Solutions to Math 51 Quiz 2 Practice A

1. (10 points) Consider the following 3-vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}.$$

Find the triple of scalars (a, b, c) so that the vector $\mathbf{x} = \begin{bmatrix} 8 \\ -7 \\ 16 \end{bmatrix}$ satisfies

$$\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3.$$

Compute the product abc.

We first check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbf{R}^3$.

Since $\dim \mathbf{R}^3$, so its only 3-dimensional subspace is itself, and any orthogonal collection of nonzero vectors has dimension equal to the number of vectors in the collection, it suffices to check that the collection of $\mathbf{v}_i$'s is orthogonal. This amounts to computing dot products: $\mathbf{v}_1 \cdot \mathbf{v}_2 = -3 + 2 + 1 = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = -(-1) + 2(-4) + 7 = 1 - 8 + 7 = 0$, and $\mathbf{v}_1 \cdot \mathbf{v}_3 = 3(-1) - 4 + 7 = -3 - 4 + 7 = 0$.

Since we're expressing things in terms of an orthogonal basis, the coefficients c_i are given by the formula

$$a = \frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}, \quad b = \frac{\mathbf{x} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}, \quad c = \frac{\mathbf{x} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}.$$

So we directly compute the dot products: first we compute the denominators

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 11, \quad \mathbf{v}_2 \cdot \mathbf{v}_2 = 6, \quad \mathbf{v}_3 \cdot \mathbf{v}_3 = 66,$$

so

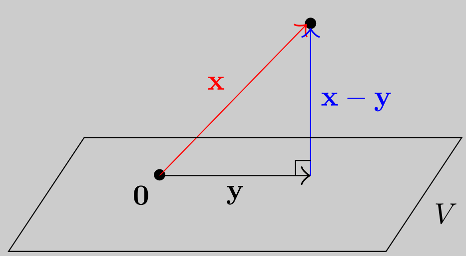
$$\mathbf{x} = \frac{24 - 7 + 16}{11}\mathbf{v}_1 + \frac{-8 - 14 + 16}{6}\mathbf{v}_2 + \frac{-8 + 28 + 112}{66}\mathbf{v}_3 = \frac{33}{11}\mathbf{v}_1 + \frac{-6}{6}\mathbf{v}_2 + \frac{132}{66}\mathbf{v}_3 = 3\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3.$$

Hence, $(a, b, c) = (3, -1, 2)$, so $abc = -6$.

2. (2 points) **True or False:** Let V be a linear subspace of $\mathbf{R}^n$ and $\mathbf{x}$ a vector in $\mathbf{R}^n$. If $\mathbf{y} = \mathrm{Proj}_V(\mathbf{x})$, then it's always the case that

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2.$$

True. Since $\mathbf{y} = \mathrm{Proj}_V(\mathbf{x})$, $\mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are orthogonal to each other. The result follows from the Pythagorean Theorem.



More explicitly, we have

$$\begin{aligned}\|\mathbf{x}\|^2 &= \|\mathbf{x}-\mathbf{y}+\mathbf{y}\|^2 \\ &= ((\mathbf{x}-\mathbf{y})+\mathbf{y})\cdot((\mathbf{x}-\mathbf{y})+\mathbf{y}) \\ &= (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) + 2(\mathbf{x}-\mathbf{y})\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y} \\ &= \|\mathbf{x}-\mathbf{y}\|^2 + 2(\mathbf{x}-\mathbf{y})\cdot\mathbf{y} + \|\mathbf{y}\|^2 .\end{aligned}$$

The vectors $\mathbf{y} = \mathrm{Proj}_V(\mathbf{x})$ and $\mathbf{x}-\mathbf{y} = \mathbf{x} - \mathrm{Proj}_V(\mathbf{x})$ are orthogonal to each other because $\mathrm{Proj}_V(\mathbf{x})$ is in V and $\mathbf{x} - \mathrm{Proj}_V(\mathbf{x})$ is orthogonal to V. Therefore,

$$2(\mathbf{x}-\mathbf{y})\cdot\mathbf{y} = 0,$$

which means

$$\|\mathbf{x}\|^2 = \|\mathbf{x}-\mathbf{y}\|^2 + \|\mathbf{y}\|^2 .$$

3. (2 points) **True or False:** Let $V = \mathrm{span}(\mathbf{v},\mathbf{w})$, where $\mathbf{v} = \begin{bmatrix}1\\2\\0\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}2\\0\\1\end{bmatrix}$. The projection of $\mathbf{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ onto V is given by the formula

$$\mathbf{Proj}_V(\mathbf{u}) = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \frac{\mathbf{u}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}.$$

Not true. Fourier's Formula $\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \frac{\mathbf{u}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$ cannot be applied here since $\mathbf{v}$ and $\mathbf{w}$ are not orthogonal.

4. (3 points) For each of the following linear subspaces V, determine its dimension.

