1. (a) A 4-vector $\mathbf{x}$ belongs to U precisely when $\mathbf{x} \cdot \mathbf{w}_1 = 0$ and $\mathbf{x} \cdot \mathbf{w}_2 = 0$; in terms of the entries of $\mathbf{x}$,

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0, \\ x_1 - 3x_2 + x_3 + x_4 &= 0. \end{aligned}$$

The two equations solved for x_4 give:

$$-x_1 - x_2 - x_3 = x_4 = -x_1 + 3x_2 - x_3.$$

Equating the outer terms gives

$$-x_1 - x_2 - x_3 = -x_1 + 3x_2 - x_3,$$

which reduces to $x_2 = 0$. Plugging this into either of the two preceding expressions for x_4 yields $x_4 = -x_1 - x_3$. Hence,

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ -x_1 - x_3 \end{bmatrix} &= \begin{bmatrix} x_1 \\ 0 \\ 0 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ -x_3 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

This expresses U as the span of $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. (Note: if you instead solve for another variable, such as x_1 in terms of x_3, x_4, then one of many other valid spanning sets arises.)

Finally, we know $\dim(U) = 2$ because these spanning vectors are not scalar multiples of each other (by inspection), so the spanning set is a basis.

(b) We can compute $\mathbf{Proj}_U(\mathbf{a})$ using an *orthogonal* basis for U. We may specify such a basis using $\mathbf{u}_1$ together with the vector $\mathbf{u}_2' = \mathbf{u}_2 - \mathbf{Proj}_{\mathbf{u}_1}(\mathbf{u}_2)$, because the latter vector lies in U and is orthogonal to $\mathbf{u}_1$ by the properties of projections.

We find that

$$\begin{aligned} \mathbf{u}_2' = \mathbf{u}_2 - \mathbf{Proj}_{\mathbf{u}_1}(\mathbf{u}_2) &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \right) \mathbf{u}_1 \\ &= \mathbf{u}_2 - \left(\frac{1}{2} \right) \mathbf{u}_1 = \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ -1/2 \end{bmatrix}. \end{aligned}$$

Then we use this basis to compute the projection of $\mathbf{a}$ onto U as follows:

$$\begin{aligned} \mathbf{Proj}_U(\mathbf{a}) &= \mathbf{Proj}_{\mathbf{u}_1}(\mathbf{a}) + \mathbf{Proj}_{\mathbf{u}_2'}(\mathbf{a}) \\ &= \left(\frac{\mathbf{a} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \right) \mathbf{u}_1 + \left(\frac{\mathbf{a} \cdot \mathbf{u}_2'}{\mathbf{u}_2' \cdot \mathbf{u}_2'} \right) \mathbf{u}_2' \\ &= \left(\frac{1}{2} \right) \mathbf{u}_1 + \left(\frac{9/2}{3/2} \right) \mathbf{u}_2' \\ &= \frac{1}{2} \mathbf{u}_1 + 3\mathbf{u}_2' = \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}. \end{aligned}$$

Our "safety check" can involve checking that our answer $\begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$ lies in U (by checking that its dot products with $\mathbf{w}_1$ and $\mathbf{w}_2$ are both zero), as well as that the difference vector $\mathbf{a} - \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \\ 5 \end{bmatrix}$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$ (as must be true by the properties of orthogonal projection onto $U = \mathrm{span}(\mathbf{u}_1, \mathbf{u}_2)$).

$\diamond$

2. There are multiple approaches (and multiple correct answers) to this problem; we will just give one approach. Our strategy will be to find *some* basis for $\mathcal{P}$, then use Theorem 7.1.1 to convert it into an orthogonal basis.

   In fact, any two non-zero vectors in $\mathcal{P}$ not on the same line through the origin constitute a basis for $\mathcal{P}$. For example, we can find a basis as follows: let $\mathbf{v}_1$ be the unique point on $\mathcal{P}$ with $x = 1, y = 0$ (so we must solve for z), while $\mathbf{v}_2$ has $x = 0, y = 1$. More specifically, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 5/2 \end{bmatrix}$.

