Solutions to Math 51 Practice problems for Quiz 6

1. (2 points) Determine whether the following statement is **true** (i.e., *always* true) or **false** (i.e., sometimes not true):

   Every orthogonal collection of seven nonzero vectors in $\mathbf{R}^{13}$ is linearly independent.

The above statement is **true**. From Chapter 5, any orthogonal collection of **nonzero** vectors forms a basis for the subspace it spans; and from Chapter 19, any basis is linearly independent. Or alternatively, for any possible relation

$$c_1\mathbf{v}_1 + \cdots + c_7\mathbf{v}_7 = \mathbf{0},$$

we may take the dot product of both sides with any vector $\mathbf{v}_i$ (i.e., for any $i = 1, \ldots, 7$) to obtain

$$\begin{aligned} 0 = \mathbf{0} \cdot \mathbf{v}_i &= (c_1\mathbf{v}_1 + \cdots + c_7\mathbf{v}_7) \cdot \mathbf{v}_i \\ &= c_1(\mathbf{v}_1 \cdot \mathbf{v}_i) + \cdots + c_7(\mathbf{v}_7 \cdot \mathbf{v}_i) \\ &= c_1(0) + \cdots + c_i\|\mathbf{v}_i\|^2 + \cdots + c_7(0) \\ &= c_i\|\mathbf{v}_i\|^2, \end{aligned}$$

and since each $\mathbf{v}_i$ is nonzero, its magnitude is nonzero, and we may conclude $c_i = 0$ for each i. So the only relation of the form $c_1\mathbf{v}_1 + \cdots + c_7\mathbf{v}_7 = \mathbf{0}$ has all c_i's equal to zero; this implies that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_7$ are linearly independent.

Now consider also the following variant, which omits one important word:

"True or False? Every orthogonal collection of four vectors in $\mathbf{R}^7$ is linearly independent."

The above statement is **false**, because the **zero vector** is orthogonal to every vector; the collection

$$\mathbf{v}_1 = \mathbf{e}_1, \quad \mathbf{v}_2 = \mathbf{e}_2, \quad \mathbf{v}_3 = \mathbf{e}_3, \quad \mathbf{v}_4 = \mathbf{0}$$

(where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the first three standard basis vectors of $\mathbf{R}^7$) is orthogonal, but it is *not* linearly independent: for example, we may obtain a valid linear dependency relation if we let $c_1 = c_2 = c_3 = 0$ and $c_4 = 17$:

$$0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3 + 17(\mathbf{0}) = \mathbf{0}.$$

(Alternatively, we may write $\mathbf{v}_4 = \mathbf{0}$ as a (not so interesting) linear combination of the first three vectors: $\mathbf{v}_4 = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$; so by the definition of linear dependence, the collection is linearly dependent.)

2. (3 points) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are unit vectors in $\mathbf{R}^{691}$, with

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \qquad \mathbf{v}_1 \cdot \mathbf{v}_3 = \frac{5}{13}, \qquad \mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{12}{13}.$$

   It is a fact that when we apply the Gram-Schmidt process to $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$, we obtain the vectors $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ in sequence, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are nonzero and $\mathbf{w}_3 = \mathbf{0}$.

   Given this information, which of the following is a valid linear dependence relation among the $\mathbf{v}_i$'s ?

   (i) $5\mathbf{v}_1 + 12\mathbf{v}_2 - 13\mathbf{v}_3 = \mathbf{0}$
   (ii) $5\mathbf{v}_1 - 12\mathbf{v}_2 + 13\mathbf{v}_3 = \mathbf{0}$
   (iii) $5\mathbf{v}_1 + 12\mathbf{v}_2 + 13\mathbf{v}_3 = \mathbf{0}$
   (iv) $5\mathbf{v}_1 - 12\mathbf{v}_2 - 13\mathbf{v}_3 = \mathbf{0}$

After completing the k-th step of the Gram-Schmidt process, we know two facts about that step:

- the nonzero vectors among $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are an orthogonal basis for $V_k = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$; and
- the next vector to compute, $\mathbf{w}_{k+1}$, equals $\mathbf{v}_{k+1} - \mathbf{Proj}_{V_k}(\mathbf{v}_{k+1})$.

