Problem 1: Matrix of a quadratic form

(a) Consider the quadratic form $q(x,y,z) = x^2 + 2y^2 - z^2 + 4xy + 6xz - 2yz$. Find the symmetric matrix A so that $q(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(b) Consider the quadratic form $q(w,x,y,z) = 2x^2 + y^2 + 3xy + 4yw - zw$ (note that w is the first coordinate, x is the second coordinate, etc.). Find the symmetric matrix A so that $q(w,x,y,z) = \begin{bmatrix} w & x & y & z \end{bmatrix} A \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$.

Solution:

(a) We can read off

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -1 \\ 3 & -1 & -1 \end{bmatrix}.$$

(b) Keeping in mind that this has been set up with x as the second coordinate and not the first, we can read off

$$A = \begin{bmatrix} 0 & 0 & 2 & -1/2 \\ 0 & 2 & 3/2 & 0 \\ 2 & 3/2 & 1 & 0 \\ -1/2 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 2: Linear system solution set, in parametric form

Let L be the line determined by the overlap of two planes in $\mathbf{R}^3$ (not passing through the origin):

$$x + y + z = 4 \quad \text{and} \quad 2x + 3y + z = 9.$$

(a) Find a point on L whose z-coordinate is 5.

(b) Find a point on L whose z-coordinate is 6.

(c) Find general formulas for x and y in terms of z for all points (x,y,z) in the line L; this should recover your answers to (a) and (b) upon plugging in $z = 5$ and $z = 6$ respectively. (Hint: think about z as a "constant" and x and y as "variables" to guide your algebraic work in the style of what you did for (a) and (b).)

Solution:

(a) We plug $z = 5$ into both of the given equations, to get the new simulaneous system just in terms of x and y: $\left\{\begin{matrix} x + y = -1 \\ 2x + 3y = 4 \end{matrix}\right\}$. This can be solved by the method of high school algebra to get $x = -7$, $y = 6$. Hence, there is one such point: $(-7, 6, 5)$.

(b) We plug $z = 6$ into both of the given equations, to get the new simulaneous system just in terms of x and y: $\left\{\begin{matrix} x + y = -2 \\ 2x + 3y = 3 \end{matrix}\right\}$. As in (a), this can be solved to get $x = -9$, $y = 7$. Hence, there is one such point: $(-9, 7, 6)$.

(c) We seek to solve the system of equations $\left\{\begin{array}{c} x+y=4-z \\ 2x+3y=9-z \end{array}\right\}$ for x and y in terms of z, which can be done by the high school algebra method of “elimination” as is implicit in the work done to solve the two equations in two unknowns in (a) and (b). First subtract twice the first equation from the second equation to eliminate x and obtain $y = 9 - z - 2(4 - z) = 1 + z$. Plugging this back into the first equation, we obtain $x = 4 - z - (1 + z) = 3 - 2z$, so

$$\begin{cases} x = 3 - 2z \\ y = 1 + z \end{cases}$$

is the answer. (We could have instead eliminated y; in the end we would obtain the same final answer.) Note that if we plug in $z = 5$ and $z = 6$, we get the points we found in (a) and (b) respectively.

Problem 3: Recognizing sets as null spaces (or not)

Often in linear algebra one builds a collection of n-vectors satisfying a variety of conditions, and it can be useful to know if the collection is the null space of an $m \times n$ matrix for some m. (in effect: is the collection describable as the simultaneous solution set for a system of m linear equations in n unknowns?)

(a) Explain why the subset $S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbf{R}^3 : v_1 + v_2 = v_3 + 1 \right\} \subset \mathbf{R}^3$ cannot be the null space of a matrix A (such an A would have to be $m \times 3$ for some m). Hint: try to find some general property of null spaces that S violates.

(b) Find a 2×3 matrix A for $N(A)$ is equal to the set $S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbf{R}^3 : v_1 + v_2 - v_3 = 0,\ v_2 + v_3 = 0 \right\} \subset \mathbf{R}^3$.

Solution:

(a) The nullspace $N(A)$ always contains the zero vector but the given set S does not, so we can never have $S = N(A)$. More generally, if we form linear combinations of solutions to the given equation for S then we generally don’t get another solution since the constant term 1 is affected. This shows that S is not “closed” under the formation of linear combinations, whereas a null space always is, so that is another way to understand why S cannot be a null space.

