Solutions to Math 51 Practice problems for Quiz 8

1. (2 points) Suppose the matrix A satisfies

$$A = ST$$

where

$$S = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Observe that the columns of S are orthonormal.

What are the dimensions of $N(A)$ and $C(A)$ (i.e., the null space and column space of A, respectively)?

(i) 0 (ii) 1 (iii) 2 (iv) 3

The null space of A is the set of solutions $\mathbf{x} \in \mathbf{R}^3$ to the system $A\mathbf{x} = \mathbf{0}$, or equivalently in this case, to $ST\mathbf{x} = \mathbf{0}$. Since the columns of S are orthonormal, S is an orthogonal matrix; in particular, S is invertible. So we may multiply both sides of $ST\mathbf{x} = \mathbf{0}$ by S^{-1} $(= S^T)$, giving

$$T\mathbf{x} = \mathbf{0} \iff \left\{ \begin{aligned} 3x_2 + 2x_3 &= 0 \\ 3x_2 + 2x_3 &= 0 \\ 5x_3 &= 0 \end{aligned} \right\}$$

The latter system has solution $x_2 = x_3 = 0$, leaving x_1 permitted to be any scalar; so the set of solutions takes the form

$$N(A) = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbf{R} \right\} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

and it follows that $\dim(N(A)) = 1$.

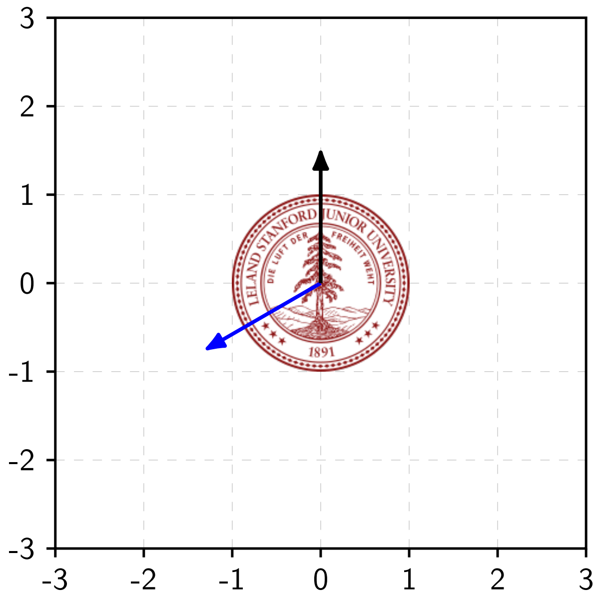
By the Rank-Nullity Theorem, $\dim(C(A)) = 3 - \dim(N(A)) = 3 - 1 = 2$. Alternatively, we may notice that if the columns of S are denoted (in order) $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, then the columns of the matrix product $A = ST$ are $\mathbf{0}$, $3\mathbf{w}_1 + 3\mathbf{w}_2$, $2\mathbf{w}_1 + 2\mathbf{w}_2 + 5\mathbf{w}_3$; and so the column space of A is

$$\begin{aligned} C(A) &= \mathrm{span}(\mathbf{0},\ 3\mathbf{w}_1 + 3\mathbf{w}_2,\ 2\mathbf{w}_1 + 2\mathbf{w}_2 + 5\mathbf{w}_3) \\ &= \mathrm{span}(3\mathbf{w}_1 + 3\mathbf{w}_2,\ 2\mathbf{w}_1 + 2\mathbf{w}_2 + 5\mathbf{w}_3) \\ &= \mathrm{span}(3(\mathbf{w}_1 + \mathbf{w}_2),\ 2(\mathbf{w}_1 + \mathbf{w}_2) + 5\mathbf{w}_3). \end{aligned}$$

The two spanning vectors $3(\mathbf{w}_1 + \mathbf{w}_2)$ and $2(\mathbf{w}_1 + \mathbf{w}_2) + 5\mathbf{w}_3$ cannot be scalings of each other without $5\mathbf{w}_3$ being a scalar multiple of $\mathbf{w}_1 + \mathbf{w}_2$, which is impossible since $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent (being orthogonal). Thus, again $\dim(C(A)) = 2$, this time by direct reasoning.

2. (4 points) Suppose A is a 2×2 matrix with eigenvalues μ_1, μ_2; and that B is a *symmetric* 2×2 matrix with eigenvalues λ_1, λ_2.

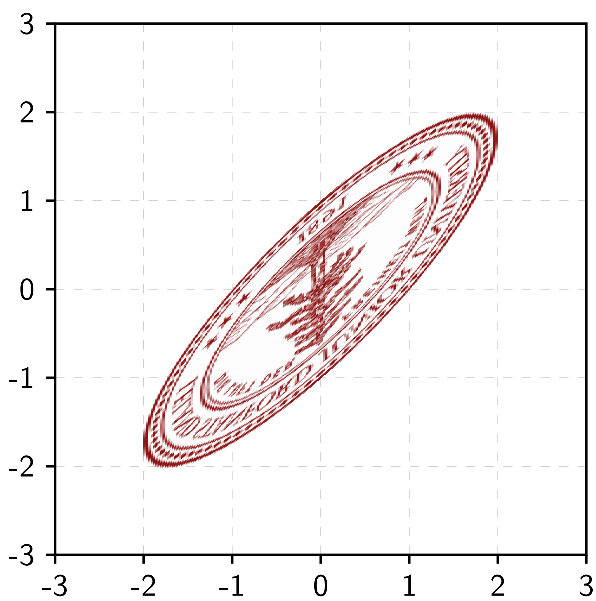
   Consider also the following picture of the Stanford seal with two vectors (one black/vertical; one blue/non-vertical) superimposed, and suppose the additional three statements about these vectors, written alongside the figure:



