Solving exact differential equations

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Abstract

Exact ODEs, also known as total differential equations, are a category of ordinary differential equations where a continuously differentiable function F, referred to as the "potential function", exists.

1 The basic idea

Given a differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

suppose there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \tag{2}$$

$$\frac{\partial F}{\partial y} = N(x,y) \tag{3}$$

Then, using eqs. (2) & (3), we can write eq. (1) as

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0 \tag{4}$$

which can be simplified to

$$dF(x,y) = 0 \tag{5}$$

whose solution is simply given by

$$F(x,y) = c \tag{6}$$

where c is some constant.

In the example you provided in your email, we have

$$M(x,y) = 4x^3y + 3 \tag{7}$$

$$N(x,y) = x^4 - 2 \tag{8}$$

and we are trying to find an $F(x,y)$ that satisfies eqs. (2) & (3) for this $M(x,y)$ & $N(x,y)$.

2 When does such an $F(x,y)$ exist?

A necessary condition for such a nice $F(x,y)$ to exist is that the mixed partial derivatives should be equal, that is,

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) \tag{9}$$

which translates to, in terms of $M(x,y)$ & $N(x,y)$ using eqs. (2) & (3), to

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{10}$$

So, given the differential eq. (1), we first check if eq. (10) is satisfied, and only then try to find such an $F(x, y)$.

In your example, we have

$$\frac{\partial M}{\partial y} = 4x^3 \tag{11}$$

$$\frac{\partial N}{\partial x} = 4x^3 \tag{12}$$

so eq. (10) is satisfied, and we can try to find an appropriate $F(x, y)$ that satisfies eqs. (2) & (3).

3 How to find $F(x, y)$ after checking that it exists?

Integrating eq. (2) gives

$$F = \int M(x, y)dx = G(x, y) + g(y) \tag{13}$$

where $G(x, y)$ is an indefinite integral of $M(x, y)$ with respect to y and $g(y)$ serves as the constant of integration (since we're integrating w.r.t. x).

Alternatively, we can integrate eq. (3), and get

$$F = \int N(x, y)dy = H(x, y) + h(x) \tag{14}$$

where $H(x, y)$ is an indefinite integral of $N(x, y)$ with respect to x and $h(x)$ serves as the constant of integration (since we're integrating w.r.t. y).

At first glance, it may look like we get two different expressions for F from eqs. (13) & (14), but they can be reconciled as follows.

In your example, if we apply eq. (13), we get

$$F = \int (4x^3y + 3)dx = x^4y + 3x + g(y) \tag{15}$$

so our $G(x, y)$ from eq. (13) becomes

$$G(x, y) = x^4y + 3x \tag{16}$$

Notice that $G(x, y)$ has two terms: one containing both x and y and another containing only x. So, we can write, using eq. (13)

$$F = G(x, y) + g(x) = A(x, y) + a(x) + g(y) \tag{17}$$

where $A(x, y) = x^4y$ and $a(x) = 3x$.

Similarly, applying eq. (14), we get

$$F = \int (x^4 - 2)dy = x^4y - 2y + h(x) \tag{18}$$

so our $H(x, y)$ from eq. (14) becomes

$$H(x, y) = x^4y - 2y \tag{19}$$

Notice again that $H(x, y)$ has two terms: one containing both x and y and another containing only y. So, we can write, using eq. (14),

$$F = H(x, y) + h(x) = A(x, y) + b(y) + h(x) \tag{20}$$

where $A(x, y) = x^4y$ (same as before) and $b(y) = -2y$.

Now, comparing eqs. (17) & (20), it's easy to see that we can get a unique expression for F by setting

$$g(y) = b(y) = -2y \tag{21}$$

$$h(x) = a(x) = 3x \tag{22}$$

which implies, from either eq. (17) or eq. (20) that

$$F = A(x,y) + a(x) + b(y) = x^4y + 3x - 2y \tag{23}$$

For completeness, I'll note that the full solution should include a constant term (see eq. (6)), that is,

$$F = A(x,y) + a(x) + b(y) = x^4y + 3x - 2y + c \tag{24}$$

4 General proof

Does this procedure work in general? In other words, how can you prove that this method would always work?

Very simply, by following the same steps as illustrated with the example above.

First, find F by integrating M w.r.t. x as in eq. (13), then group all the terms containing both x & y together into $A(x,y)$, and all the terms containing only x into $a(x)$, as in eq. (17), to get:

$$F = A(x,y) + a(x) + g(y) \tag{25}$$

where $A(x,y)$ consists only of terms containing both x & y, and $a(x)$ consists of terms containing only x.

Similarly, find another expression for F by integrating M w.r.t. x as in eq. (14), then group all the terms containing both x & y together into $A(x,y)$, and all the terms containing only y together into $b(y)$, as in eq. (20), to get:

$$F = A(x,y) + b(y) + h(x) \tag{26}$$

where $A(x,y)$ consists only of terms containing both x & y, and $b(y)$ consists of terms containing only y. Notice, that $A(x,y)$ should be identical in the last two equations (why?).

Finally, compare eqs. (25) & (26) to conclude that $a(x) = h(x)$ and $b(y) = g(y)$, which results in the following expression for F:

$$F = A(x,y) + a(x) + b(y) \tag{27}$$

This procedure amounts to the same thing that you mentioned in your email: integrate M w.r.t. x, integrate N w.r.t. y, then combine the terms containing only x and the terms containing only y with the terms containing both x & y to get the solution of the DE.