Calculation of $\langle R_{0,i}(t)R_{2,j}(t)\rangle_{\xi,\theta}$

$$\begin{aligned}
\langle R_{0,i}(t)R_{2,j}(t)\rangle_{\xi,\theta} &= \langle R_{0,i}(t)\int_0^t dt' e^{-\kappa\bar{\Gamma}(t-t')}\sum_k M_{jk}(t')R_{1,k}(t')\rangle_{\xi,\theta} \\
&= \int_0^t dt' e^{-\kappa\bar{\Gamma}(t-t')}\Big\langle \sum_k M_{jk}(t')\langle R_{0,i}(t)R_{1,k}(t')\rangle_\xi \Big\rangle_\theta \\
= \Big(\frac{K_BT}{\kappa}\Big)e^{-2\kappa\bar{\Gamma}t}&\int_0^t dt'\int_0^{t'} dt_2'(e^{2\kappa\bar{\Gamma}t_2'}-1)\Big\langle \sum_k M_{jk}(t')M_{ki}(t_2')\Big\rangle_\theta \\
+ v_p^2\int_0^t dt' e^{-\kappa\bar{\Gamma}(t-t')}&\int_0^{t'} dt'' e^{\kappa\bar{\Gamma}(t+t')}\int_0^t dt_1'\int_0^{t''} dt_2'' e^{\kappa\bar{\Gamma}(t_1'+t_2'')}\Big\langle \sum_{k,l} M_{jk}(t')M_{kl}(t'')\hat{n}_i(t_1')\hat{n}_l(t_2'')\Big\rangle_\theta
\end{aligned} \tag{1}$$

Along x and y direction, $i=j$. Also, considering for the case of $t'>t''>t_1'>t_2'$,

we have

$$\sum_{k,l}\Big\langle M_{ik}(t')M_{kl}(t'')\hat{n}_i(t_1')\hat{n}_l(t_2'')\Big\rangle_\theta = e^{-D_r(t_1'-t_2'+4t'-4t'')} \tag{2}$$

Substituting,

$$\begin{aligned}
\langle x_0(t)x_2(t)\rangle_{\xi,\theta} &= \Big(\frac{K_BT}{\kappa}\Big)e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^{t'} dt_2'(e^{2\kappa\bar{\Gamma}t_2'}-1)\Big\langle \cos 2(\theta(t\prime)-\theta(t_2\prime))\Big\rangle_\theta \\
&+ v_p^2\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\int_0^{t'} dt'' e^{4D_rt''}\int_0^t dt_1'\int_0^{t''} dt_2'' e^{(\kappa\bar{\Gamma}-D_r)t_1'}e^{(\kappa\bar{\Gamma}+D_r)t_2''}
\end{aligned} \tag{3}$$

Simplification of the 2^{nd} term of eqn. 3,

$$\begin{aligned}
&v_p^2\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\int_0^{t'} dt'' e^{4D_rt''}\int_0^t dt_1'\int_0^{t''} dt_2'' e^{(\kappa\bar{\Gamma}-D_r)t_1'}e^{(\kappa\bar{\Gamma}+D_r)t_2''} \\
&= v_p^2\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\int_0^{t'} dt'' e^{4D_rt''}\Big[\int_{t''}^t dt_1'\int_0^{t''} dt_2'' e^{(\kappa\bar{\Gamma}-D_r)t_1'}e^{(\kappa\bar{\Gamma}+D_r)t_2''} \\
&\qquad + 2\int_0^{t''} dt_1'\int_0^{t_1'} dt_2'' e^{(\kappa\bar{\Gamma}-D_r)t_1'}e^{(\kappa\bar{\Gamma}+D_r)t_2''}\Big] \\
&= v_p^2\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\int_0^{t'} dt'' e^{4D_rt''}\Big[\frac{e^{(\kappa\bar{\Gamma}-D_r)t}(e^{(\kappa\bar{\Gamma}+D_r)t''}-1)}{(\kappa\bar{\Gamma})^2-D_r^2}+\frac{(2-e^{2\kappa\bar{\Gamma}t''}-e^{(\kappa\bar{\Gamma}-D_r)t''})}{(\kappa\bar{\Gamma})^2-D_r^2} \\
&\qquad + \frac{(e^{2\kappa\bar{\Gamma}t''}-1)}{\kappa\bar{\Gamma}(\kappa\bar{\Gamma}+D_r)}\Big]
\end{aligned} \tag{4}$$

