Asymmetric [Ellipsoid] active particle:

An asymmetric active particle undergoes self-propulsion with a velocity $v_p\hat{\mathbf{n}}(t)$ along its longer axis in 2-dimension. The angle between the x-axis of the lab frame and long axis of the ellipsoid at time t is represented by $\theta(t)$. The component of the orientation vector $\hat{\mathbf{n}}(t)$ w.r.t the lab frame can be expressed as $\hat{\mathbf{n}}(t) \equiv \Big(\cos\theta(t), \sin\theta(t) \Big)$.

In the body frame, the equation of motion of the centre of mass pf the particle is given as

$$\begin{aligned} \frac{\partial \tilde{x}}{\partial t} &= v_p + \Gamma_{\parallel} F_x \cos\theta(t) + \Gamma_{\parallel} F_y \sin\theta(t) + \Gamma_{\parallel}\tilde{\eta}_x(t) \\ \frac{\partial \tilde{y}}{\partial t} &= \Gamma_{\perp} F_x \cos\theta(t) + \Gamma_{\perp} F_y \sin\theta(t) + \Gamma_{\perp}\tilde{\eta}_x(t) \\ \frac{\partial \theta}{\partial t} &= \Gamma_{\theta}\tau + \tilde{\eta}_{\theta} \end{aligned} \tag{1}$$

where F_x and F_y are the forces acting on the particle along the x and y directions (in the lab frame), respectively and τ is the torque acting on the particle.

In the lab frame, the displacements are related to the body frame as

$$\begin{aligned} \delta x &= \cos\theta\delta\tilde{x} - sin\theta\delta\tilde{y}, \\ \delta y &= \cos\theta\delta\tilde{y} + sin\theta\delta\tilde{x}. \end{aligned} \tag{2}$$

Substituting equation(1) in equation(2), we have

$$\frac{\partial \mathbf{x}_i}{\partial t} = \Gamma_{ij}\mathbf{F}_j + v_p\hat{\mathbf{n}}_i(t) + \xi_i(t) \tag{3}$$

where

$$\begin{bmatrix} \xi_x(t) \\ \xi_y(t) \end{bmatrix} = \begin{bmatrix} \Gamma_{\parallel} & \\ & \Gamma_{\perp} \end{bmatrix} \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} \tilde{\eta}_x(t) \\ \tilde{\eta}_y(t) \end{bmatrix} \tag{4}$$

$$\begin{gathered} \text{The corretation of thermal fluctuation in the body frame are} \\ \langle\tilde{\eta}\rangle = 0 \text{ and } \langle\tilde{\eta}_i(t)\tilde{\eta}_j(t')\rangle = 2D_i\delta_{ij}\delta(t-t') \\ \text{where } D_{\parallel} = K_B T\Gamma_{\parallel}, D_{\perp} = K_B T\Gamma_{\perp} \end{gathered}$$

The coupled Langevin equation in the presence of an external force F and torque τ are

$$\begin{gathered} \partial_t r_i = \Gamma_{ij}(\theta(t))\, F_j + v_p\hat{n}_i(t) + \xi_i(t), \\ where\ \Gamma_{ij}(\theta(t)) = \bar{\Gamma}\delta_{ij} + \frac{\Delta\Gamma}{2}M_{ij}(\theta(t)) \\ \partial_t\theta = \Gamma_{\theta}\tau + \xi_{\theta}(t) \end{gathered} \tag{5}$$

$$\bar{\Gamma} = \frac{\Gamma_{\parallel} + \Gamma_{\perp}}{2}, \Delta\Gamma = \Gamma_{\parallel} - \Gamma_{\perp}\ and\ M_{ij}(\theta(t)) = \begin{bmatrix} \cos 2\theta(t) & sin2\theta(t) \\ sin2\theta(t) & -\cos 2\theta(t) \end{bmatrix} \tag{6}$$

