Ordinary Differential Equations

Chapter 1.2 Notes: Solutions & Inital Value Problems

Victor C. Van

December 19, 2024

An nth order ordinary differential equation is an equality involving the independent variable x, the dependent variable y, and the first n derivative of y. Examples are:

$$\text{Second Order} : x^2 + \frac{d^2y}{dx^2} + y = x^3$$

$$\text{Second Order} : \sqrt{1 - (\frac{d^2y}{dx^2})^2} - y = 0$$

$$\text{Fourth Order} : \frac{d^4y}{dx^4} = xy$$

Thus a general form for an nth order equation would be:

$$F(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n} = 0) \tag{1}$$

where F is a function of the independent variable x, the dependent variable y, and the derivative of y up to order n; that is $x, y, \cdots, \frac{d^{n-1}y}{dx^{n-1}}$. We assume thst the equation holds for all x in an interval I (which masy or may not include its endpoints: $a \leq x \leq b, a < x \leq b, etc$). In many cases, we can isolate the highest-order term $\frac{d^ny}{dx^n}$, and write (1) as:

$$\frac{d^ny}{dx^n} = f(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}) \tag{2}$$

which is often preferable to (1) for theoretical and computational purposes.

EXPLICIT SOLUTION

Definition 1. A function $\phi(x)$ that when substituted for y in equation (1) [or (2)] satisfies the equation for all x in the interval I is called an **explicit solution** to the equation on I.

EXAMPLE 1: Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution today

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0.$$

Solution: The functions $\phi(x) = x^2 - x^{-1}, \phi'(x) = 2x + x^{-2}$, and $\phi''(x) = 2 - 2x^{-3}$ are defined for all $x \neq 0$. Substitution of $\phi(x)$ for y in the equation (3) gives:

$$(2 - 2x^{-3}) - \frac{2}{x^2}(x^2 - x^{-1}) = (2 - 2x^{-3}) - (2 - 2x^{-3}) = 0.$$

EXAMPLE 2: Show that for *any* choice of the constants c_1 and c_2, the function

$$\phi(x) = c_1e^{-1} + c_2e^{2x}$$

is an explicit solution to

$$y'' - y' - 2y = 0. \tag{4}$$

Solution: We compute $\phi'(x) = -c_1e^{-x} + 2c_2e^{2x} and \phi''(x) = c_1e^{-1} + 4c_2e^{2x}$. Substitution of ϕ, ϕ', ϕ'' for y, y', and y'' in equation (4) yields:

$$(c_1e^{-1} + 4c_2e^{2x}) - (-c_1e^{-x} + 2c_2e^{2x}) - 2(c_1e^{-x} + c_2e^{2x})$$

$$= (c_1 + c_1 - 2c_1)e^{-x} + (4c_2 - 2c_2 - 2c_2)e^{2x} = 0.$$

Since equality holds for all x in $(-\infty, \infty)$, then $\phi(x) = c_1e^{-1} + c_2e^{2x}$ is an explicit solution to (4) on the interval $(-\infty, \infty)$ for any choice of the constants c_1 and c_2.

IMPLICIT SOLUTION

Definition 2. A relation $G(x, y) = 0$ is said to be an **implicit solution** to an equation (1) on the interval I if it defines one or more explicit solutions on I.

EXAMPLE 5: Verify that $4x^2 - y^2 = C$, where C is an arbitrary constant, gives a one-parameter family of implicit solutions to:

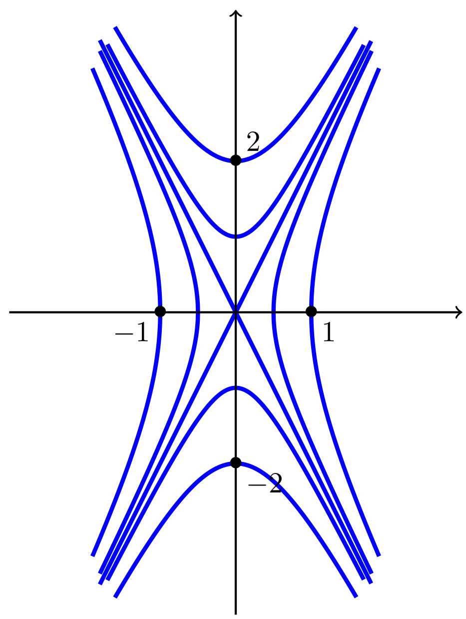
$$y\frac{dy}{dx} - 4x = 0, \tag{9}$$

and graph several of these solution curves.

Solution: When we implicitly differentiate the equation $4x^2 - y^2 = C$ with respect to x, we find

$$8x - 2y\frac{dy}{dx} = 0,$$

which is equivalent to (9). In figure 1.4 we have sketchedthe implicit solutions for $C = 0, \pm 1, \pm 4$. The curves are hyperbolas with common asymptotes $y = \pm 2x$. Notice that the implicit solution curves (with C arbitrary) fill the entire plane and are nonintersecting for $C \neq 0$. For $C = 0$, the implicit solution gives rise to the two explicit solutions $y = 2x$ and $y = -2x$, both of which passes through the origin.



(Figure 1.4 Implicit solutions $4x^2 - y^2 = C$)

INITIAL VALUE PROBLEM

Definition 3.
By an **initial value problem** for an n th order differential equation

$$F(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}) = 0,$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$y(x_0) = 0;$$

$$\frac{dy}{dx}(x_0) = y$$

$$\frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}$$

EXISTENCE & UNIQUENESS OF SOLUTION

Theorem 1. Given the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0,$$

assume that f and $\frac{\partial f}{\partial y}$ are continious functions in a rectange

$$R = \{(x, y) \mid a < x < b, c < y < d\}$$

that contains the point (x_0, y_0). Then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - h < x < x_0 + h$, where h is a positive number.