Self Organized Criticality and the Forest Fire Model

Dimitrios Loupas
Aristotle University of Thessaloniki
MSc Computational Physics - Computational Biophysics course

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Abstract

Many spatial extended systems in nature obey scaling laws (earthquakes from the motion of tectonic plates, forest fires etc), despite the fact that the physics of those systems might be very different they seem to behave in a similar manner. It turns out that these class of systems can be described by Self Organized Criticality (SOC) as introduced by P. Bak et al. [1] using a simple model called Sandpile model. In this paper we will review the idea of SOC and we will consider the forest fire model [6].

1 Introduction

In a seminal paper Per Bak et al. [1] introduced the idea of self organized criticality (SOC). The main goal of the paper was to introduce a mechanism in order to explain the occurrence of $1/f$ noise in various physical systems (resistors, luminosity of stars etc.), as well as, the appearance of self-similar fractal structures, which are observed in many spatially extended systems existing in nature (mountain landscapes, coastal lines etc.). In turbulence self-similarity appears in both space and time. The common characteristic of those systems are that the correlations, in time or space, behave as power laws, although the microscopic physics might be different. The explanation of the above phenomena is that spatially extended dissipative systems naturally evolve in a self organized critical state. Note here that the idea of criticality is very different form the criticality that one encounters in the theory of phase transitions. The difference is that the critical point in a phase transition is reached by tuning an external parameter, e.g the temperature in an Ising model, whereas here the system evolves naturally to the critical state. In this paper we start with a review of the idea of self organized criticality as introduced by P. Bak et al. in 1988 [1] via the sandpile model. We will reproduce the results of [1]. Then, we will consider the forest fire model which shows critical behaviour and we will reproduce the results of [6].

2 SOC from Sandpile Model

In order to demonstrate self organized criticality P.Bak et al. considered a very simple model called sandpile model (Figure 1).

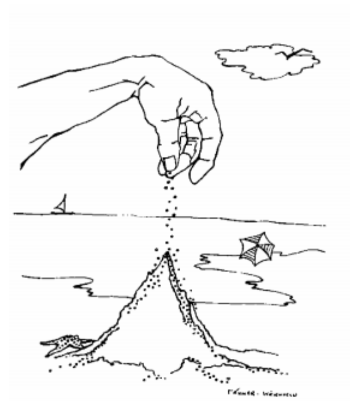

Figure 1: Physical illustration of Sandpile model. Input of new grains on the system results in the creation of avalanches. Figure taken from [2]

The model is essentially a d dimensional cellular automaton obeying some simple rules in each generation. We will consider the case of $2d$ and reproduce the results of [1]. Consider a $2d$ grid where each cell contains $z(x, y)$ number of grains. Then the idea is that when a cell exceeds a critical value z_c, a number of grains go to the neighbors of the cell. Hence each cell obeys the following rules:

If $z(x, y) > z_c$:

1. $z(x, y) \to z(x, y) - 2d$
2. $z(x \pm 1, y) \to z(x \pm 1, y) + 1$
3. $z(x, y \pm 1) \to z(x, y \pm 1) + 1$

where the update occurs for all cells and d is the dimension, in our case $d = 2$. Intuitively when a cell exceeds the critical value an avalanche will occur. When a cell spreads the grains to neighbour cells they might become unstable and spread their grains to their neighbours also, like a domino effect (Figure 2).

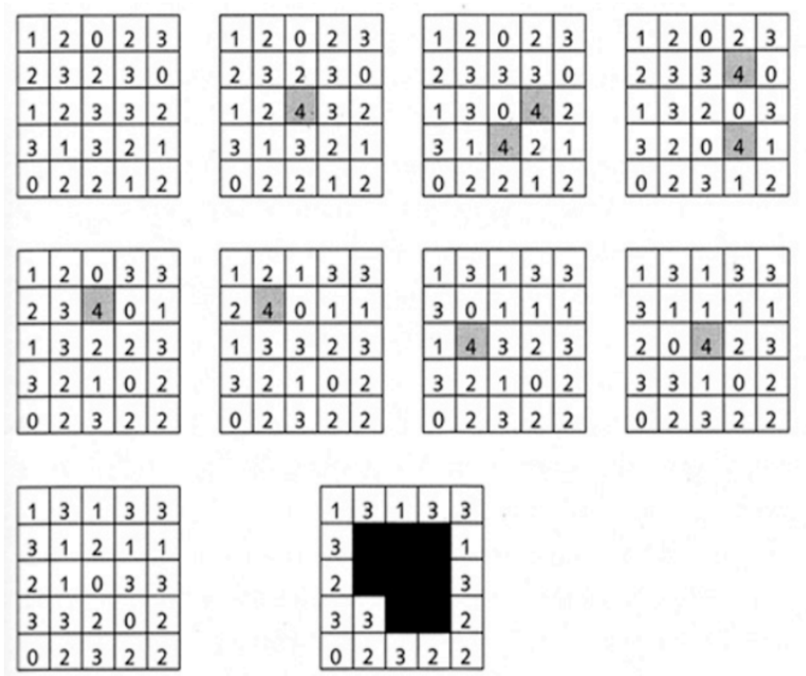

Figure 2: Example of avalanche. Figure taken from [2]

