Calculus 3 Proof

(jokingly called "The Victor Theorem", according to a classmate)

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Proof. For simplicity, we can start with a basic double integral. This principle can be applied to any arbitrary n-number of iterated integrals.

$$\iint_R f(x,y)\,dA.$$

For any iterated integral of some function $f(x,y)$ in any Euclidean coordinate system (e.g. Cartesian, polar, and cylindrical), such that $f(x,y)$ can be expressed as the product of two independent functions $g(x)$ and $h(y)$, and that the bounds of integration are constants or infinities:

$$\int_c^d \int_a^b f(x,y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dxdy$$

$$\{(a,b,c,d) \in \mathbb{R} \mid -\infty \leq (a,b,c,d) \leq \infty\}$$

$h(y)$ is a constant when you integrate with respect to x.

$g(x)$ is a constant when you integrate with respect to y.

We can represent this as the product of two definite integrals of functions that are independent of each other:

$$\int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$$

If this claim is true, it must also apply to partial derivatives due to the *Fundamental Theorem of Calculus.*

Suppose some function $f(x,y) = x^2y^3$.

$$f(x,y) = g(x) \cdot h(y); \quad g(x) = x^2, \quad h(y) = y^3$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

$$\frac{\partial g}{\partial x} = 2x$$

$$\frac{\partial h}{\partial y} = 3y^2$$

$$\frac{\partial g}{\partial x} \cdot \frac{\partial h}{\partial y} = 6xy^2$$

Thus:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial g}{\partial x} \cdot \frac{\partial h}{\partial y} = \frac{dg}{dx} \cdot \frac{dh}{dy}$$

Therefore, if and only if $f_i(x_i)$ is independent from $f_{i+1}(x_{i+1})$

$$\prod_{i=1}^{n} \int_{a_i}^{b_i} f_i(x_i)\,dx_i = \int_{a_1}^{b_1} f_1(x_1)\,dx_1 \cdot \int_{a_2}^{b_2} f_2(x_2)\,dx_2 \cdots \int_{a_n}^{b_n} f_n(x_n)\,dx_n$$

$$\{(a,b,c,d) \in \mathbb{R} \mid -\infty \leq (a,b,c,d) \leq \infty \ \mid i \in \mathbb{N}\}$$

The proof is trivial.

QED

Fubini's Theorem:
The triple integral of a continious function $f(x, y, z)$ over a box B, where $a \leq x \leq b$, $c \leq y \leq d$, and $e \leq z \leq f$ to the integral:

$$\iiint_B f(x,y,z)\, dV = \int_a^b \int_c^d \int_e^f f(x,y,z) dz dy dx$$

Below is in-class example that this proof was created for, and my work. It was debated if this method of iterated integration was correct mathematically correct. Take note about how my proof is basically just a special case of *Fubini's Theorem* when $f_1(x_1)$, $f_2(x_2)$, and $f_3(x_3)$ are all independent of each other.

Evaluate $\iiint_B xyz^2\, dV$ **where** $B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$

$$\int_0^1 \int_{-1}^2 \int_0^3 (xyz^2)\, dz dy dx$$

$$= \int_0^1 x\, dx \cdot \int_{-1}^2 y\, dy \cdot \int_0^3 z^2\, dz$$

$$= \frac{1}{2}[x^2]_0^1 \cdot \frac{1}{2}[y^2]_{-1}^2 \cdot \frac{1}{3}[z^3]_0^3$$

$$= \frac{1}{12}[x^2]_0^1 \cdot [y^2]_{-1}^2 \cdot [z^3]_0^3$$

$$= \frac{3^3}{12}((2)^2 - (-1)^2)$$

$$= \frac{3^3}{12}(4 - 1)$$

$$= \frac{27}{4}$$