Solutions Manual for

Probability and Random Processes for Electrical and Computer Engineers

John A. Gubner
University of Wisconsin–Madison

File Generated July 13, 2007

CHAPTER 1
Problem Solutions

1. $\Omega = \{1,2,3,4,5,6\}$.

2. $\Omega = \{0,1,2,\ldots,24,25\}$.

3. $\Omega = [0,\infty)$. RTT > 10 ms is given by the event $(10,\infty)$.

4. (a) $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 100\}$.

   (b) $\{(x,y) \in \mathbb{R}^2 : 4 \leq x^2+y^2 \leq 25\}$.

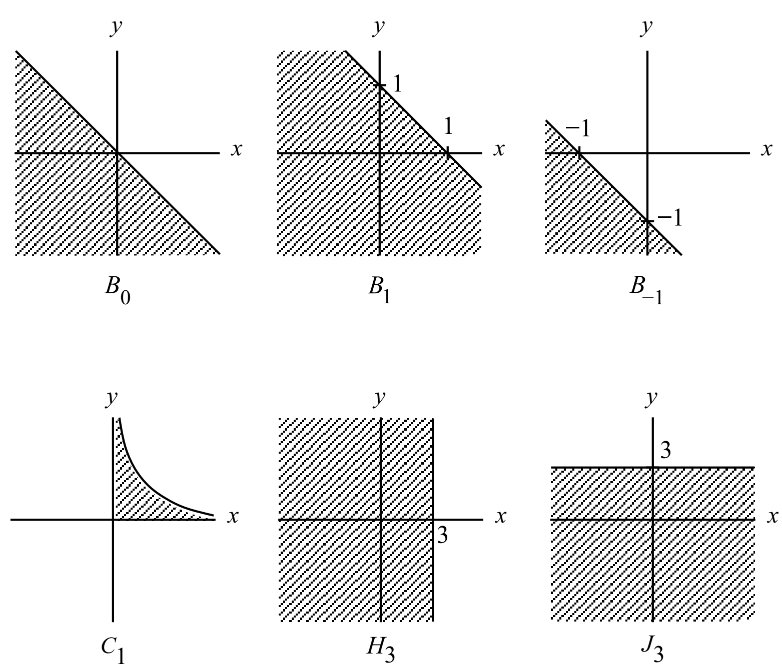
5. (a) $[2,3]^{\mathrm{c}} = (-\infty,2) \cup (3,\infty)$.

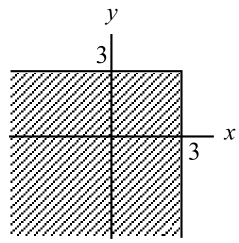
   (b) $(1,3) \cup (2,4) = (1,4)$.

   (c) $(1,3) \cap [2,4) = [2,3)$.

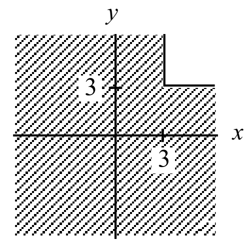
   (d) $(3,6] \setminus (5,7) = (3,5]$.

6. Sketches:

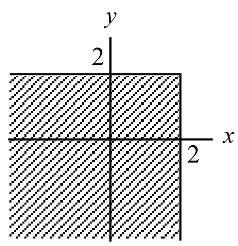




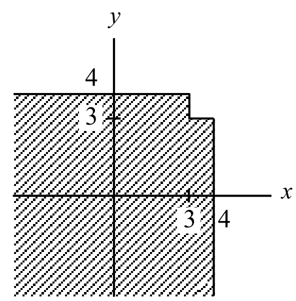
$H_3 \cap J_3 = M_3$



$H_3 \cup J_3 = N_3$



$M_2 \cap N_3 = M_2$



$M_4 \cap N_3$

7. (a) $[1,4] \cap \Big([0,2] \cup [3,5]\Big) = \Big([1,4] \cap [0,2]\Big) \cup \Big([1,4] \cap [3,5]\Big) = [1,2] \cup [3,4].$

(b)

$$\begin{aligned}
\Big([0,1] \cup [2,3]\Big)^{c} &= [0,1]^{c} \cap [2,3]^{c} \\
&= \Big[(-\infty,0) \cup (1,\infty)\Big] \cap \Big[(-\infty,2) \cup (3,\infty)\Big] \\
&= \Big((-\infty,0) \cap \Big[(-\infty,2) \cup (3,\infty)\Big]\Big) \\
&\quad \cup \Big((1,\infty) \cap \Big[(-\infty,2) \cup (3,\infty)\Big]\Big) \\
&= (-\infty,0) \cup (1,2) \cup (3,\infty).
\end{aligned}$$

(c) $\displaystyle\bigcap_{n=1}^{\infty} (-\tfrac{1}{n}, \tfrac{1}{n}) = \{0\}.$

(d) $\displaystyle\bigcap_{n=1}^{\infty} [0, 3 + \tfrac{1}{2n}) = [0,3].$

(e) $\displaystyle\bigcup_{n=1}^{\infty} [5, 7 - \tfrac{1}{3n}] = [5,7).$

(f) $\displaystyle\bigcup_{n=1}^{\infty} [0,n] = [0,\infty).$

8. We first let $C \subset A$ and show that for all B, $(A\cap B)\cup C = A\cap(B\cup C)$. Write

$$\begin{aligned} A\cap(B\cup C) &= (A\cap B)\cup(A\cap C), \quad \text{by the distributive law,} \\ &= (A\cap B)\cup C, \quad \text{since } C\subset A \Rightarrow A\cap C = C. \end{aligned}$$

For the second part of the problem, suppose $(A\cap B)\cup C = A\cap(B\cup C)$. We must show that $C\subset A$. Let $\omega\in C$. Then $\omega\in(A\cap B)\cup C$. But then $\omega\in A\cap(B\cup C)$, which implies $\omega\in A$.

9. Let $I := \{\omega\in\Omega : \omega\in A \Rightarrow \omega\in B\}$. We must show that $A\cap I = A\cap B$.

$\subset$: Let $\omega\in A\cap I$. Then $\omega\in A$ and $\omega\in I$. Therefore, $\omega\in B$, and then $\omega\in A\cap B$.

$\supset$: Let $\omega\in A\cap B$. Then $\omega\in A$ and $\omega\in B$. We must show that $\omega\in I$ too. In other words, we must show that $\omega\in A\Rightarrow\omega\in B$. But we already have $\omega\in B$.

10. The function $f\colon(-\infty,\infty)\to[0,\infty)$ with $f(x)=x^3$ is not well defined because not all values of $f(x)$ lie in the claimed co-domain $[0,\infty)$.

11. (a) The function will be invertible if $Y=[-1,1]$.

    (b) $\{x : f(x)\le 1/2\} = [-\pi/2,\pi/6]$.

    (c) $\{x : f(x)<0\} = [-\pi/2,0)$.

12. (a) Since f is not one-to-one, no choice of co-domain Y can make $f:[0,\pi]\to Y$ invertible.

    (b) $\{x : f(x)\le 1/2\} = [0,\pi/6]\cup[5\pi/6,\pi]$.

    (c) $\{x : f(x)<0\} = \varnothing$.

13. For $B\subset \mathbb{R}$,

$$f^{-1}(B) = \begin{cases} X, & 0\in B \text{ and } 1\in B, \\ A, & 1\in B \text{ but } 0\notin B, \\ A^{c}, & 0\in B \text{ but } 1\notin B, \\ \varnothing, & 0\notin B \text{ and } 1\notin B. \end{cases}$$

14. Let $f\colon X\to Y$ be a function such that f takes only n distinct values, say $y_1,\dots,y_n$. Let $B\subset Y$ be such that $f^{-1}(B)$ is nonempty. By definition, each $x\in f^{-1}(B)$ has the property that $f(x)\in B$. But $f(x)$ must be one of the values $y_1,\dots,y_n$, say y_i. Now $f(x)=y_i$ if and only if $x\in A_i := f^{-1}(\{y_i\})$. Hence,

$$f^{-1}(B) = \bigcup_{i:y_i\in B} A_i.$$

15. (a) $f(x)\in B^{c} \Leftrightarrow f(x)\notin B \Leftrightarrow x\notin f^{-1}(B) \Leftrightarrow x\in f^{-1}(B)^{c}$.

    (b) $f(x)\in\displaystyle\bigcup_{n=1}^{\infty} B_n$ if and only if $f(x)\in B_n$ for some n; i.e., if and only if $x\in f^{-1}(B_n)$ for some n. But this says that $x\in\displaystyle\bigcup_{n=1}^{\infty} f^{-1}(B_n)$.

(c) $f(x) \in \bigcap_{n=1}^{\infty} B_n$ if and only if $f(x) \in B_n$ for all n; i.e., if and only if $x \in f^{-1}(B_n)$ for all n. But this says that $x \in \bigcap_{n=1}^{\infty} f^{-1}(B_n)$.

16. If $B = \bigcup_i \{b_i\}$ and $C = \bigcup_i \{c_i\}$, put $a_{2i} := b_i$ and $a_{2i-1} := c_i$. Then $A = \bigcup_i a_i = B \cup C$ is countable.

17. Since each C_i is countable, we can write $C_i = \bigcup_j c_{ij}$. It then follows that

$$B := \bigcup_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{c_{ij}\}$$

is a doubly indexed sequence and is therefore countable as shown in the text.

18. Let $A = \bigcup_m \{a_m\}$ be a countable set, and let $B \subset A$. We must show that B is countable. If $B = \varnothing$, we're done by definition. Otherwise, there is at least one element of B in A, say a_k. Then put $b_n := a_n$ if $a_n \in B$, and put $b_n := a_k$ if $a_n \notin B$. Then $\bigcup_n \{b_n\} = B$ and we see that B is countable.

19. Let $A \subset B$ where A is uncountable. We must show that B is uncountable. We prove this by contradiction. Suppose that B is countable. Then by the previous problem, A is countable, contradicting the assumption that A is uncountable.

20. Suppose A is countable and B is uncountable. We must show that $A \cup B$ is uncountable. We prove this by contradiction. Suppose that $A \cup B$ is countable. Then since $B \subset A \cup B$, we would have B countable as well, contradicting the assumption that B is uncountable.

21. **MATLAB.** OMITTED.

22. **MATLAB.** Intuitive explanation: Using only the numbers $1, 2, 3, 4, 5, 6$, consider how many ways there are to write the following numbers:

$2 = 1+1$	1 way,	$1/36 = 0.0278$
$3 = 1+2 = 2+1$	2 ways,	$2/36 = 0.0556$
$4 = 1+3 = 2+2 = 3+1$	3 ways,	$3/36 = 0.0833$
$5 = 1+4 = 2+3 = 3+2 = 4+1$	4 ways,	$4/36 = 0.1111$
$6 = 1+5 = 2+4 = 3+3 = 4+2 = 5+1$	5 ways,	$5/36 = 0.1389$
$7 = 1+6 = 2+5 = 3+4 = 4+3 = 5+2 = 6+1$	6 ways,	$6/36 = 0.1667$
$8 = 2+6 = 3+5 = 4+4 = 5+3 = 6+2$	5 ways,	$5/36 = 0.1389$
$9 = 3+6 = 4+5 = 5+4 = 6+3$	4 ways,	$4/36 = 0.1111$
$10 = 4+6 = 5+5 = 6+4$	3 ways,	$3/36 = 0.0833$
$11 = 5+6 = 6+5$	2 ways,	$2/36 = 0.0556$
$12 = 6+6$	1 way,	$1/36 = 0.0278$
	36 ways,	$36/36 = 1$

23. Take $\Omega := \{1, \ldots, 26\}$ and put

$$\mathsf{P}(A) := \frac{|A|}{|\Omega|} = \frac{|A|}{26}.$$

The event that a vowel is chosen is $V = \{1,5,9,15,21\}$, and $\mathsf{P}(V) = |V|/26 = 5/26$.

24. Let $\Omega := \{(i,j) : 1 \le i,j \le 26 \text{ and } i \ne j\}$. For $A \subset \Omega$, put $\mathsf{P}(A) := |A|/|\Omega|$. The event that a vowel is chosen followed by a consonant is

$$B_{\text{vc}} \;=\; \big\{(i,j)\in\Omega : i = 1,5,9,15, \text{ or } 21 \text{ and } j \in \{1,\ldots,26\}\setminus\{1,5,9,15,21\}\big\}.$$

Similarly, the event that a consonant is followed by a vowel is

$$B_{\text{cv}} \;=\; \big\{(i,j)\in\Omega : i \in \{1,\ldots,26\}\setminus\{1,5,9,15,21\} \text{ and } j = 1,5,9,15, \text{ or } 21\big\}.$$

We need to compute

$$\mathsf{P}(B_{\text{vc}}\cup B_{\text{cv}}) = \frac{|B_{\text{vc}}|+|B_{\text{cv}}|}{|\Omega|} = \frac{5\cdot(26-5)+(26-5)\cdot 5}{650} = \frac{21}{65} \approx 0.323.$$

The event that two vowels are chosen is

$$B_{\text{vv}} \;=\; \big\{(i,j)\in\Omega : i,j \in \{1,5,9,15,21\} \text{ with } i \ne j\big\},$$

and $\mathsf{P}(B_{\text{vv}}) = |B_{\text{vv}}|/|\Omega| = 20/650 = 2/65 \approx .031$.

25. **MATLAB.** The code for simulating the drawing of a face card is

```
% Simulation of Drawing a Face Card
%
n = 10000;                % Number of draws.
X = ceil(52*rand(1,n));
faces = (41 <= X & X <= 52);
nfaces = sum(faces);
fprintf('There were %g face cards in %g draws.\n',nfaces,n)
```

26. Since 9 pm to 7 am is 10 hours, take $\Omega := [0,10]$. The probability that the baby wakes up during a time interval $0 \le t_1 < t_2 \le 10$ is

$$\mathsf{P}([t_1,t_2]) \;:=\; \int_{t_1}^{t_2} \frac{1}{10}\,d\omega.$$

Hence, $\mathsf{P}([2,10]^{\text{c}}) = \mathsf{P}([0,2]) = \int_0^2 1/10\,d\omega = 1/5$.

27. Starting with the equations

$$\begin{aligned} S_N &= 1 + z + z^2 + \cdots + z^{N-2} + z^{N-1} \\ zS_N &= \phantom{1 + {}} z + z^2 + \cdots + z^{N-2} + z^{N-1} + z^N, \end{aligned}$$

subtract the second line from the first. Canceling common terms leaves

$$S_N - zS_N \;=\; 1 - z^N, \quad \text{or} \quad S_N(1-z) \;=\; 1 - z^N.$$

If $z \ne 1$, we can divide both sides by $1-z$ to get $S_N = (1-z^N)/(1-z)$.

28. Let $x = p(1)$. Then $p(2) = 2p(1) = 2x$, $p(3) = 2p(2) = 2^2x$, $p(4) = 2p(3) = 2^3x$, $p(5) = 2^4x$, and $p(6) = 2^5x$. In general, $p(\omega) = 2^{\omega-1}x$ and we can write

$$1 \;=\; \sum_{\omega=1}^{6} p(\omega) \;=\; \sum_{\omega=1}^{6} 2^{\omega-1}x \;=\; x\sum_{\omega=0}^{5} 2^{\omega} \;=\; \frac{1-2^6}{1-2}x \;=\; 63x.$$

Hence, $x = 1/63$, and $p(\omega) = 2^{\omega-1}/63$ for $\omega = 1,\ldots,6$.

29. (a) By inclusion–exclusion, $\mathsf{P}(A\cup B) = \mathsf{P}(A)+\mathsf{P}(B)-\mathsf{P}(A\cap B)$, which can be rearranged as $\mathsf{P}(A\cap B) = \mathsf{P}(A)+\mathsf{P}(B)-\mathsf{P}(A\cup B)$.

(b) Since $\mathsf{P}(A) = \mathsf{P}(A\cap B)+\mathsf{P}(A\cap B^{\mathrm{c}})$,

$$\mathsf{P}(A\cap B^{\mathrm{c}}) \;=\; \mathsf{P}(A)-\mathsf{P}(A\cap B) \;=\; \mathsf{P}(A\cup B)-\mathsf{P}(B), \quad \text{by part (a).}$$

(c) Since B and $A\cap B^{\mathrm{c}}$ are disjoint,

$$\mathsf{P}(B\cup(A\cap B^{\mathrm{c}})) \;=\; \mathsf{P}(B)+\mathsf{P}(A\cap B^{\mathrm{c}}) \;=\; \mathsf{P}(A\cup B), \quad \text{by part (b).}$$

(d) By De Morgan's law, $\mathsf{P}(A^{\mathrm{c}}\cap B^{\mathrm{c}}) = \mathsf{P}([A\cup B]^{\mathrm{c}}) = 1-\mathsf{P}(A\cup B)$.

30. We must check the four axioms of a probability measure. First,

$$\mathsf{P}(\varnothing) \;=\; \lambda\mathsf{P}_1(\varnothing)+(1-\lambda)\mathsf{P}_2(\varnothing) \;=\; \lambda\cdot 0+(1-\lambda)\cdot 0 \;=\; 0.$$

Second,

$$\mathsf{P}(A) \;=\; \lambda\mathsf{P}_1(A)+(1-\lambda)\mathsf{P}_2(A) \;\geq\; \lambda\cdot 0+(1-\lambda)\cdot 0 \;=\; 0.$$

Third,

$$\begin{aligned}
\mathsf{P}\biggl(\bigcup_{n=1}^{\infty} A_n\biggr) &= \lambda\mathsf{P}_1\biggl(\bigcup_{n=1}^{\infty} A_n\biggr)+(1-\lambda)\mathsf{P}_2\biggl(\bigcup_{n=1}^{\infty} A_n\biggr) \\
&= \lambda\sum_{n=1}^{\infty}\mathsf{P}_1(A_n)+(1-\lambda)\sum_{n=1}^{\infty}\mathsf{P}_2(A_n) \\
&= \sum_{n=1}^{\infty}[\lambda\mathsf{P}_1(A_n)+(1-\lambda)\mathsf{P}_2(A_n)] \\
&= \sum_{n=1}^{\infty}\mathsf{P}(A_n).
\end{aligned}$$

Fourth, $\mathsf{P}(\Omega) = \lambda\mathsf{P}_1(\Omega)+(1-\lambda)\mathsf{P}_2(\Omega) = \lambda+(1-\lambda) = 1$.

