1. (a) The partial derivatives of f are

$$f_x = 3x^2 - 6x - 6y + 9, \;\; f_y = -6x + 6y.$$

The vanishing of the second says $x = y$, and then if we plug that into the first and set it to be 0 we get

$$0 = 3x^2 - 12x + 9 = 3(x-1)(x-3),$$

so $x = 1, 3$. Thus, the two critical points are $(1,1), (3,3)$.

(b) On $x = 1$, we get the function $f(1, y) = 7 - 6y + 3y^2$ that is a quadratic polynomial having positive quadratic coefficient, so it has a local minimum where its derivative vanishes: at $y = 1$. Thus, $(1,1)$ is a local minimum for f on this line. On $y = x$, we get the function

$$g(x) = f(x, x) = x^3 - 6x^2 + 9x$$

that satisfies $g'(x) = 3x^2 - 12x + 9$, which vanishes at $x = 1$. But $g''(x) = 6x - 12$ is negative at $x = 1$, so $x = 1$ is a local maximum for g, which is to say f on the line $y = x$ has a local maximum at $(1,1)$. These opposite behaviors along different lines through $(1,1)$ shows this point is a saddle point.

(c) The x-axis consists of points $(x, 0)$, and $f(x, 0) = x^3 - 3x^2 + 9x$ is a cubic polynomial. This is arbitrarily big and positive for large $x > 0$, and it is arbitrarily negative for very negative x, so by the Intermediate Value Theorem this polynomial (like all cubics) takes on all real values. Hence, there is no global extremum for f on $\mathbf{R}^2$.

◇

2. (a) The partial derivatives are

$$f_x = 6xy + 6y, \;\; f_y = 3x^2 + 3y^2 + 6x.$$

The vanishing of the first says $0 = 6xy + 6y = 6y(x+1)$, so $y = 0$ or $x = -1$. The vanishing of the second says (after cancelling 3 throughout) $x^2 + y^2 + 2x = 0$. In case $y = 0$, this second condition says $x^2 + 2x = 0$, or equivalently $x(x+2) = 0$, so $x = 0$ or $x = -2$. In case $x = -1$, this second condition says $1 + y^2 - 2 = 0$, or equivalently $y^2 = 1$, so $y = 1$ or $y = -1$.

Putting it all together, we have obtained 4 points: $(0,0), (-2,0), (-1,1), (-1,-1)$.

(b) Looking at $f(x, y)$ on the lines $y = x$ and $y = -x$ amounts to looking at the functions $f(x, x)$ and $f(x, -x)$. The first of these is $g_1(x) = f(x, x) = 3x^3 + x^3 + 6x^2 = 4x^3 + 6x^2$ and the second is $g_2(x) = f(x, -x) = -3x^3 - x^3 - 6x^2 = -4x^3 - 6x^2$ (which is the negative of $f(x, x)$, a coincidence for this particular f).

We will show that g_1 has a local minimum at $x = 0$ (corresponding to the point $(0,0)$) and that g_2 has a local maximum at $x = 0$ (corresponding to the point $(0, -0) = (0,0)$), so the saddle point property will then be established. Since $g_2(x) = -g_1(x)$, once we have settled the case of g_1 then the case of g_2 will follow immediately since its behavior is exactly the negative of g_1 (and when a function of x is negated, local minima are turned into local maxima because the graph flips upside-down); one could also just repeat for g_2 the calculations we are about to do for g_1, but we have explained why this isn't necessary to do.

Now for the calculations with g_1 at $x = 0$: we have $g_1'(x) = 12x^2 + 12x$, which vanishes at $x = 0$, and $g_1''(x) = 24x + 12$. Since $g_1''(0) = 12$ is positive, it follows that $x = 0$ is a local minimum for g_1 as claimed.

◇

3. (a) Here is a picture of the domain D.

