1. DIFFERENTIAL EQUATIONS SERIES SOLUTION OF A DIFFERENTIAL EQUATIONS.

A differential Equation is an equation Containing a derivative or derivatives of unknown functions.

A Solution of a differential equation is a relation between x and y which does not contain any derivative, eg.
Verify that → $y = A\cos x + B\sin x$ is a Solution of a differential equation $y'' + y = 0$.

Soln.

$$y = A\cos x + B\sin x$$
$$y' = -A\sin x + B\cos x.$$
$$y'' = -A\cos x - B\sin x.$$

Substitute in the D.E

$$y'' + y = 0$$

Then,

$$-A\cos x - B\sin x + A\cos x + B\sin x = 0.$$

Analytic Function.

A function $f(x)$ is said to be analytic if it is defined every where, Example All polynomials, e^x, $\cos x$, $\sin x$, $\cosh x$ & $\sinh x$.

A rational function is analytic at every where exception when the denominator is zero.

Ordinary and Singular Points.

A point $x = x_0$ is called an ordinary point of the differential Equation.

$$y'' + P(x)y' + Q(x)y = 0.$$

if both $P(x)$ and $Q(x)$ are analytic at x_0.

if either (or both) of these functions $P(x)$ and $Q(x)$ is not analytic at x_0 then x_0 is called a singular point.

Example.

Consider a Differential equation

$$(x-1)y'' + xy' + \frac{1}{x}y = 0.$$

In its normalized form is :-

$$y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0$$

$$P(x) = \frac{x}{x-1}, \quad Q(x) = \frac{1}{x(x-1)}$$

at $x_0 = 1$ and $x_0 = 0$.

at $x_0 = 1$ is a Singular Point of DE., also $x_0 = 0$ is a Singular Point.

When $x_0 = 2$ is analytical Point.

Both $P(x)$ and $Q(x)$ are not analytic at $x_0 = 1$, $x_0 = 0$ is a Singular Point.

Theorem;

If x_0 is an ordinary Point of the differential Equation has two non-trivial linearly independent power series solution of the form of

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Converges is some interval $|x-x_0| < R$

Example for $x_0 = 0$.

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n \text{ become } y = \sum_{n=0}^{\infty} a_n x^n,$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where,

a_0, a_1 and $a_3 \cdots \to$ are Constants.

Example 1.

Solve the differential Equation

$$y'' - xy' - y = 0 \quad \text{at} \quad x_0 = 0.$$

Soln.

$P(x) = -x, \quad Q(x) = -1.$

At $x=0$ both $P(x)$ and $Q(x)$ are analytical

Then we assume the solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Differentiate the assumed solution.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} . (x').$$

$$\to y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

Substitute y'', y' in Differential Equation.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

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Make Power of x the same.

Let $k = n-2$ in the 1^{st} term and $k = n$ in the 2^{nd} and 3^{rd} term, in the $k = n-2$, $n = k+2$.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^{k} - \sum_{k=1}^{\infty}k a_k x^{k} - \sum_{k=0}^{\infty}a_k x^{k} = 0.$$

$$2a_2 + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^{k} - \sum_{k=1}^{\infty}k a_k x^{k} - a_0\sum_{k=1}^{\infty}a_k x^{k} = 0.$$

$$2a_2 - a_0 = 0.$$

$$\sum_{k=1}^{\infty}\left[(k+2)(k+1)a_{k+2} - k a_k - a_k\right]x^{k} = 0.$$

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \tfrac{1}{2}a_0.$$

$$(k+2)(k+1)a_{k+2} - k a_k - a_k = 0$$

$$(k+2)(k+1)a_{k+2} - (k+1)a_k = 0$$

Then

$$(k+2)\cancel{(k+1)}a_{k+2} = \cancel{(k+1)}a_k$$

$$(k+2)a_{k+2} = a_k.$$

Now we have;

$$a_{k+2} = \frac{a_k}{k+2}$$

This relation is called Reccurence Relation.

$k=1$, $a_3 = a_1/3$

$k=2$ $a_4 = a_2/4 = \dfrac{\frac{1}{2}a_0}{4} = a_0/8$

$k=3$, $a_5 = a_3/5 = a_1/15$

$k=4$ $a_6 = a_4/6 = a_0/48$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + a_1/3\, x^3 + a_0/8\, x^4 + \frac{a_1 x^5}{15} + \frac{a_0}{48} x^6 + \cdots$

Then;

$$y = a_0 \underbrace{\left[1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots\right]}_{y_1} + a_1 \underbrace{\left[x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots\right]}_{y_2}$$

$\therefore$ The Solution of the given DE is:-

$$y(x) = A\left[1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots\right] + B\left[x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots\right]$$

where A and B are orbitary Constants.

Example 2;

Solve the Differential Equation.

$$y'' - 2xy = 0 \quad \text{at} \quad x_0 = 0.$$

Soln.

$P(x) = 0$, $Q(x) = -2x$ at $x=0$, both $P(x)$ and $Q(x)$ are analytic; $x=0$, is an ordinary Point of the DE then we assumed the Solution of the form of

$$y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute y'' and y in the Differential Equation.

$y'' - 2xy = 0$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Make power of x the same.

Let $k=n-2$ in the 1st term and $k=n+1$ in 2nd term.

So $k=n-2$, $n=k+2$, $k=n+1$, $n=k-1$.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - 2\sum_{k=1} a_{k-1}x^k = 0$$

Take out when $k=0$.

$$2a_2 + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k - 2\sum_{k=1}^{\infty} a_{k-1}x^k = 0.$$

$$2a_2 = 0$$

and.

$$\sum_{k=1}^{\infty}\left[(k+2)(k+1)a_{k+2} - 2\sum_{k=1}^{\infty} a_{k-1}\right]x^k = 0$$

$$(k+2)(k+1)a_{k+2} - 2a_{k-1} = 0.$$

Make a_{k+2} the subject.

$$a_{k+2} = \frac{2a_{k-1}}{(k+2)(k+1)}.$$

where; $k = 1, 2, 3, 4, \ldots\ldots$

From $2a_2 = 0$, $a_2 = 0$.

$k=1$ $\quad a_3 = \frac{2a_0}{6} = a_0/3.$

$k=2$ $\quad a_4 = \frac{2a_1}{12} = a_1/6$

$k=3 \quad a_5 = \frac{2a_2}{20} = 0$

$k=4 \quad a_6 = \frac{2a_3}{30} = a_0/45$

$k=5 \quad a_7 = 2a_4/42 = a_1/126.$

Substitute $a_1, a_2, a_3, a_4, \ldots$ in the assummed soln.

$y = a_0 + a_1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \ldots$

$= a_0 + a_1x + 0 + \frac{a_0x^3}{3} + \frac{a_1x^4}{6} + 0 + \frac{a_0x^6}{45} + \frac{a_1x^7}{126} + \ldots$

$y_1 = a_0\left[1 + x^3/3 + x^6/45 + \ldots\right] + a_1\left[x + x^4/6 + x^7/126 + \ldots\right]$

$\therefore$ The Solution of the given D.E is

$$y(x) = A\left[1 + \frac{x^3}{3} + \frac{x^4}{45} + \ldots\right] + B\left[x + \frac{x^4}{6} + \frac{x^7}{126} + \ldots\right]$$

where A and B are orbitary Constants.