In our case, take $k = 2$. Since we are given $\mathbf{w}_3 = \mathbf{0}$, the second fact above implies $\mathbf{v}_3 = \mathbf{Proj}_{V_2}(\mathbf{v}_3)$, which is equivalent to saying that $\mathbf{v}_3$ lies in $V_2 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. Thus, if we write

$$\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2,$$

we may now determine the unknown scalars a and b by taking dot products of each side by $\mathbf{v}_1$, $\mathbf{v}_2$:

$$\begin{aligned}
\frac{5}{13} &= \mathbf{v}_3 \cdot \mathbf{v}_1 = (a\mathbf{v}_1 + b\mathbf{v}_2) \cdot \mathbf{v}_1 = a\|\mathbf{v}_1\|^2 + b(\mathbf{v}_2 \cdot \mathbf{v}_1) = a(1) + b(0) = a, \\
-\frac{12}{13} &= \mathbf{v}_3 \cdot \mathbf{v}_2 = (a\mathbf{v}_1 + b\mathbf{v}_2) \cdot \mathbf{v}_2 = a(\mathbf{v}_1 \cdot \mathbf{v}_2) + b\|\mathbf{v}_2\|^2 = a(0) + b(1) = b,
\end{aligned}$$

because $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors with $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. We conclude that

$$\mathbf{v}_3 = \frac{5}{13}\mathbf{v}_1 - \frac{12}{13}\mathbf{v}_2, \text{ which is equivalent to } 5\mathbf{v}_1 - 12\mathbf{v}_2 - 13\mathbf{v}_3 = \mathbf{0}, \text{ or option (iv).}$$

Alternate solution: Based on the information given (including that the $\mathbf{v}_i$'s are unit vectors), the Gram-Schmidt process will proceed as follows:

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1, \\
\mathbf{w}_2 &= \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 \\
&= \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{w}_1 = \mathbf{v}_2 - 0\mathbf{w}_1 = \mathbf{v}_2, \\
\mathbf{w}_3 &= \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 \\
&= \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&= \mathbf{v}_3 - \left(\frac{5/13}{1}\right)\mathbf{v}_1 - \left(\frac{-12/13}{1}\right)\mathbf{v}_2 = \mathbf{v}_3 - \frac{5}{13}\mathbf{v}_1 + \frac{12}{13}\mathbf{v}_2
\end{aligned}$$

Since we are given that $\mathbf{w}_3 = \mathbf{0}$, we find that

$$\mathbf{0} = \mathbf{v}_3 - \frac{5}{13}\mathbf{v}_1 + \frac{12}{13}\mathbf{v}_2, \text{ which is equivalent to } 5\mathbf{v}_1 - 12\mathbf{v}_2 - 13\mathbf{v}_3 = \mathbf{0}, \text{ or option (iv).}$$

As a safety check on our answer, we may solve this relation to obtain $\mathbf{v}_3 = \frac{5}{13}\mathbf{v}_1 - \frac{12}{13}\mathbf{v}_2$, and then we may take dot products of this expression with $\mathbf{v}_1$ and $\mathbf{v}_2$ to reconfirm the dot-product information given (using the assumption that $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal unit vectors).

3. (3 points) Suppose the four 51-vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are all nonzero, and that

$$2\mathbf{v}_1 + 4\mathbf{v}_3 + 2\mathbf{v}_4 = \mathbf{0}.$$

Which of the following are possible numbers of nonzero vectors that lie in an orthogonal basis for $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$, obtained via the Gram-Schmidt process?

[Select all options that are possible with the information given.]

(i) zero (ii) one (iii) two (iv) three (v) four

The correct choices are (ii), (iii), and (iv).

Suppose the vectors obtained from the Gram-Schmidt process are $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ (not necessarily all nonzero). Option (i) is not possible, because $\mathbf{w}_1 = \mathbf{v}_1$ is not $\mathbf{0}$; so at least one of the $\mathbf{w}_i$'s will be nonzero.

The linear relation given implies that $\mathbf{v}_4$ lies in $V_3 = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Thus, $\mathbf{Proj}_{V_3}(\mathbf{v}_4) = \mathbf{v}_4$, which means that step 4 of the Gram-Schmidt process will compute

$$\begin{aligned} \mathbf{w}_4 &= \mathbf{v}_4 - \mathbf{Proj}_{V_3}(\mathbf{v}_4) \\ &= \mathbf{v}_4 - \mathbf{v}_4 \\ &= \mathbf{0}. \end{aligned}$$

This means that option (v) is not possible.

The other three options are all possible, because there might exist other linear relations among the $\mathbf{v}_i's$; these could in turn lead to other $\mathbf{w}_i$'s being $\mathbf{0}$. For example, if $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, then a similar reasoning to the above would find in step 3 of Gram-Schmidt that $\mathbf{w}_3 = \mathbf{0}$.