$\xi_i(t)$ and ξ_{θ} are the Gaussian random noise with zero mean and

$\langle\xi_{\theta}(t)\xi_{\theta}(t\prime)\rangle = 2K_B T\,\Gamma_{\theta}\,\delta(t-t\prime) = 2D_r\,\delta(t-t\prime)$

for a fixed angle $\theta(t)$,

$\langle\xi_i(t)\xi_j(t\prime)\rangle = 2K_B T\,\Gamma_{ij}(\theta(t))\,\delta(t-t\prime)$

Calculation of $\langle \hat{n}_i(t_1)\hat{n}_j(t_2)\rangle_{\theta_0}$, the average is taken for fixed initial angleθ_0

$\theta(t_1) \equiv \theta_1, \theta(t_2) \equiv \theta_2$

$$\langle \hat{n}_i(t_1)\hat{n}_j(t_2)\rangle_{\theta_0} = \begin{bmatrix} \langle \cos\theta_1 \cos\theta_2 \rangle & \langle \cos\theta_1 \sin\theta_2 \rangle \\ \langle \sin\theta_1 \cos\theta_2 \rangle & \langle \sin\theta_1 \sin\theta_2 \rangle \end{bmatrix} \tag{7}$$

$2\langle \cos\theta_1 \cos\theta_2 \rangle = e^{-D_r(t_1+t_2-2\min(t_1,t_2))} + \cos 2\theta_0 e^{-D_r(t_1+t_2+2\min(t_1,t_2))}$

$2\langle \sin\theta_1 \sin\theta_2 \rangle = e^{-D_r(t_1+t_2-2\min(t_1,t_2))} - \cos 2\theta_0 e^{-D_r(t_1+t_2+2\min(t_1,t_2))}$

$2\langle \cos\theta_1 \sin\theta_2 \rangle = \sin 2\theta_0 e^{-D_r(t_1+t_2+2\min(t_1,t_2))}$

$2\langle \cos\theta_2 \sin\theta_1 \rangle = \sin 2\theta_0 e^{-D_r(t_1+t_2+2\min(t_1,t_2))}$

Substituting, we have

$$\langle \hat{n}_i(t_1)\hat{n}_j(t_2)\rangle_{\theta_0} = \frac{\delta_{ij}}{2} e^{-D_r(t_1+t_2-2\min(t_1,t_2))} + \frac{M_{ij}(\theta_0)}{2} e^{-D_r(t_1+t_2+2\min(t_1,t_2))} \tag{8}$$

Langevin equation without trap:

$$\frac{\partial r_i}{\partial t} = v_p \hat{n}_i(t) + \xi_i(t) \tag{9}$$

$$\implies r_i(t) - r_i(0) = \int_0^t v_p \hat{n}_i(t\prime) dt\prime + \int_0^t \xi_i(t\prime) dt\prime$$

$$\implies \Delta r_i(t) = \int_0^t v_p \hat{n}_i(t\prime) dt\prime + \int_0^t \xi_i(t\prime) dt\prime$$

Mean Square Displacement (MSD) :-

$$\langle \Delta x^2(t)\rangle_{\theta_0} = v_p^2 \int_0^t dt_1' \int_0^t dt_2' \, \langle \hat{n}_i(t_1')\hat{n}_i(t_2')\rangle_{\theta_0} + \int_0^t dt_1' \int_0^t dt_2' \, \langle \xi_i(t_1\prime)\xi_i(t_2\prime)\rangle_{\theta_0} \tag{10}$$