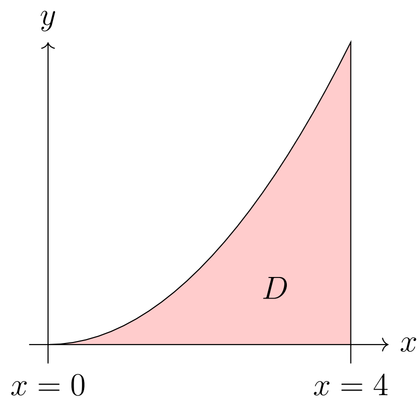

Figure 1: A region defined by a parabola and two lines

The partial derivatives are $f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$, so the vanishing of both says $x^2 = y$ and $y^2 = x$. But on the interior we have $y < x^2$, so the vanishing of f_x is impossible here.

(b) The boundary consists of three parts: the parabolic arc $C_1 = \{(x, x^2) : 0 \leq x \leq 4\}$, the segment $C_2 = \{(x, 0) : 0 \leq x \leq 4\}$ in the x-axis, and the segment $C_3 = \{(4, y) : 0 \leq y \leq 4^2 = 16\}$ inside the line $x = 4$. We analyze the behavior of f on each of these in turn, finding extrema on each part and then comparing the values of f at all of them.

On C_1 we have $g_1(x) = f(x, x^2) = x^3 + x^6 - 3x^3 = x^6 - 2x^3$ with $0 \leq x \leq 4$, on C_2 we have $g_2(x) = f(x, 0) = x^3$ with $0 \leq x \leq 4$, and on C_3 we have $g_3(y) = f(4, y) = 64 + y^3 - 12y$ with $0 \leq y \leq 16$. We need to find the extrema for each function on the indicated closed interval, which is a single-variable calculus problem (always remembering to check the endpoints too).

For C_1 we compute $g_1'(x) = 6x^5 - 6x^2 = 6x^2(x^3 - 1)$, which vanishes at $x = 0$ and $x = 1$. So on C_1 we need to check for extrema at the endpoints $x = 0, 4$ and $x = 1$. The values are $g_1(0) = 0$, $g_1(1) = -1$, and $g_1(4) = 3968$. For C_2 we either observe by inspection of x^3 (as an increasing function) or by computing the derivative $3x^2$ noting it doesn't vanish on $(0, 4)$ that the extrema are at the endpoints $x = 0, 4$, with $g_2(0) = 0$ and $g_2(4) = 64$. Finally, for C_3 we compute $g_3'(y) = 3y^2 - 12 = 3(y^2 - 4)$ which vanishes at $y = \pm 2$. Working on the interval $[0, 4]$, we only need to consider its behavior at the interior point $y = 2$ since the endpoints correspond to endpoints of C_1 and C_2 that we have already looked at. The value is $g_3(2) = 48$.

(c) Comparing these values of the g_j's and looking for the biggest and smallest, the biggest value is 3968 attained at $(4, 16)$ and the smallest is -1 attained at $(1, 1)$.

◇

4. The partial derivatives of f are

$$f_x = 4x^3 - 4x + 4y, \qquad f_y = 4y^3 + 4x - 4y.$$

Setting both to 0, we have

$$x^3 = x - y, \qquad y^3 = y - x.$$

Hence $x^3 = -y^3$, so $x = -y$ and so $x^3 = x - y = 2x$. We conclude that $0 = x^3 - 2x = x(x^2 - 2)$, so $x = 0, \pm\sqrt{2}$. Since we saw that $y = -x$ at the critical points, these points are $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$. ◇

5. (a) The gradient directions at the points P, Q, R, S, T are shown in the picture below. There is no gradient shown at T because T is a critical point.

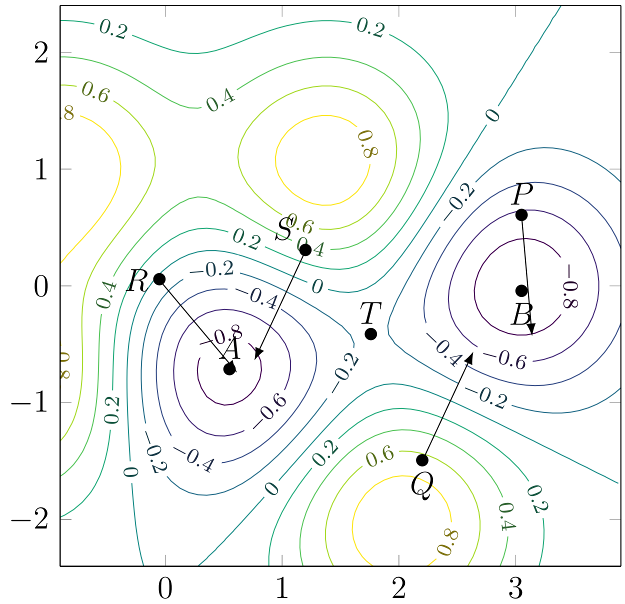

Figure 2: The negative gradient directions at P, Q, R, S, T. Note that T is a critical (saddle) point, and so the gradient at T vanishes. The points A and B are local minima.

