SERIES SOLUTION OF A DIFFERENTIAL EQUATION.

- A differential equation is an equation containing a derivative of derivatives of unknown function.
- A solution of a D.E is a relation between X and Y which does not contain any derivative. eg.

Verify that $y = A\cos X + B\sin X$ is a solution of a D.E $y'' + y = 0$.

Solution.

$$y = A\cos x + B\sin x$$

$$y' = -A\sin x + B\cos x$$

$$y'' = -A\cos x - B\sin x$$

Substitute into D.E

$$y'' + y = 0$$

$$-A\cos x - B\sin x + A\cos x + B\sin x = 0$$

$$y'' + y = 0$$

Hence verified.

ANALYTIC FUNCTION:

- A function $f(x)$ is said to be analytic if it is defined every where. eg.

All polynomials e^x, $\cos x$, $\sin x$, $\cosh x$.

- A ~~relation~~ rational function is analytic at every where except when the denominator is zero.

ORDINARY AND SINGULAR POINTS.

- A point $x = X_0$ is called an ordinary point of the D.E.

$y'' + P(x)y' + Q(x)y = 0$. If both $P(x)$ and $Q(x)$ are analytic at X_0.

- If either (or both) of these functions $P(x)$ and $Q(x)$ is not analytic at X_0 then X_0 is called a singular point.

eg.

consider a D.E;

$$(x-1)y'' + xy' + \frac{1}{x}y = 0$$

in its normalize form is

$$y'' + \frac{x}{(x-1)}y' + \frac{1}{x(x-1)}y = 0 .$$

$$y'' + y'\left(\frac{x}{x-1}\right) + \frac{y}{x(x-1)} = 0$$

$$P(x) = \frac{x}{x-1} \quad \text{and} \quad Q(x) = \frac{1}{x(x-1)}$$

at $x=1$ and $x_0 = 0$.

$\therefore$ Both $P(x)$ and $Q(x)$ are not analytic at $x=1$.

$x=1$ is a singular point.

Theorem: If x_0 is an ordinary point of the D.E then the D.E has two non-trivial linearly independent solution of the form of

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{which converges in some}$$

interval $|x-x_0| < R$.

- For $x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{becomes}$$

$$y = \sum_{n=0}^{\infty} a_n x^n .$$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$ where

$a_0, a_1, a_2, a_3 \ldots\ldots a_n$ are constants.

Example 1
Solve the D.E
$y'' - xy' - y = 0$ at $x_0 = 0$

Soln:

$P(x) = -x$, $Q(x) = -1$
at $x = 0$ both $P(x)$ and $Q(x)$ are analytic, Then we assume a solution of the form of

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots\ldots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + \ldots\ldots$$

Substitute y'', y' and y in the D.E $y'' - xy' - y = 0$

$$y'' - xy' - y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=i} a_n x^n = 0$$

- Make powers of x the same,
  let $k = n-2 \Rightarrow n = k+2$
  in 1^{st} term $k = n$, in the 2^{nd} term and 3^{rd} term

Then;

$$\underbrace{\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} k a_k x^k}_{\text{at this case } k=0} - \underbrace{\sum_{k=0} a_k x^k}_{\text{when } k=0} = 0$$

$$2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} k a_k x^k - a_0 - \sum_{k=1}^{\infty} a_k x^k = 0$$

but; $\boxed{2a_2 - a_0 = 0}$

$$\sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^{k} - \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=1}^{\infty} a_k x^k = 0$$

$$\sum_{k=1}^{\infty}\left[(k+2)(k+1)a_{k+2} - k a_k \cancel{*} a_k\right] x^k = 0$$

$$(k+2)(k+1)a_{k+2} - k a_k - a_k = 0$$

from: $2a_2 - a_0 = 0$

$$2a_2 = a_0$$

$$a_2 = \frac{a_0}{2}.$$

$$a_2 = \tfrac{1}{2}a_0.$$

$$(k+2)(k+1)a_{k+2} - k a_k - a_k = 0$$

$$(k+2)(k+1)a_{k+2} = k a_k + a_k$$

$$(k+2)(k+1)a_{k+2} = a_k(k+1).$$

$$(k+2)a_{k+2} = a_k;$$

Reference Relation; $a_{k+2} = \dfrac{a_k}{k+2}$ for $k = 1, 2, 3, 4, 5, \ldots$

$k=1$, $a_3 = \dfrac{a_1}{3}$

$k=2$, $a_4 = \dfrac{a_2}{4} = \dfrac{\frac{1}{2}a_0}{4} = \dfrac{a_0}{8}$

$k=3$, $a_5 = \dfrac{a_3}{5} = \dfrac{\frac{1}{3}a_1}{5} = \dfrac{a_1}{15}$

$k=4$, $a_6 = \dfrac{a_4}{6} = \dfrac{\frac{1}{8}a_0}{6} = \dfrac{a_0}{48}$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \frac{1}{48} a_0 x^6 + \cdots$$

$$y = \underbrace{a_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \cdots \right]}_{y_1} + \underbrace{a_1 \left[x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \cdots \right]}_{y_2}$$

- The soln of the given D.E is :-

$$y(x) = A \left[1 + \tfrac{1}{2} x^2 + \tfrac{1}{8} x^4 + \tfrac{1}{48} x^6 + \cdots \right] + B \left[x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \cdots \right]$$

Where A and B are arbitrary constants.

Example 2.

Solve the D.E $y'' - 2xy = 0$ at $x_0 = 0$.

Soln.

$$y'' - 2xy = 0$$

$P(x) = 0$, $Q(x) = -2x$ at $x = 0$.

both $P(x)$ and $Q(x)$ are analytic $x = 0$ is an ordinary point of the D.E then we assume the soln of the form of :

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

- Substitute y'' and y in the DE;

$$y'' - 2xy = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Make powers of x the same,

- let $k = n-2$ in the 1^{st} term and $k = n+1$ in the 2^{nd} term when $k = n-2$, $n = k+2$ and $k = n+1$, $n = k-1$
- Substitute the powers

$$\underbrace{\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k}_{\text{when } k=0} - 2 \sum_{k=1}^{\infty} a_{k-1} x^k = 0$$

$$2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} x^k - 2 \sum_{k+1}^{\infty} a_{k-1} x^k = 0$$

$2a_2 = 0.$ and...

$$\sum_{k=1}^{\infty} \underbrace{\left[(k+2)(k+1) a_{k+2} - 2a_{k-1} \right]}_{\text{coefficient of } x^k} x^k = 0$$

$$(k+2)(k+1) a_{k+2} - 2a_{k-1} = 0$$

$$(k+2)(k+1) a_{k+2} = 2a_{k-1}$$

Reference Relation; $a_{k+2} = \dfrac{2a_{k-1}}{(k+2)(k+1)}$ for $k = 1, 2, 3, 4, \ldots$

but $2a_2 = 0 \Rightarrow a_2 = 0$

$k=1, \quad a_3 = \frac{2a_0}{6} = \frac{1}{3}a_0$

$k=2, \quad a_4 = \frac{2a_1}{12} = \frac{a_1}{6}$

$k=3, \quad a_5 = \frac{2a_2}{20} = \frac{a_2}{10} = 0$

$k=4, \quad a_6 = \frac{2a_3}{30} = \frac{a_3}{15} = \frac{\frac{1}{3}a_0}{15} = \frac{a_0}{45}.$

$k=5, \quad a_7 = \frac{2a_4}{42} = \frac{a_4}{21} = \frac{\frac{1}{6}a_1}{21} = \frac{a_1}{126}$

$k=6, \quad a_8 = \frac{2a_5}{56} = \frac{a_5}{28} = \frac{0}{28} = 0.$

- Then substitute a_1, a_2, a_3, a_4 in the assigned solution.

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \ldots$

$y = a_0 + a_1x + 0 + \frac{1}{3}a_0x^3 + \frac{a_1}{6}x^4 + 0 + \frac{1}{45}a_0x^6 + \frac{1}{126}a_1x^7 + 0 \ldots\ldots$

$$y = a_0\underbrace{\left[1 + \frac{1}{3}x^3 + \frac{1}{45}x^6 + \ldots\right]}_{y_1} + a_1\underbrace{\left[x + \frac{1}{6}x^4 + \frac{1}{126}x^7 + \ldots\right]}_{y_2}$$

- The soln of the given D.E is :-

$$y(x) = A\left[1 + \frac{1}{3}x^3 + \frac{1}{45}x^6 + \ldots\right] + B\left[x + \frac{1}{6}x^4 + \frac{1}{126}x^7 + \ldots\right]$$

Where A and B are arbitrary constants.

SOLUTION OF DE BY FROBENIUS METHOD.

→ If functions defined by $Q_1 = (x - x_0)\, P(x)$ and $Q_2 = (x - x_0)^2 Q(x)$ are analytic at x_0, then x_0 is called a regular singular point.

→ If either (or both) Q_1 and Q_2 is not analytic at x_0, then x_0 is called an irregular singular point.

→ Example;

Find and classify singular points in the DE

$$x^2 (x-2)^2 y'' + 2(x-2) y' + (x+1) y = 0$$

Soln,

$$x^2 (x-2)^2 y'' + 2(x-2) y' + (x+1) y = 0$$

$$y'' + \frac{2}{x^2(x-2)} y' + \frac{(x+1)}{x^2(x-2)^2} y = 0$$

$$P(x) = \frac{2}{x^2(x-2)}, \quad Q(x) = \frac{x+1}{x^2(x-2)^2}$$

$$Q_1 = (x - x_0) P(x) \qquad Q_2 = (x - x_0)^2 Q(x)$$

$x^2(x-2) = 0$ and $x^2(x-2)^2 = 0$

$x = 0$ or $x = 2$ $\qquad$ $x = 0$ or $x = 2$

singular points are $x = 0$ and $x = 2$.

- for $x_0 = 0$

$$Q_1 = \frac{x(2)}{x^3(x-2)} = \frac{2}{x(x-2)} = \frac{2}{0} = \infty \quad \text{not analytic}$$

$$Q_2 = \frac{x^2(x+1)}{x^2(x-2)^2} = \frac{x+1}{(x-2)^2} = \frac{1}{4} \quad \text{is analytic}$$

Q_1 is not analytic at $x_0 = 0$, $x = 0$ is an irregular singular point.