10. $(1-x^2)y'' - 2xy' + 30y = 0$

EXERCISE.

Solve the following Differential Equation. at $x=0$.

1. $y'' + 2xy = 0$
2. $y'' + x^2y = 0$
3. $(x-1)y'' - xy' + y = 0$
4. $y'' + xy' + y = 0$
5. $y'' + y' + xy = 0$
6. $(1-x^2)y'' - 2xy' + 2y = 0$
7. $y'' + (4 + x^2)y = 0$
8. $y'' - 4xy' + (4x^2-2)y = 0$
9. $y'' + 3xy' + 2y = 0.$

SOLUTION OF DIFFERENTIAL EQUATION BY FROBENIUS METHOD.

If functions defined by $\phi_1 = (x-x_0)P(x)$ and $\phi_2 = (x-x_0)^2 Q(x)$ are analytic at x_0, then is called a <u>regular singular points.</u>

If either (both) ϕ_1 and ϕ_2 is not analytic at x_0, then is ~~x_0~~ is called an <u>irregular singular point.</u>

Example.

Find and classify singular points in the differential equation (DE)

$$x^2(x-2)^2y'' + 2(x-2)y' + (x+1)y = 0$$

Soln.

$$\frac{\cancel{x^2(x-2)^2}y''}{\cancel{x^2(x-2)^2}} + \frac{2(x-2)y'}{x^2(x-2)^2} + \frac{(x+1)y}{x^2(x-2)^2} = 0$$

From Nomallization Form we have;

$$y'' + \frac{2}{x^2(x-2)}y' + \frac{x+1}{x^2(x-2)^2}y = 0$$

$$P(x) = \frac{2}{x^2(x-2)}, \qquad Q(x) = \frac{x+1}{x^2(x-2)^2}$$

$$\phi_1 = (x-x_0)P(x), \qquad \phi_2 = (x-x_0)^2 Q(x).$$

Equate denominate equal to zero.

$$x^2(x-2) = 0 \qquad \text{and} \qquad x^2(x-2)^2 = 0$$

$$x = 0 \text{ or } x = 2, \qquad x = 0, \; x = 2.$$

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Classification

Singular points are $x=0$ and $x=2$.

For $x=0$.

$$Q_1 = \frac{x \cdot 2}{x^2(x-2)} = \frac{2}{x(x-2)} = \frac{2}{0} = \infty$$

not analytic.

Q_1 is not analytic at $x_0=0$.

$\therefore$ $x=0$ is an irregular Singular Point.

$$Q_2 = \frac{x^2(x+1)}{x^2(x-2)^2} = \frac{x+1}{(x-2)^2} = 1/4$$ is analytic.

For $x_0 = 2$.

$$Q_1 = (x-x_0)\, P_x.$$

$$Q_1 = \frac{(x-2)\times 2}{x^2(x-2)} = 2/x^2 = 1/2$$ Analytic.

For Q_2, when $x_0 = 2$.

$$Q_2 = (x-x_0)^2 Q(x).$$

$$Q_2 = \frac{(x-2)^2(x+1)}{x^2(x-2)^2}$$

$$= \frac{x+1}{x^2} = 3/4$$ is Analytic.

Since Q_1 and Q_2 are analytic at $x_0=2$; $x=2$ is a regular Point.

THEOREM.

→ If x_0 is a regular Singular Point of the D.E Then the D.E has at least one non-trivial

Soln of the form of :-

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

INDICIAL ROOTS.

→ In general when using assumption $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ is solving 2 order D.E the

indicial Equation is a quadratic equation in resulting from equating to zero the total coefficients of the Lowest power of x. The values of r obtained are called indicial roots.

Case 1: If r_1 and r_2 are distinct and do not differ by an integer, then there exist two Linearly independent

Soln of the form of

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}, \quad a_0 \neq 0$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} \quad b_0 \neq 0$$

General Solution of the D.E is

$$y(x) = y_1(x) + y_2(x)$$

Case 11:

If $r_1 - r_2 = N$ where N is an integer then there exist two Linearly independent Solutions of the form of $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1} \quad a_0 \neq 0$

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$$y_2(x) = k y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2} \qquad b_0 \neq 0$$

Case III If $r_1 = r_2 = r$ (double roots), there always exist two linearly independent solns of the form of

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1} \qquad a_0 \neq 0$$

$$y_2(x) = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2} \qquad b_0 \neq 0$$

Example.

Solve the DE.

$$2xy'' + y' + y = 0 \quad \text{at} \quad x_0 = 0.$$

Soln.

$$y'' + \frac{y'}{2x} + \frac{y}{2x} = 0$$

$$P(x) = \frac{1}{2x}, \quad Q(x) = \frac{1}{2x} \quad \text{at } x_0 = 0.$$

$P(x)$ and $Q(x)$ are not analytic i.e $x=0$ is a singular point.

$$Q_1 = (x - x_0) P(x)$$

$$= x - \frac{1}{2x}$$

$$= \frac{1}{2}. \quad \text{Is analytic.}$$

$x_0 = 0$, is a regular S. Point.

We assume a soln of the form of.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= a_0 x^r + a_1 x^{1+r} + a_2 x^{2+r} + a_3 x^{3+r} + \cdots$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} + a_4 x^{r+4} + \cdots + a_n x^{n+r} + \cdots \quad (1)$$

$$= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + a_4 x^{r+4} + \cdots + a_n x^{n+r} + \cdots$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = r a_0 x^{r-1} + (r+1) a_1 x^r + (r+2) a_2 x^{r+1} + \cdots + (n+r) a_n x^{n+r-1} + \cdots$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$= r(r-1)a_0 x^{r-2} + (r+1)r a_1 x^{r-1} + (r+2)(r+1) a_2 x^r + \cdots + (n+r)(n+r-1) a_n x^{n+r-2} \quad \text{(iii)}$$

$$2xy'' = 2\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1}$$

$$= 2r(r-1)a_0 x^{r-1} + 2(r+1)r a_1 x^r + 2(r+2)(r+1) a_2 x^{r+1} + \cdots + 2(n+r)(r-1) a_n^{n+r-1} \quad \text{(iv)}$$

Substitute eq (i) (ii) (iii) (iv) in the given D.E and find the sum of the coefficient of the lowest term of x powers of x.

$$r a_0 + 2r(r-1)a_0 = 0$$
$$a_0 (r + 2r(r-1)) = 0$$
$$a_0 (r + 2r^2 - 2r) = 0$$

Since $a_0 \neq 0$ then;

$$2r^2 - 2r + r = 0.$$
$$2r^2 - r = 0$$

By Applying Quadric Equations.

$r = 0$ or $1/2$

$r_1 = 0$ and $r_2 = 1/2$.

Take Sum of n^{th} terms in eq (i), (ii) + (iv)

Then.

$$a_n x^{n+r} + (n+r) a_n x^{n+r-1} + 2(n+r)(n+r-1) a_n x^{n+r-1} = 0$$

Substitute $n+1$ in n for eq (ii) and (iv).

$$a_n x^{n+r} + (n+1+r) a_{n+1} x^{n+r} + 2(n+1+r)(n+r) a_{n+1} x^{n+r} = 0$$

Take Sum of Coefficients.

