DIFFERENTIAL EQUATIONS:

SERIES SOLN OF A D.E

- A differential eqn is an eqn which contains derivative(s) of unknown functions. There are two classes of DE
  - Partial DE (PDE)
  - Ordinary DE (ODE)

- A solution of DE is a relation between x and y which does not contain any derivaties example

$Y = A\cos x + B\sin x$ where A and B are arbitary constants.

- A function $f(x)$ is said to be analytic if it is defined everywhere example $y = e^x$, all polynomial functions are analytic functions.

- A point $x = x_0$ is called an ordinary point of a DE $y'' + P(x)y' + Q(x)y = 0$ if both fns $P(x)$ and $Q(x)$ are analytic at x_0.

- A rational function is analytic every where except at all values of x where the denominator is zero

e.g $f(x) = \frac{x}{x^2 - 3x + 2}$.

$f(x)$ is Analytic at all values of x except $x = 1$ and $x = 2$.

- If either $P(x)$ or $Q(x)$ (or both) are not analytic at point x_0, then the point x_0 is called a ~~single~~ singular point of the DE.

→ Example: Given a DE

$(x-1)y'' + xy' + \frac{1}{x^3}y = 0$.

Classify $x_0 = 0$, $x_0 = 1$, $x_0 = 2$.

Soln

In its normal form DE

Divide by $(x-1)$

$$y'' + \frac{x}{x-1}y' + \frac{1}{x^3(x-1)}y = 0.$$

$P(x) = \frac{x}{x-1}$ $\quad Q(x) = \frac{1}{x^3(x-1)}$

at $x_0 = 0$

$P(x) = \frac{0}{-1}$, $P(x)$ is analytical

$Q(x) = \frac{1}{x^3(x-1)} = \frac{1}{0}$ - Undefined.

$Q(x)$ - Not analytical

Hence, $x_0 = 0$ is a singular point of the DE.

at $x_0 = 1$

$P(x) = \frac{x}{x-1} = \frac{1}{0} = \infty$ Not analytic

$Q_x = \frac{1}{x^3(x-1)} = \frac{1}{0} = \infty$ Not analytic

∵ Since both $P(x)$ & $Q(x)$ are not analytic at $x_0 = 1$ then $x_0 = 1$ is a singular point of the DE.

At $x_0 = 2$.

$$P(x) = \frac{x}{x-1} = \frac{2}{1} = \text{It is Analytic.}$$

$$Q(x) = \frac{1}{x^3(x-1)} = \frac{1}{8(1)} = \text{It is Analytic.}$$

$\therefore$ since both $P(x)$ and $Q(x)$ are Analytic at $x_0 = 2$, then $x_0 = 2$ is an Ordinary point of a DE.

Theorem 1. If x_0 is an Ordinary point of a DE, then the DE has two non trivial linearly independent series solution of the form of

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

if $x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

that is:

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

Power series Method.

- This Method is used to solve DE at x_0 where x_0 is and Ordinary point of the DE

Example 1:

Solve the DE $y'' - xy' - y = 0$ at $x_0 = 0$.

Soln.

$P(x) = -x$

$Q(x) = -1$

$x_0 = 0$.

Test for x_0 is singular or Ordinary:

$P(x) = -x(-0) = 0$

$Q(x) = -1 = 0$ at $x_0 = 0$

since the $P(x)$ and $Q(x)$ are Analytical then, x_0 is an Ordinary Point of the DE.

Thus; we assume the condition of $y = \sum_{n=0}^{\infty} a_n x^n$.

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

at $n=1$, $y' =$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

→ from power series

$y = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n \ldots$

differentiating $y: = y'$

$y' = a_1 + 2a_2 x + 3a_3 x^2 \ldots + n a_n x^{n-1}$

differentiang $= y' = y''$

$y'' = 2a_2 + 6a_3 x + \ldots n(n-1) a_n x^{n-2}$ eq 3

$-xy' = -a_1 x - 2a_2 x^2 - 3a_3 x^3 \ldots - n a_n x^n \ldots$ eq 4

$-y = -a_0 - a_1 x - a_2 x^2 \ldots a_n x^n$. eq 5 substitute eq 3, 4 & 5 in the DE,

$2a_2 + 6a_3 x + \dots n(n-1) a_n x^{n-2} - a_1 x - 2a_2 x^2 - 3a_3 x^3 - na_n x^n -$
$a_0 - a_1 x - a_2 x^2 - a_n x^n = 0$

$\cancel{-a_1 x - a_1 x = 0}$

$\cancel{x(-a_1 - a_1) = 0}$

Starting with constants.

