Problem 1: Checking for Linearity

For the following functions F, analyze their interaction with vector addition and scalar multiplication to determine if they are linear or not. If not linear, give an explicit pair of vectors $\mathbf{v}, \mathbf{w}$ for which $F(\mathbf{v}+\mathbf{w}) \neq F(\mathbf{v}) + F(\mathbf{w})$ or an explicit vector $\mathbf{v}$ and scalar c for which $F(c\mathbf{v}) \neq c\,F(\mathbf{v})$.

(a) $f(x) = \begin{bmatrix} 2x \\ 2x+3 \end{bmatrix}$

(b) $g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + y.$

(c) $h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \\ x^2 \end{bmatrix}$

(d) $k\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \\ 0 \end{bmatrix}$

Solution:

(a) We have $f(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So f is not linear (e.g., $f(0 \cdot 0) \neq 0\,f(0)$). Alternatively, if you compute $f(x_1 + x_2)$ and $f(x_1) + f(x_2)$ algebraically then you'll see they are *never* the same:

$$f(x_1+x_2) = \begin{bmatrix} 2(x_1+x_2) \\ 2(x_1+x_2)+3 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ 2x_1+2x_2+3 \end{bmatrix}, \quad f(x_1)+f(x_2) = \begin{bmatrix} 2x_1 \\ 2x_1+3 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 2x_2+3 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ 2x_1+2x_2+6 \end{bmatrix},$$

with the second entries always differing by 3.

(b) We check

$$g\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = g\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = (x_1+x_2) + (y_1+y_2)$$

and

$$g\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + g\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = (x_1+y_1) + (x_2+y_2),$$

whose right sides are equal. Hence, $g(\mathbf{v}+\mathbf{w}) = g(\mathbf{v}) + g(\mathbf{w})$ for every $\mathbf{v}, \mathbf{w} \in \mathbf{R}^2$. Likewise, for every scalar c we have $g(c\begin{bmatrix} x \\ y \end{bmatrix}) = g(\begin{bmatrix} cx \\ cy \end{bmatrix}) = cx + cy$ and $c\,g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c(x+y)$, so again the right sides coincide and we conclude that $g(c\mathbf{v}) = c\,g(\mathbf{v})$ for every $\mathbf{v} \in \mathbf{R}^2$.

Hence, g is linear (as we can also see from the component functions, but the point is to understand linearity through the interaction with vector operations).

(c) This does not behave well for addition (nor for scalar multiplication). If we work it out explicitly,

$$h\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = h\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1+x_2) \\ y_1+y_2 \\ (x_1+x_2)^2 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ y_1+y_2 \\ x_1^2+2x_1x_2+x_2^2 \end{bmatrix}$$

whereas

$$h\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + h\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ y_1 \\ x_1^2 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ y_2 \\ x_2^2 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ y_1+y_2 \\ x_1^2+x_2^2 \end{bmatrix},$$

which differ by $2x_1x_2$ in the final entry. To make an explicit counterexample, we just make sure $x_1, x_2 \neq 0$, such as:

$$h\left(\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\right)=h\left(\begin{bmatrix}2\\0\end{bmatrix}\right)=\begin{bmatrix}4\\0\\4\end{bmatrix},\ h\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+h\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\0\\1\end{bmatrix}+\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}4\\0\\2\end{bmatrix}.$$

(d) This goes similarly to (b):

$$k\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=k\left(\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}\right)=\begin{bmatrix}2(x_1+x_2)\\y_1+y_2\\0\end{bmatrix}$$

and

$$k\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)+k\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=\begin{bmatrix}2x_1\\y_2\\0\end{bmatrix}+\begin{bmatrix}2x_2\\y_2\\0\end{bmatrix}=\begin{bmatrix}2x_1+2x_2\\y_1+y_2\\0\end{bmatrix};$$

the right sides are equal, so $k(\mathbf{v}+\mathbf{w})=k(\mathbf{v})+k(\mathbf{w})$ for every $\mathbf{v},\mathbf{w}\in\mathbf{R}^2$.