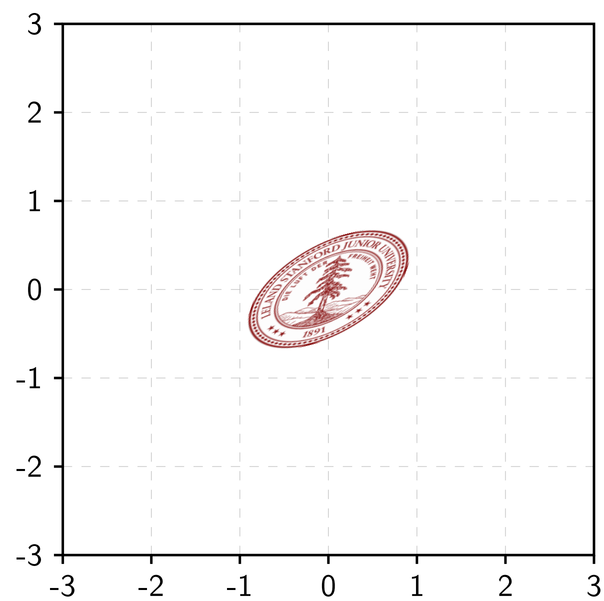
- The blue (non-vertical) vector shown above is an eigenvector of A with eigenvalue μ_1.
- The black (vertical) vector shown above is an eigenvector of A with eigenvalue μ_2.
- The blue (non-vertical) vector shown above is an eigenvector of B with eigenvalue λ_1.

Finally, shown below are the outputs when these matrices are applied to the original picture of the Stanford seal:

Effect of applying the 2×2 matrix A:



Effect of applying the *symmetric* 2×2 matrix B:

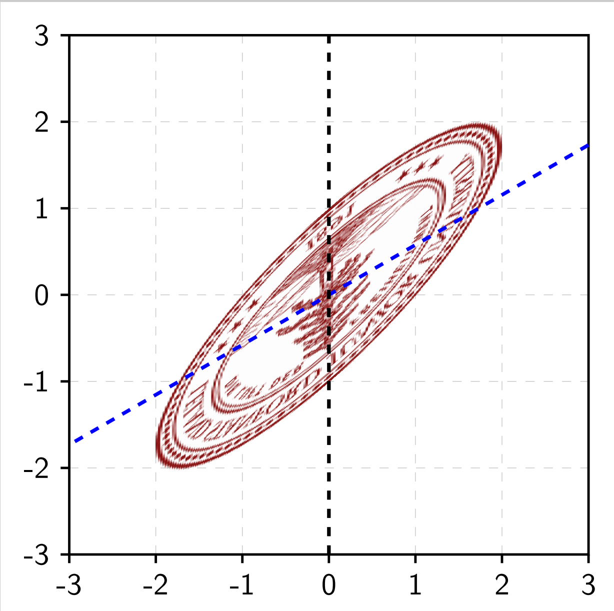


Given all of this information, estimate: the eigenvalues μ_1, μ_2, of A, and the eigenvalues λ_1, λ_2 of B.

(i) -2 (ii) -1 (iii) -0.5 (iv) 0 (v) 0.5 (vi) 1 (vii) 2

The correct answers are: $\mu_1 \approx 2$, $\mu_2 \approx -1$; and $\lambda_1 \approx 1$, $\lambda_2 \approx 0.5$.

For the matrix A, we use the grid divisions to roughly approximate the widths of the seal along the lines spanned by the two given eigenvectors (the two "eigen-lines"):



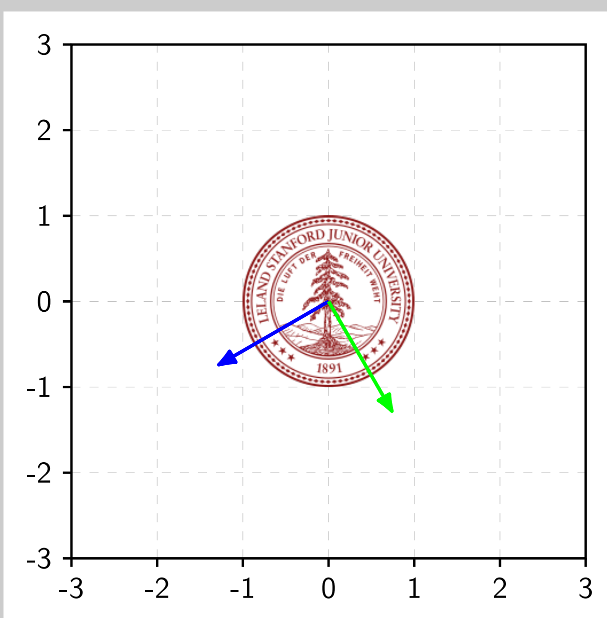
The width of the seal along the line (dashed, shown at left) spanned by the blue/non-vertical vector appears to be about twice as long as its original width, so $|\mu_1| \approx 2$. In addition, the seal is not "flipped" (reflected) along this axis (otherwise, the portion of the seal in the third quadrant near the blue line would include the word "UNIVERSITY" rather than "⋆⋆⋆ LELAND"), so the eigenvalue is positive. Thus, $\mu_1 \approx 2$.