$$\langle \Delta x^2(t)\rangle_{\theta_0} = v_p^2 \int_0^t dt_1\prime \int_0^t dt_2\prime \, [\frac{1}{2} e^{-D_r(t_1\prime+t_2\prime-2\min(t_1\prime,t_2\prime))} + \frac{M_{xx}(\theta_0)}{2} e^{-D_r(t_1\prime+t_2\prime+2\min(t_1\prime,t_2\prime))}] +$$
$$2K_B T \int_0^t dt_1\prime \int_0^t dt_2\prime \, \langle \Gamma_{xx}(\theta(t_1\prime))\rangle_{\theta_0} \, \delta(t_1\prime - t_2\prime)$$
$$\text{Considering } t_1\prime > t_2\prime,$$
$$\langle \Delta x^2(t)\rangle_{\theta_0} = 2v_p^2 \int_0^t dt_2\prime \int_{t_2\prime}^t dt_1\prime \, [\frac{1}{2} e^{-D_r(t_1\prime-t_2\prime)}] + 2v_p^2 \int_0^t dt_2\prime \int_{t_2\prime}^t dt_1\prime \, \frac{M_{xx}(\theta_0)}{2} e^{-D_r(t_1\prime+3t_2\prime)}$$
$$+ 2K_B T \int_0^t dt_1\prime \int_0^t dt_2\prime \, [\bar{\Gamma} + \frac{\Delta\Gamma}{2} \langle \cos 2\theta(t_1\prime)\rangle_{\theta_0}] \, \delta(t_1\prime - t_2\prime)$$
$$\langle \Delta x^2(t)\rangle_{\theta_0} = 2v_p^2 \int_0^t dt_2\prime \int_{t_2\prime}^t dt_1\prime \, [\frac{1}{2} e^{-D_r(t_1\prime-t_2\prime)}] + 2v_p^2 \int_0^t dt_2\prime \int_{t_2\prime}^t dt_1\prime \, \frac{M_{xx}(\theta_0)}{2} e^{-D_r(t_1\prime+3t_2\prime)}$$
$$+ 2K_B T \int_0^t dt_1\prime \int_0^t dt_2\prime \, \bar{\Gamma} \, \delta(t_1\prime - t_2\prime) + 2K_B T \frac{\Delta\Gamma}{2} \int_0^t dt_1\prime \int_0^t dt_2\prime \, \cos 2\theta_0 e^{-4D_r t_1\prime} \, \delta(t_1\prime - t_2\prime)$$
$$\text{Take } \tau_r = \frac{1}{2D_r}, \tau_1(t) = \frac{1 - e^{-D_r t}}{D_r}, \tau_4(t) = \frac{1 - e^{-4D_r t}}{4D_r}$$
$$\langle \Delta x^2(t)\rangle_{\theta_0} = 2\tau_r v_p^2 [t - \tau_1(t)] + \frac{2\tau_r v_p^2}{3} \cos 2\theta_0 [\tau_1(t) - \tau_4(t)] + 2K_B T \, \bar{\Gamma} t + K_B T \Delta\Gamma \, \cos 2\theta_0 \tau_4(t)$$

Similarly,

$$\langle \Delta y^2(t)\rangle_{\theta_0} = 2\tau_r v_p^2 [t - \tau_1(t)] - \frac{2\tau_r v_p^2}{3} \cos 2\theta_0 [\tau_1(t) - \tau_4(t)] + 2K_B T \, \bar{\Gamma} t - K_B T \Delta\Gamma \, \cos 2\theta_0 \tau_4(t \tag{11}$$

Generalising, we have

$$\langle \Delta x_i(t)\Delta x_j(t)\rangle_{\theta_0} = \{2\bar{D}t + 2\tau_r v_p^2[t-\tau_1(t)]\}\delta_{ij} + \left[\frac{2\tau_r v_p^2}{3}[\tau_1(t)-\tau_4(t)] + \Delta D\,\tau_4(t)\right] M_{ij}(\theta_0) \quad (12)$$

Therefore,

$$\begin{aligned}
\langle \Delta r^2(t)\rangle_{\theta_0} &= \langle \Delta x^2(t)\rangle_{\theta_0} + \langle \Delta y^2(t)\rangle_{\theta_0} \\
\langle \Delta r^2(t)\rangle_{\theta_0} &= 4(K_B T\,\bar{\Gamma} + \tau_r v_p^2)t - 4\tau_r v_p^2 \tau_1(t) \\
&\text{Take } \bar{D} = K_B T\,\bar{\Gamma}, \\
&\text{The MSD for the untrapped active asymmetric particle is given as} \\
\langle \Delta r^2(t)\rangle_{\theta_0} &= 4\bar{D}t + 4\tau_r v_p^2(t - 2\tau_r(1-e^{-\frac{t}{2\tau_r}}))
\end{aligned} \quad (13)$$

