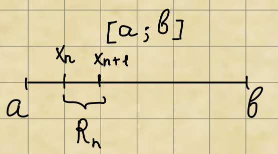
Вычматы (лекция 2)

Численное интегрирование.

$$J = \int_a^b f(x)\,dx$$

3 осн. ф-лы (квадратурные ф-лы):

- прямоугольника
- трапеции
- Симпсона



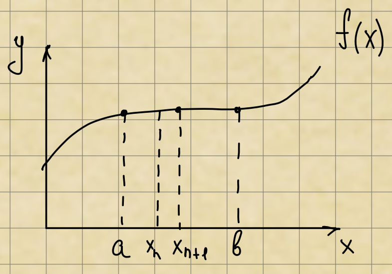
$h_n = x_{n+1} - x_n$; x_n - узлы, h_n - шаг инт-ия

$$\sum_{n=0}^{N-1} h_n = b-a$$

$$f(x_n) = f_n$$

$$J = \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} f(x)\,dx = \sum_{n=0}^{N-1} J_n \quad ; \quad J_n = \int_{x_n}^{x_{n+1}} f(x)\,dx$$

1 Ф-ла пр-ика



$f_{n+1/2} = f\left(x_n + \frac{h_n}{2}\right)$ $\quad J_n = h_n + f_{n+1/2}$ ср. прям.

$f_n = f(x_n)$ $\quad J_n = h_n \cdot f_n$ лев. прям.

$f_{n+1} = f(x_{n+1})$ $\quad J_n = h_n \cdot f_{n+1}$ прав. прям.

$$\boxed{J = \sum_{n=0}^{N-1} h_n f_{n+1/2}}$$

② Формула трапеции.

$$f(x) \approx f(x_{n+1/2}) \ ; \ J = \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} f(x_{n+1/2})\, dx = \sum_{n=0}^{N-1} f(x_{n+1/2})\, x \Big|_{x_n}^{x_{n+1}} =$$

$$= \sum_{n=0}^{N-1} f(x_{n+1/2}) \underbrace{(x_{n+1} - x_n)}_{h_n} = \sum_{n=0}^{N-1} f(x_{n+1/2})\, h_n$$

$$f(x) \approx f_n + \frac{f_{n+1} - f_n}{x_{n+1} - x_n} (x - x_n)$$

$$J_n = \int_{x_n}^{x_{n+1}} f(x)\, dx = \int_{x_n}^{x_{n+1}} \left[f_n + \frac{f_{n+1} - f_n}{x_{n+1} - x_n} (x - x_n) \right] dx = f_n \underbrace{(x_{n+1} - x_n)}_{h_n} + \frac{f_{n+1} - f_n}{x_{n+1} - x_n} \Bigg[$$

$$\frac{x_{n+1}^2 - x_n^2}{2} - x_n (x_{n+1} - x_n) \Bigg] = f_n h_n - \frac{f_{n+1} - f_n}{2 h_n} \left[x_{n+1}^2 - x_n^2 - 2 x_n x_{n+1} + \right.$$

$$\left. + 2 x_n^2 \right] = f_n h_n + \frac{f_{n+1} - f_n}{2 h_n} \left[x_{n+1} - x_n \right]^2 = f_n h_n + \frac{f_{n+1} - f_n}{2} h_n =$$

$$= \frac{1}{2} h_n \left[f_n + f_{n+1} \right]$$

$$J_n = \frac{h_n}{2} \left(f_n + f_{n+1} \right) \qquad J = \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} \frac{h_n}{2} \left(f_n + f_{n+1} \right)$$

③ Формула Симпсона.