$a_n + (n+1+r)\, a_{n+1} + 2(n+1+r)(n+r)\, a_{n+1} = 0$

factor out a_{n+1}.

$$\left\{\begin{aligned} &a_{n+1}\,(n+r+1) + 2(n+r+1)(n+r) = -a_n. \\ &a_{n+1}\,(n+r+1)\,[1+2n+2r] = -a_n \end{aligned}\right.$$

$$a_{n+1} = \frac{-a_n}{(n+r+1)(1+2n+2r)}$$

For $r=0$.

$$a_{n+1} = \frac{-a_n}{(n+1)(1+2n)}, \quad R_0 R_1 \quad \text{for } n = 0, 1, 2, 3, 4, \ldots$$

$n=0 \qquad a_1 = \dfrac{-a_0}{(0+1)(1+2(0))} = -a_0$

$n=1 \qquad a_2 = -a_1/6 = a_0/6$

$n=2 \qquad a_3 = \dfrac{-a_2}{15} = -a_0/90$

$n=3 \qquad a_4 = \dfrac{-a_3}{28} = \dfrac{a_0}{2520}$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}$$

$$= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + a_4 x^{r+4}$$

$$y_1(x) = a_0 x^r - a_0 x^{r+1} + \frac{a_0}{6} x^{r+2} - \frac{a_0}{90} x^{r+3} + \cdots$$

$$y_1(x) = a_0 \left[1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \cdots \right]$$

For $r_2 = 1/2$.

$$b_{n+1} = \frac{-b_n}{(n+1+1/2)(1+2n+2\times 1/2)}$$

$$b_{n+1} = \frac{-b_n}{(2n+3)(n+1)} \quad , R_1 R_2 \text{ for } n=0,1,2,3,4,\cdots$$

$n=0 \quad b_1 = -b_0/3$

$n=1 \quad b_2 = -b_1/10 = b_0/30$

$n=2 \quad b_3 = -b_2/21 = -b_0/630$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$
$$= b_0 x^{r_2} + b_1 x^{r_2+1} + b_2 x^{r_2+2} + b_3 x^{r_2+3} + \cdots$$

$$y_2(x) = x^{r_2}\left[b_0 - \frac{b_0}{3}x + \frac{b_0}{30}x^2 - \frac{b_0}{630}x^3 + \cdots\right]$$

$$y_2(x) = b_0 x^{1/2}\left[1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \cdots\right]$$

General Solution

$$y(x) = y_1 + y_2.$$

$$y(x) = A\left[1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \cdots\right] + Bx^{1/2}\left[1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \cdots\right]$$

where A and B are orbitary constant.

Example.

Solve the DE $x(x-1)y'' - xy' + y = 0$ at $x_0 = 0$.

Soln.

$$y'' - \frac{xy'}{x(x-1)} + \frac{y}{x(x-1)} = 0$$

$$P(x) = \frac{-1}{x-1}, \quad Q(x) = \frac{1}{x(x-1)}$$

$x_0 = 0$ is a regular S.P (Singular Point).

We assume the soln. $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

Then,

i. $y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \dots + a_n x^{n+r} + \dots$

ii. $y' = r a_0 x^{r-1} + (r+1) a_1 x^r + (r+2) a_2 x^{r+1} + \dots + (n+r) a_n x^{n+r-1} + \dots$

iii. $y'' = (r-1) r a_0 x^{r-2} + r(r+1) a_1 x^{r-1} + (r+1)(r+2) a_2 x^r + \dots + (n+r)(n+r-1) a_n x^{n+r-2}$

Then,

iv. $x^2 y'' = (r-1) r a_0 x^r + r(r+1) a_1 x^{r+1} + \dots + (n+r)(n+r-1) a_n x$

v. $-xy'' = -(r-1) r a_0 x^{r-1} - r(r+1) a_1 x^r + \dots - (n+r)(n+r-1) a_n x^{n+r-1}$

vi. $-xy' = -r a_0 x^r - (r+1) a_1 x^{r+1} + \dots - (n+r) a_n x^{n+r}$

Substitute y, y' and y'' in the DE and equate to zero the sum of coefficients of the lowest power of x.

$-(r-1) r a_0 = 0$ Since $a_0 \neq 0$

$-(r-1) r = 0 \quad \therefore \quad r_1 = 0$ and $r_2 = 1$.

Find the Sum of n^{th} terms.

$$a_n x^{n+r} + (n+r)(n+r-1)a_n x^{n+r} - (n+r)(n+r-1)a_n x^{n+r-1} - (n+r)a_n x^{n+r} = 0$$

Substitute $n+1 \to n$ in the 3^{rd} term.

Then.

$$a_n x^{n+r} + (n+r)(n+r-1)a_n x^{n+r} - (n+1+r)(n+r)a_{n+1} x^{n+r} - (n+r)a_n x^{n+r} = 0$$

$$a_n + (n+r)(n+r-1)a_n - (n+r+1)(n+r)a_{n+1} - (n+r)a_n = 0$$

$$-(n+r+1)(n+r)a_{n+1} = a_n\left[-1-(n+r)(n+r-1)+(n+r)\right]$$

The recarence relation.

$$a_{n+1} = \frac{a_n\left[1+(n+r)(n+r-2)\right]}{(n+r+1)(n+r)}$$ R.R.

For $r=1$.

$$a_{n+1} = \frac{n^2 a_n}{(n+2)(n+1)}$$ For $n=0,1,2,3,\ldots$

Now.

$n=0$, $a_1 = 0$.

$n=1$; $a_2 = a_1/6 = 0/6. = 0$.

$n=2$; $a_3 = \frac{2a_2}{4\times 3} = 0$.

So $a_1 = a_2 = a_3 = a_4 = a_5 = 0$.

$$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \ldots$$

$$= a_0 x + 0 + 0 + 0 + 0 \ldots$$

$$\therefore y(x) = a_0 x$$

y_2 is obtained by the reduction of order formular.

$$y_2(x) = y_1 V = y_1 \int \frac{e^{-\int P(x)\,dx}}{y(x)^2}\,dx.$$

$$= x \int \frac{e^{\int \frac{1}{x-1}\,dx}}{x^2}\,dx.$$

But $\int \frac{1}{x-1}\,dx = \ln(x-1) + \cancel{c}$ $\to 0$

$$y_2(x) = x \int \frac{x-1}{x^2}\,dx$$

$$= x\left[\ln x + \frac{1}{x}\right]$$

Example 3.

