1. (a) The inverse has the form

$$U^{-1} = \begin{bmatrix} 1/4 & a & b & c \\ 0 & -1/3 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

for some entries that we find by using the matrix equation $UU^{-1} = \mathrm{I}_4$. Multiplying, we have

$$\begin{aligned} UU^{-1} &= \begin{bmatrix} 4 & 2 & -4 & 8 \\ 0 & -3 & 6 & 12 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/4 & a & b & c \\ 0 & -1/3 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4a-2/3 & 4b+2d-4 & 4c+2e-4f+4 \\ 0 & 1 & -3d+6 & -3e+6f+6 \\ 0 & 0 & 1 & f+2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

so equating this with $\mathrm{I}_4$ with vanishing everywhere above the diagonal gives successive equations as we work our way into the upper right corner layer by layer from the diagonal: the layer just above the diagonal gives

$$4a - 2/3 = 0, \ \ -3d + 6 = 0, \ \ f + 2 = 0,$$

which solves to give $a = 1/6$, $d = 2$, $f = -2$. The next layer up gives

$$0 = 4b + 2d - 4 = 4b + 4 - 4 = 4b, \ \ 0 = -3e + 6f + 6 = -3e - 12 + 6 = -3e - 6,$$

so $b = 0$ and $e = -2$. Finally, the upper right corner gives

$$0 = 4c + 2e - 4f + 4 = 4c - 4 - 4(-2) + 4 = 4c + 8,$$

so $c = -2$.
Putting it all together, we have

$$U^{-1} = \begin{bmatrix} 1/4 & 1/6 & 0 & -2 \\ 0 & -1/3 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

To check this works, we multiply matrices

$$\begin{aligned} UU^{-1} &= \begin{bmatrix} 4 & 2 & -4 & 8 \\ 0 & -3 & 6 & 12 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/4 & 1/6 & 0 & -2 \\ 0 & -1/3 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2/3-2/3 & 4-4 & -8-4+8+4 \\ 0 & 1 & -6+6 & 6-12+6 \\ 0 & 0 & 1 & -2+2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \mathrm{I}_4. \end{aligned}$$

(b) The general solution is $\mathbf{x} = U^{-1}\mathbf{b}$, so using the answer in (a) we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/6 & 0 & -2 \\ 0 & -1/3 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} b_1/4 + b_2/6 - 2b_4 \\ -b_2/3 + 2b_3 - 2b_4 \\ b_3 - 2b_4 \\ b_4/2 \end{bmatrix}.$$

The entries on the right give a general formula for each $x_i$ when given $\mathbf{b}$. Taking $(b_1, b_2, b_3, b_4) = (8, -6, 2, 6)$, the general formula for the solution becomes

$$\begin{bmatrix} 2-1-12 \\ 2+4-12 \\ 2-12 \\ 3 \end{bmatrix} = \begin{bmatrix} -11 \\ -6 \\ -10 \\ 3 \end{bmatrix},$$

recovering the answer in Exercise 22.1 (a).

◇

2. (a) The matrix product is

$$\begin{bmatrix} 3 & 2 & -2 \\ 15 & 10+2 & -10+2 \\ 9 & 6-4 & -6-4+4 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ 15 & 12 & -8 \\ 9 & 2 & -6 \end{bmatrix}.$$

(b) The inverses $L^{-1}$ and $U^{-1}$ have the form

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1/2 & 0 \\ b & c & -1 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} 1/3 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & -1/4 \end{bmatrix}$$

with entries determined by the vanishing of the off-diagonal entries in

$$LL^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a & 1/2 & 0 \\ b & c & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5+2a & 1 & 0 \\ 3-4a-b & -2-c & 1 \end{bmatrix}$$

and

$$UU^{-1} = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1/3 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & -1/4 \end{bmatrix} = \begin{bmatrix} 1 & 3a'+2 & 3b'+2c'+1/2 \\ 0 & 1 & c'-1/4 \\ 0 & 0 & 1 \end{bmatrix}.$$

This gives the systems of equations

$$5+2a=0, \ , -2-c=0, \ \ 3-4a-b=0$$

and

$$3a'+2=0, \ \ c'-1/4=0, \ \ 3b'+2c'+1/2=0,$$

which we solve in succession. For the first system, we initially solve the first two equations to get $a = -5/2$ and $c = -2$, so the final equation can be solved for $b$ to give $b = 3 - 4a = 3 - (-10) = 13$. For the second system, we initially solve the first two equations to get $a' = -2/3$ and $c' = 1/4$, so the final equation can be solved for $b'$ to give $b' = -(2/3)c' - 1/6 = -1/6 - 1/6 = -1/3$.