- for $x_0 = 2$

$$Q_1 = \frac{(x-2)\,2}{x^2(x-2)} = \frac{2}{x^2} = \frac{1}{2} \quad \text{is analytic}$$

$$Q_2 = \frac{(x-2)^2(x+1)}{x^2(x-2)^2} = \frac{x+1}{x^2} = 3/4 \quad \text{is analytic}$$

Since Q_1 and Q_2 are analytic at $x_0 = 2$, then $x = 2$ is a regular singular point.

Theorem:

→ If x_0 is a regular singular point of the DE then the DE has at least one nontrivial solution of the form of $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

INDICIAL ROOTS.

→ In general when using assumption $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in

solving 2nd order DE, the indicial equation is a quadratic equation in r resulting from equating to zero. the total coefficients of the lowest power of x. The values of r obtained are called indicial roots.

Case I : If r_1 and r_2 are distinct and donot differ by an integer, then there exist two linearly independent solution of the form of

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1} \qquad a_0 \neq 0$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} \qquad b_0 \neq 0.$$

· General solution of the DE is:

$$Y(x) = y_1(x) + y_2(x)$$

Case II: If r_1 and r_2 are distinct and $r_1 - r_2 = N$ where N is an integer then there exist two linearly independent solution of the form of:

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1} \qquad a_0 \neq 0$$

$$y_2(x) = k y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2} \qquad b_0 \neq 0$$

where k is a constant that could be zero.

Case III: If $r_1 = r_2 = r$ (Double roots) there always exist two linearly independent solutions of the form of

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1} \qquad a_0 \neq 0$$

$$y_2(x) = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2} \qquad b_0 \neq 0$$

Example:

Solve the DF, $2xy'' + y' + y = 0$ at $x_0 = 0$

Soln

$$2xy'' + y' + y = 0$$

$$y'' + \frac{1}{2x} y' + \frac{1}{2x} y = 0$$

$$P(x) = \frac{1}{2x}, \quad Q(x) = \frac{1}{2x}$$

at $x_0 = 0$, $P(x)$ and $Q(x)$ are not analytic

i.e $x_0 = 0$ is a singular point

$q_1 = (x - x_0) P(x) = \frac{x(1)}{2x} = \frac{1}{2}$ is analytic

$Q_2 = (x-x_0)^2 Q(x) = \frac{x^2(1)}{2x} = \frac{x}{2} = 0$ is analytic

so, $x_0 = 0$ is a regular singular point. we assume the solution of the form of

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{—(i)}$$

$$y = a_0 x^r + a_1 x^{1+r} + a_2 x^{2+r} + a_3 x^{3+r} + \dots + a_n x^{n+r} + \dots$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \quad \text{—(ii)}$$

$$y' = r a_0 x^{r-1} + (1+r) a_1 x^r + (2+r) a_2 x^{r+1} + \dots + (n+r) a_n x^{n+r-1} + \dots$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \quad \text{—(iii)}$$

$$y'' = r(r-1) a_0 x^{r-2} + (1+r) r a_1 x^{r-1} + (2+r)(r+1) a_2 x^r + \dots + (n+r)(n+r-1) a_n x^{n+r-2} + \dots$$

Substitute into d.e.

$$2xy'' + y' + y = 0$$

$$2xy'' = 2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' = 2r(r-1) a_0 x^{r-1} + 2(1+r) r a_1 x^r + 2(2+r)(r+1) a_2 x^{r+1} + \dots + 2(n+r)(n+r-1) a_n x^{n+r-1} + \dots \quad \text{—(iv)}$$

Substitute eqn (i), (ii) & (iv) in the given DE and find sum of the coefficient of the lowest term of powers of x

$$r a_0 + 2r(r-1) a_0 = 0$$

$$a_0 (r + 2r^2 - 2r) = 0 \qquad \text{since } a_0 \neq 0$$

$$2r^2 - r = 0$$
$$r = 0 \text{ or } 1/2$$
ie $r_1 = 0$ and $r_2 = 1/2$

Take sum of n^{th} terms in eqn (i), (ii) & (iv)

$$a_n x^{n+r} + (n+r) a_n x^{n+r-1} + 2(n+r)(n+r-1) a_n x^{n+r-1} = 0$$

Substitute $n+1$ ~~in~~ n for eqn (ii) and (iv)

$$a_n x^{n+r} + (n+r+1) a_{n+1} x^{n+r} + 2(n+r+1)(n+r) a_{n+1} x^{n+r} = 0$$

$$a_n + (n+r+1) a_{n+1} + 2(n+r+1)(n+r) a_{n+1} = 0$$

$$[(n+r+1) + 2(n+r+1)(n+r)] a_{n+1} = -a_n.$$

$$[(n+r+1)[1 + 2(n+r)]] a_{n+1} = -a_n$$

$$[(n+r+1)(1 + 2n + 2r)] a_{n+1} = -a_n$$

$$a_{n+1} = \frac{-a_n}{(n+r+1)(1+2n+2r)} \quad \cancel{\text{for } n = 1, 2, 3, \ldots}$$

for $r = 0$

$$a_{n+1} = \frac{-a_n}{(n+1)(1+2n)} \quad \text{for } n = 1, 2, 3, \ldots$$

for $n = 0$, $a_1 = \frac{-a_0}{1} = -a_0$

for $n = 1$, $a_2 = \frac{-a_1}{6} = \frac{a_0}{6}$

for $n = 2$, $a_3 = \frac{-a_2}{15} = \frac{-(a_0/6)}{15} = \frac{-a_0}{80}$

for $n=3$, $a_4 = \dfrac{-a_3}{28} = \dfrac{-(-a_0/80)}{28} = \dfrac{a_0}{2240}$.

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1(x) = a_0 x^r + a_1 x^{1+r} + a_2 x^{2+r} + \cdots\cdots$$

$$y_1(x) = a_0 x^r + -a_0 x^{1+r} + \frac{a_0}{6} x^{2+r} + -\frac{a_0}{90} x^{3+r} + \cdots$$

$$y_1(x) = a_0 \left(x^r - x^{1+r} + \frac{1}{6} x^{2+r} - \frac{1}{90} x^{3+r} + \cdots\cdots \right).$$

for $r = 1/2$.

$$b_{n+1} = \frac{-b_n}{(n+r+1)(1+2n+2r)}$$

$$b_{n+1} = \frac{-b_n}{(n+3/2)(2n+2)} \qquad n = 0, 1, 2, 3, \cdots\cdots$$

for $n=0$, $b_1 = \dfrac{-b_0}{3} = \dfrac{-b_0}{3}$

for $n=1$, $b_2 = \dfrac{-b_1}{10} = \dfrac{-(-b_0/3)}{10} = \dfrac{b_0}{30}$.

for $n=2$, $b_3 = \dfrac{-b_2}{21} = \dfrac{-(b_0/30)}{21} = \dfrac{-b_0}{630}$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+1/2}$$

$$y_2(x) = b_0 x^{1/2} + b_1 x^{3/2} + b_2 x^{5/2} + b_3 x^{7/2} + \cdots\cdots$$

$$y_2(x) = b_0 x^{1/2} + -\frac{b_0}{3} x^{3/2} + \frac{b_0}{30} x^{5/2} + -\frac{b_0}{630} x^{7/2} + \cdots\cdots$$

$$y_2(x) = b_0 \left(x^{1/2} - \frac{1}{3} x^{3/2} + \frac{1}{30} x^{5/2} - \frac{1}{630} x^{7/2} + \cdots\cdots \right)$$

$y(x) = y_1(x) + y_2(x)$

$$y(x) = A\left[1 + x + \tfrac{1}{6}x^2 - \tfrac{1}{90}x^3 + \ldots\right] + B\left[1 - \tfrac{1}{3}x + \tfrac{1}{30}x^2 - \tfrac{1}{630}x^3 + \ldots\right]$$

$$y(x) = A\left[1 - x + \tfrac{1}{6}x^2 - \tfrac{1}{90}x^3 + \ldots\right] + B\left[1 - \tfrac{1}{3}x + \tfrac{1}{30}x^2 - \tfrac{1}{630}x^3 + \ldots\right]$$

where A and B are arbitrary constants.

Example.
Solve the DE $x(x-1)y'' - xy' + y = 0$ at $x_0 = 0$.

soln

$$x(x-1)y'' - xy' + y = 0$$

$$y'' - \frac{x}{x(x-1)}y' + \frac{1}{x(x-1)}y = 0$$

$$y'' - \frac{1}{(x-1)}y' + \frac{1}{x(x-1)}y = 0$$

$$P(x) = \frac{-1}{x-1} \qquad Q(x) = \frac{1}{x(x-1)}$$

by testing the singularity,
$x_0 = 0$ is a regular singular point.

We assume the solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y = a_0 x^r + a_1 x^{1+r} + a_2 x^{2+r} + a_3 x^{3+r} + \ldots + a_n x^{n+r} + \ldots$$

$$y' = r a_0 x^{r-1} + (r+1) a_1 x^r + (r+2) a_2 x^{r+1} + \ldots + (n+r) a_n x^{n+r-1} + \ldots$$

$$y'' = r(r-1) a_0 x^{r-2} + r(r+1) a_1 x^{r-1} + (r+1)(r+2) a_2 x^r + \ldots + (n+r-1)(n+r) a_n x^{n+r-2}$$

$$x(x-1)\sum_{n=0}^{\infty}(n+r-1)(n+r)a_n x^{n+r-2}$$

$$x^2y'' = r(r-1)a_0x^r + r(r+1)a_1x^{r+1} + (r+1)(r+2)a_2x^{r+2} + \dots + (n+r-1)(n+r)a_nx^{n+r}$$

~~$-xy'' = -(r(r$~~

$$-xy'' = -(r-1)ra_0x^{r-1} - r(r+1)a_1x^r + \dots - (n+r)(n+r-1)a_nx^{n+r-1}$$

$$-xy' = -ra_0x^r - (r+1)a_1x^{r+1} + \dots - (n+r)a_nx^{n+r}$$

Substitute y, y' & y'' in the given DE and find / equate to zero the sum of coefficients of the lowest power of x.