The width of the seal along the line spanned by the black/vertical vector (dashed black/vertical line shown at left) appears to be about the same as its original width, so $|\mu_2| \approx 1$. In addition, the seal is "flipped" (reflected) along this axis, since the tree is upside-down, so the eigenvalue is negative. Thus, $\mu_2 \approx -1$.

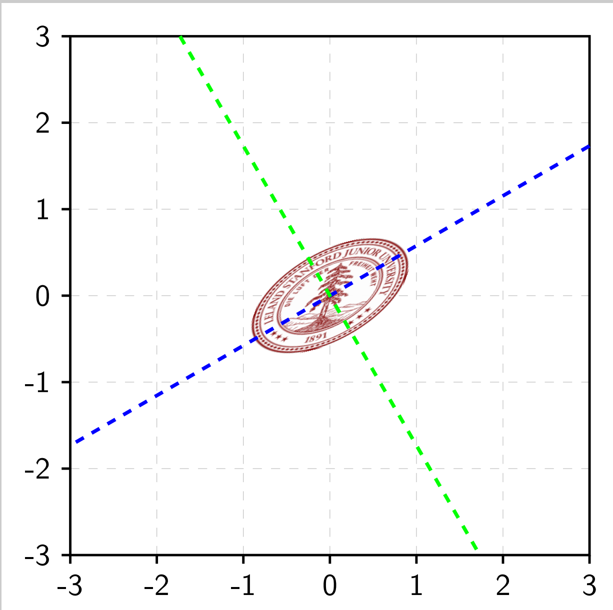
For the matrix B, we can take the same approach, once we identify the direction of the eigenvector corresponding to λ_2. We know that such a vector must exist, because B is *symmetric*: thus, by the Spectral Theorem, $\mathbf{R}^2$ has a basis consisting of eigenvectors of B (one basis vector for each eigenvalue).

Notice also that $\lambda_1 \neq \lambda_2$: for, if a 2×2 symmetric matrix has two equal eigenvalues (or in other words, has a 2-dimensional space of eigenvectors for a single eigenvalue), then it follows that *every* vector in $\mathbf{R}^2$ is an eigenvector with this eigenvalue, and that matrix's effect is simply to multiply every vector by the same scalar. From the picture given, this is certainly not the effect of our B.

Now finally, we recall (also by the Spectral Theorem) that eigenvectors of B corresponding to the different eigenvalues λ_1, λ_2 must be *orthogonal*. Thus, the green vector shown below (in the fourth quadrant, perpendicular to the blue vector) must be an eigenvector of B with eigenvalue λ_2:



With two different eigenvector directions ("eigen-lines") for the matrix B now identified, we can again use the grid divisions to roughly approximate the widths of the seal along the lines spanned by the two given eigenvectors:



The width of the seal along the line (dashed, positively-sloped, shown at left) spanned by the blue vector appears to be about the same as its original width, so $|\lambda_1| \approx 1$. In addition, the seal is not "flipped" (reflected) along this axis (otherwise, the portion of the seal in the third quadrant near the blue line would include the word "UNIVERSITY" rather than "⋆⋆⋆ LELAND"), so the eigenvalue is positive. Thus, $\lambda_1 \approx 1$.

The width of the seal along the line (dashed, negatively-sloped, shown at left) spanned by the green vector appears to be about half of its original width, so $|\lambda_2| \approx 0.5$. In addition, the seal is not "flipped" (reflected) along this axis (otherwise, the portion of the seal in the fourth quadrant near the green line would include the word "STANFORD" rather than "1891 ⋆⋆⋆"), so the eigenvalue is positive. Thus, $\lambda_2 \approx 0.5$.

3. (3 points) Suppose M is a symmetric 3×3 matrix with eigenvalues -2 and 1, and suppose additionally that:

- vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, unit-length eigenvectors for M with eigenvalue -2; and
- vector $\mathbf{c}$ is a unit-length eigenvector for M with eigenvalue 1.

If we define the vector $\mathbf{x}$ by

$$\mathbf{x} = M^{2021}(\mathbf{a} - \mathbf{c}),$$

then suppose

- the angle between $\mathbf{x}$ and $\mathbf{a}$ is A degrees; and
- the angle between $\mathbf{x}$ and $\mathbf{b}$ is B degrees; and
- the angle between $\mathbf{x}$ and $\mathbf{c}$ is C degrees.

What are the approximate values of A, B, C?

(i) 0 (ii) 45 (iii) 90 (iv) 135

(v) 180 (vi) not enough information to determine

The given information allows us to conclude that

- $M\mathbf{a} = -2\mathbf{a}$, $M\mathbf{b} = -2\mathbf{b}$, and $M\mathbf{c} = \mathbf{c}$; and
- $\mathbf{c}$ is orthogonal to each of $\mathbf{a}$ and $\mathbf{b}$ (since by the Spectral Theorem, eigenvectors of a symmetric matrix associated to different eigenvalues are orthogonal).