Putting it all together, we have

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -5/2 & 1/2 & 0 \\ 13 & -2 & -1 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} 1/3 & -2/3 & -1/3 \\ 0 & 1 & 1/4 \\ 0 & 0 & -1/4 \end{bmatrix}.$$

To check that this work, we directly compute the matrix products

$$LL^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5/2 & 1/2 & 0 \\ 13 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5-5 & 1 & 0 \\ 3+10-13 & -2+2 & 1 \end{bmatrix} = \mathrm{I}_3$$

and

$$UU^{-1} = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & -1/3 \\ 0 & 1 & 1/4 \\ 0 & 0 & -1/4 \end{bmatrix} = \begin{bmatrix} 1 & -2+2 & -1+1/2+1/2 \\ 0 & 1 & 1/4-1/4 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_3.$$

 

(c) We compute the product $U^{-1}L^{-1}$ is equal to

$$\begin{bmatrix} 1/3 & -2/3 & -1/3 \\ 0 & 1 & 1/4 \\ 0 & 0 & -1/4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5/2 & 1/2 & 0 \\ 13 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1/3+5/3-13/3 & -1/3+2/3 & 1/3 \\ -5/2+13/4 & 1/2-1/2 & -1/4 \\ -13/4 & 1/2 & 1/4 \end{bmatrix}$$
$$= \begin{bmatrix} -7/3 & 1/3 & 1/3 \\ 3/4 & 0 & -1/4 \\ -13/4 & 1/2 & 1/4 \end{bmatrix}.$$

To check this is inverse to $A$, we multiply it against $A$ on the right: this is the product

$$\begin{bmatrix} -7/3 & 1/3 & 1/3 \\ 3/4 & 0 & -1/4 \\ -13/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 \\ 15 & 12 & -8 \\ 9 & 2 & -6 \end{bmatrix}$$

that is equal to

$$\begin{bmatrix} -7+5+3 & -14/3+4+2/3 & 14/3-8/3-2 \\ 9/4-9/4 & 3/2-1/2 & -3/2+3/2 \\ -39/4+15/2+9/4 & -13/2+6+1/2 & 13/2-4-3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

◇

3. (a) The matrix product is

$$\begin{bmatrix} 1 & 4/3+2/3 & -5/3-1/3 \\ 1 & 4/3-1/3 & -5/3+1/6-1/2 \\ 1 & 4/3-1/3 & -5/3+1/6+1/2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix}.$$

(b) The matrix $R^{-1}$ has the form

$$R^{-1} = \begin{bmatrix} 1/\sqrt{3} & a & b \\ 0 & \sqrt{3/2} & c \\ 0 & 0 & \sqrt{2} \end{bmatrix},$$

determined by the vanishing of the off-diagonal entries of

$$\begin{aligned} RR^{-1} &= \begin{bmatrix} \sqrt{3} & 4/\sqrt{3} & -5/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & a & b \\ 0 & \sqrt{3/2} & c \\ 0 & 0 & \sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \sqrt{3}\,a+2\sqrt{2} & \sqrt{3}\,b+(4/\sqrt{3})c-5\sqrt{2/3} \\ 0 & 1 & \sqrt{2/3}\,c-1/\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

This amounts to the three equations

$$\sqrt{3}\,a+2\sqrt{2}=0, \quad \sqrt{2/3}\,c-1/\sqrt{3}=0, \quad \sqrt{3}\,b+(4/\sqrt{3})c-5\sqrt{2/3}=0.$$

The first two are solved to give $a=-2\sqrt{2/3}$ and $c=1/\sqrt{2}$, so the final equation says

$$\sqrt{3}\,b+2\sqrt{2/3}-5\sqrt{2/3}=0.$$

This is the same as $\sqrt{3}\,b=3\sqrt{2/3}=\sqrt{6}$, so $b=\sqrt{2}$.
Putting it all together, we obtain

$$R^{-1} = \begin{bmatrix} 1/\sqrt{3} & -2\sqrt{2/3} & \sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

To check that this works, we compute $RR^{-1}$: this is the matrix product

$$\begin{bmatrix} \sqrt{3} & 4/\sqrt{3} & -5/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -2\sqrt{2/3} & \sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

that works out to be

$$\begin{bmatrix} 1 & -2\sqrt{2}+2\sqrt{2} & \sqrt{6}+2\sqrt{2/3}-5\sqrt{2/3} \\ 0 & 1 & 1/\sqrt{3}-1/\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(where the vanishing of the upper right entry is because $2\sqrt{2/3}-5\sqrt{2/3} = -3\sqrt{2/3} = -\sqrt{6}$).

(c) The matrix $R^{-1}Q^{\top}$ is the product

$$\begin{bmatrix} 1/\sqrt{3} & -2\sqrt{2/3} & \sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

which works out to be

$$\begin{bmatrix} 1/3-4/3 & 1/3+2/3-1 & 1/3+2/3+1 \\ 1 & -1/2-1/2 & -1/2+1/2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

To check this is inverse to $A$, we multiply it on the left by $A$ to get the product

$$\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & -2+2 & 2-2 \\ -1+1 & -1+2 & 2-2 \\ -1+1 & -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

◇

4. (a) Since $Q'R' = QR$, if we multiply both sides by $Q^{-1}$ on the left and $R'^{-1}$ on the right then we get $Q^{-1}Q'R'R'^{-1} = Q^{-1}QRR'^{-1}$. The left side has the factor $R'R'^{-1} = \mathrm{I}_n$ and the right side has the factor $Q^{-1}Q = \mathrm{I}_n$, so making those substitutions gives $Q^{-1}Q'\mathrm{I}_n = \mathrm{I}_nRR'^{-1}$, or in other words $Q^{-1}Q' = RR'^{-1}$ as desired.