Solve the DE $x(x-1)y'' + (3x-1)y' + y = 0$. at $x=0$

Soln.

$$y'' + \frac{3x-1}{x(x-1)}y' + \frac{y}{x(x-1)} = 0.$$

$$P(x) = \frac{3x-1}{x(x-1)} \qquad Q(x) = \frac{1}{x(x-1)}$$

$x=0$ is a regular S.P

We assume a soln of $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \cdots + a_n x^{n+r} + \cdots$$

$$y' = r a_0 x^{r-1} + (r+1)a_1 x^r + (r+2)a_2 x^{r+1} + \cdots + (n+r)a_n x^{n+r-1} + \cdots$$

$$y'' = r(r-1)a_0 x^{r-2} + r(r+1)a_1 x^{r-1} + \cdots + (n+r)(n+r-1)a_n x^{n+r-2}$$

$$-y' = -ra_0 x^{r-1} - (r+1)a_1 x^r - (r+2)a_2 x^{r+1} + \cdots - (n+r)a_n x^{n+r-1}$$

$$3xy' = 3ra_0 x^r + 3(r+1)a_1 x^{r+1} + \cdots + 3(n+r)a_n x^{n+r}$$

$$-xy'' = -r(r-1)a_0 x^{r-1} - r(r+1)a_1 x^r + \cdots - (n+r)(n+r-1)a_n x^{n+r-1}$$

$$x^2 y'' = r(r-1)a_0 x^r + r(r+1)a_1 x^{r+1} + \cdots + (n+r)(n+r-1)a_n x^{n+r}$$

Substitute $y, y' \& y''$ in the D.E and equate to zero
Sum of coefficients of the lowest power of x.

$$-ra_0 - r(r-1)a_0 = 0$$

$$-r - r^2 + r = 0 \qquad \text{Since } a_0 \neq 0.$$

$$r_1 = r_2 = 0$$

Sum of n^{th} terms.

$$a_n x^{n+r} - (n+r)a_n x^{n+r-1} + 3(n+r)a_n x^{n+r} - (n+r)(n+r-1)a_n x^{n+r-1} + (n+r)(n+r-1)a_n x^{n+r} = 0$$

Replace n by $n+1$ in 2^{nd} and 4^{th} terms.

$$a_n x^{n+r} - (n+1+r)a_{n+1} x^{n+r} + 3(n+r)a_n x^{n+r} - (n+1+r)(n+r)a_{n+1} x^{n+r} + (n+r)(n+r-1)a_n x^{n+r} = 0$$

$$-a_{n+1}\left[(n+1+r) + (n+1+r)(n+r)\right] = -a_n\left[1 + 3(n+r) + (n+r)(n+r-1)\right]$$

$$a_{n+1} = \frac{a_n\left[1 + (n+r)(n+r+2)\right]}{(n+r+1)^2}$$

For $r=0$ $\quad a_{n+1} = \dfrac{a_n(1+n^2+2n)}{(n+1)^2}$

$a_{n+1} = a_n$ For $n = 0, 1, 2, 3, 4, \dots$

$n=0$ $a_1 = a_0$

$n=1$ $a_2 = a_1 = a_0$

$n=2$ $a_3 = a_2 = a_0$

$n=3$ $a_4 = a_3 = a_0$

$$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \dots$$

$$y = a_0 + a_0 x + a_0 x^2 + a_0 x^3 + a_0 x^4 + \dots$$

$$y_1(x) = a_0 (1 + x + x^2 + x^3 + x^4 + x^5 + \dots) \rightarrow \text{This series is } \frac{1}{1-x}$$

y_2 is obtained by the reduction of order formula.

$$y_2 = y_1 v = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$= \frac{1}{1-x} \int \frac{e^{-\int \frac{3x-1}{x(x-1)}\,dx}}{\frac{1}{(1-x)^2}}\,dx$$

But integrate.

$$-\int \frac{3x-1}{x(x-1)}\,dx = \frac{A}{x} - \frac{B}{x-1}$$

$$= -\left[\int \frac{1}{x}\,dx - \int \frac{2}{(x-1)}\,dx\right]$$

$$= -\left[\ln x + 2\ln(x-1)\right]$$

$$= e^{\frac{1}{\ln x (x-1)^2}}$$

$$= \frac{1}{1-x}\int \frac{(x-1)^{-2}}{x(1-x)^{-2}}\,dx$$

$$y_2(x) = \frac{1}{1-x}\int \frac{1}{x}\,dx = \frac{1}{1-x}\ln x.$$

$\therefore$ G.S $\Rightarrow$ $y(x) = y(1) + y_2(x)$

$$\Rightarrow \frac{1}{1-x}\left[1 + \ln x\right]$$

TEST Q.

Solve the D.E $x^2y'' + (x^2-3x)y' + 3y = 0$.

Soln.

$$y'' + \frac{(x^2-3x)}{x^2}y' + \frac{3}{x^2}y = 0.$$

$$P(x) = \frac{x^2-3x}{x^2} = \frac{x-3x}{x}, \quad Q(x) = 3/x^2$$

At $x=0$ is regular Singular Point.

We assume the soln of the DE as $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + a_3x^{r+3} + \dots + a_nx^{n+r} + \dots$$

$$y' = ra_0x^{r-1} + (r+1)a_1x^r + (r+2)a_2x^{r+1} + (r+3)a_3x^{r+2} + (n+r)a_nx^{n+r-1} + \dots$$

$$y'' = (r-1)ra_0x^{r-2} + r(r+1)a_1x^{r-1} + (r+2)(r+1)a_2x^r + (r+3)(r+2)a_3x^{r+1} + (n+r)(n+r-1)a_nx$$

$$x^2y'' + x^2y' - 3xy' + 3y = 0$$

$$3y = 3a_0x^r + 3a_1x^{r+1} + 3a_2x^{r+2} + 3a_3x^{r+3} + \dots + 3a_nx^{n+r} + \dots$$

$$-3xy' = -3ra_0x^r - 3(r+1)a_1x^{r+1} - 3(r+2)a_2x^{r+2} + \dots + 3(n+r)a_nx^{n+r} + \dots$$

$$x^2y' = ra_0x^{r-1} + (r+1)a_1x^{r+2} + \dots + (a+r)a_nx^{r+n+1}$$

$$x^2y'' = (r-1)ra_0x^r + r(r+1)a_1x^{r+1} + \dots + (n+r)(n+r-1)a_nx^{n+r} + \dots$$

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Substitute y, y' and y'' in the D.E and Sum the lowest powers of x Coefficients to zero.

Then;

$$x^r 3a_0 - 3r a_0 x^r + (r-1)r a_0 x^r = 0$$

$$[3a_0 - 3r a_0 + (r-1)r a_0] x^r = 0$$

$$[3 - 3r + r(r-1)] a_0 x^r = 0$$

$$(3 - 3r + r^2 - r) a_0 = 0$$

$$(r^2 - 4r + 3) a_0 = 0 \qquad \text{Since } a_0 \neq 0.$$

$$r^2 - 4r + 3 = 0$$

$$r(r-1) - 3(r-1) = 0$$

$$(r-3)(r-1) = 0. \qquad r_1 = 3, \quad r_2 = 1.$$

The Sum of n^{th} term to zero

$$3a_n x^{n+r} - 3(n+r) a_n x^{n+r} + (r+n) a_n x^{r+n+1} + (n+r)(n+r-1) a_n x^{n+r} = 0$$