1st term of eqn. 4,

$$\begin{aligned}
&v_p^2\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\int_0^{t'} dt'' e^{4D_rt''}\Big[\frac{e^{(\kappa\bar{\Gamma}-D_r)t}(e^{(\kappa\bar{\Gamma}+D_r)t''}-1)}{(\kappa\bar{\Gamma})^2-D_r^2}\Big] \\
&= \frac{v_p^2e^{(\kappa\bar{\Gamma}-D_r)t}}{(\kappa\bar{\Gamma})^2-D_r^2}\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\Big[\frac{e^{(\kappa\bar{\Gamma}+5D_r)t'}-1)}{(\kappa\bar{\Gamma}+5D_r)}-\frac{e^{4D_rt'}-1)}{(4D_r)}\Big] \\
&= \frac{v_p^2e^{(\kappa\bar{\Gamma}-D_r)t}}{(\kappa\bar{\Gamma})^2-D_r^2}\Big[\frac{1}{(\kappa\bar{\Gamma}+5D_r)}\Big(\frac{e^{(3\kappa\bar{\Gamma}+D_r)t}-1)}{(3\kappa\bar{\Gamma}+D_r)}-\frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1)}{(2\kappa\bar{\Gamma}-4D_r)}\Big) \\
&\qquad - \frac{1}{4D_r}\Big(\frac{e^{2\kappa\bar{\Gamma}t}-1)}{2\kappa\bar{\Gamma}}-\frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1)}{(2\kappa\bar{\Gamma}-4D_r)}\Big)\Big]
\end{aligned} \tag{9}$$

2^{nd} term of eqn. 4,

$$
\begin{aligned}
&\frac{v_p^2}{(\kappa\bar{\Gamma})^2 - D_r^2}\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\int_0^{t'} dt'' e^{4D_r t''}(2 - e^{2\kappa\bar{\Gamma}t''} - e^{(\kappa\bar{\Gamma}-D_r)t''}) \\
&= \frac{v_p^2}{(\kappa\bar{\Gamma})^2 - D_r^2}\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\left[\frac{2(e^{4D_r t'}-1)}{4D_r} - \frac{(e^{(2\kappa\bar{\Gamma}+4D_r)t'}-1)}{(2\kappa\bar{\Gamma}+4D_r)} - \frac{(e^{(\kappa\bar{\Gamma}+3D_r)t'}-1)}{(\kappa\bar{\Gamma}+3D_r)}\right] \\
&= \frac{v_p^2}{(\kappa\bar{\Gamma})^2 - D_r^2}\left[\frac{1}{2D_r}\int_0^t dt'(e^{2\kappa\bar{\Gamma}t'} - e^{(2\kappa\bar{\Gamma}-4D_r)t'}) - \frac{1}{(2\kappa\bar{\Gamma}+4D_r)}\int_0^t dt'(e^{4\kappa\bar{\Gamma}t'} - e^{(2\kappa\bar{\Gamma}-4D_r)t'})\right. \\
&\qquad\left. - \frac{1}{(\kappa\bar{\Gamma}+3D_r)}\int_0^t dt'(e^{(3\kappa\bar{\Gamma}-D_r)t'} - e^{(2\kappa\bar{\Gamma}-4D_r)t'})\right] \\
&= \frac{v_p^2}{(\kappa\bar{\Gamma})^2 - D_r^2}\left[\frac{1}{2D_r}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right) - \frac{1}{(2\kappa\bar{\Gamma}+4D_r)}\left(\frac{e^{4\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right)\right. \\
&\qquad\left. - \frac{1}{(\kappa\bar{\Gamma}+3D_r)}\left(\frac{e^{(3\kappa\bar{\Gamma}-D_r)t}-1}{(3\kappa\bar{\Gamma}-D_r)} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right)\right]
\end{aligned} \tag{6}
$$