The inverse of an orthogonal matrix is known to be orthogonal (it is just the transpose, and we know that an orthogonal matrix has row vectors that are also mutually orthogonal unit vectors), and the inverse of an upper triangular matrix with nonzero diagonal entries is upper triangular with diagonal entries given by the reciprocals of the original diagonal entries (so those are positive if the original diagonal entries are positive). Hence, $Q^{-1}$ is orthogonal and $R'^{-1}$ is upper triangular with positive diagonal entries. We know that products of orthogonal matrices are orthogonal, so $Q^{-1}Q'$ is orthogonal. Likewise, $RR'^{-1}$ is a product of upper triangular matrices, so it is also upper triangular due to how one multiplies such matrices (look at the $3 \times 3$ case to see how the pattern goes), and its diagonal entries are the products of the corresponding diagonal entries of $R$ and $R'^{-1}$ (again look at products of $3 \times 3$ upper triangular matrices to see how the diagonal works out in a product of two such matrices). So $RR'^{-1}$ has its diagonal entries that are products of two positive numbers each and hence are positive.

(b) If $Q^{-1}Q' = \mathrm{I}_n$ then multiplying both sides by $Q$ on the left gives $Q(Q^{-1}Q') = Q\mathrm{I}_n = Q$. But $Q(Q^{-1}Q') = (QQ^{-1})Q' = \mathrm{I}_nQ' = Q'$, so we have $Q' = Q$ as desired.

Likewise, if $RR'^{-1} = \mathrm{I}_n$ then multiplying both sides by $R'$ on the right gives $(RR'^{-1})R' = \mathrm{I}_nR' = R'$. But $(RR'^{-1})R' = R(R'^{-1}R') = R\mathrm{I}_n = R$, so we have $R = R'$ as desired.

(c) Once we establish the general assertion about such matrices $M$, let's see how we can conclude by using (a) and (b). By (a), we have the matrix $Q^{-1}Q' = RR'^{-1}$ that is both orthogonal and upper triangular with positive entries. Hence, by the general fact about such matrices $M$ that we're going to establish below, this common matrix must be $\mathrm{I}_n$. Then the hypothesis of (b) holds, so by (b) we get $Q = Q'$ and $R = R'$ as desired.

It remains to show that if an orthogonal $n \times n$ matrix $M$ is also upper triangular with positive diagonal entries then it equals $\mathrm{I}_n$. We will work one step at a time, showing that the $j$th column of $M$ is $\mathbf{e}_j$ for $j = 1, 2, 3, \dots, n$ in turn. Since $M$ is upper triangular with positive diagonal entry, its $j$th column $\mathbf{m}_j$ has vanishing entries below the $j$th position, with positive $j$th entry. In other words:

$$\mathbf{m}_j = \begin{bmatrix} m_{1j} \\ \vdots \\ m_{jj} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = m_{1j}\mathbf{e}_1 + \cdots + m_{jj}\mathbf{e}_j$$

with $m_{jj} > 0$.

For $j = 1$ we have $\mathbf{m}_1 = m_{11}\mathbf{e}_1$ with $m_{11} > 0$. But $\mathbf{m}_1$ is a unit vector by orthogonality of $M$, so $|m_{11}| = 1$, yet $m_{11} > 0$, so $m_{11} = |m_{11}| = 1$. Hence, $\mathbf{m}_1 = 1\mathbf{e}_1 = \mathbf{e}_1$ as desired. Next, for $j = 2$ we have $\mathbf{m}_2 = m_{12}\mathbf{e}_1 + m_{22}\mathbf{e}_2$ with $m_{22} > 0$. But $\mathbf{m}_2$ is orthogonal to $\mathbf{m}_1 = \mathbf{e}_1$ by orthogonality of $M$, so $0 = \mathbf{m}_1 \cdot \mathbf{m}_2 = \mathbf{e}_1 \cdot \mathbf{m}_2 = m_{12}$. Hence, $\mathbf{m}_2 = m_{22}\mathbf{e}_2$. But $\mathbf{m}_2$ is a unit vector (by orthogonality of $M$) and $m_{22} > 0$, so $m_{22} = 1$ just as we showed $m_{11} = 1$; this yields that $\mathbf{m}_2 = 1\mathbf{e}_2 = \mathbf{e}_2$, as desired.

The pattern persists. Suppose for some $1 \le j \le n-1$ that we have established $\mathbf{m}_r = \mathbf{e}_r$ for all $r \le j$; we want to deduce that $\mathbf{m}_{j+1} = \mathbf{e}_{j+1}$ and then we can continue all the way to the final column and finish. By the orthogonality of distinct columns of the orthogonal matrix $M$, we have for $r \le j$ that

$$0 = \mathbf{m}_r \cdot \mathbf{m}_{j+1} = \mathbf{e}_r \cdot \mathbf{m}_{j+1} = m_{r,j+1}.$$

This say that all entries in $\mathbf{m}_{j+1}$ above its diagonal entry vanish, so we are left with $\mathbf{m}_{j+1} = m_{j+1,j+1}\mathbf{e}_{j+1}$ where $m_{j+1,j+1} > 0$. Now we argue as we did for $m_{11}$ and $m_{22}$: this $m_{j+1,j+1}\mathbf{e}_{j+1} = \mathbf{m}_{j+1}$ is a unit vector, we have $|m_{j+1,j+1}| = 1$. But $m_{j+1,j+1} > 0$, so $m_{j+1,j+1} = |m_{j+1,j+1}| = 1$ and hence $\mathbf{m}_{j+1} = 1\mathbf{e}_{j+1} = \mathbf{e}_{j+1}$ as desired. The pattern carries us through all of the columns, so $M$ and $\mathrm{I}_n$ agree in every column and hence are equal as matrices.

