Problem 1: Determining the nature of a span

For each collection of 3-vectors, determine whether its span is a point, a line, a plane, or all of $\mathbf{R}^3$. Give a basis of the span in each case. (Keep in mind that if a vector in the collection is a linear combination of others then it can be dropped without affecting the span.)

(a) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}$ (b) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}$ (c) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}$

Solution:

(a) Neither of those vectors is a scalar multiple of the other one, so the span is a plane and the two given vectors are a basis for it.

(b) We see $\begin{bmatrix}-1\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}2\\1\\2\end{bmatrix}$, so the span of all three given vectors coincides with the span of the first two. Neither of those two is a scalar multiple of the other, so their span is a plane and those first two vectors are a basis of this plane (as are any two of the three vectors, if you'd like to think through why that should hold).

(c) None of these three nonzero vectors can be written as a linear combination of the other two, as is checked by setting up a hypothetical such expression with unknown coefficients x and y and checking that the resulting 3 equations on x and y (one per vector entry) has no solution. Since none of these given vectors is a scalar multiple of the other, if any one can be expressed as a linear combination of the others then each can be so expressed. So to rule it out, we just need to rule out one of them having such an expression.

For example, if we try to express the third vector as a linear combination of the other two then we are seeking scalars x and y satisfying

$$\begin{bmatrix}-1\\1\\0\end{bmatrix} = x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}x+2y\\2x+y\\3x+2y\end{bmatrix},$$

which amounts to *three simultaneous equations* (one per vector entry) in these two unknowns:

$$x + 2y = -1, \;\; 2x + y = 1, \;\; 3x + 2y = 0.$$

The first two equations together are a situation from high school algebra, which one works out to have exactly one solution (corresponding to two non-parallel lines crossing at a point): $(x, y) = (1, -1)$. But this violates the third equation, as $3(1) + 2(-1) = 1 \neq 0$. So there is no solution, and hence the third vector in the given collection cannot be expressed as a linear combination of the other two.

So the span is all of $\mathbf{R}^3$, and hence it cannot be spanned by less than three nonzero vectors (a span of less than three would be a plane or line through the origin, so it would not coincide with $\mathbf{R}^3$). Thus, any three spanning vectors are a basis, such as the three that are given.

Problem 2: More recognizing and describing linear subspaces

Which of the following subsets S of $\mathbf{R}^3$ are linear subspaces? If a set S is a linear subspace, exhibit it as a span. If it is not a linear subspace, describe it geometrically and explain why it is not a linear subspace.

(a) The set S_1 of points (x, y, z) in $\mathbf{R}^3$ with both $z = x + 2y$ and $z = 5x$.

(b) The set S_2 of points (x, y, z) in $\mathbf{R}^3$ with either $z = x + 2y$ or $z = 5x$.

(c) The set S_3 of points (x, y, z) in $\mathbf{R}^3$ of the form $t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t' \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ for some scalars t and t' (which are allowed to be anything, depending on the point (x, y, z)).

Solution: Points in S_1 are those satisfying $z = 5x$ and $5x = x + 2y$, where the second equation says $4x = 2y$ or equivalently $y = 2x$. Hence, these are points of the form $\begin{bmatrix} x \\ 2x \\ 5x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, so this is the span of the vector $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$. Geometrically, this is the overlap of two planes through the origin (meeting in a line through the origin).

The subset S_2 not a subspace. Informally, it is the collection of points on either of two planes through the origin, and this cannot be a span: any span is "closed" under forming linear combinations, but if we add a point in one plane to a point in the other plane then the vector sum (in terms of the parallelogram law) is generally outside both planes (apart from the special case when we begin with vectors on the line from (a) along which the planes meet).

Finally, S_3 is obtained by shifting in space by a point $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ some plane through the origin. But this point by which we shift is in the initial plane through the origin:

$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

In other words, we can rewrite the expressions giving S_3 as vectors of the form

$$t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t' \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = (t+1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (t'+1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

with arbitrary t and t' (so equivalently, arbitrary $t + 1$ and $t' + 1$). Hence, this is a span of two vectors in $\mathbf{R}^3$ that aren't scalar multiples of each other, so it is a plane through the origin. In particular, we have exhibited it as a linear subspace.

Problem 3: Multiple descriptions as a span (Extra)

Let $\mathbf{v}, \mathbf{w}$ be two vectors in $\mathbf{R}^{12}$. Show that $\mathrm{span}(\mathbf{v}, \mathbf{w}) = \mathrm{span}(\mathbf{v} + \mathbf{w}, \mathbf{v} - \mathbf{w})$. (Hint: You can show that two sets S and T are equal in two steps: everything belong to S also belong to T, and everything belonging to T also belongs to S.)