31. First, since $\omega_0 \notin \varnothing$, $\mu(\varnothing) = 0$. Second, by definition, $\mu(A) \geq 0$. Third, for disjoint A_n, suppose $\omega_0 \in \bigcup_n A_n$. Then $\omega_0 \in A_m$ for some m, and $\omega_0 \notin A_n$ for $n \neq m$. Then $\mu(A_m) = 1$ and $\mu(A_n) = 0$ for $n \neq m$. Hence, $\mu\Bigl(\bigcup_n A_n\Bigr) = 1$ and $\sum_n \mu(A_n) = \mu(A_m) = 1$. A similar analysis shows that if $\omega_0 \notin \bigcup_n A_n$ then $\mu\Bigl(\bigcup_n A_n\Bigr)$ and $\sum_n \mu(A_n)$ are both zero. Finally, since $\omega_0 \in \Omega$, $\mu(\Omega) = 1$.

32. Starting with the assumption that for any two disjoint events A and B, $\mathsf{P}(A\cup B) = \mathsf{P}(A)+\mathsf{P}(B)$, we have that for $N=2$,

$$\mathsf{P}\biggl(\bigcup_{n=1}^N A_n\biggr) = \sum_{n=1}^N \mathsf{P}(A_n). \tag{$*$}$$

Now we must show that if $(*)$ holds for any $N\geq 2$, then $(*)$ holds for $N+1$. Write

$$\begin{aligned}\mathsf{P}\biggl(\bigcup_{n=1}^{N+1} A_n\biggr) &= \mathsf{P}\biggl(\biggl[\bigcup_{n=1}^N A_n\biggr]\cup A_{N+1}\biggr)\\ &= \mathsf{P}\biggl(\bigcup_{n=1}^N A_n\biggr)+\mathsf{P}(A_{N+1}), \quad \text{additivity for two events,}\\ &= \sum_{n=1}^N \mathsf{P}(A_n)+\mathsf{P}(A_{N+1}), \quad \text{by } (*),\\ &= \sum_{n=1}^{N+1}\mathsf{P}(A_n).\end{aligned}$$

33. Since $A_n := F_n\cap F_{n-1}^{\mathrm{c}}\cap\cdots\cap F_1^{\mathrm{c}}\subset F_n$, it is easy to see that

$$\bigcup_{n=1}^N A_n \subset \bigcup_{n=1}^N F_n.$$

The hard part is to show the reverse inclusion $\supset$. Suppose $\omega\in\bigcup_{n=1}^N F_n$. Then $\omega\in F_n$ for some n in the range $1,\ldots,N$. However, ω may belong to F_n for several values of n since the F_n may not be disjoint. Let

$$k := \min\{n:\omega\in F_n \text{ and } 1\leq n\leq N\}.$$

In other words, $1\leq k\leq N$ and $\omega\in F_k$, but $\omega\notin F_n$ for $n<k$; in symbols,

$$\omega\in F_k\cap F_{k-1}^{\mathrm{c}}\cap\cdots\cap F_1^{\mathrm{c}} =: A_k.$$

Hence, $\omega\in A_k\subset\bigcup_{n=1}^N A_n$. The proof that $\bigcup_{n=1}^\infty A_n\subset\bigcup_{n=1}^\infty F_n$ is similar except that $k:=\min\{n:\omega\in F_n \text{ and } n\geq 1\}$.

34. For arbitrary events F_n, let A_n be as in the preceding problem. We can then write

$$\begin{aligned}\mathsf{P}\biggl(\bigcup_{n=1}^\infty F_n\biggr) = \mathsf{P}\biggl(\bigcup_{n=1}^\infty A_n\biggr) &= \sum_{n=1}^\infty \mathsf{P}(A_n), \quad \text{since the } A_n \text{ are disjoint,}\\ &= \lim_{N\to\infty}\sum_{n=1}^N \mathsf{P}(A_n), \quad \text{by def. of infinite sum,}\\ &= \lim_{N\to\infty}\mathsf{P}\biggl(\bigcup_{n=1}^N A_n\biggr)\\ &= \lim_{N\to\infty}\mathsf{P}\biggl(\bigcup_{n=1}^N F_n\biggr).\end{aligned}$$

35. For arbitrary events G_n, put $F_n := G_n^c$. Then

$$\begin{aligned}
\mathsf{P}\bigg(\bigcap_{n=1}^{\infty} G_n\bigg) &= 1-\mathsf{P}\bigg(\bigcup_{n=1}^{\infty} F_n\bigg), \quad \text{by De Morgan's law,} \\
&= 1-\lim_{N\to\infty}\mathsf{P}\bigg(\bigcup_{n=1}^{N} F_n\bigg), \quad \text{by the preceding problem,} \\
&= 1-\lim_{N\to\infty}\bigg[1-\mathsf{P}\bigg(\bigcap_{n=1}^{N} G_n\bigg)\bigg], \quad \text{by De Morgan's law,} \\
&= \lim_{N\to\infty}\mathsf{P}\bigg(\bigcap_{n=1}^{N} G_n\bigg).
\end{aligned}$$

36. By the inclusion–exclusion formula,

$$\mathsf{P}(A\cup B) = \mathsf{P}(A)+\mathsf{P}(B)-\mathsf{P}(A\cap B) \le \mathsf{P}(A)+\mathsf{P}(B).$$

This establishes the union bound for $N=2$. Now suppose the union bound holds for some $N\ge 2$. We must show it holds for $N+1$. Write

$$\begin{aligned}
\mathsf{P}\bigg(\bigcup_{n=1}^{N+1} F_n\bigg) &= \mathsf{P}\bigg(\bigg[\bigcup_{n=1}^{N} F_n\bigg]\cup F_{N+1}\bigg) \\
&\le \mathsf{P}\bigg(\bigcup_{n=1}^{N} F_n\bigg)+\mathsf{P}(F_{N+1}), \quad \text{by the union bound for two events,} \\
&\le \sum_{n=1}^{N}\mathsf{P}(F_n)+\mathsf{P}(F_{N+1}), \quad \text{by the union bound for } N \text{ events,} \\
&= \sum_{n=1}^{N+1}\mathsf{P}(F_n).
\end{aligned}$$

37. To establish the union bound for a countable sequence of events, we proceed as follows. Let $A_n := F_n\cap F_{n-1}^c\cap\cdots\cap F_1^c \subset F_n$ be disjoint with $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} F_n$. Then

$$\begin{aligned}
\mathsf{P}\bigg(\bigcup_{n=1}^{\infty} F_n\bigg) &= \mathsf{P}\bigg(\bigcup_{n=1}^{\infty} A_n\bigg) \\
&= \sum_{n=1}^{\infty}\mathsf{P}(A_n), \quad \text{since the } A_n \text{ are disjoint,} \\
&\le \sum_{n=1}^{\infty}\mathsf{P}(F_n), \quad \text{since } A_n\subset F_n.
\end{aligned}$$

38. Following the hint, we put $G_n := \bigcup_{k=n}^{\infty} B_k$ so that we can write

$$\mathsf{P}\bigg(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_k\bigg) = \mathsf{P}\bigg(\bigcap_{n=1}^{\infty} G_n\bigg) = \lim_{N\to\infty}\mathsf{P}\bigg(\bigcap_{n=1}^{N} G_n\bigg), \quad \text{limit property of } \mathsf{P},$$

$$\begin{aligned}
&= \lim_{N\to\infty} \mathsf{P}(G_N), \quad \text{since } G_n \supset G_{n+1}, \\
&= \lim_{N\to\infty} \mathsf{P}\biggl(\bigcup_{k=N}^{\infty} B_k\biggr), \quad \text{definition of } G_N, \\
&\le \lim_{N\to\infty} \sum_{k=N}^{\infty} \mathsf{P}(B_k), \quad \text{union bound.}
\end{aligned}$$

This last limit must be zero since $\sum_{k=1}^{\infty} \mathsf{P}(B_k) < \infty$.

39. In this problem, the probability of an interval is its length.

(a) $\mathsf{P}(A_0) = 1$, $\mathsf{P}(A_1) = 2/3$, $\mathsf{P}(A_2) = 4/9 = (2/3)^2$, and $\mathsf{P}(A_3) = 8/27 = (2/3)^3$.

(b) $\mathsf{P}(A_n) = (2/3)^n$.

(c) Write

$$\begin{aligned}
\mathsf{P}(A) = \mathsf{P}\biggl(\bigcap_{n=1}^{\infty} A_n\biggr) &= \lim_{N\to\infty} \mathsf{P}\biggl(\bigcap_{n=1}^{N} A_n\biggr), \quad \text{limit property of } \mathsf{P}, \\
&= \lim_{N\to\infty} \mathsf{P}(A_N), \quad \text{since } A_n \supset A_{n+1}, \\
&= \lim_{N\to\infty} (2/3)^N = 0.
\end{aligned}$$

40. Consider the collection consisting of the empty set along with all unions of the form $\bigcup_i A_{k_i}$ for some finite subsequence of distinct elements from $\{1,\ldots,n\}$. We first show that this collection is a σ-field. First, it contains $\varnothing$ by definition. Second, since $A_1,\ldots,A_n$ is a partition,

$$\biggl(\bigcup_i A_{k_i}\biggr)^{\mathsf{c}} = \bigcup_i A_{m_i},$$

where m_i is the subsequence $\{1,\ldots,n\} \setminus \{k_i\}$. Hence, the collection is closed under complementation. Third,

$$\bigcup_{n=1}^{\infty}\biggl(\bigcup_i A_{k_{n,i}}\biggr) = \bigcup_j A_{m_j},$$

where an integer $l \in \{1,\ldots,n\}$ is in $\{m_j\}$ if and only if $k_{n,i} = l$ for some n and some i. This shows that the collection is a σ-field. Finally, since every element in our collection must be contained in every σ-field that contains $A_1,\ldots,A_n$, our collection must be $\sigma(A_1,\ldots,A_n)$.

41. We claim that $\mathscr{A}$ is not a σ-field. Our proof is by contradiction: We assume $\mathscr{A}$ is a σ-field and derive a contradiction. Consider the set

$$\bigcap_{n=1}^{\infty} [0, 1/2^n) = \{0\}.$$

Since $[0,1/2^n) \in \mathscr{C}_n \subset \mathscr{A}_n \subset \mathscr{A}$, the intersection must be in $\mathscr{A}$ since we are assuming $\mathscr{A}$ is a σ-field. Hence, $\{0\} \in \mathscr{A}$. Now, any set in $\mathscr{A}$ must belong to some $\mathscr{A}_n$. By the preceding problem, every set in A_n must be a finite union of sets from $\mathscr{C}_n$. However, the singleton set $\{0\}$ cannot be expressed as a finite union of sets from any $\mathscr{C}_n$. Hence, $\{0\} \notin \mathscr{A}$.

42. Let $A_i := X^{-1}(\{x_i\})$ for $i = 1,\ldots,n$. By the problems mentioned in the hint, for any subset B, if $X^{-1}(B) \neq \varnothing$, then

$$X^{-1}(B) \;=\; \bigcup_{i:x_i \in B} A_i \;\in\; \sigma(A_1,\ldots,A_n).$$

It follows that the smallest σ-field containing all the $X^{-1}(B)$ is $\sigma(A_1,\ldots,A_n)$.

43. (a) $\mathscr{F} = \big\{\varnothing, A, B, \{3\}, \{1,2\}, \{4,5\}, \{1,2,4,5\}, \Omega\big\}$

(b) The corresponding probabilities are $0, 5/8, 7/8, 1/2, 1/8, 3/8, 1/2, 1$.

(c) Since $\{1\} \notin \mathscr{F}$, $\mathsf{P}(\{1\})$ is not defined.

44. Suppose that a σ-field $\mathscr{A}$ contains an infinite sequence F_n of sets. If the sequence is not disjoint, we can construct a new sequence A_n that is disjoint with each $A_n \in \mathscr{A}$. Let $\mathbf{a} = a_1, a_2, \ldots$ be an infinite sequence of zeros and ones. Then $\mathscr{A}$ contains each union of the form

$$\bigcup_{i:a_i=1} A_i.$$

Furthermore, since the A_i are disjoint, each sequence $\mathbf{a}$ gives a different union, and we know from the text that the number of infinite sequences $\mathbf{a}$ is uncountably infinite.

45. (a) First, since $\varnothing$ is in each $\mathscr{A}_\alpha$, $\varnothing \in \bigcap_\alpha \mathscr{A}_\alpha$. Second, if $A \in \bigcap_\alpha \mathscr{A}_\alpha$, then $A \in \mathscr{A}_\alpha$ for each α, and so $A^{\mathrm{c}} \in \mathscr{A}_\alpha$ for each α. Hence, $A^{\mathrm{c}} \in \bigcap_\alpha \mathscr{A}_\alpha$. Third, if $A_n \in \mathscr{A}$ for all n, then for each n and each α, $A_n \in \mathscr{A}_\alpha$. Then $\bigcup_n A_n \in \mathscr{A}_\alpha$ for each α, and so $\bigcup_n A_n \in \bigcap_\alpha \mathscr{A}_\alpha$.

(b) We first note that

$$\mathscr{A}_1 = \{\varnothing, \{1\}, \{2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2\}, \Omega\}$$

and

$$\mathscr{A}_2 = \{\varnothing, \{1\}, \{3\}, \{2,4\}, \{2,3,4\}, \{1,2,4\}, \{1,3\}, \Omega\}.$$

It is then easy to see that

$$\mathscr{A}_1 \cap \mathscr{A}_2 \;=\; \{\varnothing, \{1\}, \{2,3,4\}, \Omega\}.$$

(c) First note that by part (a), $\bigcap_{\mathscr{A}:\mathscr{C}\subset\mathscr{A}} \mathscr{A}$ is a σ-field, and since $\mathscr{C} \subset \mathscr{A}$ for each $\mathscr{A}$, the σ-field $\bigcap_{\mathscr{A}:\mathscr{C}\subset\mathscr{A}} \mathscr{A}$ contains $\mathscr{C}$. Finally, if $\mathscr{D}$ is any σ-field that contains $\mathscr{C}$, then $\mathscr{D}$ is one of the $\mathscr{A}$s in the intersection. Hence,

$$\mathscr{C} \;\subset\; \bigcap_{\mathscr{A}:\mathscr{C}\subset\mathscr{A}} \mathscr{A} \;\subset\; \mathscr{D}.$$

Thus $\bigcap_{\mathscr{A}:\mathscr{C}\subset\mathscr{A}} \mathscr{A}$ is the smallest σ-field that contains $\mathscr{C}$.

46. The union of two σ-fields is not always a σ-field. Here is an example. Let $\Omega := \{1,2,3,4\}$, and put

$$\mathscr{F} := \big\{\varnothing,\{1,2\},\{3,4\},\Omega\big\} \quad\text{and}\quad \mathscr{G} := \big\{\varnothing,\{1,3\},\{2,4\},\Omega\big\}.$$

Then

$$\mathscr{F}\cup\mathscr{G} = \big\{\varnothing,\{1,2\},\{3,4\},\{1,3\},\{2,4\},\Omega\big\}$$

is not a σ-field since it does not contain $\{1,2\}\cap\{1,3\}=\{1\}$.

47. Let Ω denote the positive integers, and let $\mathscr{A}$ denote the collection of subsets A such that either A or A^c is finite.

(a) Let E denote the subset of even integers. Then E does not belong to $\mathscr{A}$ since neither E nor E^c (the odd integers) is a finite set.

(b) To show that $\mathscr{A}$ is closed under finite unions, we consider two cases. First suppose that $A_1,\ldots,A_n$ are all finite. Then

$$\bigg|\bigcup_{i=1}^n A_i\bigg| \le \sum_{i=1}^n |A_i| < \infty,$$

and so $\bigcup_{i=1}^n A_i \in \mathscr{A}$. In the second case, suppose that some A_j^c is finite. Then

$$\bigg(\bigcup_{i=1}^n A_i\bigg)^c = \bigcap_{i=1}^n A_i^c \subset A_j^c.$$

Hence, the complement of $\bigcup_{i=1}^n A_i$ is finite, and so the union belongs to $\mathscr{A}$.

(c) $\mathscr{A}$ is not a σ-field. To see this, put $A_i := \{2i\}$ for $i=1,2,\ldots.$ Then $\bigcup_{i=1}^\infty A_i = E \notin \mathscr{A}$ by part (a).

48. Let Ω be an uncountable set. Let $\mathscr{A}$ denote the collection of all subsets A such that either A is countable or A^c is countable. We show that $\mathscr{A}$ is a σ-field. First, the empty set is countable. Second, if $A\in\mathscr{A}$, we must show that $A^c\in\mathscr{A}$. There are two cases. If A is countable, then the complement of A^c is A, and so $A^c\in\mathscr{A}$. If A^c is countable, then $A^c\in\mathscr{A}$. Third, let $A_1,A_2,\ldots$ belong to $\mathscr{A}$. There are two cases to consider. If all A_n are countable, then $\bigcup_n A_n$ is also countable by an earlier problem. Otherwise, if some A_m^c is countable, then write

$$\bigg(\bigcup_{n=1}^\infty A_n\bigg)^c = \bigcap_{n=1}^\infty A_n^c \subset A_m^c.$$

Since the subset of a countable set is countable, we see that the complement of $\bigcup_{n=1}^\infty A_n$ is countable, and thus the union belongs to $\mathscr{A}$.

49. (a) Since $(a,b] = \bigcap_{n=1}^\infty (a,b+\frac{1}{n})$, and since each $(a,b+\frac{1}{n})\in\mathscr{B}$, $(a,b]\in\mathscr{B}$.

(b) Since $\{a\} = \bigcap_{n=1}^\infty (a-\frac{1}{n},a+\frac{1}{n})$, and since each $(a-\frac{1}{n},a+\frac{1}{n})\in\mathscr{B}$, the singleton $\{a\}\in\mathscr{B}$.

(c) Since by part (b), singleton sets are Borel sets, and since A is a countable union of Borel sets, $A \in \mathscr{B}$; i.e., A is a Borel set.

