Calculation of $\langle R_{0,i}(t_1)R_{1,j}(t_2)\rangle_{\xi,\theta}$

$$\langle R_{0,i}(t_1)R_{1,j}(t_2)\rangle_\xi = \langle R_{0,i}(t_1)\int_0^{t_2} dt_2' e^{-\kappa\bar{\Gamma}(t_2-t_2')}\sum_k M_{jk}(t_2')R_{0,k}(t_2')\rangle_\xi \tag{1}$$
$$= \int_0^{t_2} dt_2' e^{-\kappa\bar{\Gamma}(t_2-t_2')}\sum_k M_{jk}(t_2')\langle R_{0,i}(t_1)R_{0,k}(t_2')\rangle_\xi$$
$$= \int_0^{t_2} dt_2' e^{-\kappa\bar{\Gamma}(t_2-t_2')}\sum_k M_{jk}(t_2')\Bigg[\left(\frac{K_BT}{\kappa}\right)\delta_{ik}\left(e^{-\kappa\bar{\Gamma}(t_1-t_2')} - e^{-\kappa\bar{\Gamma}(t_1+t_2')}\right) + K_BT\Delta\Gamma e^{-\kappa\bar{\Gamma}(t_1+t_2')}$$
$$\int_0^{min(t_1,t_2')} dt_1' e^{2\kappa\bar{\Gamma}t_1'} M_{ik}(t_1') + v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2')}\int_0^{t_1} dt_1'\int_0^{t_2'} dt'' e^{\kappa\bar{\Gamma}(t_1'+t'')}\langle\hat{n}_i(t_1')n_k(t'')\rangle_\xi\Bigg]$$
$$= \left(\frac{K_BT}{\kappa}\right)e^{-\kappa\bar{\Gamma}(t_1+t_2)}\int_0^{t_2} dt_2' e^{-\kappa\bar{\Gamma}t_2'} M_{ji}(t_2')\left(e^{-\kappa\bar{\Gamma}t_2'} - e^{-\kappa\bar{\Gamma}t_2'}\right) + K_BT\Delta\Gamma e^{-\kappa\bar{\Gamma}(t_1+t_2)}$$
$$\int_0^{t_2} dt_2'\int_0^{min(t_1,t_2')} dt_1' e^{2\kappa\bar{\Gamma}t_1'}\sum_k M_{jk}(t_2')M_{ik}(t_1') +$$
$$v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)}\int_0^{t_2} dt_2'\int_0^{t_1} dt_1'\int_0^{t_2'} dt'' e^{\kappa\bar{\Gamma}(t_1'+t'')}\sum_k M_{jk}(t_2')\hat{n}_i(t_1')n_k(t'')$$

$$\langle R_{0,i}(t_1)R_{1,j}(t_2)\rangle_{\xi,\theta_0} = \left(\frac{K_BT}{\kappa}\right)e^{-\kappa\bar{\Gamma}(t_1+t_2)}\int_0^{t_2} dt_2' e^{-\kappa\bar{\Gamma}t_2'}\langle M_{ji}\rangle_{\theta_0}(t_2')\left(e^{-\kappa\bar{\Gamma}t_2'} - e^{-\kappa\bar{\Gamma}t_2'}\right) + \tag{2}$$
$$K_BT\Delta\Gamma e^{-\kappa\bar{\Gamma}(t_1+t_2)}\int_0^{t_2} dt_2'\int_0^{min(t_1,t_2')} dt_1' e^{2\kappa\bar{\Gamma}t_1'}\sum_k\langle M_{jk}(t_2')M_{ik}(t_1')\rangle_{\theta_0} +$$
$$v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)}\int_0^{t_2} dt_2'\int_0^{t_1} dt_1'\int_0^{t_2'} dt'' e^{\kappa\bar{\Gamma}(t_1'+t'')}\sum_k\left\langle M_{jk}(t_2')\hat{n}_i(t_1')n_k(t'')\right\rangle_{\theta_0}$$

Along the x and y direction, $i=j$. Also, Considering $t_2' > t_1' > t''$

we have

$$\sum_k\left\langle M_{jk}(t_2')\hat{n}_i(t_1')n_k(t'')\right\rangle_{\theta_0} = e^{-D_r(4t_2'-3t_1'-t'')} \tag{3}$$