Thus, since matrix powers have the "composition" effect $M^k\mathbf{w} = \lambda^k\mathbf{w}$ whenever $M\mathbf{w} = \lambda\mathbf{w}$, we have

$$\begin{aligned}\mathbf{x} = M^{2021}(\mathbf{a}-\mathbf{c}) &= M^{2021}\mathbf{a} - M^{2021}\mathbf{c}\\ &= (-2)^{2021}\mathbf{a} - (1)^{2021}\mathbf{c}\\ &= -2^{2021}\mathbf{a} - \mathbf{c}.\end{aligned}$$

To find each angle, we will use the dot product formula; this will require computing $\|\mathbf{x}\|$, which can be done as follows:

$$\begin{aligned}\|\mathbf{x}\|^2 = \mathbf{x}\cdot\mathbf{x} &= (-2^{2021}\mathbf{a}-\mathbf{c})\cdot(-2^{2021}\mathbf{a}-\mathbf{c})\\ &= 2^{4042}(\mathbf{a}\cdot\mathbf{a}) + (2)(2^{2021})(\mathbf{a}\cdot\mathbf{c}) + (\mathbf{c}\cdot\mathbf{c})\\ &= 2^{4042} + (2)(2^{2021})(0) + 1 \qquad [\mathbf{a},\mathbf{c}\text{ orthogonal \& unit-length}]\\ &= 2^{4042}+1\end{aligned}$$

so $\|\mathbf{x}\| = \sqrt{2^{4042}+1} \approx 2^{2021}$.

(A) We have

$$\begin{aligned}\cos(A) = \frac{\mathbf{x}\cdot\mathbf{a}}{\|\mathbf{x}\|\|\mathbf{a}\|} &= \frac{(-2^{2021}\mathbf{a}-\mathbf{c})\cdot\mathbf{a}}{\|\mathbf{x}\|\|\mathbf{a}\|}\\ &= \frac{-2^{2021}(\mathbf{a}\cdot\mathbf{a}) - (\mathbf{c}\cdot\mathbf{a})}{\|\mathbf{x}\|\|\mathbf{a}\|}\\ &= \frac{-(2^{2021})(1)-0}{\|\mathbf{x}\|(1)} \qquad [\mathbf{a},\mathbf{c}\text{ orthogonal \& unit-length}]\\ &= -\frac{2^{2021}}{\|\mathbf{x}\|}\\ &\approx -\frac{2^{2021}}{2^{2021}} = -1,\end{aligned}$$

so A is approximately 180 (degrees).

Note: the approximation $\frac{2^{2021}}{\|\mathbf{x}\|} \approx 1$ is equivalent to the approximation $2^{-2021} \approx 0$ (which is correct to many hundreds of decimal places), as shown by factoring 2^{2021} out of the radical in the exact expression for $\|\mathbf{x}\|$:

$$\frac{2^{2021}}{\|\mathbf{x}\|} = \frac{2^{2021}}{\sqrt{2^{4042}+1}} = \frac{2^{2021}}{\sqrt{2^{4042}}\sqrt{1+2^{-4042}}} = \frac{1}{\sqrt{1+2^{-4042}}} \approx \frac{1}{\sqrt{1+0}} = 1$$

(B) We have

$$\begin{aligned}\cos(B) = \frac{\mathbf{x}\cdot\mathbf{b}}{\|\mathbf{x}\|\|\mathbf{b}\|} &= \frac{(-2^{2021}\mathbf{a}-\mathbf{c})\cdot\mathbf{b}}{\|\mathbf{x}\|\|\mathbf{b}\|} \\ &= \frac{-2^{2021}(\mathbf{a}\cdot\mathbf{b})-(\mathbf{c}\cdot\mathbf{b})}{\|\mathbf{x}\|\|\mathbf{b}\|} \\ &= \frac{-(2^{2021})(\mathbf{a}\cdot\mathbf{b})-0}{\|\mathbf{x}\|(1)} \qquad [\mathbf{b} \text{ is unit-length and orthogonal to } \mathbf{c}] \\ &= -(\mathbf{a}\cdot\mathbf{b})\frac{2^{2021}}{\|\mathbf{x}\|} \\ &\approx -(\mathbf{a}\cdot\mathbf{b}),\end{aligned}$$

which is not a known quantity; so we do not have enough information to determine B.

Note: even for a symmetric matrix, two linearly independent eigenvectors associated to the *same* eigenvalue are not necessarily orthogonal! As an example, consider the following symmetric 3×3 matrix and eigenvectors:

$$M = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{a} = \mathbf{e}_1, \quad \mathbf{b} = \frac{1}{\sqrt{2}}(\mathbf{e}_1+\mathbf{e}_2), \quad \mathbf{c} = \mathbf{e}_3.$$

These unit-length eigenvectors for M satisfy all of the hypotheses of the problem, and yet $\mathbf{a}\cdot\mathbf{b} = \frac{1}{\sqrt{2}} \neq 0$. In this case, $\cos(B) \approx -(\mathbf{a}\cdot\mathbf{b}) = -\frac{1}{\sqrt{2}}$, so B is approximately 135 (degrees); but in another situation we might find $B \approx 90$, or some other value (in fact any value strictly between 0 and 180 is possible for B).

(C) We have

$$\begin{aligned}\cos(C) = \frac{\mathbf{x}\cdot\mathbf{c}}{\|\mathbf{x}\|\|\mathbf{c}\|} &= \frac{(-2^{2021}\mathbf{a}-\mathbf{c})\cdot\mathbf{c}}{\|\mathbf{x}\|\|\mathbf{c}\|} \\ &= \frac{-2^{2021}(\mathbf{a}\cdot\mathbf{c})-(\mathbf{c}\cdot\mathbf{c})}{\|\mathbf{x}\|\|\mathbf{c}\|} \\ &= \frac{-(2^{2021})(0)-1}{\|\mathbf{x}\|(1)} \qquad [\mathbf{a},\mathbf{c} \text{ orthogonal \& unit-length}] \\ &\approx \frac{-1}{2^{2021}} \approx 0,\end{aligned}$$

so C is approximately 90 (degrees).

