Calculation of $\langle R_{1,i}(t)R_{1,j}(t)\rangle$

$$\langle R_{1,i}(t)R_{1,j}(t)\rangle = \int_0^t dt' \int_0^t dt'' e^{-\kappa\bar{\Gamma}(t-t')} e^{-\kappa\bar{\Gamma}(t-t'')} \langle \sum_{k,l} M_{ik}(t')M_{jl}(t'')\langle R_{0,k}(t')R_{0,l}(t'')\rangle_\xi \rangle_\theta \tag{1}$$

$$= 2K_BT \int_0^t dt' \int_0^t dt'' e^{-\kappa\bar{\Gamma}(t-t')} e^{-\kappa\bar{\Gamma}(t-t'')} \langle \sum_{k,l} M_{ik}(t')M_{jl}(t'') \int_0^{t'} dt_1' \int_0^{t''} dt_2' e^{-\kappa\bar{\Gamma}(t'-t_1')} e^{-\kappa\bar{\Gamma}(t''-t_2')}$$

$$\left[\bar{\Gamma}\delta_{kl} + \frac{\Delta\Gamma}{2} M_{kl}(t_1')\right] \delta(t_1'-t_2')\rangle_\theta + v_p^2 e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \int_0^t dt'' e^{\kappa\bar{\Gamma}(t'+t'')} \left\langle \sum_{k,l} M_{ik}(t')M_{jl}(t'')\right.$$

$$\left. \int_0^{t'} dt_1' \int_0^{t''} dt_2' e^{-\kappa\bar{\Gamma}(t'-t_1')} e^{-\kappa\bar{\Gamma}(t''-t_2')} \hat{n}_k(t_1')\hat{n}_l(t_2') \right\rangle_\theta$$

Ignoring the term proportional to $\Delta\Gamma$, we have

$$\langle R_{1,i}(t)R_{1,j}(t)\rangle = 2K_BT\bar{\Gamma} e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \int_0^t dt'' \int_0^{min(t',t'')} dt_1' e^{2\kappa\bar{\Gamma}t_1'} \langle \sum_{k,l} M_{ik}(t')M_{jl}(t'')\delta_{kl}\rangle_\theta \tag{2}$$

$$+ v_p^2 e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \int_0^t dt'' \int_0^{t'} dt_1' \int_0^{t''} dt_2' e^{\kappa\bar{\Gamma}(t_1'+t_2')} \left\langle \sum_{k,l} M_{ik}(t')M_{jl}(t'')\hat{n}_k(t_1')\hat{n}_l(t_2') \right\rangle_\theta$$

Along x and y direction, $i=j$. Considering the case for $t' > t'' > t_1' > t_2'$,

$$\langle \sum_{k,l} M_{ik}(t')M_{ik}(t'')\rangle_\theta = \langle \cos 2[\theta(t') - \theta(t'')]\rangle_{\theta_0} = e^{-4D_r(t'-t'')} \tag{3}$$

$$\left\langle \sum_{k,l} M_{ik}(t')M_{jl}(t'')\hat{n}_k(t_1')\hat{n}_l(t_2') \right\rangle_\theta = e^{-D_r(4t'-4t''+t_1'-t_2')}$$

Substituting we have,

$$\langle x_1(t)x_1(t)\rangle = 2K_BT\bar{\Gamma} e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \int_0^t dt'' \int_0^{t''} dt_1' e^{2\kappa\bar{\Gamma}t_1'} e^{-4D_r(t'-t'')} + \tag{4}$$

$$v_p^2 e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \int_0^t dt'' \int_0^{t'} dt_1' \int_0^{t''} dt_2' e^{\kappa\bar{\Gamma}(t_1'+t_2')} e^{-D_r(4t'-4t''+t_1'-t_2')}$$

Evaluation of first term:

$$2K_BT\bar{\Gamma} e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \int_0^t dt'' \int_0^{t''} dt_1' e^{2\kappa\bar{\Gamma}t_1'} e^{-4D_r(t'-t'')} \tag{5}$$

$$= 2K_BT\bar{\Gamma} e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \int_0^t dt'' e^{-4D_r(t'-t'')} \frac{e^{2\kappa\bar{\Gamma}t''}-1}{2\kappa\bar{\Gamma}}$$

$$= \frac{K_BT}{\kappa} e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' e^{-4D_rt'} \int_0^t dt'' \left(e^{(2\kappa\bar{\Gamma}+4D_r)t''} - e^{4D_rt''} \right)$$

$$= \frac{K_BT}{\kappa} e^{-2\kappa\bar{\Gamma}t} \times 2 \int_0^t dt' e^{-4D_rt'} \int_0^{t'} dt'' \left(e^{(2\kappa\bar{\Gamma}+4D_r)t''} - e^{4D_rt''} \right)$$