Substitute $n = n-1$ in the 3^{rd} term

$$3a_n x^{n+r} - 3(n+r) a_n x^{n+r} + (r+n-1) a_{n-1} x^{r+n-1+1} + (n\cancel{-1}+r)(n-1+r-1) a_{n\cancel{-1}} x^{n+r\cancel{-1}} = 0$$

$$3a_n \cancel{x^{n+r}} - 3(n+r) a_n \cancel{x^{n+r}} + (r-1+n) a_{n-1} \cancel{x^{r+n}} + (r+n\cancel{-1})(r+n-\cancel{1}) a_{n\cancel{-1}} \cancel{x^{n+r}} = 0$$

$$a_n [3 - 3(r+n) + (r+n-1)(r+n)] = -(n+r-1) a_{n-1}$$

$$a_n [3 - (n+r)(4-r-n)] = -(n+r-1) a_{n-1}$$

$$a_n = \frac{-(n+r-1) a_{n-1}}{3 - (n+r)(4-n-r)}$$

x^{n+r-2} $+ \dots$

$$a_n = \frac{-(n+r-1) a_{n-1}}{3 - (4n - n^2 - nr + 4r - nr - r^2)}$$

$$a_n = \frac{-(n+r-1) a_{n-1}}{3 - 4n + n^2 + nr - 4r + nr + r^2}$$

$$a_n = \frac{(-n+r-1)\,a_{n-1}}{3-4n+n^2+2nr-4r+r^2}$$

For $r=3$.

$$a_n = \frac{-(n+2)a_{n-1}}{3-4n+n^2+6n-12+9}$$

$$= \frac{-(n+2)\,a_{n-1}}{3-(n+3)(1-n)}$$

$$= \frac{-(n+2)\,a_{n-1}}{n(n+2)}$$

$$a_n = \frac{-a_{n-1}}{n}.$$ R.R for $n=1,2,3$ for $r=3$

(Do not start with $n=0$, start with $n=1$ interm of a_0)

For $r_2=1$

$$a_n = \frac{-(n+(r-1))\,a_{n-1}}{3-4n+n^2+2nr-4r+r^2}$$

$$= \frac{-n\,a_{n-1}.}{3-4n+n^2+2n-4+1}$$

$$= \frac{-n\,a_{n-1}}{3-2n+n^2-3}$$

$$a_n = \frac{-n\,a_{n-1}}{n(-2+n)}, \quad a_n = \frac{-a_{n-1}}{-2+n} = \frac{-a_{n-1}}{-2+n}$$

Now

$n=1 \qquad a_1 = -\frac{a_0}{1} = -a_0$

$n=2 \qquad a_2 = -\frac{a_1}{2} = \frac{a_0}{2}$

$n=3 \quad a_3 = -\frac{a_2}{3} = -\frac{a_0}{6}$

$n=4 \quad a_4 = -\frac{a_3}{4} = \frac{a_0}{24}$

$y(x) = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \ldots\ldots$

$= x^r \left[a_0 x + a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots \right]$

$= x^3 \left[a_0 - a_0 x + \frac{a_0}{2} x^2 - \frac{a_0}{6} x^3 + \frac{a_0}{24} x^4 + \ldots \right]$

$= a_0 x^3 \left[1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots \right]$

$= a_0 x^3 e^{-x}$

Leting $a_0 = 1$ we have

$y_1(x) = x^3 e^{-x}$

$y_2(x)$ is obtained from the reduction fermular.

$$y = y_1 v = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= x^3 e^{-x} \int \frac{e^{-\int \frac{x-3x}{x^2} dx}}{(x^3 e^{-x})^2} dx$$

But $\int \frac{x-3}{x} dx = \int \left(1 - \frac{3}{x}\right) dx$

$= x - 3\ln x$

$$e^{-\int \frac{x-3}{x} dx} = e^{-(x - 3\ln x)} = e^{-x + 3\ln x}$$

$$= e^{-x} \cdot e^{3\ln x}$$

$$= x^3 e^{-x}$$

$$y_2(x) = x^3 e^{-x} \int \frac{x^3 e^{-x}}{(x^3 e^{-x})^2} dx = x^3 e^{-x} \int x^{-3} e^{x} dx$$

$\Rightarrow \int x^{-3} e^{x} dx$

Let $u = e^{x}$, $dv = x^{-3} dx$, $du = e^{x} dx$ $v = x^{-2}/-2$.

By Part.

$$\int x^{-3} e^{-x} dx = uv - \int v\, du$$

$$= \frac{e^{x} x^{-2}}{-2} - \int \frac{x^{-2}}{-2} e^{x} dx$$

$$= -\frac{e^{x} x^{-2}}{2} + \frac{1}{2}\int \frac{e^{x}}{x^{2}} dx.$$

Not the series will repeated non stop.

Then

$$\int x^{-3} e^{-x} dx = \int x^{-3}\left(1 + x + \frac{x^{2}}{2} + \frac{x^{3}}{6} + \frac{x^{4}}{24} + \cdots\right) dx$$

$$= x^{-3} + x^{2} + \frac{x^{-1}}{2} + \frac{1}{6} + \frac{x}{24} + \frac{x^{2}}{5!}$$

$$= \frac{x^{-2}}{-2} + \frac{x^{-1}}{-1} + \frac{x}{2} + \frac{x}{6} + \frac{x^{2}}{48} + \frac{x^{3}}{3.5!} + \cdots$$

$$y_2(x) = x^{3} e^{-x}\left[\frac{x^{-2}}{-2} - x^{-1} + \frac{\ln x}{2} + \frac{x}{6} + \frac{x^{2}}{48} + \frac{x^{3}}{3.5!} + \cdots\right]$$

when $r_1 - r_2 = N$.

$$y_2 = k y_1 \ln x + \frac{1}{2} x^{3} e^{-x} \ln x + x^{3} e^{-x}\left[-\frac{x^{-2}}{2} - x^{-1} + \frac{x}{6} + \frac{x^{2}}{48} + \frac{x^{3}}{3.5!} + \cdots\right]$$

Note. The Mauclarias Series.

$$f(x) = f(0) + x\frac{f'(0)}{1!} + x^{2}\frac{f''(0)}{2!} + x^{3}\frac{f'''(0)}{3!} + \cdots$$

for

$f(x) = e^{-x} \rightarrow f(0) = e^{-0} = 1$

$f'(x) = -e^{-x} \rightarrow f'(0) = -e^{-0} = -1$

$f''(x) = e^{-x} \rightarrow f''(0) = e^{-0} = 1$

$f'''(x) = -e^{-x} \rightarrow f'''(0) = -e^{-0} = -1.$

The series is.

$$\therefore f(x) = \left\{1 - x + \frac{x^{2}}{2!} - \frac{x^{3}}{3!} + \frac{x^{4}}{4!} - \cdots\right\}$$

BESSEL'S EQUATION.

