Calculus 3 Proof

(jokingly called "The Victor Theorem", according to a classmate)

Written for: Professor Habib Maagoul

Victor C. Van

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Proof. For simplicity, we can start with a basic double integral. This principle can be applied to any arbitrary n-number of iterated integrals.

$$\iint_R f(x,y)\,dA.$$

For any iterated integral of some function $f(x,y)$ in any Euclidean coordinate system (e.g. Cartesian, polar, and cylindrical), such that $f(x,y)$ can be expressed as the product of two independent functions $g(x)$ and $h(y)$, and that the bounds of integration are constants or infinities:

$$\int_c^d \int_a^b f(x,y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dxdy$$

$$\{(a,b,c,d) \in \mathbb{R} \mid -\infty \le (a,b,c,d) \le \infty\}$$

$h(y)$ is a constant when you integrate with respect to x.

$g(x)$ is a constant when you integrate with respect to y.

We can represent this as the product of two definite integrals of functions that are independent of each other:

$$\int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$$

If this claim is true, it must also apply to partial derivatives due to the *Fundamental Theorem of Calculus.*

Suppose some function $f(x,y) = x^2y^3$.

$$f(x,y) = g(x)\cdot h(y); \quad g(x) = x^2, \quad h(y) = y^3$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

$$\frac{\partial g}{\partial x} = 2x$$

$$\frac{\partial h}{\partial y} = 3y^2$$

$$\frac{\partial g}{\partial x}\cdot\frac{\partial h}{\partial y} = 6xy^2$$

Thus:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial g}{\partial x}\cdot\frac{\partial h}{\partial y} = \frac{dg}{dx}\cdot\frac{dh}{dy}$$

Therefore, if and only if $f_i(x_i)$ is independent from $f_{i+1}(x_{i+1})$

$$\prod_{i=1}^{n}\int_{a_i}^{b_i} f_i(x_i)\,dx_i = \int_{a_1}^{b_1} f_1(x_1)\,dx_1 \cdot \int_{a_2}^{b_2} f_2(x_2)\,dx_2 \cdots \int_{a_n}^{b_n} f_n(x_n)\,dx_n$$

$$\{(a,b,c,d) \in \mathbb{R} \mid -\infty \le (a,b,c,d) \le \infty \ \mid i \in \mathbb{N}\}$$

The proof is trivial.

QED

Fubini's Theorem:
The triple integral of a continious function $f(x,y,z)$ over a box B, where $a \leq x \leq b$, $c \leq y \leq d$, and $e \leq z \leq f$ to the integral:

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z) dz dy dx$$

Below is in-class example that this proof was created for, and my work. It was contested if this method of iterated integration was mathematically correct. I was unable to find a proof that sufficed to prove my claim online; therefore, this became my motivation to create a proof myself. Take note about how my proof is basically just a special case of *Fubini's Theorem* when $f_1(x_1)$, $f_2(x_2)$, and $f_3(x_3)$ are all independent of each other.

Evaluate $\iiint_B xyz^2\,dV$ **where** $B = \{(x,y,z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$

$$\int_0^1 \int_{-1}^2 \int_0^3 (xyz^2)\,dz dy dx$$

$$= \int_0^1 x\,dx \cdot \int_{-1}^2 y\,dy \cdot \int_0^3 z^2\,dz$$

$$= \frac{1}{2}[x^2]_0^1 \cdot \frac{1}{2}[y^2]_{-1}^2 \cdot \frac{1}{3}[z^3]_0^3$$

$$= \frac{1}{12}[x^2]_0^1 \cdot [y^2]_{-1}^2 \cdot [z^3]_0^3$$

$$= \frac{3^3}{12}((2)^2 - (-1)^2)$$

$$= \frac{3^3}{12}(4-1)$$

$$= \frac{27}{4}$$