$$= \frac{2K_BT}{\kappa} e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' e^{-4D_rt'} \left(\frac{e^{(2\kappa\bar{\Gamma}+4D_r)t'}-1}{(2\kappa\bar{\Gamma}+4D_r)} - \frac{e^{4D_rt'}-1}{4D_r} \right)$$

$$= \frac{2K_BT}{\kappa} e^{-2\kappa\bar{\Gamma}t} \int_0^t dt' \left(\frac{e^{2\kappa\bar{\Gamma}t'} - e^{-4D_rt'}}{(2\kappa\bar{\Gamma}+4D_r)} - \frac{1-e^{-4D_rt'}}{4D_r} \right)$$

$$= \frac{2K_BT}{\kappa} e^{-2\kappa\bar{\Gamma}t} \left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}(2\kappa\bar{\Gamma}+4D_r)} - \frac{1-e^{-4D_rt}}{4D_r(2\kappa\bar{\Gamma}+4D_r)} - \frac{t}{4D_r} + \frac{1-e^{-4D_rt}}{16D_r^2} \right)$$

$$= \frac{K_BT}{\kappa} \left(\frac{1-e^{-2\kappa\bar{\Gamma}t}}{\kappa\bar{\Gamma}(2\kappa\bar{\Gamma}+4D_r)} + \kappa\bar{\Gamma} \frac{e^{-2\kappa\bar{\Gamma}t} - e^{-(2\kappa\bar{\Gamma}+4D_r)t}}{4D_r^2(2\kappa\bar{\Gamma}+4D_r)} - \frac{te^{-2\kappa\bar{\Gamma}t}}{2D_r} \right)$$

Evaluation of the 2^{nd} term:

$$v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt''\int_0^{t'} dt_1'\int_0^{t''} dt_2' e^{\kappa\bar{\Gamma}(t_1'+t_2')} e^{-D_r(4t'-4t''+t_1'-t_2')} \tag{6}$$
$$= v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt'' e^{-4D_r(t'-t'')}\left(2\int_0^{t''} dt_1'\int_0^{t'} dt_2' + \int_{t''}^{t'} dt_1'\int_0^{t''} dt_2'\right) e^{(\kappa\bar{\Gamma}-D_r)t_1'} e^{(\kappa\bar{\Gamma}+D_r)t_2'}$$
$$= 2v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt'' e^{-4D_r(t'-t'')}\int_0^{t''} dt_1' e^{(\kappa\bar{\Gamma}-D_r)t_1'}\int_0^{t'} dt_2' e^{(\kappa\bar{\Gamma}+D_r)t_2'} +$$
$$v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt'' e^{-4D_r(t'-t'')}\int_{t''}^{t'} dt_1' e^{(\kappa\bar{\Gamma}-D_r)t_1'}\int_0^{t''} dt_2' e^{(\kappa\bar{\Gamma}+D_r)t_2'}$$
$$= 2v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt'' e^{-4D_r(t'-t'')}\left(\frac{e^{(\kappa\bar{\Gamma}-D_r)t''}-1}{(\kappa\bar{\Gamma}-D_r)}\right)\left(\frac{e^{(\kappa\bar{\Gamma}+D_r)t'}-1}{(\kappa\bar{\Gamma}+D_r)}\right) +$$
$$v_p^2 e^{-2\kappa\bar{\Gamma}t}\int_0^t dt'\int_0^t dt'' e^{-4D_r(t'-t'')}\left(\frac{e^{(\kappa\bar{\Gamma}-D_r)t'}-e^{(\kappa\bar{\Gamma}-D_r)t''}}{(\kappa\bar{\Gamma}-D_r)}\right)\left(\frac{e^{(\kappa\bar{\Gamma}+D_r)t''}-1}{(\kappa\bar{\Gamma}+D_r)}\right)$$
$$= \frac{2v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\int_0^t dt'\int_0^t dt''(e^{(\kappa\bar{\Gamma}+3D_r)t''}-e^{4D_rt''})(e^{(\kappa\bar{\Gamma}-3D_r)t'}-e^{-4D_rt'}) +$$
$$\frac{v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\int_0^t dt'\int_0^t dt''\left(e^{(\kappa\bar{\Gamma}+5D_r)t''}-e^{4D_rt''}\right)e^{(\kappa\bar{\Gamma}-5D_r)t'}-\left(e^{(2\kappa\bar{\Gamma}+4D_r)t''}-e^{(\kappa\bar{\Gamma}+3D_r)t''}\right)e^{-4D_rt'}$$