(A) $V = \mathrm{Span}\left(\begin{bmatrix}2\\2\\-4\end{bmatrix}, \begin{bmatrix}3\\3\\-6\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right)$

(B) $V = \mathrm{Span}\left(\begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\4\end{bmatrix}\right)$

(C) $V = \mathrm{Span}\left(\begin{bmatrix}9\\-2\\1\\-1\end{bmatrix}, \begin{bmatrix}3\\-1\\2\\1\end{bmatrix}, \begin{bmatrix}-3\\-1\\8\\7\end{bmatrix}\right)$

(A) $\mathbf{v}_1 = \begin{bmatrix}2\\2\\-4\end{bmatrix}$, $\mathbf{v}_2 = \frac{2}{3}\begin{bmatrix}3\\3\\-6\end{bmatrix}$, $\mathbf{v}_1 = \frac{2}{3}\mathbf{v}_2$, so $\mathrm{span}(\mathbf{v}_1,\mathbf{v}_2)$ is 1-dimensional. Since $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is in $\mathrm{span}(\mathbf{v}_1,\mathbf{v}_2)$, V must be 1-dimensional.

(B) Since $\mathbf{v}_1 = \begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$ are nonzero and (by inspection) not scalar multiples of each other, their span is 2-dimensional.
Since $\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2$,

$$\mathrm{span}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) = \mathrm{span}(\mathbf{v}_1,\mathbf{v}_2)$$

is 2-dimensional.

(C) Since $\mathbf{v}_1 = \begin{bmatrix} 9 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ are nonzero and (by inspection) not scalar multiples of each other, their span is 2-dimensional. We just have to check if $\mathbf{v}_3 = \begin{bmatrix} -3 \\ -1 \\ 8 \\ 7 \end{bmatrix}$ belong to span$(\mathbf{v}_1, \mathbf{v}_2)$.

That is, we seek scalars a and b so that $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$. Plugging in the actual 4-vectors, this equality says

$$\begin{bmatrix} -3 \\ -1 \\ 8 \\ 7 \end{bmatrix} = a \begin{bmatrix} 9 \\ -2 \\ 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9a+3b \\ -2a-b \\ a+2b \\ -a+b \end{bmatrix},$$

which is the system of 4 simultaneous equations

$$9a + 3b = -3, \quad -2a - b = -1, \quad a + 2b = 8, \quad -a + b = 7$$

in a and b.

We'll solve the second and fourth equations simultaneously (we try this since its coefficients are "smallest") and then check the solution works for the other two equations. Adding the second and fourth gives $-3a = 6$, so $a = -2$, and then from the fourth equation we get $b = 7+a = 5$. The pair $(a, b) = (-2, 5)$ also satisfies the first and third equations $(9(-2) + 3(5) = -18 + 15 = -3$, $-2 + 2(5) = -2 + 10 = 8)$. Thus, $\mathbf{v}_3 = -2\mathbf{v}_1 + 5\mathbf{v}_2$. So span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ = span$(\mathbf{v}_1, \mathbf{v}_2)$ is 2-dimensional.

5. (3 points) Let $\mathbf{u}$ be a fixed nonzero vector in $\mathbf{R}^3$, $V = \{\mathbf{x} \in \mathbf{R}^3 : \mathbf{x} \cdot \mathbf{u} = 0\}$, and $\mathbf{v}$ a fixed nonzero vector in V. Geometrically, the collection of vectors $\mathbf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbf{R}^3$ satisfying the condition

$$\mathbf{Proj}_V(\mathbf{w}) = \mathbf{v}$$

is

a) line.

b) plane.

c) point.

d) $\mathbf{R}^3$.

e) a linear subspace.

f) might take different shapes depending on what $\mathbf{u}$ and $\mathbf{v}$ are.

V is plane through the origin orthogonal to $\mathbf{u}$, $\mathbf{v}$ is a fixed nonzero vector in V. Suppose V has an orthogonal basis $\{\mathbf{v}, \mathbf{v}'\}$,

$$\mathbf{Proj}_V(\mathbf{w}) = \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} + \frac{\mathbf{w} \cdot \mathbf{v}'}{\mathbf{v}' \cdot \mathbf{v}'}\mathbf{v}'.$$

For $\mathbf{Proj}_V(\mathbf{w}) = \mathbf{v}$, we must have

$$\frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = 1, \quad \mathbf{w} \cdot \mathbf{v}' = 0. \tag{1}$$

$\{\mathbf{v}, \mathbf{v}', \mathbf{u}\}$ is an orthogonal basis for $\mathbf{R}^3$ since $\mathbf{u}$ is orthogonal to every vector in V. Suppose $\mathbf{w} = a\mathbf{v} + b\mathbf{v}' + c\mathbf{u}$, then

$$\mathbf{w} \cdot \mathbf{v} = a\mathbf{v} \cdot \mathbf{v}, \quad \mathbf{w} \cdot \mathbf{v}' = b\mathbf{v}' \cdot \mathbf{v}', \quad \mathbf{w} \cdot \mathbf{u} = c\mathbf{u} \cdot \mathbf{u}.$$

It follows from Equation (1) that

$$a = \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = 1, \quad b = \frac{\mathbf{w} \cdot \mathbf{v}'}{\mathbf{v}' \cdot \mathbf{v}'} = 0.$$

So $\mathbf{w} = \mathbf{v} + c\mathbf{u}$ for any $c \in \mathbf{R}$. This is a line in $\mathbf{R}^3$ through $\mathbf{v}$ in the direction of $\mathbf{u}$, it is not a linear subspace since $\mathbf{0}$ is not in the line.