Likewise, for every scalar c we have

$$k\left(c\begin{bmatrix}x\\y\end{bmatrix}\right)=k\left(\begin{bmatrix}cx\\cy\end{bmatrix}\right)=\begin{bmatrix}2(cx)\\cy\\0\end{bmatrix},\ c\,k\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=c\begin{bmatrix}2x\\y\\0\end{bmatrix}=\begin{bmatrix}c(2x)\\cy\\0\end{bmatrix},$$

and the outcomes of both calculations agree. Hence, $k(c\mathbf{v})=c\,k(\mathbf{v})$ for every $\mathbf{v}\in\mathbf{R}^2$ and every scalar c.

We conclude that k is linear (as can also be seen from the component functions).

Problem 2: Composition of linear maps and matrix multiplication

(a) Let $R:\mathbf{R}^2\to\mathbf{R}^2$ be the operation that carries any vector $\mathbf{v}$ to its reflection across the y-axis. Explain in words or with a picture why $R\left(\begin{bmatrix}a\\b\end{bmatrix}\right)=\begin{bmatrix}-a\\b\end{bmatrix}$ for any a,b, so R is linear. Then calculate the 2×2 matrix A for which $R(\mathbf{v})=A\mathbf{v}$ for every $\mathbf{v}\in\mathbf{R}^2$.

(b) Find the matrix corresponding to the linear transformation that first rotates a vector in $\mathbf{R}^2$ by an angle of α counterclockwise and then reflects across the y-axis. Then find the matrix corresponding to the opposite order of these operations (it is a different matrix when $\alpha\neq 0,\pi$, which is to say when $\sin\alpha\neq 0$).

(c) Draw a picture to illustrate visually (so without any appeal to matrices) why the composition of the two operations in (b) (rotation and reflection) depends on the order in which they are carried out when $\alpha\neq 0,\pi$.

Solution:

(a) The reflection of $\mathbf{v}$ across the y-axis corresponds to drawing a perpendicular line segment from the tip of $\mathbf{v}$ to the y-axis, so really a horizontal segment, and then continuing horizontally across the y-axis for the same distance on the other side. This process stays on the same horitzonal line, so the y-coordinate does not change. However, the x-coordinate is flipped across 0 with the same magnitude, so it is negated. These two conclusions express together exactly the desired formula for R. From this formula, the matrix is $A=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$ (or separately calculate $R(\mathbf{e}_1)$ and $R(\mathbf{e}_2)$ to compute the columns of A).

(b) We compute the matrix product $\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{bmatrix}=\begin{bmatrix}-\cos\alpha&\sin\alpha\\\sin\alpha&\cos\alpha\end{bmatrix}$. Composing the operations in the other order corresponds to the matrix product in the other order: $\begin{bmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{bmatrix}\begin{bmatrix}-1&0\\0&1\end{bmatrix}=\begin{bmatrix}-\cos\alpha&-\sin\alpha\\-\sin\alpha&\cos\alpha\end{bmatrix}$.

By inspection of the upper right and lower left entries, these two matrix products are different precisely when $\sin\alpha \neq 0$, which is to say $\alpha \neq 0, \pi$.

(c) Figure **??** demonstrates the transformation of rotating first and then reflecting:

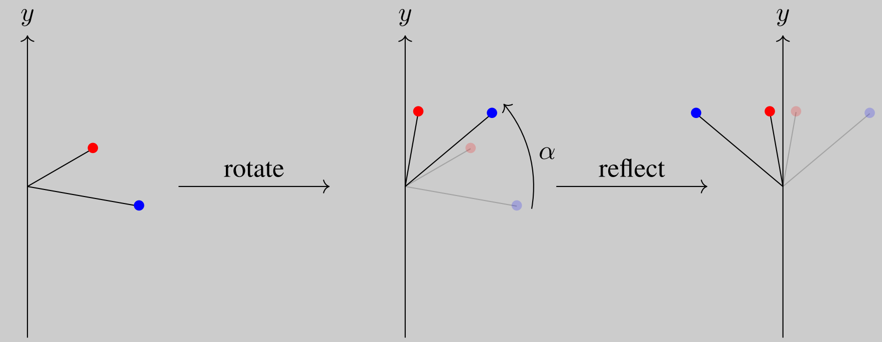

Figure 1: First rotating by α, and then reflecting.