4. (2 points) Suppose A and B are 2×2 symmetric matrices. In which of the following situations can we conclude that the collection of A's eigenvalues must be the same as the collection of B's eigenvalues? Select all that apply. (Note that each situation is to be considered separately.)

(i) A and B have the same determinant, and each has 5 as an eigenvalue.

(ii) A and B have the same trace, and each has $\begin{bmatrix}1\\2\end{bmatrix}$ as an eigenvector.

(iii) A and B have the same trace and determinant.

Among the options given, the correct answers are (i) and (iii).

The key points pertain to the characteristic polynomial of a 2×2 matrix M:

- the eigenvalues of M are the roots of $\lambda^2 - \mathrm{tr}(M)\lambda + \det(M) = 0$;
- since $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$, then whenever M has eigenvalues λ_1, λ_2 we know that $\mathrm{tr}(M) = \lambda_1 + \lambda_2$ and $\det(M) = \lambda_1\lambda_2$.

Since A and B are symmetric, they each have two eigenvalues (noting that here we are thinking of a repeated root as two occurrences of the same eigenvalue; e.g., I_2 has the two eigenvalues 1 and 1); say μ_1, μ_2 for A and λ_1, λ_2 for B.

(i) Always true: if, say, $\mu_1 = \lambda_1 = 5$ and $\mu_1\mu_2 = \det(A) = \det(B) = \lambda_1\lambda_2$, then we may divide both sides by 5 $(= \mu_1 = \lambda_1)$ to obtain $\mu_2 = \lambda_2$.

(ii) Not always true: the fact that a matrix has $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as an eigenvector provides no information at all about the corresponding eigenvalue. (As one concrete counterexample, consider the symmetric matrix $A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ of trace 5, whose rows are perpendicular to $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so that $A\mathbf{w} = \mathbf{0}$ or equivalently that $\mathbf{w}$ is an eigenvector of A with eigenvalue 0. The other eigenvalue of A is therefore $\mathrm{tr}(A) - 0 = 5 - 0 = 5$. Meanwhile, the matrix $B = (5/2)I_2 = \begin{bmatrix} 5/2 & 0 \\ 0 & 5/2 \end{bmatrix}$, which also has trace 5, satisfies $B\mathbf{x} = (5/2)\mathbf{x}$ for all 2-vectors $\mathbf{x}$ (including $\mathbf{w}$). The eigenvalues of B are 5/2 and 5/2, different from A's eigenvalues.)

(iii) Always true: the characteristic polynomials of A and B are the same, so their roots (the eigenvalues) are also the same.

5. (3 points) Let A be a 3×3 matrix and let λ be an eigenvalue of A. Which of the following are possible dimensions of $C(A - \lambda I_3)$?
Select all that apply.

(a) 0 (b) 1 (c) 2 (d) 3

(a), (b), and (c) are possible.

If $A = \lambda I_3$, $A - \lambda I_3$ is the zero matrix, so $C(A - \lambda I_3) = \{\mathbf{0}\}$, and $\dim(C(A - \lambda I_3)) = 0$.

If A is a diagonal matrix with diagonal entries $(\lambda, \lambda, \alpha)$ where $\alpha \neq \lambda$, then $A - \lambda I_3$ has two zero columns and 1 nonzero column, so $\dim(C(A - \lambda I_3)) = 1$.

If A is a diagonal matrix with diagonal entries (λ, α, β) where $\alpha \neq \lambda$ and $\beta \neq \lambda$, then $A - \lambda I_3$ has two linearly independent nonzero columns, so $\dim(C(A - \lambda I_3)) = 2$.

(d) is not possible: λ being an eigenvalue of A means that $N(A - \lambda I)$ contains a nonzero vector (namely an eigenvector of A with eigenvalue λ), and hence $N(A - \lambda I)$ has dimension at least 1. By the Rank-Nullity theorem, this implies that $C(A - \lambda I)$ has rank at most $3 - \dim(N(A - \lambda I)) = 2$.

6. (2 points) Recall from Exam 4 the Markov matrix

$$M = \begin{bmatrix} 3/4 & 1/6 & 0 \\ 1/4 & 2/3 & 1/2 \\ 0 & 1/6 & 1/2 \end{bmatrix}$$

describing the weekly dynamics of universities' opening and closing. M has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ with corresponding eigenvalues $1, 2/3, 1/4$.

$M^{2021}(\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3)$ is approximately

(a) $\mathbf{v}_1$ (b) $2\mathbf{v}_2$ (c) $3\mathbf{v}_3$ (d) $\mathbf{0}$

Since

$$M\mathbf{v}_1 = \mathbf{v}_1, \qquad M\mathbf{v}_2 = (2/3)\mathbf{v}_2, \qquad M\mathbf{v}_3 = (1/4)\mathbf{v}_3,$$

we have

$$M^{2021}\mathbf{v}_1 = \mathbf{v}_1, \qquad M^{2021}\mathbf{v}_2 = (2/3)^{2021}\mathbf{v}_2 \approx \mathbf{0}, \qquad M^{2021}\mathbf{v}_3 = (1/4)^{2021}\mathbf{v}_3 \approx \mathbf{0},$$

and so

$$M^{2021}(\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3) = M^{2021}\mathbf{v}_1 + 2M^{2021}\mathbf{v}_2 + 3M^{2021}\mathbf{v}_3 \approx \mathbf{v}_1.$$

7. (2 points) Let L be a lower triangular matrix, and $A = LL^\top$. Which of the following statements are always true? Select all that apply.

