Problem 1: Geometry with dot products

(a) Using that perpendicularity is governed by the dot products being equal to 0, find a nonzero vector in $\mathbf{R}^3$ that is perpendicular to $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Then find another that is not a scalar multiple of that one.

(b) Find an equation in x, y, z that characterizes when $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. What does this collection of vectors look like?

(c) (**Extra**) What does the collection of nonzero vectors $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$ making an angle of at most $60°$ against $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ look like? Using the relation of dot products and cosines to describe this region with a pair of conditions of the form $ax^2 + bxy + cy^2 \geq 0$ and $y \leq (3/4)x$ (away from the origin).

Problem 2: Algebra with dot products

(a) For $\mathbf{a} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$ show that $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}$.

(b) Give an example of 2-vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ for which $(\mathbf{a} \cdot \mathbf{b})\mathbf{c} \neq (\mathbf{a} \cdot \mathbf{c})\mathbf{b}$. (Hint: what if $\mathbf{b}$ and $\mathbf{c}$ are not on the same line through the origin?)

(c) (**Extra**) Explain in terms of variables why $\mathbf{v} \cdot (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \cdot \mathbf{w}_1 + \mathbf{v} \cdot \mathbf{w}_2$ for any 3-vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$, and $\mathbf{w}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$. If you then replace $\mathbf{v}$ with $\mathbf{v}_1 + \mathbf{v}_2$ for 3-vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and apply another instance of the same general identity, why does it follow without any extra work with algebra in vector entries that

$$(\mathbf{v}_1 + \mathbf{v}_2) \cdot (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v}_1 \cdot \mathbf{w}_1 + \mathbf{v}_1 \cdot \mathbf{w}_2 + \mathbf{v}_2 \cdot \mathbf{w}_1 + \mathbf{v}_2 \cdot \mathbf{w}_2?$$

(This is showing the analogue for vectors of the fact for numbers that the distributive law $r(s+t) = rs+rt$ is what makes the identity $(a+b)(c+d) = ac+ad+bc+bd$ hold, since $(a+b)(c+d) = (a+b)c+(a+b)d = ac+bc+ad+bd$.)

Do your arguments work for n-vectors for any n?

(d) For n-vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, verify that $\|\mathbf{w}_1+\mathbf{w}_2\|^2 = \|\mathbf{w}_1\|^2+2(\mathbf{w}_1 \cdot \mathbf{w}_2)+\|\mathbf{w}_2\|^2$ by using the relation $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ and general properties of dot products as stated in (c) (even if you didn't do (c)), *not* by writing out big formulas for lengths and dot products in terms of vector entries.

Problem 3: A correlation coefficient

Consider the collection of 5 data points: $(-2, 5)$, $(-1, 3)$, $(0, 0)$, $(1, -2)$, $(2, -6)$.

(a) Plot the points to see if they look close to a line.

(b) Compute the compute correlation coefficient exactly. Plug that into a calculator to approximate it to three decimal digits to see if its nearness to ± 1 fits well with the visual quality of fit of the line to the data plot in (a).

Problem 4: More convex combinations (Extra)

(a) For the 2-vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, describe the set of all possible vectors $r\mathbf{a} + s\mathbf{b} + t\mathbf{c}$ where $r + s + t = 1$ with $0 \leq r, s, t \leq 1$. Which points in your description correspond to the case $t = 0$. How about $s = 0$? Or $r = 0$? (Hint: plot points for a variety of triples $(r, s, t) = (r, s, 1 - r - s)$ with $0 \leq r, s, 1 - (r + s) \leq 1$.)

(b) Try the same using the 3-vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (Hint: first sketch the points you get with $t = 0$, then with $s = 0$, then with $r = 0$, and finally with $r = s = t = 1/3$.)

(c) Can you explain why your description in (a) applies to any three 2-vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ not on a common line? Use whatever physical or mathematical idea comes to mind. (Here is one approach: for $0 \leq t < 1$ check the equality $r\mathbf{a} + s\mathbf{b} + t\mathbf{c} = (1 - t)\mathbf{d}_{r,s} + t\mathbf{c}$ with a convex combination on the right where $\mathbf{d}_{r,s}$ is defined to be the convex combination $(r/(r + s))\mathbf{a} + (s/(r + s))\mathbf{b}$; this algebra works because $r + s = 1 - t > 0$. Interpret these convex combinations geometrically.)

(d) Is there a version for a triple of 3-vectors not all on a common line in space? Can you explain why it works?