On the other hand, Figure **??** demonstrates the transformation of *reflecting* first and then rotating:

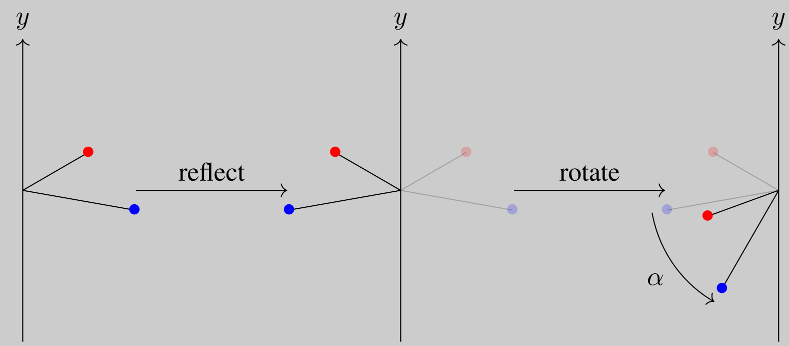

Figure 2: First reflecting, and then rotating by α.

It is clear from the figures that the two operations are not the same.

Problem 3: Algebra and geometry with hyperbolas

This problem studies hyperbolas, using some algebra and thinking (more instructive than a computer) to determine the basic geometry of a hyperbola from an equation. This will be useful later when relating contour plots to the multivariable second derivative test. You do *not* need a calculator for this exercise; human brainpower is sufficient!

(a) For $A, B, C > 0$, the equations $Ax^2 - By^2 = \pm C$ (one for each sign) are hyperbolas with asymptotes $y = \pm\sqrt{A/B}\,x$. For the pair of hyperbolas $H_\pm$ in each of (i) and (iii) below (treating each $\pm$ case separately), or else in each of (ii) and (iv) below (treating each $\pm$ case separately), compute:

- which of the coordinate axes each crosses (the x-axis consists of points $(u, 0)$, the y-axis consists of points $(0, v)$),
- the slopes $\pm m$ of the asymptotes,
- which coordinate axis is "nearer" to the asymptotes (note: if the slope c of a line $y = cx$ satisfies $|c| < 1$ then the line is "closer" to the x-axis, whereas if $|c| > 1$ then the line is "closer" to the y-axis; draw the cases $c = \pm 2, \pm 1/2$ to see why this is reasonable).

(i) $x^2 - 6y^2 = \pm 10$, (ii) $3x^2 - 5y^2 = \pm 13$, (iii) $7x^2 - 2y^2 = \pm 18$, (iv) $-5x^2 + y^2 = \pm 21$

(b) Use the information found in (a) to approximately draw each pair of hyperbolas (both signs) on a common coordinate grid (one grid for each pair), indicating the approximate axis crossings and the asymptotes drawn "closer" to the appropriate coordinate axis (a qualitatively correct picture is sufficient).

(c) Based on your work, for a general hyperbola $Ax^2 - By^2 = C$ with $C \neq 0$ and both A and B with the same sign, how does the coordinate axis (x or y) to which the asymptotes are "closer" related to how $|A|$ compares to $|B|$ (in terms of which is bigger and which is smaller)?

Solution:

(a) We treat each pair of hyperbolas in turn.

(i) The x-intercept corresponds to setting $y = 0$, so it is $x^2 = \pm 10$, which has no solution for -10. Thus, H_- does not meet the x-axis, whereas H_+ meets the x-axis at solutions of $x^2 = 10$, which is to say $\pm\sqrt{10}$. The y-intercept corresponds to setting $x = 0$, so it is $-6y^2 = \pm 10$. This has no solution for 10, so H_+ does not meet the y-axis, whereas H_- meets the y-axis at solutions of $-6y^2 = -10$, which is to say $y^2 = (-10)/(-6) = 5/3$, or equivalently $y = \pm\sqrt{5/3}$.

Since $9 < 10 < 16$ and $1 < 5/3 < 4$ (as $5/3$ is between 1 and 2), we have $\pm\sqrt{10}$ lies between ± 3 and ± 4 and likewise $\pm\sqrt{5/3}$ lies between ± 1 and ± 2. The asymptotes are $x^2 - 6y^2 = 0$, which is to say $y^2 = (1/6)x^2$, so $y = \pm\sqrt{1/6}\,x$. These asymptotes have slopes $\pm\sqrt{1/6}$ with absolute value < 1, so they are "closer" to the x-axis.