(d) Using part (a), write

$$\begin{aligned}
\lambda\big((a,b]\big) &= \lambda\bigg(\bigcap_{n=1}^{\infty}(a,b+\tfrac{1}{n})\bigg) \\
&= \lim_{N\to\infty}\lambda\bigg(\bigcap_{n=1}^{N}(a,b+\tfrac{1}{n})\bigg), \quad \text{limit property of probability,} \\
&= \lim_{N\to\infty}\lambda\big((a,b+\tfrac{1}{N})\big), \quad \text{decreasing sets,} \\
&= \lim_{N\to\infty}(b+\tfrac{1}{N})-a, \quad \text{characterization of } \lambda, \\
&= b-a.
\end{aligned}$$

Similarly, using part (b), we can write

$$\begin{aligned}
\lambda\big(\{a\}\big) &= \lambda\bigg(\bigcap_{n=1}^{\infty}(a-\tfrac{1}{n},a+\tfrac{1}{n})\bigg) \\
&= \lim_{N\to\infty}\lambda\bigg(\bigcap_{n=1}^{N}(a-\tfrac{1}{n},a+\tfrac{1}{n})\bigg), \quad \text{limit property of probability,} \\
&= \lim_{N\to\infty}\lambda\big((a-\tfrac{1}{N},a+\tfrac{1}{N})\big), \quad \text{decreasing sets,} \\
&= \lim_{N\to\infty} 2/N, \quad \text{characterization of } \lambda, \\
&= 0.
\end{aligned}$$

50. Let $\mathscr{I}$ denote the collection of open intervals, and let $\mathscr{O}$ denote the collection of open sets. We need to show that $\sigma(\mathscr{I}) = \sigma(\mathscr{O})$. Since $\mathscr{I} \subset \mathscr{O}$, every σ-field containing $\mathscr{O}$ also contains $\mathscr{I}$. Hence, the smallest σ-field containing $\mathscr{O}$ contains $\mathscr{I}$; i.e., $\mathscr{I} \subset \sigma(\mathscr{O})$. By the definition of the smallest σ-field containing $\mathscr{I}$, it follows that $\sigma(\mathscr{I}) \subset \sigma(\mathscr{O})$. Now, if we can show that $\mathscr{O} \subset \sigma(\mathscr{I})$, then it will similarly follow that $\sigma(\mathscr{O}) \subset \sigma(\mathscr{I})$. Recall that in the problem statement, it was shown that every open set U can be written as a countable union of open intervals. This means $U \in \sigma(\mathscr{I})$. This proves that $\mathscr{O} \subset \sigma(\mathscr{I})$ as required.

51. **MATLAB.** Chips from S1 are 80% reliable; chips from S2 are 70% reliable.

52. Observe that

$$N(O_{\mathrm{w,S1}}) = N(O_{\mathrm{S1}}) - N(O_{\mathrm{d,S1}}) = N(O_{\mathrm{S1}})\bigg(1 - \frac{N(O_{\mathrm{d,S1}})}{N(O_{\mathrm{S1}})}\bigg)$$

and

$$N(O_{\mathrm{w,S2}}) = N(O_{\mathrm{S2}}) - N(O_{\mathrm{d,S2}}) = N(O_{\mathrm{S2}})\bigg(1 - \frac{N(O_{\mathrm{d,S2}})}{N(O_{\mathrm{S2}})}\bigg).$$

53. First write

$$\mathsf{P}(A|B\cap C)\,\mathsf{P}(B|C) = \frac{\mathsf{P}(A\cap[B\cap C])}{\mathsf{P}(B\cap C)}\cdot\frac{\mathsf{P}(B\cap C)}{\mathsf{P}(C)} = \frac{\mathsf{P}([A\cap B]\cap C)}{\mathsf{P}(C)} = \mathsf{P}(A\cap B|C).$$

From this formula, we can isolate the equation

$$\mathsf{P}(A|B\cap C)\,\mathsf{P}(B|C) \;=\; \frac{\mathsf{P}([A\cap B]\cap C)}{\mathsf{P}(C)}.$$

Multiplying through by $\mathsf{P}(C)$ yields $\mathsf{P}(A|B\cap C)\,\mathsf{P}(B|C)\,\mathsf{P}(C) = \mathsf{P}(A\cap B\cap C)$.

54. (a) $\mathsf{P}(\mathrm{MM}) = 140/(140+60) = 140/200 = 14/20 = 7/10 = 0.7$. Then $\mathsf{P}(\mathrm{HT}) = 1-\mathsf{P}(\mathrm{MM}) = 0.3$.

(b) Let D denote the event that a workstation is defective. Then

$$\begin{aligned}\mathsf{P}(D) &= \mathsf{P}(D|\mathrm{MM})\mathsf{P}(\mathrm{MM})+\mathsf{P}(D|\mathrm{HT})\mathsf{P}(\mathrm{HT})\\ &= (.1)(.7)+(.2)(.3)\\ &= .07+.06 \;=\; 0.13.\end{aligned}$$

(c) Write

$$\mathsf{P}(\mathrm{MM}|D) \;=\; \frac{\mathsf{P}(D|\mathrm{MM})\mathsf{P}(\mathrm{MM})}{\mathsf{P}(D)} \;=\; \frac{.07}{.13} \;=\; \frac{7}{13}.$$

55. Let O denote the event that a cell is overloaded, and let B denote the event that a call is blocked. The problem statement tells us that

$$\mathsf{P}(O) \;=\; 1/3, \quad \mathsf{P}(B|O) \;=\; 3/10, \quad \text{and} \quad \mathsf{P}(B|O^{\mathrm{c}}) \;=\; 1/10.$$

To find $\mathsf{P}(O|B)$, first write

$$\mathsf{P}(O|B) \;=\; \frac{\mathsf{P}(B|O)\mathsf{P}(O)}{\mathsf{P}(B)} \;=\; \frac{3/10\cdot 1/3}{\mathsf{P}(B)}.$$

Next compute

$$\begin{aligned}\mathsf{P}(B) &= \mathsf{P}(B|O)\mathsf{P}(O)+\mathsf{P}(B|O^{\mathrm{c}})\mathsf{P}(O^{\mathrm{c}})\\ &= (3/10)(1/3)+(1/10)(2/3) \;=\; 5/30 \;=\; 1/6.\end{aligned}$$

We conclude that

$$\mathsf{P}(O|B) \;=\; \frac{1/10}{1/6} \;=\; \frac{6}{10} \;=\; \frac{3}{5} \;=\; 0.6.$$

56. The problem statement tells us that $\mathsf{P}(R_1|T_0) = \varepsilon$ and $\mathsf{P}(R_0|T_1) = \delta$. We also know that

$$1 \;=\; \mathsf{P}(\Omega) \;=\; \mathsf{P}(T_0\cup T_1) \;=\; \mathsf{P}(T_0)+\mathsf{P}(T_1).$$

The problem statement tells us that these last two probabilities are the same; hence they are both equal to $1/2$. To find $\mathsf{P}(T_1|R_1)$, we begin by writing

$$\mathsf{P}(T_1|R_1) \;=\; \frac{\mathsf{P}(R_1|T_1)\mathsf{P}(T_1)}{\mathsf{P}(R_1)}.$$

Next, we note that $\mathsf{P}(R_1|T_1) = 1 - \mathsf{P}(R_0|T_1) = 1 - \delta$. By the law of total probability,

$$\begin{aligned}\mathsf{P}(R_1) &= \mathsf{P}(R_1|T_1)\mathsf{P}(T_1) + \mathsf{P}(R_1|T_0)\mathsf{P}(T_0) \\ &= (1-\delta)(1/2) + \varepsilon(1/2) = (1-\delta+\varepsilon)/2.\end{aligned}$$

So,

$$\mathsf{P}(T_1|R_1) = \frac{(1-\delta)(1/2)}{(1-\delta+\varepsilon)/2} = \frac{1-\delta}{1-\delta+\varepsilon}.$$

57. Let H denote the event that a student does the homework, and let E denote the event that a student passes the exam. Then the problem statement tells us that

$$\mathsf{P}(E|H) = .8, \quad \mathsf{P}(E|H^{\mathrm{c}}) = .1, \quad \text{and} \quad \mathsf{P}(H) = .6.$$

We need to compute $\mathsf{P}(E)$ and $\mathsf{P}(H|E)$. To begin, write

$$\begin{aligned}\mathsf{P}(E) &= \mathsf{P}(E|H)\mathsf{P}(H) + \mathsf{P}(E|H^{\mathrm{c}})\mathsf{P}(H^{\mathrm{c}}) \\ &= (.8)(.6) + (.1)(1-.6) = .48 + .04 = .52.\end{aligned}$$

Next,

$$\mathsf{P}(H|E) = \frac{\mathsf{P}(E|H)\mathsf{P}(H)}{\mathsf{P}(E)} = \frac{.48}{.52} = \frac{12}{13}.$$

58. The problem statement tells us that

$$\mathsf{P}(A_F|C_F) = 1/3, \quad \mathsf{P}(A_F|C_F^{\mathrm{c}}) = 1/10, \quad \text{and} \quad \mathsf{P}(C_F) = 1/4.$$

We must compute

$$\mathsf{P}(C_F|A_F) = \frac{\mathsf{P}(A_F|C_F)\mathsf{P}(C_F)}{\mathsf{P}(A_F)} = \frac{(1/3)(1/4)}{\mathsf{P}(A_F)} = \frac{1/12}{\mathsf{P}(A_F)}.$$

To compute the denominator, write

$$\begin{aligned}\mathsf{P}(A_F) &= \mathsf{P}(A_F|C_F)\mathsf{P}(C_F) + \mathsf{P}(A_F|C_F^{\mathrm{c}})\mathsf{P}(C_F^{\mathrm{c}}) \\ &= (1/3)(1/4) + (1/10)(1-1/4) = 1/12 + 3/40 = 19/120.\end{aligned}$$

It then follows that

$$\mathsf{P}(C_F|A_F) = \frac{1}{12} \cdot \frac{120}{19} = \frac{10}{19}.$$

59. Let F denote the event that a patient receives a flu shot. Let S, M, and R denote the events that Sue, Minnie, or Robin sees the patient. The problem tells us that

$$\mathsf{P}(S) = .2,\ \mathsf{P}(M) = .4,\ \mathsf{P}(R) = .4,\ \mathsf{P}(F|S) = .6,\ \mathsf{P}(F|M) = .3, \text{ and } \mathsf{P}(F|R) = .1.$$

We must compute

$$\mathsf{P}(S|F) = \frac{\mathsf{P}(F|S)\mathsf{P}(S)}{\mathsf{P}(F)} = \frac{(.6)(.2)}{\mathsf{P}(F)} = \frac{.12}{\mathsf{P}(F)}.$$

Next,

$$\begin{aligned} \mathsf{P}(F) &= \mathsf{P}(F|S)\mathsf{P}(S)+\mathsf{P}(F|M)\mathsf{P}(M)+\mathsf{P}(F|R)\mathsf{P}(R) \\ &= (.6)(.2)+(.3)(.4)+(.1)(.4) \;=\; .12+.12+.04 \;=\; 0.28. \end{aligned}$$

Thus,

$$\mathsf{P}(S|F) \;=\; \frac{12}{100}\cdot\frac{100}{28} \;=\; \frac{3}{7}.$$

60. (a) Let $\Omega = \{1,2,3,4,5\}$ with $\mathsf{P}(A) := |A|/|\Omega|$. Without loss of generality, let 1 and 2 correspond to the two defective chips. Then $D := \{1,2\}$ is the event that a defective chip is tested. Hence, $\mathsf{P}(D) = |D|/5 = 2/5$.

(b) Your friend's information tells you that of the three chips you may test, one is defective and two are not. Hence, the conditional probability that the chip you test is defective is $1/3$.

(c) Yes, your intuition is correct. To prove this, we construct a sample space and probability measure and compute the desired conditional probability. Let

$$\Omega \;:=\; \{(i,j,k) : i<j \text{ and } k\neq i, k\neq j\},$$

where $i,j,k\in\{1,2,3,4,5\}$. Here i and j are the chips taken by the friend, and k is the chip that you test. We again take 1 and 2 to be the defective chips. The 10 possibilities for i and j are

12	13	14	15
23	24	25	
34	35		
45			

For each pair in the above table, there are three possible values of k:

345	245	235	235
145	135	134	
125	124		
123			

Hence, there are 30 triples in Ω. For the probability measure we take $\mathsf{P}(A) := |A|/|\Omega|$. Now let F_{ij} denote the event that the friend takes chips i and j with $i<j$. For example, if the friend takes chips 1 and 2, then from the second table, k has to be 3 or 4 or 5; i.e.,

$$F_{12} \;=\; \{(1,2,3),(1,2,4),(1,2,5)\}.$$

The event that the friend takes two chips is then

$$T \;:=\; F_{12}\cup F_{13}\cup F_{14}\cup F_{15}\cup F_{23}\cup F_{24}\cup F_{25}\cup F_{34}\cup F_{35}\cup F_{45}.$$

Now the event that you test a defective chip is

$$D \;:=\; \{(i,j,k) : k=1 \text{ or } 2 \text{ and } i<j \text{ with } i,j\neq k\}.$$

We can now compute

$$\mathsf{P}(D|T) = \frac{\mathsf{P}(D\cap T)}{\mathsf{P}(T)}.$$

Since the F_{ij} that make up T are disjoint, $|T| = 10\cdot 3 = 30$ and $\mathsf{P}(T) = |T|/|\Omega| = 1$. We next observe that

$$\begin{aligned} D\cap T &= \varnothing \cup [D\cap F_{13}] \cup [D\cap F_{14}] \cup [D\cap F_{15}] \\ &\quad \cup [D\cap F_{23}] \cup [D\cap F_{24}] \cup [D\cap F_{25}] \\ &\quad \cup [D\cap F_{34}] \cup [D\cap F_{35}] \cup [D\cap F_{45}]. \end{aligned}$$

Of the above intersections, the first six intersections are singleton sets, and the last three are pairs. Hence, $|D\cap T| = 6\cdot 1 + 3\cdot 2$ and so $\mathsf{P}(D\cap T) = 12/30 = 2/5$. We conclude that $\mathsf{P}(D|T) = \mathsf{P}(D\cap T)/\mathsf{P}(T) = (2/5)/1 = 2/5$, which is the answer in part (a).

Remark. The model in part (c) can be used to solve part (b) by observing that the probability in part (b) is

$$\mathsf{P}(D|F_{12}\cup F_{13}\cup F_{14}\cup F_{15}\cup F_{23}\cup F_{24}\cup F_{25}),$$

which can be similarly evaluated.

61. (a) If two sets A and B are disjoint, then by definition, $A\cap B = \varnothing$.

(b) If two events A and B are independent, then by definition, $\mathsf{P}(A\cap B) = \mathsf{P}(A)\mathsf{P}(B)$.

(c) If two events A and B are disjoint, then $\mathsf{P}(A\cap B) = \mathsf{P}(\varnothing) = 0$. In order for them to be independent, we must have $\mathsf{P}(A)\mathsf{P}(B) = 0$; i.e., at least one of the two events must have zero probability. If two disjoint events both have positive probability, then they cannot be independent.

62. Let W denote the event that the decoder outputs the wrong message. Of course, W^{c} is the event that the decoder outputs the correct message. We must find $\mathsf{P}(W) = 1 - \mathsf{P}(W^{\mathrm{c}})$. Now, W^{c} occurs if only the first bit is flipped, or only the second bit is flipped, or only the third bit is flipped, or if no bits are flipped. Denote these disjoint events by F_{100}, F_{010}, F_{001}, and F_{000}, respectively. Then

$$\begin{aligned} \mathsf{P}(W^{\mathrm{c}}) &= \mathsf{P}(F_{100}\cup F_{010}\cup F_{001}\cup F_{000}) \\ &= \mathsf{P}(F_{100}) + \mathsf{P}(F_{010}) + \mathsf{P}(F_{001}) + \mathsf{P}(F_{000}) \\ &= p(1-p)^2 + (1-p)p(1-p) + (1-p)^2p + (1-p)^3 \\ &= 3p(1-p)^2 + (1-p)^3. \end{aligned}$$

Hence,

$$\mathsf{P}(W) = 1 - 3p(1-p)^2 - (1-p)^3 = 3p^2 - 2p^3.$$

If $p = 0.1$, then $\mathsf{P}(W) = 0.03 - 0.002 = 0.028$.

63. Let A_i denote the event that your phone selects channel i, $i = 1,\ldots,10$. Let B_j denote the event that your neighbor's phone selects channel j, $j = 1,\ldots,10$. Let $\mathsf{P}(A_i) =$

$\mathsf{P}(B_j) = 1/10$, and assume A_i and B_j are independent. Then

$$\mathsf{P}\bigg(\bigcup_{i=1}^{10}[A_i \cup B_j]\bigg) = \sum_{i=1}^{10} \mathsf{P}(A_i \cap B_j) = \sum_{i=1}^{10} \mathsf{P}(A_i)\mathsf{P}(B_j) = \sum_{i=1}^{10} \frac{1}{10}\cdot\frac{1}{10} = 0.1.$$

64. Let L denote the event that the left airbag works properly, and let R denote the event that the right airbag works properly. Assume L and R are independent with $\mathsf{P}(L^c) = \mathsf{P}(R^c) = p$. The probability that at least one airbag works properly is

$$\mathsf{P}(L \cup R) = 1 - \mathsf{P}(L^c \cap R^c) = 1 - \mathsf{P}(L^c)\mathsf{P}(R^c) = 1 - p^2.$$

65. The probability that the dart never lands within 2 cm of the center is

$$\begin{aligned}
\mathsf{P}\bigg(\bigcap_{n=1}^{\infty} A_n^c\bigg) &= \lim_{N\to\infty} \mathsf{P}\bigg(\bigcap_{n=1}^{N} A_n^c\bigg), \quad \text{limit property of } \mathsf{P}, \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \mathsf{P}(A_n^c), \quad \text{independence}, \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} (1-p) \\
&= \lim_{N\to\infty} (1-p)^N = 0.
\end{aligned}$$

66. Let W_i denote the event that you win on your ith play of the lottery. The probability that you win at least once in n plays is

$$\begin{aligned}
\mathsf{P}\bigg(\bigcup_{i=1}^{n} W_i\bigg) &= 1 - \mathsf{P}\bigg(\bigcap_{i=1}^{n} W_i^c\bigg) = 1 - \prod_{i=1}^{n} \mathsf{P}(W_i^c), \quad \text{by independence}, \\
&= 1 - \prod_{i=1}^{n}(1-p) = 1-(1-p)^n.
\end{aligned}$$

We need to choose n so that $1-(1-p)^n > 1/2$, which happens if and only if

$$1/2 > (1-p)^n \quad \text{or} \quad -\ln 2 > n\ln(1-p) \quad \text{or} \quad \frac{-\ln 2}{\ln(1-p)} < n,$$

where the last step uses the fact that $\ln(1-p)$ is negative. For $p = 10^{-6}$, we need $n > 693147$.