The time dependent displacement diffusion tensor for a fixed initial angle θ_0 are

$$\begin{aligned}
&D_{ij}(t,\theta_0) = \frac{\langle \Delta x_i(t)\Delta x_j(t)\rangle_{\theta_0}}{2t} \\
&\text{Asymmetric Brownian particle} \\
&D_{ij}(t,\theta_0) = \bar{D}\delta_{ij} + \frac{\Delta D}{2}\frac{\tau_4(t)}{t} M_{ij}(\theta_0) \\
&\text{Asymmetric Active particle without harmonic trap} \\
&D_{ij}(t,\theta_0) = [\bar{D} + \tau_r v_p^2 - \frac{\tau_r v_p^2}{t}\tau_1(t)]\delta_{ij} + \left[\frac{2\tau_r v_p^2}{3}[\tau_1(t)-\tau_4(t)] + \Delta D\,\tau_4(t)\right]\frac{M_{ij}(\theta_0)}{2t}
\end{aligned} \quad (14)$$

If we average D_{ij} over all the initial angles θ_0,

$$\begin{aligned}
&\text{Asymmetric Brownian particle} \\
&D_{ij}(t) = \frac{1}{2\pi}\int_0^{2\pi} d\theta_0 D_{ij}(t,\theta_0) = \bar{D}\delta_{ij} \\
&\text{Asymmetric Active particle without harmonic trap} \\
&D_{ij}(t) = \frac{1}{2\pi}\int_0^{2\pi} d\theta_0 D_{ij}(t,\theta_0) = [\bar{D} + \tau_r v_p^2 - \frac{\tau_r v_p^2}{t}\tau_1(t)]\delta_{ij}
\end{aligned} \quad (15)$$

For $t \gg \tau_r$, $D_{ij}(t) \sim [\bar{D} + \tau_r v_p^2]\delta_{ij}$, which means that the asymmetry is lost at long times and the system behave like a passive brownian particle but with an enhanced diffusion.

For $t \ll \tau_r$, $D_{ij}(t) \sim [\bar{D} + \frac{v_p^2}{4}]\delta_{ij}$, $\langle \Delta r^2(t)\rangle_{\theta_0} = 4\bar{D}t + v_p^2 t^2$, which shows ballistic behaviour at short times.

The correlation $\langle \Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0}$

$$\langle\Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} = v_p^2\int_0^{t_1} dt_1\prime\int_0^{t_2} dt_2\prime\,\langle\hat{n}_x(t_1\prime)\hat{n}_x(t_2\prime)\rangle_{\theta_0} + \int_0^{t_1} dt_1\prime\int_0^{t_2} dt_2\prime\,\langle\xi_x(t_1\prime)\xi_x(t_2\prime)\rangle_{\theta_0} \tag{16}$$

