Problem 1: Determining linear independence

For each of the following collections of vectors, determine if it is linearly independent (think in terms of expressing a vector as a linear combination of others or studying if "$\sum c_j \mathbf{v}_j = \mathbf{0}$" can happen with some nonzero c_j, not by using Gram–Schmidt), and give a basis of their span in each case.

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix}.$

(b) $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$

(c) $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}.$

(d) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Problem 2: Computing the Gram–Schmidt process

Run the Gram–Schmidt process on the following collection of vectors, and obtain an orthogonal basis for its span.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -4 \\ 5 \\ -2 \\ -3 \end{bmatrix}.$$

Using the outcome of your calculations, also compute the dimension of the span and if you encounter any $\mathbf{w}_i$ equal to $\mathbf{0}$ then use this to produce a linear dependence relation among the $\mathbf{v}_j$'s (in which case, and as a safety check, confirm such a linear dependence relation by direct computation once you have found one).

As a safety check on your work, make sure at each step that each $\mathbf{w}_i$ is orthogonal to the previous $\mathbf{w}_j$'s. (The $\mathbf{v}_i$'s have been designed so that you only have to work with integers throughout, and in particular the $\mathbf{w}_i$'s have integer entries. If you find yourself at any step grappling with things like $-5/3$ or $11/4$ and so on, you have made a mistake.)

Problem 3: Determining independence with vector algebra (Extra)

If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbf{R}^{1000}$ are linearly independent, show that $\mathbf{u} + \mathbf{v}, 2\mathbf{u} + \mathbf{v} + \mathbf{w}, -\mathbf{u} + \mathbf{v} + \mathbf{w}$ are linearly independent. (Hint: don't think in terms of vector entries! Think in terms of the formulation of linear independence as: "$\sum c_i \mathbf{v}_i = \mathbf{0}$ implies all c_i vanish".)