67. Let A denote the event that Anne catches no fish, and let B denote the event that Betty catches no fish. Assume A and B are independent with $\mathsf{P}(A) = \mathsf{P}(B) = p$. We must compute

$$\mathsf{P}(A|A \cup B) = \frac{\mathsf{P}(A \cap [A \cup B])}{\mathsf{P}(A \cup B)} = \frac{\mathsf{P}(A)}{\mathsf{P}(A \cup B)},$$

where the last step uses the fact that $A \subset A \cup B$. To compute the denominator, write

$$\mathsf{P}(A \cup B) = 1 - \mathsf{P}(A^c \cap B^c) = 1 - \mathsf{P}(A^c)\mathsf{P}(B^c) = 1-(1-p)^2 = 2p - p^2 = p(2-p).$$

Then
$$\mathsf{P}(A|A\cup B) \;=\; \frac{p}{p(2-p)} \;=\; \frac{1}{2-p}.$$

68. We show that A and $B\setminus C$ are independent as follows. First, since $C\subset B$,
$$\mathsf{P}(B) \;=\; \mathsf{P}(C)+\mathsf{P}(B\setminus C).$$
Next, since A and B are independent and since A and C are independent,
$$\mathsf{P}(A\cap B) \;=\; \mathsf{P}(A)\mathsf{P}(B) \;=\; \mathsf{P}(A)[\mathsf{P}(C)+\mathsf{P}(B\setminus C)] \;=\; \mathsf{P}(A\cap C)+\mathsf{P}(A)\mathsf{P}(B\setminus C).$$
Again using the fact that $C\subset B$, we now write
$$\mathsf{P}(A\cap B) \;=\; \mathsf{P}(A\cap[C\cup B\setminus C]) \;=\; \mathsf{P}(A\cap C)+\mathsf{P}(A\cap B\setminus C).$$
It follows that $\mathsf{P}(A\cap B\setminus C)=\mathsf{P}(A)\mathsf{P}(B\setminus C)$, which establishes the claimed independence.

69. We show that A, B, and C are mutually independent. To begin, note that $\mathsf{P}(A) = \mathsf{P}(B)=\mathsf{P}(C)=1/2$. Next, we need to identify the events
$$\begin{aligned} A\cap B &= [0,1/4) \\ A\cap C &= [0,1/8)\cup[1/4,3/8) \\ B\cap C &= [0,1/8)\cup[1/2,5/8) \\ A\cap B\cap C &= [0,1/8) \end{aligned}$$
so that we can compute
$$\mathsf{P}(A\cap B)=\mathsf{P}(A\cap C)=\mathsf{P}(B\cap C)=1/4 \quad\text{and}\quad \mathsf{P}(A\cap B\cap C)=1/8.$$
We find that
$$\mathsf{P}(A\cap B)=\mathsf{P}(A)\mathsf{P}(B),\quad \mathsf{P}(A\cap C)=\mathsf{P}(A)\mathsf{P}(C),\quad \mathsf{P}(B\cap C)=\mathsf{P}(B)\mathsf{P}(C),$$
and
$$\mathsf{P}(A\cap B\cap C)=\mathsf{P}(A)\mathsf{P}(B)\mathsf{P}(C).$$

70. From a previous problem we have that $\mathsf{P}(A\cap C|B)=\mathsf{P}(A|B\cap C)\mathsf{P}(C|B)$. Hence, $\mathsf{P}(A\cap C|B)=\mathsf{P}(A|B)\mathsf{P}(C|B)$ if and only if $\mathsf{P}(A|B\cap C)=\mathsf{P}(A|B)$.

71. We show that the probability of the complementary event is zero. By the union bound,
$$\mathsf{P}\biggl(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}B_k^{\mathrm{c}}\biggr) \;\le\; \sum_{n=1}^{\infty}\mathsf{P}\biggl(\bigcap_{k=n}^{\infty}B_k^{\mathrm{c}}\biggr).$$
We show that every term on the right is zero. Write
$$\mathsf{P}\biggl(\bigcap_{k=n}^{\infty}B_k^{\mathrm{c}}\biggr) \;=\; \lim_{N\to\infty}\mathsf{P}\biggl(\bigcap_{k=n}^{N}B_k^{\mathrm{c}}\biggr), \quad \text{limit property of } \mathsf{P},$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \prod_{k=n}^{N} \mathsf{P}(B_k^{\mathrm{c}}), \quad \text{independence},\\
&= \lim_{N\to\infty} \prod_{k=n}^{N} [1-\mathsf{P}(B_k)]\\
&\leq \lim_{N\to\infty} \prod_{k=n}^{N} \exp[-\mathsf{P}(B_k)], \quad \text{the hint},\\
&= \lim_{N\to\infty} \exp\left[-\sum_{k=n}^{N} \mathsf{P}(B_k)\right]\\
&= \exp\left[-\lim_{N\to\infty} \sum_{k=n}^{N} \mathsf{P}(B_k)\right], \quad \text{since exp is continuous},\\
&= \exp\left[-\sum_{k=n}^{\infty} \mathsf{P}(B_k)\right]\\
&= e^{-\infty} \;=\; 0,
\end{aligned}
$$

where the second-to-last step uses the fact that $\sum_{k=1}^{\infty} \mathsf{P}(B_k) = \infty \Rightarrow \sum_{k=n}^{\infty} \mathsf{P}(B_k) = \infty$.

72. There are $3\cdot 5\cdot 7 = 105$ possible systems.

73. There are 2^n n-bit numbers.

74. There are 100! different orderings of the 100 message packets. In order that the first header packet to be received is the 10th packet to arrive, the first 9 packets to be received must come from the 96 data packets, the 10th packet must come from the 4 header packets, and the remaining 90 packets can be in any order. More specifically, there are 96 possibilities for the first packet, 95 for the second, ..., 88 for the ninth, 4 for the tenth, and 90! for the remaining 90 packets. Hence, the desired probability is

$$\frac{96\cdots 88\cdot 4\cdot 90!}{100!} \;=\; \frac{96\cdots 88\cdot 4}{100\cdots 91} \;=\; \frac{90\cdot 89\cdot 88\cdot 4}{100\cdot 99\cdot 98\cdot 97} \;=\; 0.02996.$$

75. $\binom{5}{2} = 10$ pictures are needed.

76. Suppose the player chooses distinct digits *wxyz*. The player wins if any of the $4! = 24$ permutations of *wxyz* occurs. Since each permutation has probability $1/10\,000$ of occurring, the probability of winning is $24/10\,000 = 0.0024$.

77. There are $\binom{8}{3} = 56$ 8-bit words with 3 ones (and 5 zeros).

78. The probability that a random byte has 4 ones and 4 zeros is $\binom{8}{4}/2^8 = 70/256 = 0.2734$.

79. In the first case, since the prizes are different, order is important. Hence, there are $41\cdot 40\cdot 39 = 63\,960$ outcomes. In the second case, since the prizes are the same, order is not important. Hence, there are $\binom{41}{3} = 10\,660$ outcomes.

80. There are $\binom{52}{14}$ possible hands. Since the deck contains 13 spades, 13 hearts, 13 diamonds, and 13 clubs, there are $\binom{13}{2}\binom{13}{3}\binom{13}{4}\binom{13}{5}$ hands with 2 spades, 3 hearts, 4

diamonds, and 5 clubs. The probability of such a hand is

$$\frac{\binom{13}{2}\binom{13}{3}\binom{13}{4}\binom{13}{5}}{\binom{52}{14}} = 0.0116.$$

81. All five cards are of the same suit if and only if they are all spades or all hearts or all diamonds or all clubs. These are four disjoint events. Hence, the answer is four times the probability of getting all spades:

$$4\frac{\binom{13}{5}}{\binom{52}{5}} = 4\frac{1287}{2\,598\,960} = 0.00198.$$

82. There are $\binom{n}{k_1,\ldots,k_m}$ such partitions.

83. The general result is

$$\binom{n}{k_0,\ldots,k_{m-1}}\bigg/m^n.$$

When $n = 4$ and $m = 10$ and a player chooses *xxyz*, we compute $\binom{4}{2,1,1}/10\,000 = 0.0012$. For *xxyy*, we compute $\binom{4}{2,2}/10\,000 = 0.0006$. For *xxxy*, we compute $\binom{4}{3,1}/10\,000 = 0.0004$.

84. Two apples and three carrots corresponds to $(0,0,1,1,0,0,0)$. Five apples corresponds to $(0,0,0,0,0,1,1)$.

CHAPTER 2
Problem Solutions

1. (a) $\{\omega : X(\omega) \leq 3\} = \{1,2,3\}$.

   (b) $\{\omega : X(\omega) > 4\} = \{5,6\}$.

   (c) $\mathsf{P}(X \leq 3) = \mathsf{P}(X = 1) + \mathsf{P}(X = 2) + \mathsf{P}(X = 3) = 3(2/15) = 2/5$, and $\mathsf{P}(X > 4) = \mathsf{P}(X = 5) + \mathsf{P}(X = 6) = 2/15 + 1/3 = 7/15$.

2. (a) $\{\omega : X(\omega) = 2\} = \{1,2,3,4\}$.

   (b) $\{\omega : X(\omega) = 1\} = \{41,42,\ldots,52\}$.

   (c) $\mathsf{P}(X = 1 \text{ or } X = 2) = \mathsf{P}(\{1,2,3,4\} \cup \{41,42,\ldots,52\})$. Since these are disjoint events, the probability of their union is $4/52 + 12/52 = 16/52 = 4/13$.

3. (a) $\{\omega \in [0,\infty) : X(\omega) \leq 1\} = [0,1]$.

   (b) $\{\omega \in [0,\infty) : X(\omega) \leq 3\} = [0,3]$.

   (c) $\mathsf{P}(X \leq 1) = \int_0^1 e^{-\omega}\,d\omega = 1 - e^{-1}$. $\mathsf{P}(X \leq 3) = 1 - e^{-3}$, $\mathsf{P}(1 < X \leq 3) = \mathsf{P}(X \leq 3) - \mathsf{P}(X \leq 1) = e^{-1} - e^{-3}$.

4. First, since $X^{-1}(\varnothing) = \varnothing$, $\mu(\varnothing) = \mathsf{P}(X^{-1}(\varnothing)) = \mathsf{P}(\varnothing) = 0$. Second, $\mu(B) = \mathsf{P}(X^{-1}(B)) \geq 0$. Third, for disjoint B_n,

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mathsf{P}\left(X^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right)\right) = \mathsf{P}\left(\bigcup_{n=1}^{\infty} X^{-1}(B_n)\right) = \sum_{n=1}^{\infty} \mathsf{P}(X^{-1}(B_n)) = \sum_{n=1}^{\infty} \mu(B_n).$$

   Fourth, $\mu(\mathbb{R}) = \mathsf{P}(X^{-1}(\mathbb{R})) = \mathsf{P}(\Omega) = 1$.

5. Since

$$\mathsf{P}(Y > n-1) = \sum_{k=n}^{\infty} \mathsf{P}(Y = k) = \mathsf{P}(Y = n) + \sum_{k=n+1}^{\infty} \mathsf{P}(Y = k) = \mathsf{P}(Y = n) + \mathsf{P}(Y > n),$$

   it follows that $\mathsf{P}(Y = n) = \mathsf{P}(Y > n-1) - \mathsf{P}(Y > n)$.

6. $\mathsf{P}(Y = 0) = \mathsf{P}(\{\text{TTT,THH,HTH,HHT}\}) = 4/8 = 1/2$, and $\mathsf{P}(Y = 1) = \mathsf{P}(\{\text{TTH,THT,HTT,HHH}\}) = 4/8 = 1/2$.

7. $\mathsf{P}(X = 1) = \mathsf{P}(X = 2) = \mathsf{P}(X = 3) = \mathsf{P}(X = 4) = \mathsf{P}(X = 5) = 2/15$ and $\mathsf{P}(X = 6) = 1/3$.

8. $\mathsf{P}(X = 2) = \mathsf{P}(\{1,2,3,4\}) = 4/52 = 1/13$. $\mathsf{P}(X = 1) = \mathsf{P}(\{41,42,\ldots,52\}) = 12/52 = 3/13$. $\mathsf{P}(X = 0) = 1 - \mathsf{P}(X = 2) - \mathsf{P}(X = 1) = 9/13$.

9. The possible values of X are $0,1,4,9,16$. We have $\mathsf{P}(X = 0) = \mathsf{P}(\{0\}) = 1/7$, $\mathsf{P}(X = 1) = \mathsf{P}(\{-1,1\}) = 2/7$, $\mathsf{P}(X = 4) = \mathsf{P}(\{-2,2\}) = 2/7$, $\mathsf{P}(X = 9) = \mathsf{P}(\{3\}) = 1/7$, and $\mathsf{P}(X = 16) = \mathsf{P}(\{4\}) = 1/7$.

10. We have

$$\begin{aligned}\mathsf{P}(X>1) &= 1-\mathsf{P}(X\le 1) = 1-[\mathsf{P}(X=0)+\mathsf{P}(X=1)]\\ &= 1-[e^{-\lambda}+\lambda e^{-\lambda}] = 1-e^{-\lambda}(1+\lambda).\end{aligned}$$

When $\lambda=1$, $\mathsf{P}(X>1) = 1-e^{-2}(2) = 1-2/e = 0.264$.

11. The probability that the sensor fails to activate is

$$\mathsf{P}(X<4) = \mathsf{P}(X\le 3) = \mathsf{P}(X=0)+\cdots+\mathsf{P}(X=3) = e^{-\lambda}(1+\lambda+\lambda^2/2!+\lambda^3/3!).$$

If $\lambda=2$, $\mathsf{P}(X<4) = e^{-2}(1+2+2+4/3) = e^{-2}(19/3) = 0.857$. The probability that the sensor activates is $1-\mathsf{P}(X<4) = 0.143$.

12. Let $\{X_k=1\}$ correspond to the event that the kth student gets an A. This event has probability $\mathsf{P}(X_k=1)=p$. Now, the event that only the kth student gets an A is

$$\{X_k=1 \text{ and } X_l=0 \text{ for } l\neq k\}.$$

Hence, the probability that exactly one student gets an A is

$$\begin{aligned}\mathsf{P}\bigg(\bigcup_{k=1}^{15}\{X_k=1 \text{ and } X_l=0 \text{ for } l\neq k\}\bigg) &= \sum_{k=1}^{15}\mathsf{P}(\{X_k=1 \text{ and } X_l=0 \text{ for } l\neq k\})\\ &= \sum_{k=1}^{15} p(1-p)^{14}\\ &= 15(.1)(.9)^{14} = 0.3432.\end{aligned}$$

13. Let X_1,X_2,X_3 be the random digits of the drawing. Then $\mathsf{P}(X_i=k)=1/10$ for $k=0,\ldots,9$ since each digit has probability $1/10$ of being chosen. Then if the player chooses $d_1d_2d_3$, the probability of winning is

$$\begin{aligned}\mathsf{P}\Big(&\{X_1=d_1,X_2=d_2,X_3=d_3\}\cup\{X_1=d_1,X_2=d_2,X_3\neq d_3\}\\ &\cup\{X_1=d_1,X_2\neq d_2,X_3=d_3\}\cup\{X_1\neq d_1,X_2=d_2,X_3=d_3\}\Big),\end{aligned}$$

which is equal to $.1^3+3[.1^2(.9)] = 0.028$ since the union is disjoint and since X_1,X_2,X_3 are independent.

14.

$$\begin{aligned}\mathsf{P}\bigg(\bigcup_{k=1}^{m}\{X_k<2\}\bigg) &= 1-\mathsf{P}\bigg(\bigcap_{k=1}^{m}\{X_k\ge 2\}\bigg)\\ &= 1-\prod_{k=1}^{m}\mathsf{P}(X_k\ge 2) = 1-\prod_{k=1}^{m}[1-\mathsf{P}(X_k\le 1)]\\ &= 1-[1-\{e^{-\lambda}+\lambda e^{-\lambda}\}]^m = 1-[1-e^{-\lambda}(1+\lambda)]^m.\end{aligned}$$

15. (a)

$$\mathsf{P}\bigg(\bigcup_{i=1}^{n}\{X_i \geq 2\}\bigg) = 1-\mathsf{P}\bigg(\bigcap_{i=1}^{n}\{X_i \leq 1\}\bigg) = 1-\prod_{i=1}^{n}\mathsf{P}(X_i \leq 1)$$
$$= 1-\prod_{i=1}^{n}[\mathsf{P}(X_i=0)+\mathsf{P}(X_i=1)]$$
$$= 1-[e^{-\lambda}+\lambda e^{-\lambda}]^n = 1-e^{-n\lambda}(1+\lambda)^n.$$

(b) $\mathsf{P}\bigg(\bigcap_{i=1}^{n}\{X_i \geq 1\}\bigg) = \prod_{i=1}^{n}\mathsf{P}(X_i \geq 1) = \prod_{i=1}^{n}[1-\mathsf{P}(X_i=0)] = (1-e^{-\lambda})^n.$

(c) $\mathsf{P}\bigg(\bigcap_{i=1}^{n}\{X_i = 1\}\bigg) = \prod_{i=1}^{n}\mathsf{P}(X_i = 1) = (\lambda e^{-\lambda})^n = \lambda^n e^{-n\lambda}.$

16. For the geometric$_0$ pmf, write

$$\sum_{k=0}^{\infty}(1-p)p^k = (1-p)\sum_{k=0}^{\infty}p^k = (1-p)\cdot\frac{1}{1-p} = 1.$$

For the geometric$_1$ pmf, write

$$\sum_{k=1}^{\infty}(1-p)p^{k-1} = (1-p)\sum_{k=1}^{\infty}p^{k-1} = (1-p)\sum_{n=0}^{\infty}p^n = (1-p)\cdot\frac{1}{1-p} = 1.$$

17. Let X_i be the price of stock i, which is a geometric$_0(p)$ random variable. Then

$$\mathsf{P}\bigg(\bigcup_{i=1}^{29}\{X_i > 10\}\bigg) = 1-\mathsf{P}\bigg(\bigcap_{i=1}^{29}\{X_i \leq 10\}\bigg) = 1-\prod_{i=1}^{29}[(1-p)(1+p+\cdots+p^{10})]$$
$$= 1-\bigg((1-p)\frac{1-p^{11}}{1-p}\bigg)^{29} = 1-(1-p^{11})^{29}.$$

Substituting $p=.7$, we have $1-(1-.7^{11})^{29} = 1-(.98)^{29} = 1-.560 = 0.44.$

18. For the first problem, we have

$$\mathsf{P}(\min(X_1,\ldots,X_n) > \ell) = \mathsf{P}\bigg(\bigcap_{k=1}^{n}\{X_k > \ell\}\bigg) = \prod_{k=1}^{n}\mathsf{P}(X_k > \ell) = \prod_{k=1}^{n}p^{\ell} = p^{n\ell}.$$

Similarly,

$$\mathsf{P}(\max(X_1,\ldots,X_n) \leq \ell) = \mathsf{P}\bigg(\bigcap_{k=1}^{n}\{X_k \leq \ell\}\bigg) = \prod_{k=1}^{n}\mathsf{P}(X_k \leq \ell) = \prod_{k=1}^{n}(1-p^{\ell}) = (1-p^{\ell})^n.$$

19. Let X_k denote the number of coins in the pocket of the kth student. Then the X_k are independent and is uniformly distributed from 0 to 20; i.e., $\mathsf{P}(X_k = i) = 1/21$.

