2 - x_3 = 0$, which says $x_3 = -2x_2$. Hence,

$$\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ -2x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},$$

so a basis for this null space is given by the vector $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, or any nonzero scalar multiple of that. Direct matrix-vector multiplication establishes the eigenvector condition:

$$A\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ -2-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -6 \end{bmatrix} = 3\mathbf{w}.$$

Problem 5: Quadratic forms and definiteness I

(a) For each of the following 2×2 symmetric matrices M, compute the quadratic form $q_M(x, y) = \begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix}$:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 17 & 4 \\ 4 & 2 \end{bmatrix}, D = \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}.$$

(b) For each M in (a), use its characteristic polynomial to find its eigenvalues (they are all integers in these cases), and from that determine if $q_M(x, y)$ is positive-definite, negative-definite, or indefinite.

Solution:

(a) We compute $q_A(x, y) = 3x^2 + 4xy$, $q_B(x, y) = 2x^2 - 2xy + 2y^2$, $q_C(x, y) = 17x^2 + 8xy + 2y^2$, and $q_D(x, y) = -6x^2 + 4xy - 3y^2$.

(b) The characteristic polynomials $p_M(\lambda) = \lambda^2 - \mathrm{tr}(M)\lambda + \det(M)$ are given by

$$p_A(\lambda) = \lambda^2 - 3\lambda - 4, \;\; p_B(\lambda) = \lambda^2 - 4\lambda + 3, \;\; p_C(\lambda) = \lambda^2 - 19\lambda + 18, \;\; p_D(\lambda) = \lambda^2 + 9\lambda + 14.$$

These all factor, enabling us to see the roots without needing to haul out the quadratic formula: $p_A(\lambda) = (\lambda-4)(\lambda+1)$, $p_B(\lambda) = (\lambda - 1)(\lambda - 3)$, $p_C(\lambda) = (\lambda - 1)(\lambda - 18)$, and $p_D(\lambda) = (\lambda + 2)(\lambda + 7)$.

Hence, the respective eigenvalues are $\{4, -1\}$ for A, $\{1, 3\}$ for B, $\{1, 18\}$ for C, and $\{-2, -7\}$ for D. Thus, q_B and q_C are positive-definite, q_D is negative-definite, and q_A is indefinite.

Problem 6: Quadratic forms and definiteness II

For each of the following symmetric 3×3 matrices M and given nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, carry out two tasks:

(i) Compute the associated quadratic form $q_M(x, y, z)$, and verify that $\mathbf{v}_i$'s are pairwise orthogonal and eigenvectors, determining the eigenvalue for each.

(ii) Use your answer to (i) to write down the quadratic form $q(u, v, w) = q_M(u\mathbf{v}_1' + v\mathbf{v}_2' + w\mathbf{v}_3')$ when everything is described in terms of the basis of orthonormal eigenvectors $\mathbf{v}_i' = \mathbf{v}_i/\|\mathbf{v}_i\|$, from which you should determine if q_M is positive-definite, negative-definite, indefinite, positive-semidefinite (but not positive-definite), or negative-semidefinite (but not negative-definite). You *do not* need to compute the lengths $\|\mathbf{v}_i\|$.

(a) $\begin{bmatrix} 5 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$

(b) $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: In all cases the dot products $\mathbf{v}_i \cdot \mathbf{v}_j$ for $i \neq j$ are checked to vanish.

(a) We have $q_M(x, y, z) = 5x^2 - 2y^2 + 2z^2 - 4xz$. One calculates that $M\mathbf{v}_1 = 6\mathbf{v}_1$, $M\mathbf{v}_2 = -2\mathbf{v}_2$, and $M\mathbf{v}_3 = \mathbf{v}_3$, so the corresponding eigenvalues are $6, -2, 1$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u, v, w) = 6u^2 - 2v^2 + w^2$. This is indefinite.

(b) We have
$$q_M(x, y, z) = 3x^2 + 3y^2 + 5z^2 + 2xy - 2xz - 2yz.$$
One calculates that $M\mathbf{v}_1 = 3\mathbf{v}_1$, $M\mathbf{v}_2 = 6\mathbf{v}_2$, and $M\mathbf{v}_3 = 2\mathbf{v}_3$, so the corresponding eigenvalues are $3, 6, 2$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u, v, w) = 3u^2 + 6v^2 + 2w^2$. This is positive-definite.

(c) We have $q_M(x, y, z) = x^2 + 2y^2 + z^2 - 2xy - 2yz$. One calculates that $M\mathbf{v}_1 = \mathbf{v}_1$, $M\mathbf{v}_2 = 3\mathbf{v}_2$, and $M\mathbf{v}_3 = \mathbf{0} = 0\mathbf{v}_3$, so the corresponding eigenvalues are $1, 3, 0$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u, v, w) = u^2 + 3v^2$. This is positive-semidefinite but not positive-definite.

(d) We have $q_M(x, y, z) = 3x^2 + 3z^2 + 4xy + 8xz + 4yz$. One calculates that $M\mathbf{v}_1 = -\mathbf{v}_1$, $M\mathbf{v}_2 = -\mathbf{v}_2$, and $M\mathbf{v}_3 = 8\mathbf{v}_3$, so the corresponding eigenvalues are $-1, -1, 8$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u, v, w) = -u^2 - v^2 + 8w^2$. This is indefinite.

(e) We have $q_M(x, y, z) = 2xy + 2yz + 2xz$. One calculates that $M\mathbf{v}_1 = -\mathbf{v}_1$, $M\mathbf{v}_2 = -\mathbf{v}_2$, and $M\mathbf{v}_3 = 2\mathbf{v}_3$, so the corresponding eigenvalues are $-1, -1, 2$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u, v, w) = -u^2 - v^2 + 2w^2$. This is indefinite.