Problem 1: Linear approximation

Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = 4x + y^3 + xy$.

(a) Compute the gradient $(\nabla f)(x, y)$, and then use it to give the linear approximation to f at $(1, 1)$.

(b) Using your answer to (a), estimate $f(0.9, 1.2)$. Compare your answer to the exact result on a calculator, and compare the effort in computing the approximation by hand versus the exact answer by hand.

(c) Give the linear approximation to f at $(2, -2)$. Use it to estimate $f(3, -1)$, and then compare this estimate to the exact value using a calculator. Why is the approximation so bad?

Solution:

(a) By computing partial derivatives of f, we have $(\nabla f)(x, y) = \begin{bmatrix} 4 + y \\ 3y^2 + x \end{bmatrix}$, so $(\nabla f)(1, 1) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Hence, for s, t near 0 we have

$$f(1 + s, 1 + t) \approx f(1, 1) + \begin{bmatrix} 5 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} s \\ t \end{bmatrix} = 6 + 5s + 4t,$$

or equivalently

$$f(x, y) \approx 6 + 5(x - 1) + 4(y - 1) = 5x + 4y - 3$$

for (x, y) near $(1, 1)$.

(b) Using $s = -0.1$ and $t = .2$, we have

$$f(0.9, 1.2)) \approx 6 + 5(-0.1) + 4(.2) = 6 - 0.5 + 0.8 = 6.3.$$

The exact value on a calculator is 6.408, so we got a pretty good approximation considering that carrying out the exact calculation by hand is rather more tedious due to the cubing involved.

(c) By (a), $(\nabla f)(2, -2) = \begin{bmatrix} 2 \\ 14 \end{bmatrix}$, so the linear approximation at $(2, -2)$ says that for s, t near 0 we have

$$f(2 + s, -2 + t) \approx f(2, -2) + \begin{bmatrix} 2 \\ 14 \end{bmatrix} \cdot \begin{bmatrix} s \\ t \end{bmatrix} = -4 + 2s + 14t.$$

If we plug in $s = 1$ and $t = 1$ (which are not close to 0!) then the “approximating” expression is

$$-4 + 2(1) + 14(1) = -4 + 2 + 14 = 12.$$

But $f(3, -1) = 12 - 1 - 3 = 8$, so the value 12 is not a good approximation. The reason it is not surprising that this is a bad approximation is that we are using values s and t that are not close to 0.

Problem 2: Finding tangents to implicit curves and surfaces

For each of the following, find the equation of the tangent line to the given curve or the tangent plane to the given surface at the specified point $\mathbf{a}$.

(a) $x^3 + y^2 = 31$ at $\mathbf{a} = (3, 2)$.

(b) $xz^2 + y^2 z^5 = 19$ at $\mathbf{a} = (3, 4, 1)$.

Solution:

(a) For $f(x, y) = x^3 + y^2$ we have $f_x = 3x^2$ and $f_y = 2y$, so $f_x(3, 2) = 27$ and $f_y(3, 2) = 4$. Hence, the equation of the tangent line is $0 = 27(x - 3) + 4(y - 2) = 27x - 81 + 4y - 8 = 27x + 4y - 89$. In other words, this is the line $27x + 4y = 89$.

(b) For $g(x, y, z) = xz^2 + y^2z^5$ we have

$$g_x = z^2, \;\; g_y = 2yz^5, \;\; g_z = 2xz + 5y^2z^4.$$

Hence, the equation of the tangent plane at $(3, 4, 1)$ is

$$\begin{aligned} 0 = g_x(3,4,1)(x-3) + g_y(3,4,1)(y-4) + g_z(3,4,1)(z-1) &= 1(x-3) + 8(y-4) + (6+80)(z-1) \\ &= x - 3 + 8y - 32 + 86z - 86 \\ &= x + 8y + 86z - 121. \end{aligned}$$

In other words, the plane is $x + 8y + 86z = 121$.

Problem 3: Tangent planes: graphs versus level sets

Let S be the sphere of radius 3 centered at the origin in $\mathbf{R}^3$. Let's consider two approaches to finding the equation of the tangent plane to S at the point $(2, 2, 1)$.

(a) For the surface graph $z = f(x, y)$ of a function $f(x, y)$, its tangent plane at a point $(a, b, f(a, b))$ is given by the equation $z = L(x, y)$ where $L(x, y)$ is the linear approximation to f at (a, b). Describe the upper half ($z > 0$) of the sphere S as a graph of a function of x and y, and use this to compute the equation of the tangent plane to S at the point $(2, 2, 1)$ in that upper hemisphere.

(b) The surface S is also a level set of $g(x, y, z) = x^2 + y^2 + z^2$ at a certain level c (what is the value of c?). Use the approach via gradients to compute the tangent plane to S at $(2, 2, 1)$. Verify that this is the same plane as you found in (a). (Note: the equation might not literally be the same as in (a) even though the solution sets to the equations – which are what actually matter – are the same, much as $2x - 2y + 2z = 0$ and $x - y + z = 0$ define the same plane; why?)

(c) Which method was easier? Do you have any thoughts about which method should usually be easier?

Solution:

(a) When $z > 0$ we have $z = \sqrt{9 - x^2 - y^2}$ on the sphere S, so the upper hemisphere is the graph of $f(x, y) = \sqrt{9 - x^2 - y^2}$. We want to compute its linear approximation at $(2, 2)$, so first we calculate

$$(\nabla f)(x, y) = \begin{bmatrix} -x/\sqrt{9 - x^2 - y^2} \\ -y/\sqrt{9 - x^2 - y^2} \end{bmatrix}.$$

Hence, $(\nabla f)(2, 2) = \begin{bmatrix} -2 & -2 \end{bmatrix}$, so the linear approximation at $(2, 2)$ is

$$f(2, 2) + \begin{bmatrix} -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} x - 2 \\ y - 2 \end{bmatrix} = 1 - 2(x - 2) - 2(y - 2) = 9 - 2x - 2y.$$

The tangent plane to S at $(2, 2, 1) = (2, 2, f(2, 2))$ is therefore given by $z = 9 - 2x - 2y$, or equivalently

$$2x + 2y + z = 9.$$

(b) By the 3-dimensional Pythagorean theorem, S is the level set $g(x, y, z) = 3^2 = 9$. The equation of its tangent plane at $(2, 2, 1)$ is given by the vanishing of

$$(\nabla g)(x, y, z) \cdot \begin{bmatrix} x - 2 \\ y - 2 \\ z - 1 \end{bmatrix},$$

and we calculate

$$(\nabla g)(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix},$$

so $(\nabla g)(2, 2, 1) = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$. Hence, the equation of the tangent plane is

$$0 = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x-2 \\ y-2 \\ z-1 \end{bmatrix} = 4(x-2) + 4(y-2) + 2(z-1) = 4x + 4y + 2z - 18,$$

which is to say

$$4x + 4y + 2z = 18.$$

This is not literally the same equation as obtained in (a), but rather is related to it via multiplying both sides by 2. Multiplying both sides of an equation by a nonzero scalar has no impact on where the equation holds, so the two equations do indeed define the same plane (as we know they must).

(c) The method based on the gradient of a level set is probably easier in general since to use the graph description we usually have to "solve for z" in some implicit equation $h(x, y, z) = c$, and such implicit solutions are usually algebraically a bit of a mess (involving inverse functions such as square roots, cube roots, inverse trig functions, and so on).