(a) $\mathsf{P}\bigg(\bigcap_{k=1}^{25}\{X_k \geq 5\}\bigg) = \prod_{k=1}^{25} 16/21 = (16/21)^{25} = 1.12 \times 10^{-3}$.

(b) $\mathsf{P}\bigg(\bigcup_{k=1}^{25}\{X_k \geq 19\}\bigg) = 1 - \mathsf{P}\bigg(\bigcap_{k=1}^{25}\{X_k \leq 18\}\bigg) = 1-(1-2/21)^{25} = 0.918$.

(c) The probability that only student k has 19 coins in his or her pocket is

$$\mathsf{P}\bigg(\{X_k = 10\} \cap \bigcap_{l \neq k}\{X_l \neq 19\}\bigg) \;=\; (1/21)(20/21)^{24} \;=\; 0.01477.$$

Hence, the probability that exactly one student has 19 coins is

$$\mathsf{P}\bigg(\bigcup_{k=1}^{25}\{X_k = 10\} \cap \bigcap_{l \neq k}\{X_l \neq 19\}\bigg) \;=\; 25(0.01477) \;=\; 0.369.$$

20. Let $X_i = 1$ if block i is good. Then $\mathsf{P}(X_i = 1) = p$ and

$$\mathsf{P}(Y = k) = \mathsf{P}\big(\{X_1 = 1\} \cap \cdots \cap \{X_{k-1} = 1\} \cap \{X_k = 0\}\big) = p^{k-1}(1-p), \quad k = 1,2,\ldots.$$

Hence, $Y \sim \text{geometric}_1(p)$.

21. (a) Write

$$\begin{aligned}\mathsf{P}(X > n) \;&=\; \sum_{k=n+1}^{\infty}(1-p)p^{k-1} \;=\; \sum_{\ell=0}^{\infty}(1-p)p^{\ell+n} \\ &=\; (1-p)p^n \sum_{\ell=0}^{\infty} p^{\ell} \;=\; (1-p)p^n \frac{1}{1-p} \;=\; p^n.\end{aligned}$$

(b) Write

$$\mathsf{P}(X > n+k | X > n) = \frac{\mathsf{P}(X > n+k, X > n)}{\mathsf{P}(X > n)} = \frac{\mathsf{P}(X > n+k)}{\mathsf{P}(X > n)} = \frac{p^{n+k}}{p^n} = p^k.$$

22. Since $\mathsf{P}(Y > k) = \mathsf{P}(Y > n+k | Y > n)$, we can write

$$\mathsf{P}(Y > k) \;=\; \mathsf{P}(Y > n+k | Y > n) \;=\; \frac{\mathsf{P}(Y > n+k, Y > n)}{\mathsf{P}(Y > n)} \;=\; \frac{\mathsf{P}(Y > n+k)}{\mathsf{P}(Y > n)}.$$

Let $p := \mathsf{P}(Y > 1)$. Taking $k = 1$ above yields $\mathsf{P}(Y > n+1) = \mathsf{P}(Y > n)p$. Then with $n = 1$ we have $\mathsf{P}(Y > 2) = \mathsf{P}(Y > 1)p = p^2$. With $n = 2$ we have $\mathsf{P}(Y > 3) = \mathsf{P}(Y > 2)\mathsf{P}(Y > 1) = p^3$. In general then $\mathsf{P}(Y > n) = p^n$. Finally, $\mathsf{P}(Y = n) = \mathsf{P}(Y > n-1) - \mathsf{P}(Y > n) = p^{n-1} - p^n = p^{n-1}(1-p)$, which is the $\text{geometric}_1(p)$ pmf.

23. (a) To compute $p_X(i)$, we sum row i of the matrix. This yields $p_X(1) = p_X(3) = 1/4$ and $p_X(2) = 1/2$. To compute $p_Y(j)$, we sum column j to get $p_Y(1) = p_Y(3) = 1/4$ and $p_Y(2) = 1/2$.

(b) To compute $\mathsf{P}(X < Y)$, we sum $p_{XY}(i,j)$ over i and j such that $i < j$. We have

$$\mathsf{P}(X < Y) \;=\; p_{XY}(1,2) + p_{XY}(1,3) + p_{XY}(2,3) \;=\; 0 + 1/8 + 0 \;=\; 1/8.$$

(c) We claim that X and Y are not independent. For example, $p_{XY}(1,2) = 0$ is not equal to $p_X(1)p_Y(2) = 1/8$.

24. (a) To compute $p_X(i)$, we sum row i of the matrix. This yields $p_X(1) = p_X(3) = 1/4$ and $p_X(2) = 1/2$. To compute $p_Y(j)$, we sum column j to get $p_Y(1) = p_Y(3) = 1/6$ and $p_Y(2) = 2/3$.

(b) To compute $\mathsf{P}(X < Y)$, we sum $p_{XY}(i,j)$ over i and j such that $i < j$. We have

$$\mathsf{P}(X < Y) \;=\; p_{XY}(1,2) + p_{XY}(1,3) + p_{XY}(2,3) \;=\; 1/6 + 1/24 + 1/12 \;=\; 7/24.$$

(c) Using the results of part (a), it is easy to verify that $p_X(i)p_Y(j) = p_{XY}(i,j)$ for $i,j = 1,2,3$. Hence, X and Y are independent.

25. To compute the marginal of X, write

$$p_X(1) \;=\; \frac{e^{-3}}{3}\sum_{j=0}^{\infty}\frac{3^j}{j!} \;=\; \frac{e^{-3}}{3}e^3 \;=\; 1/3.$$

Similarly,

$$p_X(2) \;=\; \frac{4e^{-6}}{6}\sum_{j=0}^{\infty}\frac{6^j}{j!} \;=\; \frac{4e^{-6}}{6}e^6 \;=\; 2/3.$$

Alternatively, $p_X(2) = 1 - p_X(1) = 1 - 1/3 = 2/3$. Of course $p_X(i) = 0$ for $i \neq 1,2$. We clearly have $p_Y(j) = 0$ for $j < 0$ and

$$p_Y(j) \;=\; \frac{3^{j-1}e^{-3}}{j!} + 4\frac{6^{j-1}e^{-6}}{j!}, \quad j \geq 0.$$

Since $p_X(1)p_Y(j) \neq p_{XY}(1,j)$, X and Y are not independent.

26. (a) For $k \geq 1$,

$$\begin{aligned} p_X(k) \;&=\; \sum_{n=0}^{\infty}\frac{(1-p)p^{k-1}k^n e^{-k}}{n!} \\ &=\; (1-p)p^{k-1}e^{-k}\sum_{n=0}^{\infty}\frac{k^n}{n!} \;=\; (1-p)p^{k-1}e^{-k}e^k \;=\; (1-p)p^{k-1}, \end{aligned}$$

which we recognize as the geometric$_1(p)$ pmf.

(b) Next,

$$\begin{aligned} p_Y(0) \;&=\; \sum_{k=1}^{\infty}(1-p)p^{k-1}e^{-k} \;=\; \frac{1-p}{e}\sum_{k=1}^{\infty}(p/e)^{k-1} \\ &=\; \frac{1-p}{e}\sum_{m=0}^{\infty}(p/e)^m = \frac{1-p}{e}\cdot\frac{1}{1-p/e} \;=\; \frac{1-p}{e-p}. \end{aligned}$$

(c) Since $p_X(1)p_Y(0) = (1-p)^2/(e-p)$ is not equal to $p_{XY}(1,0) = (1-p)/e$, X and Y are not independent.

27. **MATLAB.** Here is a script:

```
p = ones(1,51)/51;
k=[0:50];
i = find(g(k) >= -16);
fprintf('The answer is %g\n',sum(p(i)))
```

where

```
function y = g(x)
y = 5*x.*(x-10).*(x-20).*(x-30).*(x-40).*(x-50)/1e6;
```

28. **MATLAB.** If you modified your program for the preceding problem only by the way you compute $\mathsf{P}(X=k)$, then you may get only $0.5001 = \mathsf{P}(g(X) \geq -16 \text{ and } X \leq 50)$. Note that $g(x) > 0$ for $x > 50$. Hence, you also have to add $\mathsf{P}(X \geq 51) = p^{51} = 0.0731$ to 0.5001 to get 0.5732.

29. **MATLAB.** OMITTED.

30. **MATLAB.** OMITTED.

31. **MATLAB.** OMITTED.

32. $\mathsf{E}[X] = 2(1/3) + 5(2/3) = 12/3 = 4$.

33. $\mathsf{E}[I_{(2,6)}(X)] = \sum_{k=3}^{5} \mathsf{P}(X=k) = (1-p)[p^3+p^4+p^5]$. For $p = 1/2$, we get $\mathsf{E}[I_{(2,6)}(X)] = 7/64 = 0.109375$.

34. Write

$$\begin{aligned}
\mathsf{E}[1/(X+1)] &= \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{\lambda^n e^{-\lambda}}{n!} \;=\; \frac{e^{-\lambda}}{\lambda} \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{(n+1)!} \;=\; \frac{e^{-\lambda}}{\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \\
&= \frac{e^{-\lambda}}{\lambda} \left[\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} - 1 \right] \;=\; \frac{e^{-\lambda}}{\lambda}[e^{\lambda} - 1] \;=\; \frac{1-e^{-\lambda}}{\lambda}.
\end{aligned}$$

35. Since $\mathsf{var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2$, $\mathsf{E}[X^2] = \mathsf{var}(X) + (\mathsf{E}[X])^2$. Hence, $\mathsf{E}[X^2] = 7 + 2^2 = 7 + 4 = 11$.

36. Since $Y = cX$, $\mathsf{E}[Y] = \mathsf{E}[cX] = cm$. Hence,

$$\begin{aligned}
\mathsf{var}(Y) &= \mathsf{E}[(Y-cm)^2] \;=\; \mathsf{E}[(cX-cm)^2] \;=\; \mathsf{E}[c^2(X-m)^2] \\
&= c^2\mathsf{E}[(X-m)^2] \;=\; c^2\,\mathsf{var}(X) \;=\; c^2\sigma^2.
\end{aligned}$$

37. We begin with

$$\begin{aligned}
\mathsf{E}[(X+Y)^3] &= \mathsf{E}[X^3 + 3X^2Y + 3XY^2 + Y^3] \\
&= \mathsf{E}[X^3] + 3\mathsf{E}[X^2Y] + 3\mathsf{E}[XY^2] + \mathsf{E}[Y^3] \\
&= \mathsf{E}[X^3] + 3\mathsf{E}[X^2]\mathsf{E}[Y] + 3\mathsf{E}[X]\mathsf{E}[Y^2] + \mathsf{E}[Y^3], \quad \text{by independence.}
\end{aligned}$$

Now, as noted in the text, for a Bernoulli(p) random variable, $X^n = X$, and so $\mathsf{E}[X^n] = \mathsf{E}[X] = p$. Similarly $\mathsf{E}[Y^n] = q$. Thus,

$$\mathsf{E}[(X+Y)^3] \;=\; p+3pq+3pq+q \;=\; p+6pq+q.$$

38. The straightforward approach is to put

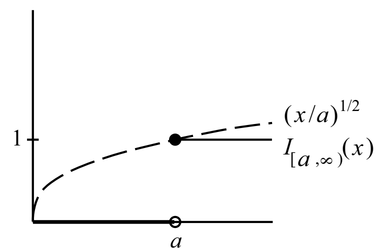
$$f(c) \;:=\; \mathsf{E}[(X-c)^2] \;=\; \mathsf{E}[X^2]-2mc+c^2$$

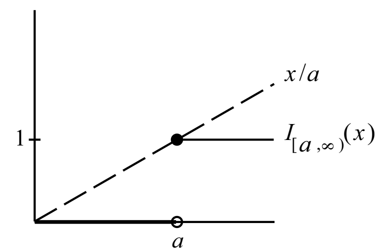
and differentiate with respect to c to get $f'(c) = -2m+2c$. Solving $f'(c) = 0$ results in $c = m$. An alternative approach is to write

$$\mathsf{E}[(X-c)^2] \;=\; \mathsf{E}[\{(X-m)+(m-c)\}^2] \;=\; \sigma^2+(m-c)^2.$$

From this expression, it is obvious that $c = m$ minimizes the expectation.

39. The two sketches are:





40. The right-hand side is easy: $\mathsf{E}[X]/2 = (3/4)/2 = 3/8 = 0.375$. The left-hand side is more work:

$$\mathsf{P}(X \geq 2) \;=\; 1-\mathsf{P}(X \leq 1) \;=\; 1-[\mathsf{P}(X=0)+\mathsf{P}(X=1)] \;=\; 1-e^{-\lambda}(1+\lambda).$$

For $\lambda = 3/4$, $\mathsf{P}(X \geq 2) = 0.1734$. So the bound is a little more than twice the value of the probability.

41. The Chebyshev bound is $(\lambda+\lambda^2)/4$. For $\lambda = 3/4$, the bound is 0.3281, which is a little better than the Markov inequality bound in the preceding problem. The true probability is 0.1734.

42. Comparing the definitions of ρ_{XY} and $\mathsf{cov}(X,Y)$, we find $\rho_{XY} = \mathsf{cov}(X,Y)/(\sigma_X\sigma_Y)$. Hence, $\mathsf{cov}(X,Y) = \sigma_X\sigma_Y\rho_{XY}$. Since $\mathsf{cov}(X,Y) := \mathsf{E}[(X-m_X)(Y-m_Y)$, if $Y = X$, we see that $\mathsf{cov}(X,X) = \mathsf{E}[(X-m_X)^2] =: \mathsf{var}(X)$.

43. Put

$$f(a) \;:=\; \mathsf{E}[(X-aY)^2] \;=\; \mathsf{E}[X^2]-2a\mathsf{E}[XY]+a^2\mathsf{E}[Y^2] \;=\; \sigma_X^2-2a\rho\sigma_X\sigma_Y+a^2\sigma_Y^2.$$

Then

$$f'(a) \;=\; -2\rho\sigma_X\sigma_Y+2a\sigma_Y^2.$$

Setting this equal to zero and solving for a yields $a = \rho(\sigma_X/\sigma_Y)$.

44. Since $\mathsf{P}(X = \pm 1) = \mathsf{P}(X = \pm 2) = 1/4$, $\mathsf{E}[X] = 0$. Similarly, since $\mathsf{P}(XY = \pm 1) = 1/4$ and $\mathsf{P}(XY = \pm 4) = 1/4$, $\mathsf{E}[XY] = 0$. Thus, $\mathsf{E}[XY] = 0 = \mathsf{E}[X]\mathsf{E}[Y]$ and we see that X and Y are uncorrelated. Next, since $X = 1$ implies $Y = 1$, $\mathsf{P}(X = 1, Y = 1) = \mathsf{P}(X = 1) = 1/4$ while $\mathsf{P}(Y = 1) = \mathsf{P}(X = 1 \text{ or } X = -1) = 1/2$. Thus,

$$\mathsf{P}(X=1)\mathsf{P}(Y=1) \;=\; (1/4)(1/2) \;=\; 1/8, \quad \text{but} \quad \mathsf{P}(X=1,Y=1) \;=\; 1/4.$$

45. As discussed in the text, for uncorrelated random variables, the variance of the sum is the sum of the variances. Since independent random variables are uncorrelated, the same results holds for them too. Hence, for $Y = X_1 + \cdots + X_M$,

$$\mathsf{var}(Y) \;=\; \sum_{k=1}^{M} \mathsf{var}(X_k).$$

We also have $\mathsf{E}[Y] = \sum_{k=1}^{M} \mathsf{E}[X_k]$. Next, since the X_k are i.i.d. geometric$_1(p)$, $\mathsf{E}[X_k] = 1/(1-p)$ and $\mathsf{var}(X_k) = p/(1-p)^2$. It follows that $\mathsf{var}(Y) = Mp/(1-p)^2$ and $\mathsf{E}[Y] = M/(1-p)$. We conclude by writing

$$\mathsf{E}[Y^2] \;=\; \mathsf{var}(Y) + (\mathsf{E}[Y])^2 \;=\; \frac{Mp}{(1-p)^2} + \frac{M^2}{(1-p)^2} \;=\; \frac{M(p+M)}{(1-p)^2}.$$

46. From $\mathsf{E}[Y] = \mathsf{E}[dX - s(1-X)] = dp - s(1-p) = 0$, we find that $d/s = (1-p)/p$.

47. (a) $p = 1/1000$.

(b) Since $(1-p)/p = (999/1000)/(1/1000) = 999$, the fair odds against are 999 :1.

(c) Since the fair odds of 999 :1 are not equal to the offered odds of 500 :1, the game is not fair. To make the game fair, the lottery should pay \$900 instead of \$500.

48. First note that

$$\int_1^\infty \frac{1}{t^p}\,dt \;=\; \begin{cases} \dfrac{1}{1-p}\cdot\dfrac{1}{t^{p-1}}\Big|_1^\infty, & p \neq 1, \\ \ln t\Big|_1^\infty, & p = 1. \end{cases}$$

For $p > 1$, the integral is equal to $1/(p-1)$. For $p \leq 1$, the integral is infinite.

For $0 < p \leq 1$, write

$$\sum_{k=1}^\infty \frac{1}{k^p} \;\geq\; \sum_{k=1}^\infty \int_k^{k+1} \frac{1}{t^p}\,dt \;=\; \int_1^\infty \frac{1}{t^p}\,dt \;=\; \infty.$$

For $p > 1$, it suffices to show that $\sum_{k=2}^\infty 1/k^p < \infty$. To this end, write

$$\sum_{k=2}^\infty \frac{1}{k^p} \;=\; \sum_{k=1}^\infty \frac{1}{(k+1)^p} \leq \sum_{k=1}^\infty \int_k^{k+1} \frac{1}{t^p}\,dt \;=\; \int_1^\infty \frac{1}{t^p}\,dt \;<\; \infty.$$

49. First write

$$\mathsf{E}[X^n] \;=\; \sum_{k=1}^\infty k^n \frac{C_p^{-1}}{k^p} \;=\; C_p^{-1} \sum_{k=1}^\infty \frac{1}{k^{p-n}}.$$

By the preceding problem this last sum is finite for $p-n>1$, or equivalently, $n<p-1$. Otherwise the sum is infinite; the case $1 \geq p-n>0$ being handled by the preceding problem, and the case $0 \geq p-n$ being obvious.

