Asymmetric [Ellipsoid] active particle:

An asymmetric active particle undergoes self-propulsion with a velocity $v_p\mathbf{\hat{n}}(t)$ along its longer axis in 2-dimension. The angle between the x-axis of the lab frame and long axis of the ellipsoid at time t is represented by $\theta(t)$. The component of the orientation vector $\mathbf{\hat{n}}(t)$ w.r.t the lab frame can be expressed as $\mathbf{\hat{n}}(t) \equiv \Big(\cos\theta(t), \sin\theta(t)\Big)$. The coupled Langevin equation in the presence of an external force F and torque τ are

$$\begin{gathered}\partial_t r_i = \Gamma_{ij}(\theta(t))\, F_j + v_p \hat{n}_i(t) + \xi_i(t), \\ where\ \Gamma_{ij}(\theta(t)) = \bar{\Gamma}\delta_{ij} + \frac{\Delta\Gamma}{2} M_{ij}(\theta(t)) \\ \partial_t\theta = \Gamma_\theta \tau + \xi_\theta(t)\end{gathered} \tag{1}$$

$$\bar{\Gamma} = \frac{\Gamma_\parallel + \Gamma_\perp}{2}, \Delta\Gamma = \Gamma_\parallel - \Gamma_\perp\ and\ M_{ij}(\theta(t)) = \begin{bmatrix}\cos 2\theta(t) & sin2\theta(t) \\ sin2\theta(t) & -\cos 2\theta(t)\end{bmatrix} \tag{2}$$

$\xi_i(t)$ and ξ_θ are the Gaussian random noise with zero mean and

$\langle \xi_\theta(t)\xi_\theta(t\prime)\rangle = 2K_B T\, \Gamma_\theta\ \delta(t-t\prime) = 2D_r\ \delta(t-t\prime)$

for a fixed angle $\theta(t)$,

$\langle \xi_i(t)\xi_j(t\prime)\rangle = 2K_B T\, \Gamma_{ij}(\theta(t))\ \delta(t-t\prime)$

1. **Calculation of $\langle \hat{n}_i(t_1)\hat{n}_j(t_2)\rangle_{\theta_0}$, the average is taken for fixed initial angleθ_0**

$\theta(t_1) \equiv \theta_1, \theta(t_2) \equiv \theta_2$

$$\langle \hat{n}_i(t_1)\hat{n}_j(t_2)\rangle_{\theta_0} = \begin{bmatrix}\langle\cos\theta_1\,\cos\theta_2\rangle & \langle\cos\theta_1\,\sin\theta_2\rangle \\ \langle\sin\theta_1\,\cos\theta_2\rangle & \langle\sin\theta_1\,\sin\theta_2\rangle\end{bmatrix} \tag{3}$$

$2\langle\cos\theta_1\,\cos\theta_2\rangle = e^{-D_r(t_1+t_2-2\,\min(t_1,t_2))} + \cos 2\theta_0 e^{-D_r(t_1+t_2+2\,\min(t_1,t_2))}$

$2\langle\sin\theta_1\,\sin\theta_2\rangle = e^{-D_r(t_1+t_2-2\,\min(t_1,t_2))} - \cos 2\theta_0 e^{-D_r(t_1+t_2+2\,\min(t_1,t_2))}$

$2\langle\cos\theta_1\,\sin\theta_2\rangle = \sin 2\theta_0 e^{-D_r(t_1+t_2+2\,\min(t_1,t_2))}$

$2\langle\cos\theta_2\,\sin\theta_1\rangle = \sin 2\theta_0 e^{-D_r(t_1+t_2+2\,\min(t_1,t_2))}$

Substituting, we have

$$\langle \hat{n}_i(t_1)\hat{n}_j(t_2)\rangle_{\theta_0} = \frac{\delta_{ij}}{2} e^{-D_r(t_1+t_2-2\,\min(t_1,t_2))} + \frac{M_{ij}(\theta_0)}{2} e^{-D_r(t_1+t_2+2\,\min(t_1,t_2))} \tag{4}$$

