Exact Differential Equations Proof

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Proof. For some differential equation in the form

$$F(x, y) = M(x, y)dx + N(x, y) = 0$$

It is an exact differential equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

We will establish that an alternative method of solving an exact differential equation is equivalent to the method that we learned in class.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies F(x, y)\text{is an exact differential equation.}$$

Therefore, we can partially integrate $M(x, y)$ in respect to x and partially integrate $N(x, y)$ in respect to y. We assume that the constants of integration are equal to 0.

$$\int M(x, y)\, dx; \int N(x, y)dy$$

Then we will set each term in these integrals in to sets $S_M = \{M_1, M_2, M_n...\}$ and $S_N = \{N_1, N_2, N_n...\}$
We can solve the exact differential equation by taking the union of S_M and S_N and summing all of its elements

$$S_T = S_M \cup S_N = \{M_1, M_2, \ldots, M_n, N_1, N_2, \ldots, N_n\}$$

$$\sum_{x \in S^c} x = S_T^c$$

Suppose an exact differential equation

$$(4x^3y + 3)dx + (x^4 + 2)dy = 0; \{M(x, y) = 4x^3y + 3, N(x, y) = x^4 + 2 \mid \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\}$$

We will solve this exact differential equation using our proposed method.

$$\int M(x, y)\, dx \equiv \int (4x^3y + 3)\, dx = x^4y + 3x; C = 0 \therefore S_M = \{x^4y, 3x\}$$

$$\int N(x, y)\, dy \equiv \int (x^4 + 2)\, y = x^4y + 2y; C = 0 \therefore S_N = \{x^4y, 2y\}$$

$$S_T = S_M \cup S_N$$

$$S_T = \{x^4y, 3x, 2y\}$$

$$S_T^C = x^4y + 3x + 2y$$

$$C = x^4y + 3x + 2y$$

The method taught in class

$$(4x^3y+3)dx+(x^4+2)dy=0;\{M(x,y)=4x^3y+3, N(x,y)=x^4+2 \mid \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\}$$

$$F(x,y)=C \implies \int M(x,y)dx \equiv \int (4x^3y+3)\,dx = x^4y+3x+h(y)$$

$$\frac{\partial F}{\partial y}=x^4+h'(y)$$

$$\frac{\partial F}{\partial y}=\frac{\partial N}{\partial x} \therefore x^4+h'(y)=x^4+2$$

$$h'(y)=2; \int h'(y)=2y$$

$$C=x^4y+3x+2y$$

With the class's method,

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=4x^3$$

The proof is trivial.

QED