$$\langle x_0(t_1)x_1(t_2)\rangle_{\xi,\theta_0} = \left(\frac{K_BT}{\kappa}\right)\cos\theta_0 e^{-\kappa\bar{\Gamma}t_1}\left(\frac{e^{(\kappa\bar{\Gamma}-4D_r)t_2} - e^{-\kappa\bar{\Gamma}t_2}}{2\kappa\bar{\Gamma}-4D_r} - \frac{e^{-\kappa\bar{\Gamma}t_2} - e^{-(\kappa\bar{\Gamma}+4D_r)t_2}}{4D_r}\right) + \tag{4}$$
$$\left(\frac{K_BT}{\kappa}\right)\left(\frac{\Delta\Gamma}{2\bar{\Gamma}}\right)e^{-\kappa\bar{\Gamma}t_1}\left[\frac{e^{\kappa\bar{\Gamma}t_2} - e^{-\kappa\bar{\Gamma}t_2}}{2\kappa\bar{\Gamma}+4D_r} - \left(\frac{2\kappa\bar{\Gamma}}{4D_r}\right)\frac{e^{-\kappa\bar{\Gamma}t_2} - e^{-(\kappa\bar{\Gamma}+4D_r)t_2}}{\kappa\bar{\Gamma}+4D_r}\right] +$$
$$v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)}\int_0^{t_2} dt_2' e^{-4D_rt_2'}\int_0^{t_1} dt_1' e^{(\kappa\bar{\Gamma}+3D_r)t_1'}\int_0^{t_2'} dt'' e^{(\kappa\bar{\Gamma}+D_r)t''}$$

Simplification of the 3^{rd} term of eqn. 4,

$$
\begin{aligned}
&v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)} \int_0^{t_2} dt_2' e^{-4D_r t_2'} \int_0^{t_1} dt_1' e^{(\kappa\bar{\Gamma}+3D_r)t_1'} \int_0^{t_2'} dt'' e^{(\kappa\bar{\Gamma}+D_r)t''} \\
&= v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)} \int_0^{t_2} dt_2' e^{-4D_r t_2'} \Bigg[2\int_0^{t_2'} dt_1' e^{(\kappa\bar{\Gamma}+3D_r)t_1'} \int_0^{t_1'} dt'' e^{(\kappa\bar{\Gamma}+D_r)t''} + \\
&\qquad \int_{t_2'}^{t_1} dt_1' e^{(\kappa\bar{\Gamma}+3D_r)t_1'} \int_0^{t_2'} dt'' e^{(\kappa\bar{\Gamma}+D_r)t''} \Bigg] \\
&= v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)} \int_0^{t_2} dt_2' e^{-4D_r t_2'} \Bigg[\frac{2}{\kappa\bar{\Gamma}+D_r} \left(\frac{e^{(2\kappa\bar{\Gamma}+4D_r)t_2'}-1}{(2\kappa\bar{\Gamma}+4D_r)} - \frac{e^{(\kappa\bar{\Gamma}+3D_r)t_2'}-1}{(\kappa\bar{\Gamma}+3D_r)} \right) + \\
&\qquad \frac{(e^{(\kappa\bar{\Gamma}+3D_r)t_1} - e^{(\kappa\bar{\Gamma}+3D_r)t_2'})(e^{(\kappa\bar{\Gamma}+D_r)t_2'}-1)}{(\kappa\bar{\Gamma}+D_r)(\kappa\bar{\Gamma}+3D_r)} \Bigg] \\
&= v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)} \int_0^{t_2} dt_2' \Bigg[\frac{2}{\kappa\bar{\Gamma}+D_r} \left(\frac{e^{(2\kappa\bar{\Gamma}+4D_r)t_2'} - e^{-4D_r t_2'}}{2\kappa\bar{\Gamma}} - \frac{e^{(\kappa\bar{\Gamma}-D_r)t_2'} - e^{-4D_r t_2'}}{(\kappa\bar{\Gamma}+3D_r)} \right) + \\
&\qquad \frac{(e^{(\kappa\bar{\Gamma}+3D_r)t_1} - e^{(\kappa\bar{\Gamma}+3D_r)t_2'})(e^{(\kappa\bar{\Gamma}-3D_r)t_2'} - e^{-4D_r t_2'})}{(\kappa\bar{\Gamma}+D_r)(\kappa\bar{\Gamma}+3D_r)} \Bigg]
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&= v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)} \Bigg[\frac{2}{2\kappa\bar{\Gamma}(\kappa\bar{\Gamma}+D_r)} \left(\frac{e^{(2\kappa\bar{\Gamma}+4D_r)t_2}-1}{(2\kappa\bar{\Gamma}+4D_r)} - \frac{1-e^{-4D_r t_2}}{4D_r} \right) \\
&\quad - \frac{2}{(\kappa\bar{\Gamma}+D_r)(\kappa\bar{\Gamma}+3D_r)} \left(\frac{e^{(\kappa\bar{\Gamma}-D_r)t_2}-1}{(\kappa\bar{\Gamma}-D_r)} - \frac{1-e^{-4D_r t_2}}{4D_r} \right) + \\
&\quad \int_0^{t_2} dt_2' \frac{e^{(\kappa\bar{\Gamma}+3D_r)t_1}\left(e^{(\kappa\bar{\Gamma}-3D_r)t_2'} - e^{-4D_r t_2'}\right) - \left(e^{2\kappa\bar{\Gamma}t_2'} - e^{(\kappa\bar{\Gamma}-D_r)t_2'}\right)}{(\kappa\bar{\Gamma}+D_r)(\kappa\bar{\Gamma}+3D_r)} \Bigg] \\
&= v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)} \Bigg[\frac{2}{2\kappa\bar{\Gamma}(\kappa\bar{\Gamma}+D_r)} \left(\frac{e^{(2\kappa\bar{\Gamma}+4D_r)t_2}-1}{(2\kappa\bar{\Gamma}+4D_r)} - \frac{1-e^{-4D_r t_2}}{4D_r} \right) \\
&\quad - \frac{2}{(\kappa\bar{\Gamma}+D_r)(\kappa\bar{\Gamma}+3D_r)} \left(\frac{e^{(\kappa\bar{\Gamma}-D_r)t_2}-1}{(\kappa\bar{\Gamma}-D_r)} - \frac{1-e^{-4D_r t_2}}{4D_r} \right) + \\
&\quad \frac{1}{(\kappa\bar{\Gamma}+D_r)(\kappa\bar{\Gamma}+3D_r)} \Bigg\{ e^{(\kappa\bar{\Gamma}+3D_r)t_1} \left(\frac{e^{(\kappa\bar{\Gamma}-3D_r)t_2}-1}{(\kappa\bar{\Gamma}-3D_r)} - \frac{1-e^{-4D_r t_2}}{4D_r} \right) - \\
&\quad \left(\frac{e^{2\kappa\bar{\Gamma}t_2}-1}{2\kappa\bar{\Gamma}} - \frac{e^{(\kappa\bar{\Gamma}-D_r)t_2}-1}{(\kappa\bar{\Gamma}-D_r)} \right) \Bigg\}
\end{aligned} \tag{6}
$$