$-(r-1)ra_0 = 0$ since $a_0 \neq 0$

$-(r-1)r = 0$ (indicial equation)

$r_1 = 0$ and $r_2 = 1$ (indicial roots)

⇒ find the sum of n^{th} terms -

$$a_nx^{n+r} + (n+r)(n+r-1)a_nx^{n+r} - (n+r)(n+r-1)a_nx^{n+r-1} - (n+r)a_nx^{n+r} = 0$$

Put n+1 in the 3rd term.

$$a_nx^{n+r} + (n+r)(n+r-1)a_nx^{n+r} - (n+r+1)(n+r)a_{n+1}x^{n+r} - (n+r)a_nx^{n+r} = 0$$

$$a_n + (n+r)(n+r-1)a_n - (n+r+1)(n+r)a_{n+1} - (n+r)a_n = 0$$

$$(n+r+1)(n+r)a_{n+1} = (n+r)a_n - a_n - (n+r)(n+r-1)a_n$$

$$(n+r+1)(n+r)a_{n+1} = a_n\left((n+r) - 1 - (n+r)(n+r-1)\right)$$

$$(n+r)(n+r+1)\, a_{n+1} = a_n\left[-1-(n+r)(n+r)\right]$$

$$a_{n+1} = \frac{a_n\left(1+(n+r)(n+r-2)\right)}{(n+r+1)(n+r)}$$ → Reference recurrence Relation.

For $r = 1$

$$a_{n+1} = \frac{a_n\left(1+(n+1)(n-1)\right)}{(n+2)(n+1)}$$

$$a_{n+1} = \frac{n^2 a_n}{(n+2)(n+1)} \quad \text{for } n = 0, 1, 2, 3, \ldots$$

$n=0,\quad a_1 = 0$

$n=1,\quad a_2 = \frac{a_1}{6} = 0$

$n=2,\quad a_3 = \frac{4a_2}{12} = 0$

$$a_1 = a_2 = a_3 = a_4 = a_5 = \ldots = 0.$$

$$y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + a_3x^{r+3} + \ldots$$

$r=1$

$$y = a_0x + 0 + 0 + 0 + \ldots$$

$$y_{(x)} = a_0x$$

y_2 is obtained by the Reduction of order formula.

$$y_{(2)}(x) = y_1 v = y_1 \int \frac{e^{-\int p\,dx}}{y_1(x)^2}\, dx.$$

in $y_{(x)} = a_0 x$ let $a_0 = 1$

$$y_2(x) = x \int \frac{e^{-\int \frac{1}{x-1} dx}}{x^2}$$

$$y_2(x) = x \int \frac{e^{\ln x - 1}}{x^2} dx$$

$$y_2(x) = x \int \frac{x-1}{x^2} dx$$

$$y_2(x) = x \left[\ln x + \frac{1}{x} \right].$$

$$y_2(x) = x \ln x + 1 .$$

General solution;

$$y(x) = y_1(x) + y_2(x)$$

$$y(x) = x + x \ln x + 1$$

Example :

Solve the DE $x(x-1)y'' - (3x-1)y' + y = 0$ at $x_0 = 0$

Soln

$$x(x-1)y'' - (3x-1)y' + y = 0$$

$$y'' - \frac{(3x-1)}{x(x-1)} y' + \frac{1}{x(x-1)} y = 0$$

$$P(x) = \frac{-(3x-1)}{x(x-1)}, \quad Q(x) = \frac{1}{x(x-1)}$$

By testing the singularity

$x = 0$ is a regular singular point

we assume a soln of;

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \dots + a_n x^{n+r} + \dots$$

$$y' = r a_0 x^{r-1} + (r+1) a_1 x^r + (r+2) a_2 x^{r+1} + \dots + (n+r) a_n x^{n+r-1}$$

$$y'' = r(r-1) a_0 x^{r-2} + r(r+1) a_1 x^{r-1} + (r+1)(r+2) a_2 x^r + \dots + (n+r-1)(n+r) a_n x^{n+r-2}$$

Expanding the given DE.

$$x^2 y'' - x y'' + 3 x y' - y' + y = 0$$

The series are;

✓ $-y' = -r a_0 x^{r-1} - (r+1) a_1 x^r - (r+2) a_2 x^{r+1} + \dots - (n+r) a_n x^{n+r-1}$

✓ $3xy' = 3r a_0 x^r + 3(r+1) a_1 x^{r+1} + 3(r+2) a_2 x^{r+2} + \dots + 3(n+r) a_n x^{n+r}$

✓ $-xy'' = -r(r-1) a_0 x^{r-1} - r(r+1) a_1 x^r - (r+1)(r+2) a_2 x^{r+1} + \dots - (n+r-1)(n+r) a_n x^{n+r-1}$

✓ $x^2 y'' = r(r-1) a_0 x^r + r(r+1) a_1 x^{r+1} + \dots + (n+r)(n+r-1) a_n x^{n+r}$

Substitute y, y' & y'' in the DE and equate to zero the sum of coefficients of the lowest powers of x.

$$-r a_0 - r(r-1) a_0 = 0$$

Since $a_0 \neq 0$

$$-r - r^2 + r = 0$$

$$-r^2 = 0$$

$$r_1 = r_2 = 0.$$

Sum of n^{th} terms

$$a_n x^{n+r} - (n+r) a_n x^{n+r-1} + 3(n+r) a_n x^{n+r} - (n+r-1)(n+r) a_n x^{n+r-1} +$$
$$(n+r)(n+r-1) a_n x^{n+r} = 0$$

Replace n by n+1 in 2nd and 4th terms.

$$a_n x^{n+r} - (n+1+r) a_{n+1} x^{n+r} + 3(n+r) a_n x^{n+r} - (n+r)(n+1+r) a_{n+1} x^{n+r} +$$
$$(n+r)(n+r-1) a_n x^{n+r} = 0$$

$$a_n - (n+r+1) a_{n+1} + 3(n+r) a_n - (n+r)(n+r+1) a_{n+1} + (n+r)(n+r+1) a_n = 0$$

$$- a_{n+1}\left[(n+r+1) + (n+r+1)(n+r)\right] = -a_n \left[1 + 3(n+r) + (n+r)(n+r+1)\right]$$

$$a_{n+1}\left[(n+r+1)(n+r+1)\right] = a_n\left[1 + (n+r)(n+r+2)\right]$$

$$a_{n+1} = \frac{a_n\left[1 + (n+r)(n+r+2)\right]}{(n+r+1)^2}$$

For $r=0$

$$a_{n+1} = \frac{a_n(1+n^2+2n)}{(n+1)^2}$$

$$a_{n+1} = a_n \quad \text{for } n = 0, 1, 2, 3, \dots$$

$n=0, \quad a_1 = a_0$

$n=1, \quad a_2 = a_1 = a_0$

$n=2, \quad a_3 = a_2 = a_0$

$n=3, \quad a_4 = a_3 = a_0$

The assumed soln;

$$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \dots$$

$r=0$

$$y = a_0 + a_0 x + a_0 x^2 + a_0 x^3 + \dots$$

$$y = a_0(1 + x + x^2 + x^3 + x^4 + x^5 + \dots)$$

$y_1(x) = a_0 (1 + x + x^2 + x^3 + x^4 + x^5 + \dots)$.

y_2 is obtained by the reduction of order formula.

$$y_2 = y_1 v = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

Note: $\sum x^n = \frac{1}{1-x}$.

$$y_2 = \frac{1}{1-x} \int \frac{e^{-\int \frac{3x-1}{x(x-1)} dx}}{\left(\frac{1}{1-x}\right)^2} dx$$

$$-\int \frac{3x-1}{x(x-1)} dx = -(\ln x + 2\ln(x-1)) = -\ln x - 2\ln(x-1)$$
$$= \ln x^{-1}(x-1)^{-2}$$

$$e^{-\int \frac{3x-1}{x(x-1)} dx} = e^{\ln x^{-1}(x-1)^{-2}} = \frac{(x-1)^{-2}}{x}.$$

$$y_2 = \frac{1}{1-x} \int \frac{(x-1)^{-2}}{x(1-x)^{-2}} dx$$

$$y_2 = \frac{1}{1-x} \ln x$$

$$y_2(x) = \frac{1}{1-x} \ln x.$$

G.S

$y(x) = y_1(x) + y_2(x)$

$y(x) =$

BESSEL'S EQUATION.

Bessel's equation is an ordinary differential equation of of $x^2y'' + xy' + (x^2 - v^2)y = 0$ where v is a real number in its normalized form is.

$$y'' + \frac{y'}{x} + \left(1 - \frac{v^2}{x^2}\right)y = 0$$

$$P(x) = \frac{1}{x} \quad \text{and} \quad Q(x) = 1 - \frac{v^2}{x^2}$$

by testing ;

$x_0 = 0$ is a regular singular point.

$\Rightarrow$ we assume the solution of the form of:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \dots + a_n x^{n+r} + \dots,$$

$$y' = r a_0 x^{r-1} + (r+1) a_1 x^r + (r+2) a_2 x^{r+1} + \dots + (n+r) a_n x^{n+r-1} + \dots,$$

$$y'' = r(r-1) a_0 x^{r-2} + (r+1) r a_1 x^{r-1} + (r+2)(r+1) a_2 x^r + \dots + (n+r)(n+r-1) a_n x^{n+r-2} + \dots$$

so $\quad x^2y'' + xy' + x^2y - v^2y = 0$

$$-v^2y = -v^2 a_0 x^r - v^2 a_1 x^{r+1} - v^2 a_2 x^{r+2} + \dots - v^2 a_n x^{n+r} + \dots,$$

$$x^2y = a_0 x^{r+2} + a_1 x^{r+3} + a_2 x^{r+4} + a_3 x^{r+5} + \dots + a_n x^{n+r+2} + \dots$$

$$xy' = r a_0 x^r + (r+1) a_1 x^{r+1} + (r+2) a_2 x^{r+2} + \dots + (n+r) a_n x^{n+r} + \dots$$

$$x^2y'' = r(r-1) a_0 x^r + (r+1) r a_1 x^{r+1} + (r+2)(r+1) a_2 x^{r+2} + \dots + (n+r)(n+r-1) a_n x^{n+r} + \dots$$

The lowest power of x is x^r, take the sum of coefficients of x^r and equate to zero.