$$J_n = \int_{x_n}^{x_{n+1}} \left[f_{n+1/2} + \frac{f_{n+1} - f_n}{h_n} (x - x_{n+1/2}) + 2 \cdot \frac{f_{n+1} - 2 f_{n+1/2} + f_n}{h_n^2} (x - x_{n+1/2})^2 \right] dx =$$

$$= f_{n+1/2}\, h_n + \frac{f_{n+1} - f_n}{h_n} \left(\frac{x_{n+1} - x_n^2}{2} - x_{n+1/2}\, h_n \right) + \frac{2}{h_n^2} \left(f_{n+1} - 2 f_{n+1/2} + f_n \right) \cdot$$

$$\cdot \frac{(x - x_{n+1/2})^3}{3} \Bigg|_{x_n}^{x_{n+1}} = f_{n+1/2}\, h_n + \frac{f_{n+1} - f_n}{h_n} \left(\frac{x_{n+1}^2 - x_n^2}{2} - x_{n+1/2}\, h_n \right) + \frac{2}{3 h_n^2} \left(f_{n+1} - 2 f_{n+1/2} + \right.$$

$$+ f_n)\left[\left(x_{n+1} - x_{n+1/2}\right)^3 - \left(x_n + x_{n+1/2}\right)^3\right] = f_{n+1/2} \cdot h_n + \frac{f_{n+1} - f_n}{2h_n}\left[x_{n+1}^2 - x_n^2 -\right.$$

$$\left. - 2h_n\left(\frac{h_n}{2} + x_n\right)\right] + \frac{2}{3h_n^2}\left(f_{n+1} - 2f_{n+1/2} + f_n\right)\left[\frac{h_n^3}{8} - \left(-\frac{h_n}{2}\right)^3\right] =$$

$$= f_{n+1/2}\, h_n + \frac{2}{3h_n^2}\left(f_{n+1} - 2f_{n+1/2} + f_n\right)\frac{h_n^3}{4} = f_{n+1/2} h_n + \frac{1}{6}\left(f_{n+1} - 2f_{n+1/2} +\right.$$

$$\left. + f_n\right) h_n = \frac{h_n}{6}\left(f_{n+1} + 4f_{n+1/2} + f_n\right)$$

$$\boxed{J_n = \frac{h_n}{6}\left(f_{n+1} + 4f_{n+1/2} + f_n\right)}$$

$$J = \sum_{n=0}^{N-1} \frac{h_n}{6}\left(f_{n+1} + 4f_{n+1/2} + f_n\right)$$

Погрешности квадратурных формул

Прав. прям.

$$f(x) = f(x_{n+1}) + f'(x_{n+1})(x - x_{n+1}) + \frac{f''(x_{n+1})}{2}(x - x_{n+1})^2 + \ldots$$

$$J_n = \int_{x_n}^{x_{n+1}} f(x)\,dx = f(x_{n+1})h_n + f'(x_{n+1})\frac{(x - x_{n+1})^2}{2}\Bigg|_{x_n}^{x_{n+1}} + \frac{f''(x_{n+1})}{6}\cdot$$

$$\cdot (x - x_{n+1})^3\Bigg|_{x_n}^{x_{n+1}} + \ldots = f(x_{n+1})h_n + \frac{f'(x_{n+1})}{2}\left[(x_{n+1} - x_{n+1})^2 - (\underbrace{x_n - x_{n+1}}_{-h_n})^2\right] +$$

$$+ \frac{f''(x_{n+1})}{6}\left[(x_{n+1} - x_{n+1})^3 - (\underbrace{x_n - x_{n+1}}_{-h_n})^3\right] + \ldots = f(x_{n+1})h_n - \frac{f'(x_{n+1})}{2}h_n^2 +$$

$$+ \frac{f''(x_{n+1})}{6}h_n^3 + \ldots \qquad [f(x_{n+1}) = f_{n+1}]$$

прибл. $\tilde{J}_n = h_n \cdot f_{n+1}$

$$\left|\tilde{J}_n - J_n\right| = \left|h_n f_{n+1} - h_n f_{n+1} + \frac{f'_{n+1} h_n^2}{2} - \frac{f''_{n+1} h_n^3}{6} + \ldots\right| = \overbrace{\frac{f'_{n+1} h_n^2}{2}}^{\text{Гл. член погр-ти}} + \ldots$$

$$|\tilde{J} - J_n| = |\tilde{J}_n - J_h| N = \left[\underset{h=const}{N = \frac{b-a}{h}}\right] = |\tilde{J}_n - J_n| \cdot \frac{b-a}{h} = f'_{n-1}\frac{h^2}{2} \cdot \frac{(b-a)}{h} =$$

$$= f'_{n-1}\frac{h(b-a)}{2}$$ – ф-ла первого порядка точности

Оценка погр-ти

Ф-ла лев. и прав. прям.