For completeness, here are explicit ways that each of options (ii),(iii),(iv) are possible, in terms of the standard basis vectors of $\mathbf{R}^{51}$:

(ii) Let $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{e}_1$. If we set $\mathbf{v}_4 = -\frac{1}{2}(2\mathbf{v}_1 + 4\mathbf{v}_3) = -\mathbf{v}_1 - 2\mathbf{v}_3$ $(= -3\mathbf{e}_1)$; then the given linear relation is automatically satisfied. Meanwhile, $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{span}(\mathbf{e}_1)$, which has dimension 1, so the Gram-Schmidt process will yield one nonzero vector.

(iii) Let $\mathbf{v}_1 = \mathbf{v}_3 = \mathbf{e}_1$ and $\mathbf{v}_2 = \mathbf{e}_2$; again set $\mathbf{v}_4 = -\mathbf{v}_1 - 2\mathbf{v}_3$ $(= -3\mathbf{e}_1)$ to satisfy the given linear relation. Then $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$, which has dimension 2, so the Gram-Schmidt process will yield two nonzero vectors.

(iv) Let $\mathbf{v}_1 = \mathbf{e}_1$, $\mathbf{v}_2 = \mathbf{e}_2$, and $\mathbf{v}_3 = \mathbf{e}_3$; again set $\mathbf{v}_4 = -\mathbf{v}_1 - 2\mathbf{v}_3$ $(= -\mathbf{e}_1 - 2\mathbf{e}_3)$ to satisfy the given linear relation. Then $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, which has dimension 3, so the Gram-Schmidt process will yield three nonzero vectors.

4. (3 points) Consider the following 3×3 matrix:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & a \\ -1/\sqrt{2} & 1/\sqrt{3} & b \\ 0 & -1/\sqrt{3} & c \end{bmatrix}$$

where $a, b, c \in \mathbf{R}$. How many 3×3 orthogonal matrices exist that have the above form?

(i) none (ii) one (iii) two

(iv) more than two, but finitely many (v) infinitely many

An orthogonal matrix consists of orthonormal columns; the first two columns of the given matrix are confirmed to be unit vectors that are orthogonal to each other, so in order to determine the number of such matrices we need the following conditions to hold:

(a) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a unit vector; and

(b) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is orthogonal to each of $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$.

This may be determined geometrically, using what we know about $\mathbf{R}^3$. Condition (b) requires the vector $\left[\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right]$ to be orthogonal to the plane spanned by the other two column vectors; this places $\left[\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right]$ on the normal line to this plane. However, the unit-length condition (a) implies that we may select only two vectors on this line. Since the choice we make for its third column determines the matrix, it follows that the number of possible matrices is two (i.e., option (iii)).

Alternate solution: Conditions (a) and (b) may be converted to algebraic conditions:

(a) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ must satisfy $a^2 + b^2 + c^2 = 1$; and

(b) $0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \dfrac{a}{\sqrt{2}} - \dfrac{b}{\sqrt{2}}$, and $0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} = \dfrac{a}{\sqrt{3}} + \dfrac{b}{\sqrt{3}} - \dfrac{c}{\sqrt{3}}$.

After clearing denominators, we find the first condition of (b) leads to the equation $a - b = 0$, or $a = b$; similarly, the second condition leads to the equation $a + b - c = 0$, or $c = a + b$. Together, we find that $(a, b, c) = (a, a, a + a) = (a, a, 2a)$, in terms of a. Now the first condition requires that

$$1 = a^2 + b^2 + c^2 = a^2 + a^2 + (2a)^2 = a^2 + a^2 + 4a^2 = 6a^2,$$

so $a = \pm\dfrac{1}{\sqrt{6}}$. These two choices for a lead to two ordered triples $(a, b, c) = (a, a, 2a)$, and corresponding to the two matrices

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}, \quad \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & -1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$$

As a safety check, we can confirm directly that these two matrices have orthonormal columns. We can thus again conclude that the answer is (iii) (i.e., two matrices).

5. (4 points) Suppose that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbf{R}^{2021}$ are linearly independent. Which of the following sets are linearly independent?

(a) $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$

(b) $\{\mathbf{u} + \mathbf{v} + \mathbf{w}, \mathbf{u} - \mathbf{v} - 2\mathbf{w}, \mathbf{u} + 3\mathbf{v} + 4\mathbf{w}\}$

(c) $\{\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{w}, \mathbf{v} + \mathbf{w}\}$

(d) $\{\mathbf{u} + \mathbf{v} + \mathbf{w}, \mathbf{u} - \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - \mathbf{w}, \mathbf{u} - \mathbf{v} + \mathbf{w}\}$.