Thus we can have small or big avalanches depending on the state of the system. At the critical point we will have avalanches of all length scales limited only by the size of the system. For the simulations we considered a $2d$ array with size 50×50 and fixed boundary conditions, i.e $z = 0$ at the boundaries. The critical value was $z_c = 4$. The algorithm of the simulations is as follows:

1. For each t choose a random cell (x, y) and add a grain, $z(x, y) \to z(x, y) + 1$
2. if $z(x, y) > z_c$ use the rules described above for every cell and go to 1
3. else go to 1

The above algorithm is repeated for $T = 2000$ time steps. Initially we run the algorithm for $t = 10000$ in order to thermallize the system and reach the critical state. Then we actually begin to run the algorithm in order to collect statistics. We measured the response of the systems state to local perturbations (adding grains) by measuring the size of avalanches and fluctuation lifetimes (for how long the avalanche exists). The results are averaged over 200 samples. Below we plot the distribution of cluster (avalanche) sizes $D(s)$ in a log-log plot.

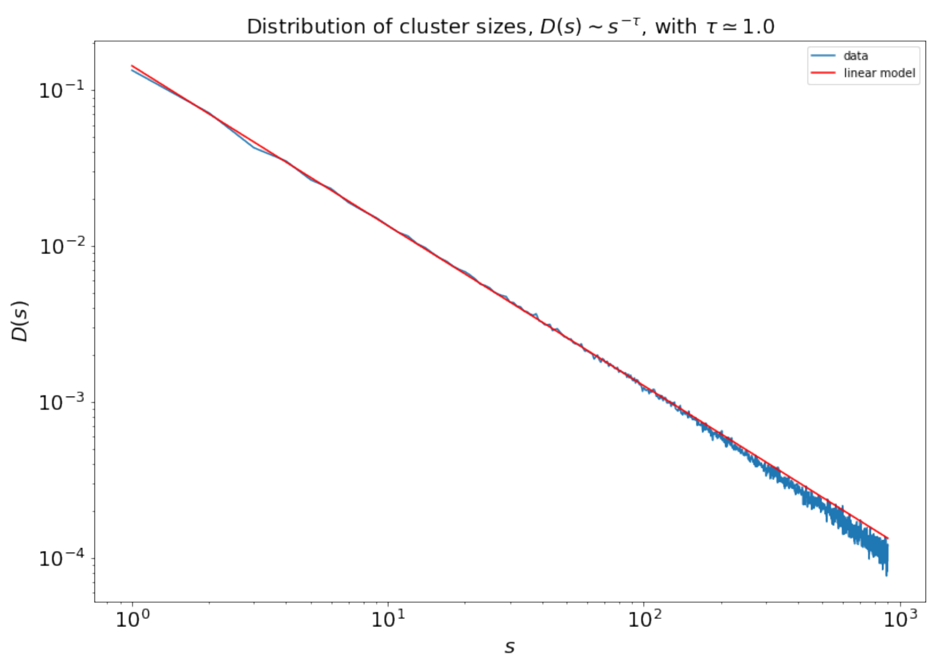

Figure 3

From the above figure we see that the distribution of clusters obeys a power law $D(s) \sim s^{-\tau}$, $\tau \simeq 1.0$. The fact that the curve is linear over two decades indicates that the system is at a critical point with a scaling distribution of clusters. The deviation of the straight line at the end is due to finite-size effects. Next we consider the distribution of fluctuation lifetimes, weighted by the averaged response s/t.

