In the presence of harmonic trap

Potential confinement, $U(x,y) = \kappa(x^2 + y^2)/2$

Langevin Equation in the presence of harmonic trap,

$$\begin{aligned}\frac{\partial x}{\partial t} &= -\kappa x[\bar{\Gamma} + \frac{\Delta\Gamma}{2}\cos 2\theta(t)] - \frac{\kappa y}{2}\Delta\Gamma \sin 2\theta(t) + v_p \hat{n}_x(t) + \xi_x(t) \\ \frac{\partial y}{\partial t} &= -\frac{\kappa x}{2}\Delta\Gamma \sin 2\theta(t) - \kappa y[\bar{\Gamma} - \frac{\Delta\Gamma}{2}\cos 2\theta(t)] + v_p \hat{n}_y(t) + \xi_y(t)\end{aligned} \tag{1}$$

Perturbative Expansion:

Define $R \equiv (x,y)^T$, the above equation can be reduce as

$$\begin{gathered}\dot{R}(t) = -\kappa\Big[\bar{\Gamma}\mathbb{I} + \frac{\Delta\Gamma}{2}\mathbb{M}(t)\Big]R(t) + v_p\hat{\mathbf{n}}(t) + \xi(t) \\ \text{where } \mathbb{M}(t) = \begin{bmatrix} \cos 2\theta(t) & sin 2\theta(t) \\ sin 2\theta(t) & -\cos 2\theta(t)\end{bmatrix}\end{gathered} \tag{2}$$

Perturbative expansion is

$$R(t) = R_0(t) - \left(\frac{\kappa\Delta\Gamma}{2}\right)R_1(t) + \left(\frac{\kappa\Delta\Gamma}{2}\right)^2 R_2(t) + \mathcal{O}\left(\frac{\kappa\Delta\Gamma}{2}\right)^3 \tag{3}$$

Substituting equation 12 in 11, we obtain

$$\begin{aligned}\dot{R}_0(t) &= -\kappa\bar{\Gamma}R_0(t) + \xi(t) + v_p\hat{\mathbf{n}}(t) \\ \dot{R}_1(t) &= -\kappa\bar{\Gamma}R_1(t) + \mathbb{M}(t)R_0(t) \\ \dot{R}_2(t) &= -\kappa\bar{\Gamma}R_2(t) + \mathbb{M}(t)R_1(t)\end{aligned} \tag{4}$$

Taking R(0)=0, solving the above equations

$$\begin{aligned}R_0(t) &= \int_0^t e^{-\kappa\bar{\Gamma}(t-t')}\xi(t')dt' + v_p\int_0^t e^{-\kappa\bar{\Gamma}(t-t')}\hat{\mathbf{n}}(t')dt' \\ R_1(t) &= \int_0^t e^{-\kappa\bar{\Gamma}(t-t')}\mathbb{M}(t')R_0(t')dt' \\ R_2(t) &= \int_0^t e^{-\kappa\bar{\Gamma}(t-t')}\mathbb{M}(t')R_1(t')dt'\end{aligned} \tag{5}$$

The equal time correlation matrix

$$\begin{aligned}\langle R_i(t)R_j(t)\rangle_{\xi,\theta_0} = \langle R_{0,i}(t)R_{0,j}(t)\rangle_{\xi,\theta_0} - \left(\frac{\kappa\Delta\Gamma}{2}\right)\langle R_{0,i}(t)R_{1,j}(t)\rangle_{\xi,\theta_0} + \left(\frac{\kappa\Delta\Gamma}{2}\right)^2\Big[\langle R_{1,i}(t)R_{1,j}(t)\rangle_{\xi,\theta_0} \\ + 2\langle R_{0,i}(t)R_{2,j}(t)\rangle_{\xi,\theta_0}\Big] + \mathcal{O}\left(\frac{\kappa\Delta\Gamma}{2}\right)^3\end{aligned} \tag{6}$$