(a) Linearly independent. Suppose there are c_1, c_2 such that $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v}) = \mathbf{0}$. Then, we have $(c_1 + c_2)\mathbf{u} + (c_1 - c_2)\mathbf{v} = \mathbf{0}$. Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, we must have $c_1 + c_2 = c_1 - c_2 = 0$. Solving this system, we get $c_1 = c_2 = 0$, so $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is a linearly independent set.

(b) Linearly dependent. We have in fact

$$2(\mathbf{u}+\mathbf{v}+\mathbf{w}) - (\mathbf{u}-\mathbf{v}-2\mathbf{w}) = \mathbf{u}+3\mathbf{v}+4\mathbf{w}.$$

To find these coefficients, note that a set of 3 vectors are linearly dependent if we can express one as a linear combination of the other two. As such, we can write $a(\mathbf{u}+\mathbf{v}+\mathbf{w}) + b(\mathbf{u}-\mathbf{v}-2\mathbf{w}) = \mathbf{u}+3\mathbf{v}+4\mathbf{w}$ and solve for a, b.

(c) Linearly independent. Suppose there are c_1, c_2, c_3 such that $c_1(\mathbf{u}+\mathbf{v})+c_2(\mathbf{u}+\mathbf{w})+c_3(\mathbf{v}+\mathbf{w}) = \mathbf{0}$. Then, we have

$$(c_1+c_2)\mathbf{u} + (c_1+c_3)\mathbf{v} + (c_2+c_3)\mathbf{w} = \mathbf{0}.$$

Since $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent, we need $c_1+c_2 = c_1+c_3 = c_2+c_3 = 0$. Solving this, we have $c_1 = c_2 = c_3 = 0$, so $\{\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}\}$ is a linearly independent set.

(d) Linearly dependent. We have that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent, so $\mathrm{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is 3 dimensional. Note that $\mathbf{u}+\mathbf{v}+\mathbf{w}, \mathbf{u}-\mathbf{v}-\mathbf{w}, \mathbf{u}+\mathbf{v}-\mathbf{w}, \mathbf{u}-\mathbf{v}+\mathbf{w}$ are all linear combinations of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ so they belong to $\mathrm{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$. As a result, they can't be linearly independent as a set of four vectors in a 3 dimensional subspace.

6. (4 points) Let

$$\mathbf{v}_1 = \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}1\\-4\\-5\\0\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}2\\-6\\-6\\6\end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix}0\\-3\\-6\\-9\end{bmatrix}$$

and $V = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. What is the dimension of V?
Hint: You can use the Gram-Schmidt process.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

We will go through the Gram-Schmidt procedure. First let $\mathbf{w}_1 = \mathbf{v}_1$. Next let

$$\mathbf{w}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{w}_1}(\mathbf{v}_2) = \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix}1\\-4\\-5\\0\end{bmatrix} - \frac{-14}{14}\begin{bmatrix}0\\1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\-4\\-5\\0\end{bmatrix} + \begin{bmatrix}0\\1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\-3\\-3\\3\end{bmatrix}.$$

At this point, we could continue the Gram-Schmidt procedure and we would arrive at $\mathbf{w}_3 = \mathbf{w}_4 = 0$, or we can simply observe that $\mathbf{v}_3 = 2\mathbf{w}_2$ and $\mathbf{v}_4 = -3\mathbf{w}_1$, so that $V = \mathrm{span}\{\mathbf{w}_1, \mathbf{w}_2\}$. Since $\mathbf{w}_1$ and $\mathbf{w}_2$ are not scalar multiples of each other, the dimension of V is 2.

7. (2 points) **True or False:**
If the Gram-Schmidt process is applied to

$$\mathbf{v}_1 = \begin{bmatrix}2\\0\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}3\\5\\0\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}7\\11\\13\end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix}17\\19\\23\end{bmatrix}$$

to yield $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ sequentially, then $\mathbf{w}_4 = \mathbf{0}$.
Hint: Do not use brute force to actually carry out the Gram-Schmidt process.

Always true:
Note that we have 4 vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^3$ here, so they must be linearly dependent. By a direct computation, you can check that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent and thus forming a basis of $\mathbf{R}^3$. Thus, the last vector $\mathbf{v}_4$ is redundant, so the Gram-Schmidt process will give $\mathbf{w}_4 = \mathbf{0}$.

8. (3 points) Let A and B be two invertible 3×3 matrices, and P be the matrix for projection onto a plane in $\mathbf{R}^3$ through the origin. Which of the following matrices must **always** be invertible?

   (a) $A + B$. (b) $AB^\top$. (c) AP.

(a) $A + B$ is not always invertible. Just let $A = I_3$ and $B = -I_3$, then $A + B = 0$ is not invertible.