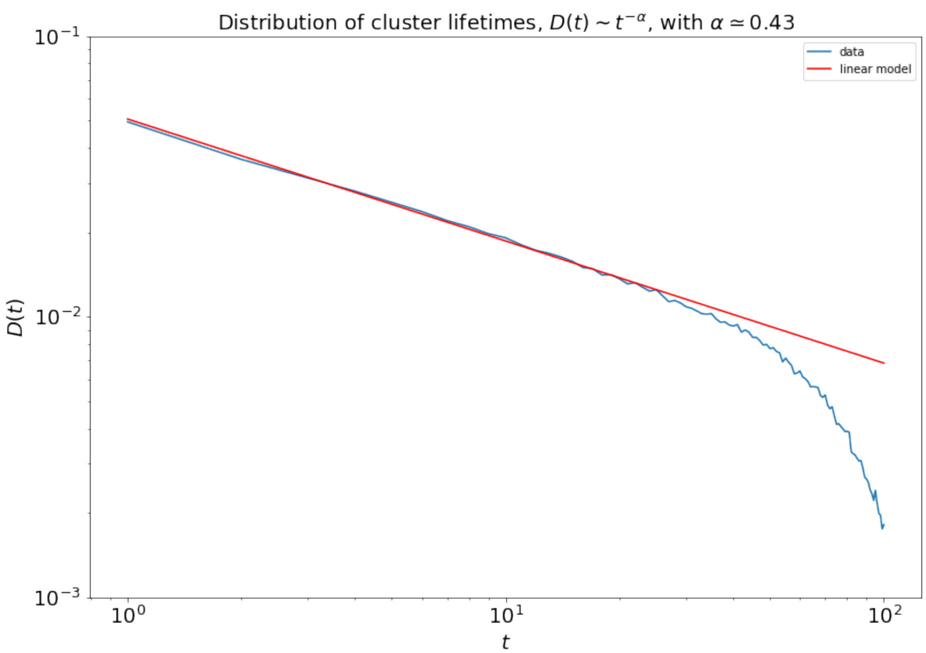

Figure 4

Again we observe a power law for the distribution of fluctuation lifetimes, $D(t) \sim t^{-\alpha}$, $\alpha \simeq -0.43$. Note that the curves for the lifetime distribution fit a power law only over a decade or so, whereas the cluster size distributions fit for at least two decades. This is due to the fact that the lifetime of a cluster is much smaller than its size. Our simulations agree with those of [1]. Per Bak's conclusion was the following:

"*We believe that the new concept of self-organized crticality can be taken much further and might be the underlying concept for temporal and spatial scaling in a wide class of dissipative systems with extended degrees of freedom*".

We will see in the next section another application by investigating the forest fire model.

3 Forest fire model

In this section we will introduce the Drossel-Schwabl forest fire model, [6], which shows SOC behaviour. Initially we will see some aspects of the forest fire model and and then we will perform simulations based on the Drossel-Schwabl forest fire model.

3.1 P. Bak Forest Fire model

The forest fire model introduced again by P.Bak in 1990, [3], as an attempt to answer the following question: "*How uniform energy injection results in a fractal energy dissipation?*" which is seen in turbulence. Motivated by the concept of SOC they speculated that fractal energy dissipation is a manifestation of a critical state. In order to demonstrate fractal energy dissipation, under uniform energy input, they considered a simple forest fire model. The forest fire model is again a cellular automaton, defined in a d-dimensional lattice and each cell is occupied by either a tree, a burning tree or it is empty. The model evolves according to the following rules:

1. trees grow with a small probability p from empty sites at each time step.

2. trees on fire will burn down at the next time step.

3. the fire on a site will spread to trees at its nearest neighbor sites at the next time step.

They introduced a correlation length $\xi(p)$ and based on an energy conservation argument they found that

$$\xi(p) \sim p^{-\nu}, \nu = 1/(d - D)$$

where d is the dimension of the lattice and D is the fractal dimension of the fire. Performing simulations in 2 and 3 dimensions they found the fractal dimension D and then the exponent ν. Having this relation of the correlation length $\xi(p)$ they concluded that the forest fire model is critical for $p \to 0$ and energy dissipates in a fractal. However, further computer simulations [4], [5], showed that the model is rather deterministic than critical in the limit $p \to 0$. Thus, the forest fire model is not a self organized critical system. In the last sentence of [5] they mentioned that the addition of a second parameter - a lightning probability f - to the system will make it critical. This model is the The Drossel-Schwabl Forest fire model [6] and will be described in the next section.

3.2 The Drossel-Schwabl Forest fire model

In order for the forest fire model to be critical we must observe fires in all sizes, up to the correlation length. The model proposed by P. Bak does not have this property as shown in [4], [5], instead there is a typical length $(1/p)$ which determines the size of fires and the model behaviour is deterministic. Adding one more parameter, a lightning probability f will make the system critical. Hence the rules for the Drossel-Schwabl Forest fire model [6] are the following:

1. trees grow with a small probability p from empty sites at each time step.