$$|\tilde{J} - J| \leq \frac{M_1(b-a)h}{2}, \quad M_1 = \max_{[x_n, x_{n+1}]} |f'|$$

Ф-ла центр. прямоуг.

$$|\tilde{J} - J| \leq \frac{M_2(b-a)h^2}{24}$$

Ф-ла трапеций:

$$|\tilde{J} - J| \leq \frac{M_2(b-a)h^2}{12}$$

Ф-ла Симпсона:

$$|\tilde{J} - J| \leq \frac{M_4(b-a)h^4}{180}$$

Примеры:

№1

$$\int_{-1}^{1} \frac{x^2}{2} \operatorname{arcctg} x \, dx \quad h - ? \text{ (м-д ср. пр.)} \quad \varepsilon = 10^{-4}$$

$$|\tilde{J} - J| \leq \frac{M_2(b-a)h^2}{24} < \varepsilon$$ – для центр. пр.

$$f' = x \cdot \operatorname{arcctg} x + \frac{x^2}{2} \cdot - \frac{1}{1+x^2} = x \cdot \operatorname{arcctg} x - \frac{x^2}{2(1+x^2)}$$

$$f'' = \operatorname{arcctg} x - \frac{1}{1+x^2} \cdot x + \frac{x^3}{(1+x^2)^2} - \frac{x}{(1+x^2)}$$

$$= \operatorname{arcctg} x + \frac{x^3 - 2x - 2x^3}{(1+x^2)^2} = \operatorname{arcctg} x - \frac{x^3 + 2x}{(1+x^2)^2}$$

$$|f''| < \frac{5\sqrt{3}}{16} - \frac{\pi}{6} = M_2$$

<u>№2</u>

$$f(x) = \frac{1}{4}x^6 - \frac{1}{2}x^5 + 12x^4 - 8x^2$$ метод Симпсона

$a = 1$; $b = 2$; $\varepsilon = 10^{-4}$ $\qquad N = \frac{b-a}{h}$

$$|\tilde{J} - J| \leq \frac{M_4 (b-a) h^4}{180}$$

$$f' = \frac{3}{2}x^5 - \frac{5}{2}x^4 + 48x^3 - 16x$$

$$f'' = \frac{15}{2}x^4 - 10x^3 + 144x^2 - 16$$

$$f''' = 30x^3 - 30x^2 + 288x$$

$$f'''' = 90x^2 - 60x + 288$$

$$f^{V} = 180x - 60 = 0 \Rightarrow x = 1/3.$$

$$f^{IV}(1) = 318 \qquad f^{IV}(2) = 528 = M_4$$

$$N \geq 13{,}1 = 14$$

Контроль за точностью вычислений.
(практический)

Метод Симпсона

$$J_T = \tilde{J}_h + c \cdot h^4$$

$$J_T = \tilde{J}_{2h} + 16ch^4$$

$$|J_{2h} - J_h| \approx 15\underbrace{ch^4}_{\tilde{J}_T - \tilde{J}_h}$$

$$|J_{2h} - J_h| \approx 15|J_T - \tilde{J}_h|$$

$$|J_T - \tilde{J}_h| \approx \frac{|J_{2h} - J_h|}{15} < \varepsilon$$

Метод оценки Рунге

$$\int_0^5 \frac{7}{x^2+1}\,dx$$

метод трапеций

$N = 10$