Math 51

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You are encouraged to turn on your video ☺

I recommend looking over the worksheet before each section.

Problem 1

These are 2-vectors

Let $\vec{a}=\begin{bmatrix}2\\1\end{bmatrix}$, $\vec{b}=\begin{bmatrix}0\\3\end{bmatrix}$, $\vec{c}=\begin{bmatrix}-1\\-1\end{bmatrix}$.

Recall: Add vectors componentwise.

Must have same number of components! (cannot add a 3-vector with 2-vector)

Multiply by real numbers ("scalars") componentwise.

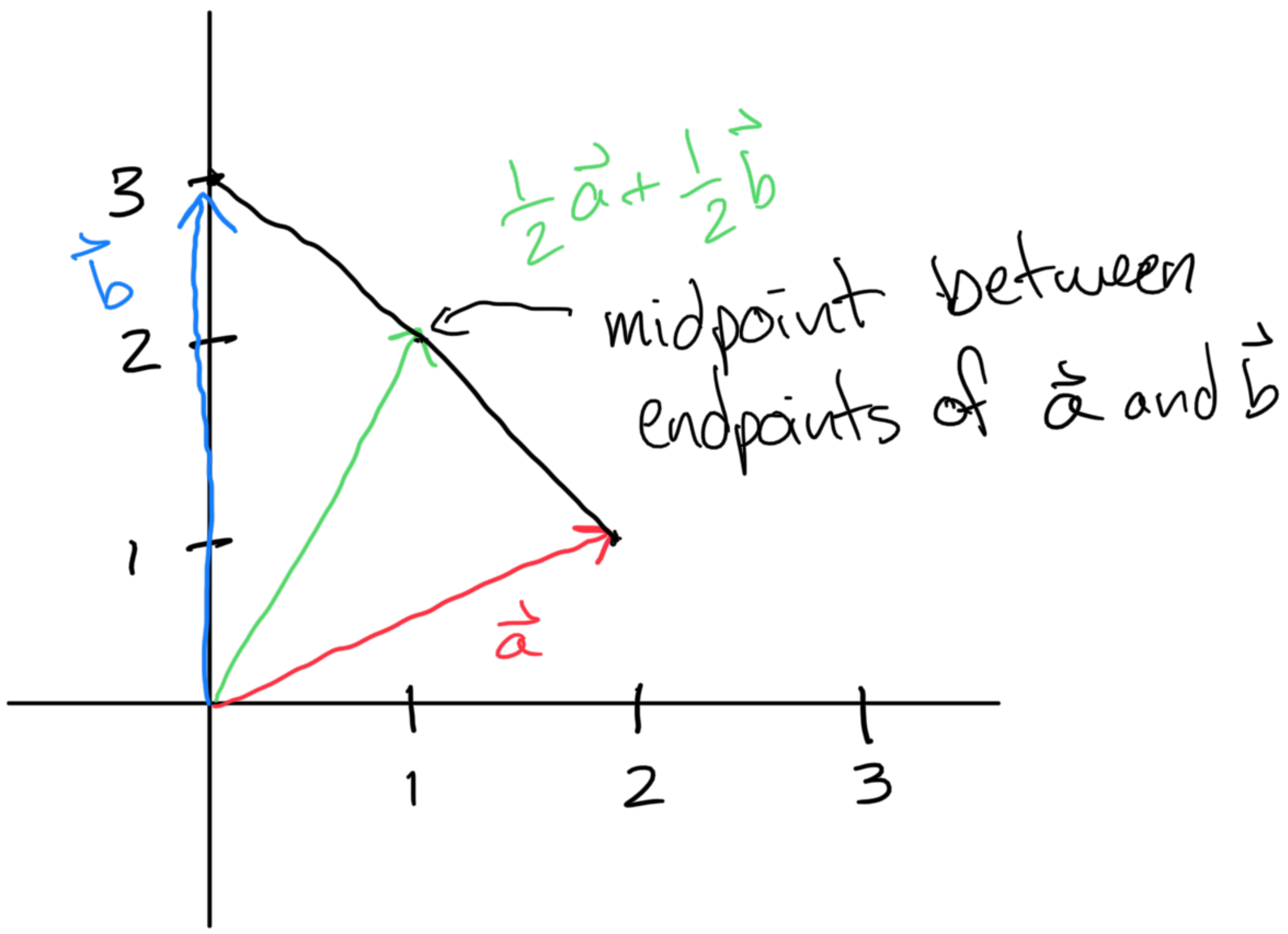
(a) Compute $\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}$. Draw $\vec{a}$, $\vec{b}$, and $\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}$ in a coordinate plane, and describe geometrically where the sum lies relative to $\vec{a}$ and $\vec{b}$.

