Solutions to Math 51 Quiz 4 Practice A

1. (10 points) Suppose $g(x, y)$ is a function where $f(1, 1) = 4$ and linear approximation of $f(x, y)$ near $(1, 1)$ yields

$$f(1.1, 1.1) \approx 4.2, \qquad f(0.9, 1.1) \approx 3.4$$

Estimate $A = f_x(1, 1)$ and $B = f_y(1, 1)$ from these data.

The linear approximation for $f(x, y)$ near $(1, 1)$ is

$$f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = 4 + A(x - 1) + B(y - 1)$$

where we can determine (or at least sensibly estimate!) A and B. The two numerical values give the approximations

$$4.2 \approx f(1.1, 1.1) \approx 4 + A(0.1) + B(0.1)$$

and

$$3.4 \approx f(0.9, 1.1) \approx 4 + A(-0.1) + B(0.1),$$

so we can set up the system of equations

$$(0.1)A + (0.1)B = 0.2, \;\; (-0.1)A + (0.1)B = -0.6.$$

Multiplying through by 10 for each equation turns this into the pair of equations

$$A + B = 2, \;\; -A + B = -6$$

This is readily solved: adding the first equation to the second gives $2B = -4$, so $B = \boxed{-2}$; and thus $A = 2 - B = 2 - (-2) = \boxed{4}$.

2. (4 points) Suppose $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ is the unit disk, and let $g\colon D \to \mathbf{R}$ be the function given by $g(x, y) = \cos(x^2 + y^2)$. Complete the following two statements:

   (a) (2 points) g has a critical point at ____________ point(s) in the interior of D .

   (b) (2 points) g has a constrained local extremum at ____________ point(s) on the boundary of D.

   Choices for fill-ins:

   a) zero   b) one   c) two   d) four   e) infinitely many

(a) The gradient $\nabla g = \begin{bmatrix} -2x\sin(x^2 + y^2) \\ -2y\sin(x^2 + y^2) \end{bmatrix}$ vanishes only if

$$-2x\sin(x^2 + y^2) = 0 = -2y\sin(x^2 + y^2),$$

which in turn occurs when either $\sin(x^2 + y^2) = 0$ or $-2x = -2y = 0$. The latter case implies $(x, y) = (0, 0)$ is a critical point of g; the former implies g has another critical point whenever $\sin(x^2 + y^2) = 0$, or equivalently $x^2 + y^2 = k\pi$ for some (necessarily non-neqative) integer k. The case $k = 0$ gives only $(x, y) = (0, 0)$, which we already had; the case $k \geq 1$ yields points for which $x^2 + y^2 = k\pi > 1$, so which do not lie in the interior of D (or even on the boundary). Thus, g has a critical point at $\boxed{\text{one}}$ point in the interior or D (namely, at $(0, 0)$).

(b) On the boundary of D, points satisfy $x^2 + y^2 = 1$, so $g(x, y) = \cos(1)$ at every point (x, y) on the boundary of D. This means that at every point on the boundary, the value of g is greater than or equal to the value of g at every other point on the boundary; so every point on the boundary is a constrained local maximum (in fact, global maximum) for g. (Similarly, every such point is a minimum for g as well.) That is, g has a constrained local extremum at $\boxed{\text{infinitely many}}$ point(s) on the boundary of D.

Alternatively, we may compute the gradient of x^2+y^2 (it is $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$, which does not vanish on the boundary $x^2+y^2=1$) and set up the system of equations for the Lagrange condition:

$$\begin{aligned} -2x\sin(x^2+y^2) &= \lambda(2x) \\ -2y\sin(x^2+y^2) &= \lambda(2y) \\ x^2+y^2 &= 1 \end{aligned}$$

This has infinitely many solutions, because we could always let $\lambda = -\sin(1)$ for any (x,y) on the boundary circle. (Solve the first two equations for λ in the usual way and then combine with the third equation.) So every point on the boundary is a candidate local extremum; we then evaluate g at these points to find that the value of g is constant at these points — so every one is an extremum (both maximum and minimum).

3. (3 points) Suppose C is the curve in $\mathbf{R}^2$ given by the equation

$$xy^3 - yx^4 = -6$$

At which of the following points P is the tangent line to C at P parallel to the x-axis? Choose all that apply.

a) $(1,-2)$ b) $(2,-3)$ c) $(3,-3)$ d) none of these

The tangent to C at P is parallel to the x-axis precisely when the normal vector to it is parallel to the y-axis, which is to say it has vanishing x-component. Let $f(x,y) = xy^3 - yx^4$; the normal vector to $f=-6$ is given by the gradient of f, so we want

$$0 = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xy^3 - yx^4\right) = y^3 - 4yx^3 = y(y^2-4x^3),$$

which is only possible on the curve C if $y^2 = 4x^3$ (since if $y=0$, then the equation for C becomes $0=-6$). Among the choices provided, only point (i) both lies on C and has $y^2=4x^3$.

4. (3 points) Let

$$g(x,y,z) = x^2+y^4+z^6, \qquad f(x,y,z) = 4 + g(x,y,z)$$

How many solution(s) does the Lagrange Multiplier system for maximizing $f(x,y,z)$ under the constraint

$$g(x,y,z) = 2021$$

have?

You may assume that the number of solution(s) is at least 1.

a) 1 b) 2 c) 4 d) 6

e) 2021 f) infinitely many g) not enough information to tell

Note that the constraint equation $x^2 + y^4 + z^6 = 2021$ has infinitely many solutions. For example, set $z = 0$, $x^2 + y^4 = x^2 + (y^2)^2 = 2021$ has infinitely many solutions where (x, y^2) are points on the top semicircle with radius $\sqrt{2021}$ centered at $(0, 0)$ on the xy-plane.

The Lagrange Multiplier system for maximizing $f(x, y, z)$ under the constraint $g(x, y, z) = 2021$ is given by

$$\nabla f = \lambda \nabla g, \;\; g(x, y, z) = 2021.$$

Since $f(x, y, z) = 4 + g(x, y, z)$, $\nabla f = \nabla g$ for all (x, y, z) on the 2021-level surface S of $g(x, y, z)$, with $\lambda = 1$. The Lagrange Multiplier system has infinitely many solutions, i.e. all the points on the surface given by $x^2 + y^4 + z^6 = 2021$. The value of $f(x, y, z) = 4 + g(x, y, z)$ on S is constant, it is equal to 2025, which is the global minimum and global maximum of $f(x, y, z)$ on S.