3^{rd} term of eqn. 4,

$$
\begin{aligned}
&\frac{v_p^2}{\kappa\bar{\Gamma}(\kappa\bar{\Gamma}+D_r)}\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\int_0^{t'} dt''(e^{(2\kappa\bar{\Gamma}+4D_r)t''} - e^{4D_r t''}) \\
&= \frac{v_p^2}{\kappa\bar{\Gamma}(\kappa\bar{\Gamma}+D_r)}\int_0^t dt' e^{(2\kappa\bar{\Gamma}-4D_r)t'}\left[\frac{e^{(2\kappa\bar{\Gamma}+4D_r)t'}-1}{(2\kappa\bar{\Gamma}+4D_r)} - \frac{e^{4D_r t'}-1}{4D_r}\right] \\
&= \frac{v_p^2}{\kappa\bar{\Gamma}(\kappa\bar{\Gamma}+D_r)}\left[\frac{1}{(2\kappa\bar{\Gamma}+4D_r)}\int_0^t dt'(e^{4\kappa\bar{\Gamma}t'} - e^{(2\kappa\bar{\Gamma}-4D_r)t'}) - \frac{1}{4D_r}\int_0^t dt'(e^{2\kappa\bar{\Gamma}t'} - e^{(2\kappa\bar{\Gamma}-4D_r)t'})\right] \\
&\qquad = \frac{v_p^2}{\kappa\bar{\Gamma}(\kappa\bar{\Gamma}+D_r)}\left[\frac{1}{(2\kappa\bar{\Gamma}+4D_r)}\left(\frac{e^{4\kappa\bar{\Gamma}t}-1}{4\kappa\bar{\Gamma}} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right)\right. \\
&\qquad\qquad\left. - \frac{1}{4D_r}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right)\right]
\end{aligned} \tag{7}
$$

The final expression for $\langle x_0(t)x_2(t)\rangle_{\xi,\theta}$ is

$$
\begin{aligned}
\langle x_0(t)x_2(t)\rangle_{\xi,\theta} &= \left(\frac{K_B T}{\kappa}\right)\left[\frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}(2\kappa\bar{\Gamma}+4D_r)} - \frac{te^{-2\kappa\bar{\Gamma}t}}{4D_r} + \frac{2\kappa\bar{\Gamma}(1-e^{-4D_r t})}{4D_r}\right] \\
&+ \frac{v_p^2}{(\kappa\bar{\Gamma})^2 - D_r^2}\left[\frac{e^{(\kappa\bar{\Gamma}-D_r)t}}{(\kappa\bar{\Gamma}+5D_r)}\left(\frac{(e^{(3\kappa\bar{\Gamma}+D_r)t}-1)}{(3\kappa\bar{\Gamma}+D_r)} - \frac{(e^{(2\kappa\bar{\Gamma}-4D_r)t}-1)}{(2\kappa\bar{\Gamma}-4D_r)}\right)\right. \\
&\qquad - \frac{e^{(\kappa\bar{\Gamma}-D_r)t}}{4D_r}\left(\frac{(e^{2\kappa\bar{\Gamma}t}-1)}{2\kappa\bar{\Gamma}} - \frac{(e^{(2\kappa\bar{\Gamma}-4D_r)t}-1)}{(2\kappa\bar{\Gamma}-4D_r)}\right) \\
&+ \frac{1}{2D_r}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right) - \frac{1}{(2\kappa\bar{\Gamma}+4D_r)}\left(\frac{e^{4\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right) \\
&\qquad\left. - \frac{1}{(\kappa\bar{\Gamma}+3D_r)}\left(\frac{e^{(3\kappa\bar{\Gamma}-D_r)t}-1}{(3\kappa\bar{\Gamma}-D_r)} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right)\right] \\
&+ \frac{v_p^2}{\kappa\bar{\Gamma}(\kappa\bar{\Gamma}+D_r)}\left[\frac{1}{(2\kappa\bar{\Gamma}+4D_r)}\left(\frac{e^{4\kappa\bar{\Gamma}t}-1}{4\kappa\bar{\Gamma}} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right)\right. \\
&\qquad\left. - \frac{1}{4D_r}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}} - \frac{e^{(2\kappa\bar{\Gamma}-4D_r)t}-1}{(2\kappa\bar{\Gamma}-4D_r)}\right)\right]
\end{aligned} \tag{8}
$$