Problem 1: LU-decomposition

Let $A = \begin{bmatrix} 12 & 9 & 3 \\ -4 & 1 & 7 \\ 4 & 3 & 2 \end{bmatrix}$ and $L = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Verify that $LU = A$, so this is an LU-decomposition of A.

(b) Let $\mathbf{b} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$. Find all solutions to $L\mathbf{y} = \mathbf{b}$. (You should get that $\mathbf{y}_0 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is the only solution.)

(c) Find all solutions to $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b}$ as in (b). (Hint: This means solving $LU\mathbf{x} = \mathbf{b}$, which is the same as $U\mathbf{x} = \mathbf{y}_0$. Why?)

Solution:

(a) We calculate

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 9 & 3 \\ -4 & 1 & 7 \\ 4 & 3 & 2 \end{bmatrix}.$$

(b) We solve the system

$$\begin{aligned} 3y_1 \qquad\qquad &= 6 \\ -y_1 + 2y_2 \qquad &= 2 \\ y_1 + \qquad y_3 &= 1 \end{aligned}$$

Forward substitution gives $y_1 = 2$, then $y_2 = 2$, and finally $y_3 = -1$.

(c) We already know that $L\mathbf{y} = \mathbf{b}$ has exactly one solution, $\mathbf{y}_0 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. Hence, the system $\mathbf{b} = A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x})$ says exactly that $U\mathbf{x} = \mathbf{y}_0$. We express the latter as a system of linear equations:

$$\begin{aligned} 4x_1 + 3x_2 + x_3 &= 2 \\ 2x_2 + 4x_3 &= 2 \\ x_3 &= -1 \end{aligned}$$

Backward substitution gives $x_3 = -1$, then $x_2 = 3$, and finally $x_1 = -\frac{3}{2}$.

Problem 2: QR-decomposition

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 5 \\ 1 & 5 & 3 \end{bmatrix}$, and define $\mathbf{v}_i$ to be the ith column of A.

(a) Apply the Gram–Schmidt process to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The output vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ that you obtain should all be nonzero, and as a check on your work make sure that they are pairwise orthogonal.

(b) Examine your calculations from (a) to express each $\mathbf{v}_i$ as a linear combination of the orthogonal basis of $\mathbf{w}_j$'s. (This should be found from the work already done in (a); do *not* directly compute the projections of $\mathbf{v}_i$ onto each $\mathbf{w}_j$, as that would be defeating the point of the work in (a).) Then compute the unit vectors $\mathbf{w}_j' = \mathbf{w}_j / \|\mathbf{w}_j\|$ and express $\mathbf{v}_i$ as a linear combination of the $\mathbf{w}_j'$'s.

(c) Use (b) to find a decomposition $A = QR$ where Q is an orthogonal matrix and R is an upper triangular matrix. Check your answer is correct by computing the product QR of the Q and R that you find.

(d) Use (c) to find A^{-1} as an explicit 3×3 matrix (with entries that are fractions with denominator that is a factor of 10, no $\sqrt{5}$ anywhere), and check that its product against A on the left or the right is equal to I_3; it is fine to compute just one of those products.

Hint: when computing R^{-1}, you may find it convenient to first extract $\sqrt{5}$ as a factor from every entry of R (i.e., write $R = \sqrt{5}R'$ for an upper triangular matrix R', so $R^{-1} = (1/\sqrt{5})R'^{-1}$; it is easier to find R'^{-1}.)