   As an orthonormal basis, we can take $\mathbf{v}_1$ and $\mathbf{v}_2' = \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{v}_2) = \mathbf{v}_2 - \dfrac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$, so we need to compute some dot products:

$$\mathbf{v}_2 \cdot \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 5/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = -5, \quad \mathbf{v}_1 \cdot \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 5,$$

   so

$$\mathbf{v}_2' = \mathbf{v}_2 - \frac{5}{5}\mathbf{v}_1 = \mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 5/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1/2 \end{bmatrix}.$$

   We can also rescale $\mathbf{v}_2'$ by 2 without changing the orthogonality (call the result $\mathbf{v}_2''$ for clarity). Summarizing, the orthogonal basis we've found is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \mathbf{v}_2'' = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

$\diamond$

3. (a) The vectors are

$$\mathbf{X} = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -4 \\ 0 \\ 2 \\ 2 \\ 5 \end{bmatrix}.$$

   (b) The vector $\mathbf{v}$ can be taken to be $\mathbf{X} - \mathbf{Proj}_{\mathbf{1}}(\mathbf{X}) = \mathbf{X} - \overline{x}\mathbf{1}$ with $\overline{x}$ equal to the average of the entries x_i in $\mathbf{X}$. This average is

$$\frac{-3-2-1+0+1}{5} = \frac{-5}{5} = -1,$$

   so

$$\mathbf{v} = \mathbf{X} + \mathbf{1} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

(This $\mathbf{v}$ is what is called $\widehat{\mathbf{X}}$ in the main text.) The projection of $\mathbf{Y}$ into V is then given by

$$\begin{aligned} \frac{\mathbf{Y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \frac{\mathbf{Y}\cdot\mathbf{1}}{\mathbf{1}\cdot\mathbf{1}}\mathbf{1} &= \frac{8+0+0+2+10}{10}\mathbf{v} + \frac{-4+0+2+2+5}{5}\mathbf{1} \\ &= \frac{20}{10}\mathbf{v} + \frac{5}{5}\mathbf{1} \\ &= 2\mathbf{v} + (1)\mathbf{1}. \end{aligned}$$

Hence, $t = 2$ and $s = 1$.

(c) We have $\mathbf{v} = \mathbf{X} + \mathbf{1}$, so

$$\begin{aligned} \mathbf{Proj}_V(\mathbf{Y}) = 2\mathbf{v} + (1)\mathbf{1} &= 2(\mathbf{X}+\mathbf{1}) + \mathbf{1} \\ &= 2\mathbf{X} + (2+1)\mathbf{1} \\ &= 2\mathbf{X} + (3)\mathbf{1}. \end{aligned}$$

Hence, the line of best fit is $y = 2x + 3$.

(d) The line of best fit and the original data are shown in Figure 1 ; it looks like a very good fit.

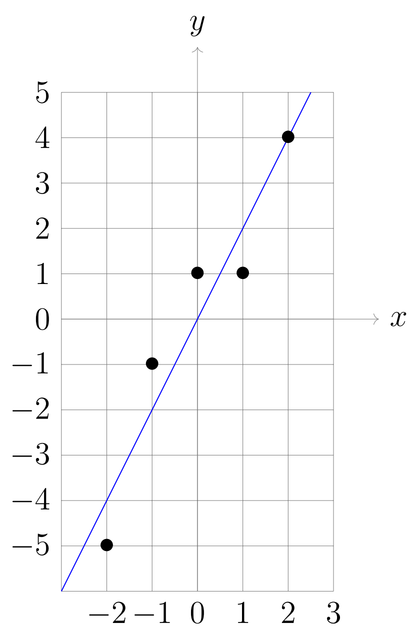

Figure 1: The best-fit line $y = 2x + 3$ compared with the data.

$\diamond$

4. (a) By the parametric form of L, all displacement vectors between points of L are scalar multiples of $\mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ -1 \\ -2 \end{bmatrix}$. We want all of these to belong to $\mathcal{P}$, so it is the same to show $\mathbf{x}$ belongs to $\mathcal{P} = \mathrm{span}(\mathbf{v}, \mathbf{w})$. That is, we seek $c, d \in \mathbf{R}$ for which $\mathbf{x} = c\mathbf{v} + d\mathbf{w}$. That is, we seek c, d satisfying

$$\begin{bmatrix} -1 \\ 4 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2c + 11d \\ -c + 5d \\ -c - 10d \\ c + d \end{bmatrix}.$$

By equating coordinates, this is a collection of 4 equations in 2 unknowns:

$$2c + 11d = -1, \quad -c + 5d = 4, \quad -c - 10d = -1, \quad c + d = -2.$$

If we consider the first two equations, we can use the usual method from high-school algebra to find that those two have one simultaneous solution: $c = -7/3$ and $d = 1/3$. Direct evaluation confirms that these values also satisfy the third and fourth equations, so $\mathbf{x} = -(7/3)\mathbf{v} + (1/3)\mathbf{w} \in \mathcal{P}$ as desired.