$$\frac{1}{2}\vec{a}=\frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1/2\cdot 2\\1/2\cdot 1\end{bmatrix}=\begin{bmatrix}1\\1/2\end{bmatrix}$$

$$\frac{1}{2}\vec{b}=\frac{1}{2}\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}0\\3/2\end{bmatrix}$$

$$\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}=\begin{bmatrix}1\\1/2\end{bmatrix}+\begin{bmatrix}0\\3/2\end{bmatrix}=\begin{bmatrix}1+0\\1/2+3/2\end{bmatrix}$$

$$=\begin{bmatrix}1\\2\end{bmatrix}$$



Do you expect such a relationship for any 2-vectors $\vec{a}$ and $\vec{b}$? For 3-vectors?

In general:

$$\frac{1}{2}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{a_1+b_1}{2} \\ \frac{a_2+b_2}{2} \end{bmatrix}$$

averages of each component

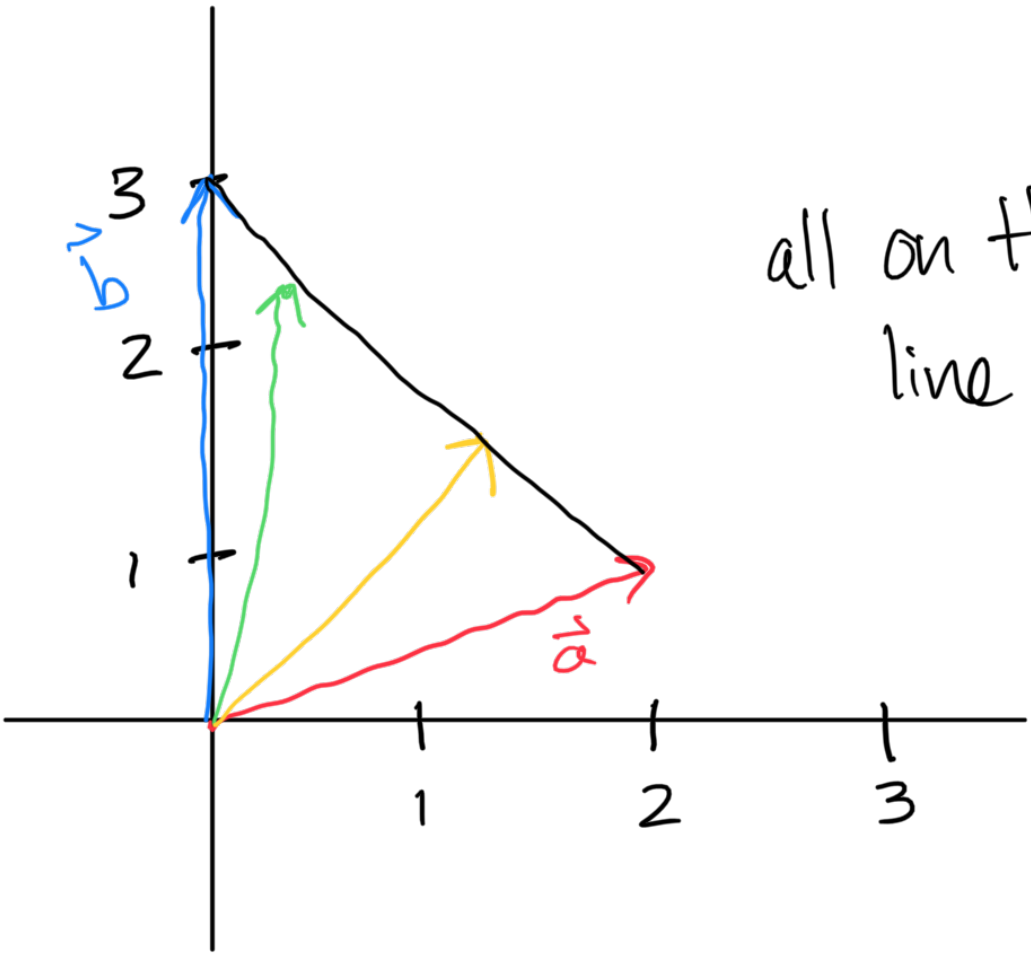
So this is always the midpoint between $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. The same is true for 3-vectors.