Solution:

(a) We have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Then

$$\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_2) = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 = \mathbf{v}_2 - \frac{5}{5}\mathbf{w}_1 = \mathbf{v}_2 - \mathbf{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}.$$

Finally,

$$\begin{aligned} \mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_3) - \mathbf{Proj}_{\mathbf{w}_2}(\mathbf{v}_3) &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \mathbf{w}_2 \\ &= \mathbf{v}_3 - \frac{5}{5}\mathbf{w}_1 - \frac{10}{20}\mathbf{w}_2 \\ &= \mathbf{v}_3 - \mathbf{w}_1 - \frac{1}{2}\mathbf{w}_2 \\ &= \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}. \end{aligned}$$

The dot products $\mathbf{w}_i \cdot \mathbf{w}_j$ for $i \neq j$ are all checked to be 0 by direct calculation,

(b) We see from the work in (a) that

$$\mathbf{v}_1 = \mathbf{w}_1, \ \ \mathbf{v}_2 = \mathbf{w}_2 + \mathbf{w}_1 = \mathbf{w}_1 + \mathbf{w}_2, \ \ \mathbf{v}_3 = \mathbf{w}_3 + \mathbf{w}_1 + (1/2)\mathbf{w}_2 = \mathbf{w}_1 + (1/2)\mathbf{w}_2 + \mathbf{w}_3.$$

Dividing each $\mathbf{w}_i$ by its length gives that

$$\mathbf{w}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \ \ \mathbf{w}_2' = \frac{1}{\sqrt{20}}\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \ \ \mathbf{w}_3' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and substituting $\mathbf{w}_i = \|\mathbf{w}_1\|\mathbf{w}'_i$ into the expression for each $\mathbf{v}_i$ in terms of the $\mathbf{w}_j$'s yields

$$\mathbf{v}_1 = \sqrt{5}\,\mathbf{w}'_1,\ \ \mathbf{v}_2 = \sqrt{5}\,\mathbf{w}'_1 + \sqrt{20}\,\mathbf{w}'_2,\ \ \mathbf{v}_3 = \sqrt{5}\,\mathbf{w}'_1 + \sqrt{5}\,\mathbf{w}'_2 + 5\mathbf{w}'_3.$$

(c) From the work in (b) we can read off Q from the $\mathbf{w}'_j$'s and R from the coefficients of each $\mathbf{v}_i$ in terms of the $\mathbf{w}'_j$'s:

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{5} & \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{20} & \sqrt{5} \\ 0 & 0 & 5 \end{bmatrix}.$$

(You can write the middle entry in R as $2\sqrt{5}$ if you wish, but this is entirely unnecessary.) Direct multiplication confirms that QR is indeed equal to A.

(d) We know $A^{-1} = R^{-1}Q^{-1}$. We have

$$Q^{-1} = Q^{\top} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}.$$

To find R^{-1}, we write $R = \sqrt{5}R'$ for

$$R' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix},$$

so $R^{-1} = (1/\sqrt{5})R'^{-1}$. We seek numbers a, b, c for which

$$R'^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & \frac{1}{2} & c \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

To find these numbers we calculate

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R'^{-1}R' = \begin{bmatrix} 1 & 1+2a & 1+a+\sqrt{5}\,b \\ 0 & 1 & (1/2)+\sqrt{5}\,c \\ 0 & 0 & 1 \end{bmatrix}.$$

So we have $a = -\frac{1}{2}, b = -\frac{1}{2\sqrt{5}}$, and $c = -\frac{1}{2\sqrt{5}}$. Hence,

$$R^{-1} = \frac{1}{\sqrt{5}}R'^{-1} = \begin{bmatrix} 1/\sqrt{5} & -1/(2\sqrt{5}) & -1/10 \\ 0 & 1/(2\sqrt{5}) & -1/10 \\ 0 & 0 & 1/5 \end{bmatrix}.$$

Altogether this gives

$$\begin{aligned} A^{-1} = R^{-1}Q^{\top} &= \begin{bmatrix} 1/\sqrt{5} & -1/(2\sqrt{5}) & -1/10 \\ 0 & 1/(2\sqrt{5}) & -1/10 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2/5+1/10 & -1/10 & 1/5-1/5 \\ -1/10 & -1/10 & 1/5 \\ 0 & 1/5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & -1/10 & 0 \\ -1/10 & -1/10 & 1/5 \\ 0 & 1/5 & 0 \end{bmatrix} \end{aligned}$$

Direct multiplication of this against A on either the left or the right (take your pick) is seen to yield I_3, as desired.