$2a_2 - a_0 = 0.$

$2a_2 = a_0$

$a_2 = \frac{1}{2} a_0$ ---- *.

Coefficient of x^1.

$6a_3 x - a_1 x - a_1 x = 0.$

$6a_3 x = 2a_1 x$

$a_3 = \frac{2a_1}{6}$

$a_3 = \frac{1}{3} a_1$ ---- **

Coefficient of x^2.

$\cancel{-2a_2 x^2 - a_2 x^2 = 0}$

$\cancel{-3a_2 = 0}$

$\cancel{a_2 = 0.}$

* take;

$n(n-1) a_n x^{n-2} - na_n x^n - a_n x^n = 0$

Make powers of x the same

let $n-2 = k$ in the 1st term

and let $n = k$ in the 2nd and 3rd term.

$\therefore n = k+2$ and $n = k$

$\cancel{n(n-1)\, a_n\, x^{k+2}}$

$$\Rightarrow (k+2)(k+1)\, a_{k+2}\, x^k - k\, a_k\, x^k - a_k\, x^k = 0.$$

$$\Rightarrow (k+2)\,(k+1)\, a_{k+2} - a_k\,(k+1) = 0$$

Make a_{k+2} the subject.

$$a_{k+2} = \frac{a_k\,(k+1)}{(k+2)\,(k+1)}$$

$$\boxed{a_{k+2} = \frac{a_k}{(k+2)}}$$ ÷ Reoccurrence Relation for $k=0,1,2,3$.

When: $k=0$

$$a_2 = \frac{a_0}{2}.$$

$$k=1,\ a_3 = \frac{a_1}{3}$$

$$k=2,\ a_4 = \frac{a_2}{4}$$

$$k=3,\ a_5 = \frac{a_3}{5}$$

$$k=4,\ a_6 = \frac{a_4}{6}$$

: Now our assumed soln $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \dots$

$$y = a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3} x^3 + \frac{a_0}{8} x^3 + \frac{a_0}{8} x^4 + \frac{a_1}{15} x^3 + \frac{a_0}{48} x^6 + \dots$$

$$y = a_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \dots \right] + a_1 \left[x + \frac{1}{3} x^3 + \frac{1}{15} x^3 + \dots \right]$$

General soln: $y(x) = Ay_1 + By_2$.

$$y(x) = A \left[1 + \frac{1}{2} x^2 + \frac{x^4}{8} + \frac{1}{48} x^6 + \dots \right] + B \left[x + \frac{1}{3} x^3 + \frac{1}{15} x^3 + \dots \right]$$

where A and B are arbitrary constants.

NOTE:

$$n(n-1)a_n x^{n-2} + na_n x^{n+1} + a_n x^{n} = 0$$

$n-2=k \qquad n+1=k \qquad n=k.$

$n=k+2 \qquad n=k-1$

$$\rightarrow (k+2)(k+1)a_{k+2}x^k + (k-1)a_{k-1}x^k + a_k x^k = 0.$$

Make a_{k+2} the subject

$$a_{k+2} = \frac{-(k-1)a_{k-1} - a_k x^k}{(k+2)(k+1)}$$

from $k=1$

Also; For a qn. like; $(x-1)y'' + (1-x)y' - y = 0.$

— 1st right in its normal form to test $P(x)$ & $Q(x)$ then open brackets

$$xy'' - y'' + y' - xy' - y = 0$$

→ In its normal form.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots a_n x^n.$$

Exercise,

1. Show that the soln of the DE $y'' + y = 0$ at $x_0 = 0$ is $y = A\sin x + B\cos x$.

2. Write the following DE at $x_0 = 0$

a) $y'' + 2xy = 0$.

b) $y'' + xy' + y = 0$

c) $(x^2 - 1)y'' + 4xy' + 2y = 0$

d) $(1 + x^2)y'' - xy' + 9y = 0$.