(b) We are looking for a matrix A for which $A\mathbf{v} = \mathbf{0}$ precisely when $\left\{ \begin{array}{ccc} v_1 + v_2 - v_3 & = & 0 \\ v_2 + v_3 & = & 0 \end{array} \right.$. We can think of each equation as an entry of the vector $A\mathbf{v}$ that we want to vanish, and “read off” the entries of A from the coefficients of v_1, v_2, v_3 in these equations. In particular the first row of A will be given by the coefficients in the first equation and the second row by the coefficients in the second equation (where the second equation involve v_1 with a coefficient of 0: $0v_1 + v_2 + v_3$ on the left side). We thereby obtain $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, and indeed the vanishing of $A\mathbf{v}$ for this A expresses exactly the two vanishing conditions that define S, so $S = N(A)$.

Problem 4: Basis and dimension

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) Write the third column of A as a linear combination of the first two columns. Use this to find a basis for $C(A)$.

(b) Find a basis for $C(A)$ that contains $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$.

(c) (**Extra**) How can having a basis for the column space $C(M)$ of an $m \times n$ matrix M help in figuring out for a given m-vector $\mathbf{b}$ if $M\mathbf{x} = \mathbf{b}$ has a solution? (Hint: think about $\mathbf{Proj}_{C(M)}$.)

Solution:

(a) (a) We want $a, b \in \mathbf{R}$ for which $a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$. That is, we want to simultaneously solve the system

$$\begin{cases} a + 3b &= 5 \\ 2a - b &= 3 \\ a &= 2 \end{cases}.$$

The last condtion says $a = 2$, so the second condition tells us $b = 1$. The resulting pair $(a, b) = (2, 1)$ is checked to satisfy the first condition. The upshot is that the first two columns, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, span $C(A)$ and so are a basis since neither is a scalar multiple of the other.

(b) For a basis for the 2-dimensional subspace $C(A)$, we merely need a pair of non-zero vectors in $C(A)$ that are not scalar multiples of each other, since this pair will automatically span $C(A)$ for dimension reasons (and therefore form a basis). If we want the basis to contain the third column $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$, we note that (say) the first column $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is not a scalar multiple of this (by inspection), so $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$ is a linearly independent spanning set for $C(A)$ and hence is a basis. (The second and third vectors also consistute a basis, for the same reasons.)

(c) We want to be able to figure out if $\mathbf{b} \in C(M)$. This happens exactly when $\mathbf{b} = \mathbf{Proj}_{C(M)}(\mathbf{b})$ (quite generally, for a linear subspace $V \subset \mathbf{R}^m$ and $\mathbf{w} \in \mathbf{R}^m$, we have $\mathbf{w} \in V$ precisely when $\mathbf{w} = \mathbf{Proj}_V(\mathbf{w})$; why?). If we have a basis for the column space $C(M)$ then we can use Gram–Schmidt to extract from that an orthogonal basis and so can actually compute $\mathbf{Proj}_{C(M)}(\mathbf{b})$ (and so can check if it is or is not equal to $\mathbf{b}$, thereby determining if $\mathbf{b} \in C(M)$).

Problem 5: Column spaces and an overdetermined linear system

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix}$, and let the 5-vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ denote the columns of A (from left to right). For a 5-vector $\mathbf{b}$, the vector equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} \in \mathbf{R}^3$ encodes a system of 5 scalar linear equations in 3 unknowns (the entries of $\mathbf{x}$), so it is overdetermined (more equations than unknowns) and hence the rule of thumb is that only for rather special $\mathbf{b}$ should a solution exist. This problem works out a description of those special $\mathbf{b}$ for this specific A by thinking in terms of column spaces.

(a) If a vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \in \mathbf{R}^5$ belongs to the column space of A (i.e., $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ for some scalars x_1, x_2, x_3, or equivalently $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3$) then by turning this vector condition into a collection of 5 simultaneous scalar conditions explain why $b_3 - b_1 - b_2 = 0$, $b_4 + b_1 - b_2 = 0$, and $b_5 - 2b_2 + b_1 = 0$. (Hint: use that each b_i can be expressed in a specific way in terms of the x_j's).

(b) The three equations in the b_i's at the end of (a) can be rewritten to say $b_3 = b_1 + b_2$, $b_4 = -b_1 + b_2$, and $b_5 = -b_1 + 2b_2$, so collectively they say exactly that $\mathbf{b}$ is a 5-vector of the special form $\begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \\ -b_1 + b_2 \\ -b_1 + 2b_2 \end{bmatrix}$ with $b_1, b_2 \in \mathbf{R}$. Check explicitly that any $\mathbf{b}$ of this form actually is in $C(A)$. (Hint: for $\mathbf{b}$ of this special form in terms of b_1 and b_2, adapt the type of argument in Problem 2(c) to find an $\mathbf{x} \in \mathbf{R}^3$ with $x_3 = 0$ for which the first two entries of $A\mathbf{x}$ are b_1, b_2 respectively, and then use the assumed three equations on the b_i's to check that actually $A\mathbf{x} = \mathbf{b}$.). This says that three conditions on $\mathbf{b}$ at the end of (a) are not merely a consequence of membership in $C(A)$ but even exactly characterize it.