(b) Gradient descent starting at R or S likely will end at the local minimum A. The gradient descent starting at P or Q likely will end at the local minimum B. The gradient descent starting at T will stay at T forever (since T is a critical point). $\diamond$

6. The tangent at P to C is parallel to the y-axis precisely when the normal vector to it is parallel to the x-axis, which is to say it has vanishing y-component and nonvanishing x-component. The gradient at P (when nonzero) specifies the direction of the normal vector at P, so in particular we want

$$0 = \frac{\partial}{\partial y}(3yx^2 + 3xy^2 + y^3 + 2x^3) = 3(x^2 + 2xy + y^2) = 3(x+y)^2.$$

This means that at such points P, we must have $x = -y$. We also need these points P on C to satisfy

$$0 \neq \frac{\partial}{\partial x}(3yx^2 + 3xy^2 + y^3 + 2x^3) = 6yx + 3y^2 + 6x^2.$$

This indeed holds if $x = -y$, because then $6yx + 3y^2 + 6x^2 = -6y^2 + 3y^2 + 6y^2 = 3y^2$, which cannot vanish unless both y and x $(= -y)$ vanish; yet the origin doesn't satisfy the equation defining C.

To find the points P on C of the form $(-y, y)$, we plug $x = -y$ into the equation defining C to get $3y^3 - 3y^3 + y^3 - 2y^3 = 27$, which says $y^3 = -27$ or equivalently $y = -3$, so $x = 3$. Hence, there is exactly one such point: $(3, -3)$. $\diamond$

7. (a) The normal vector to the ellipsoid at P is given by the gradient $\begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix}$ of the defining equation evaluated at $P = (1, -1, 2)$, which is $\begin{bmatrix} 4 \\ -6 \\ 4 \end{bmatrix}$. The normal vector to the hyperboloid at P is given by the gradient $\begin{bmatrix} -6x \\ 12y \\ 2z \end{bmatrix}$ of the defining equation evaluated at $P = (1, -1, 2)$, which is $\begin{bmatrix} -6 \\ -12 \\ 4 \end{bmatrix}$.

(b) A nonzero vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ along the tangent direction is perpendicular to both normal directions in (a) (since the tangent line is where those two tangent planes meet). Hence,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -6 \\ 4 \end{bmatrix} = 0, \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} -6 \\ -12 \\ 4 \end{bmatrix} = 0.$$

Evaluating the dot products, this says

$$4x - 6y + 4z = 0$$

and

$$-6x - 12y + 4z = 0.$$

Solving these yields

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 6 \\ -10 \\ -21 \end{bmatrix}$$

for a scalar k, so we can use $\mathbf{v} = \begin{bmatrix} 6 \\ -10 \\ -21 \end{bmatrix}$. The tangent line in parametric form is therefore

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ -10 \\ -21 \end{bmatrix} = \begin{bmatrix} 1+6t \\ -1-10t \\ 2-21t \end{bmatrix}$$

for $t \in \mathbf{R}$.

◇

8. Equality of tangent planes through a common point P is the same as having a common normal direction (by the point-normal form for a plane). The normal direction is given by the gradient of the defining equation (when the gradient is nonzero), so we want the gradients to be nonzero and scalar multiples of each other.

   We have

$$(\nabla f)(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}, \quad (\nabla g)(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix}.$$

   The latter is never zero since its final entry is always 1, and the former only vanishes at the origin, which is not on S (so it cannot be a point in common with S and a level set of f). So both gradients are nonzero at any possible point in common, so the condition of a common tangent plane is the same as saying that ∇f is a scalar multiple of ∇g at such a point.