Without Trap

2. **Langevin equation**:

$$\begin{gathered}\frac{\partial r_i}{\partial t} = v_p\hat{n}_i(t) + \xi_i(t) \\ \implies r_i(t) - r_i(0) = \int_0^t v_p\hat{n}_i(t\prime)dt\prime + \int_0^t \xi_i(t\prime)dt\prime \\ \implies \Delta r_i(t) = \int_0^t v_p\hat{n}_i(t\prime)dt\prime + \int_0^t \xi_i(t\prime)dt\prime\end{gathered} \tag{5}$$

Mean Square Displacement (MSD) :-

$$
\begin{aligned}
\langle \Delta x^2(t)\rangle_{\theta_0} &= v_p^2 \int_0^t dt_1' \int_0^t dt_2' \; \langle \hat{n}_i(t_1')\hat{n}_i(t_2')\rangle_{\theta_0} + \int_0^t dt_1' \int_0^t dt_2' \; \langle \xi_i(t_1')\xi_i(t_2')\rangle_{\theta_0} \qquad (6) \\
\langle \Delta x^2(t)\rangle_{\theta_0} &= v_p^2 \int_0^t dt_1' \int_0^t dt_2' \, [\frac{1}{2} e^{-D_r(t_1'+t_2'-2\min(t_1',t_2'))} + \frac{M_{xx}(\theta_0)}{2} e^{-D_r(t_1'+t_2'+2\min(t_1',t_2'))}] + \\
& 2K_BT \int_0^t dt_1' \int_0^t dt_2' \, \langle \Gamma_{xx}(\theta(t_1'))\rangle_{\theta_0} \; \delta(t_1'-t_2') \\
& \text{Considering } t_1' > t_2', \\
\langle \Delta x^2(t)\rangle_{\theta_0} &= 2v_p^2 \int_0^t dt_2' \int_{t_2'}^t dt_1' \; [\frac{1}{2} e^{-D_r(t_1'-t_2')}] + 2v_p^2 \int_0^t dt_2' \int_{t_2'}^t dt_1' \, \frac{M_{xx}(\theta_0)}{2} e^{-D_r(t_1'+3t_2')} \\
& + 2K_BT \int_0^t dt_1' \int_0^t dt_2' \, [\bar{\Gamma} + \frac{\Delta\Gamma}{2} \langle \cos 2\theta(t_1')\rangle_{\theta_0}] \; \delta(t_1'-t_2') \\
\langle \Delta x^2(t)\rangle_{\theta_0} &= 2v_p^2 \int_0^t dt_2' \int_{t_2'}^t dt_1' \; [\frac{1}{2} e^{-D_r(t_1'-t_2')}] + 2v_p^2 \int_0^t dt_2' \int_{t_2'}^t dt_1' \, \frac{M_{xx}(\theta_0)}{2} e^{-D_r(t_1'+3t_2')} \\
& + 2K_BT \int_0^t dt_1' \int_0^t dt_2' \, \bar{\Gamma} \; \delta(t_1'-t_2') + 2K_BT \frac{\Delta\Gamma}{2} \int_0^t dt_1' \int_0^t dt_2' \; \cos 2\theta_0 e^{-4D_r t_1'} \; \delta(t_1'-t_2') \\
& \text{Take } \tau_r = \frac{1}{2D_r}, \tau_1(t) = \frac{1-e^{-D_r t}}{D_r}, \tau_4(t) = \frac{1-e^{-4D_r t}}{4D_r} \\
\langle \Delta x^2(t)\rangle_{\theta_0} &= 2\tau_r v_p^2 [t-\tau_1(t)] + \frac{2\tau_r v_p^2}{3} \cos 2\theta_0 [\tau_1(t)-\tau_4(t)] + 2K_BT\,\bar{\Gamma} t + K_BT\Delta\Gamma \; \cos 2\theta_0 \tau_4(t)
\end{aligned}
$$