$r(r-1)a_0 + ra_0 - v^2 a_0 = 0$

$a_0(r^2 - r + r - v^2) = 0$

since $a_0 \neq 0$

$r^2 - v^2 = 0$

$r_1 = +v$ and $r_2 = -v$

Take sum of coefficients of x^{r+1} and equate to zero.

$r(r+1)a_1 + (r+1)a_1 - v^2 a_1 = 0$

$a_1(r^2 + r + r + 1 - v^2) = 0$

$a_1(r^2 + 2r + 1 - v^2) = 0$

For $r = v$

$a_1(2v+1) = 0$

$a_1 = 0$

Take sum of n^{th} terms.

$(n+r)(n+r-1)a_n x^{n+r} + (n+r)a_n x^{n+r} + a_n x^{n+r+2} - v^2 a_n x^{n+r} = 0$

Replace n by $n-2$ in the 3rd term.

$\cancel{(n+r)(n+r-1)a_n x^{n+r} + (n+r-2)a_{n-2} x}$

$(n+r)(n+r-1)a_n x^{n+r} + (n+r)a_n x^{n+r} + a_{n-2} x^{n+r} - v^2 a_n x^{n+r} = 0$

$a_n\left[(n+r)(n+r-1) + (n+r) - v^2\right] = -a_{n-2}$

$a_n\left[(n+r)(n+r-1+1) - v^2\right] = -a_{n-2}$

$a_n\left[(n+r)(n+r) - v^2\right] = -a_{n-2}$

$$\boxed{a_n = \frac{-a_{n-2}}{(n+r)^2 - v^2}}$$ Reference Recurrence eqn

For $r = v$

$$a_n = \frac{-a_{n-2}}{n(n+2v)} \quad \text{for } n = 2, 3, 4, 5, \dots$$

$n = 2$, $a_2 = \dfrac{-a_0}{2(2+2v)}$ $\qquad a_1 = 0$

$n = 3$, $a_3 = \dfrac{-a_1}{3(3+2v)} = 0$

$n = 4$, $a_4 = \dfrac{-a_2}{4(4+2v)}$

Since $a_1 = 0$, then $a_1 = a_3 = a_5 = a_{odd} = 0$.

For even index replace n by $2n$

$$a_{2n} = \frac{-a_{2n-2}}{2n(2n+2v)}$$

$$a_{2n} = \frac{-a_{2n-2}}{2^2 n(n+v)} \quad \text{for } n = 1, 2, 3, 4, \dots$$

$n = 1$, $a_2 = \dfrac{-a_0}{2^2 \times 1(1+v)}$

$n = 2$, $a_4 = \dfrac{-a_2}{2^2 \times 2(2+v)} = \dfrac{a_0}{2^4 \times 1 \times 2 \times (1+v)(2+v)}$

$n = 3$, $a_6 = \dfrac{-a_4}{2^2 \times 3(3+v)} = \dfrac{-a_0}{2^6 \times 3!\,(1+v)(2+v)(3+v)}$

$\vdots$

$n = n$, $a_{2n} = \dfrac{(-1)^n a_0}{2^{2n} \times n!\,(1+v)(2+v)(3+v)\dots(n+v)}$

A simpler series can be found by absorbing the growing product $(1+v)(2+v)(3+v)\dots(n+v)$ into $(n+v)!$ so we let $a_0 = \dfrac{1}{2^v v!}$

Since $v!(1+v)(2+v)\dots(n+v) = (n+v)!$

So,

$$a_n = \frac{(-1)^n}{2^{2n+v}\, n!\,(n+v)!}$$

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+v}\, n!\,(n+v)!}\, x^{n+r} \qquad n = 2n,\ r = v$$

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v}\, n!\,(n+v)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+v)!} \cdot \left(\frac{x}{2}\right)^{2n+v}$$

$$y_1(x) = \left(\frac{x}{2}\right)^v \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+v)!} \left(\frac{x}{2}\right)^{2n}$$

$y_1(x) = J_v(x)$ is called Bessel's function of the first kind of order v

i.e
$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+v)!} \left(\frac{x}{2}\right)^{2n}$$

$J_v(x)$ is called Bessel's function of the first kind of order v. The series converges for all values of x. The function $J_v(x)$ and $J_{-v}(x)$ are the solutions of Bessel's eqn. If v is not an integer they are linearly independent and the general soln is given by $y(x) = C_1 J_v(x) + C_2 J_{-v}(x)$ where $J_{-v}(x)$ is ~~integer~~ obtained by substituting $-v$ in $J_v(x)$. If v is an integer the two solns are linearly dependent because $J_{-N}(x) = (-1)^N J_N(x)$

Bessel's function: $J_0(x)$ and $J_1(x)$.

- When $v=0$, $J_0(x)$ is called a Bessel function of the first kind of order 0. Substituting $v=0$ in

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n!(n+v)!}$$ we have;

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! n!}$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4 \times 4} - \frac{x^6}{2^6 \times 6 \times 6} + \frac{x^8}{2^8 \times 4! \times 4!} + \dots$$

It looks similar to cosine function

+ When $v=1$, $J_1(x)$ is called a Bessel's function of the first kind of the order 1. Substituting $v=1$ in

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n!(n+v)!}$$ we have;

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n!(1+n)!}$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 \times 2} + \frac{x^5}{2^5 \times 2! 3!} - \frac{x^7}{2^7 \times 3! 4!} + \dots$$

It looks similar to sine function.

BESSEL'S FUNCTIONS INTERMS OF GAMMA FUNCTIONS.

Recall. $K! = \Gamma{K+1}$, then $(v+n)! = \Gamma{v+n+1}$,

from;

$$a_{2n} = \frac{(-1)^n}{2^{2n+v}.n!(n+v)!} \quad \text{we have.}$$

$$a_{2n} = \frac{(-1)^n}{2^{2n+v}\, n!\, \Gamma{n+v+1}}, \text{ then}$$

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n!(n+v)!}, \text{ becomes}$$

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n . x^{2n+v}}{2^{2n+v} n!\, \Gamma{n+v+1}} = \left(\frac{x}{2}\right)^v \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\, \Gamma{n+v+1}}$$

Similarly;

$$I_v(x) = \left(\frac{x}{2}\right)^{-v} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(n-v)!} = \left(\frac{x}{2}\right)^{-v} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!\, \Gamma{n-v+1}}$$

Question;

UE prove that

(a) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

(b) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

EQUATIONS REDUCIBLE TO BESSEL'S EQUATION.

There are some differential equations which can be reduced (reducible) to Bessel's equation and then can be solved

Example!

Use change of variable $y = x^2 u$ to reduce the DE

$$9x^2y'' - 27xy' + (9x^2+35)y = 0$$ into Bessel's eqn.

Soln

$$9x^2y'' - 27xy' + (9x^2+35)y = 0$$

Given

$y = x^2 u$ $\qquad \dfrac{dy}{dx} = 2xu + x^2\dfrac{du}{dx}$

$$\frac{d^2y}{dx^2} = 2u + 2x\frac{du}{dx} + 2x\frac{du}{dx} + x^2\frac{d^2u}{dx^2}$$

Substitute y, y' & y'' in the DE.

$$9x^2\left(2u + 4x\frac{du}{dx} + x^2\frac{d^2u}{dx^2}\right) - 27x\left(2xu + x^2\frac{du}{dx}\right) + (9x^2+35)x^2u = 0$$

$$18x^2u + 36x^3\frac{du}{dx} + 9x^4\frac{d^2u}{dx^2} - 54x^2u - 27x^3\frac{du}{dx} + 9x^4u + 35x^2u = 0$$

$$9x^4\frac{d^2u}{dx^2} + 9x^3\frac{du}{dx} - x^2u + 9x^4u = 0$$

$$9x^2\frac{d^2u}{dx^2} + 9x\frac{du}{dx} - u + 9x^2u = 0$$

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - \frac{u}{9} + x^2u = 0$$

$$x^2u'' + xu' + x^2u - \frac{1}{9}u = 0$$

$$x^2u'' + xu' + (x^2 - 1/9)u = 0$$

$$x^2u'' + xu' + (x^2 - (1/3)^2)u = 0$$

This is a Bessel's equation of order $v = 1/3$

General soln;

$$y(x) = AJ_v(x) + BJ_{-v}(x)$$

Since $v = 1/3$ not an integer, then the general solution of the Bessel's eqn is:

$u(x) = AJ_{1/3}(x) + BJ_{-1/3}(x)$ where A and B are arbitrary constants;

The original DE will have the general solution of

$$y(x) = \left[A J_{1/3}(x) + B J_{-1/3}(x)\right] x^2$$

$$y(x) = A x^2 J_{1/3}(x) + B x^2 J_{-1/3}(x)$$

Example 2.

Reduce $x^2y'' + xy' + (k^2x^2 - v^2)y = 0$ into Bessel's equation. Where k and v are constant. Use $t = kx$

soln:

$$x^2y'' + xy' + (k^2x^2 - v^2)y = 0$$

Gives

$$t = kx \quad , \quad \frac{dt}{dx} = k \quad , \quad \frac{dy}{dx} = \frac{dt}{dx} \cdot \frac{dy}{dt} = k\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(k\frac{dy}{dt}\right) = \frac{d}{dt}\left(k\frac{dy}{dt}\right) \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = k^2\frac{d^2y}{dt^2}.$$

Substitute into the DE

$$k^2x^2\frac{d^2y}{dt^2} + kx\frac{dy}{dt} + (k^2x^2 - v^2)y = 0$$

from $t = kx$, $k = t/x$

$$x^2\left(\frac{t^2}{x^2}\right)\frac{d^2y}{dt^2} + x\left(\frac{t}{x}\right)\frac{dy}{dt} + \left(x^2\left(\frac{t^2}{x^2}\right) - v^2\right)y = 0$$

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - v^2)y = 0$$

This is a Bessel's equation of the order v. If v is not an integer, the general soln is:

$$y(t) = A J_v(t) + B J_{-v}(t)$$

The original DE will have the general solution of:

$$y(x) = A J_v(kx) + B J_{-v}(kx)$$

Example 3.

