Problem 1: Visualizing vectors and convex combinations in $\mathbf{R}^2$

Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

(a) Compute $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$. Draw $\mathbf{a}, \mathbf{b}$ and $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$ in a coordinate plane, and describe geometrically where the sum lies relative to $\mathbf{a}$ and $\mathbf{b}$. Do you expect such a geometric relationship is true for any 2-vectors $\mathbf{a}, \mathbf{b}$? How about for 3-vectors?

(b) Compute $\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$ and $\frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$, and plot these. Do you notice a pattern that should hold for any 2-vectors $\mathbf{a}$ and $\mathbf{b}$?

(c) Compute $\frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c}$, plot this point and draw segments joining it to each of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, and describe geometrically where this lies relative to $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(d) Find a nonzero vector that is perpendicular to $\mathbf{a}$. (Hint: draw a picture.)

Problem 2: Linear combinations

(a) Express $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Express $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (This amounts to solving 2 equations in 2 unknowns.)

(c) Write a general 2-vector $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The coefficients of the linear combination will depend on x and y. Make sure that for $x = 5$ and $y = 4$ it agrees with your answer to (b)!

(d) (**Extra**) Draw a picture for each of (a) and (b). Then draw all points $n\mathbf{v}+m\mathbf{w}$ for integers n and m with $-2 \leq n, m \leq 3$, and draw lines through these parallel to each of $\mathbf{v}$ and $\mathbf{w}$, which should yield a tiling of the plane with copies of the parallelogram P whose corners are at the tips of the vectors $\mathbf{0}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$.

Interpret geometrically (without any calculations) the meaning of the answer to (b) in terms of these parallelograms, and do the same for the result in (c) that such a linear combination always exists. (Hint: for (c), mark $(6, 5)$ and $(3, 4)$ and compare where they lie among the parallelograms with the general formula in (c) for these two cases.)

Problem 3: Length and distance

(a) Compute the distance between $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$. (The answer is an integer.)

(b) Compute the distance between $\begin{bmatrix} 4 \\ -1 \\ 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -6 \\ 1 \\ -3 \end{bmatrix}$. (The answer is an integer.)

(c) If a nonzero vector $\mathbf{v}$ lies at an angle 30° counterclockwise from the positive x-axis, what is the unit vector in the same direction as $\mathbf{v}$? (Draw a picture to get an idea.) What if 30° is replaced with a general angle θ?

Problem 4: Vector operations with data

Suppose there are three students in Math 51 with the following components for their course grades:
Student 1: 81/100 on homework, 83/100 on midterm A, 70/100 on midterm B, 75/100 on the final.
Student 2: 73/100 on homework, 75/100 on midterm A, 74/100 on midterm B, 88/100 on the final.
Student 3: 90/100 on homework, 95/100 on midterm A, 88/100 on midterm B, 92/100 on the final.

(a) Write down vectors $\mathbf{v}_{\mathrm{HW}}, \mathbf{v}_{\mathrm{A}}, \mathbf{v}_{\mathrm{B}}, \mathbf{v}_{\mathrm{Final}}$ (all in $\mathbf{R}^3$) representing respectively the grades as percentages on homework, midterm A, midterm B, and the final exam (e.g., for a score of $83/100$, the vector entry should be 83 rather than .83).

(b) Give a general formula as a linear combination of those four vectors in $\mathbf{R}^3$ for a 3-vector $\mathbf{v}_{\mathrm{CG}}$ whose entries are the course grades of the three students in order from student 1 to student 3, assuming the breakdown of the total grade for the course is 16% homework, 36% final, 24% each midterm, and then compute it (you may use a calculator).