Similarly,

$$
\langle \Delta y^2(t)\rangle_{\theta_0} = 2\tau_r v_p^2 [t-\tau_1(t)] - \frac{2\tau_r v_p^2}{3} \cos 2\theta_0 [\tau_1(t)-\tau_4(t)] + 2K_BT\,\bar{\Gamma} t - K_BT\Delta\Gamma \; \cos 2\theta_0 \tau_4(t) \qquad (7)
$$

Therefore,

$$
\begin{aligned}
\langle \Delta r^2(t)\rangle_{\theta_0} &= \langle \Delta x^2(t)\rangle_{\theta_0} + \langle \Delta y^2(t)\rangle_{\theta_0} \qquad (8) \\
\langle \Delta r^2(t)\rangle_{\theta_0} &= 4(K_BT\,\bar{\Gamma} + \tau_r v_p^2)t - 4\tau_r v_p^2 \tau_1(t)
\end{aligned}
$$

3. **The correlation** $\langle \Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0}$

$$
\begin{aligned}
\langle \Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} &= v_p^2 \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \, \langle \hat{n}_x(t_1')\hat{n}_x(t_2')\rangle_{\theta_0} + \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \, \langle \xi_x(t_1')\xi_x(t_2')\rangle_{\theta_0} \qquad (9) \\
\langle \Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} &= v_p^2 \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \, [\frac{1}{2} e^{-D_r(t_1'+t_2'-2\min(t_1',t_2'))} + \frac{M_{xx}(\theta_0)}{2} e^{-D_r(t_1'+t_2'+2\min(t_1',t_2'))}] \\
& + 2K_BT \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \, \langle \Gamma_{xx}(\theta(t_1'))\rangle_{\theta_0} \; \delta(t_1'-t_2') \\
\langle \Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} &= v_p^2 \int_0^{t_2} dt_2' \int_{t_2'}^{t_2} dt_1' \, \frac{1}{2} e^{-D_r(t_1'-t_2')} + v_p^2 \int_0^{t_2} dt_2' \int_{t_2'}^{t_1} dt_1' \, \frac{1}{2} e^{-D_r(t_1'-t_2')} \\
& + v_p^2 \frac{M_{xx}(\theta_0)}{2} \int_0^{t_2} dt_2' \int_{t_2'}^{t_2} dt_1' \, e^{-D_r(t_1'+3t_2')} + v_p^2 \frac{M_{xx}(\theta_0)}{2} \int_0^{t_2} dt_2' \int_{t_2'}^{t_1} dt_1' \, e^{-D_r(t_1'+3t_2')} \\
& + 2K_BT \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \, [\bar{\Gamma} + \frac{\Delta\Gamma}{2} \langle \cos 2\theta(t_1')\rangle_{\theta_0}] \delta(t_1'-t_2') \\
\langle \Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} &= v_p^2 \tau_r \left[2t_2 - \tau_1(t_2) - 2\tau_r \left(e^{\frac{-(t_1-t_2)}{2\tau_r}} - e^{\frac{-t_1}{2\tau_r}} \right) \right] + \cos 2\theta_0 \tau_r v_p^2 \left[\frac{2\tau_4(t_2)}{3} - \frac{\tau_1(t_2)}{3} - e^{\frac{-t_1}{2\tau_r}} \tau_3(t_2) \right] \\
& + 2K_BT\,\bar{\Gamma} t_2 + K_BT\Delta\Gamma \; \cos 2\theta_0 \tau_4(t_2)
\end{aligned}
$$

In the presence of harmonic trap

Potential confinement, $U(x,y) = \kappa(x^2+y^2)/2$

Langevin Equation in the presence of harmonic trap,

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= -\kappa x [\bar{\Gamma} + \frac{\Delta\Gamma}{2} \cos 2\theta(t)] - \frac{\kappa y}{2} \Delta\Gamma \sin 2\theta(t) + v_p \hat{n}_x(t) + \xi_x(t) \qquad (10) \\
\frac{\partial y}{\partial t} &= -\frac{\kappa x}{2} \Delta\Gamma \sin 2\theta(t) - \kappa y [\bar{\Gamma} - \frac{\Delta\Gamma}{2} \cos 2\theta(t)] + v_p \hat{n}_y(t) + \xi_y(t)
\end{aligned}
$$