(b) Compute $\frac{1}{3}\vec{a} + \frac{2}{3}\vec{b}$ and $\frac{3}{4}\vec{a} + \frac{1}{4}\vec{b}$, and plot these. Do you notice a pattern that should hold for any 2-vectors $\vec{a}$ and $\vec{b}$?

$$\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} + \frac{2}{3}\begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}1/3 \cdot 2 + 2/3 \cdot 0\\ 1/3 \cdot 1 + 2/3 \cdot 3\end{bmatrix}$$

$$= \boxed{\begin{bmatrix}2/3\\7/3\end{bmatrix}}$$

$$\frac{3}{4}\begin{bmatrix}2\\1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}3/4 \cdot 2\\ 3/4 \cdot 1 + 1/4 \cdot 3\end{bmatrix} = \boxed{\begin{bmatrix}3/2\\3/2\end{bmatrix}}$$



all on the same line

In general, if $0 \leq t \leq 1$, then

$$t\vec{a} + (1-t)\vec{b}$$ ("convex combination")

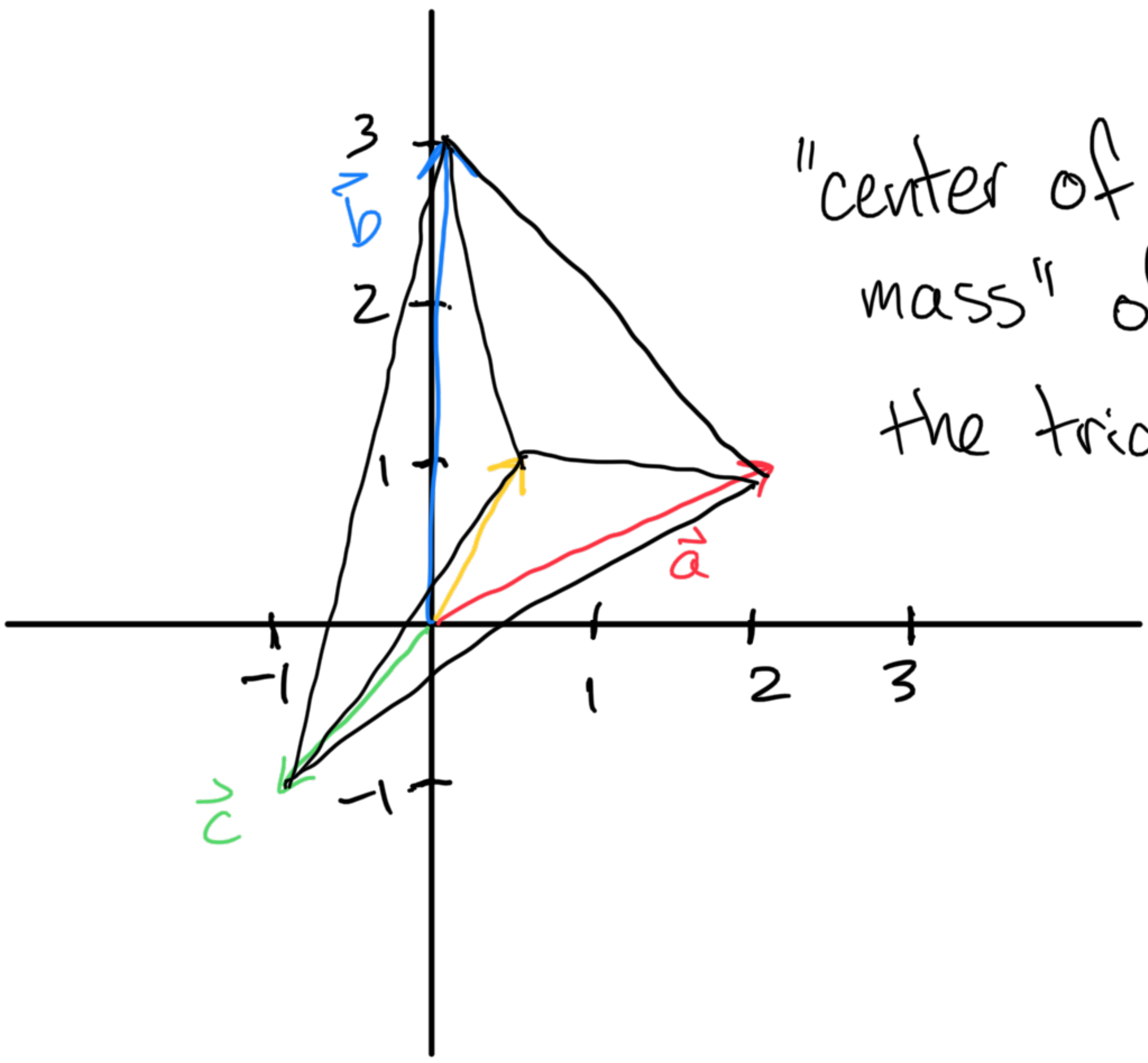
lies on the line between endpoints of $\vec{a}$ and $\vec{b}$.