Solution: Anything in the second span can be written as

$$a(\mathbf{v} + \mathbf{w}) + b(\mathbf{v} - \mathbf{w}) = a\mathbf{v} + a\mathbf{w} + b\mathbf{v} - b\mathbf{w} = (a + b)\mathbf{v} + (a - b)\mathbf{w},$$

so it lies in the first span. In the other direction, we want to show that anything in the first span lies in the second: can we rewrite any $a\mathbf{v} + b\mathbf{w}$ in the form $x(\mathbf{v} + \mathbf{w}) + y(\mathbf{v} - \mathbf{w})$? We have seen that expressions of the latter sort are exactly $(x + y)\mathbf{v} + (x - y)\mathbf{w}$, so by comparing coefficients of $\mathbf{v}$ and of $\mathbf{w}$ it is enough to show for any given a, b that we can find x, y so that $x + y = a$ and $x - y = b$.

But this is "solving 2 equations in 2 unknowns" of a sort done in high school algebra: we add the equations to get $2x = a + b$, so $x = (a + b)/2$, and we subtract the second equation from the first to get $2y = a - b$, so $y = (a - b)/2$. The pair $(x, y) = ((a + b)/2, (a - b)/2)$ is readily seen to be an actual solution to that pair of equations, so we have

$$a\mathbf{v} + b\mathbf{w} = \frac{a + b}{2}(\mathbf{v} + \mathbf{w}) + \frac{a - b}{2}(\mathbf{v} - \mathbf{w})$$

(as is readily verified directly, if we want to do so; this is entirely unnecessary). This shows that the first span is contained in the second, so the two spans coincide.

Problem 4: Linear subspaces and orthogonality (computations)

Let V be the set of vectors in $\mathbf{R}^4$ orthogonal to both $\begin{bmatrix}1\\0\\4\\2\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$. Find a pair of vectors that span V, so it is a linear subspace.

Solution: Membership of a 4-vector $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$ in V amounts to 2 equations in these 4 variables (one equation for each orthogonality condition):

$$x + 4z + 2w = 0, \quad y + z + w = 0.$$

We can rewrite the first as expressing x in terms of z and w, and rewrite the second as expressing y in terms of z and w: $x = -4z - 2w$ and $y = -z - w$. So these are vectors of the form

$$\begin{bmatrix}-4z-2w\\-z-w\\z\\w\end{bmatrix} = \begin{bmatrix}-4z\\-z\\z\\0\end{bmatrix} + \begin{bmatrix}-2w\\-w\\0\\w\end{bmatrix} = z\begin{bmatrix}-4\\-1\\1\\0\end{bmatrix} + w\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix},$$

which is exactly the span of $\begin{bmatrix}-4\\-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$.

Problem 5: An orthogonal basis

Let V be the set of vectors $\mathbf{v} \in \mathbf{R}^3$ satisfying $\mathbf{v} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix}2\\3\\4\end{bmatrix}$ (this says that both of these explicit 3-vectors have the same projection onto $\mathbf{v}$, or in other words make the same "shadow" onto the line spanned by $\mathbf{v}$).

(a) Express V as the collection of 3-vectors orthogonal to a single nonzero 3-vector.

(b) By fiddling with orthogonality equations, build an orthogonal basis of V. There are many possible answers.

(c) Use your answer to (b) to give an orthonormal basis for V.

Solution:

(a) Some vector algebra simplifies the condition for belonging to V: the given condition

$$\mathbf{v} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix}2\\3\\4\end{bmatrix}$$

says exactly that $\mathbf{v} \cdot \left(\begin{bmatrix}2\\3\\4\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} \right) = 0$, which is to say

$$\mathbf{v} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 0.$$

That is, V is the collection of 3-vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) The vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ belongs to V precisely when

$$x + y + z = 0,$$

which is the equation defining a plane through the origin (i.e., its dimension is 2) having normal vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. So to find an orthogonal basis for V, we need to find two nonzero vectors $\mathbf{v}_1, \mathbf{v}_2$ perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* perpendicular to each other. For $\mathbf{v}_1$, we can pick any nonzero vector in the given plane; that is, any solution to the defining equation other than $(0, 0, 0)$. One such vector is $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (and again, there are many other choices). Having chosen $\mathbf{v}_1$, the condition on $\mathbf{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is that it is a solution to the pair of equations

$$\begin{aligned} x + y + z &= 0 \\ y - z &= 0 \end{aligned}$$

other than the zero-vector solution $(0, 0, 0)$. We have $y = z$, and $x = -y - z$, so $x = -z - z = -2z$. One solution is given by choosing $z = 1$, so then $y = 1$ and $x = -2$. That is, we can use $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ (and any nonzero scalar multiple of this $\mathbf{v}_2$ works just as well, given the choice we made for $\mathbf{v}_1$). In other words, the pair of vectors

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is an orthogonal basis for the plane V.

(c) Now dividing by lengths gives an orthonormal basis:

$$\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$