Last time

- computing partial derivatives
- critical points and extrema

Today

- the gradient
- tangent lines and planes

Review: Gradients

Problem 1: Linear approximation

Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = 4x + y^3 + xy$.

(a) Compute the gradient $(\nabla f)(x, y)$, and then use it to give the linear approximation to f at $(1, 1)$.

(b) Using your answer to (a), estimate $f(0.9, 1.2)$. Compare your answer to the exact result on a calculator, and compare the effort in computing the approximation by hand versus the exact answer by hand.

(c) Give the linear approximation to f at $(2, -2)$. Use it to estimate $f(3, -1)$, and then compare this estimate to the exact value using a calculator. Why is the approximation so bad?

Problem 2: Finding tangents to implicit curves and surfaces

For each of the following, find the equation of the tangent line to the given curve or the tangent plane to the given surface at the specified point $\mathbf{a}$.

(a) $x^3 + y^2 = 31$ at $\mathbf{a} = (3, 2)$.

(b) $xz^2 + y^2z^5 = 19$ at $\mathbf{a} = (3, 4, 1)$.

Problem 3: Tangent planes: graphs versus level sets

Let S be the sphere of radius 3 centered at the origin in $\mathbf{R}^3$. Let's consider two approaches to finding the equation of the tangent plane to S at the point $(2, 2, 1)$.

(a) For the surface graph $z = f(x, y)$ of a function $f(x, y)$, its tangent plane at a point $(a, b, f(a, b))$ is given by the equation $z = L(x, y)$ where $L(x, y)$ is the linear approximation to f at (a, b). Describe the upper half ($z > 0$) of the sphere S as a graph of a function of x and y, and use this to compute the equation of the tangent plane to S at the point $(2, 2, 1)$ in that upper hemisphere.

(b) The surface S is also a level set of $g(x, y, z) = x^2 + y^2 + z^2$ at a certain level c (what is the value of c?). Use the approach via gradients to compute the tangent plane to S at $(2, 2, 1)$. Verify that this is the same plane as you found in (a). (Note: the equation might not literally be the same as in (a) even though the solution sets to the equations – which are what actually matter – are the same, much as $2x - 2y + 2z = 0$ and $x - y + z = 0$ define the same plane; why?)

(c) Which method was easier? Do you have any thoughts about which method should usually be easier?

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