   From inspecting the first two entries in each of the vectors $(\nabla f)(x, y, z)$ and $(\nabla g)(x, y, z)$, as long as one of x or y is nonzero we see that the scalar multiplier must be 1, and then comparing third entries in these gradient vectors forces $z = 1/2$. The equation of S then says $x^2 + y^2 = 1/2$, in which case

$$f(x, y, z) = x^2 + y^2 + z^2 = (x^2 + y^2) + 1/4 = 1/2 + 1/4 = 3/4.$$

   The collection of such points is a circle of radius $1/\sqrt{2}$ in the plane $z = 1/2$ centered at the "origin" $(0, 0, 1/2)$ in that plane.

   Suppose instead that $x = y = 0$, so the equation of S forces $z = 1$, and hence $P = (0, 0, 1)$. But we forbade this possibility for P in the initial setup. ◇

9. (a) Setting $g(x, y, z) = x^2 + y^2 - 4z^2$, the constraint is given by the level set $g(x, y, z) = 1$. Thus, using Theorem 12.2.1 , the extrema of f subject to $g(x, y, z) = 1$ occur only when $\nabla g = \mathbf{0}$ or $\nabla f = \lambda(\nabla g)$. Now, $(\nabla g)(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ -8z \end{bmatrix}$. This is zero only at the origin, which does not satisfy $g(x, y, z) = 1$. So we can focus on points where the second case happens.

We have $\nabla f = \begin{bmatrix} z \\ z \\ x+y \end{bmatrix}$. Thus, we have the vector equation $\nabla f = \lambda \nabla g$ for some unknown scalar λ; this encodes three scalar equations which together with the constraint equation gives a combined system of 4 equations:

$$z = 2\lambda x, \;\; z = 2\lambda y, \;\; x + y = -8\lambda z, \;\; x^2 + y^2 - 4z^2 = 1.$$

To carry out the method of "solving for λ" with each of the first three equations and setting the resulting expressions equal to each other, as always we have to be careful about division by zero.

If we don't worry about division by zero, the expressions we get are:

$$\lambda = \frac{z}{2x}, \;\; \lambda = \frac{z}{2y}, \;\; \lambda = \frac{x+y}{-8z},$$

so all three right sides are equal to each other provided we avoid situations in which any of their denominators vanish. So let's first dispose of cases of such vanishing denominators: this means that at least one of x, y, or z vanishes. We treat these one at a time.

If $x = 0$ then the first equation in our combined system forces $z = 0$, and then the third equation in the combined system forces $x+y = 0$ and hence $y = 0$ (since we're assuming $x = 0$), but $(0, 0, 0)$ violates the constraint equation $g(x, y, z) = 1$. Hence, the possibility $x = 0$ cannot occur.

Likewise, if $y = 0$ then the second equation in our combined system forces $z = 0$, so the third equation in the combined system forces $x + y = 0$ once again, so now $x = 0$ (as we are assuming $y = 0$), but we already ruled out the possibility $x = 0$. So now the possibility $y = 0$ has been ruled out.

Finally, if $z = 0$ then the third equation in the combined system forces $x + y = 0$, so $y = -x$, and the first and second equations in the combined system becomes $0 = 2\lambda x$ and $0 = -2\lambda x$. But we already ruled out the possibility $x = 0$, so necessarily $\lambda = 0$. The constraint equation gives us more information: $1 = x^2 + y^2 - 4(0)^2 = 2x^2$ (since $y = -x$), so $x = \pm 1/\sqrt{2}$ and $y = -x = \mp 1/\sqrt{2}$. So in this way we have arrived at two special points that will require separate consideration: $(1/\sqrt{2}, -1/\sqrt{2}, 0)$ and $(-1/\sqrt{2}, 1/\sqrt{2}, 0)$ (with $\lambda = 0$, but that is not going to matter).