◇

5. (a) By looking at how the columns are multiples of each other and the rows are multiples of each other, we can pick $\mathbf{c}_i$ to be the $i$th column of $B_i$ and then choose the entries of $\mathbf{r}_i$ to make the multiplication work out correctly:

$$A_1 = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} = \mathbf{c}_1\mathbf{r}_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 3/2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} = \mathbf{c}_2\mathbf{r}_2 = \begin{bmatrix} 0 \\ -5 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

(b) We multiply $LU = \begin{bmatrix} 2 & 0 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix} = A$.

(c) By looking at how the columns are multiples of each other and the rows are multiples of each other, we can pick $\mathbf{c}_i$ to be a nonzero multiple of the $i$th column of $B_i$ and then choose the entries of $\mathbf{r}_i$ to make the multiplication work out correctly:

$$B_1 = \begin{bmatrix} 2 & 5 & 3 \\ -4 & -10 & -6 \\ 6 & 15 & 9 \end{bmatrix} = \mathbf{c}_1\mathbf{r}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 3 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 7 & 14 \\ 0 & -14 & -28 \end{bmatrix} = \mathbf{c}_2\mathbf{r}_2 = \begin{bmatrix} 0 \\ 7 \\ -14 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 28 \end{bmatrix} = \mathbf{c}_3\mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 28 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

(We chose $\mathbf{c}_1$ to be as above rather than to be the first column of $B_1$ solely to make $\mathbf{r}_1$ have integer entries; it wasn't necessary to do this, but that makes the computations come out more cleanly.)

Finally, we can directly multiply

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 7 & 0 \\ 3 & -14 & 28 \end{bmatrix} \begin{bmatrix} 2 & 5 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 3 \\ -4 & -3 & 8 \\ 6 & 1 & 9 \end{bmatrix} = B.$$

◇

6. For both parts, the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 1$, so in each case we will compute a basis of $N(A - 4\mathrm{I}_3)$ and $N(A - \mathrm{I}_3)$, and verify directly that the basis vectors we compute are indeed eigenvectors with the desired eigenvalue.

(a) We have

$$A - 4\mathrm{I}_3 = \begin{bmatrix} 0 & 3 & -6 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix},$$

 

so a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ belongs to $N(A - 4\mathrm{I}_3)$ precisely when

$$3x_2 - 6x_3 = 0, \;\; -3x_2 + 6x_3 = 0, \;\; 0 = 0.$$

The first two equations each say $x_2 = 2x_3$ and the last equation contributes nothing, so there is no condition on $x_1$ and we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

with arbitrary $x_1, x_3 \in \mathbf{R}$. Hence, a basis of $N(A - 4\mathrm{I}_3)$ is given by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e}_1$ and $\mathbf{v}' = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ (which are clearly linearly independent: these two vectors are nonzero and not multiples of each other); of course, other bases are possible.

To check that $\mathbf{v}$ and $\mathbf{v}'$ are indeed eigenvectors for $A$ with eigenvalue 4, we compute

$$A\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = 4\mathbf{v}, \;\; A\mathbf{v}' = \begin{bmatrix} 6-6 \\ 2+6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} = 4\mathbf{v}'$$

as desired.

Next, for the eigenvalue 1 we compute

$$A - \mathrm{I}_3 = \begin{bmatrix} 3 & 3 & -6 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix},$$

whose null space consists of vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying the simultaneous equations

$$3x_1 + 3x_2 - 6x_3 = 0, \;\; 6x_3 = 0, \;\; 3x_3 = 0.$$

The last two equations says $x_3 = 0$, and plugging this into the first turns that into the condition $3x_1 + 3x_2 = 0$, or equivalently $x_2 = -x_1$. Hence,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

so $N(A - \mathrm{I}_3)$ is the line spanned by $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (or any nonzero scalar multiple of this). To check that $\mathbf{w}$ really is an eigenvector with eigenvalue 1, we compute

$$A\mathbf{w} = \begin{bmatrix} 4-3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \mathbf{w}$$

as desired.

(b) We have

$$A - 4\mathrm{I}_3 = \begin{bmatrix} 0 & 3 & a \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix},$$

so a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ belongs to $N(A - 4\mathrm{I}_3)$ precisely when

$$3x_2 + ax_3 = 0, \;\; -3x_2 + 6x_3 = 0, \;\; 0 = 0.$$

 

The first equation says $x_2 = -(a/3)x_3$ and the second says $x_2 = 2x_3$. But $a \neq -6$, s0 $-(a/3) \neq 2$. Hence, the combined equation $2x_3 = x_3 = -(a/3)x_3$ has two *different* scalars 2 and $-(a/3)$ each multiplying $x_3$ to give the same output. This is impossible if $x_3 \neq 0$ (as otherwise we could cancel $x_3$ to get $2 = -(a/3)$, forcing $a = -6$, but we are assuming $a \neq -6$), so $x_3 = 0$. Then also $x_3 = 0$. The third equation "$0 = 0$" tells us nothing, so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x_1 \mathbf{e}_1.$$

This says that the 4-eigenspace is the line spanned by $\mathbf{e}_1$ (or any nonzero scalar multiple of this). To check directly that $\mathbf{e}_1$ is an eigenvector of $A$ with eigenvalue 4 is the same (short!) calculation we did in the solution to part (a).