Reduce $x^2y'' + xy' + (4x^4 - 1/4)y = 0$ to Bessel's eqn

Use $z = x^2$

Soln:

$$x^2y'' + xy' + (4x^4 - 1/4)y = 0$$

Given;

$$z = x^2 \qquad \frac{dz}{dx} = 2x \quad , \quad \frac{dy}{dx} = \frac{dz}{dx} \cdot \frac{dy}{dz} = 2x\frac{dy}{dz}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(2x\frac{dy}{dz}\right) = 2x\frac{d}{dx}\left(\frac{dy}{dz}\right) + 2\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = 2x\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} + 2\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = 4x^2\frac{d^2y}{dz^2} + 2\frac{dy}{dz}.$$

Substitute into the DE

$$x^2\left(4x^2\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + x\left(2x\frac{dy}{dz}\right) + (4x^4 - 1/4)y = 0$$

$$4x^4\frac{d^2y}{dz^2} + 2x^2\frac{dy}{dz} + 2x^2\frac{dy}{dz} + (4x^4 - 1/4)y = 0.$$

$$4x^4\frac{d^2y}{dz^2} + 4x^2\frac{dy}{dz} + (4x^4 - 1/4)y = 0$$

Div by 4

$$x^4\frac{d^2y}{dz^2} + x^2\frac{dy}{dz} + (x^4 - 1/16)y = 0$$

but $z = x^2$

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - 1/16)y = 0$$

Since $v = 1/4$, the general solution is:

$$y(z) = A\,J_{1/4}(z) + B\,J_{-1/4}(z)$$

Since v is not an integer, the original general soln is

$$y(x) = A\,J_{1/4}(x^2) + B\,J_{-1/4}(x^2).$$

LEGENDRE'S EQUATION.

Legendre's equation is an ODE of the form of $(1-x^2)y'' - 2xy' + k(k+1)y = 0$ where k is a real constant. This one of the most important in engineering problems. It involves a parameter K whose value depends on physical or engineering problem.

→ In its normalized form is

$$y'' - \frac{2x}{(1-x^2)} y' + \frac{k(k+1)}{(1-x^2)} y = 0$$

$$P(x) = \frac{-2}{1-x^2}, \quad Q(x) = \frac{k(k+1)}{(1-x^2)}$$

at $x_0 = 0$, P(x) and Q(x) is analytic, so $x_0 = 0$ is an ordinary point

→ We assume a solution of the form of

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute y, y', y'' in the DE

↓↓↓

The recurrence relation is

$$a_{n+2} = \frac{-(k-n)(k+n+1)a_n}{(n+2)(n+1)} \quad \text{for } n = 0, 1, 2, 3 \ldots$$

For $n=0$, $a_2 = \frac{-k(k+1)}{2 \times 1} a_0$

for $n=1$, $a_3 = \frac{-(k-1)(k+2)}{3 \times 2} a_1$

for $n=2$, $a_4 = \frac{-(k-2)(k+3)a_2}{4\times 3} = \frac{k(k+1)(k-2)(k+3)}{4!}a_0$

for $n=3$, $a_5 = \frac{-(k-3)(k+4)a_3}{5\times 4} = \frac{(k-1)(k+2)(k-3)(k+4)a_1}{5!}$

for $n=4$, $a_6 = \frac{-(k-4)(k+5)a_4}{6\times 5} = \frac{-k(k+1)(k+3)(k-2)(k-4)a_0}{6!}$

Substitute a's in the assumed solution.

$$y = a_0 + a_1 x - \frac{k(k+1)a_0 x^2}{2!} - \frac{(k-1)(k+2)a_1 x^3}{3!} + \frac{k(k+1)(k-2)(k+3)a_0 x^4}{4!} + \frac{(k-1)(k+2)(k-3)(k+4)a_1 x^5}{5!} + \dots$$

$$y = a_0\left[1 - \frac{k(k+1)x^2}{2!} + \frac{k(k+1)(k-2)(k+3)x^4}{4!} + \dots\right] + a_1\left[x - \frac{(k-1)(k+2)x^3}{3!} + \frac{(k-1)(k+2)(k-3)(k+4)x^5}{5!} + \dots\right]$$

The general solution is $y(x) = y_1(x) + y_2(x)$. This means

$$y_1(x) = a_0\left(1 - \frac{k(k+1)x^2}{2!} + \frac{k(k+1)(k-2)(k+3)x^4}{4!} + \dots\right)$$

$$y_2(x) = a_1\left(x - \frac{(k-1)(k+2)x^3}{3!} + \frac{(k-1)(k+2)(k-3)(k+4)x^5}{5!} + \dots\right)$$

$y_1(x)$ and $y_2(x)$ are Linearly independent. The series converges for all $|x|<1$.

LEGENDRE'S POLYNOMIAL $P_n(x)$.

When k is a positive integer n one of the solution series terminates after a finite number of terms. The resulting polynomial in x denoted by $P_n(x)$ is called Legendre polynomial with a_0 and a_1 choosen so that the polynomial has a unit value when $x=1$

For n is even, $y_1(x)$ reduces to a polynomial of degree n.

For n is odd, $y_2(x)$ reduces to a polynomial of degree n.

→ For n is even, $n = 0, 2, 4 \ldots$

$P_0(x) = a_0 \Rightarrow P_0(x) = 1$.

→ For n is odd, $n = 1, 3, 5, \ldots$

$P_1(x) = a_1 x \Rightarrow P_1(x) = x$

$P_2(x) = a_0(1-3x^2) \Rightarrow P_2(x) = -\frac{1}{2}(1-3x^2) \Rightarrow P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = a_1(1-5/3x^2) \Rightarrow P_3(x) = -3/2(x-5/3x^3) \Rightarrow P_3(x) = 1/2(5x^3-3x)$

$P_4(x) = a_0(1-10x^2+35/3x^4) \Rightarrow P_4(x) = 3/8(1-10x^2+35/3x^4)$

$\Rightarrow P_4(x) = \frac{1}{8}(3-30x^2+35x^4)$

$\Rightarrow P_4(x) = 1/8(35x^4-30x^2+3)$

RODRIGUES FORMULA.

Legendre polynomials can be derived using Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$$

Eg. $P_4(x)$

$P_4(x) \Rightarrow n=4$

$$P_4(x) = \frac{1}{2^4 4!} \frac{d^4(x^2-1)^4}{dx^4}$$

$$P_4(x) = \frac{1}{384} \frac{d^4(x^8-4x^6+6x^4-4x^2+1)}{dx^4}$$

$$= \frac{1}{384} \frac{d^3}{dx^3}(8x^7-24x^5+24x^3-8x)$$

$$P_4(x) = \frac{1}{384}\frac{d^2}{dx^2}(56x^6 - 120x^4 + 72x^2 - 8)$$

$$= \frac{1}{384}\frac{d}{dx}(336x^5 - 480x^3 + 144x)$$

$$= \frac{1}{384}(1680x^4 - 1440x^2 + 144)$$

$$= \frac{1}{8}(35x^4 - 30x^2 + 3)$$

GENERATING FUNCTION.

The function $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \quad |t| < 1$ is called the generating function for Legendre polynomials. It is used to obtain some properties of Legendres polynomial.

Example. 1

Use generating function to find $P_n(1)$.

soln.

$P_n(1) \Rightarrow x = 1$.

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Considering LHS., substituting $x = 1$.

$$\frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{\sqrt{(1-t)^2}} = \frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \dots + t^n + \dots$$

$\Rightarrow$ The series is the same as $\sum_{n=0}^{\infty} t^n$

Compare to R.H.S.

$$\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

⇒ Then by taking the coefficients of t^n in both sides, we have;

$$1 = P_n(1)$$

$$\therefore \; P_n(1) = 1.$$

Example 2.

Use generating function to find $P_n(-1)$.

soln.

$P_n(-1) \Rightarrow x = -1$.

$$\frac{1}{\sqrt{1-2xt+t^2}} = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{\sqrt{(1+t)^2}} = \frac{1}{1+t}$$

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 + \dots = \sum_{n=0}^{\infty}(-1)^n t^n$$

Compare to RHS.

$$\sum_{n=0}^{\infty}(-1)^n t^n = \sum_{n=0}^{\infty} P_n(-1)\, t^n$$

By taking the coefficients of t^n

$$(-1)^n = P_n(-1).$$

$$\therefore \; P_n(-1) = (-1)^n$$

POLYNOMIAL AS A SUM FINITE SERIES OF LEGENDRE POLYNOMIALS.

Any polynomial can be written as a sum of finite series of Legendre polynomials as

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

Example 1

$f(x) = x^3 + x^2 + x + 1$ write as a sum of finite series of Legendre polynomials.

soln.

$f(x) = x^3 + x^2 + x + 1$

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$x^3 + x^2 + x + 1 = \sum_{n=0}^{\infty} a_n P_n(x)$$

$x^3 + x^2 + x + 1 = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + a_4 P_4(x) + \cdots$

$x^3 + x^2 + x + 1 = a_0 + a_1 x + a_2 \tfrac{1}{2}(3x^2 - 1) + a_3 \tfrac{1}{2}(5x^3 + 3x) + a_4 \tfrac{1}{8}(35x^4 - 30x^2 + 3) + \cdots$

$x^3 + x^2 + x + 1 = a_0 + a_1 x + \tfrac{3}{2} a_2 x^2 - \tfrac{1}{2} a_2 + \tfrac{5}{2} a_3 x^3 - \tfrac{3}{2} a_3 x$:

compare coefficients of x.

$x^0 : 1 = a_0 - \tfrac{1}{2} a_2 \Rightarrow a_0 = 4/3$

$x^1 : 1 = a_1 - \tfrac{3}{2} a_3 \Rightarrow a_1 = 8/5$

$x^2 : 1 = \tfrac{3}{2} a_2 \Rightarrow a_2 = 2/3$

$x^3 : 1 = \tfrac{5}{2} a_3 \Rightarrow a_3 = 2/5$

$\therefore x^3 + x^2 + x + 1 = \tfrac{4}{3} P_0(x) + \tfrac{8}{5} P_1(x) + \tfrac{2}{3} P_2(x) + \tfrac{2}{5} P_3(x) + \cdots$

Example 2.