50. If all outcomes are equally likely,

$$H(X) \;=\; \sum_{i=1}^{n} p_i \log \frac{1}{p_i} \;=\; \frac{1}{n} \sum_{i=1}^{n} \log n \;=\; \log n.$$

If X is a constant random variable with $p_i = 0$ for $i \neq j$, then

$$H(X) \;=\; \sum_{i=1}^{n} p_i \log \frac{1}{p_i} \;=\; p_j \log \frac{1}{p_j} \;=\; 1 \log 1 \;=\; 0.$$

51. Let $\mathsf{P}(X = x_i) = p_i$ for $i = 1,\ldots,n$. Then

$$\mathsf{E}[g(X)] \;=\; \sum_{i=1}^{n} g(x_i) p_i \quad \text{and} \quad g(\mathsf{E}[X]) \;=\; g\biggl(\sum_{i=1}^{n} x_i p_i\biggr).$$

For $n = 2$, Jensen's inequality says that

$$p_1 g(x_1) + p_2 g(x_2) \;\geq\; g(p_1 x_1 + p_2 x_2).$$

If we put $\lambda = p_1$, then $1-\lambda = p_2$ and the above inequality becomes

$$\lambda g(x_1) + (1-\lambda) g(x_2) \;\geq\; g(\lambda x_1 + (1-\lambda) x_2),$$

which is just the definition of a convex function. Hence, if g is convex, Jensen's inequality holds for $n = 2$. Now suppose Jensen's inequality holds for some $n \geq 2$. We must show it holds for $n+1$. The case of n is

$$\sum_{i=1}^{n} g(x_i) p_i \;\geq\; g\biggl(\sum_{i=1}^{n} x_i p_i\biggr), \quad \text{if } p_1 + \cdots + p_n = 1.$$

Now suppose that $p_1 + \cdots + p_{n+1} = 1$, and write

$$\sum_{i=1}^{n+1} g(x_i) p_i \;=\; (1-p_{n+1}) \biggl[\sum_{i=1}^{n} g(x_i) \frac{p_i}{1-p_{n+1}}\biggr] + p_{n+1} g(x_{n+1}).$$

Let us focus on the quantity in brackets. Since

$$\sum_{i=1}^{n} \frac{p_i}{1-p_{n+1}} \;=\; \frac{p_1 + \cdots + p_n}{p_{n+1}} \;=\; \frac{1-p_{n+1}}{1-p_{n+1}} \;=\; 1,$$

Jensen's inequality for n terms yields

$$\sum_{i=1}^{n} g(x_i) \frac{p_i}{1-p_{n+1}} \;\geq\; g\biggl(\sum_{i=1}^{n} x_i \frac{p_i}{1-p_{n+1}}\biggr).$$

Hence,

$$\sum_{i=1}^{n+1} g(x_i)p_i \;\geq\; (1-p_{n+1})g\left(\sum_{i=1}^{n} x_i \frac{p_i}{1-p_{n+1}}\right) + p_{n+1}g(x_{n+1}).$$

Now apply the two-term Jensen inequality to get

$$\begin{aligned}\sum_{i=1}^{n+1} g(x_i)p_i \;&\geq\; g\left((1-p_{n+1})\left[\sum_{i=1}^{n} x_i \frac{p_i}{1-p_{n+1}}\right] + p_{n+1}x_{n+1}\right)\\ &=\; g\left(\sum_{i=1}^{n+1} p_i x_i\right).\end{aligned}$$

52. With $X = |Z|^\alpha$ and $g(x) = x^{\beta/\alpha}$, we have

$$\mathsf{E}[g(X)] \;=\; \mathsf{E}[X^{\beta/\alpha}] \;=\; \mathsf{E}[(|Z|^\alpha)^{\beta/\alpha}] \;=\; \mathsf{E}[|Z|^\beta]$$

and

$$\mathsf{E}[X] \;=\; \mathsf{E}[|Z|^\alpha].$$

Then Jensen's inequality tells us that

$$\mathsf{E}[|Z|^\beta] \;\geq\; (\mathsf{E}[|Z|^\alpha])^{\beta/\alpha}.$$

Raising both sides the $1/\beta$ power yields Lyapunov's inequality.

53. (a) For all discrete random variables, we have $\sum_i \mathsf{P}(X = x_i) = 1$. For a nonnegative random variable, if $x_k < 0$, we have

$$1 \;=\; \sum_i \mathsf{P}(X = x_i) \;\geq\; \sum_i I_{[0,\infty)}(x_i)\mathsf{P}(X = x_i) + \mathsf{P}(X = x_k) \;=\; 1 + \mathsf{P}(X = x_k).$$

From this it follows that $0 \geq \mathsf{P}(X = x_k) \geq 0$, and so $\mathsf{P}(X = x_k) = 0$.

(b) Write

$$\mathsf{E}[X] \;=\; \sum_i x_i \mathsf{P}(X = x_i) \;=\; \sum_{i:x_i \geq 0} x_i \mathsf{P}(X = x_i) + \sum_{k:x_k<0} x_i \mathsf{P}(X = x_k).$$

By part (a), the last sum is zero. The remaining sum is obviously nonnegative.

(c) By part (b), $0 \leq \mathsf{E}[X - Y] = \mathsf{E}[X] - \mathsf{E}[Y]$. Hence, $\mathsf{E}[Y] \leq \mathsf{E}[X]$.

CHAPTER 3
Problem Solutions

1. First, $\mathsf{E}[X] = G_X'(1) = \left(\frac{1}{6} + \frac{4}{3}z\right)\Big|_{z=1} = \frac{1}{6} + \frac{4}{3} = \frac{9}{6} = \frac{3}{2}$. Second, $\mathsf{E}[X(X-1)] = G_X''(1) = 4/3$. Third, from $\mathsf{E}[X(X-1)] = \mathsf{E}[X^2] - \mathsf{E}[X]$, we have $\mathsf{E}[X^2] = 4/3 + 3/2 = 17/6$. Finally, $\mathsf{var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 17/6 - 9/4 = (34-27)/12 = 7/12$.

2. $p_X(0) = G_X(0) = \left(\frac{1}{6} + \frac{1}{6}z + \frac{2}{3}z^2\right)\Big|_{z=0} = 1/6$, $p_X(1) = G_X'(0) = \left(\frac{1}{6} + \frac{4}{3}z\right)\Big|_{z=0} = 1/6$, and $p_X(2) = G_X''(0)/2 = (4/3)/2 = 2/3$.

3. To begin, note that $G_X'(z) = 5((2+z)/3)^4/3$, and $G_X''(z) = 20((2+z)/3)^3/9$. Hence, $\mathsf{E}[X] = G_X'(1) = 5/3$ and $\mathsf{E}[X(X-1)] = 20/9$. Since $\mathsf{E}[X(X-1)] = \mathsf{E}[X^2] - \mathsf{E}[X]$, $\mathsf{E}[X^2] = 20/9 + 5/3 = 35/9$. Finally, $\mathsf{var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 35/9 - 25/9 = 10/9$.

4. For $X \sim \text{geometric}_0(p)$,
$$G_X(z) = \sum_{n=0}^{\infty} z^n \mathsf{P}(X=n) = \sum_{n=0}^{\infty} z^n (1-p)p^n = (1-p)\sum_{n=0}^{\infty} (zp)^n = (1-p)\frac{1}{1-pz}.$$
Then
$$\mathsf{E}[X] = G_X'(1) = \frac{1-p}{(1-pz)^2}p\Big|_{z=1} = \frac{1-p}{(1-p)^2}p = \frac{p}{1-p}.$$
Next,
$$\begin{aligned}\mathsf{E}[X^2] - \mathsf{E}[X] \;=\; \mathsf{E}[X(X-1)] \;=\; G_X''(1) \;&=\; \frac{(1-p)p}{(1-pz)^4}\cdot 2p(1-pz)\Big|_{z=1} \\ &=\; \frac{(1-p)p\cdot 2p}{(1-p)^3} \;=\; \frac{2p^2}{(1-p)^2}.\end{aligned}$$
This implies that
$$\mathsf{E}[X^2] \;=\; \frac{2p^2}{(1-p)^2} + \frac{p}{1-p} \;=\; \frac{2p^2 + p(1-p)}{(1-p)^2} \;=\; \frac{p+p^2}{(1-p)^2}.$$
Finally,
$$\mathsf{var}(X) \;=\; \mathsf{E}[X^2] - (\mathsf{E}[X])^2 \;=\; \frac{p+p^2}{(1-p)^2} - \frac{p^2}{(1-p)^2} \;=\; \frac{p}{(1-p)^2}.$$

   For $X \sim \text{geometric}_1(p)$,
$$\begin{aligned}G_X(z) \;&=\; \sum_{n=1}^{\infty} z^n \mathsf{P}(X=n) \;=\; \sum_{n=1}^{\infty} z^n (1-p)p^{n-1} \;=\; \frac{1-p}{p}\sum_{n=1}^{\infty}(zp)^n \\ &=\; \frac{1-p}{p}\frac{pz}{1-pz} \;=\; \frac{(1-p)z}{1-pz}.\end{aligned}$$

Now

$$G'_X(z) = \frac{(1-pz)(1-p)+(1-p)pz}{(1-pz)^2} = \frac{1-p}{(1-pz)^2}.$$

Hence,

$$\mathsf{E}[X] = G'_X(1) = \frac{1}{1-p}.$$

Next,

$$G''_X(z) = \frac{(1-p)}{(1-pz)^4} \cdot 2(1-pz)p = \frac{2p(1-p)}{(1-pz)^3}.$$

We then have

$$\mathsf{E}[X^2] - \mathsf{E}[X] = \mathsf{E}[X(X-1)] = G''_X(1) = \frac{2p(1-p)}{(1-p)^3} = \frac{2p}{(1-p)^2},$$

and

$$\mathsf{E}[X^2] = \frac{2p}{(1-p)^2} + \frac{1}{1-p} = \frac{2p+1-p}{(1-p)^2} = \frac{1+p}{(1-p)^2}.$$

Finally,

$$\mathsf{var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = \frac{1+p}{(1-p)^2} - \frac{1}{(1-p)^2} = \frac{p}{(1-p)^2}.$$

5. Since the X_i are independent Poisson(λ_i), we use probability generating functions to find $G_Y(z)$, which turns out to be Poisson(λ) with $\lambda := \lambda_1 + \cdots + \lambda_n$. It then follows that $\mathsf{P}(Y=2) = \lambda^2 e^{-\lambda}/2$. It remains to write

$$\begin{aligned} G_Y(z) &= \mathsf{E}[z^Y] = \mathsf{E}[z^{X_1+\cdots+X_n}] = \mathsf{E}[z^{X_1}\cdots z^{X_n}] = \prod_{i=1}^n \mathsf{E}[z^{X_i}] = \prod_{i=1}^n e^{\lambda_i(z-1)} \\ &= \exp\left[\sum_{i=1}^n \lambda_i(z-1)\right] = e^{\lambda(z-1)}, \end{aligned}$$

which is the Poisson(λ) pgf. Hence, $Y \sim$ Poisson(λ) as claimed.

6. If $G_X(z) = \sum_{k=0}^\infty z^k \mathsf{P}(X=k)$, then $G_X(1) = \sum_{k=0}^\infty \mathsf{P}(X=k) = 1$. For the particular formula in the problem, we must have $1 = G_X(1) = (a_0+a_1+a_2+\cdots+a_n)^m/D$, or $D = (a_0+a_1+a_2+\cdots+a_n)^m$.

7. From the table on the inside of the front cover of the text, $\mathsf{E}[X_i] = 1/(1-p)$. Thus,

$$\mathsf{E}[Y] = \sum_{i=1}^n \mathsf{E}[X_i] = \frac{n}{1-p}.$$

Second, since the variance of the sum of uncorrelated random variables is the sum of the variances, and since $\mathsf{var}(X_i) = p/(1-p)^2$, we have

$$\mathsf{var}(Y) = \sum_{i=1}^n \mathsf{var}(X_i) = \frac{np}{(1-p)^2}.$$

It then easily follows that

$$\mathsf{E}[Y^2] \;=\; \mathsf{var}(Y) + (\mathsf{E}[Y])^2 \;=\; \frac{np}{(1-p)^2} + \frac{n^2}{(1-p)^2} \;=\; \frac{n(p+n)}{(1-p)^2}.$$

Since Y is the sum of i.i.d. geometric$_1(p)$ random variables, the pgf of Y is the product of the individual pgfs. Thus,

$$G_Y(z) \;=\; \prod_{i=1}^{n} \frac{(1-p)z}{1-pz} \;=\; \left[\frac{(1-p)z}{1-pz}\right]^n.$$

8. Starting with $G_Y(z) = [(1-p)+pz]^n$, we have

$$\mathsf{E}[Y] \;=\; G_Y'(1) = n[(1-p)+pz]^{n-1}p\Big|_{z=1} \;=\; np.$$

Next,

$$\mathsf{E}[Y(Y-1)] \;=\; G_Y''(1) \;=\; n(n-1)[(1-p)+pz]^{n-2}p^2\Big|_{z=1} \;=\; n(n-1)p^2.$$

It then follows that

$$\mathsf{E}[Y^2] \;=\; \mathsf{E}[Y(Y-1)] + \mathsf{E}[Y] \;=\; n(n-1)p^2 + np \;=\; n^2p^2 - np^2 + np.$$

Finally,

$$\mathsf{var}(Y) \;=\; \mathsf{E}[Y^2] - (\mathsf{E}[Y])^2 \;=\; n^2p^2 - np^2 + np - (np)^2 \;=\; np(1-p).$$

9. For the binomial(n,p) random variable,

$$G_Y(z) \;=\; \sum_{k=0}^{n} \mathsf{P}(Y=k)z^k \;=\; \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} z^k.$$

Hence,

$$1 \;=\; G_Y(1) \;=\; \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}.$$

10. Starting from

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \;=\; 1,$$

we follow the hint and replace p with $a/(a+b)$. Note that $1-p = b/(a+b)$. With these substitutions, the above equation becomes

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{a+b}\right)^k \left(\frac{b}{a+b}\right)^{n-k} \;=\; 1 \quad\text{or}\quad \sum_{k=0}^{n} \binom{n}{k} \frac{a^k b^{n-k}}{(a+b)^k (a+b)^{n-k}} \;=\; 1.$$

Multiplying through by $(a+b)^n$ we have

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \;=\; (a+b)^n.$$

11. Let $X_i = 1$ if bit i is in error, $X_i = 0$ otherwise. Then $Y_n := X_1 + \cdots + X_n$ is the number of errors in n bits. Assume the X_i are independent. Then

$$\begin{aligned} G_{Y_n}(z) &= \mathsf{E}[z^{Y_n}] \;=\; \mathsf{E}[z^{X_1+\cdots+X_n}] \;=\; \mathsf{E}[z^{X_1}\cdots z^{X_n}] \\ &= \mathsf{E}[z^{X_1}]\cdots\mathsf{E}[z^{X_n}], \quad \text{by independence}, \\ &= [(1-p)+pz]^n, \quad \text{since the } X_i \text{ are i.i.d. Bernoulli}(p). \end{aligned}$$

We recognize this last expression as the binomial(n,p) probability generating function. Thus, $\mathsf{P}(Y_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

12. Let $X_i \sim \text{binomial}(n_i, p)$ denote the number of students in the ith room. Then $Y = X_1 + \cdots + X_M$ is the total number of students in the school. Next,

$$\begin{aligned} G_Y(z) &= \mathsf{E}[z^Y] \;=\; \mathsf{E}[z^{X_1+\cdots+X_M}] \\ &= \mathsf{E}[z^{X_1}]\cdots\mathsf{E}[z^{X_M}], \quad \text{by independence}, \\ &= \prod_{i=1}^{M}[(1-p)+pz]^{n_i}, \quad \text{since } X_i \sim \text{binomial}(n_i,p), \\ &= [(1-p)+pz]^{n_1+\cdots+n_M}. \end{aligned}$$

Setting $n := n_1 + \cdots + n_M$, we see that $Y \sim \text{binomial}(n,p)$. Hence, $\mathsf{P}(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

13. Let $Y = X_1 + \cdots + X_n$, where the X_i are i.i.d. with $\mathsf{P}(X_i = 1) = 1-p$ and $\mathsf{P}(X_i = 2) = p$. Observe that $X_i - 1 \sim \text{Bernoulli}(p)$. Hence,

$$Y \;=\; n + \underbrace{\sum_{i=1}^{n}(X_i - 1)}_{=:\,Z}.$$

Since Z is the sum of i.i.d. Bernoulli(p) random variables, $Z \sim \text{binomial}(n,p)$. Hence,

$$\begin{aligned} \mathsf{P}(Y = k) &= \mathsf{P}(n+Z = k) \;=\; \mathsf{P}(Z = k-n) \\ &= \binom{n}{k-n} p^{k-n}(1-p)^{2n-k}, \quad k = n,\ldots,2n. \end{aligned}$$

14. Let X_i be i.i.d. Bernoulli(p), where $X_i = 1$ means bit i is flipped. Then $Y := X_1 + \cdots + X_n$ $(n = 10)$ is the number of bits flipped. A codeword cannot be decoded if $Y > 2$. We need to find $\mathsf{P}(Y > 2)$. Observe that

$$G_Y(z) \;=\; \mathsf{E}[z^Y] \;=\; \mathsf{E}[z^{X_1+\cdots+X_n}] \;=\; \mathsf{E}[z^{X_1}\cdots z^{X_n}] \;=\; \prod_{i=1}^{n}\mathsf{E}[z^{X_i}] \;=\; [(1-p)+pz]^n.$$

This is the pgf of a binomial(n,p) random variable. Hence,

$$\begin{aligned} \mathsf{P}(Y > 2) &= 1 - \mathsf{P}(Y \le 2) \;=\; 1 - [\mathsf{P}(Y=0) + \mathsf{P}(Y=1) + \mathsf{P}(Y=2)] \\ &= 1 - \left[\binom{10}{0}(1-p)^{10} + \binom{10}{1}p(1-p)^9 + \binom{10}{2}p^2(1-p)^8\right] \\ &= 1 - (1-p)^8[(1-p)^2 + 10p(1-p) + 45p^2]. \end{aligned}$$

15. For $n = 150$ and $p = 1/100$, we have

k	$\mathsf{P}\Big(\text{Binomial}(n,p) = k\Big)$	$\mathsf{P}\Big(\text{Poisson}(np) = k\Big)$
0	0.2215	0.2231
1	0.3355	0.3347
2	0.2525	0.2510
3	0.1258	0.1255
4	0.0467	0.0471
5	0.0138	0.0141

16. If the X_i are i.i.d. with mean m, then

$$\mathsf{E}[M_n] \;=\; \mathsf{E}\left[\frac{1}{n}\sum_{i=1}^n X_i\right] \;=\; \frac{1}{n}\sum_{i=1}^n \mathsf{E}[X_i] \;=\; \frac{1}{n}\sum_{i=1}^n m \;=\; \frac{nm}{n} \;=\; m.$$

If X is any random variable with mean m, then $\mathsf{E}[cX] = c\mathsf{E}[X] = cm$, and

$$\mathsf{var}(cX) \;=\; \mathsf{E}[(cX - cm)^2] \;=\; \mathsf{E}[c^2(X-m)^2] = c^2\mathsf{E}[(X-m)^2] \;=\; c^2\,\mathsf{var}(X).$$

17. In general, we have

$$\mathsf{P}(|M_n - m| < \varepsilon) \;\geq\; 0.9 \quad\Leftrightarrow\quad \mathsf{P}(|M_n - m| \geq \varepsilon) \;<\; 0.1.$$

By Chebyshev's inequality,

$$\mathsf{P}(|M_n - m| \geq \varepsilon) \;\leq\; \frac{\sigma^2}{n\varepsilon^2} \;<\; 0.1$$

if $n > \sigma^2/(.1)\varepsilon^2$. For $\sigma^2 = 1$ and $\varepsilon = 0.25$, we require $n > 1/(.1)(.25)^2 = 1/.00625 = 160$ students. If instead $\varepsilon = 1$, we require $n > 0.1 = 10$ students.