Let $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

(c) Compute $\frac{1}{3}\vec{a} + \frac{1}{3}\vec{b} + \frac{1}{3}\vec{c}$, plot and draw segments joining it to each of $\vec{a}, \vec{b}$, and $\vec{c}$. Describe geometrically where this lies.

$$\frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \cdot 2 + 1/3(-1) \\ 1/3 + 1/3 \cdot 3 + 1/3(-1) \end{bmatrix}$$

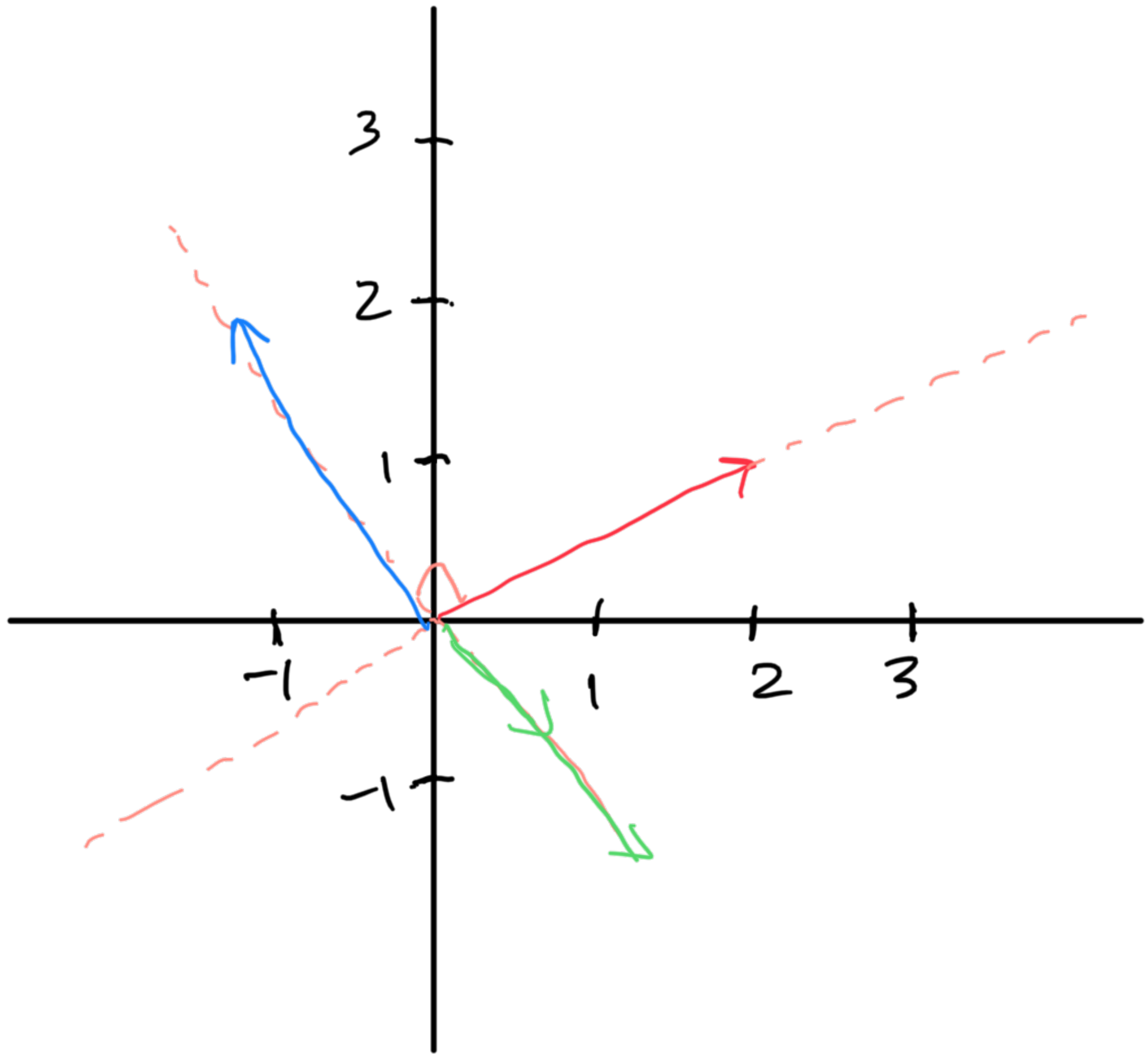
$$= \boxed{\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}}$$