(b) Since A and B have inverses A^{-1} and B^{-1}, $(B^\top)^{-1} = (B^{-1})^\top$, $AB^\top$ is invertible with inverse given by $(AB^\top)^{-1} = (B^\top)^{-1}A^{-1} = (B^{-1})^\top A^{-1}$ is invertible.

(c) Just let $A = I_3$, and $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ be the matrix for projection onto the xy-plane, then $AP = P$ is not invertible. In fact, AP is never invertible. Recall that a matrix P representing projection from $\mathbf{R}^3$ to a plane through the origin is never invertible. If AP were invertible, then the product of two invertible matrix A^{-1} and AP, $A^{-1}(AP) = (A^{-1}A)P = P$, would be invertible, which would be a contradiction.

9. (2 points) **True or False:**
   If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set in $\mathbf{R}^n$, then $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$ is also a linearly independent set in $\mathbf{R}^n$.

Always true: To show $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$ is a linearly independent set, we need to show that the only linear relation

$$a(\mathbf{u} + \mathbf{v}) + b(\mathbf{u} - \mathbf{v}) + c(\mathbf{u} + \mathbf{v} + \mathbf{w}) = \mathbf{0}$$

must have $a = b = c = 0$.

$$a(\mathbf{u} + \mathbf{v}) + b(\mathbf{u} - \mathbf{v}) + c(\mathbf{u} + \mathbf{v} + \mathbf{w}) = (a + b + c)\mathbf{u} + (a - b + c)\mathbf{v} + c\mathbf{w} = \mathbf{0}$$

Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set, we must have

$$a + b + c = a - b + c = c = 0.$$

In particular, $c = 0$, and $a + b = a - b = 0$. It follows that $a = b = 0$ as well. So we have shown that $a = b = c = 0$ is the only solution, hence $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$ is a linearly independent set.

10. (4 points) If we apply the Gram-Schmidt process to the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix},$$

   Which of the following expressions is used to compute $\mathbf{w}_4$?
   Note that $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

(a) $\begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{12}{18}\begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix} - \frac{17}{27}\begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{7}{15}\begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$

(c) $\frac{5}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{12}{18}\begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix} + \frac{17}{27}\begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{7}{15}\begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{17}{27}\begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

The answer is (b). We set

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix},$$

and then compute

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}.$$

Since $\mathbf{v}_3$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ ($\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$), we will have $\mathbf{w}_3 = 0$. Alternatively, $\mathbf{w}_3$ could also be computed directly. This means that we can eliminate this vector from the process, so

$$\mathbf{w}_4 = \mathbf{v}_4 - \frac{\mathbf{v}_4 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_4 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{7}{15}\begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}.$$

Alternatively, the final step in the Gram Schmidt process involves taking $\mathbf{v}_4$, and subtracting its projections onto an *orthogonal* set of vectors. Note that the vectors being subtracted from $\mathbf{v}_4$ in answers (a) and (d) are not orthogonal, so the answer must be (b).

11. (3 points) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ be a set of n-vectors. Suppose that the Gram-Schmidt process applies to this list yields, in order, the vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ where

$$\mathbf{w}_3 = \mathbf{w}_4 = \mathbf{0}$$

and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_5$ are all non-zero. Then a basis for $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5)$ is given by:

Select all that apply.

(a) $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$

(b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$

(c) $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{w}_5\}$

(d) $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_5\}$

(e) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$

The answer is (d) and (e). A basis for $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5)$ is given by the non-zero vectors that are output by the Gram-Schmidt process. Thus, (d) is one basis. Note that this implies that the dimension of $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5)$ is 3. $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, otherwise $\mathbf{w}_2$ would be equal to zero. Similarly, $\mathbf{v}_5$ is not in $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$, otherwise $\mathbf{w}_5$ would be equal to zero. Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$ is a linearly-independent list. Since its length is 3, it is a basis for $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5)$. The answers (a), (b), and (c) cannot be bases because they have length greater than 3.

12. (2 points) Let $A : \mathbf{R}^{51} \to \mathbf{R}^{51}$ be a linear transformation. Suppose that the null space of A is a plane. Fix $\mathbf{b} \in \mathbf{R}^{51}$. What could the set $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$ be?

(a) the empty set: there are no solutions

(b) a line

(c) a plane

(d) all of $\mathbf{R}^{51}$

(e) none of the other choices.