18. (a) $\mathsf{E}[X_i] = \mathsf{E}[I_B(Z_i)] = \mathsf{P}(Z_i \in B)$. Setting $p := \mathsf{P}(Z_i \in B)$, we see that $X_i = I_B(Z_i) \sim$ Bernoulli(p). Hence, $\mathsf{var}(X_i) = p(1-p)$.

(b) In fact, the X_i are independent. Hence, they are uncorrelated.

19. $M_n = 0$ if and only if all the X_i are zero. Hence,

$$\mathsf{P}(M_n = 0) \;=\; \mathsf{P}\left(\bigcap_{i=1}^n \{X_i = 0\}\right) \;=\; \prod_{i=1}^n \mathsf{P}(X_i = 0) \;=\; (1-p)^n.$$

In particular, if $p = 1/1000$, then $\mathsf{P}(M_{100} = 0) = (1-p)^{100} = 0.999^{100} = 0.905$. Hence, the chances are more than 90% that when we run a simulation, $M_{100} = 0$ and we learn nothing!

20. If $X_i = Z \sim$ Bernoulli$(1/2)$ for all i, then

$$M_n \;=\; \frac{1}{n}\sum_{i=1}^n X_i \;=\; \frac{1}{n}\sum_{i=1}^n Z \;=\; Z,$$

and $m = \mathsf{E}[X_i] = \mathsf{E}[Z] = 1/2$. So,

$$\mathsf{P}(|M_n - m| \geq 1/4) \;=\; \mathsf{P}(|Z - 1/2| \geq 1/4).$$

Now, $Z - 1/2 = \pm 1/2$, and $|Z - 1/2| = 1/2$ with probability one. Thus,

$$\mathsf{P}(|Z - 1/2| \geq 1/4) \;=\; \mathsf{P}(1/2 \geq 1/4) \;=\; 1 \;\not\to\; 0.$$

21. From the discussion of the weak law in the text, we have

$$\mathsf{P}(|M_n - m| \geq \varepsilon_n) \;\leq\; \frac{\sigma^2}{n\varepsilon_n^2}.$$

If $n\varepsilon_n^2 \to \infty$ as $n \to \infty$, then probability on the left will go to zero as $n \to \infty$.

22. We have from the example that with $p := \lambda/(\lambda+\mu)$, $p_{X|Z}(i|j) = \binom{j}{i} p^i (1-p)^{j-i}$ for $i = 0, \ldots, j$. In other words, as a function of i, $p_{X|Z}(i|j)$ is a binomial(j,p) pmf. Hence,

$$\mathsf{E}[X|Z = j] \;=\; \sum_{i=0}^{j} i p_{X|Z}(i|j)$$

is just the mean of a binomial(j,p) pmf. The mean of such a pmf is jp. Hence, $\mathsf{E}[X|Z = j] = jp = j\lambda/(\lambda+\mu)$.

23. The problem is telling us that $\mathsf{P}(Y = k|X = i) = \binom{n}{k} p_i^k (1-p_i)^{n-k}$. Hence

$$\mathsf{P}(Y < 2|X = i) = \mathsf{P}(Y = 0|X = i) + \mathsf{P}(Y = 1|X = i) = (1-p_i)^n + np_i(1-p_i)^{n-1}.$$

24. The problem is telling us that $\mathsf{P}(X = k|Y = j) = \lambda_j^k e^{-\lambda_j}/k!$. Hence,

$$\begin{aligned}
\mathsf{P}(X > 2|Y = j) \;&=\; 1 - \mathsf{P}(X \leq 2|Y = j) \\
&=\; 1 - [\mathsf{P}(X = 0|Y = j) + \mathsf{P}(X = 1|Y = j) + \mathsf{P}(X = 2|Y = j)] \\
&=\; 1 - [e^{-\lambda_j} + \lambda_j e^{-\lambda_j} + \lambda_j^2 e^{-\lambda_j}/2] \\
&=\; 1 - e^{-\lambda_j}[1 + \lambda_j + \lambda_j^2/2].
\end{aligned}$$

25. For the first formula, write

$$p_{X|Y}(x_i|y_j) := \frac{\mathsf{P}(X = x_i, Y = y_j)}{\mathsf{P}(Y = y_j)} = \frac{\mathsf{P}(X = x_i)\mathsf{P}(Y = y_j)}{\mathsf{P}(Y = y_j)} = \mathsf{P}(X = x_i) = p_X(x_i).$$

Similarly, for the other formula,

$$p_{Y|X}(y_j|x_i) := \frac{\mathsf{P}(Y = y_j, X = x_i)}{\mathsf{P}(X = x_i)} = \frac{\mathsf{P}(Y = y_j)\mathsf{P}(X = x_i)}{\mathsf{P}(X = x_i)} = \mathsf{P}(Y = y_j) = p_Y(y_j).$$

26. We use the law of total probability to write

$$\begin{aligned}
\mathsf{P}(T=n) &= \mathsf{P}(X-Y=n) = \sum_{k=0}^{\infty}\mathsf{P}(X-Y=n|Y=k)\mathsf{P}(Y=k) \\
&= \sum_{k=0}^{\infty}\mathsf{P}(X-k=n|Y=k)\mathsf{P}(Y=k), \quad \text{by the substitution law,} \\
&= \sum_{k=0}^{\infty}\mathsf{P}(X=n+k|Y=k)\mathsf{P}(Y=k) \\
&= \sum_{k=0}^{\infty}\mathsf{P}(X=n+k)\mathsf{P}(Y=k), \quad \text{by independence,} \\
&= \sum_{k=0}^{\infty}(1-p)p^{n+k}\cdot(1-q)q^k \\
&= (1-p)(1-q)p^n\sum_{k=0}^{\infty}(pq)^k = \frac{(1-p)(1-q)p^n}{1-pq}.
\end{aligned}$$

27. The problem is telling us that

$$\mathsf{P}(Y=n|X=1) = \frac{\mu^n e^{-\mu}}{n!} \quad \text{and} \quad \mathsf{P}(Y=n|X=2) = \frac{\nu^n e^{-\nu}}{n!}.$$

The problem also tells us that $\mathsf{P}(X=1)=\mathsf{P}(X=2)=1/2$. We can now write

$$\mathsf{P}(X=1|Y=2) = \frac{\mathsf{P}(X=1,Y=2)}{\mathsf{P}(Y=2)} = \frac{\mathsf{P}(Y=2|X=1)\mathsf{P}(X=1)}{\mathsf{P}(Y=2)} = \frac{(\mu^2e^{-\mu}/2)(1/2)}{\mathsf{P}(Y=2)}.$$

It remains to use the law of total probability to compute

$$\begin{aligned}
\mathsf{P}(Y=2) &= \sum_{i=1}^{2}\mathsf{P}(Y=2|X=i)\mathsf{P}(X=i) \\
&= [\mathsf{P}(Y=2|X=1)+\mathsf{P}(Y=2|X=2)]/2 \\
&= [\mu^2e^{-\mu}/2+\nu^2e^{-\nu}/2]/2 = [\mu^2e^{-\mu}+\nu^2e^{-\nu}]/4.
\end{aligned}$$

We conclude by writing

$$\mathsf{P}(X=1|Y=2) = \frac{\mu^2e^{-\mu}/4}{[\mu^2e^{-\mu}+\nu^2e^{-\nu}]/4} = \frac{1}{1+(\nu/\mu)^2e^{\mu-\nu}}.$$

28. Let $X=0$ or $X=1$ according to whether message zero or message one is sent. The problem tells us that $\mathsf{P}(X=0)=\mathsf{P}(X=1)=1/2$ and that

$$\mathsf{P}(Y=k|X=0) = (1-p)p^k \quad \text{and} \quad \mathsf{P}(Y=k|X=1) = (1-q)q^k,$$

where $q \neq p$. We need to compute

$$\mathsf{P}(X=1|Y=k) = \frac{\mathsf{P}(Y=k|X=1)\mathsf{P}(X=1)}{\mathsf{P}(Y=k)} = \frac{(1-q)q^k(1/2)}{\mathsf{P}(Y=k)}.$$

We next use the law of total probability to compute

$$\mathsf{P}(Y=k) = [\mathsf{P}(Y=k|X=0)+\mathsf{P}(Y=k|X=1)]/2 = [(1-p)p^k+(1-q)q^k]/2.$$

We can now compute

$$\mathsf{P}(X=1|Y=k) \;=\; \frac{(1-q)q^k(1/2)}{[(1-p)p^k+(1-q)q^k]/2} \;=\; \frac{1}{1+\frac{(1-p)p^k}{(1-q)q^k}}.$$

29. Let R denote the number of red apples in a crate, and let G denote the number of green apples in a crate. The problem is telling us that $R \sim \text{Poisson}(\rho)$ and $G \sim \text{Poisson}(\gamma)$ are independent. If $T = R+G$ is the total number of apples in the crate, we must compute

$$\mathsf{P}(G=0|T=k) \;=\; \frac{\mathsf{P}(T=k|G=0)\mathsf{P}(G=0)}{\mathsf{P}(T=k)}.$$

We first use the law of total probability, substitution, and independence to write

$$\mathsf{P}(T=k|G=0) \;=\; \mathsf{P}(R+G=k|G=0) \;=\; \mathsf{P}(R=k|G=0) \;=\; \mathsf{P}(R=k) \;=\; \rho^k e^{-\rho}/k!.$$

We also note from the text that the sum of two independent Poisson random variables is a Poisson random variable whose parameter is the sum of the individual parameters. Hence, $\mathsf{P}(T=k) = (\rho+\gamma)^k e^{-(\rho+\gamma)}/k!$. We can now write

$$\mathsf{P}(G=0|T=k) \;=\; \frac{\rho^k e^{-\rho}/k! \cdot e^{-\gamma}}{(\rho+\gamma)^k e^{-(\rho+\gamma)}/k!} \;=\; \left(\frac{\rho}{\rho+\gamma}\right)^k.$$

30. We begin with

$$\mathsf{P}(X=n|Y=1) \;=\; \frac{\mathsf{P}(Y=1|X=n)\mathsf{P}(X=n)}{\mathsf{P}(Y=1)} \;=\; \frac{\frac{1}{n+1}\cdot\frac{\lambda^n e^{-\lambda}}{n!}}{\mathsf{P}(Y=1)}.$$

Next, we compute

$$\begin{aligned}
\mathsf{P}(Y=1) \;&=\; \sum_{n=0}^{\infty} \mathsf{P}(Y=1|X=n)\mathsf{P}(X=n) \;=\; \sum_{n=0}^{\infty} \frac{1}{n+1}\cdot\frac{\lambda^n e^{-\lambda}}{n!} \\
&=\; \frac{e^{-\lambda}}{\lambda}\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{(n+1)!} \;=\; \frac{e^{-\lambda}}{\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!} \;=\; \frac{e^{-\lambda}}{\lambda}\left[\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}-1\right] \\
&=\; \frac{e^{-\lambda}}{\lambda}[e^{\lambda}-1] \;=\; \frac{1-e^{-\lambda}}{\lambda}.
\end{aligned}$$

We conclude with

$$\mathsf{P}(X=n|Y=1) \;=\; \frac{\frac{1}{n+1}\cdot\frac{\lambda^n e^{-\lambda}}{n!}}{\mathsf{P}(Y=1)} \;=\; \frac{\frac{1}{n+1}\cdot\frac{\lambda^n e^{-\lambda}}{n!}}{(1-e^{-\lambda})/\lambda} \;=\; \frac{\lambda^{n+1}}{(e^{\lambda}-1)(n+1)!}.$$

31. We begin with

$$\mathsf{P}(X=n|Y=k) \;=\; \frac{\mathsf{P}(Y=k|X=n)\mathsf{P}(X=n)}{\mathsf{P}(Y=k)} \;=\; \frac{\binom{n}{k}p^k(1-p)^{n-k}\cdot\lambda^n e^{-\lambda}/n!}{\mathsf{P}(Y=k)}.$$

Next,

$$\begin{aligned}\mathsf{P}(Y=k) \;&=\; \sum_{n=0}^{\infty}\mathsf{P}(Y=k|X=n)\mathsf{P}(X=n) \;=\; \sum_{n=k}^{\infty}\binom{n}{k}p^k(1-p)^{n-k}\lambda^n e^{-\lambda}/n! \\ &=\; \frac{p^k\lambda^k e^{-\lambda}}{k!}\sum_{n=k}^{\infty}\frac{[(1-p)\lambda]^{n-k}}{(n-k)!} \;=\; \frac{p^k\lambda^k e^{-\lambda}}{k!}\sum_{m=0}^{\infty}\frac{[(1-p)\lambda]^m}{m!} \\ &=\; \frac{p^k\lambda^k e^{-\lambda}}{k!}e^{(1-p)\lambda} \;=\; \frac{(p\lambda)^k e^{-p\lambda}}{k!}.\end{aligned}$$

Note that $Y \sim \text{Poisson}(p\lambda)$. We continue with

$$\begin{aligned}\mathsf{P}(X=n|Y=k) \;&=\; \frac{\binom{n}{k}p^k(1-p)^{n-k}\cdot\lambda^n e^{-\lambda}/n!}{\mathsf{P}(Y=k)} \;=\; \frac{\binom{n}{k}p^k(1-p)^{n-k}\cdot\lambda^n e^{-\lambda}/n!}{(p\lambda)^k e^{-p\lambda}/k!} \\ &=\; \frac{[(1-p)\lambda]^{n-k}e^{-(1-p)\lambda}}{(n-k)!}.\end{aligned}$$

32. First write

$$\mathsf{P}\big(X>k\big|\max(X,Y)>k\big) = \frac{\mathsf{P}\big(\{X>k\}\cap\{\max(X,Y)>k\}\big)}{\mathsf{P}\big(\max(X,Y)>k\big)} = \frac{\mathsf{P}(X>k)}{\mathsf{P}\big(\max(X,Y)>k\big)},$$

since $\{X>k\}\subset\{\max(X,Y)>k\}$. We next compute

$$\mathsf{P}\big(\max(X,Y)>k\big) \;=\; 1-\mathsf{P}\big(\max(X,Y)\le k\big) \;=\; 1-\mathsf{P}(X\le k)\mathsf{P}(Y\le k).$$

If we put $\theta_k := \mathsf{P}(X\le k)$ and use the fact that X and Y have the same pmf, then

$$\mathsf{P}\big(X>k\big|\max(X,Y)>k\big) \;=\; \frac{1-\theta_k}{1-\theta_k^2} \;=\; \frac{1-\theta_k}{(1-\theta_k)(1+\theta_k)} \;=\; \frac{1}{1+\theta_k}.$$

With $n=100$ and $p=.01$, we compute

$$\theta_1 = \mathsf{P}(X\le 1) = \mathsf{P}(X=0)+\mathsf{P}(X=1) = .99^{100}+.99^{99} = .366+.370 = .736.$$

It follows that the desired probability is $1/(1+\theta_1) = 1/1.736 = 0.576$.

33. (a) Observe that

$$\begin{aligned}\mathsf{P}(XY=4) \;&=\; \mathsf{P}(X=1,Y=4)+\mathsf{P}(X=2,Y=2)+\mathsf{P}(X=4,Y=1) \\ &=\; (1-p)(1-q)[pq^4+p^2q^2+p^4q].\end{aligned}$$

(b) Write

$$\begin{aligned} p_Z(j) &= \sum_i p_Y(j-i)p_X(i) \\ &= \sum_{i=0}^{\infty} p_Y(j-i)p_X(i), \quad \text{since } p_X(i)=0 \text{ for } i<0, \\ &= \sum_{i=0}^{j} p_Y(j-i)p_X(i), \quad \text{since } p_Y(k)=0 \text{ for } k<0, \\ &= (1-p)(1-q)\sum_{i=0}^{j} p^i q^{j-i} \;=\; (1-p)(1-q)q^j \sum_{i=0}^{j}(p/q)^i. \end{aligned}$$

Now, if $p=q$,

$$p_Z(j) = (1-p)^2 p^j (j+1).$$

If $p \neq q$,

$$p_Z(j) = (1-p)(1-q)q^j \frac{1-(p/q)^{j+1}}{1-p/q} = (1-p)(1-q)\frac{q^{j+1}-p^{j+1}}{q-p}.$$

34. For $j=0,1,2,3$,

$$p_Z(j) = \sum_{i=0}^{j}(1/16) = (j+1)/16.$$

For $j=4,5,6$,

$$p_Z(j) = \sum_{i=j-3}^{3}(1/16) = (7-j)/16.$$

For other values of j, $p_Z(j)=0$.

35. We first write

$$\frac{\mathsf{P}(Y=j|X=1)}{\mathsf{P}(Y=j|X=0)} \geq \frac{\mathsf{P}(X=0)}{\mathsf{P}(X=1)}$$

as

$$\frac{\lambda_1^j e^{-\lambda_1}/j!}{\lambda_0^j e^{-\lambda_0}/j!} \geq \frac{1-p}{p}.$$

We can further simplify this to

$$\left(\frac{\lambda_1}{\lambda_0}\right)^j \geq \frac{1-p}{p}e^{\lambda_1-\lambda_0}.$$

Taking logarithms and rearranging, we obtain

$$j \geq \left[\lambda_1-\lambda_0+\ln\frac{1-p}{p}\right]\Big/\ln(\lambda_1/\lambda_0).$$

Observe that the right-hand side is just a number (threshold) that is computable from the problem data. If we observe $Y=j$, we compare j to the threshold. If j is greater than or equal to this number, we decide $X=1$; otherwise, we decide $X=0$.