Next, for the eigenvalue 1 we compute

$$A - \mathrm{I}_3 = \begin{bmatrix} 3 & 3 & a \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix},$$

whose null space consists of vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying the simultaneous equations

$$3x_1 + 3x_2 + ax_3 = 0, \;\; 6x_3 = 0, \;\; 3x_3 = 0.$$

The last two equations says $x_3 = 0$, and plugging this into the first turns that into the condition $3x_1 + 3x_2 = 0$ (the contribution of $a$ has disappeared!), or equivalently $x_2 = -x_1$. Hence,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

so $N(A - \mathrm{I}_3)$ is the line spanned by $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (or any nonzero scalar multiple of this). To check that $\mathbf{w}$ really is an eigenvector with eigenvalue 1, we compute

$$A\mathbf{w} = \begin{bmatrix} 4-3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \mathbf{w}$$

as desired.

$\diamond$

7. (a) The eigenvalues of $A$ are given as solutions to the equation

$$\lambda^2 - (a+d)\lambda + ad - cb = 0.$$

We also have

$$A^\top = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

So the eigenvalues of $A^\top$ are solutions to the equation

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

Both equations are the same, hence $A$ and $A^\top$ have the same eigenvalues.

(b) We have $\mathrm{tr}(A') + \mathrm{tr}(A) = (a' + d') + (a + d) = a' + d' + a + d$ and

$$A' + A = \begin{bmatrix} a' + a & b' + b \\ c' + c & d' + d \end{bmatrix},$$

so $\mathrm{tr}(A' + A) = (a' + a) + (d' + d) = a' + a + d' + d = a' + d' + a + d$. The product of the determinants is

$$\det(A')\det(A) = (a'd' - b'c')(ad - bc) = a'd'ad - a'd'bc - b'c'ad + b'c'bc$$

whereas the product matrix

$$A'A = \begin{bmatrix} a'a+b'c & a'b+b'd \\ c'a+d'c & c'b+d'd \end{bmatrix}$$

has determinant $(a'a+b'c)(c'b+d'd)-(a'b+b'd)(c'a+d'c)$ that expands out to be

$$(a'ac'b+a'ad'd+b'cc'b+b'cd'd)-(a'bc'a+a'bd'c+b'dc'a+b'dd'c),$$

and distributing the subtraction turns this into

$$a'c'ab+a'd'ad+b'c'cb+b'd'cd-a'c'ba-a'd'bc-b'c'da-b'd'dc.$$

The terms $a'c'ab$ and $-a'c'ba$ cancel out, as do the terms $b'd'cd$ and $-b'd'dc$, leaving us with

$$a'd'ad+b'c'cb-a'd'bc-b'c'da.$$

Rearranging some factors in the products and rearranging the terms being added converts this into the exactly the same expression $a'd'ad-a'd'bc-b'c'ad+b'c'bc$ that was obtained above for the product of the determinants.

$\diamond$

8. (a) From the definition, if we swap the roles of $\mathbf{v}$ and $\mathbf{w}$ then the effect is to swap the order of subtraction in each vector entry of the cross product. Hence, $\mathbf{w}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{w})$. Likewise, in the definition of $\mathbf{v}\times\mathbf{v}$ each vector entry is a difference of the form $v_iv_j-v_jv_i=0$ with $i\neq j$, so it is the zero vector in $\mathbf{R}^3$.

   (b) For the calculations,

$$\begin{aligned}
\mathbf{v}\times\mathbf{w} &= \det\begin{bmatrix}-1 & 3\\ 2 & 3\end{bmatrix}\mathbf{e}_1-\det\begin{bmatrix}2 & 3\\ 1 & 3\end{bmatrix}\mathbf{e}_2+\det\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\mathbf{e}_3\\
&= (-3-6)\mathbf{e}_1-(6-3)\mathbf{e}_2+(4-(-1))\mathbf{e}_3\\
&= -9\mathbf{e}_1-3\mathbf{e}_2+5\mathbf{e}_3\\
&= \begin{bmatrix}-9\\-3\\5\end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathbf{w}\times\mathbf{u} &= \det\begin{bmatrix}2 & 3\\ 3 & -2\end{bmatrix}\mathbf{e}_1-\det\begin{bmatrix}1 & 3\\ 4 & -2\end{bmatrix}\mathbf{e}_2+\det\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}\mathbf{e}_3\\
&= (-4-9)\mathbf{e}_1-(-2-12)\mathbf{e}_2+(3-8)\mathbf{e}_3\\
&= -13\mathbf{e}_1+14\mathbf{e}_2-5\mathbf{e}_3\\
&= \begin{bmatrix}-13\\14\\-5\end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
(\mathbf{v}\times\mathbf{w})\times\mathbf{u} &= \det\begin{bmatrix}-3 & 5\\ 3 & -2\end{bmatrix}\mathbf{e}_1-\det\begin{bmatrix}-9 & 5\\ 4 & -2\end{bmatrix}\mathbf{e}_2+\det\begin{bmatrix}-9 & -3\\ 4 & 3\end{bmatrix}\mathbf{e}_3\\
&= (6-15)\mathbf{e}_1-(18-20)\mathbf{e}_2+(-27-(-12))\mathbf{e}_3\\
&= -9\mathbf{e}_1+2\mathbf{e}_2-15\mathbf{e}_3\\
&= \begin{bmatrix}-9\\2\\-15\end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}\times(\mathbf{w}\times\mathbf{u}) &= \det\begin{bmatrix}-1 & 3\\ 14 & -5\end{bmatrix}\mathbf{e}_1-\det\begin{bmatrix}2 & 3\\ -13 & -5\end{bmatrix}\mathbf{e}_2+\det\begin{bmatrix}2 & -1\\ -13 & 14\end{bmatrix}\mathbf{e}_3\\
&= (5-42)\mathbf{e}_1-(-10-(-39))\mathbf{e}_2+(28-13)\mathbf{e}_3\\
&= -37\mathbf{e}_1-29\mathbf{e}_2+15\mathbf{e}_3\\
&= \begin{bmatrix}-37\\-29\\15\end{bmatrix}.
\end{aligned}$$