(b) Based on the parametric form for L, we pick $\mathbf{y} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -7 \end{bmatrix}$ (corresponding to $t = 0$), though any choice of point on L will work just as well below.

The closest distance from $\mathbf{y}$ to $\mathcal{P}$ is the length of $\mathbf{y} - \mathbf{Proj}_{\mathcal{P}}(\mathbf{y})$: the displacement vector between $\mathbf{y}$ and its closest point $\mathbf{Proj}_{\mathcal{P}}(\mathbf{y})$ on $\mathcal{P}$. In order to compute the projection, we would like an orthogonal basis for $\mathcal{P}$. By Theorem 7.1.1 , one such basis is $\mathbf{v}, \mathbf{w}' = \mathbf{w} - \mathbf{Proj}_{\mathbf{v}}(\mathbf{w}) = \mathbf{w} - \dfrac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$, so we need to compute some dot products:

$$\mathbf{w} \cdot \mathbf{v} = \begin{bmatrix} 11 \\ 5 \\ -10 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 22 - 5 + 10 + 1 = 28, \quad \mathbf{v} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 7,$$

so

$$\mathbf{w}' = \mathbf{w} - 4\mathbf{v} = \begin{bmatrix} 11 \\ 5 \\ -10 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ -10 \\ 1 \end{bmatrix} - \begin{bmatrix} 8 \\ -4 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ -6 \\ -3 \end{bmatrix}.$$

Noting the common factor of 3 in all entries of this final expression, it does us no harm to replace $\mathbf{w}'$ by $\mathbf{w}'' = \dfrac{1}{3}\mathbf{w}' = \begin{bmatrix} 1 \\ 3 \\ -2 \\ -1 \end{bmatrix}$ (though if you don't make this change then it has no impact on the final answer below).

As we explained above, the shortest distance is the length of the vector

$$\mathbf{y} - \mathbf{Proj}_{\mathcal{P}}(\mathbf{y}) = \mathbf{y} - \mathbf{Proj}_{\mathbf{v}}(\mathbf{y}) - \mathbf{Proj}_{\mathbf{w}''}(\mathbf{y}) = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} - \frac{\mathbf{y} \cdot \mathbf{w}''}{\mathbf{w}'' \cdot \mathbf{w}''}\mathbf{w}'',$$

which is

$$\mathbf{y} - \left(\frac{4 \cdot 2 + 4 \cdot (-1) + 4 \cdot (-1) + (-7) \cdot 1}{7}\right)\mathbf{v} - \left(\frac{4 \cdot 1 + 4 \cdot 3 + 4 \cdot (-2) + (-7) \cdot (-1)}{1 + 9 + 4 + 1}\right)\mathbf{w}'',$$

that in turn is equal to

$$\mathbf{y} - \frac{-7}{7}\mathbf{v} - \frac{15}{15}\mathbf{w}'' = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -7 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{15}{15}\begin{bmatrix} 1 \\ 3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 + 2 - 1 \\ 4 - 1 - 3 \\ 4 - 1 + 2 \\ -7 + 1 + 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \\ -5 \end{bmatrix}.$$

The length of this vector is $5\sqrt{3} = \sqrt{75}$.

$\diamond$

5. (a) The region S is exactly the level set $f(x, y, z) = 0$ for

$$f(x, y, z) = x^3 + z^3 + 3y^2z^3 + 5xy.$$

It is also the level set $h(x, y, z) = 7$ for $h(x, y, z) = x^3 + z^3 + 3y^2z^3 + 5xy + 7$, and the level set $F(x, y, z) = 0$ for $F = 13f$ (or with any other nonzero scalar in place of 13).

(b) Computing, we see that S is described alternatively by the equation

$$z = \sqrt[3]{-\frac{x^3 + 5xy}{3y^2 + 1}},$$

so if $g(x, y)$ denotes the expression on the right side then S is the graph of this g.