4. **Calculation of** $\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0}$

$$\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0} = \int_0^t dt' \int_0^t dt'' e^{-\kappa\bar{\Gamma}(t-t')} e^{-\kappa\bar{\Gamma}(t-t'')} \langle \xi(t')\xi(t'')\rangle_{\xi,\theta_0} \tag{7}$$

$$+ v_p^2 \int_0^t dt' \int_0^t dt'' e^{-\kappa\bar{\Gamma}(t-t')} e^{-\kappa\bar{\Gamma}(t-t'')} \langle \hat{\mathbf{n}}(t')\hat{\mathbf{n}}(t'')\rangle_{\xi,\theta_0}$$

$$\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0} = 2K_BTe^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \int_0^t dt'' e^{\kappa\bar{\Gamma}(t'+t'')} \langle \bar{\Gamma}\mathbb{I} + \frac{\Delta\Gamma}{2}\mathbb{M}(\theta(t'))\rangle_{\xi,\theta_0} \delta(t'-t'')$$

$$+ v_p^2 e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \int_0^t dt'' e^{\kappa\bar{\Gamma}(t'+t'')} \left[\frac{\mathbb{I}}{2} e^{-D_r(t'+t''-2\min(t',t''))} + \frac{\mathbb{M}(\theta_0)}{2} e^{-D_r(t'+t''+2\min(t',t''))}\right]$$

Considering the case of $t' > t''$,

$$\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}\mathbb{I}\left(1-e^{-2\kappa\bar{\Gamma}t}\right) + K_BT\Delta\Gamma\mathbb{M}(\theta_0)\left(\frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r}\right)$$

$$+ \frac{v_p^2 e^{-2\kappa\bar{\Gamma}t}}{2}\mathbb{I} \times 2 \int_0^t dt'' \int_{t''}^t dt' \; e^{\kappa\bar{\Gamma}(t'+t'')} e^{-D_r(t'-t'')}$$

$$+ \frac{v_p^2 e^{-2\kappa\bar{\Gamma}t}}{2}\mathbb{M}(\theta_0) \times 2 \int_0^t dt'' \int_{t''}^t dt' \; e^{\kappa\bar{\Gamma}(t'+t'')} e^{-D_r(t'+3t'')}$$

$$\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}\mathbb{I}\left(1-e^{-2\kappa\bar{\Gamma}t}\right) + K_BT\Delta\Gamma\mathbb{M}(\theta_0)\left(\frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r}\right)$$

$$+ v_p^2\mathbb{I}\left[\frac{1-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma})^2-D_r^2} - \frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}(\kappa\bar{\Gamma}-D_r)}\right] + \frac{v_p^2\mathbb{M}(\theta_0)}{(\kappa\bar{\Gamma}-D_r)}\left[\frac{e^{-4D_rt}-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma}-3D_r)} - \frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2(\kappa\bar{\Gamma}-2D_r)}\right]$$

The x and y components are

$$\langle x_0^2(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}\left(1-e^{-2\kappa\bar{\Gamma}t}\right) + K_BT\Delta\Gamma\cos 2\theta_0\left(\frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r}\right) \tag{8}$$

$$+ v_p^2\left[\frac{1-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma})^2-D_r^2} - \frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}(\kappa\bar{\Gamma}-D_r)}\right] + \frac{v_p^2\cos 2\theta_0}{(\kappa\bar{\Gamma}-D_r)}\left[\frac{e^{-4D_rt}-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma}-3D_r)} - \frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2(\kappa\bar{\Gamma}-2D_r)}\right]$$

and

$$\langle y_0^2(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}\left(1-e^{-2\kappa\bar{\Gamma}t}\right) - K_BT\Delta\Gamma\cos 2\theta_0\left(\frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r}\right)$$

$$+ v_p^2\left[\frac{1-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma})^2-D_r^2} - \frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}(\kappa\bar{\Gamma}-D_r)}\right] - \frac{v_p^2\cos 2\theta_0}{(\kappa\bar{\Gamma}-D_r)}\left[\frac{e^{-4D_rt}-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma}-3D_r)} - \frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2(\kappa\bar{\Gamma}-2D_r)}\right]$$

The cross correlation function

$$\langle x_0(t)y_0(t)\rangle_{\xi,\theta_0} = \Delta D\sin 2\theta_0\left(\frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r}\right) + \tag{9}$$

$$\frac{v_p^2\sin 2\theta_0}{(\kappa\bar{\Gamma}-D_r)}\left[\frac{e^{-4D_rt}-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma}-3D_r)} - \frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2(\kappa\bar{\Gamma}-2D_r)}\right]$$

where $\Delta D = K_BT\Delta\Gamma$