Bessels equation is an ODE in the form of

$$x^2y'' + xy' + (x^2 - v^2)y = 0.$$

where x is a real number. In its normalized form is

$$y'' + \frac{y'}{x} + \left(1 - \frac{v^2}{x^2}\right)y = 0$$

$P(x) = 1/x$ and $Q(x) = 1 - \frac{v^2}{x^2}$, $x_0 = 0$ is r.s.p

We Assume the Soln of form of $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} \cdots + a_n x^{n+r} + \cdots$$

$$y' = r a_0 x^{r-1} + (r+1) a_1 x^r + (r+2) a_2 x^{r+1} + \cdots + (n+r) a_n x^{n+r-1}$$

$$y'' = r(r-1) a_0 x^{r-2} + r(r+1) a_1 x^{r-1} + (r+1)(r+2) a_2 x^r + \cdots + (n+r)(n+r-1) a_n x^{n+r-2}$$

$$-v^2 y = v^2 a_0 x^r - v^2 a_1 x^{r+1} - v^2 a_2 x^{r+2} - v^2 a_3 x^{r+3} + \cdots - v^2 a_n x^{n+r}$$

$$x^2 y = a_0 x^{r+2} + a_1 x^{r+3} + a_2 x^{r+4} + \cdots + a_n x^{n+r+2}$$

$$xy' = r a_0 x^r + (r+1) a_1 x^{r+1} + (r+2) a_2 x^{r+2} + (n+r) a_n x^{n+r}$$

$$x^2 y'' = r(r-1) a_0 x^r + r(r+1) a_1 x^{r+1} + (r+1)(r+2) a_2 x^{r+2} + \cdots + (n+r)(n+r-1) a_n x^{n+r}$$

Lowest Power of x is x^1

Sum of Coefficient of x^1

$$(r-1) r a_0 + r a_0 - v^2 a_0 = 0$$

$$a_0 (r^2 - r + r - v^2) = 0 \qquad \text{since } a_0 \neq 0$$

$$r^2 - v^2 = 0$$

$$r_1 = +v \quad \text{and} \quad r_2 = -v$$

Take Sum of Coefficient of x^{r+1}

$$-v^2 a_1 + (r+1) a_1 + (r+1) r a_1 = 0$$
$$a_1 [-v^2 + r + 1 + r^2 + r] = 0$$

For $r = v$

$$a_1 [2v+1] = 0$$
$$a_1 = 0 \qquad v = -1/2 .$$

Sum of n^{th} terms
$$-v^2 a_n x^{n+r} + a_n x^{n+r+2} + (n+r) a_n x^{n+r} + (n+r)(n+r-1) a_n x^{n+r} = 0$$
replace n by n-2 in the 2^{nd} terms.

$$-v^2 a_n x^{n+r} + a_{n-2} x^{n-r} + (n+r) a_n x^{n+r} + (n+r)(n+r-1) a_n x^{n+r} = 0$$

$$a_n \left[-v^2 + (n+r) + (n+r)(n+r-1)\right] = -a_{n-2}$$
$$a_n \left[-v^2 + (n+r)^2\right] = -a_{n-2} .$$

$$a_n = \frac{-a_{n-2}}{(n+1)^2 - v^2} \qquad \text{R.R.}$$

For $r = v$

$$a_n = \frac{-a_{n-2}}{n(n+2v)} \qquad \text{for } n = 2, 3, 4, \ldots$$

Since
$a_1 = 0$, then $a_1 = a_3 = a_5 = a_{odd} = 0$.
For even index replace n by 2n.

$$a_{2n} = \frac{-a_{2n-2}}{2n(2n+2v)} = -\frac{a_{2n-2}}{2^2 n(n+v)} \qquad \text{for } n = 1, 2, 3, 4 \ldots$$

For

$n=1$; $a_2 = \frac{-a_0}{2^2 \times 1(1+v)}$

$n=2$; $a_4 = -\frac{a_2}{2^2 \cdot 2(2+v)} = \frac{a_0}{2^2 \times 1 \times 2(1+v)(2+v)}$

$n=3$; $a_6 = -\frac{a_4}{2^2 \cdot 3(3+v)} = -\frac{a_0}{2^6 \cdot 3!(1+v)(2+v)(3+v)}$

$n=n$;

$$a_n = \frac{(-1)^n a_0}{2^{2n} \cdot n!(1+v)(2+v)(3+v)\cdots\cdots(n+v)}$$

A Simpler series can be found by absorbing the growing product $(1+v)(2+v)(3+v)\cdots(n+v)$ in to $(n+v)!$

So we let $a_0 = \frac{1}{2^v v!}$

Since

$$v!(1+v)(2+v)\cdots\cdots(n+v) = (n+v)!$$

Then.

$$a_n = \frac{(-1)^n}{2^{2n+v} n(n+v)!}$$

and

$$y_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n!(n+v)!}$$

$$= \left(\frac{x}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+v)} \left(\frac{x}{2}\right)^{2n}$$

$y_1(x) = J_v(x)$ is called Bessel's form of the first kind of order v.

The series Converges for all values of x.

The functions $J_v(x)$ and $J_{-v}(x)$ are the Solutions of Bessele's equation. If V is not an integer. they are Linealy independent and the general Soln is given by
$y(x) = C_1 J_v(x) + C_2 J_{-v}(x)$
where
$J_{-v}(x)$ is obtained by Substituting $-V$ in $J_v(x)$.
If V is an integer the two Solns are Linearly dependent because.

$$J_{-n}(x) = -(-1)^n J_n(x)$$

Bessel functions $J_0(x)$ and $J_1(x)$

When $V=0$, $J_0(x)$ is called a Bessel function of the first kind of order 0.

Substitutionaly $V=0$ in.

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n! (v+n)!}$$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! n!}$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4 \cdot 4} - \frac{x^6}{2^6 \cdot 6 \times 6} + \frac{x^8}{2^8 \cdot 4! 4!} + \cdots$$

When $V=1$ $J(x)$ is called a Bessel function of the first kind of order 1.

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Substituting $v = 1$ in.

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n! (n+v)!}$$

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n! (1+n)!}$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 \times 2} + \frac{x^5}{2^5 \times 2! \times 3!} - \frac{x^7}{2^7 3! 4!}$$

03rd 12. 2018

BESSEL'S FUNCTIONS INTERMS OF GAMMA FUNCTIONS.

Recall $k! = \Gamma k+1$ then $(v+n)! = \Gamma v+n + 1$.

from $a_{2n} = \frac{(-1)^n}{2^{2n+v} n! (n+v)!}$ we have.

$$a_{2n} = \frac{(-1)^n}{2^{2n+v} n! \Gamma n+v+1} \quad \text{Then.}$$

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n! (n+v)!}$$

Becomes $J_v(x) = \left(\frac{x}{2}\right)^v \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! \Gamma n+v+1}$

$$J_v(x) = \left(\frac{x}{2}\right)^{-v} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! (n-v)!}$$

become $J_v(x) = \left(\frac{x}{2}\right)^{-v} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! \Gamma n-v+1}$

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UE

Q. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ b) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

EQUATIONS REDUCIBLE TO BESSEL'S EQUATIONS.

There are some differential Equation which can be reducable to bessel's equation and then can be solved.

Example 01;

Use change of variable $y = x^2 u$ to reduce DE

$$9x^2y'' - 27xy + (9x^2 + 35)y = 0$$

Into Bessels Equation;

$$y = x^2 u \qquad \frac{dy}{dx} = 2xu + x^2 \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2u + 2x\frac{du}{dx} + 2x\frac{du}{dx} + x^2\frac{d^2u}{dx^2}$$

Substitute y, y' and y'' in the D.E

$$9x^2\left(2u + 4x\,du/dx + x^2 d^2u/dx^2\right) - 27x\left(2xu + x^2 dy/dx\right) + (9x^2 + 35)x^2u = 0$$

$$18x^2u + 36x^3\frac{dy}{dx} + 9x^4\frac{d^2y}{dx^2} - 54x^2u - 27x^3\frac{dy}{dx} + 9x^4u + 35x^2u = 0$$

$$9x^4\frac{d^2y}{dx^2} + 9x^3\frac{du}{dx} - x^2u + 9x^4u = 0$$

divide by $9x^2$ both sides.

$$x^2u'' + xu' + x^2u - \tfrac{1}{9}u = 0$$

$$x^2u'' + xu' + (x^2 - \tfrac{1}{9})u = 0$$

This is a Bessels equation of order $V = 1/3$

General Solution $y(x) = A J_V(x) + B J_V(x)$.