Putting that into the fridge for later, now we can look into equating the preceding various fraction expressions we obtained for λ when all the denominators are nonzero. Equating them all yields the triple equality

$$\frac{z}{2x} = \frac{z}{2y} = \frac{x+y}{-8z}.$$

Since the denominators are all nonzero, so $z \neq 0$, the first equality tells us that $2x = 2y$, so $x = y$. Then the equating the second and third expressions (or equivalently the first and third) gives upon cross-multiplying

$$(-8z)z = 2y(x + y) = 2x(x + x) = 4x^2,$$

so $-2z^2 = x^2$. But $x, z \neq 0$, so the left side is negative and the right side is positive, so this is impossible.

The upshot is that the only candidate points for local extrema, let along global extrema, on the constraint surface $g = 1$ are $(1/\sqrt{2}, -1/\sqrt{2}, 0)$ and $(-1/\sqrt{2}, 1/\sqrt{2}, 0)$.

(b) Evaluating at the two candidates from (a), $f(1/\sqrt{2}, -1/\sqrt{2}, 0) = 0$ and $f(-1/\sqrt{2}, 1/\sqrt{2}, 0) = 0$. On the other hand, we can find points on the hyperboloid such as $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ where f is positive (since $f(1, 2, 1) = 3$). Similarly, we can find points on the hyperboloid such as $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ where f is negative (since $f(-2, 1, 1) = -1$). Thus, the value 0 is not a global maximum nor minimum. But these were the only possibilities, so f has no global extrema on the hyperboloid.

◇

10. We have the (budget) constraint

$$20 = (7/16)x + 2y + (4/3)z,$$

which we can think of as a level set $g(x, y, z) = 20$, where $g(x, y, z) = (7/16)x + 2y + (4/3)z$. Our aim is to optimize $U(x, y, z)$ subject to this constraint. The domain of U is $x, y, z > 0$.

The Lagrange multipliers method says that the maximum occurs when either $(\nabla g)(x,y,z) = \mathbf{0}$ or $(\nabla U)(x,y,z) = \lambda \cdot (\nabla g)(x,y,z)$. We can compute both gradients:

$$(\nabla g)(x,y,z) = \begin{bmatrix} 7/16 \\ 2 \\ 4/3 \end{bmatrix}$$

and

$$(\nabla U)(x,y,z) = (1/7)\begin{bmatrix} 4x^{-3/7}y^{1/7}z^{2/7} \\ x^{4/7}y^{-6/7}z^{2/7} \\ 2x^{4/7}y^{1/7}z^{-5/7} \end{bmatrix}.$$

Thus, $(\nabla g)(x,y,z) \neq \mathbf{0}$ everywhere. Hence, the maximum satisfaction (x,y,z) must satisfy the system of equations

$$\begin{cases} (4/7)x^{-3/7}y^{1/7}z^{2/7} & = (7/16)\lambda \\ (1/7)x^{4/7}y^{-6/7}z^{2/7} & = 2\lambda \\ (2/7)x^{4/7}y^{1/7}z^{-5/7} & = (4/3)\lambda \\ (7/16)x + 2y + (4/3)z & = 20 \end{cases}$$

for some $\lambda \in \mathbf{R}$. $\diamond$

11. (a) We first compute the gradient of f:

$$(\nabla f)(x,y,z) = \begin{bmatrix} 4x+5y \\ 5x+2y \\ 4 \end{bmatrix}.$$

Because the third coordinate doesn't vanish, f has no critical points in $\mathbf{R}^3$. Hence, it has none in the interior of the R.

(b) Our constraint is $g(x,y,z) = 1$, where $g(x,y,z) = x+y+z$. Thus, $(\nabla g)(x,y,z) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This doesn't vanish anywhere, so the method of Lagrange multipliers says that our candidate extrema on the given region all satisfy

$$\begin{bmatrix} 4x+5y \\ 5x+2y \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

From equality in the third coordinate, we see that $\lambda = 4$. We can then solve for x, y based on equality in the first two coordinates:

$$\begin{cases} 4x+5y = 4 \\ 5x+2y = 4 \end{cases}.$$

Multiply the first equation by 2 and the second by 5 to obtain $8x+10y = 8$ and $25x+10y = 20$. Subtracting these, we obtain $17x = 12$, so $x = 12/17$. Plugging this back in to the second equation, we have $y = 4/17$. For our point to satisfy the constraint $x+y+z = 1$, we must therefore have $z = 1/17$. So we have a single candidate extremum on this region: $(12/17, 4/17, 1/17)$.