The answers are (a) and (c).
If $\mathbf{b}$ lies in $C(A)$ then the set $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$ is a plane parametrized by $\mathbf{x}_0 + t\mathbf{v} + t'\mathbf{v}'$ where $A\mathbf{x}_0 = \mathbf{b}$ for some $\mathbf{x}_0$ and $\{\mathbf{v}, \mathbf{v}'\}$ is a basis for $N(A)$. If $\mathbf{b}$ does not lie in $C(A)$ then $A\mathbf{x} = \mathbf{b}$ has no solution and the set $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$ is the empty set.

13. (3 points) Suppose that A is an **invertible** 3×3 matrix such that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

What is the solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$?

Note that $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ *are orthogonal to each other.*

(a) $\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

(e) There is not enough information to determine this.

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. $A\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2$ can be solved uniquely as

$$\mathbf{x} = A^{-1}(a\mathbf{v}_2 + b\mathbf{v}_2) = aA^{-1}\mathbf{v}_1 + bA^{-1}\mathbf{v}_2 = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Let $V = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$. Suppose $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbf{R}^3$. Without knowing what $A^{-1}\mathbf{v}_3$ is, we cannot determine $A^{-1}(a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3)$. We can compute $\mathbf{x} = A^{-1}\mathbf{w}$ exactly when $\mathbf{w} \in V$.

To check if $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ is in V, since $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthogonal basis for V, we can compute

$$\begin{aligned} \mathbf{Proj}_V(\mathbf{w}) &= \frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{w} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\ &= \frac{-2}{3}\mathbf{v}_1 + \frac{15}{14}\mathbf{v}_2 \\ &\neq \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \end{aligned}$$

So $\mathbf{w}$ is not in V. So we cannot compute $A^{-1}\mathbf{w}$, and the answer is (e).

14. (2 points) **True or False:**
If two $m \times n$ matrices A and B have the same column space and same null space, i.e.

$$C(A) = C(B), \;\; N(A) = N(B),$$

then $A = B$ always.

Not always true: Let $A = I_n$ and $B = 2I_n$, then $A \neq B$ but $C(A) = C(B) = \mathbf{R}^n$ and $N(A) = N(B) = \{\mathbf{0}\}$.

15. (2 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ -1 & 3 \end{bmatrix}.$$

Suppose that we append two columns to the right of second column of the matrix A, obtaining a 3×4 matrix A'. Which of the following are the possible value(s) for the dimension of $N(A')$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

By the Rank Nullity theorem, $\dim C(A') + \dim N(A') = 4$. The dimension of $C(A')$ is at least 2, because the two columns of A are linearly independent, and adding extra columns can only increase the dimension of $C(A')$. The dimension of $C(A')$ is at most 3, because $C(A')$ is contained in $\mathbf{R}^3$. Thus, the dimension of $N(A')$ is between 1 and 2, so (a), (d), and (e) are incorrect.

To see that the dimension of $C(A')$ can be either 2 or 3, we give explicit examples where each case occurs. To get 2, we can add the first column of A twice and the dimension of the column space does not increase. To get 3, we add any vector not already in the column space as one of the two added columns, and $C(A')$ will be 3-dimensional. Thus, (b) and (c) are correct.

16. (3 points) Let M be a 5×4 matrix and $\mathbf{b} \in \mathbf{R}^5$ a vector. Assume that there is at least one solution $\mathbf{x} \in \mathbf{R}^4$ to the equation $M\mathbf{x} = \mathbf{b}$.

If A is an invertible 5×5 matrix, does $M\mathbf{y} = A\mathbf{b}$ have a solution $\mathbf{y} \in \mathbf{R}^4$?

(a) Yes: No matter what A is, there is some $\mathbf{y} \in \mathbf{R}^4$ so that $M\mathbf{y} = A\mathbf{b}$.
(b) Maybe: Depending on what A is, there may or may not be some $\mathbf{y} \in \mathbf{R}^4$ so that $M\mathbf{y} = A\mathbf{b}$.
(c) No: No matter what A is, there is no $\mathbf{y} \in \mathbf{R}^4$ so that $M\mathbf{y} = A\mathbf{b}$.

The correct answer is "Maybe."

For some choices of A, there is a solution. For example, if $A = I_5$ is the identity, then $A\mathbf{b} = I_5\mathbf{b} = \mathbf{b}$, which means that $\mathbf{y} = \mathbf{x}$ is a solution to the equation $M\mathbf{y} = A\mathbf{b}$. (Depending on what M and $\mathbf{b}$ are, there may be other choices of A which work as well.)

However, depending on what M, $\mathbf{b}$, and A are, we can't guarantee that there is a solution. For example, consider the following:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then the equation $M\mathbf{x} = \mathbf{b}$ is equivalent to the system of equations

$$\begin{cases} 1 = x_1 \\ 1 = x_1 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

There is a solution to $M\mathbf{x} = \mathbf{b}$; for instance,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a solution.