(c) Give an explicit 5-vector $\mathbf{b}$ for which the linear system "$A\mathbf{x} = \mathbf{b}$" does *not* have a solution. (There are many answers.)

Solution:

(a) A vector $\mathbf{b}$ is in the column space $C(A)$ precisely when it can be written in the form $A\mathbf{x} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, where $x_1, x_2, x_3 \in \mathbf{R}$. Working out $A\mathbf{x}$ explicitly, this says

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \mathbf{b} = A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 3x_2 + 4x_3 \\ 3x_1 + 5x_2 + 7x_3 \\ x_1 + x_2 + x_3 \\ 3x_1 + 4x_2 + 5x_3 \end{bmatrix}.$$

This single vector equation amounts to says that (x_1, x_2, x_3) is a solution to the system of five simultaneous scalar equations given by equating corresponding entries in the 5-vectors on the far left and the far right. This expresses each b_i as a linear combination of x_1, x_2, x_3. We are going to substitute those expressions for the b_i's into each of $b_3 - b_1 - b_2$, $b_4 + b_1 - b_2$, and $b_5 - 2b_2 + b_1$ to see that we get 0 for all of them!

If we substitute the expressions for b_1, b_2, and b_3 into $b_3 - b_1 - b_2$ we get

$$b_3 - b_1 - b_2 = (3x_1 + 5x_2 + 7x_3) - (x_1 + 2x_2 + 3x_3) - (2x_1 + 3x_2 + 4x_3) = 0x_1 + 0x_2 + 0x_3 = 0,$$

as desired. Likewise, substituting the expressions for b_1, b_2, and b_4 into $b_4 + b_1 - b_2$ yields

$$b_4 + b_1 - b_2 = (x_1 + x_2 + x_3) + (x_1 + 2x_2 + 3x_3) - (2x_1 + 3x_2 + 4x_3) = 0x_1 + 0x_2 + 0x_3 = 0.$$

Finally, substituting in the expressions for b_1, b_2, b_5 gives

$$b_5 - 2b_2 + b_1 = (3x_1 + 4x_2 + 5x_3) - 2(2x_1 + 3x_2 + 4x_3) + (x_1 + 2x_2 + 3x_3) = 0x_1 + 0x_2 + 0x_3 = 0,$$

as desired.

(b) As suggested in the hint, for $\mathbf{b}$ of the given special form (in terms of b_1 and b_2), rather than aim to find straight away a solution to the equality of 5-vectors $A\mathbf{x} = \mathbf{b}$ (a system of 5 scalar conditions), we first seek a solution to the more limited system of two scalar equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 & = b_1 \\ 2x_1 + 3x_2 + 4x_3 & = b_2 \end{cases}$$

arising from equating the corresponding first two vector entries of $A\mathbf{x}$ and $\mathbf{b}$, subject to the further condition $x_3 = 0$. Under this vanishing condition on x_3, the pair of equations becomes two simultaneous equations

$$x_1 + 2x_2 = b_1, \;\; 2x_1 + 3x_2 = b_2$$

in two unknowns x_1 and x_2.

We now solve this for x_1 and x_2 in terms of b_1 and b_2 by the elimination method from high school algebra, in the spirit of what we did in Problem 2(c). By subtracting twice the first equation from the second to eliminate x_1 we get

$-x_2 = b_2 - 2b_1$, so $x_2 = 2b_1 - b_2$. Then by plugging this back into the first equation, we get $x_1 = -3b_1 + 2b_2$. Alternatively, this pair of equations in x_1 and x_2 can be written in vector language as $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and so can be solved by multiplying both sides on the left by the inverse of the indicated 2×2 matrix:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -3b_1 + 2b_2 \\ 2b_1 - b_2 \end{bmatrix},$$

yielding the same expressions for x_1 and x_2 in terms of b_1 and b_2 that we found when working by hand with the elimination method.