Since our $V = 1/3$ not an integer then;

The general Soln of the Bessel's Equations is

$u(x) = A J_{1/3}(x) + B J_{-1/3}(x)$ where A and B are orbitary constant. The original DE will have the general Soln of $y(x) = A x^2 J_{1/3}(x) + B x^2 J_{-1/3}(x)$.

Example 02;

Reduce $x^2 y'' + xy' + (k^2 x^2 - v^2) y = 0$ into Bessels eqn where k and v is constant. Use $t = kx$

Soln

$t = kx$, $dt/dx = k$, $dy/dx = dy/dt \cdot dt/dx = k\, dy/dt$.

$$d^2y/dx^2 = d/dx\left(dy/dx\right) = d/dx\left(k\, dy/dt\right) = d/dt\left(k\, dy/dt\right)\cdot \frac{dt}{dx} = k^2 \frac{d^2y}{dt^2}$$

Substitute in the D.E

$$x^2 k^2 \frac{d^2y}{dt^2} + xk\frac{dy}{dt} + (k^2x^2 - v^2)y = 0$$

from $t = kx$

$k = t/x$

$$x^2 \cdot \left(\frac{t^2}{x^2}\right)\frac{d^2y}{dt^2} + x \cdot \frac{t}{x}\frac{dy}{dt} + \left(\frac{t^2}{x^2}\cdot x^2 - v^2\right)y = 0$$

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - v^2)y = 0$$

Compare with General Bessel's Equation.

This is a Bessel's equation of order V. if V is not an integer, the general

Soln. $y(t) = A J_v(t) + B J_{-v}(t)$

The origional DE will have the general equation of

$y(x) = A J_v(kx) + B J_{-v}(kx)$

Reduce $x^2y'' + xy' + (4x^4 - \frac{1}{4})y = 0$ to Bessels eq use $z = x^2$.

Soln.

$$\frac{dz}{dx} = 2x.$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 2x\frac{dy}{dz}$$

$$\frac{dy}{dx} = 2x\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(2x\frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = 2x\frac{d}{dx}\left(\frac{dy}{dz}\right) + 2\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = 2x\left(\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}\right) + 2\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = 4x^2\frac{d^2y}{dz^2} + 2\frac{dy}{dz}$$

Substitute into DE.

$$x^2\left(4x^2\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + x\left(2x\frac{dy}{dz}\right) + (4x^4 - \frac{1}{4})y = 0$$

$$4x^4\frac{d^2y}{dz^2} + 2x^2\frac{dy}{dz} + 2x^2\frac{dy}{dz} + (4x^4 - \frac{1}{4})y = 0$$

$$4x^4\frac{d^2y}{dz^2} + 4x^2\frac{dy}{dz} + (4x^4 - \frac{1}{4})y = 0$$

Divide by 4 both sides.

$$x^4\frac{d^2y}{dz^2} + x^2\frac{dy}{dz} + (x^4 - \frac{1}{16})y = 0 \quad \text{but } z = x^2$$

$$z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz} + (z^2 - 1/16)y = 0$$

This is a Bessel's equation of order $V = 1/4$ and if V is not integer, the general soln is :-

$$y(z) = AJ_v(z) + BJ_{-v}(z)$$
$$y(z) = AJ_{1/4}(z) + BJ_{-1/4}(z)$$

The Origional DE will have the general Equation of

$$y(x) = AJ_{1/4}(x^2) + BJ_{-1/4}(x^2)$$

LEGENDRE'S EQUATION

Lengendre's eqn is an ODE of the form $(1-x^2)y'' - 2xy' +$ where k is a real constant. This ore of the most important in Engineering problems. It involves ~~important in~~ a parameter K whose value depends on physical or Engineering problem. In its nomalized form is

$$y'' - \frac{2x}{1-x^2} y' + \frac{k(k+1)}{1-x^2} y = 0$$

$$P(x) = \frac{-2x}{1-x^2}, \quad Q(x) = \frac{k(k+1)}{1-x^2}$$

$x = 0$ is an ordinary point we ~~Assume~~ a Soln of the form of

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n x^{n-2}$$

Substitute y, y' & y'' in the DE the recurrence relations is

$$a_{n+2} = -\frac{(k-n)(k+n+1)a_n}{(n+2)(n+1)} \quad \text{for } n=0,1,2,3, \ldots$$

for $n=0$

$$a_2 = -\frac{k(k+1)}{2\times1} a_0$$

$n=1 \quad a_3 = -\frac{(k-1)(k+2)}{3\times2} a_1$

$n=2 \quad a_4 = -\frac{(k-2)(k+3)}{4\times3} a_2$

$$= \frac{k(k+1)(k-2)(k+3)a_0}{4!}$$

$n=3$, $a_5 = \frac{-(k-2)(k+4)a_2}{5\times4}$

$= \frac{(k-1)(k+1)(k-3)(k+4)a_1}{5!}$

$n=4$

$a_6 = \frac{-(k-1)(k+5)a_2}{6\times5}$

$= \frac{-(k+k+1)(k-2)(k+5)(k+5)(k+5)}{6!}$

Substitute as in the assumed soln.

$$y = a_0 + a_1x - \frac{k(k+1)a_n x^2}{2!} - \frac{(k-1)(k+2)a_1x^3}{3!} + \frac{k(k+1)(k-2)(k+3)}{4!}a_0x^4$$
$$+ \frac{(k-1)(k+2)(k-3)(k+4)a_1x^5}{5!} + \ldots\ldots$$

$$y = a_0\left(1 - \frac{k(k+1)x^2}{2!}\right) + \frac{k(k+1)(k^3-4k+3)x^4}{4!} + \ldots$$
$$a_1\left(x - \frac{(k+1)(k+2)x^3}{3!}\right) + \frac{(k-1)(k+2)(k-3)(k+4)x^5}{5!} + \ldots$$

The general soln is $y(x) = y_1(x) + y_2(x)$ This means

$$y_1(x) = a_0\left[1 - k(1 - \frac{k(k+1)}{2!}x^2 + \frac{k(k+1)(k-2)(k+1)x^4}{4!} + \ldots\right]$$

$$y_2(x) = a_1\left[x - \left(\frac{k-1(k+2)x^3}{3!}\right) + \frac{(k+1)(k+2)(k-3)(k+4)x^5}{5!} + \ldots\right]$$

$y_1(x)$ and $y_2(x)$ are linearly independent. The series converges for all $|x| < 1$ LEGENDRE POLYNOMIAL $P_n(x)$

When k is a +ve integer n are of the soln Series terminate after a finite number of terms. The resulting polynomial in x denoted by $P_n(x)$ is called Legendre polynomial with a_0 & a_1 choosen so that the polynomial has a Unit Value where $x = 1$. for n is even y(x) reduces to a polynomial of degree n for n is odd $y_2(x)$ reduces to a polynomial of degree n.

$$P_0(x) = a_0, \quad P_0(x) = 1$$

$$P_1(x) = a_1 x, \quad P_1(x) = x.$$

$$P_2(x) = a_0[1-3x^2]$$

$$1 = a_0(-2)$$

$$a_0 = -1/2$$

$$P_2(x) = -1/2(1-3x^2)$$

$$= \frac{1}{2}(3x^2-1)$$

$$P_3(x) = a_1(x - 5/3 x^3)$$

$$1 = a_1(1-5/3)$$

$$a_1 = -3/2$$

$$P_3(x) = -3/2\left(x - 5x^3/3\right)$$

$$= 1/2(5x^3-3x)$$

$$P_4(x) = a_0\left(1 - \frac{4\times5x^2}{2} + \frac{4\times5\times2\times2x^4}{4\times2\times2}\right)$$

$$= a_0\left(1-10x^2+35/3\, x^4\right)$$

$$1 = a_0\, 8/3$$

$$a_0 = 3/8$$

$$P_4(x) = \frac{3}{8}\left(1 - 10x^2 + \frac{35}{3}x^4\right)$$

$$= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

RODRIGUES FORMULA.