Perturbative Expansion:

Define $R \equiv (x,y)^T$, the above equation can be reduce as

$$\dot{R}(t) = -\kappa\Big[\bar{\Gamma}\mathbb{I} + \frac{\Delta\Gamma}{2}\mathbb{M}(t)\Big]R(t) + v_p\hat{\mathbf{n}}(t) + \xi(t) \tag{11}$$
$$\text{where } \mathbb{M}(t) = \begin{bmatrix} \cos 2\theta(t) & sin2\theta(t) \\ sin2\theta(t) & -\cos 2\theta(t) \end{bmatrix}$$

Perturbative expansion is

$$R(t) = R_0(t) - \bigg(\frac{\kappa\Delta\Gamma}{2}\bigg)R_1(t) + \bigg(\frac{\kappa\Delta\Gamma}{2}\bigg)^2 R_2(t) + \mathcal{O}\bigg(\frac{\kappa\Delta\Gamma}{2}\bigg)^3 \tag{12}$$

Substituting equation 12 in 11, we obtain

$$\dot{R}_0(t) = -\kappa\bar{\Gamma}R_0(t) + \xi(t) + v_p\hat{\mathbf{n}}(t) \tag{13}$$
$$\dot{R}_1(t) = -\kappa\bar{\Gamma}R_1(t) + \mathbb{M}(t)R_0(t)$$
$$\dot{R}_2(t) = -\kappa\bar{\Gamma}R_2(t) + \mathbb{M}(t)R_1(t)$$

Taking R(0)=0, solving the above equations

$$R_0(t) = \int_0^t e^{-\kappa\bar{\Gamma}(t-t')}\xi(t')dt' + v_p\int_0^t e^{-\kappa\bar{\Gamma}(t-t')}\hat{\mathbf{n}}(t')dt' \tag{14}$$
$$R_1(t) = \int_0^t e^{-\kappa\bar{\Gamma}(t-t')}\mathbb{M}(t')R_0(t')dt'$$
$$R_2(t) = \int_0^t e^{-\kappa\bar{\Gamma}(t-t')}\mathbb{M}(t')R_1(t')dt'$$

The equal time correlation matrix

$$\langle R_i(t)R_j(t)\rangle_{\xi,\theta_0} = \langle R_{0,i}(t)R_{0,j}(t)\rangle_{\xi,\theta_0} - \bigg(\frac{\kappa\Delta\Gamma}{2}\bigg)\langle R_{0,i}(t)R_{1,j}(t)\rangle_{\xi,\theta_0} + \bigg(\frac{\kappa\Delta\Gamma}{2}\bigg)^2\Big[\langle R_{1,i}(t)R_{1,j}(t)\rangle_{\xi,\theta_0} \tag{15}$$
$$+ 2\langle R_{0,i}(t)R_{2,j}(t)\rangle_{\xi,\theta_0}\Big] + \mathcal{O}\bigg(\frac{\kappa\Delta\Gamma}{2}\bigg)^3$$

4. **Calculation of** $\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0}$

$$\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0} = \int_0^t dt'\int_0^t dt'' e^{-\kappa\bar{\Gamma}(t-t')}e^{-\kappa\bar{\Gamma}(t-t'')}\langle\xi(t')\xi(t'')\rangle_{\xi,\theta_0} \tag{16}$$
$$+ v_p^2\int_0^t dt'\int_0^t dt'' e^{-\kappa\bar{\Gamma}(t-t')}e^{-\kappa\bar{\Gamma}(t-t'')}\langle\hat{\mathbf{n}}(t')\hat{\mathbf{n}}(t'')\rangle_{\xi,\theta_0}$$
$$\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0} = 2K_BTe^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt'' e^{\kappa\bar{\Gamma}(t'+t'')}\langle\bar{\Gamma}\mathbb{I} + \frac{\Delta\Gamma}{2}\mathbb{M}(\theta(t'))\rangle_{\xi,\theta_0}\delta(t'-t'')$$
$$+ v_p^2e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt'' e^{\kappa\bar{\Gamma}(t'+t'')}\bigg[\frac{\mathbb{I}}{2}e^{-D_r(t'+t''-2\min(t',t''))} + \frac{\mathbb{M}(\theta_0)}{2}e^{-D_r(t'+t''+2\min(t',t''))}\bigg]$$