Solving the 1^{st} term of equation. 6,

$$\frac{2v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\int_0^t dt'\int_0^t dt''(e^{(\kappa\bar{\Gamma}+3D_r)t''}-e^{4D_rt''})(e^{(\kappa\bar{\Gamma}-3D_r)t'}-e^{-4D_rt'}) \tag{7}$$
$$= \frac{4v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\int_0^t dt'(e^{(\kappa\bar{\Gamma}-3D_r)t'}-e^{-4D_rt'})\int_0^{t'} dt''(e^{(\kappa\bar{\Gamma}+3D_r)t''}-e^{4D_rt''})$$
$$= \frac{4v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\int_0^t dt'(e^{(\kappa\bar{\Gamma}-3D_r)t'}-e^{-4D_rt'})\left(\frac{e^{(\kappa\bar{\Gamma}+3D_r)t'}-1}{(\kappa\bar{\Gamma}+3D_r)}-\frac{e^{4D_rt'}-1}{4D_r}\right)$$
$$= \frac{4v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\left[\frac{1}{(\kappa\bar{\Gamma}+3D_r)}\left(\int_0^t dt'(e^{2\kappa\bar{\Gamma}t'}-e^{(\kappa\bar{\Gamma}-3D_r)t'}-e^{(\kappa\bar{\Gamma}-D_r)t'}-e^{-4D_rt'})\right)\right.$$
$$\left.-\frac{1}{4D_r}\left(\int_0^t dt'(e^{(\kappa\bar{\Gamma}-3D_r)t'})-e^{(\kappa\bar{\Gamma}+D_r)t'}+1-e^{-4D_rt'})\right)\right]$$
$$= \frac{4v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\left[\frac{1}{(\kappa\bar{\Gamma}+3D_r)}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}}-\frac{e^{(\kappa\bar{\Gamma}-3D_r)t}-1}{(\kappa\bar{\Gamma}-3D_r)}-\frac{e^{(\kappa\bar{\Gamma}-D_r)t}-1}{(\kappa\bar{\Gamma}-D_r)}-\frac{1-e^{-4D_r}}{4D_r}\right)\right.$$
$$\left.-\frac{1}{4D_r}\left(\frac{e^{(\kappa\bar{\Gamma}-3D_r)t}-1}{(\kappa\bar{\Gamma}-3D_r)}-\frac{e^{(\kappa\bar{\Gamma}+D_r)t}-1}{(\kappa\bar{\Gamma}+D_r)}+t-\frac{1-e^{-4D_rt}}{4D_r}\right)\right]$$

Solving the 2^{nd} term of equation. 6,

$$\frac{v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\int_0^t dt'\int_0^t dt''\left(e^{(\kappa\bar{\Gamma}+5D_r)t''}-e^{4D_rt''}\right)e^{(\kappa\bar{\Gamma}-5D_r)t'}-\left(e^{(2\kappa\bar{\Gamma}+4D_r)t''}-e^{(\kappa\bar{\Gamma}+3D_r)t''}\right)e^{-4D_rt'} \tag{8}$$
$$= \frac{2v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\left[\int_0^t dt' e^{(\kappa\bar{\Gamma}-5D_r)t'}\int_0^{t'} dt''\left(e^{(\kappa\bar{\Gamma}+5D_r)t''}-e^{4D_rt''}\right)-\right.$$
$$\left.\int_0^t dt' e^{-4D_rt'}\int_0^{t'} dt''\left(e^{(2\kappa\bar{\Gamma}+4D_r)t''}-e^{(\kappa\bar{\Gamma}+3D_r)t''}\right)\right]$$
$$= \frac{2v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\left[\int_0^t dt' e^{(\kappa\bar{\Gamma}-5D_r)t'}\left(\frac{e^{(\kappa\bar{\Gamma}+5D_r)t'}-1}{(\kappa\bar{\Gamma}+5D_r)}-\frac{e^{4D_rt'}-1}{4D_r}\right)-\right.$$
$$\left.\int_0^t dt' e^{-4D_rt'}\left(\frac{e^{(2\kappa\bar{\Gamma}+4D_r)t'}-1}{(2\kappa\bar{\Gamma}+4D_r)}-\frac{e^{(\kappa\bar{\Gamma}+3D_r)t'}-1}{(\kappa\bar{\Gamma}+3D_r)}\right)\right]$$
$$= \frac{2v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\left[\frac{1}{(\kappa\bar{\Gamma}+5D_r)}\int_0^t dt'\left(e^{2\kappa\bar{\Gamma}t'}-e^{(\kappa\bar{\Gamma}-5D_r)t'}\right)-\frac{1}{4D_r}\int_0^t dt'\left(e^{(\kappa\bar{\Gamma}-D_r)t'}-e^{(\kappa\bar{\Gamma}-5D_r)t'}\right)\right.$$
$$\left.-\frac{1}{(2\kappa\bar{\Gamma}+4D_r)}\int_0^t dt'\left(e^{2\kappa\bar{\Gamma}t'}-e^{-4D_rt'}\right)+\frac{1}{(\kappa\bar{\Gamma}+3D_r)}\int_0^t dt'\left(e^{(\kappa\bar{\Gamma}-D_r)t'}-e^{-4D_rt'}\right)\right]$$
$$= \frac{2v_p^2 e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\left[\frac{1}{(\kappa\bar{\Gamma}+5D_r)}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}}-\frac{e^{(\kappa\bar{\Gamma}-5D_r)t}-1}{(\kappa\bar{\Gamma}-5D_r)}\right)-\frac{1}{4D_r}\left(\frac{e^{(\kappa\bar{\Gamma}-D_r)t}-1}{(\kappa\bar{\Gamma}-D_r)}-\frac{e^{(\kappa\bar{\Gamma}-5D_r)t}-1}{(\kappa\bar{\Gamma}-5D_r)}\right)\right.$$
$$\left.-\frac{1}{(2\kappa\bar{\Gamma}+4D_r)}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}}-\frac{1-e^{-4D_rt}}{4D_r}\right)+\frac{1}{(\kappa\bar{\Gamma}+3D_r)}\left(\frac{e^{(\kappa\bar{\Gamma}-D_r)t}}{(\kappa\bar{\Gamma}-D_r)}-\frac{1-e^{-4D_rt}}{4D_r}\right)\right]$$