"center of mass" of the triangle

Its components are the averages of the components of $\vec{a}, \vec{b}, \vec{c}$.

(d) Find a nonzero vector that is perpendicular to $\vec{a}$.



e.g. $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is perpendicular to $\vec{a}$

Reasoning: $\vec{a}$ lies on the line of slope $\frac{1}{2}$ through the origin.
The perpendicular line has slope -2 and formula $y=-2x$.

$\boxed{\text{Problem 2}}$ Linear combinations

Recall: A linear combination of n-vectors $\vec{a}_1, \dots, \vec{a}_k$ is a vector

$$c_1\vec{a}_1 + \dots + c_k\vec{a}_k,$$

where $c_1, \dots, c_k$ are scalars.

(a) Express $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as a linear combination of $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Want $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} c_1 \cdot 1 + c_2 \cdot 0 \\ c_1 \cdot 0 + c_2 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\boxed{c_1 = 5, \ c_2 = 4}$$

(b) Express $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as a linear combination of $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Want $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 2c_1 + c_2 \end{bmatrix}$$

$$\begin{bmatrix} \\ c_1+2c_2 \end{bmatrix}$$

So need to solve

$$\begin{cases} 5 = 2c_1 + c_2 \\ 4 = c_1 + 2c_2 \end{cases}$$

Substitute $c_1 = 4 - 2c_2$:

$$\begin{aligned} 5 &= 2(4-2c_2) + c_2 \\ &= 8 - 4c_2 + c_2 \\ &= 8 - 3c_2 \end{aligned}$$

$\Rightarrow 3c_2 = 3$

$\boxed{c_2 = 1, \ c_1 = 2}$ $\boxed{\begin{bmatrix} 5 \\ 4 \end{bmatrix} = 2\vec{v} + \vec{w}}$

(c) Write a general 2-vector $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Want

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix}$$

So solve

$$\begin{cases} x = 2c_1 + c_2 \\ y = c_1 + 2c_2 \end{cases}$$

for c_1, c_2. Solve, e.g. by substituting

$c_2 = x - 2c_1$:

$y = c_1 + 2(x - 2c_1)$

$$= c_1 - 4c_1 + 2x = -3c_1 + 2x$$

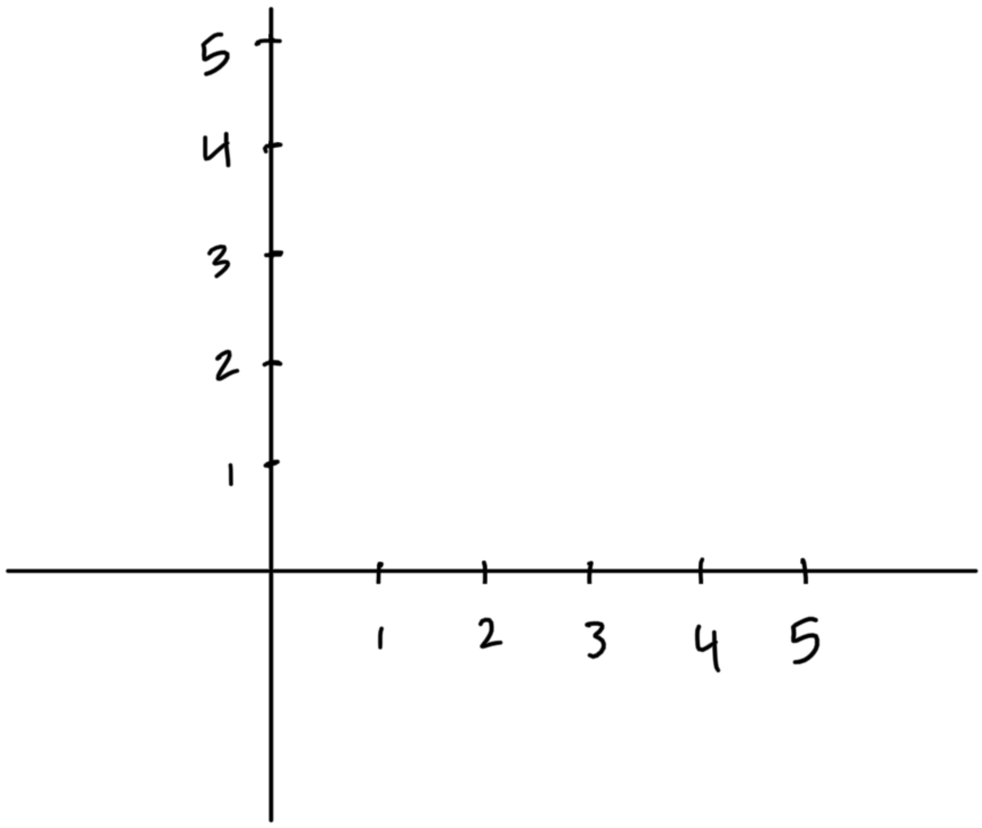
$$3c_1 = 2x - y$$

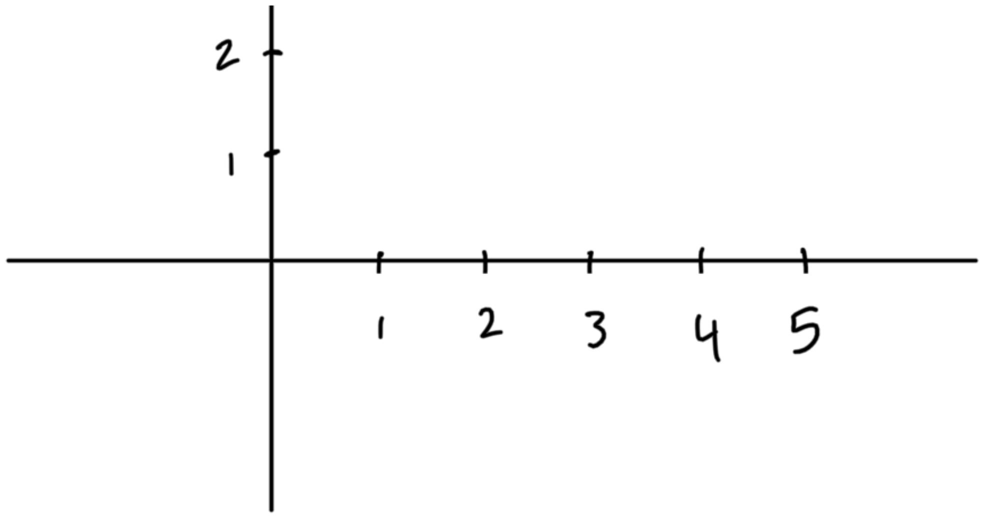
$$\Rightarrow c_1 = \frac{2x-y}{3}, \quad c_2 = \frac{2y-x}{3}$$