Considering the case of $t' > t''$,

$$\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}\mathbb{I}\Big(1 - e^{-2\kappa\bar{\Gamma}t}\Big) + K_BT\Delta\Gamma\mathbb{M}(\theta_0)\bigg(\frac{e^{-4D_rt} - e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma} - 4D_r}\bigg)$$
$$+ \frac{v_p^2e^{-2\kappa\bar{\Gamma}t}}{2}\mathbb{I}\times 2\int_0^t dt''\int_{t''}^t dt'\ e^{\kappa\bar{\Gamma}(t'+t'')}e^{-D_r(t'-t'')}$$
$$+ \frac{v_p^2e^{-2\kappa\bar{\Gamma}t}}{2}\mathbb{M}(\theta_0)\times 2\int_0^t dt''\int_{t''}^t dt'\ e^{\kappa\bar{\Gamma}(t'+t'')}e^{-D_r(t'+3t'')}$$
$$\langle R_0(t)R_0(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}\mathbb{I}\Big(1 - e^{-2\kappa\bar{\Gamma}t}\Big) + K_BT\Delta\Gamma\mathbb{M}(\theta_0)\bigg(\frac{e^{-4D_rt} - e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma} - 4D_r}\bigg)$$
$$+ v_p^2\mathbb{I}\bigg[\frac{1 - e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma})^2 - D_r^2} - \frac{1 - e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}(\kappa\bar{\Gamma} - D_r)}\bigg] + \frac{v_p^2\mathbb{M}(\theta_0)}{(\kappa\bar{\Gamma} - D_r)}\bigg[\frac{e^{-4D_rt} - e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma} - 3D_r)} - \frac{e^{-4D_rt} - e^{-2\kappa\bar{\Gamma}t}}{2(\kappa\bar{\Gamma} - 2D_r)}\bigg]$$

The x and y components are

$$\langle x_0^2(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}\left(1-e^{-2\kappa\bar{\Gamma}t}\right) + K_BT\Delta\Gamma\cos 2\theta_0\left(\frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r}\right) \tag{17}$$

$$+ v_p^2\left[\frac{1-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma})^2-D_r^2} - \frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}(\kappa\bar{\Gamma}-D_r)}\right] + \frac{v_p^2\cos 2\theta_0}{(\kappa\bar{\Gamma}-D_r)}\left[\frac{e^{-4D_rt}-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma}-3D_r)} - \frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2(\kappa\bar{\Gamma}-2D_r)}\right]$$

and

$$\langle y_0^2(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}\left(1-e^{-2\kappa\bar{\Gamma}t}\right) - K_BT\Delta\Gamma\cos 2\theta_0\left(\frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r}\right)$$

$$+ v_p^2\left[\frac{1-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma})^2-D_r^2} - \frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}(\kappa\bar{\Gamma}-D_r)}\right] - \frac{v_p^2\cos 2\theta_0}{(\kappa\bar{\Gamma}-D_r)}\left[\frac{e^{-4D_rt}-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma}-3D_r)} - \frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2(\kappa\bar{\Gamma}-2D_r)}\right]$$