 

(c) From the definition,

$$(c\mathbf{v}) \times \mathbf{w} = \begin{bmatrix} (cv_2)w_3 - (cv_3)w_2 \\ (cv_3)w_1 - (cv_1)w_3 \\ (cv_1)w_2 - (cv_2)w_1 \end{bmatrix} = \begin{bmatrix} c(v_2w_3 - v_3w_2) \\ c(v_3w_1 - v_1w_3) \\ c(v_1w_2 - v_2w_1) \end{bmatrix} = c \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} = c(\mathbf{v} \times \mathbf{w}).$$

Finally,

$$\begin{aligned} (\mathbf{v} + \mathbf{v}') \times \mathbf{w} &= \begin{bmatrix} (v_2 + v_2')w_3 - (v_3 + v_3')w_2 \\ (v_3 + v_3')w_1 - (v_1 + v_1')w_3 \\ (v_1 + v_1')w_2 - (v_2 + v_2')w_1 \end{bmatrix} \\ &= \begin{bmatrix} v_2w_3 + v_2'w_3 - v_3w_2 - v_3'w_2 \\ v_3w_1 + v_3'w_1 - v_1w_3 - v_1'w_3 \\ v_1w_2 + v_1'w_2 - v_2w_1 - v_2'w_1 \end{bmatrix} \\ &= \begin{bmatrix} (v_2w_3 - v_3w_2) + (v_2'w_3 - v_3'w_2) \\ (v_3w_1 - v_1w_3) + (v_3'w_1 - v_1'w_3) \\ (v_1w_2 - v_2w_1) + (v_1'w_2 - v_2'w_1) \end{bmatrix} \\ &= \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} + \begin{bmatrix} v_2'w_3 - v_3'w_2 \\ v_3'w_1 - v_1'w_3 \\ v_1'w_2 - v_2'w_1 \end{bmatrix} \\ &= \mathbf{v} \times \mathbf{w} + \mathbf{v}' \times \mathbf{w}. \end{aligned}$$

(d) The length of $\mathbf{v} \times \mathbf{w} = \begin{bmatrix} -9 \\ -3 \\ 5 \end{bmatrix}$ is $\sqrt{81 + 9 + 25} = \sqrt{115}$ and

$$\|\mathbf{v}\| \, \|\mathbf{w}\| \sin(\theta) = \sqrt{14}\sqrt{14}\sin(\theta) = 14\sin(\theta).$$

To compute $\sin(\theta)$, the dot product formula "$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos(\theta)$" says

$$9 = \sqrt{14}\sqrt{14}\cos(\theta),$$

so $\cos(\theta) = 9/14$. Hence, $\sin(\theta) = \sqrt{1 - (\cos\theta)^2} = \sqrt{1 - 81/196} = \sqrt{115/196} = \sqrt{115}/14$. Hence, finally, $\|\mathbf{v}\| \, \|\mathbf{w}\| \sin(\theta) = 14\sin(\theta) = 14(\sqrt{115}/14) = \sqrt{115}$, which matches the calculation of $\|\mathbf{v} \times \mathbf{w}\|$.

◇

9. (a) Using our rules of matrix algebra, $(M^\top M)^\top = M^\top (M^\top)^\top = M^\top M$, so $M^\top M$ is indeed symmetric.

(b) We can write $q(\mathbf{x}) = \mathbf{x}^\top (M^\top M\mathbf{x})$ as a matrix product. Using the associativity of matrix multiplication, we see that this is equal to $(\mathbf{x}^\top M^\top)(M\mathbf{x}) = (M\mathbf{x}) \cdot (M\mathbf{x}) = \|M\mathbf{x}\|^2$. Since the squared-length of a vector (e.g. $M\mathbf{x}$) is always non-negative, we have $q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbf{R}^n$. That is, $q$ is positive-semidefinite.