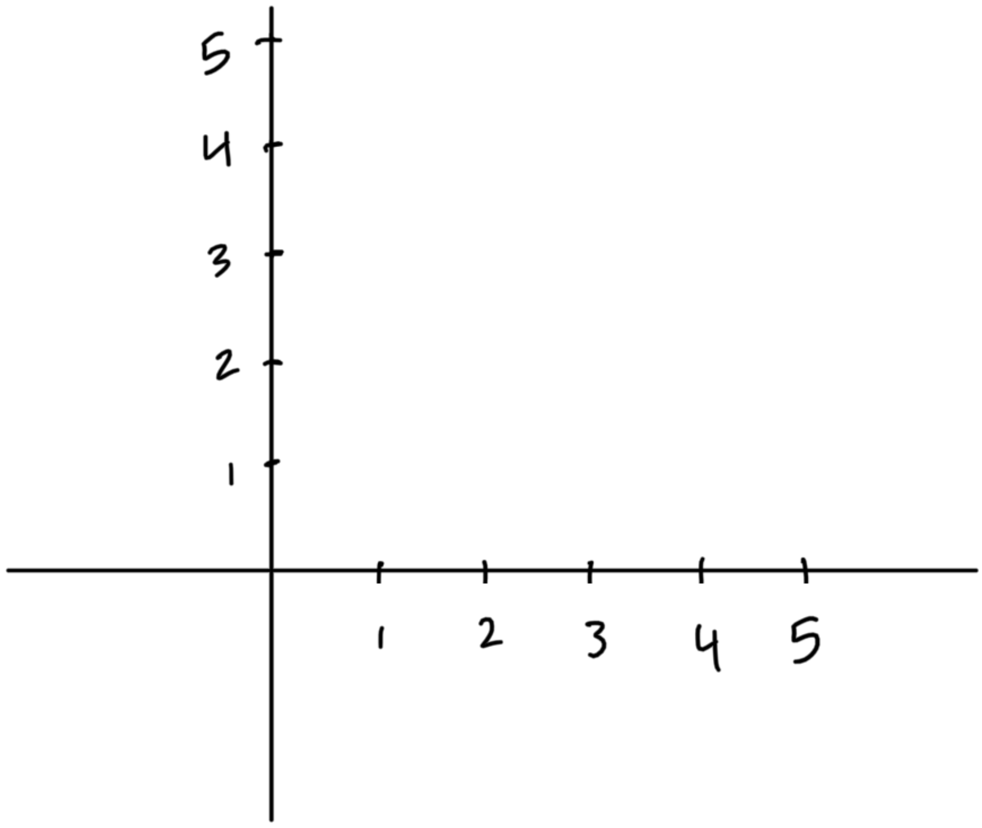
Check: if $x=5, y=4$, then

$$c_1 = \frac{2\cdot 5-4}{3} = 2, \quad c_2 = \frac{2\cdot 4-5}{3} = 1$$

(d) (Extra) Draw pictures and interpret geometrically




2
1
1
2
3
4
5


5
4
3
2
1
1
2
3
4
5

$\boxed{\text{Problem 3}}$

Recall: The length of an n-vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

is

$$\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2}.$$

The distance between two n-vectors $\vec{v}, \vec{w}$ is

$$\|\vec{v} - \vec{w}\| = \|\vec{w} - \vec{v}\|.$$

(a) Compute the distance between $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

$$\begin{bmatrix} 7 \\ -2 \end{bmatrix} - \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -5 \end{bmatrix}$$

$$\left\| \begin{bmatrix} 12 \\ -5 \end{bmatrix} \right\| = \sqrt{12^2 + (-5)^2} = \sqrt{144 + 25}$$

$$= \sqrt{169}$$

$$= \boxed{13}$$

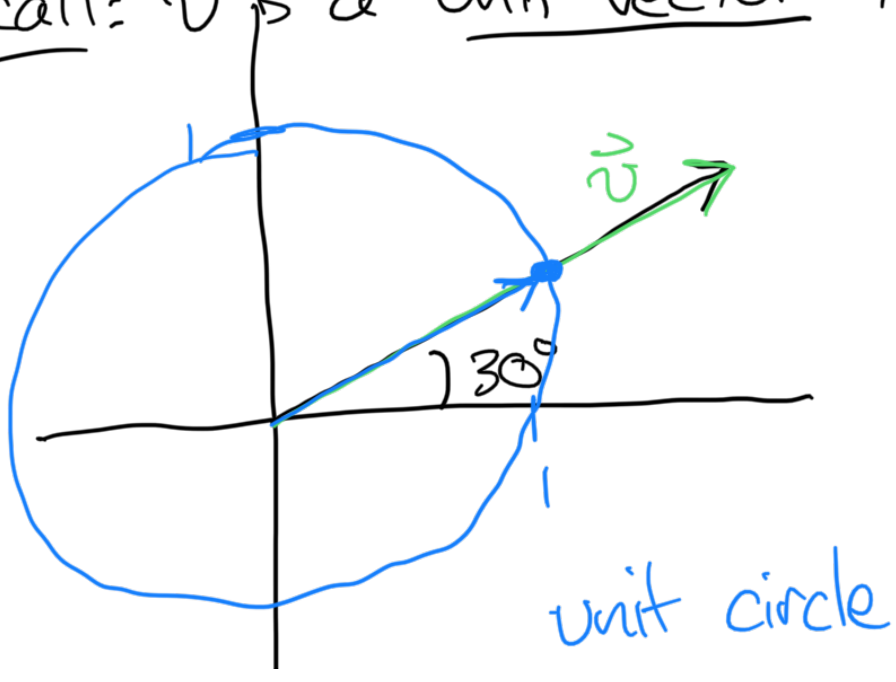
(b) Compute the distance between $\begin{bmatrix}4\\-1\\0\\-2\end{bmatrix}$ and $\begin{bmatrix}7\\-6\\1\\-3\end{bmatrix}$.

