Last time

- linearity & matrices
- the derivative matrix

Today

- matrices for projections and more complicated linear transformations
- matrix "algebra" (addition & multiplication, plugging matrices into polynomials)

Problem 1: Matrix of a projection

Let V be the plane $x+y+z=0$ in $\mathbf{R}^3$ through the origin, so V has an orthogonal basis $\{\mathbf{v}, \mathbf{w}\}$ for $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be the function $L(\mathbf{x}) = \mathbf{Proj}_V(\mathbf{x})$.

(a) Compute the 3×3 matrix A for L; the entries should be fractions with denominator 3. (Hint: what is the meaning of each column?)

Recall: the columns of A are $L(\vec{e}_1), L(\vec{e}_2), L(\vec{e}_3)$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$L(\vec{e}_1) = \mathbf{Proj}_V(\vec{e}_1) = \left(\frac{\vec{v}\cdot\vec{e}_1}{\vec{v}\cdot\vec{v}}\right)\vec{v} + \left(\frac{\vec{w}\cdot\vec{e}_1}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

$$\left[\vec{v}\cdot\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2, \quad \vec{w}\cdot\vec{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 6\right]$$

$$L(\vec{e}_1) = \frac{1}{2}\vec{v} + \frac{1}{6}\vec{w} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

$$L(\vec{e}_2) = \left(\frac{\vec{v}\cdot\vec{e}_2}{2}\right)\vec{v} + \left(\frac{\vec{w}\cdot\vec{e}_2}{6}\right)\vec{w}$$

$$= -\frac{1}{2}\vec{v} + \frac{1}{6}\vec{w} = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$L(\vec{e}_3) = 0\cdot\vec{v} + \frac{-2}{6}\vec{w} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

(b) For $\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, compute $\mathbf{Proj}_V(\mathbf{a})$ in two ways: using the orthogonal basis $\{\mathbf{v}, \mathbf{w}\}$ for V, and using the matrix-vector product against your answer in (a). (You should get the same answer both ways, a vector with integer entries.)

$$\mathrm{Proj}_V(\vec{a}) = \left(\frac{\vec{v}\cdot\vec{a}}{2}\right)\vec{v} + \left(\frac{\vec{w}\cdot\vec{a}}{6}\right)\vec{w}$$

$$= \frac{-2}{2}\vec{v} + \frac{-6}{6}\vec{w}$$

$$= -\vec{v} - \vec{w} = \underline{\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}}$$

$\vec{v}\cdot\vec{a} = 1-3$
$= -2$

$\vec{w}\cdot\vec{a} = 1+3-10$
$= -6$

$$A\vec{a} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -6 \\ 0 \\ 6 \end{bmatrix} = \underline{\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}} \quad \checkmark$$

(c) The geometric definition of $\mathbf{Proj}_V$ gives that its output lies in V, on which $\mathbf{Proj}_V$ has no effect, so $\mathbf{Proj}_V \circ \mathbf{Proj}_V = \mathbf{Proj}_V$. Check that your answer A in (a) satisfies the corresponding matrix equality $A^2 = A$. (Hint: if you write $A = (1/3)B$ for a matrix B with integer entries then the calculation will be cleaner.)

$$A^2 = \frac{1}{9}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$= A \quad \checkmark$$

Problem 2: Matrix multiplication

(a) Compute the following matrix products.

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 9 & 11 \\ 2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 9 & 11 \\ 0 & -13 & -16 \end{bmatrix}$$

$$\begin{pmatrix} \text{for } \mathbf{v}, \mathbf{w} \text{ two} \\ \text{vectors in } \mathbf{R}^n \end{pmatrix} \quad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 18 \\ 20 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 9 & 11 \\ 2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 6 \\ 1 & 9 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 9 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 90 \\ 4 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -d & -e & -f \\ 3g & 3h & 3i \end{bmatrix}$$

matrix with 1 entry

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1w_1 + v_2w_2 + \cdots + v_nw_n \end{bmatrix}$$

(b) Let $q(x, y, z) = x^2 + 2y^2 - z^2 - 3xy + 4xz + yz$. Find values of a, b, c, d, e, f that satisfy

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = q(x, y, z)$$

for every x, y, z. Strictly speaking, the left side multiplies out to be a 1×1 matrix and the equality means that the scalar $q(x, y, z)$ on the right side is the unique entry in that matrix. (Hint: multiply the left side fully, and compare coefficients on the two sides, such as for x^2, yz, etc.)