(ii) The x-intercept corresponds to setting $y = 0$, so it is $3x^2 = \pm 13$, which has no solution for -13. Thus, H_- does not meet the x-axis, whereas H_+ meets the x-axis at solutions of $3x^2 = 13$, which is to say $x = \pm\sqrt{13/3}$. The y-intercept corresponds to setting $x = 0$, so it is $-5y^2 = \pm 13$. This has no solution for 13, so H_+ does not meet the y-axis, whereas H_- meets the y-axis at solutions of $-5y^2 = -13$, which is to say $y^2 = 13/5$; i.e., $y = \pm\sqrt{13/5}$.

Since $4 < 13/3 < 9$ (as $13/3$ lies between 4 and 5) and $1 < 13/5 < 4$ (as $13/5$ lies between 2 and 3), we have $\pm\sqrt{13/3}$ lies between ± 2 and ± 3, and likewise $\pm\sqrt{13/5}$ lies between ± 1 and ± 2. The asymptotes are $3x^2 - 5y^2 = 0$, which is to say $y^2 = (3/5)x^2$, so $y = \pm\sqrt{3/5}\,x$. These asymptotes have slopes $\pm\sqrt{3/5}$ with absolute value < 1, so "closer" to the x-axis.

(iii) The x-intercept corresponds to setting $y = 0$, so it is $-7x^2 = \pm 18$, which has no solution for 18. Thus, H_+ does not meet the x-axis, whereas H_- meets the x-axis at solutions of $-7x^2 = -18$; i.e., $x = \pm\sqrt{18/7}$. The y-intercept corresponds to setting $x = 0$, so it is $2y^2 = \pm 18$. This has no solution for -18, so H_- does not meet the y-axis, whereas H_+ meets the y-axis at solutions of $2y^2 = 18$, which is to say $y^2 = 9$; i.e., $y = \pm 3$.

Since $1 < 18/7 < 4$ (as $18/7$ lies between 2 and 3), we have $\pm\sqrt{18/7}$ lies between ± 1 and ± 2. The asymptotes are $-7x^2 + 2y^2 = 0$, which is to say $y^2 = (7/2)x^2$, so $y = \pm\sqrt{7/2}\,x$. These asymptotes have slopes $\pm\sqrt{7/2}$ with absolute value > 1, so "closer" to the y-axis.

(iv) The x-intercept corresponds to setting $y = 0$, so it is $-5x^2 = \pm 21$, which has no solution for 21. Thus, H_+ does not meet the x-axis, whereas H_- meets the x-axis at solutions of $-5x^2 = -21$, which is to say $x^2 = (-21)/(-5) = 21/5$; i.e., $x = \pm\sqrt{21/5}$. The y-intercept corresponds to setting $x = 0$, so it is $y^2 = \pm 21$. This has no solution for -21, so H_- does not meet the y-axis, whereas H_+ meets the y-axis at solutions of $y^2 = 21$, which is to say $y = \pm\sqrt{21}$.

Since $4 < 21/5 < 9$ (as $21/5$ lies between 4 and 5), we have $\pm\sqrt{21/5}$ lies between ± 2 and ± 3. Since $16 < 21 < 25$, we have $\pm\sqrt{21}$ lies between ± 4 and ± 5. The asymptotes are $-5x^2 + y^2 = 0$, which is to say $y^2 = 5x^2$, so $y = \pm\sqrt{5}\,x$. These asymptotes have slopes $\pm\sqrt{5}$ with absolute value > 1, so "closer" to the y-axis.

(b) The pictures for (i) and (ii) are respectively on the left and right in Figure **??** (all that is expected is approximate accuracy in the axis crossings and qualitative accuracy with the asymptotes being "closer" to the correct coordinate axis).