The cross correlation function

$$\langle x_0(t)y_0(t)\rangle_{\xi,\theta_0} = \Delta D\sin 2\theta_0\left(\frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r}\right) + \frac{v_p^2\sin 2\theta_0}{(\kappa\bar{\Gamma}-D_r)}\left[\frac{e^{-4D_rt}-e^{-(D_r+\kappa\bar{\Gamma})t}}{(\kappa\bar{\Gamma}-3D_r)} - \frac{e^{-4D_rt}-e^{-2\kappa\bar{\Gamma}t}}{2(\kappa\bar{\Gamma}-2D_r)}\right] \tag{18}$$

where $\Delta D = K_BT\Delta\Gamma$

5. **Calculation of** $\langle R_{0,i}(t)R_{1,j}(t)\rangle_{\xi,\theta_0}$

$$\langle R_{0,i}(t)R_{1,j}(t)\rangle_{\xi,\theta_0} = \left\langle R_{0,i}(t)\int_0^t e^{-\kappa\bar{\Gamma}(t-t')}\sum_k M_{jk}(t')R_0(t')dt'\right\rangle_{\xi,\theta_0} \tag{19}$$

$$\langle R_{0,i}(t)R_{1,j}(t)\rangle_{\xi,\theta_0} = \left\langle \int_0^t dt' e^{-\kappa\bar{\Gamma}(t-t')}\sum_k M_{jk}(t')\langle R_{0,i}(t)R_{0,j}(t')\rangle_\xi\right\rangle_{\theta_0}$$

$$\langle R_{0,i}(t)R_{0,j}(t')\rangle_\xi = \frac{K_BT}{\kappa}\delta_{ij}\left[e^{-\kappa\bar{\Gamma}|t-t'|} - e^{-\kappa\bar{\Gamma}(t+t')}\right] \tag{20}$$

$$+ K_BT\Delta\Gamma e^{-\kappa\bar{\Gamma}(t+t')}\int_0^{\min(t,t\prime)} dt_1' e^{-\kappa\bar{\Gamma}t_1'}M_{ij}(t_1') + v_p^2 e^{-\kappa\bar{\Gamma}(t+t')}\int_0^t dt_1'\int_0^{t'} dt_2' e^{\kappa\bar{\Gamma}(t_1'+t_2')}\langle\hat{n}_i(t_1')\hat{n}_j(t_2')\rangle_\xi$$

Substitute the value of $\langle R_{0,i}(t)R_{0,j}(t')\rangle_\xi$ in equation 19,

$$\langle R_{0,i}(t)R_{1,j}(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}M_{ji}(\theta_0)e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\left[e^{(2\kappa\bar{\Gamma}-4D_r)t'} - e^{-4D_rt'}\right] \tag{21}$$

$$+ K_BT\Delta\Gamma e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^{t'} dt_1'\left\langle\sum_k M_{jk}(t')M_{ik}(t_1')\right\rangle_{\theta_0}$$

$$+ v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt_1'\int_0^{t'} dt_2' e^{\kappa\bar{\Gamma}(t_1'+t_2')}\left\langle\sum_k M_{jk}(t')\hat{n}_i(t_1')\hat{n}_k(t_2')\right\rangle_{\theta_0}$$

For the MSD along x and y directions, i=j,

$$\left\langle\sum_k M_{ik}(t')M_{ik}(t_1')\right\rangle_{\theta_0} = e^{-4D_r(t'-t_1')} \tag{22}$$

$$\left\langle\sum_k M_{ik}(t')\hat{n}_i(t_1')\hat{n}_k(t_2')\right\rangle_{\theta_0} = e^{-D_r(4t'-3t_1'-t_2')}$$

where we use the case of $t' > t_1' > t_2'$ to solve the above equation.