Write $f(x) = 3x^3 - 4x^2 + 8$ as a sum of finite series of Legendre polynomials.

Soln.

$f(x) = 3x^3 - 4x^2 + 8$

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$3x^3 - 4x^2 + 8 = \sum_{n=0}^{\infty} a_n P_n(x)$$

$3x^3 + 4x^2 + 8 = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x)$

$3x^3 - 4x^2 + 8 = a_0 + a_1 x + a_2 \tfrac{1}{2}(3x^2 - 1) + a_3 \tfrac{1}{2}(5x^3 - 3x)$

$$3x^3 - 4x^2 + 8 = a_0 + a_1 x + \tfrac{3}{2} a_2 x^2 - \tfrac{1}{2} a_2 + \tfrac{5}{2} a_3 x^3 - \tfrac{3}{2} a_3 x$$

Compare the coefficients of x.

$x^0 : 8 = a_0 - \tfrac{1}{2} a_2 \Rightarrow a_0 = 20/3$

$x^1 : 0 = a_1 - \tfrac{3}{2} a_3 \Rightarrow a_1 = 9/5$

$x^2 : -4 = \tfrac{3}{2} a_2 \Rightarrow a_2 = -8/3$

$x^3 : 3 = \tfrac{5}{2} a_3 \Rightarrow a_3 = 6/5$

Then substituting:

$$3x^3 - 4x^2 + 8 = \tfrac{20}{3} P_0(x) + \tfrac{9}{5} P_1(x) - \tfrac{8}{3} P_2(x) + \tfrac{6}{5} P_3(x)$$

ORTHOGONAL.

Two functions $f(x)$ and $g(x)$ OR $P_m(x)$ and $P_n(x)$ where $m \neq n$ are orthogonal to each other on the interval $a \leq x \leq b$ if $\int_a^b f(x)\, g(x)\, dx = 0$ or $\int_a^b P_m(x)\, P_n(x)\, dx = 0$

Check $P_2(x)$ and $P_3(x)$ on $[-1\ 1]$

$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$, $P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$

from $\int_a^b P_2(x)\, P_3(x) = 0$

LHS.

$$\int_{-1}^{1} \tfrac{1}{2}(3x^2 - 1)\, \tfrac{1}{2}(5x^3 - 3x)\, dx.$$

$$\frac{1}{4} \int_{-1}^{1} (3x^2 - 1)(5x^3 - 3x)\, dx.$$

$$\frac{1}{4} \int_{-1}^{1} (15x^5 - 9x^3 - 5x^3 + 3x)\, dx$$

$$\frac{1}{4} \times 0$$

$$= 0.$$

Since LHS = RHS, then $P_2(x)$ and $P_3(x)$ are orthogonal.

Qn:
Write $f(x) = x^4 + 2x^2 - 1$ as a series of Legendre polynomials.
Soln.

$f(x) = x^4 + 2x^2 - 1$

$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$

$x^4 + 2x^2 - 1 = \sum_{n=0}^{\infty} a_n P_n(x)$

$x^4 + 2x^2 - 1 = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + a_4 P_4(x) + \dots$

$x^4 + 2x^2 - 1 = a_0 + a_1 x + a_2 \frac{1}{2}(3x^2 - 1) + a_3 \frac{1}{2}(5x^3 - 3x) + a_4 \frac{1}{8}(35x^4 - 30x^2 + 3)$

$x^4 + 2x^2 - 1 = a_0 + a_1 x + \frac{3}{2} a_2 x^2 - \frac{1}{2} a_2 + \frac{5}{2} a_3 x^3 - \frac{3}{2} a_3 x + \frac{35}{8} a_4 x^4 - \frac{30}{8} a_4 x^2 + \frac{3}{8} a_4$

Comparing the coefficients of x.

$x^0 : 1 = a_0 - \frac{1}{2} a_2 + \frac{3}{8} a_4 \Rightarrow a_0 = \frac{2}{15}$

$x^1 : 0 = a_1 - \frac{3}{2} a_3 \Rightarrow a_1 = 0$

$x^2 : 2 = \frac{3}{2} a_2 - \frac{30}{8} a_4 \Rightarrow a_2 = \frac{40}{21}.$

$x^3 : 0 = \frac{5}{2} a_3 \Rightarrow a_3 = 0$

$x^4 : 1 = \frac{35}{8} a_4 \Rightarrow a_4 = \frac{8}{35}$

Then substituting:

$x^4 - 4x^2 + 8 = \frac{2}{15} P_0(x) + \frac{40}{21} P_2(x) + \frac{8}{35} P_4(x) + \dots$

Qn
Write $f(x) = 2x^3 + x^2 - 1$ as a series of Legendre polynomials.
Soln

$f(x) = 2x^3 + x^2 - 1$

$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$

$2x^3 + x^2 - 1 = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + \dots$

$2x^3 + x^2 - 1 = a_0 + a_1 x + a_2 \frac{1}{2}(3x^2 - 1) + a_3 \frac{1}{2}(5x^3 - 3x)$

$2x^3 + x^2 - 1 = a_0 + a_1 x + \frac{3}{2} a_2 x^2 - \frac{1}{2} a_2 + \frac{5}{2} a_3 x^3 - \frac{3}{2} a_3 x$

Comparing the coefficients of x.

$x^0 : -1 = a_0 - \frac{1}{2} a_2 \Rightarrow a_0 = -\frac{2}{3}$

$x^1 : 0 = a_1 - \frac{3}{2} a_3 \Rightarrow a_1 = \frac{6}{5}$

$x^2 : 1 = \frac{3}{2} a_2 \quad \Rightarrow \quad a_2 = 2/3$

$x^3 : 2 = \frac{5}{2} a_3 \quad \Rightarrow \quad a_3 = 4/5$

Then substituting:

$$2x^3 + x^2 - 1 = -\frac{2}{3} P_0(x) + \frac{6}{5} P_1(x) + \frac{2}{3} P_2(x) + \frac{4}{5} P_3(x) + \dots$$

COMPLEX ANALYSIS.

INTRODUCTION:

A complex number is any number which has real and imaginary parts.

Let z be a complex number, then

$z = x + iy$ for all $x, y \in \mathbb{R}$ and i is a magionary.

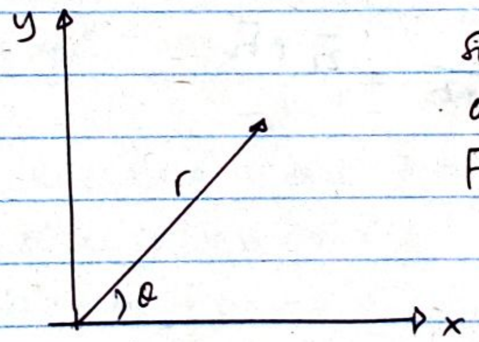
A complex number z can be represented in different forms.

(i) Cartesian form : $z = x + iy$.

(ii) Exponential form ; $z = e^{i\theta}$ / Euler's form.

(iii) Polar form : $z = r\cos\theta + ir\sin\theta$.

Consider:



$\sin\theta = y/r \implies y = r\sin\theta$

$\cos\theta = x/r \implies x = r\cos\theta$

From;

$r^2 = x^2 + y^2 \implies r = \sqrt{x^2+y^2}$

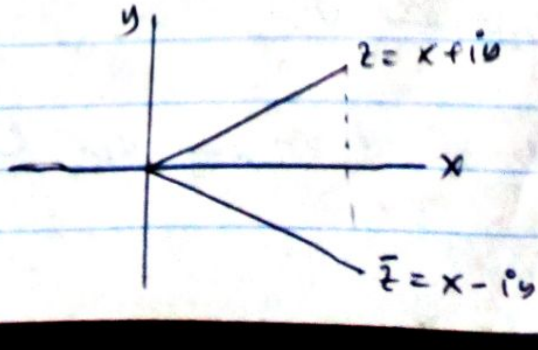
$r = \sqrt{x^2+y^2}$

r is called Modulus of z, denoted by $|z|$.

Then Mod $z = |z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$.

H/W. Read about addition, subtraction, multiplication and division of complex numbers z_1 and z_2 ($z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$)

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is given by $\bar{z} = x - iy$, it is obtained geometrically by reflection of point z in the real axis.



eg. $z = 5 + 2i$

$\bar{z} = 5 - 2i$

Let $z = x + iy$ and the conjugate $\bar{z} = x - iy$.

Then (i) $z + \bar{z} = 2x = 2 \operatorname{Re} z$

(ii) $z - \bar{z} = 2iy = 2 \operatorname{im} z$

(iii) $z\bar{z} = x^2 + y^2$

(iv) $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$

(v) $\overline{z_1 z_2} = \bar{z_1} \cdot \bar{z_2}$

(vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$

(vii) $\overline{z_1 - z_2} = \bar{z_1} - \bar{z_2}$

Example (eg)

$z_1 = 2 + 3i$

$z_2 = 1 + 2i$ prove $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$

Soln

$\bar{z_1} = 2 - 3i$

$\bar{z_2} = 1 - 2i$

$z_1 + z_2 = 3 + 5i$

$\overline{z_1 + z_2} = 3 - 5i$ —(i)

also

$\bar{z_1} + \bar{z_2} = 3 - 5i$ —(ii)

Thus

$\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$ proved.

Functions of a Complex Numbers / variables.

The function of a complex number is denoted by $f(z)$

Then,

$f(z) = w = u(x,y) + iv(x,y)$, we have four real variables $x, y, u,$ and v.

Example 1

Let $w = f(z) = z^2$ find u and v and then find $f(z)$ at $z = 4 + 3i$.