The final expression for $\langle x_0(t_1)x_1(t_2)\rangle_{\xi,\theta_0}$ is

$$
\begin{aligned}
\langle x_0(t_1)x_1(t_2)\rangle_{\xi,\theta_0} &= \left(\frac{K_BT}{\kappa}\right) \cos\theta_0 e^{-\kappa\bar{\Gamma}t_1} \left(\frac{e^{(\kappa\bar{\Gamma}-4D_r)t_2} - e^{-\kappa\bar{\Gamma}t_2}}{2\kappa\bar{\Gamma}-4D_r} - \frac{e^{-\kappa\bar{\Gamma}t_2} - e^{-(\kappa\bar{\Gamma}+4D_r)t_2}}{4D_r} \right) + \\
&\left(\frac{K_BT}{\kappa}\right)\left(\frac{\Delta\Gamma}{2\bar{\Gamma}}\right) e^{-\kappa\bar{\Gamma}t_1} \left[\frac{e^{\kappa\bar{\Gamma}t_2} - e^{-\kappa\bar{\Gamma}t_2}}{2\kappa\bar{\Gamma}+4D_r} - \left(\frac{2\kappa\bar{\Gamma}}{4D_r}\right) \frac{e^{-\kappa\bar{\Gamma}t_2} - e^{-(\kappa\bar{\Gamma}+4D_r)t_2}}{\kappa\bar{\Gamma}+4D_r} \right] + \\
&v_p^2 e^{-\kappa\bar{\Gamma}(t_1+t_2)} \Bigg[\frac{2}{2\kappa\bar{\Gamma}(\kappa\bar{\Gamma}+D_r)} \left(\frac{e^{(2\kappa\bar{\Gamma}+4D_r)t_2}-1}{(2\kappa\bar{\Gamma}+4D_r)} - \frac{1-e^{-4D_r t_2}}{4D_r} \right) \\
&- \frac{2}{(\kappa\bar{\Gamma}+D_r)(\kappa\bar{\Gamma}+3D_r)} \left(\frac{e^{(\kappa\bar{\Gamma}-D_r)t_2}-1}{(\kappa\bar{\Gamma}-D_r)} - \frac{1-e^{-4D_r t_2}}{4D_r} \right) + \\
&\frac{1}{(\kappa\bar{\Gamma}+D_r)(\kappa\bar{\Gamma}+3D_r)} \Bigg\{ e^{(\kappa\bar{\Gamma}+3D_r)t_1} \left(\frac{e^{(\kappa\bar{\Gamma}-3D_r)t_2}-1}{(\kappa\bar{\Gamma}-3D_r)} - \frac{1-e^{-4D_r t_2}}{4D_r} \right) - \\
&\left(\frac{e^{2\kappa\bar{\Gamma}t_2}-1}{2\kappa\bar{\Gamma}} - \frac{e^{(\kappa\bar{\Gamma}-D_r)t_2}-1}{(\kappa\bar{\Gamma}-D_r)} \right) \Bigg\}
\end{aligned} \tag{7}
$$