The third term of equation 21:

$$
\begin{aligned}
&v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt_1'\int_0^{t'} dt_2' e^{\kappa\bar{\Gamma}(t_1'+t_2')}e^{-D_r(4t'-3t_1'-t_2')} = v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt' e^{-4D_r t'}\int_0^t dt_1'\int_0^{t'} dt_2' e^{(\kappa\bar{\Gamma}+3D_r)t_1'}e^{(\kappa\bar{\Gamma}+D_r)t_2'} \qquad (23)\\
&= v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt' e^{-4D_r t'}\left[\int_0^{t'} dt_2' e^{(\kappa\bar{\Gamma}+D_r)t_2'}\int_{t_2'}^t dt_1' e^{(\kappa\bar{\Gamma}+3D_r)t_1'} + \int_0^{t'} dt_2' e^{(\kappa\bar{\Gamma}+D_r)t_2'}\int_{t_2'}^{t'} dt_1' e^{(\kappa\bar{\Gamma}+3D_r)t_1'}\right]\\
&= v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt' e^{-4D_r t'}\left[\frac{1}{(\kappa\bar{\Gamma}+3D_r)}\left(\frac{(e^{(\kappa\bar{\Gamma}+D_r)t'}-1)(e^{(\kappa\bar{\Gamma}+3D_r)t}+e^{(\kappa\bar{\Gamma}+3D_r)t'})}{(\kappa\bar{\Gamma}+D_r)} - \frac{(e^{(2\kappa\bar{\Gamma}+4D_r)t'}-1)}{(\kappa\bar{\Gamma}+2D_r)}\right)\right]\\
&= \frac{v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma}+3D_r)(\kappa\bar{\Gamma}+D_r)}\left[(e^{(\kappa\bar{\Gamma}+3D_r)t}\int_0^t dt'\left(e^{(\kappa\bar{\Gamma}-3D_r)t'} - e^{-4D_r t'}\right) + \int_0^t dt'\left(e^{2\kappa\bar{\Gamma}t'} - e^{(\kappa\bar{\Gamma}-D_r)t'}\right)\right]\\
&\qquad - \frac{v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma}+3D_r)(\kappa\bar{\Gamma}+2D_r)}\int_0^t dt'\left(e^{2\kappa\bar{\Gamma}t'} - e^{-4D_r t'}\right)\\
&= \frac{v_p^2}{(\kappa\bar{\Gamma}+3D_r)(\kappa\bar{\Gamma}+D_r)}\left[\frac{3(\kappa\bar{\Gamma}-D_r)}{(\kappa\bar{\Gamma}-3D_r)2\kappa\bar{\Gamma}} - \frac{e^{(-\kappa\bar{\Gamma}+3D_r)t}(\kappa\bar{\Gamma}+D_r)}{(\kappa\bar{\Gamma}-3D_r)4D_r} + \frac{e^{-(\kappa\bar{\Gamma}+D_r)t}(\kappa\bar{\Gamma}-5D_r)}{(\kappa\bar{\Gamma}-D_r)4D_r} + \frac{e^{-\kappa\bar{\Gamma}t}(\kappa\bar{\Gamma}+D_r)}{(\kappa\bar{\Gamma}-D_r)2\kappa\bar{\Gamma}}\right]\\
&\qquad - \frac{v_p^2}{(\kappa\bar{\Gamma}+3D_r)(\kappa\bar{\Gamma}+2D_r)}\left[\frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}} - \frac{e^{-2\kappa\bar{\Gamma}t} - e^{-2(\kappa\bar{\Gamma}+2D_r)t}}{4D_r}\right]\\
&= \frac{v_p^2}{(\kappa\bar{\Gamma}+3D_r)}\left[\frac{1}{(\kappa\bar{\Gamma}-3D_r)}\left(\frac{3(\kappa\bar{\Gamma}-D_r)}{(\kappa\bar{\Gamma}+D_r)2\kappa\bar{\Gamma}} - \frac{e^{-(\kappa\bar{\Gamma}-3D_r)t}}{4D_r}\right) + \frac{1}{2(\kappa\bar{\Gamma}-D_r)}\left(\frac{e^{-(\kappa\bar{\Gamma}+D_r)t}(\kappa\bar{\Gamma}-5D_r)}{2D_r(\kappa\bar{\Gamma}+D_r)}\right)\right.\\
&\qquad \left. - \frac{1}{(\kappa\bar{\Gamma}+2D_r)}\left(\frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}} - \frac{e^{-2\kappa\bar{\Gamma}t} - e^{-2(\kappa\bar{\Gamma}+2D_r)t}}{4D_r}\right)\right]
\end{aligned}
$$