Now trying the vector $\mathbf{x} = \begin{bmatrix} -3b_1 + 2b_2 \\ 2b_1 - b_2 \\ 0 \end{bmatrix}$ as suggested in the hint, we see it works as desired:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -3b_1 + 2b_2 \\ 2b_1 - b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3b_1 + 2b_2 + 4b_1 - 2b_2 \\ -6b_1 + 4b_2 + 6b_1 - 3b_2 \\ -9b_1 + 6b_2 + 10b_1 - 5b_2 \\ -3b_1 + 2b_2 + 2b_1 - b_2 \\ -9b_1 + 6b_2 + 8b_1 - 4b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \\ -b_1 + b_2 \\ -b_1 + 2b_2 \end{bmatrix} = \mathbf{b},$$

the final equality being exactly our assumption on the special form of $\mathbf{b}$. Hence, all such $\mathbf{b}$ really do belong to $C(A)$.

(c) By (a) and (b), the vector equation $A\mathbf{x} = \mathbf{b}$ (which says exactly that the 5-vector $\mathbf{b}$ belongs to $C(A)$) has a solution $\mathbf{x} \in \mathbf{R}^3$ precisely when the 5-vector $\mathbf{b}$ has the special form considered in (b).

Hence, any 5-vector $\mathbf{b}$ not of that special form must lie outside $C(A)$ and so $A\mathbf{x} = \mathbf{b}$ has no solution for such $\mathbf{b}$. So we just need to pick a 5-vector $\mathbf{b}$ whose third, fourth, or fifth entry violates the corresponding special form. That is, we just need to have at least one of the following: $b_3 \neq b_1 + b_2$, $b_4 \neq -b_1 + b_2$, or $b_5 \neq b_1 + 2b_2$ (we don't need *all* of these violations to happen: just one violation is enough!). There are many such vectors (even violating all three conditions on (b_3, b_4, b_5), but that is more than we need). An example is $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, which violates the conditions on b_3 and b_5 (though it satisfies the condition on b_4).

Problem 6: Normal matrices (Extra)

An $n \times n$ matrix A is called *normal* if A commutes with its transpose $A^\top$ (i.e., $AA^\top = A^\top A$). For example, every symmetric matrix (and in particular every diagonal matrix) is normal, since it is even equal to its own transpose.

(a) Check that every matrix of the form $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (which is not symmetric when $b \neq 0$) is normal, and that $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is normal. (When computing $MM^\top$ and $M^\top M$ you only need to compute the part on or above the diagonal, since the product is symmetric; saving work in this way assumes one doesn't make a miscalculation.)

(b) Explain why any orthogonal $n \times n$ matrix is normal, and that if M is normal then so are M^2 and M^3. (How about M^r for general $r \geq 1$?)

(c) In (a) we give a type of non-symmetric normal 2×2 matrix (with $b \neq 0$). Show by explicit example that for a symmetric (hence normal) 2×2 matrix M and normal 2×2 matrix M' of the type in (a), the product $A = MM'$ can fail to be normal. (Nearly anything you try for M and M' should work, as long as you avoid too many matrix entries equal to 0.)

In connection with (c), it is a fact that if M and M' are normal $n \times n$ matrices that *commute* (i.e., $MM' = M'M$) then the product MM' is normal. However, this has no short explanation; it requires a big generalization of an upcoming important result called the Spectral Theorem.

Solution:

(a) Direct computation shows $AA^\top$ and $A^\top A$ both equal $\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$. Direct computation also shows that $BB^\top$ and $B^\top B$ are both equal to $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(b) For an orthogonal $n \times n$ matrix A, by definition A is invertible with $A^\top = A^{-1}$. Hence, normality for such A amounts to saying A commutes with A^{-1}. But in the definition of invertibiity we have that both products AA^{-1} and $A^{-1}A$ are equal to I_n.

We know that $(M^r)^\top = (M^\top)^r$ for any $r \geq 1$, so normality of M^r for normal M means that M^r and $(M^\top)^r$ commute when M and $M^\top$ commutes. For $r = 2$ such commuting of those rth powers for normal M is seen directly: since $MM^\top = M^\top M$ we can slide $M^\top$'s all the way to the left:

$$(M^2)(M^\top)^2 = MMM^\top M^\top = MM^\top MM^\top = M^\top MMM^\top = M^\top MM^\top M = M^\top M^\top MM = (M^\top)^2 M^2.$$

The same sliding pattern works for $r = 3$ (with more sliding), and for any r. (The same argument even shows that if $n \times n$ matrices M and M' commute then M^r and M'^r commute for every $r \geq 1$.)

(c) Nearly anything will work, but here is an example. Take M to be $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $M' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Then $A = MM' = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}$, and we compute

$$AA^\top = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ -6 & 18 \end{bmatrix}, \quad A^\top A = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}.$$