Ordinary Differential Equations

Chapter 1.2 Notes: Solutions & Inital Value Problems

Victor C. Van

December 19, 2024

An nth order ordinary differential equation is an equality involving the independent variable x, the dependent variable y, and the first n derivative of y. Examples are:

$$\text{Second Order} : x^2 + \frac{d^2y}{dx^2} + y = x^3$$

$$\text{Second Order} : \sqrt{1 - (\frac{d^2y}{dx^2})^2} - y = 0$$

$$\text{Fourth Order} : \frac{d^4y}{dx^4} = xy$$

Thus a general form for an nth order equation would be:

$$F(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n} = 0) \tag{1}$$

where F is a function of the independent variable x, the dependent variable y, and the derivative of y up to order n; that is $x, y, \cdots, \frac{d^{n-1}y}{dx^{n-1}}$. We assume thst the equation holds for all x in an interval I (which masy or may not include its endpoints: $a \leq x \leq b, a < x \leq b, etc$). In many cases, we can isolate the highest-order term $\frac{d^ny}{dx^n}$, and write (1) as:

$$\frac{d^ny}{dx^n} = f(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}) \tag{2}$$

which is often preferable to (1) for theoretical and computational purposes.

EXPLICIT SOLUTION

Definition 1. A function $\phi(x)$ that when substituted for y in equation (1) [or (2)] satisfies the equation for all x in the interval I is called an **explicit solution** to the equation on I.

EXAMPLE 1: Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution today

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0.$$

Solution: The functions $\phi(x) = x^2 - x^{-1}, \phi'(x) = 2x + x^{-2}$, and $\phi''(x) = 2 - 2x^{-3}$ are defined for all $x \neq 0$. Substitution of $\phi(x)$ for y in the equation (3) gives:

$$(2 - 2x^{-3}) - \frac{2}{x^2}(x^2 - x^{-1}) = (2 - 2x^{-3}) - (2 - 2x^{-3}) = 0.$$

EXAMPLE 2: Show that for *any* choice of the constants c_1 and c_2, the function

$$\phi(x) = c_1e^{-1} + c_2e^{2x}$$

is an explicit solution to

$$y'' - y' - 2y = 0. \tag{4}$$

$a \wedge b, c \vee d, \exists$