The contribution to the mean square displacement along the x direction is

$$
\begin{aligned}
&\langle x_0(t)x_1(t)\rangle_{\xi,\theta_0} = \frac{K_BT}{\kappa}\cos 2\theta_0\left(\frac{e^{-4D_r t} - e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r} - \frac{e^{-2\kappa\bar{\Gamma}t} - e^{-(2\kappa\bar{\Gamma}+4D_r)t}}{4D_r}\right) \qquad (24)\\
&+\Delta D\left(\frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}(2\kappa\bar{\Gamma}+4D_r)} - \frac{e^{-2\kappa\bar{\Gamma}t} - e^{-(2\kappa\bar{\Gamma}+4D_r)t}}{4D_r(2\kappa\bar{\Gamma}+4D_r)}\right) + \frac{v_p^2}{(\kappa\bar{\Gamma}+3D_r)}\left[\frac{1}{(\kappa\bar{\Gamma}-3D_r)}\left(\frac{3(\kappa\bar{\Gamma}-D_r)}{(\kappa\bar{\Gamma}+D_r)2\kappa\bar{\Gamma}} - \frac{e^{-(\kappa\bar{\Gamma}-3D_r)t}}{4D_r}\right)\right.\\
&\qquad \left. + \frac{1}{2(\kappa\bar{\Gamma}-D_r)}\left(\frac{e^{-(\kappa\bar{\Gamma}+D_r)t}(\kappa\bar{\Gamma}-5D_r)}{2D_r(\kappa\bar{\Gamma}+D_r)}\right) - \frac{1}{(\kappa\bar{\Gamma}+2D_r)}\left(\frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}} - \frac{e^{-2\kappa\bar{\Gamma}t} - e^{-2(\kappa\bar{\Gamma}+2D_r)t}}{4D_r}\right)\right]
\end{aligned}
$$

and to the y direction is

$$
\begin{aligned}
&\langle y_0(t)y_1(t)\rangle_{\xi,\theta_0} = -\frac{K_BT}{\kappa}\cos 2\theta_0\left(\frac{e^{-4D_r t} - e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}-4D_r} - \frac{e^{-2\kappa\bar{\Gamma}t} - e^{-(2\kappa\bar{\Gamma}+4D_r)t}}{4D_r}\right) \qquad (25)\\
&+\Delta D\left(\frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}(2\kappa\bar{\Gamma}+4D_r)} - \frac{e^{-2\kappa\bar{\Gamma}t} - e^{-(2\kappa\bar{\Gamma}+4D_r)t}}{4D_r(2\kappa\bar{\Gamma}+4D_r)}\right) + \frac{v_p^2}{(\kappa\bar{\Gamma}+3D_r)}\left[\frac{1}{(\kappa\bar{\Gamma}-3D_r)}\left(\frac{3(\kappa\bar{\Gamma}-D_r)}{(\kappa\bar{\Gamma}+D_r)2\kappa\bar{\Gamma}} - \frac{e^{-(\kappa\bar{\Gamma}-3D_r)t}}{4D_r}\right)\right.\\
&\qquad \left. + \frac{1}{2(\kappa\bar{\Gamma}-D_r)}\left(\frac{e^{-(\kappa\bar{\Gamma}+D_r)t}(\kappa\bar{\Gamma}-5D_r)}{2D_r(\kappa\bar{\Gamma}+D_r)}\right) - \frac{1}{(\kappa\bar{\Gamma}+2D_r)}\left(\frac{1-e^{-2\kappa\bar{\Gamma}t}}{2\kappa\bar{\Gamma}} - \frac{e^{-2\kappa\bar{\Gamma}t} - e^{-2(\kappa\bar{\Gamma}+2D_r)t}}{4D_r}\right)\right]
\end{aligned}
$$

6. **Calculation of $\langle R_{1,i}(t)R_{1,j}(t)\rangle_{\xi,\theta_0}$**