Now consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now

$$A\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The equation $M\mathbf{y} = A\mathbf{b}$ is equivalent to the system of equations

$$\begin{cases} 2 = y_1 \\ 1 = y_1 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

This system does not have a solution.

17. (3 points) Suppose that A is an **invertible** 3×3 matrix such that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

What is the solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$?

Note that $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ *are orthogonal to each other.*

(a) $\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ (e) There is not enough information to determine this.

Note that

$$\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

This can also be found by computing a projection as follows: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. $A\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2$ can be solved uniquely as

$$\mathbf{x} = A^{-1}(a\mathbf{v}_2 + b\mathbf{v}_2) = aA^{-1}\mathbf{v}_1 + bA^{-1}\mathbf{v}_2 = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Let $V = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$. Let $V = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$. We can compute $\mathbf{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ exactly when $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \in V$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthogonal basis for V, we can compute

$$\begin{aligned} \mathbf{Proj}_V(\mathbf{w}) &= \frac{\mathbf{w}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{w}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 \\ &= \frac{-3}{3}\mathbf{v}_1 + \frac{14}{14}\mathbf{v}_2 \\ &= -\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \end{aligned}$$

So we have

$$
\begin{aligned}
A^{-1}\begin{bmatrix}1\\0\\4\end{bmatrix} &= A^{-1}\left(-\begin{bmatrix}1\\1\\-1\end{bmatrix}+\begin{bmatrix}2\\1\\3\end{bmatrix}\right)\\
&= -A^{-1}\begin{bmatrix}1\\1\\-1\end{bmatrix}+A^{-1}\begin{bmatrix}2\\1\\3\end{bmatrix}\\
&= -\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\-1\\1\end{bmatrix}\\
&= \begin{bmatrix}0\\-2\\0\end{bmatrix}.
\end{aligned}
$$

So the answer is (a).

18. (3 points) Let A be an $m \times n$ matrix where $m > n$. Let B be the $m \times (m+n)$ matrix as follows:

$$B = \begin{bmatrix} A & I_m \end{bmatrix},$$

where the $m \times m$ identity matrix I_m is appended to the right of A. Which of the following statements are true? Select all that apply.

(a) $N(A) = N(B)$.
(b) $N(B)$ is n-dimensional.
(c) $N(B) = \{\mathbf{0}\}$.
(d) $C(A) = C(B)$.

The m columns of I_m in B are linearly independent, and they span all of $\mathbf{R}^m$. $C(A)$ and $C(B)$ are linear subspaces of $\mathbf{R}^m$. A has n columns and $n < m$, so $C(A)$ is at most n-dimensional, while $C(B) = \mathbf{R}^m$ since B contains the m columns of I_m, which are standard basis vectors of $\mathbf{R}^m$.

(a) Always false: $N(A)$ is a subspace of $\mathbf{R}^n$ while $N(B)$ is a subspace of $\mathbf{R}^{m+n}$. $N(A)$ and $N(B)$ live in different spaces, they cannot be equal.

(b) Always true: By the Rank-Nullity Theorem, $\dim(N(B)) = m+n-\dim(C(B)) = m+n-m = n$.

(c) Always false: $N(B)$ is n-dimensional, so $N(B)$ cannot be a single point, i.e. $N(B) \neq \{\mathbf{0}\}$.

(d) Always false: $C(A)$ is at most n-dimensional, while $C(B)$ is m-dimensional, and $m > n$, so $C(A) \neq C(B)$.

19. (3 points) Suppose A is a 3×3 matrix, and that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent collection of vectors in $\mathbf{R}^3$ for which

- $\mathbf{u}$, $\mathbf{v}$ lie in the null space of A; and
- $\mathbf{w}$ lies in the column space of A.

Which of the following systems of equations must have at least one solution? Select all that apply.

(i) $A\mathbf{x} = \mathbf{0}$ (ii) $A\mathbf{x} = 2\mathbf{u} + 3\mathbf{v}$ (iii) $A\mathbf{x} = 3\mathbf{w}$ (iv) $A\mathbf{x} = \mathbf{u} + 3\mathbf{v} + 2\mathbf{w}$

Among the four choices, only systems (i),(iii) must have at least one solution; in fact, systems (ii),(iv) must have no solution.