$\diamond$

6. (a) Computing, we see that

$$\left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = \frac{2 - (-2)}{4} = 1,$$

so every point in the output of g lies on the hyperbola $x^2 - y^2 = 1$.

(b) Not every point on this hyperbola is of the form $g(t)$, because the x-coordinate $(e^t + e^{-t})/2$ of the formula for $g(t)$ is always positive. For example, $(-1, 0)$ lies on the hyperbola but is not in the output of g (and likewise for any point $(a, \pm\sqrt{a^2 - 1})$ for any $a \leq -1$).

◇

7. (a) The level set $h(x, y) = 0$ is the set of all points (x, y) for which $\sin(x) = 0$. Notice that this is true for $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ and for any value of y. Consequently, this level set is the collection of the vertical lines at $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$. Similarly, the level set $\sin(x) = 1$ is the collection of vertical lines at $\ldots, -\frac{3}{2}\pi, \frac{1}{2}\pi, \frac{5}{2}\pi, \ldots$ and the level set $\sin(x) = -1$ is the collection of vertical lines at $\ldots - \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{7}{2}\pi, \ldots$ The other level sets are also collections of vertical lines varying between the indicated values, some of which are shown in Figure 2 . There are no level sets for $h(x, y) = c$ when $|c| > 1$.

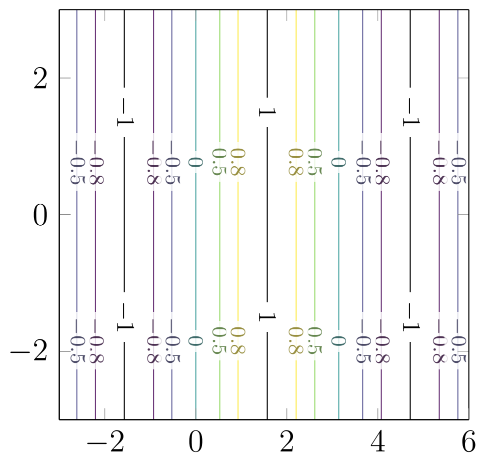

Figure 2: A contour plot for $h(x, y) = \sin x$.

(b) This function has non-empty level sets for precisely those c satisfying $-1 \leq c \leq 1$.

Similarly to (a), the level sets are collections of parallel lines. For $f(x, y) = 0$ we are now looking at $\sin(x - 3y) = 0$. This is the collection of conditions $x - 3y = \ldots, \pi, 0, \pi, 2\pi, \ldots$. These are parallel lines all of slope $\frac{1}{3}$ with various displacements (they meets the x-axis at $\ldots, \pi, 0, \pi, 2\pi, \ldots$ The level set $f(x, y) = 1$ is the collection of parallel lines all of slope $\frac{1}{3}$ given by $x - 3y = \ldots, -\frac{3}{2}\pi, \frac{1}{2}\pi, \frac{5}{2}\pi, \ldots$, and the level set $f(x, y) = -1$ is the collection of parallel lines all of slope $\frac{1}{3}$ given by $x - 3y = \ldots - \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{7}{2}\pi, \ldots$ The other level sets are collections of parallel lines all of slope $\frac{1}{3}$ varying between these, some of which are shown in Figure 3 .

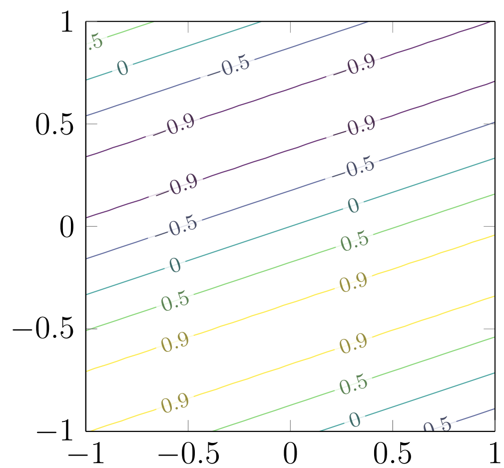

Figure 3: A contour plot of $f(x, y) = \sin(x - 3y)$.

