Last time:

- powers of symmetric matrices
- Hessian matrix and quadratic approximation of functions
- level sets of quadratic forms

Today:

- using the Hessian to analyze critical points
- critical points & contour plots 2.0

Problem 1: Unconstrained local extrema via Hessian

For each of the following functions $\mathbf{R}^2 \to \mathbf{R}$, use the gradient to find all critical points and characterize each critical point (i.e., local maximum, local minimum, saddle point, or otherwise) by computing the Hessian in general and analyzing it at each critical point.

(a) $x^4y^4 - 2x^2 - 2y^2$

Recall: If $\vec{a}$ is a critical pt of f, then

- if $(Hf)(\vec{a})$ is pos. def $\Rightarrow$ $\vec{a}$ is a local min
- if $(Hf)(\vec{a})$ is neg. def $\Rightarrow$ $\vec{a}$ is a local max

• if $(Hf)(\vec{a})$ is indefinite $\Rightarrow \vec{a}$ is a saddle pt.

If $(Hf)(\vec{a})$ pos.-semidef / neg-semidef (and not pos def. / neg. def), need more info.

$$\nabla f = \begin{bmatrix} 4x^3y^4 - 4x \\ 4x^4y^3 - 4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightsquigarrow$ $4x(x^2y^4 - 1) = 0$

$x = 0$ or $x^2y^4 = 1$

$\rightsquigarrow$ $4y(x^4y^2 - 1) = 0$

$y = 0$ or $x^4y^2 = 1$

$x = \pm y = \pm 1$

Crit pts: $(0,0), (1,1), (-1,-1), (1,-1), (-1,1)$

$$(Hf)(x,y) = \begin{bmatrix} 12x^2y^4 - 4 & 16x^3y^3 \\ 16x^3y^3 & 12x^4y^2 - 4 \end{bmatrix}$$

$(0,0)$:

$$(Hf)(0,0) = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

one e'val: -4

$\Rightarrow$ neg. def

$\Rightarrow \boxed{(0,0) \text{ local max}}$

$\underline{\pm(1,1):}$

$$(Hf)(\pm(1,1)) = \begin{bmatrix} 8 & 16 \\ 16 & 8 \end{bmatrix}$$

$\det = 8^2 - 16^2 < 0$

$\rightsquigarrow$ two e'vals have opposite signs

$\Rightarrow$ indefinite

$\Rightarrow$ $\boxed{(1,1), (-1,-1) \text{ saddle pts}}$

In general: $\lambda^2 - \mathrm{tr}(Hf)\lambda + \det(Hf) = (\lambda-\lambda_1)(\lambda-\lambda_2)$

$\det(Hf)$ = product of e'vals

$\mathrm{tr}(Hf)$ = sum of e'vals

" $f_{xx} + f_{yy}$

$\underline{\pm(1,-1):}$

$$(Hf)(\pm(1,-1)) = \begin{bmatrix} 8 & -16 \\ -16 & 8 \end{bmatrix}$$

$\det = 8^2 - 16^2 < 0$

$\Rightarrow$ indefinite

$\Rightarrow$ $\boxed{(1,-1), (-1,1) \text{ saddle pts}}$

(b) $-3x^2 + 2xy - (3/2)y^2 + y^3$

$$Df = \begin{bmatrix} -6x + 2y \\ 2x - 3y + 3y^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{aligned} 6x &= 2y \\ 3x &= y \end{aligned}$$

$\leadsto 2x - 9x + 27x^2 = 0$

$x(27x - 7) = 0$

$x = 0$ or $x = \frac{7}{27}$

<u>Crit pts:</u> $(0,0), (7/27, 7/9)$

$$Hf = \begin{bmatrix} -6 & 2 \\ 2 & -3+6y \end{bmatrix}$$

$$(Hf)(0,0) = \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}$$

det $= 18 - 4 > 0$

$\leadsto$ product of e'vals is > 0

$\Rightarrow$ e'vals have same sign

$\text{tr} \underset{}{=} -6-3 = -9 < 0$

$\Rightarrow$ sum of e'vals is < 0

$\Rightarrow$ both e'vals negative $\Rightarrow$ neg. def

$\Rightarrow$ $\boxed{(0,0) \text{ local max}}$

$$(Hf)(7/27, 7/9) = \begin{bmatrix} -6 & 2 \\ 2 & -3+\frac{14}{3} \end{bmatrix}$$