(c) We have

$$f(t, 1-t, 0) = 2t^2 + 5t(1-t) + (1-t)^2 = 2t^2 + 5t - 5t^2 + 1 - 2t + t^2 = -2t^2 + 3t + 1$$

with $0 \le t \le 1$. This function has graph that is an upside-down parabola with vertex at $t = 3/4$ where it attains its maximum $-2(3/4)^2 + 3(3/4) + 1 = 34/16 = 17/8$ (as could also be deduced using calculus, but that is overkill for parabolas). So on this edge the maximal value $17/8$ is attained at $(3/4, 1/4, 0)$.

(d) Since there are no critical points in the interior of the R, we just have to compare the two values $f(3/4, 1/4, 0) = 17/8$ (slightly bigger than 2) and

$$\begin{aligned} f\left(\frac{12}{17}, \frac{4}{17}, \frac{1}{17}\right) &= 2\left(\frac{12}{17}\right)^2 + 5\left(\frac{12}{17} \cdot \frac{4}{17}\right) + \left(\frac{4}{17}\right)^2 + 4 \cdot \frac{1}{17} \\ &= \frac{2 \cdot 12^2 + 5 \cdot 12 \cdot 4 + 4^2 + 4 \cdot 17}{17^2} \\ &= \frac{612}{289} \end{aligned}$$

(which happens to equal 36/17, but we don't expect you to notice that, though it makes more visible that this also is slightly bigger than 2). There are two ways to determine which of these two values is bigger: use a calculator to compute $17/8 = 2.125$ whereas $612/289 \approx 2.1176$ which is slightly smaller, or you can cross-multiply: to check if $17/8 > 36/17$ or not it is the same to check if $17^2 > 8 \cdot 36$ or not, and $17^2 = 289$ whereas $8 \cdot 36 = 288$, just barely (!) less than 289. So either way, the maximal value for f on R is $17/8$, occurring at $(3/4, 1/4, 0)$.

◇

12. (a) Substitute $y = z = 0$ into the plane equation to get $ax_0 = d$, so $x_0 = d/a$. Similarly, substitute in $x = z = 0$ to get $y_0 = d/b$, and substitute in $x = y = 0$ to get $z_0 = d/c$.

(b) Since $x_0, y_0, z_0 > 0$, the answers in (a) give that $d \neq 0$ too. The volume of the tetrahedron is

$$f(a,b,c,d) = \frac{x_0 y_0 z_0}{6} = \frac{d^3}{6abc}.$$

We want to minimize $f(a,b,c,d)$ subject to the constraint $g(a,b,c,d) = a + 2b + 3c - d = 0$.
The gradients are

$$\nabla f = -\frac{d^3}{6abc}\begin{bmatrix} 1/a \\ 1/b \\ 1/c \\ -3/d \end{bmatrix}, \qquad \nabla g = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}.$$

Constrained extrema of $f(a,b,c,d)$ can only occur where $\nabla g(a,b,c,d) = \mathbf{0}$ or $\nabla f = \lambda \nabla g$ for some λ. The first option is impossible, so we have to study $\nabla f = \lambda \nabla g$. This expresses three scalar equations

$$-\frac{d^3}{6a^2bc} = \lambda, \qquad -\frac{d^3}{6ab^2c} = 2\lambda, \qquad -\frac{d^3}{6abc^2} = 3\lambda, \qquad \frac{3d^3}{6abcd} = -\lambda.$$

Solving for λ in each gives four expressions for λ:

$$-\frac{d^3}{6a^2bc}, \quad -\frac{d^3}{12ab^2c}, \quad -\frac{d^3}{18abc^2}, \quad -\frac{3d^3}{6abcd}.$$

Setting these equal to each other, we can cancel $d^3/(6abc)$ throughout to get $a = 2b = 3c = d/3$. Hence, the volume is

$$f(a,b,c,d) = f(a, a/2, a/3, 3a) = \frac{(3a)^3}{6a^3/6} = 27;$$

the unknown a has canceled out (as is reasonable, since the planar equation could have always been scaled anyway). This must then be the minimal volume.

◇