2. trees on fire will burn down at the next time step.

3. the fire on a site will spread to trees at its nearest neighbor sites at the next time step.

4. a tree without a burning nearest neighbor becomes a burning tree during one time step with probability f.

When the system reaches the steady state the properties of it depend on the values of the parameters. If ρ_t, ρ_e and ρ_f denote the mean density of trees, of empty sites, and of burning trees in the steady state we have

$$\rho_t + \rho_e + \rho_f = 1 \tag{1}$$

and

$$\rho_f = p\rho_e \tag{2}$$

with the last equation stating that the mean number of growing trees equals the mean number of burning trees in the steady state. The forest fire model can have large scale structures when the fire density is small and therefore p approaches zero. Moreover the lighting probability must satisfy

$$f \ll p \tag{3}$$

Large scale structures are important because at the critical point we want clusters of all sizes. Additionally the model becomes critical when small and large forest clusters burn down in the same way. Hence, again p has to be small, that even the largest forest cluster burns down rapidly, before new trees grow at its edge. The dynamics of the model depends only on the ratio f/p but not on f and p separately. Suppose that we decrease f and p by the same factor, the overall time scale of the system is also changed by this factor, but not the number of trees that grow between two lightnings and therefore not the size distribution of forest clusters and of fires. The condition that forest clusters burn down rapidly is

$$p \ll T(s_{max})^{-1} \tag{4}$$

where $T(s_{max})$ is the time the fire needs to burn down a large forest cluster. Putting those together we obtain a double separation of time scales

$$T(s_{max}) \ll p^{-1} \ll f^{-1} \tag{5}$$

which is the condition for self organized criticality in the forest fire model.

3.2.1 Scaling laws

In this section we present some quantities that obey scaling laws and we will simulate them in the next section in order to determine the exponent of the power law. Following [6] we get the following scaling law of the number of clusters of size s

$$N(s) \sim s^{-\tau}$$

In [6] they introduce cut off functions due to the finite size effects, we keep it simple here. Moreover we introduce the cluster radius $R(s)$ (radius of gyration) which is the mean distance of the trees in a cluster from their center of mass. It is related to the cluster size s via

$$R(s) \sim s^{1/\mu} \tag{6}$$

where $1/\mu$ is the fractal dimension. In the next section we perform simulations on a $2d$ lattice in order to obtain the critical exponents τ and μ,

3.2.2 Simulation results

In this section we present our simulation results. We define our forest fire model in 2 dimensions using the following rules introduced in [7] :

1. Choose an arbitrary site in the system. If it is not occupied by a tree, proceed with rule 2. If it is occupied by a tree, then ignite the tree and burn down the forest cluster to which the tree belongs. While burning the trees, evaluate the properties of the clusters size and radius. Proceed with rule 2

2. Choose p/f arbitrary sites in the system and grow a tree at all chosen empty sites. Proceed with rule 1

By these rules, time scale separation is perfectly realized. We consider a lattice with length, $L = 100$ and $p/f = 125$. In order to reach the steady state initially we run the simulation for $t = 1000$ time steps and then for a total of $T = 40000$ time steps to collect statistics. The quantities that we measure are the number of clusters $N(s)$ and the averaged cluster radius $R(s)$ for each cluster with size s. Actually we measured the forest clusters struck by lightning, meaning $sN(s)$, so the exponent that we get is $\tau - 1$ instead of τ.
To measure $R(s)$ of cluster with size s we did the following:

- Given a cluster with s trees we compute the center of mass coordinates x_{cm}, y_{cm} of the cluster from

$$x_{cm} = \frac{1}{s}\sum_{i=1}^{s} x_i$$

$$y_{cm} = \frac{1}{s}\sum_{i=1}^{s} y_i$$

- Then we defined the cluster squared radius as the following average:

$$R(s)^2 = \frac{1}{s}\sum_{i=1}^{s}\left[(x_i - x_{cm})^2 + (y_i - y_{cm})^2\right]$$

  and took the square root.

Below we plot the distribution of burned clusters sizes $sN(s)$ in a log-log plot.