The final expression is

$$
\begin{aligned}
\langle x_1(t)x_1(t)\rangle &= \frac{K_BT}{\kappa}\left(\frac{1-e^{-2\kappa\bar{\Gamma}t}}{\kappa\bar{\Gamma}(2\kappa\bar{\Gamma}+4D_r)}+\kappa\bar{\Gamma}\frac{e^{-2\kappa\bar{\Gamma}t}-e^{-(2\kappa\bar{\Gamma}+4D_r)t}}{4D_r^2(2\kappa\bar{\Gamma}+4D_r)}-\frac{te^{-2\kappa\bar{\Gamma}t}}{2D_r}\right)+ \qquad (9)\\
&\frac{4v_p^2e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\left[\frac{1}{(\kappa\bar{\Gamma}+3D_r)}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}}-\frac{e^{(\kappa\bar{\Gamma}-3D_r)t}-1}{(\kappa\bar{\Gamma}-3D_r)}-\frac{e^{(\kappa\bar{\Gamma}-D_r)t}-1}{(\kappa\bar{\Gamma}-D_r)}-\frac{1-e^{-4D_r}}{4D_r}\right)\right.\\
&\left.-\frac{1}{4D_r}\left(\frac{e^{(\kappa\bar{\Gamma}-3D_r)t}-1}{(\kappa\bar{\Gamma}-3D_r)}-\frac{e^{(\kappa\bar{\Gamma}+D_r)t}-1}{(\kappa\bar{\Gamma}+D_r)}+t-\frac{1-e^{-4D_rt}}{4D_r}\right)\right]+\\
&\frac{2v_p^2e^{-2\kappa\bar{\Gamma}t}}{(\kappa\bar{\Gamma})^2-D_r^2}\left[\frac{1}{(\kappa\bar{\Gamma}+5D_r)}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}}-\frac{e^{(\kappa\bar{\Gamma}-5D_r)t}-1}{(\kappa\bar{\Gamma}-5D_r)}\right)-\frac{1}{4D_r}\left(\frac{e^{(\kappa\bar{\Gamma}-D_r)t}-1}{(\kappa\bar{\Gamma}-D_r)}-\frac{e^{(\kappa\bar{\Gamma}-5D_r)t}-1}{(\kappa\bar{\Gamma}-5D_r)}\right)\right.\\
&\left.-\frac{1}{(2\kappa\bar{\Gamma}+4D_r)}\left(\frac{e^{2\kappa\bar{\Gamma}t}-1}{2\kappa\bar{\Gamma}}-\frac{1-e^{-4D_rt}}{4D_r}\right)+\frac{1}{(\kappa\bar{\Gamma}+3D_r)}\left(\frac{e^{(\kappa\bar{\Gamma}-D_r)t}}{(\kappa\bar{\Gamma}-D_r)}-\frac{1-e^{-4D_rt}}{4D_r}\right)\right]
\end{aligned}
$$