(a) $LL^\top$ is an LU-decomposition of A.
(b) $L^\top L$ is an LU-decomposition of A.
(c) Solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L\mathbf{x} = \mathbf{b}$.
(d) Solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L^\top\mathbf{x} = \mathbf{b}$.
(e) If L is invertible, then solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L^\top\mathbf{x} = L^{-1}\mathbf{b}$.
(f) Solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L\mathbf{x} = L^\top\mathbf{b}$.
(g) Solutions to $A\mathbf{x} = \mathbf{b}$ are the same as solutions to $L^\top\mathbf{x} = L\mathbf{b}$.

(a) and (e) are correct.

- When L is lower triangular, $L^\top$ is upper triangular. Since $A = LL^\top$, the product $LL^\top$ is an LU-decomposition of A.
- In general, $LL^\top \neq L^\top L$, so $L^\top L \neq A$ in general.
- $A\mathbf{x} = (LL^\top)\mathbf{x} = \mathbf{b}$. Since L is invertible, we multiply both sides of the equation $(LL^\top)\mathbf{x} = \mathbf{b}$ by L^{-1} to obtain $L^\top\mathbf{x} = L^{-1}\mathbf{b}$. So the solutions to $A\mathbf{x} = \mathbf{b}$ and $L^\top\mathbf{x} = L^{-1}\mathbf{b}$ are the same.

The rest are all not sensible.

8. (3 points) The symmetric matrix A has the property that

$$A\begin{bmatrix}1\\2\\4\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\1\\2\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}2\\1\\-1\end{bmatrix} = \begin{bmatrix}-14\\-7\\7\end{bmatrix}.$$

Which of the following vectors $\mathbf{v}$ satisfies $q_A(\mathbf{v}) = \mathbf{v}^T A\mathbf{v} = 0$?

(a) $\begin{bmatrix}0\\3\\9\end{bmatrix} = 2\begin{bmatrix}1\\2\\4\end{bmatrix} - \begin{bmatrix}2\\1\\-1\end{bmatrix}$

(b) $\begin{bmatrix} -27 \\ -12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 14 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 12 \\ 27 \\ 57 \end{bmatrix} = 14 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} -36 \\ -9 \\ 45 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 21 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

The answer is (a). Let $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. Then $\mathbf{w}_1$ is an eigenvector for A with eigenvalue $\frac{1}{2}$, and $\mathbf{w}_2$ is an eigenvector for A with eigenvalue -7. Note that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal. This implies that

$$\begin{aligned} q_A(a\mathbf{w}_1 + b\mathbf{w}_2) &= (a\mathbf{w}_1 + b\mathbf{w}_2)^T A(a\mathbf{w}_1 + b\mathbf{w}_2) \\ &= (a\mathbf{w}_1 + b\mathbf{w}_2)^T (\frac{1}{2}a\mathbf{w}_1 - 7b\mathbf{w}_2) \\ &= \frac{1}{2}a^2(\mathbf{w}_1 \cdot \mathbf{w}_1) - 7b^2(\mathbf{w}_2 \cdot \mathbf{w}_2) \\ &= \frac{21}{2}a^2 - 42b^2. \end{aligned}$$

Setting this to equal 0, we must have $\frac{1}{2}a^2 = 2b^2$, so $a = \pm 2b$.

Alternatively, note that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal eigenvectors. By the Spectral Theorem, there exists a third eigenvector $\mathbf{w}_3$ which is orthogonal to $\mathbf{w}_1$ and $\mathbf{w}_2$. In terms of the orthonormal basis $\mathbf{w}'_1, \mathbf{w}'_2, \mathbf{w}'_3$ obtained by dividing each $\mathbf{w}_i$ by its length, the quadratic form takes the form

$$q_A(c_1\mathbf{w}'_1 + c_2\mathbf{w}'_2 + c_3\mathbf{w}'_3) = \frac{1}{2}c_1^2 - 7c_2^2 + \lambda_3 c_3^2,$$

where λ_3 is the eigenvalue of $\mathbf{w}_3$.

Now observe that all the answers are given in the format $a\mathbf{w}_1 + b\mathbf{w}_2$, which equals $a\sqrt{21}\mathbf{w}'_1 + b\sqrt{6}\mathbf{w}'_2$. So for the quadratic form to equal 0, we must have

$$\begin{aligned} 0 &= \frac{1}{2}(a\sqrt{21})^2 - 7(b\sqrt{6})^2 + \lambda_3(0)^2 \\ &= \frac{21}{2}a^2 - 42b^2. \end{aligned}$$

So (as in the previous solution) $a = \pm 2b$.

9. (2 points) Suppose a 3×3 invertible matrix A has QR-decomposition given by $A = QR$. Suppose A_1 is obtained from A by scaling its second column by a factor of c, while the other two columns are the same as the corresponding columns of A. What can you say about the QR decomposition of A_1?

(a) $A_1 = Q_1R_1$ where $Q_1 = Q$ and R_1 is obtained from R by scaling its second column by a factor of c, while the other two columns of R_1 are the same as the corresponding columns of R.