Soln:

$w = f(z) = z^2$ but $z = x + iy$

$= (x+iy)(x+iy)$

$= x^2 - y^2 + 2ixy.$

$u = x^2 - y^2$ and $v = 2xy.$

At $z = 4 + 3i \implies x = 4, y = 3.$

on substituting

$u = 7$ and $v = 24$

$w = f(z) = 7 + 24i.$

Example 2.

let $w = f(z) = 2iz + 6\bar{z}$, find u and v and then find $f(z)$ at $z = 1/2 + 4i$.

Soln:

$w = f(z) = 2iz + 6\bar{z}$ but $z = x + iy.$

$= 2i(x+iy) + 6(x - iy)$

$= 2ix - 2y + 6x - 6iy$

$= 6x - 2y + i(2x - 6y)$

Then $u = 6x - 2y$ and $v = 2x - 6y.$

At $z = 1/2 + 4i \implies x = 1/2$ and $y = 4$

on substituting:

$u = -5$ and $v = -23.$

Thus

$w = f(z) = -5 - 23i.$

Example 3.

Let $f(z) = z^2 + 3z$, find u and v and then find $f(z)$ at $z = 1 + 3i.$

COMPLEX MAPPING AND TRANSFORMATIONS.

Any point in z-plane will similarly be transformed into corresponding point in w-plane depending on the relationship $w = f(z)$ is called transformation function.

The point P on z-plane can be mapped onto P' in w-plane under $w = f(z)$

Example 1

Determine the image of P : $z = 4 + 3i$ on the w-plane under transformation $w = z^2$.

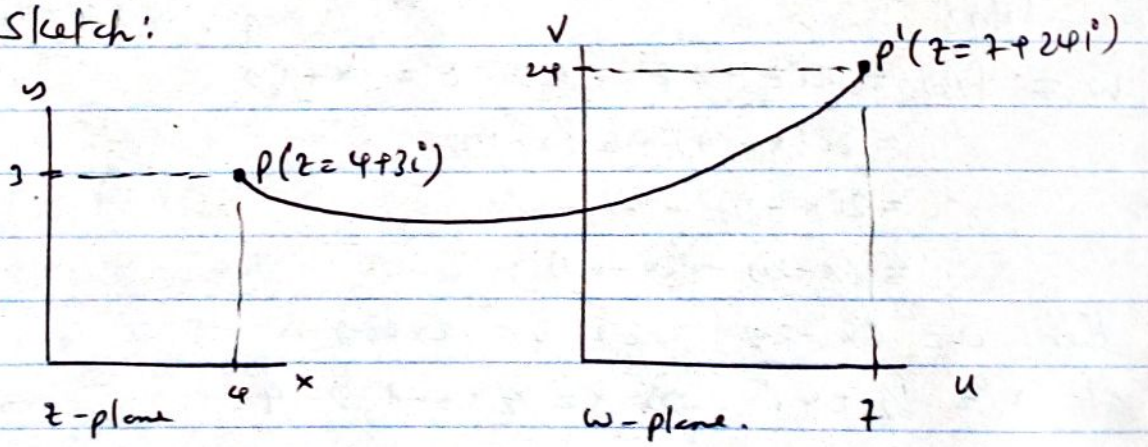
Soln,

$w = f(z) = u + iv$, $u = x^2 - y^2$, $v = 2xy$

$f(z) = 7 + 24i$.

$\therefore$ P' $z = 7 + 24i$,

Sketch:



Example 2.

Determine the image of P $z = 3 + 2i$ on w-plane under $w = 3z + 2 - i$

Soln

$w = f(z) = 3z + 2 - i$ but $z = x + iy$.

$= 3(x + iy) + 2 - i$

$= 3x + 3iy + 2 - i$

$= 3x + 2 + i(3y - 1)$

Then,

$u = 3x + 2$, and $v = 3y - 1$

At $z = 3 + 2i \implies x = 3$ and $y = 2$

On substituting

$u = 11$ and $v = 5$

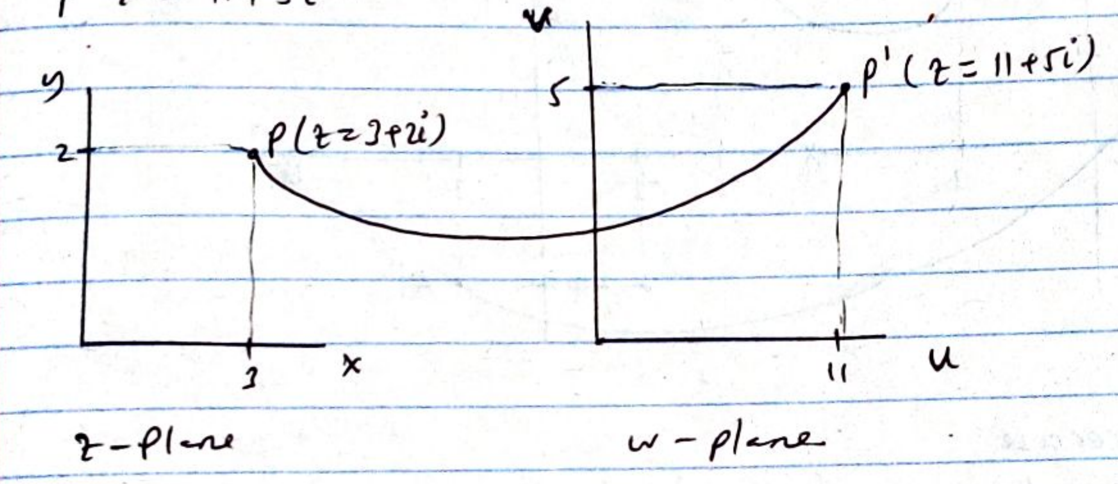
Thus,

$w = 11 + 5i$

$\therefore$

$P' \; z = 11 + 5i$.

Sketch



Example 3.

Map points $A(z = -2 + i)$ and $B(z = 3 + 4i)$ onto w-plane under $w = 2iz + 3$.

Soln.

$w = 2iz + 3$ but $z = x + iy$

$w = 2i(x + iy) + 3$

$w = 2ix - 2y + 3$.

$w = 3 - 2y + 2ix$

Then,

$u = 3 - 2y$ and $v = 2x$.

At A $z = -2 + i \implies x = -2$ and $y = 1$

on substituting, $u = 1$ and $v = -4$

So, $f(z) = 1 - 4i$

$\therefore A'(z = 1 - 4i)$

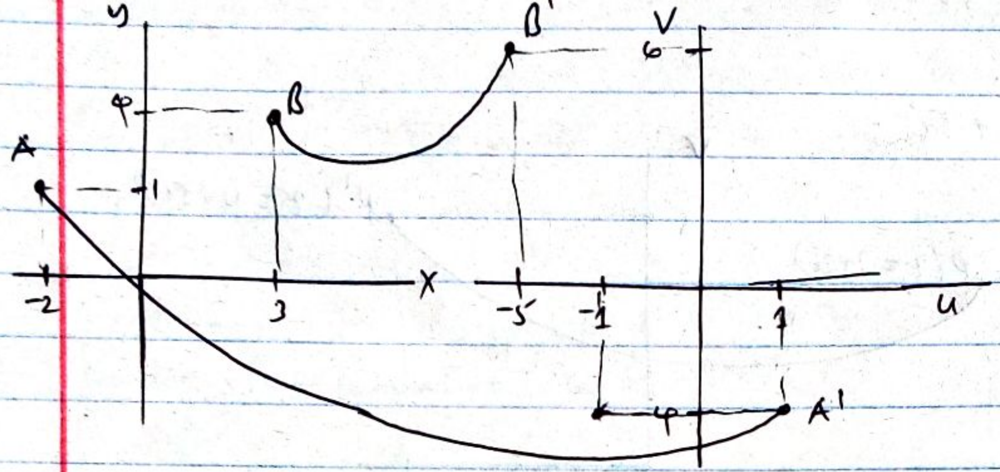
At B $z = 3 + 4i \implies x = 3$ and $y = 4$

on substituting, $u = -5$ and $v = 6$.

So, $f(z) = -5 + 6i$

$\therefore B'(z = -5 + 6i)$

Sketch:



Exercise

Map the following points in z-plane onto w-plane under the transformation $w = f(z)$ stated in each.

1. $z = 4 - 2i$ under $w = 3zi + 2i$
2. $z = -2 - i$ under $w = zi + 3$
3. $z = 3 + 2i$ under $w = (1 + i)z - 2$
4. $z = 2 + i$ under $w = z^2$
5. $z = 1 + 3i$ under $w = z^2 + 3\bar{z}$

DIFFERENTIATION OF A COMPLEX FUNCTION.

The derivative of a complex function f' at a point z_0 is given / written as $f'(z_0)$ and is defined by.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$ provided limits exists

Then f is said to be differentiable at z_0.

If we write $\Delta z = z - z_0$ we have $z = z_0 + \Delta z$, then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Example

The function, $f(z) = z^2$ is differentiable for all z, show!

soln :

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + \Delta z^2 - z^2}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} (2z + \Delta z)$$

$$f'(z) = 2z.$$

The differentiation rules are the same as in real calculus i.e. for any differentiable functions f and g and a constant c we have;

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(fg)' = gf' + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$ as well as the chain rule

$z^n = n z^{n-1}$ where n is an integer.

$\bar{z}$ is not differentiable! prove!

given $f(z) = \bar{z} = x - iy \Rightarrow \Delta z = \Delta x + i\Delta y.$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

as $\Delta y = 0$ (real part)

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

as $\Delta x = 0$ (imaginary part)

$$= \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

$\therefore$ $f(\bar{z})$ does not exist.

HOLOMORPHIC/ANALYTIC FUNCTIONS.

Let $f(z)$ be a complex function in a domain D. A function $f(z)$ is said to be analytic/holomorphic in D if $f(z)$ is defined and differentiable at all points in D.

Example.

prove that $f(z) = z^2$ is analytic at $z_0 = 5$.

Soln:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(5 + \Delta z)^2 - 5^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{5^2 + 10\Delta z + \Delta z^2 - 5^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 10 + \Delta z$$

$$f'(z) = 10$$ proved!

Example 2.

prove that $f(z) = |z|^2$ is not analytic every where but only at the original.