$$\begin{bmatrix}7\\-6\\1\\-3\end{bmatrix}-\begin{bmatrix}4\\-1\\0\\-2\end{bmatrix}=\begin{bmatrix}3\\-5\\1\\-1\end{bmatrix}$$

$$\left\|\begin{bmatrix}3\\-5\\1\\-1\end{bmatrix}\right\| = \sqrt{3^2+(-5)^2+1^2+(-1)^2}$$
$$= \sqrt{9+25+1+1}$$
$$= \sqrt{36} = \boxed{6}$$

(c) If a nonzero vector $\vec{v}$ lies at an angle 30° counterclockwise from the positive x-axis, what is the unit vector in the same direction as $\vec{v}$? For a general angle θ?

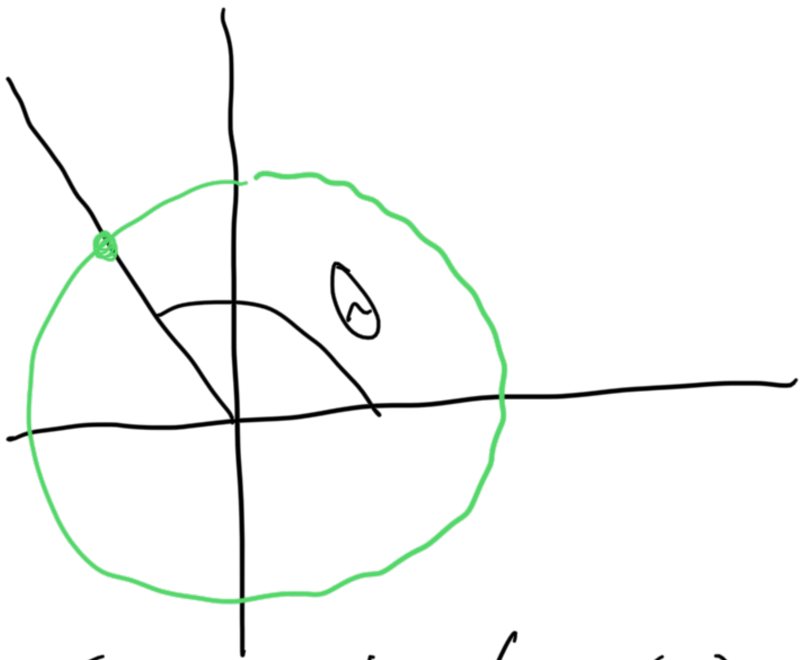
Recall: $\vec{v}$ is a unit vector if $\|\vec{v}\|=1$.



Use knowledge of unit circle:

This point is $(\cos(30), \sin(30))$

$$\boxed{\begin{bmatrix} \cos(30) \\ \sin(30) \end{bmatrix}}$$



θ = Greek letter theta

This point $(\cos(\theta), \sin(\theta))$

Problem 4

Suppose there are 3 students in Math 51 with the following grades:

Student 1: 81/100 on homework, 83/100 on midterm A, 70/100 on midterm B, 75/100 on the final

Student 2: 73/100 homework, 75/100 midterm A, 74/100 midterm B, 88/100 final

Student 3: 90/100 homework, 95/100 midterm A, 88/100 midterm B, 92/100 final

(a) Write down vectors $\vec{v}_{HW}$, $\vec{v}_A$, $\vec{v}_B$, $\vec{v}_{Final}$ representing the grades as percentages.

$$\vec{v}_{HW} = \begin{bmatrix} 81 \\ 73 \\ 90 \end{bmatrix}, \quad \vec{v}_A = \begin{bmatrix} 83 \\ 75 \\ 95 \end{bmatrix}, \quad \vec{v}_B = \begin{bmatrix} 70 \\ 74 \\ 88 \end{bmatrix}, \quad \vec{v}_{Final} = \begin{bmatrix} 75 \\ 88 \\ 92 \end{bmatrix}$$

(b) Give a general formula for a 3-vector $\vec{v}_{CG}$ whose entries are the course grades for the 3 students, where the total grade is 16% homework, 36% final, 24% each midterm.