(b) $A_1 = Q_1R_1$ where $Q_1 = Q$ and R_1 is obtained from R by scaling its second row by a factor of c, while the other two rows of R_1 are the same as the corresponding rows of R.

(c) $A_1 = Q_1R_1$ where $R_1 = R$ and Q_1 is obtained from Q by scaling its second column by a factor of c, while the other two columns of Q_1 are the same as the corresponding columns of Q.

(d) $A_1 = Q_1R_1$ where $R_1 = R$ and Q_1 is obtained from Q by scaling its second row by a factor of c, while the other two rows of Q_1 are the same as the corresponding rows of Q.

The answer is (a).

$A_1 = AD$ where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$A_1 = AD = QRD = Q(RD)$$

RD is still upper triangular and Q still orthogonal, so $Q(RD)$ is a QR decomposition of $A = Q_1R_1$, with $Q_1 = Q$ and $R_1 = RD$, so R_1 is obtained from R by scaling its second column by a factor of c, while the other two columns of R_1 are the same as the corresponding columns of R.

Note that if the columns or rows of Q are scaled by $c \neq \pm 1$, then Q will no longer be an orthogonal matrix.

10. (3 points) Suppose A is a 2×2 matrix with eigenvalues 2 and 3. Let

$$B = A^2 - 5A + 6I_2.$$

Which of the following statements are correct? Select all that apply.

(a) $\mathbf{R}^2$ has a basis consisting of eigenvectors of A.

(b) If $\mathbf{v}$ is an eigenvector for A, then $\mathbf{v}$ is a also an eigenvector for B.

(c) If $\mathbf{v}$ is an eigenvector for B, then $\mathbf{v}$ is a also an eigenvector for A.

(d) $B = 0$, i.e. B is a 2×2 matrix all of whose entries are 0.

(e) $B \neq 0$, i.e. B is not a 2×2 matrix all of whose entries are 0.

The correct answers are (a), (b) and (d).

By Theorem 24.1.1, the two eigenvectors $\mathbf{v}$ and $\mathbf{w}$ of A corresponding to eigenvalues 2 and 3 respectively must be linearly independent, and $\{\mathbf{v}, \mathbf{w}\}$ forms a basis for $\mathbf{R}^2$, so (a) is correct.

If $A\mathbf{u} = \lambda\mathbf{u}$, then

$$B\mathbf{u} = (A^2 - 5A + 6I_2)\mathbf{u} = A^2\mathbf{u} - 5A\mathbf{u} + 6\mathbf{u} = (\lambda^2 - 5\lambda + 6)\mathbf{u}.$$

So $\mathbf{u}$ is also an eigenvector of B with eigenvalue $\lambda^2 - 5\lambda + 6$. So (b) is correct.

Since A's eigenvalues 2 and 3 are roots of the polynomial $f(\lambda) = \lambda^2 - 5\lambda + 6$, i.e. $f(2) = f(3) = 0$. We see that any eigenvector $\mathbf{u}$ of A is an eigenvector of B with eigenvalue 0. In particular, the eigenbasis $\{\mathbf{v}, \mathbf{w}\}$ of A is also an eigenbasis for B where $B\mathbf{v} = \mathbf{0}$ and $B\mathbf{w} = \mathbf{0}$.

For any vector $\mathbf{x} \in \mathbf{R}^2$ where $\mathbf{x} = a\mathbf{v} + b\mathbf{w}$, we have

$$B\mathbf{x} = B(a\mathbf{v} + b\mathbf{w}) = aB\mathbf{v} + bB\mathbf{w} = a\mathbf{0} + b\mathbf{0} = \mathbf{0}.$$

Since $B\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}^2$, B must be the zero matrix. So (d) is correct, and (e) is not correct.

(c) is not correct. For example if $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, then $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are respective 2− and 3− eigenvecctors of A. $\mathbf{v}+\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 0-eigenvector of B since $B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but it is not an eigenvector of A since $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

More generally, we know that $B(\mathbf{v}+\mathbf{w}) = \mathbf{0}$ since $B = 0$, so $\mathbf{v}+\mathbf{w}$ is a 0-eigenvector of B, but $\mathbf{v}+\mathbf{w}$ can not be an eigenvector of A. Indeed, if $A(\mathbf{v}+\mathbf{w}) = \lambda(\mathbf{v}+\mathbf{w})$, since $A\mathbf{v} = 2\mathbf{v}$ and $A\mathbf{w} = 3\mathbf{w}$, then we have

$$\lambda(\mathbf{v}+\mathbf{w}) = A(\mathbf{v}+\mathbf{w}) = A\mathbf{v} + A\mathbf{w} = 2\mathbf{v} + 3\mathbf{w},$$

$$(\lambda - 2)\mathbf{v} + (\lambda - 3)\mathbf{w} = \mathbf{0}.$$

Since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, $(\lambda - 2)\mathbf{v} + (\lambda - 3)\mathbf{w} = \mathbf{0}$ implies $\lambda - 2 = 0$ and $\lambda - 3 = 0$. This is a contradiction, since λ cannot be both 2 and 3!

11. (3 points) A population of 60 goats moves between three meadows, labeled A, B, and C. Their daily movement is modeled by a symmetric Markov matrix M (whose rows and columns correspond to meadows A, B, and C, in that order).

Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Assume that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of M with eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{5}{9}$, and $\lambda_3 = \frac{1}{3}$, respectively.