$$\begin{aligned}
\langle\Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} &= v_p^2\int_0^{t_1} dt_1\prime\int_0^{t_2} dt_2\prime\,[\frac{1}{2}e^{-D_r(t_1\prime+t_2\prime-2\min(t_1\prime,t_2\prime))} + \frac{M_{xx}(\theta_0)}{2}e^{-D_r(t_1\prime+t_2\prime+2\min(t_1\prime,t_2\prime))}] \\
&+ 2K_BT\int_0^{t_1} dt_1\prime\int_0^{t_2} dt_2\prime\,\langle\Gamma_{xx}(\theta(t_1\prime))\rangle_{\theta_0}\ \delta(t_1\prime - t_2\prime) \\
\langle\Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} &= v_p^2\int_0^{t_2} dt_2\prime\int_{t_2\prime}^{t_2} dt_1\prime\,\frac{1}{2}e^{-D_r(t_1\prime-t_2\prime)} + v_p^2\int_0^{t_2} dt_2\prime\int_{t_2\prime}^{t_1} dt_1\prime\,\frac{1}{2}e^{-D_r(t_1\prime-t_2\prime)} \\
&+ v_p^2\frac{M_{xx}(\theta_0)}{2}\int_0^{t_2} dt_2\prime\int_{t_2\prime}^{t_2} dt_1\prime\, e^{-D_r(t_1\prime+3t_2\prime)} + v_p^2\frac{M_{xx}(\theta_0)}{2}\int_0^{t_2} dt_2\prime\int_{t_2\prime}^{t_1} dt_1\prime\, e^{-D_r(t_1\prime+3t_2\prime)} \\
&+ 2K_BT\int_0^{t_1} dt_1\prime\int_0^{t_2} dt_2\prime\,[\bar{\Gamma} + \frac{\Delta\Gamma}{2}\langle\cos 2\theta(t_1\prime)\rangle_{\theta_0}]\delta(t_1\prime - t_2\prime) \\
\langle\Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} &= v_p^2\tau_r\left[2t_2 - \tau_1(t_2) - 2\tau_r\left(e^{\frac{-(t_1-t_2)}{2\tau_r}} - e^{\frac{-t_1}{2\tau_r}}\right)\right] \\
&+ \cos 2\theta_0\,\tau_r v_p^2\left[\frac{2\tau_4(t_2)}{3} - \frac{\tau_1(t_2)}{3} - e^{\frac{-t_1}{2\tau_r}}\tau_3(t_2)\right] \\
&+ 2K_BT\,\bar{\Gamma}t_2 + K_BT\Delta\Gamma\ \cos 2\theta_0\tau_4(t_2)
\end{aligned}$$

The two time correlation for a fixed initial angle θ_0,

$$\begin{aligned}
&\text{Asymmetric brownian particle} \\
&\qquad\text{Taking } t_1 > t_2 \\
\langle\Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} &= 2\bar{D}t_2[1 + \frac{\Delta\Gamma}{2\bar{\Gamma}}\cos 2\theta_0\ \tau_4(t_2)] \\
&\text{Asymmetric Active brownian particle} \\
\langle\Delta x(t_1)\Delta x(t_2)\rangle_{\theta_0} &= 2\bar{D}t_2[1 + \frac{\Delta\Gamma}{2\bar{\Gamma}}\cos 2\theta_0\ \tau_4(t_2)] + v_p^2\tau_r\left[2t_2 - \tau_1(t_2) - 2\tau_r\left(e^{\frac{-(t_1-t_2)}{2\tau_r}} - e^{\frac{-t_1}{2\tau_r}}\right)\right] \\
&+ \cos 2\theta_0\,\tau_r v_p^2\left[\frac{2\tau_4(t_2)}{3} - \frac{\tau_1(t_2)}{3} - e^{\frac{-t_1}{2\tau_r}}\tau_3(t_2)\right]
\end{aligned} \tag{17}$$

Average over all the initial angles θ_0, we have a non stationary correlation,

$$\begin{aligned}
&\text{Asymmetric brownian particle} \\
&\langle\Delta x(t_1)\Delta x(t_2)\rangle = 2\bar{D}\min(t_1,t_2) \\
&\text{Asymmetric Active brownian particle,} \\
&\langle\Delta x(t_1)\Delta x(t_2)\rangle = 2\bar{D}\min(t_1,t_2) + v_p^2\tau_r\min(t_1,t_2)(2-\tau_1) - 2v_p^2\tau_r^2\left(e^{-D_r|t_1-t_2|} - e^{-D_rt_1}\right)
\end{aligned} \tag{18}$$

when $t_1, t_2 \to \infty$, $|t_1 - t_2| \to \text{finite}$,

$$\langle\Delta x(t_1)\Delta x(t_2)\rangle = 2\bar{D}t_2 + v_p^2\tau_rt_2(2 + t_1 + \frac{t_1^2D_r}{2}) - 2v_p^2\tau_r^2\left(e^{\frac{-|t_1-t_2|}{2\tau_r}} - \left[1 - D_rt_1 + \frac{(D_rt_1)^2}{2}\right]\right) \tag{19}$$

Putting $t_1 = t, t_2 = 0$,

$$\langle\Delta x(t)\Delta x(0)\rangle = 2v_p^2\tau_r^2\left(\left[1 - D_rt + \frac{(D_rt)^2}{2}\right] - e^{-D_rt}\right)$$