$$\det = -6\left(-3+\frac{14}{3}\right) - 4$$

$$= 18 - 28 - 4 < 0$$

$\rightsquigarrow$ e'vals have opposite signs

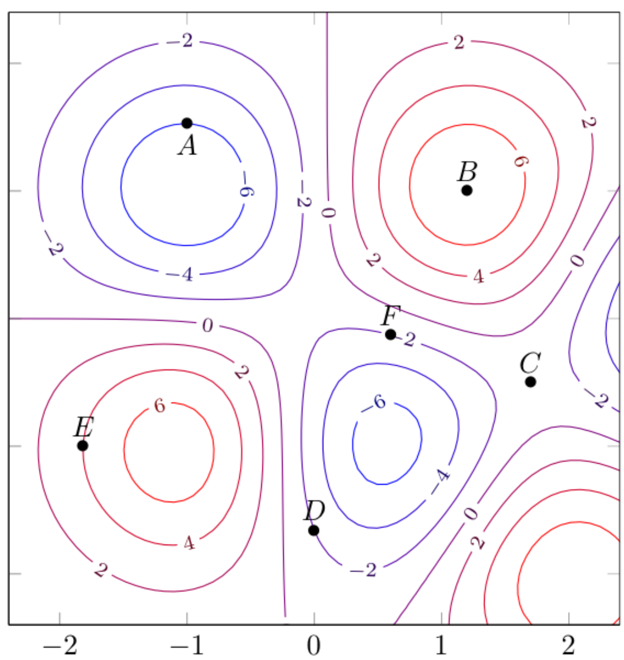
$\Rightarrow$ indefinite $\Rightarrow$ $\boxed{(7/27, 7/9) \text{ saddle pt}}$

Problem 2: Visually interpreting critical points

Consider the given contour plot for a function $f : \mathbf{R}^2 \to \mathbf{R}$.

(a) Assuming B is a critical point, is the Hessian matrix of f at B: (i) positive-definite, $\boxed{\text{(ii)}}$ negative-definite, or (iii) indefinite? (Assume it is one of these.)

(b) Assuming C is a critical point, is the Hessian matrix of f at C: (i) positive-definite, (ii) negative-definite, or $\boxed{\text{(iii)}}$ indefinite? (Assume it is one of these.)



B local max $\Rightarrow$ $(Hf)(B)$ neg. def

C saddle pt $\Rightarrow$ $(Hf)(C)$ indefinite

Recall: max or min looks like 

saddle pt looks like 

Problem 3: Using Hessian eigenvalues to characterize critical points

Consider a critical point $\mathbf{a}$ of $f : \mathbf{R}^n \to \mathbf{R}$ whose Hessian $(\mathrm{H}f)(\mathbf{a})$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ for some orthogonal basis (as we are guaranteed always happens, by the Spectral Theorem). For each of the following possibilities for the list of eigenvalues, is the behavior of f at $\mathbf{a}$ a local maximum, local minimum, or saddle point? (It is one of these in each case below.)

(a) eigenvalues $43, 5, 1$ all $>0 \Rightarrow$ pos. def $\Rightarrow$ local min

(b) eigenvalues $5\,, -3, -7$ some >0, some $<0 \Rightarrow$ indefinite $\Rightarrow$ saddle pt

(c) eigenvalues $1, 0, -1$ some >0, some $<0 \Rightarrow$ indefinite $\Rightarrow$ saddle pt

(d) eigenvalues $1, 1, 1, 1$ all $>0 \Rightarrow$ pos. def $\Rightarrow$ local min

(e) eigenvalues $-1, -5$ all $<0 \Rightarrow$ neg. def $\Rightarrow$ local max

(f) $-1, -5, 0$ not enough info