The first bullet-point tells us that the null space of A has dimension at least 2, since $\mathbf{u}$, $\mathbf{v}$ are given to be linearly independent. The second bullet-point tells us that the column space of A has dimension at least 1 (since $\mathbf{w}$ is nonzero; it is part of a linearly independent collection). But by the Rank-Nullity Theorem, $\dim(N(A)) + \dim(C(A)) = 3$, so in fact we must have $\dim(N(A)) = 2$ and $\dim(C(A)) = 1$.

In particular, this implies that $C(A) = \text{span}(\mathbf{w})$; so the linear systems "$A\mathbf{x} = \mathbf{b}$" that have at least one solution are precisely those which have $\mathbf{b}$ equal to a scalar multiple of $\mathbf{w}$. So systems (i) and (iii) are correct choices. In addition, since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent collection, we know that neither of $2\mathbf{u} + 3\mathbf{v}$ nor $\mathbf{u} + 3\mathbf{v} + 2\mathbf{w}$ can be a scalar multiple of $\mathbf{w}$; so systems (ii) and (iv) cannot have any solutions.

20. (3 points) Suppose that A is a 3×3 matrix such that:

$$A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \qquad A\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix}, \qquad A\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

What statement describes the set of solutions to $A\mathbf{x} = \begin{bmatrix}4\\3\\1\end{bmatrix}$?

Note that $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ *and* $\begin{bmatrix}2\\1\\3\end{bmatrix}$ *are orthogonal to each other.*

(i) There is not enough information to choose a single answer here.
(ii) There are no solutions (i.e., empty set).
(iii) There is a unique solution (i.e., set consisting of exactly one point in $\mathbf{R}^3$)
(iv) There are infinitely many solutions, and graphically the solution set forms a line in $\mathbf{R}^3$.
(v) There are infinitely many solutions, and graphically the solution set forms a plane in $\mathbf{R}^3$.

The correct answer is (iv).

With the given information, we know that the column space of A contains two linearly independent orthogonal vectors $\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}$ (as they are nonzero, orthogonal vectors), so $C(A)$ has dimension at least 2. We also know that the null space of A contains the (nonzero) vector $\begin{bmatrix}1\\0\\-1\end{bmatrix}$, so $N(A)$ has dimension at least 1. But by the Rank-Nullity Theorem, $\dim(C(A)) + \dim(N(A)) = 3$, so in fact we must have $\dim(C(A)) = 2$ and $\dim(N(A)) = 1$. In particular, the nullity being 1 immediately implies that any system "$A\mathbf{x} = \mathbf{b}$" that has at least one solution must have infinitely many solutions which graphically form a line in $\mathbf{R}^3$.

It remains to determine whether the system $A\mathbf{x} = \begin{bmatrix}4\\3\\1\end{bmatrix}$ has at least one solution, or equivalently whether $\mathbf{b} = \begin{bmatrix}4\\3\\1\end{bmatrix}$ lies in $C(A)$. But since $C(A)$ has dimension 2, it has basis $\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}$; and we simply need determine whether $\mathbf{b}$ is a linear combination of these vectors. There are many ways to do this; for example, since these vectors in fact form an orthogonal basis, we could use the orthogonal projection formula to quickly determine whether $\mathbf{b} = \mathbf{Proj}_{C(A)}(\mathbf{b})$:

$$\begin{aligned}
\mathbf{Proj}_{C(A)}(\mathbf{b}) &= \left(\frac{\mathbf{b}\cdot\begin{bmatrix}1\\1\\-1\end{bmatrix}}{\begin{bmatrix}1\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-1\end{bmatrix}}\right)\begin{bmatrix}1\\1\\-1\end{bmatrix} + \left(\frac{\mathbf{b}\cdot\begin{bmatrix}2\\1\\3\end{bmatrix}}{\begin{bmatrix}2\\1\\3\end{bmatrix}\cdot\begin{bmatrix}2\\1\\3\end{bmatrix}}\right)\begin{bmatrix}2\\1\\3\end{bmatrix} = \left(\frac{6}{3}\right)\begin{bmatrix}1\\1\\-1\end{bmatrix} + \left(\frac{14}{14}\right)\begin{bmatrix}2\\1\\3\end{bmatrix} \\
&= 2\begin{bmatrix}1\\1\\-1\end{bmatrix} + 1\begin{bmatrix}2\\1\\3\end{bmatrix} \\
&= \begin{bmatrix}4\\3\\1\end{bmatrix} = \mathbf{b}
\end{aligned}$$

We conclude that since $\mathbf{b} = \begin{bmatrix}4\\3\\1\end{bmatrix}$ lies in $C(A)$, and since the null space of A is a line in $\mathbf{R}^3$, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions that graphically form a line in $\mathbf{R}^3$.