$\diamond$

8. (a) We calculate

$$\begin{aligned}(g \circ f)(\theta, \phi) &= (\cos\theta \sin\phi)^2 + (\sin\theta \sin\phi)^2 + (\cos\phi)^2 \\ &= \sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi \\ &= \sin^2\phi + \cos^2\phi \\ &= 1.\end{aligned}$$

   (b) By the formula for distance from the origin in $\mathbf{R}^3$, the unit sphere is exactly the level surface $g = 1$. Since the composition $g \circ f$ always has value 1, all output of f lies in that level set.

$\diamond$

9. (a) The partial derivative $f_x(1, 1)$ should be the rate of change of f as we travel horizontally past the point $(1, 1)$. Thus,

$$f(1.01, 1) \approx f(1, 1) + (0.01)f_x(1, 1),$$

   which says $22 \approx 25 + (0.01)f_x(1, 1)$, so $f_x(1, 1) \approx -3/(0.01) = -300$.
   Similarly, $f(1, 0.99) \approx f(1, 1) - (0.01)f_y(1, 1)$, which says $23 \approx 25 - (0.01)f_y(1, 1)$, so $f_y(1, 1) \approx -2/(-0.01) = 200$.

   (b) Imagining x to be fixed (but not yet specifying $x = 1$), we can compute the derivative $g_y(x, y)$ using the Chain Rule: $g_y(x, y) = (1/(x^2y^2)) \cdot (2yx^2) = 2/y$. Thus, $g_y(1, 1) = 2$.

   (c) By working symbolically with the product rule, we can compute

$$\frac{\partial h}{\partial y}(x, y) = f_y(x, y) \cdot g(x, y) + f(x, y) \cdot g_y(x, y)$$

   as functions. Thus, at the point $(1, 1)$, we have

$$\frac{\partial h}{\partial y}(1, 1) = f_y(1, 1)g(1, 1) + f(1, 1)g_y(1, 1) \approx 200 \cdot (1 + \ln(1)) + 25 \cdot 2 = 250.$$

$\diamond$

10. (a) If we move to the right (west-to-east) past the point $(1, 1)$ while being on the line $y = 1$ then we cross lines going from larger values to smaller values (the shading is getting darker). Thus, $f_x(1, 1) < 0$.

   (b) Traveling vertically near $(2, 0)$, our path of motion appears to be tangent to the contour lines. That is, if we fix $x = 2$ and vary y near 0, the value of f will change very little. In fact, if the vertical line is exactly tangent to the contour line through that point then the instantaneous change there will be zero. In any case, we can say that $\dfrac{\partial f}{\partial y}(2, 0) \approx 0$.

(c) If we move north very near the point $(1,2)$ then we cross contour lines going from smaller values to larger values. Alternatively, the shading is getting darker. Thus, $f_y(1,2) > 0$.

(d) Near the point $(-2,-1)$, if we move in the x-direction (staying on the line $y = -1$) then the function's values increase (the shading is getting lighter). Therefore, $\dfrac{\partial f}{\partial x}(-2,-1) > 0$.

◇

11. (a) We compute symbolically

$$f_x = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}, \quad f_y = \frac{-(x+y)-(x-y)}{(x+y)^2} = -\frac{2x}{(x+y)^2},$$

so $f_x(2,1) = 2/9$ and $f_y(2,1) = -4/9$.

(b) We compute symbolically

$$f_x = \frac{\sqrt{x^2+y^2} - 2x^2/(2\sqrt{x^2+y^2})}{x^2+y^2} = \frac{(x^2+y^2)-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}, \quad f_y = -\frac{xy}{(x^2+y^2)^{3/2}},$$

so $f_x(1,1) = 1/(2\sqrt{2})$ and $f_y(1,1) = -1/(2\sqrt{2})$.

(c) We compute symbolically

$$f_x = \frac{y}{xy+z}, \quad f_y = \frac{x}{xy+z}, \quad f_z = \frac{1}{xy+z},$$

so $f_x(1,2,3) = 2/5$, $f_y(1,2,3) = 1/5$, and $f_x(1,2,3) = 1/5$.

◇

12. (a) We compute first that $f_x = B + 2Dx + Ey + 3Gx^2 + 2Hxy + Iy^2$, and then computing its y-partial yields

$$f_{xy} = E + 2Hx + 2Iy.$$

This vanishes identically precisely when $E = H = I = 0$, as desired.

(b) We compute that $f_x = g'(x)$ since $h(y)$ is independent of x. But this in turn is independent of y, so its y-partial f_{xy} vanishes.

◇