Legendre polynomials can be derived using Rodrigues formula.

$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n (x^n - 1)^n}{dx^n} \qquad \text{e.g } P_4(x)$$

$$P_4(x) \Rightarrow n = 4$$

$$P_4(x) = \frac{1}{2^4 4!} \frac{d^4 (x^2 - 1)^4}{dx^4}$$

But

$$(x^2-1)^4 = (x^2-1)(x^2-1)$$

$$= (x^4 - 2x^2 - 1).(x^4 - 2x^2 + 1)$$

$$= x^2 - 2x^4 + x^4 - 2x^4 + 4x^4 - 2x^2$$

Then

$$P_4(x) = \frac{1}{384} \frac{d^4}{dx}\left(x^8 - 4x^6 + 6x^2 - 4x^2 + 1\right)$$

$$= \frac{1}{384} \frac{d^4}{dx}\left(1688x^4 - 1440x^2 + 144\right)$$

$$\therefore P_4(x) = \frac{1}{8} \cdot 25x^4 - 30x^2 - 3.$$

GENERATING FUNCTION.

The fn $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \quad |t|<1$ is called the generating fn for Legendre polynomials. It is used to obtain some properties of Legendre polynomials.

Example.

Use generating soln to find $P_n(1)$.

Soln.

$P_n(1) \Rightarrow x = 1$

$$\sqrt{\frac{1}{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

$$\frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{\sqrt{(1-t)^2}} = \frac{1}{1-t} = 1+t+t^2+t^3+t^4 \ldots$$

$\sum_{n=0}^{\infty} t^n$ Compare to RHS.

$$\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)t^n$$

$$1 = P_n(1)$$

$$\therefore P_n(1) = 1$$

Example.

Use GF find $P_n(-1)$

Soln

$P_n(-1) \Rightarrow x = -1$

L.H.S

$$\frac{1}{\sqrt{1-2xt+t^2}} \quad \frac{1}{\sqrt{1-2xt+t^2}} = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{\sqrt{(1+t)^2}}$$

From

$$\frac{1}{1+t} = 1-t+t^2-t^3+t^4+\dots = \sum_{n=0}^{\infty} (-1)^n t^n$$

Compare R.H.S

$$\sum_{n=0}^{\infty} (-1)^n t^n = \sum_{n=0}^{\infty} P_n(-1) t^n$$

$$\therefore \quad \underline{P_n(-1) = (-1)^n}$$

POLYNOMIAL AS A FINITE SERIES OF LEGENRE POLYNOMIALS. (Sum inserted above)

Any Polynomial can be written as a sum of finite series of Legendre polynomials as $f(x) = \sum_{n=0}^{\infty} a_n P(x)$

Example.

$f(x) = x^3+x^2+x+1$ write as a sum of finite series of Legendre polynomials.

Soln

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$x^3+x^2+x+1 = \sum_{n=1}^{\infty} a_n P_n(x)$$

$$= a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \dots$$

$$= a_0 1 + a_1 x + a_2 \cdot \frac{1}{2}(3x^2-1) + a_3 \cdot \frac{1}{2}(5x^3-3x) + a_4 \cdot \frac{1}{8}(35x^4-30x^2+3) + \dots$$

$$= a_0 + a_1 x + \frac{3}{2} a_2 x^2 - \frac{a_2}{2} + \frac{5}{2} a_3 x^3 - \frac{3a_3 x}{2} + \frac{35}{8} a_4 x^4 - \frac{30}{8} a_4 x^2 + \frac{3}{8} a_4 + \dots$$

Compare Coefficiat of x L.H.S & R.H.S

$1+\frac{3}{2}$ $\frac{2+1}{2} = \frac{3}{2}$

	L.H.S		R.H.S	
x^0 :	1	$=$	$a_0 - \frac{1}{2}a_2$	$\Rightarrow a_0 = 4/3$
x^1 :	1	$=$	$a_1 - \frac{3}{2}a_3$	$\Rightarrow a_1 = 8/5$
x^2 :	1	$=$	$\frac{3}{2}a_2$	$\Rightarrow a_2 = 2/3$
x^3 :	1	$=$	$\frac{5}{2}a_3$	$\Rightarrow a_3 = 2/5$

$\therefore$ $x^3+x^2+x+1 = \frac{4}{3}P_0(x) + \frac{8}{5}P_1(x) + \frac{2}{3}P_2(x) + \frac{2}{5}P_3(x)$

Example.

Write $f(x) = 3x^3 - 4x^2 + 8$ as a sum of finite series of Legendre polynomials.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$3x^3 - 4x^2 + 8 = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + a_3P_3(x)$$

$$= a_0 + a_1x + \frac{1}{2}a_2(3x^2-1) + \frac{1}{2}a_3(5x^3-3x)$$

$$= a_0 + a_1x + \frac{3}{2}a_2x^2 - \frac{a_2}{2} + \frac{5}{2}a_3x^3 - \frac{3}{2}a_3x$$

Compared the coefficient of x in L.H.S & R.H.S.

	L.H.S		R.H.S	
x^0 :	8	$=$	$a_0 - \frac{1}{2}a_2$	$\Rightarrow a_0 = 20/3$
x^1 :	0	$=$	$a_1 - \frac{3}{2}a_3$	$\Rightarrow a_1 = 9/5$
x^2 :	-4	$=$	$\frac{3}{2}a_2$	$\Rightarrow a_2 = -8/3$
x^3 :	1	$=$	$\frac{5}{2}a_3$	$\Rightarrow a_3 = 6/5$

$\therefore$ $3x^3 - 4x^2 + 8 = \frac{20}{3}P_0(x) + \frac{9}{5}P_1(x) + -\frac{8}{3}P_2(x) + \frac{6}{5}P_3(x)$

ORTHOGONAL.

Two functions $f(x)$ and $g(x)$ OR $P_m(x)$ and $P_n(x)$ where $m \neq n$ are orthogonal to each other on the interval $a \leq x \leq b$ if

$$\int_a^b f(x)\,g(x)\,dx = 0 \quad \text{OR} \quad \int_a^b P_m(x)\,P_n(x)\,dx = 0$$

Check the orthogonality of $P_2(x)$ and $P_3(x)$ on $[-1, 1]$