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + dy + ez \\ dx + by + fz \\ ex + fy + cz \end{bmatrix}$$

$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} ax+dy+ez \\ dx+by+fz \\ ex+fy+cz \end{bmatrix} = ax^2 + dxy + exz + dxy + by^2 + fyz + exz + fyz + cz^2$$

$$= ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz$$

$$\boxed{a=1,\ b=2,\ c=-1,\ d=-\frac{3}{2},\ e=2,\ f=\frac{1}{2}}$$

(c) (**Extra**) Is there a version of (b) for any $q(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz$ in general?

Yes.

Problem 3: Some more matrix algebra

Consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by projecting a vector $\mathbf{v} \in \mathbf{R}^3$ onto its first two components (viewed as a 2-vector), then reflecting that projection across the line $x + y = 0$ in $\mathbf{R}^2$, and finally adding to this the $45°$ clockwise rotation of the projection of $\mathbf{v}$ onto its last two components. Find the 2×3 matrix A that computes T.

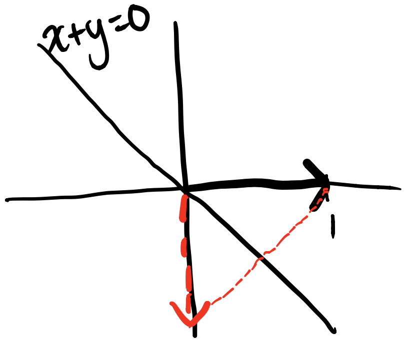
$f : \mathbb{R}^3 \to \mathbb{R}^2$ projects onto first 2 components

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

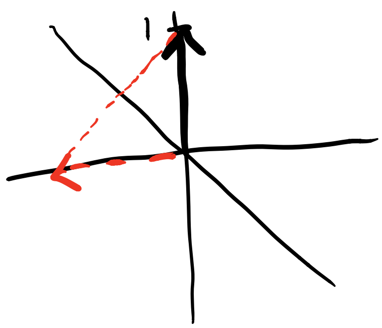
$$f(\vec{e}_1)=\begin{bmatrix}1\\0\end{bmatrix},\ f(\vec{e}_2)=\begin{bmatrix}0\\1\end{bmatrix},\ f(\vec{e}_3)=\begin{bmatrix}0\\0\end{bmatrix}$$

matrix: $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$

$g:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ reflect across $x+y=0$



$$g\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\-1\end{bmatrix}$$



$$g\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\0\end{bmatrix}$$

matrix: $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

first $\overset{f}{\text{project}}$, then $\overset{g}{\text{reflect}}$:

$$\underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$(g \circ f)(\vec{v}) = g(\underline{f(\vec{v})})$$

$$\underline{B}(\underline{A\vec{v}})$$

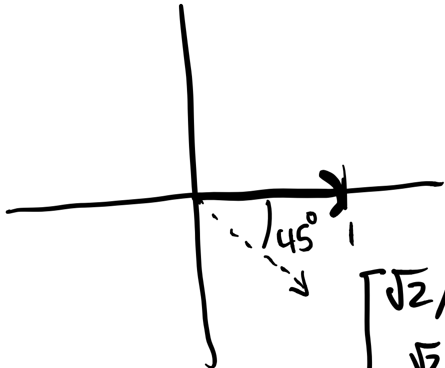
$h: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ project onto last two components

$$h\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y \\ z \end{bmatrix}$$

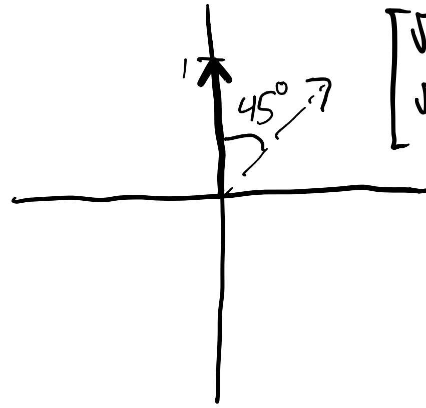
$$h(\vec{e}_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; h(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, h(\vec{e}_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

matrix: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$i: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotate 45° clockwise



$$\begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$



$$\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

matrix: $\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

ioh :

$$\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1+\sqrt{2}/2 & \sqrt{2}/2 \\ -1 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}$$