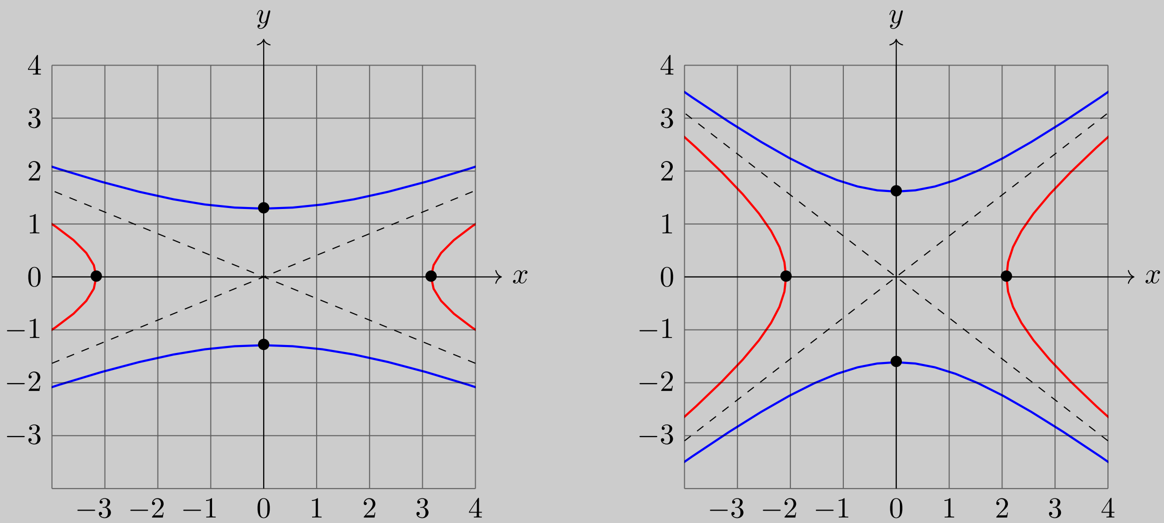

Figure 3: The hyperbolas $x^2 - 6y^2 = 10$ (red) and $x^2 - 6y^2 = -10$ (blue) on the left, and $3x^2 - 5y^2 = 13$ (red) and $3x^2 - 5y^2 = -13$ (blue) on the right

The picture for (iii) and (iv) are respectively on the left and right in Figure **??** (all that is expected is approximate accuracy in the axis crossings and qualitative accuracy with the asymptotes being "closer" to the correct coordinate axis).

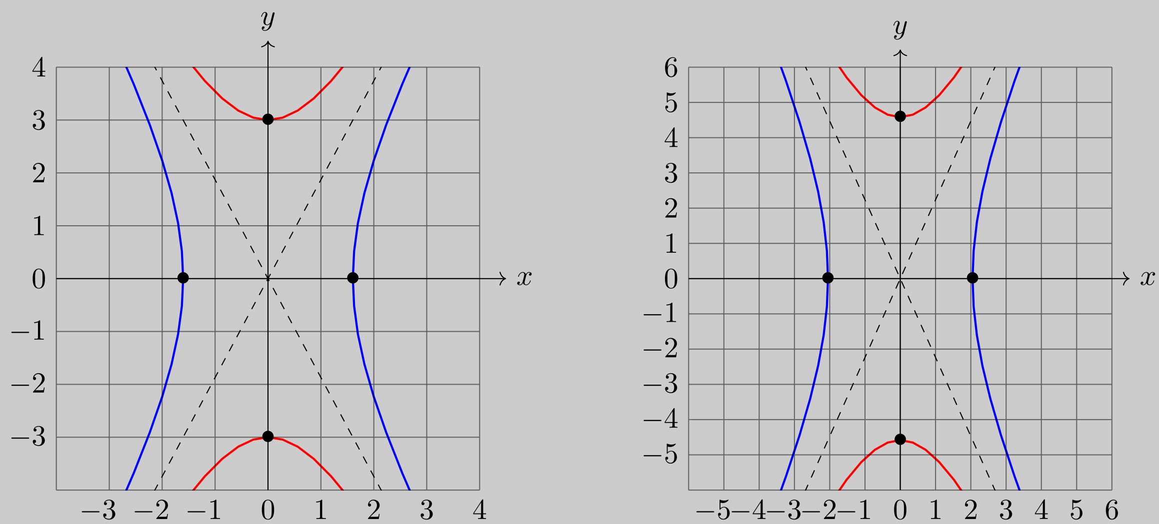

Figure 4: The hyperbolas $-7x^2 + 2y^2 = 18$ (red) and $-7x^2 + 2y^2 = -18$ (blue) on the left, and $-5x^2 + y^2 = 21$ (red) and $-5x^2 + y^2 = -21$ (blue) on the right.

(c) The pattern is that in all cases, whichever of x^2 or y^2 has its coefficient with the *smaller* absolute value corresponds to the coordinate axis to which the asymptotes are *closer*.