(c) Since we can write $q(\mathbf{x}) = \|M\mathbf{x}\|^2$, we know that $q(\mathbf{x}) = 0$ when the length of the vector $M\mathbf{x}$ is zero. But a vector's length is zero precisely when it is the zero vector. That is, $q(\mathbf{x}) = 0$ precisely when $\mathbf{x} \in N(M)$. The condition that $q$ be positive-definite (given that it is already positive-semidefinite) is the same as the condition that $q(\mathbf{x}) = 0$ only when $\mathbf{x} = 0$. Thus, this is equivalent to the condition that $N(M) = \{\mathbf{0}\}$.

◇

10. (a) By part (b) of Exercise 24.5 , $q_B$ is positive-semidefinite. Because $A$ has more columns than rows, $N(A) \neq \{\mathbf{0}\}$ (this expresses that $A\mathbf{x} = \mathbf{0}$ is an underdetermined linear system). Therefore, by part (c) of Exercise 24.5 , $q_B$ is not positive-definite.

(b) We compute $C = \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}$. The characteristic polynomial of $C$ is $\lambda^2 - 16\lambda + 35$. We can check that $8 + \sqrt{29}$ is a root of this polynomial: either use the quadratic formula or compute

$$(8 + \sqrt{29})^2 - 16(8 + \sqrt{29}) + 35 = 64 + 16\sqrt{29} + 29 - 128 - 16\sqrt{29} + 35 = 128 - 128 + 16\sqrt{29} - 16\sqrt{29} = 0.$$

◇

 

11. (a) We compute

$$A\mathbf{v}_1 = A\begin{bmatrix}1\\-2\\2\end{bmatrix} = \begin{bmatrix}-5+28+4\\-14-8-32\\2+32+20\end{bmatrix} = \begin{bmatrix}27\\-54\\54\end{bmatrix} = 27\begin{bmatrix}1\\-2\\2\end{bmatrix} = 27\mathbf{v}_1,$$

$$A\mathbf{v}_2 = A\begin{bmatrix}2\\2\\1\end{bmatrix} = \begin{bmatrix}-10-28+2\\-28+8-16\\4-32+10\end{bmatrix} = \begin{bmatrix}-36\\-36\\-18\end{bmatrix} = -18\begin{bmatrix}2\\2\\1\end{bmatrix} = -18\mathbf{v}_2,$$

and

$$A\mathbf{v}_3 = A\begin{bmatrix}-2\\1\\2\end{bmatrix} = \begin{bmatrix}10-14+4\\28+4-32\\-4-16+20\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = 0\begin{bmatrix}-2\\1\\2\end{bmatrix} = 0\mathbf{v}_3.$$

Thus, the respective eigenvalues are $\lambda_1 = 27$, $\lambda_2 = -18$, $\lambda_3 = 0$.

(b) Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis of $\mathbf{R}^3$, for any $\mathbf{x} \in \mathbf{R}^3$ we have

$$\mathbf{x} = \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{x}) + \mathbf{Proj}_{\mathbf{v}_2}(\mathbf{x}) + \mathbf{Proj}_{\mathbf{v}_3}(\mathbf{x})$$

with

$$\mathbf{Proj}_{\mathbf{v}_i}(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}\right)\mathbf{v}_i.$$

Conveniently, in this case all $\mathbf{v}_i \cdot \mathbf{v}_i$'s are equal to 9, and of course $\mathbf{e}_j \cdot \mathbf{v}_i$ is the $j$th entry of $\mathbf{v}_i$, so the coefficients of $\mathbf{v}_i$ for each $\mathbf{Proj}_{\mathbf{v}_i}(\mathbf{e}_j)$ is read off from staring at the entries for $\mathbf{v}_i$. This yields:

$$\mathbf{e}_1 = \frac{1}{9}\mathbf{v}_1 + \frac{2}{9}\mathbf{v}_2 + \frac{-2}{9}\mathbf{v}_3,$$

$$\mathbf{e}_2 = \frac{-2}{9}\mathbf{v}_1 + \frac{2}{9}\mathbf{v}_2 + \frac{1}{9}\mathbf{v}_3,$$

$$\mathbf{e}_3 = \frac{2}{9}\mathbf{v}_1 + \frac{1}{9}\mathbf{v}_2 + \frac{2}{9}\mathbf{v}_3.$$

(c) We need to compute $A^{10}\mathbf{e}_i$ for each $i = 1, 2, 3$. Using the answer to (b) and the relations $A^{10}\mathbf{v}_i = \lambda_i^{10}\mathbf{v}_i$ with the $\lambda_i$'s all known, we have

$$A^{10}\mathbf{e}_1 = \frac{27^{10}}{9}\mathbf{v}_1 + \frac{2\cdot(-18)^{10}}{9}\mathbf{v}_2 + 0\mathbf{v}_3 = 3^{28}\begin{bmatrix}1\\-2\\2\end{bmatrix} + 2^{11}3^{18}\begin{bmatrix}2\\2\\1\end{bmatrix},$$

$$A^{10}\mathbf{e}_2 = \frac{-2\cdot 27^{10}}{9}\mathbf{v}_1 + \frac{2\cdot(-18)^{10}}{9}\mathbf{v}_2 + 0\mathbf{v}_3 = -2\cdot 3^{28}\begin{bmatrix}1\\-2\\2\end{bmatrix} + 2^{11}3^{18}\begin{bmatrix}2\\2\\1\end{bmatrix},$$

and

$$A^{10}\mathbf{e}_3 = \frac{2\cdot 27^{10}}{9}\mathbf{v}_1 + \frac{(-18)^{10}}{9}\mathbf{v}_2 + 0\mathbf{v}_3 = 2\cdot 3^{28}\begin{bmatrix}1\\-2\\2\end{bmatrix} + 2^{10}3^{18}\begin{bmatrix}2\\2\\1\end{bmatrix}.$$

