Solutions to Math 51 Quiz 4 Practice B

1. (10 points) What is the minimum value attained by the function $f(x,y) = y^2 - x + 5$ on the curve $x^3 + 3x - y^2 = 0$?

   You may assume that a global minimum exists.

   Let $g(x,y) = x^3 + 3x - y^2$. We compute the gradients

   $$\nabla f(x,y) = \begin{bmatrix} -1 \\ 2y \end{bmatrix}, \qquad \nabla g(x,y) = \begin{bmatrix} 3x^2 + 3 \\ -2y \end{bmatrix}.$$

   $\nabla g(x,y)$ is never zero because $3x^2 + 3$ is always positive. Thus any maximum must satisfy the following equations for some constant λ:

   $$-1 = \lambda(3x^2 + 3), \qquad 2y = \lambda(-2y), \qquad x^3 + 3x - y^2 = 0.$$

   If $y \neq 0$, then we can divide the second equation by $-2y$ to obtain $\lambda = -1$. Plugging this into the first equation implies $-1 = -3x^2 - 3$, or $0 = 3x^2 + 2$, which has no solutions.

   If $y = 0$, then the constraint equation implies $x(x^2+3) = 0$, so $x = 0$. This means $(0,0)$ is a candidate.

   Hence $(0,0)$ must be the minimum, and so the solution is $f(0,0) = 5$.

   Alternative solution: Note that the constraint equation implies $y^2 = x^3 + 3x$. Since $y^2 \geq 0$, we have $x^3 + 3x = x(x^2 + 3) \geq 0$. Since $x^2 + 3$ is always positive, $x \geq 0$ must hold on the constraint curve.

   To minimize $f(x,y) = y^2 - x + 5$ where $y^2 = x^3 + 3x$ and $x \geq 0$, we replace y^2 with $x^3 + 3x$, so $f(x,y) = x^3 + 3x - x + 5 = x^3 + 2x + 5$.

   Let $g(x) = x^3 + 2x + 5$. $g'(x) = 3x^2 + 2$ is always positive, so there is no critical point for $g(x)$. Since $x \geq 0$, we check for global min and max of $g(x)$ on the boundary. When $x = 0$, $g(0) = f(0,0) = 5$ is the global min; global max does not exist, as when $x \to \infty$, $g(x) = f(x,y) \to \infty$.

2. (2 points) **True or False:** All scalar-valued functions $f(x,y,z)$ has at least one critical point.

   Not always true: for example, the linear function $f(x,y,z) = x + y + z$ has no critical point.

3. (2 points) **True or False:** The circle $x^2 + y^2 = 4$ has two points at which the tangent line is vertical.

   This statement is **TRUE**. If the tangent line at a point (a,b) is vertical, the gradient at the point must be horizontal. Taking $f(x,y) = x^2 + y^2$, the circle is the level set of f at the level 4. Hence, the gradient is

   $$\nabla f(a,b) = \begin{bmatrix} 2a \\ 2b \end{bmatrix},$$

   and so, the gradient is horizontal (i.e. the second component is zero while the first component is nonzero) when $b = 0$. There are indeed two such points on the circle – $(-2,0)$ and $(2,0)$.

4. (3 points) Let

   $$f(x,y) = x^2 y - y^2 - 2x^2.$$

   $f(x,y)$ has at least one critical point (a,b) that is not $(0,0)$. Find the value of $f(a,b)$.

   a) -8 b) -4 c) 0 d) 4 e) 8

The critical points are where the gradient

$$\nabla f = \begin{bmatrix} 2xy - 4x \\ x^2 - 2y \end{bmatrix}$$

vanishes. The vanishing of the first entry says $0 = 2xy - 4x = 2x(y-2)$, so $x = 0$ or $y = 2$. Then using the vanishing of $x^2 - 2y$, when $x = 0$ we have $y = 0$, and when $y = 2$ we have $x^2 = 4$, so $x = \pm 2$. Hence, the critical points of f are $(0,0)$ and $(\pm 2, 2)$.

$$f(\pm 2, 2) = 8 - 4 - 8 = -4.$$

5. (3 points) Let S be the graph of the function $f(x,y) = \ln\left(\frac{y}{x}\right)$ where $x, y > 0$. Suppose the tangent plane to S at the point $P = (a,b,c)$ on S is parallel to the plane $2x - y + 2z = 0$. What are a, b and c?

S can also be viewed as the 0-level surface of $g(x,y,z) = f(x,y) - z$. Since

$$\nabla g = \begin{bmatrix} -1/x \\ 1/y \\ -1 \end{bmatrix},$$

the normal to the tangent plane at $P = (a,b,c)$ is given by

$$\nabla g(a,b,c) = \begin{bmatrix} -1/a \\ 1/b \\ -1 \end{bmatrix}.$$

The normal $\mathbf{n}$ to the plane $2x - y + 2z = 0$ is $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. $\mathbf{n}$ and $\nabla g(a,b,c)$ must be scalar multiple of each other if the tangent plane to S at P is parallel to $2x - y + 2z = 0$. So

$$\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = k \begin{bmatrix} -1/a \\ 1/b \\ -1 \end{bmatrix}.$$

We have

$$2 = -k/a, \;\; -1 = k/b, \;\; 2 = -k.$$

Since $k = -2$, $b = -k = 2$, and $a = -k/2 = 1$.

Since $P = (a,b,c)$ is on S, we can solve for c from the fact that $c = \ln \frac{b}{a} = \ln(2)$. So $P = (1, 2, \ln(2))$.