Soln

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$ use $\Delta z = h$.

$$= \lim_{h \to 0} \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{z\bar{z}} + z\bar{h} + \bar{z}h + \cancel{h\bar{h}} - \cancel{z\bar{z}}}{h}$$

$$= \lim_{h \to 0} \frac{z\bar{h} + \bar{z}h}{h}$$

Case I: $z = 0$ (at. origin)

$f(z) = 0$ (Exist)

Case II $z \neq 0$

For real $h \to$ real.

~~= lim~~ $z = x$; $\bar{z} = x$ ie $z = \bar{z}$, $h = \bar{h}$

Then,

$$f'(z) = \lim_{h \to 0} \frac{2zh}{h} = 2z \qquad 2 \operatorname{Re} z.$$

Case III. $(z \neq 0)$

For imaginary part ie $h =$ imaginary

$h = ih$, $z = iy$; $\bar{z} = -iy \Rightarrow \bar{z} = -z$ also $\bar{h} = -ih$.

$$f'(z) = \lim_{h \to 0} \frac{-zh - zh}{h} = -2z \qquad -2 \operatorname{Im} z.$$

Since case II $\neq$ case III

Then $f(z) = |z|^2$ where $z \neq 0$ does not exist.

Note: All analytic functions are continuous
All analytic functions are integrable.

THE CAUCHY - RIEMANN EQNS (C-R eqns)

Let $f(z) = u(x,y) + iv(x,y)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

but $\Delta z = \Delta x + i\Delta y$, then,

$z + \Delta z = u(x,y) + i\,v(x,y) + \Delta z$

$$f'(z) = \cancel{\lim_{\Delta z \to 0} [u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)]}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)]}{\Delta x + i\Delta y} + i\lim_{\Delta z \to 0} \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}$$

$\Rightarrow$ As $\Delta y \to 0$, $\Delta z = \Delta x$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u((x+\Delta x), y) - u(x,y)}{\Delta x} + i\lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{———} (i)$$

$\Rightarrow$ As $\Delta x \to 0$, $\Delta z = i\Delta y$

$\frac{1}{i} = -i$

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{———} (ii)$$

Then,

eqn (i) = eqn (ii)

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

These are C-R eqns.

C-R eqns are used to show whether the given functions are analytic or not.

So, $f(z)$ is analytic in a domain D iff the first partial derivatives of v and u satisfies the C-R eqns.

ie. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$U_x = V_y$ and $V_x = -U_y$

Example 1

Show that $f(z) = z^2$ is analytic for all z.

Soln

$f(z) = z^2 = x^2 + 2ixy - y^2$

$u = x^2 - y^2$, $v = 2xy$.

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2y$

$\frac{\partial v}{\partial x} = 2y$, $\frac{\partial v}{\partial y} = 2x$.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$ and $\frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$

Since $\partial u/\partial x = \partial v/\partial y$ and $\partial v/\partial x = -\partial u/\partial y$ Then $f(z) = z^2$ is analytic for all values of z.

Example 2.

Is $f(z) = e^x(\cos y + i\sin y)$ analytic ?

Soln:

$f(z) = e^x(\cos y + i\sin y)$

$u = e^x \cos y$, $v = e^x \sin y$.

$\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial v}{\partial x} = e^x \sin y$, $\frac{\partial v}{\partial y} = e^x \cos y$.

Then,

$u_x = v_y = e^x \cos y$ and $v_x = e^x \sin y = -u_y$

Since $u_x = v_y$ and $v_x = -u_y$ then $f(z) = e^x(\cos y + i\sin y)$ is analytic.

Example 3.

Is $f(z) = \bar{z}$ analytic ?

Soln:

$f(z) = \bar{z} = x - iy$

$u = x$, $v = -y$.

$\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = -1$

Since

$u_x \neq v_y$ and $v_x = u_y$ Then,

$f(z)$ is not analytic.

Exercise

Are the following functions analytic?

1. $f(z) = iz\bar{z}$
2. $f(z) = e^{-2x}(\cos 2y + i\sin 2y)$
3. $f(z) = e^{x}(\cos y - i\sin y)$
4. $f(z) = Re(z^2) - Im(z^2)$
5. $f(z) = \cos x \cosh y - i\sin x \sinh y$.

HARMONIC FUNCTIONS.

Let $f(z)$ be a complex function in the Domain D such that $f'(z)$ exist, then $f(z)$ is said to be harmonic if $f(z) = u(x,y) + iv(x,y)$ then,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Harmonic property is used to show that the functions are analytic or not.

Example.

Show that $f(z) = |z|^2$ is harmonic only at the origin (analytic at origin).

Soln

$$f(z) = |z|^2 = x^2 + y^2$$

$$u = x^2 + y^2, \quad v = 0$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$

At $z = 0$, $(x, y) = (0, 0)$.

$$\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial^2 u}{\partial y^2} = 0$$

Then,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow f(z) \text{ is harmonic at origin.}$$

• At $z \neq 0$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad , \quad \frac{\partial^2 u}{\partial y^2} = 2$$

Then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2 = 4 \neq 0.$$

$\therefore$ $f(z)$ is not harmonic at $z \neq 0$ but only harmonic at $z = 0$ (origin).

HARMONIC CONJUGATE.

Let $f(z) = u(x,y) + iv(x,y)$ whereby $u(x,y)$ is a real component of $f(z)$ and $v(x,y)$ is an imaginary component of $f(z)$. The two components are harmonic conjugate of each other.

Suppose $u(x,y)$ is known, we can find $v(x,y)$ and vice versa.

When given u or v, the first step is to test whether it is harmonic or not. If it is harmonic then proceed and if it is not harmonic stop.

Example. 1

Given that $x^2 - y^2 - y$ is a real component of $f(z)$ Find its harmonic conjugate and write $f(z)$ in term of $u(x,y)$ and $v(x,y)$

Soln:

$u = x^2 - y^2 - y$

$u_x = 2x \quad , \quad u_{xx} = 2$

$u_y = -2y - 1 \quad , \quad u_{yy} = -2$

$u_{xx} + u_{yy} = 2 + -2 = 0$

So, u is harmonic

$U_x = 2x$, $U_y = -2y - 1$ must satisfy the C-R eqns:
Then,
$U_x = V_y = 2x$ and $V_x = -U_y = 2y + 1$

$$\frac{\partial v}{\partial y} = 2x \Rightarrow \int \partial v = \int 2x \, \partial y$$

$$\Rightarrow v = 2xy + g(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = 2y + \frac{\partial g}{\partial x} = 2y + 1$$

$$\frac{\partial g}{\partial x} = 1 \Rightarrow \int \partial g = \int \partial x$$

$$\Rightarrow g = x + c$$

$\therefore$ The harmonic conjugate

$$V(x,y) = 2xy + x + c$$

Thus,

$$f(z) = x^2 - y^2 - y + i(2xy + x + c).$$

Example 2

Given that $2xy$ is an imaginary component of $f(z)$. Find its harmonic conjugate and then write $f(z)$ in terms of u and v.

Soln.

$V = 2xy$

$V_x = 2y$, $V_{xx} = 0$

$V_y = 2x$, $V_{yy} = 0$

$V_{xx} + V_{yy} = 0$

V is harmonic.

$V_x = 2y$, $V_y = 2x$ must satisfy the C-R eqn.

Then

$V_x = -U_y = 2y$ and $V_y = U_x = 2x$

$$\frac{\partial u}{\partial x} = 2x \Rightarrow \int \partial u = \int 2x \, \partial x$$

$$\Rightarrow u = x^2 + g(y).$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial g}{\partial y} = -2y.$$

$$\Rightarrow \int \partial g = \int -2y \, \partial y$$

$$\Rightarrow g = -y^2 + c$$

The harmonic conjugate

$$u(x, y) = x^2 - y^2 + c$$

Thus,

$$f(z) = x^2 - y^2 + c + i(2xy)$$

Example 3.

Given $x^2 + y^2$ is a real component of $f(z)$. Find its harmonic conjugate and then write $f(z)$ in terms of u and v.

Soln

$u = x^2 + y^2$.

$u_x = 2x$, $u_{xx} = 2$

$u_y = 2y$, $u_{yy} = 2$

$u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$

$\therefore u$ is not harmonic.

$f(z)$ does not exist.

CAUCHY'S THEOREM.

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C and D

$$\int_C f(z)\,dz = 0$$

Proof:

Let $f(z)$ is analytic in D

$$f(z) = u(x,y) + iv(x,y)$$

$$dz = dx + i\,dy$$

$$f(z)\,dz = (u+iv)(dx+i\,dy)$$

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

From Green's Theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Then,

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

From C-R eqns.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

$$\int_C f(z)\,dz = 0 + i(0)$$

$$\int_C f(z)\,dz = 0$$

Hence proved!

CAUCHY'S INTEGRAL FORMULA.

Let $f(z)$ be analytic in a simply connected domain then for any point z_0 in D and any simple closed path c,

$$\int_c \frac{fz}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

i.e

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{fz}{z-z_0}\,dz$$

DERIVATIVES OF ANALYTIC FUNCTIONS.

Let $f(z)$ is analytic in D, then it has derivatives of all orders in D which are then also analytic in D. They are given by:

$$f'(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-z_0)^2}$$

$$f''(z_0) = \frac{2!}{2\pi i}\int_c \frac{f(z)\,dz}{(z-z_0)^3}$$

In General.

$$f^n(z_0) = \frac{n!}{2\pi i}\int_c \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

Example.

Evaluate $\int_c \frac{\sin \pi z \cos \pi z\, dz}{z^3}$

Soln:

$$\int_c \frac{\sin \pi z \cos \pi z \, dz}{z^3}$$

$f(z) = \sin \pi z \cos \pi z$

$z_0 = 0 \Rightarrow (z-0)^3 = z^3.$

$n = 2.$

$$\int_c \frac{\sin \pi z \cos \pi z}{z^3} dz = \frac{2\pi i}{2!} f''(0).$$

$$\int_c \frac{\sin \pi z \cos \pi z \, dz}{z^3} = 0$$

Exercise

Evaluate:

a $\displaystyle\int_c \frac{\sin z \, dz}{(z-\pi)^4}$ (b) $\displaystyle\int_c \frac{e^{4z}}{z^2} dz.$