If k is large enough that $\left(\frac{5}{9}\right)^k \approx 0$ and $\left(\frac{1}{3}\right)^k \approx 0$, how many goats are there in meadow B after k days? (Your answer should be an integer.)

Because M is symmetric, the Spectral Theorem applies. Let

$$\mathbf{v}_i' = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}.$$

Now $\{\mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3'\}$ is an orthonormal basis for $\mathbf{R}^3$ consisting of eigenvectors of M. If W is the orthogonal matrix whose i^{th} column is $\mathbf{v}_i'$ and D is the diagonal matrix whose i^{th} diagonal entry is λ_i, then the Spectral Theorem implies $M = WDW^T$. Since $W^TW = I$, we have

$$M^k = WD^kW^T = W \begin{bmatrix} (1)^k & 0 & 0 \\ 0 & \left(\frac{5}{9}\right)^k & 0 \\ 0 & 0 & \left(\frac{1}{3}\right)^k \end{bmatrix} W^T$$

for all k. If k is large enough that $\left(\frac{5}{9}\right)^k \approx 0$ and $\left(\frac{1}{3}\right)^k \approx 0$, then

$$\begin{aligned} M^k \approx W \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W^T &= \begin{bmatrix} | & | & | \\ \mathbf{v}_1' & \mathbf{v}_2' & \mathbf{v}_3' \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} - & (\mathbf{v}_1')^T & - \\ - & (\mathbf{v}_2')^T & - \\ - & (\mathbf{v}_3')^T & - \end{bmatrix} \\ &= \begin{bmatrix} | & | & | \\ \mathbf{v}_1' & \mathbf{0} & \mathbf{0} \\ | & | & | \end{bmatrix} \begin{bmatrix} - & (\mathbf{v}_1')^T & - \\ - & (\mathbf{v}_2')^T & - \\ - & (\mathbf{v}_3')^T & - \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ - & (\mathbf{v}_2')^T & - \\ - & (\mathbf{v}_3')^T & - \end{bmatrix} \\ &= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}. \end{aligned}$$

Alternatively, you can quote textbook Proposition 24.4.2 to conclude that

$$M^k \approx \frac{1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \mathbf{v}_1^\top = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

since the dominant eigenvalue $\lambda_1 = 1$.

If the initial populations in meadows A, B, and C are p_A, p_B, and p_C (so $p_A + p_B + p_C = 60$), then the population distribution at time k is

$$M^k \begin{bmatrix} p_A \\ p_B \\ p_C \end{bmatrix} \approx \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} p_A \\ p_B \\ p_C \end{bmatrix} = \begin{bmatrix} \frac{1}{3}p_A + \frac{1}{3}p_B + \frac{1}{3}p_C \\ \frac{1}{3}p_A + \frac{1}{3}p_B + \frac{1}{3}p_C \\ \frac{1}{3}p_A + \frac{1}{3}p_B + \frac{1}{3}p_C \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}.$$

Therefore we expect there to be 20 goats in meadow B after k days.

12. (4 points) Suppose A is a symmetric matrix with eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and corresponding eigenvalues $1, 2, 3$. Match each of the following inputs, (a)–(d), to its output (among (1)–(6); some inputs may match to the same output).

(a) $(\mathbf{v}_2 \cdot \mathbf{v}_3)\mathbf{v}_1$
(b) $(A\mathbf{v}_2 \cdot A\mathbf{v}_3)\mathbf{v}_1$
(c) $\mathbf{Proj}_{\mathbf{v}_2} A(\mathbf{v}_2 + \mathbf{v}_3)$
(d) $A\left(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{Proj}_{\mathbf{v}_2}(\mathbf{v}_3)\right)$

(1) $\mathbf{0}$ (2) $6\mathbf{v}_1$ (3) $2\mathbf{v}_2$ (4) $2\mathbf{v}_1$ (5) $\mathbf{v}_1 + 2\mathbf{v}_2$ (6) $\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$

(a) The answer is $\mathbf{0}$. Since $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors for distinct eigenvalues of a symmetric matrix, they are orthogonal. Therefore $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

(b) The answer is $\mathbf{0}$. Since $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors for distinct eigenvalues of a symmetric matrix, they are orthogonal. Therefore

$$A\mathbf{v}_2 \cdot A\mathbf{v}_3 = 2\mathbf{v}_2 \cdot 3\mathbf{v}_3 = 2 \cdot 3(\mathbf{v}_2 \cdot \mathbf{v}_3) = 0.$$

(c) The answer is $2\mathbf{v}_2$. First, $A(\mathbf{v}_2 + \mathbf{v}_3) = 2\mathbf{v}_2 + 3\mathbf{v}_3$. Since $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors for distinct eigenvalues of a symmetric matrix, they are orthogonal. So

$$\mathbf{Proj}_{\mathbf{v}_2}(2\mathbf{v}_2 + 3\mathbf{v}_3) = 2\,\mathbf{Proj}_{\mathbf{v}_2}\mathbf{v}_2 + 3\,\mathbf{Proj}_{\mathbf{v}_2}\mathbf{v}_3 = 2\mathbf{v}_2 + \mathbf{0} = 2\mathbf{v}_2.$$

(d) The answer is $\mathbf{v}_1 + 2\mathbf{v}_2$. Since $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors for distinct eigenvalues of a symmetric matrix, they are orthogonal. So $\mathbf{Proj}_{\mathbf{v}_2} \mathbf{v}_3 = \mathbf{0}$, and $A(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1 + 2\mathbf{v}_2$.