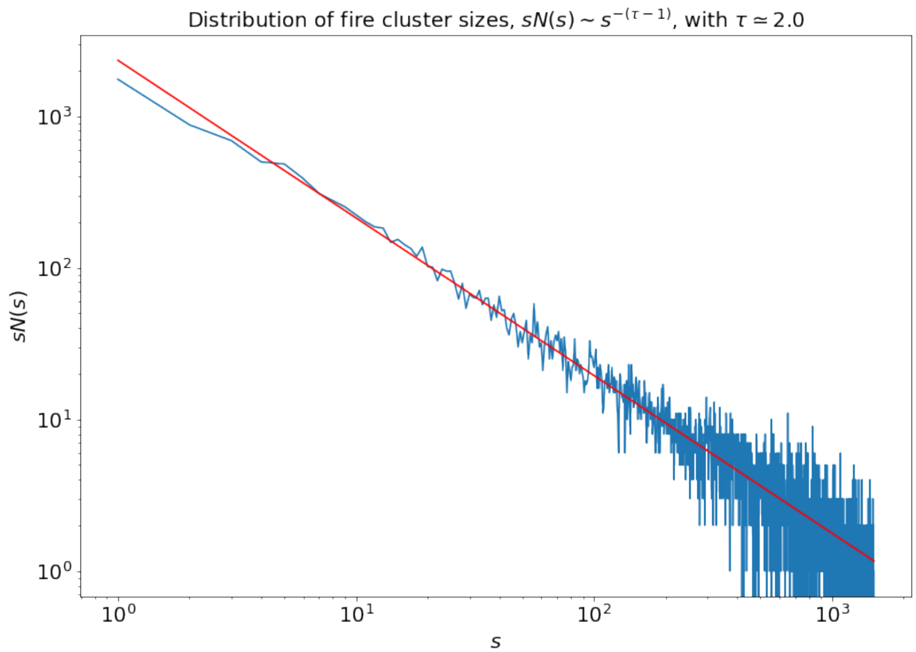

Figure 5

From the above figure we see that the distribution of clusters obeys a power law $sN(s) \sim s^{-(\tau-1)}$, $\tau \simeq 2.0$

Next we consider the distribution of mean cluster radius with size s.

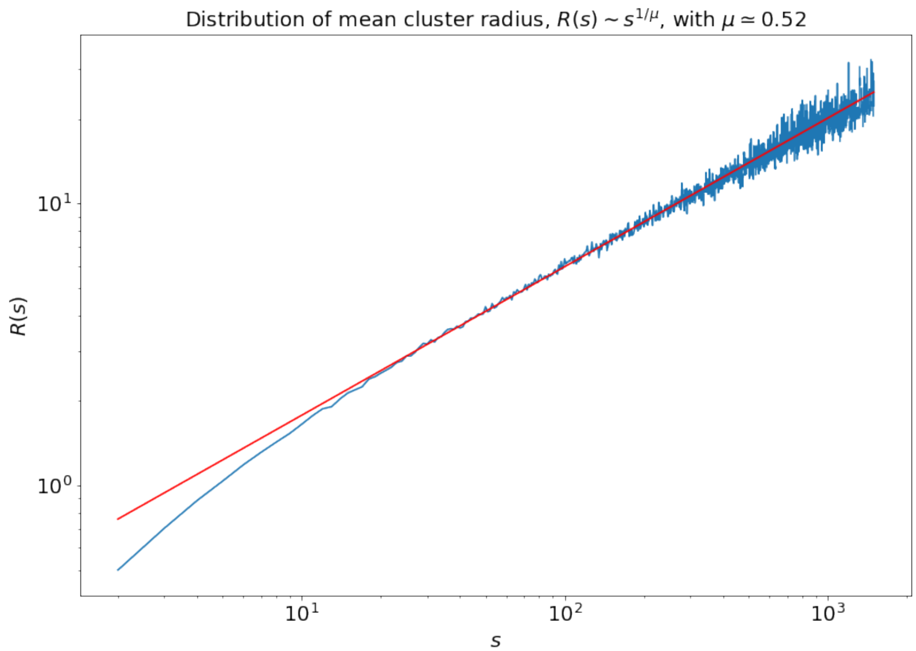

Figure 6

Again we observe a power law for the distribution of mean cluster radius, $R(s) \sim s^{1/\mu}$, $\mu \simeq 0.52$ and the fractal dimension $d = 1/\mu \simeq 1.89$. Our result for the exponent τ is in agreement with [8] but for the fractal dimension they found $d \simeq 1.96$. Maybe this has to do with the grid size of our simulations. They used $L = 16384$ whereas we used only $L = 100$ due to low computational power.

4 Conclusion

Self organized criticality is a very powerful idea which can be used to describe a large variety of physical systems that exhibit spatio-temporal scaling behaviour. By performing numerical simulations we try to reproduce the results from the two papers, [1] and [8]. Our results are the same for [1] but for [8] we found a different fractal dimension, probably due to the small grid that we performed the simulation. The simulations performed using C++ and Python programming languages.

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