Putting everything together, we have

$$A^{10} = \begin{bmatrix}3^{28}+2^{12}3^{18} & -2\cdot 3^{28}+2^{12}3^{18} & 2\cdot 3^{28}+2^{11}3^{28}\\ -2\cdot 3^{28}+2^{12}3^{18} & 2^2 3^{28}+2^{12}3^{18} & -2^2 3^{28}+2^{11}3^{18}\\ 2\cdot 3^{28}+2^{11}3^{18} & -2^2 3^{28}+2^{11}3^{18} & 2^2 3^{28}+2^{10}3^{18}\end{bmatrix}.$$

As expected, this is symmetric (as we know it had to be, due to symmetry of $A$ and the behavior of transpose on matrix products).

(d) For the second approach, we will work out the formula $A^{10} = QD^{10}Q^\top$. This says

$$\begin{aligned} A^{10} &= \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 27^{10} & 0 & 0 \\ 0 & (-18)^{10} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{bmatrix} \\ &= \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3^{30} & 0 & 0 \\ 0 & 2^{10}3^{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3^{28} & 0 & 0 \\ 0 & 2^{10}3^{18} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3^{28} & -2\cdot 3^{28} & 2\cdot 3^{28} \\ 2^{11}3^{18} & 2^{11}3^{18} & 2^{10}3^{18} \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3^{28} + 2^{12}3^{18} & -2\cdot 3^{28} + 2^{12}3^{18} & 2\cdot 3^{28} + 2^{11}3^{28} \\ -2\cdot 3^{28} + 2^{12}3^{18} & 2^2 3^{28} + 2^{12}3^{18} & -2^2 3^{28} + 2^{11}3^{18} \\ 2\cdot 3^{28} + 2^{11}3^{18} & -2^2 3^{28} + 2^{11}3^{18} & 2^2 3^{28} + 2^{10}3^{18} \end{bmatrix}. \end{aligned}$$

Indeed this agrees with the answer to (c).

◇

12. Since $-3$ is a "dominant eigenvalue," we can use the formula from Proposition 24.4.2 :

$$\begin{aligned} A^{10} &\approx \frac{(-3)^{10}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}\mathbf{w}^\top \\ &= \frac{3^{10}}{9+16+4+1}\begin{bmatrix} -3 \\ 4 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} -3 & 4 & 2 & 1 \end{bmatrix} \\ &= \frac{3^{10}}{30}\begin{bmatrix} 9 & -12 & -6 & -3 \\ -12 & 16 & 8 & 4 \\ -6 & 8 & 4 & 2 \\ -3 & 4 & 2 & 1 \end{bmatrix} \\ &= \frac{3^{9}}{10}\begin{bmatrix} 9 & -12 & -6 & -3 \\ -12 & 16 & 8 & 4 \\ -6 & 8 & 4 & 2 \\ -3 & 4 & 2 & 1 \end{bmatrix} \end{aligned}$$

◇

13. (a) The vectors in $C(M^\top)$ are those of the form $M^\top\mathbf{x}$. We have $\mathbf{v}\cdot M^\top\mathbf{x} = (M\mathbf{v})\cdot\mathbf{x} = \mathbf{0}\cdot\mathbf{x} = 0$.

(b) For linear independence, we want to show that if $\sum_{j=1}^n c_j B\mathbf{e}_j$ then the $c_j$'s all vanish. This sum is equal to $B(\sum_{j=1}^n c_j\mathbf{e}_j)$, so $\sum_{j=1}^n c_j\mathbf{e}_j \in N(B) = \{\mathbf{0}\}$. This says

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{0},$$

so all $c_j$'s vanish, as desired.

(c) Suppose $N(M)$ is nonzero, so $C(M^\top)$ is a proper subspace of $\mathbf{R}^n$ by (a). But (b) gives that if $N(M^\top) = \{\mathbf{0}\}$ then $C(M^\top) = \mathbf{R}^n$. This conclusion is already ruled out, so $N(M^\top)$ can't vanish and so must be nonzero.

(d) By the rules for transpose, $(A - \lambda\mathrm{I}_n)^\top = A^\top - \lambda\mathrm{I}_n^\top = A^\top - \lambda\mathrm{I}_n$. Since $\lambda$ is an eigenvalue for $A$ exactly when $A - \lambda\mathrm{I}_n$ has nonzero null space, applying (c) to $M = A - \lambda\mathrm{I}_n$ we see from our computation of $M^\top$ that if $\lambda$ is an eigenvalue for $A$ then it is an eigenvalue for $A^\top$. The argument works in the reverse direction too because $(A^\top)^\top = A$.

For an $n \times n$ matrix $B$ with all column sums equal to 1, to show 1 is an eigenvalue it is the same as for 1 to be an eigenvalue of the matrix $B^\top$ whose *row sums* are all equal to 1. But the row sum condition says exactly that

$$B^\top \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

so we have exhibited an eigenvector for $B^\top$ with eigenvalue 1.

◇