36. We first write

$$\frac{\mathsf{P}(Y=j|X=1)}{\mathsf{P}(Y=j|X=0)} \;\geq\; \frac{\mathsf{P}(X=0)}{\mathsf{P}(X=1)}$$

as

$$\frac{(1-q_1)q_1^j}{(1-q_0)q_0^j} \;\geq\; \frac{1-p}{p}.$$

We can further simplify this to

$$\left(\frac{q_1}{q_0}\right)^j \;\geq\; \frac{(1-p)(1-q_0)}{p(1-q_1)}.$$

Taking logarithms and rearranging, we obtain

$$j \;\leq\; \left[\ln\frac{(1-p)(1-q_0)}{p(1-q_1)}\right] \Big/ \ln(q_1/q_0),$$

since $q_1 < q_0$ implies $\ln(q_1/q_0) < 0$.

37. Starting with $\mathsf{P}(X=x_i|Y=y_j) = h(x_i)$, we have

$$\mathsf{P}(X=x_i, Y=y_j) \;=\; \mathsf{P}(X=x_i|Y=y_j)\mathsf{P}(Y=y_j) \;=\; h(x_i)p_Y(y_j).$$

If we can show that $h(x_i) = p_X(x_i)$, then it will follow that X and Y are independent. Now observe that the sum over j of the left-hand side reduces to $\mathsf{P}(X=x_i) = p_X(x_i)$. The sum over j of the right-hand side reduces to $h(x_i)$. Hence, $p_X(x_i) = h(x_i)$ as desired.

38. First write

$$p_{XY}(1,j) \;=\; (1/3)3^j e^{-3}/j! \quad\text{and}\quad p_{XY}(2,j) \;=\; (4/6)6^j e^{-6}/j!$$

Notice that $3^j e^{-3}/j!$ is a Poisson(3) pmf and $6^j e^{-6}/j!$ is a Poisson(6) pmf. Hence, $p_X(1) = \sum_{j=0}^{\infty} p_{XY}(1,j) = 1/3$ and $p_X(2) = \sum_{j=0}^{\infty} p_{XY}(2,j) = 2/3$. It then follows that $p_{Y|X}(j|1)$ is Poisson(3) and $p_{Y|X}(j|2)$ is Poisson(6). With these observations, it is clear that

$$\mathsf{E}[Y|X=1] \;=\; 3 \quad\text{and}\quad \mathsf{E}[Y|X=2] \;=\; 6,$$

and

$$\mathsf{E}[Y] \;=\; \mathsf{E}[Y|X=1](1/3) + \mathsf{E}[Y|X=2](2/3) \;=\; 3(1/3)+6(2/3) \;=\; 1+4 \;=\; 5.$$

To obtain $\mathsf{E}[X|Y=j]$, we first compute

$$p_Y(j) \;=\; p_{XY}(1,j)+p_{XY}(2,j) \;=\; (1/3)3^j e^{-3}/j! + (2/3)6^j e^{-6}/j!$$

and

$$p_{X|Y}(1|j) \;=\; \frac{(1/3)3^j e^{-3}/j!}{(1/3)3^j e^{-3}/j! + (2/3)6^j e^{-6}/j!} \;=\; \frac{1}{1+2^{j+1}e^{-3}}$$

and

$$p_{X|Y}(2|j) \;=\; \frac{(2/3)6^j e^{-6}/j!}{(1/3)3^j e^{-3}/j! + (2/3)6^j e^{-6}/j!} \;=\; \frac{1}{1+2^{-(j+1)}e^{3}}.$$

We now have

$$\begin{aligned}\mathsf{E}[X|Y=j] &= 1\cdot\frac{1}{1+2^{j+1}e^{-3}}+2\cdot\frac{1}{1+2^{-(j+1)}e^{3}}\\ &= \frac{1}{1+2^{j+1}e^{-3}}+2\cdot\frac{2^{j+1}e^{-3}}{1+2^{j+1}e^{-3}} \;=\; \frac{1+2^{j+2}e^{-3}}{1+2^{j+1}e^{-3}}.\end{aligned}$$

39. Since Y is conditionally Poisson(k) given $X=k$, $\mathsf{E}[Y|X=k]=k$. Hence,

$$\mathsf{E}[Y] \;=\; \sum_{k=1}^{\infty}\mathsf{E}[Y|X=k]\mathsf{P}(X=k) \;=\; \sum_{k=1}^{\infty}k\mathsf{P}(X=k) \;=\; \mathsf{E}[X] \;=\; \frac{1}{1-p},$$

since $X\sim$ geometric$_1(p)$. Next

$$\begin{aligned}\mathsf{E}[XY] &= \sum_{k=1}^{\infty}\mathsf{E}[XY|X=k]\mathsf{P}(X=k) \;=\; \sum_{k=1}^{\infty}\mathsf{E}[kY|X=k]\mathsf{P}(X=k),\text{ by substitution,}\\ &= \sum_{k=1}^{\infty}k\mathsf{E}[Y|X=k]\mathsf{P}(X=k) \;=\; \sum_{k=1}^{\infty}k^2\mathsf{P}(X=k) \;=\; \mathsf{E}[X^2]\\ &= \mathsf{var}(X)+(\mathsf{E}[X])^2 \;=\; \frac{p}{(1-p)^2}+\frac{1}{(1-p)^2} \;=\; \frac{1+p}{(1-p)^2}.\end{aligned}$$

Since $\mathsf{E}[Y^2|X=k]=k+k^2$,

$$\begin{aligned}\mathsf{E}[Y^2] &= \sum_{k=1}^{\infty}\mathsf{E}[Y^2|X=k]\mathsf{P}(X=k) \;=\; \sum_{k=1}^{\infty}(k+k^2)\mathsf{P}(X=k) \;=\; \mathsf{E}[X]+\mathsf{E}[X^2]\\ &= \frac{1}{1-p}+\frac{1+p}{(1-p)^2} \;=\; \frac{2}{(1-p)^2}.\end{aligned}$$

Finally, we can compute

$$\mathsf{var}(Y) \;=\; \mathsf{E}[Y^2]-(\mathsf{E}[Y])^2 \;=\; \frac{2}{(1-p)^2}-\frac{1}{(1-p)^2} \;=\; \frac{1}{(1-p)^2}.$$

40. From the solution of the example, it is immediate that $\mathsf{E}[Y|X=1]=\lambda$ and $\mathsf{E}[Y|X=0]=\lambda/2$. Next,

$$\mathsf{E}[Y] \;=\; \mathsf{E}[Y|X=0](1-p)+\mathsf{E}[Y|X=1]p \;=\; (1-p)\lambda/2+p\lambda.$$

Similarly,

$$\begin{aligned}\mathsf{E}[Y^2] &= \mathsf{E}[Y^2|X=0](1-p)+\mathsf{E}[Y^2|X=1]p\\ &= (\lambda/2+\lambda^2/4)(1-p)+(\lambda+\lambda^2)p.\end{aligned}$$

To conclude, we have

$$\mathsf{var}(Y)=\mathsf{E}[Y^2]-(\mathsf{E}[Y])^2=(\lambda/2+\lambda^2/4)(1-p)+(\lambda+\lambda^2)p-[(1-p)\lambda/2+p\lambda]^2.$$

41. Write

$$\begin{aligned}
\mathsf{E}[(X+1)Y^2] &= \sum_{i=0}^{1} \mathsf{E}[(X+1)Y^2|X=i]\mathsf{P}(X=i) = \sum_{i=0}^{1} \mathsf{E}[(i+1)Y^2|X=i]\mathsf{P}(X=i) \\
&= \sum_{i=0}^{1} (i+1)\mathsf{E}[Y^2|X=i]\mathsf{P}(X=i).
\end{aligned}$$

Now, since given $X=i$, Y is conditionally Poisson$(3(i+1))$,

$$\mathsf{E}[Y^2|X=i] = (\lambda+\lambda^2)\Big|_{\lambda=3(i+1)} = 3(i+1)+9(i+1)^2.$$

It now follows that

$$\begin{aligned}
\mathsf{E}[(X+1)Y^2] &= \sum_{i=0}^{1} (i+1)[3(i+1)+9(i+1)^2]\mathsf{P}(X=i) \\
&= \sum_{i=0}^{1} (i+1)^2[3+9(i+1)]\mathsf{P}(X=i) \\
&= 12(1/3)+84(2/3) = 4+56 = 60.
\end{aligned}$$

42. Write

$$\begin{aligned}
\mathsf{E}[XY] &= \sum_{n=0}^{\infty} \mathsf{E}[XY|X=n]\mathsf{P}(X=n) = \sum_{n=0}^{\infty} \mathsf{E}[nY|X=n]\mathsf{P}(X=n) \\
&= \sum_{n=0}^{\infty} n\mathsf{E}[Y|X=n]\mathsf{P}(X=n) = \sum_{n=0}^{\infty} n\frac{1}{n+1}\frac{\lambda^n e^{-\lambda}}{n!} \\
&= \mathsf{E}\left[\frac{X}{X+1}\right] = \mathsf{E}\left[\frac{X+1-1}{X+1}\right] = 1-\mathsf{E}\left[\frac{1}{X+1}\right].
\end{aligned}$$

By a problem in the previous chapter, this last expectation is equal to $(1-e^{-\lambda})/\lambda$. Hence,

$$\mathsf{E}[XY] = 1-\frac{1-e^{-\lambda}}{\lambda}.$$

43. Write

$$\begin{aligned}
\mathsf{E}[XY] &= \sum_{n=1}^{\infty} \mathsf{E}[XY|X=n]\mathsf{P}(X=n) = \sum_{n=1}^{\infty} \mathsf{E}[nY|X=n]\mathsf{P}(X=n) \\
&= \sum_{n=1}^{\infty} n\mathsf{E}[Y|X=n]\mathsf{P}(X=n) = \sum_{n=1}^{\infty} n\frac{n}{1-q}\mathsf{P}(X=n) \\
&= \frac{1}{1-q}\mathsf{E}[X^2] = \frac{1}{1-q}\Big[\mathsf{var}(X)+(\mathsf{E}[X])^2\Big] \\
&= \frac{1}{1-q}\left[\frac{p}{(1-p)^2}+\frac{1}{(1-p)^2}\right] = \frac{1+p}{(1-q)(1-p)^2}.
\end{aligned}$$

44. Write

$$
\begin{aligned}
\mathsf{E}[X^2] &= \sum_{k=1}^{\infty} \mathsf{E}[X^2|Y=k]\mathsf{P}(Y=k) = \sum_{n=1}^{\infty}(k+k^2)\mathsf{P}(Y=k) = \mathsf{E}[Y+Y^2] \\
&= \mathsf{E}[Y]+\mathsf{E}[Y^2] = m+(r+m^2) = m+m^2+r.
\end{aligned}
$$

45. Using probability generating functions, we see that

$$
\begin{aligned}
G_V(z) &= \mathsf{E}[z^{X+Y}] = \mathsf{E}[z^X z^Y] = \mathsf{E}[z^X]\mathsf{E}[z^Y] \\
&= [(1-p)+pz]^n[(1-p)+pz]^m = [(1-p)+pz]^{n+m}.
\end{aligned}
$$

Thus, $V \sim \text{binomial}(n+m,p)$. We next compute

$$
\begin{aligned}
\mathsf{P}(V=10|X=4) &= \mathsf{P}(X+Y=10|X=4) = \mathsf{P}(4+Y=10|X=4) \\
&= \mathsf{P}(Y=6|X=4) = \mathsf{P}(Y=6) = \binom{m}{6}p^6(1-p)^{m-6}.
\end{aligned}
$$

46. Write

$$
\begin{aligned}
G_Y(z) &= \mathsf{E}[z^Y] = \sum_{k=1}^{\infty} \mathsf{E}[z^Y|X=k]\mathsf{P}(X=k) = \sum_{k=1}^{\infty} e^{k(z-1)}\mathsf{P}(X=k) \\
&= \sum_{k=1}^{\infty}(e^{z-1})^k\mathsf{P}(X=k) = G_X(e^{z-1}) = \frac{(1-p)e^{z-1}}{1-pe^{z-1}}.
\end{aligned}
$$

CHAPTER 4
Problem Solutions

1. Let V_i denote the input voltage at the ith sampling time. The problem tells us that the V_i are independent and uniformly distributed on $[0,7]$. The alarm sounds if $V_i > 5$ for $i = 1,2,3$. The probability of this is

$$\mathsf{P}\bigg(\bigcap_{i=1}^{3}\{V_i > 5\}\bigg) \;=\; \prod_{i=1}^{3}\mathsf{P}(V_i > 5).$$

Now, $\mathsf{P}(V_i > 5) = \int_5^7 (1/7)\,dt = 2/7$. Hence, the desired probability is $(2/7)^3 = 8/343 = 0.0233$.

2. We must solve $\int_t^\infty f(x)\,dx = 1/2$ for t. Now,

$$\int_t^\infty 2x^{-3}\,dx \;=\; -\frac{1}{x^2}\bigg|_t^\infty \;=\; \frac{1}{t^2}.$$

Solving $1/t^2 = 1/2$, we find that $t = \sqrt{2}$.

3. To find c, we solve $\int_0^1 cx^{-1/2}\,dx = 1$. The left-hand side of this equation is $2cx^{1/2}|_0^1 = 2c$. Solving $2c = 1$ yields $c = 1/2$. For the median, we must solve $\int_t^1 (1/2)x^{-1/2}\,dx = 1/2$ or $x^{1/2}|_t^1 = 1/2$. We find that $t = 1/4$.

4. (a) For $t \geq 0$, $\mathsf{P}(X > t) = \int_t^\infty \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}|_t^\infty = e^{-\lambda t}$.

   (b) First, $\mathsf{P}(X > t+\Delta t|X > t) = \mathsf{P}(X > t+\Delta t, X > t)/\mathsf{P}(X > t)$. Next, observe that

$$\{X > t+\Delta t\} \cap \{X > t\} \;=\; \{X > t+\Delta t\},$$

   and so $\mathsf{P}(X > t+\Delta t|X > t) = \mathsf{P}(X > t+\Delta t)/\mathsf{P}(X > t) = e^{-\lambda(t+\Delta t)}/e^{-\lambda t} = e^{-\lambda\Delta t}$.

5. Let X_i denote the voltage output by regulator i. Then the X_i are i.i.d. $\exp(\lambda)$ random variables. Now put

$$Y \;:=\; \sum_{i=1}^{10} I_{(v,\infty)}(X_i)$$

so that Y counts the number of regulators that output more than v volts. We must compute $\mathsf{P}(Y = 3)$. Now, the $I_{(v,\infty)}(X_i)$ are i.i.d. Bernoulli(p) random variables, where

$$p \;=\; \mathsf{P}(X_i > v) \;=\; \int_v^\infty \lambda e^{-\lambda x}\,dx \;=\; -e^{-\lambda x}\bigg|_v^\infty \;=\; e^{-\lambda v}.$$

Next, we now from the previous chapter that a sum of n i.i.d. Bernoulli(p) random variables is a binomial(n,p). Thus,

$$\mathsf{P}(Y = 3) = \binom{n}{3}p^3(1-p)^{n-3} = \binom{10}{3}e^{-3\lambda v}(1-e^{-\lambda v})^7 = 120e^{-3\lambda v}(1-e^{-\lambda v})^7.$$

6. First note that $\mathsf{P}(X_i > 2) = \int_2^\infty \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}|_2^\infty = e^{-2\lambda}$.

   (a) $\mathsf{P}\big(\min(X_1,\ldots,X_n) > 2\big) = \mathsf{P}\big(\bigcap_{i=1}^n \{X_i > 2\}\big) = \prod_{i=1}^n \mathsf{P}(X_i > 2) = e^{-2n\lambda}$.

   (b) Write

$$\begin{aligned}\mathsf{P}\big(\max(X_1,\ldots,X_n) > 2\big) &= 1 - \mathsf{P}\big(\max(X_1,\ldots,X_n) \le 2\big)\\ &= 1 - \mathsf{P}\bigg(\bigcap_{i=1}^n \{X_i \le 2\}\bigg)\\ &= 1 - \prod_{i=1}^n \mathsf{P}(X_i \le 2) \;=\; 1 - [1 - e^{-2\lambda}]^n.\end{aligned}$$

7. (a) $\mathsf{P}(Y \le 2) = \int_0^2 \mu e^{-\mu y}\,dy = 1 - e^{-2\mu}$.

   (b) $\mathsf{P}(X \le 12, Y \le 12) = \mathsf{P}(X \le 12)\mathsf{P}(Y \le 12) = (1 - e^{-12\lambda})(1 - e^{-12\mu})$.

   (c) Write

$$\begin{aligned}\mathsf{P}(\{X \le 12\} \cup \{Y \le 12\}) &= 1 - \mathsf{P}(X > 12, Y > 12)\\ &= 1 - \mathsf{P}(X > 12)\mathsf{P}(Y > 12)\\ &= 1 - e^{-12\lambda}e^{-12\mu} \;=\; 1 - e^{-12(\lambda+\mu)}.\end{aligned}$$

8. (a) Make the change of variable $y = \lambda x^p$, $dy = \lambda p x^{p-1}\,dx$ to get

$$\int_0^\infty \lambda p x^{p-1} e^{-\lambda x^p}\,dx \;=\; \int_0^\infty e^{-y}\,dy \;=\; -e^{-y}\bigg|_0^\infty \;=\; 1.$$

   (b) The same change of variables also yields

$$\mathsf{P}(X > t) \;=\; \int_t^\infty \lambda p x^{p-1} e^{-\lambda x^p}\,dx \;=\; \int_{\lambda t^p}^\infty e^{-y}\,dy \;=\; e^{-\lambda t^p}.$$

   (c) The probability that none of the X_i exceeds 3 is

$$\mathsf{P}\bigg(\bigcap_{i=1}^n \{X_i \le 3\}\bigg) \;=\; \prod_{i=1}^n \mathsf{P}(X_i \le 3) \;=\; [1 - \mathsf{P}(X_1 > 3)]^n \;=\; [1 - e^{-\lambda 3^p}]^n.$$

   The probability that at least one of them exceeds 3 is

$$\mathsf{P}\bigg(\bigcup_{i=1}^n \{X_i > 3\}\bigg) \;=\; 1 - \mathsf{P}\bigg(\bigcap_{i=1}^n \{X_i \le 3\}\bigg) \;=\; 1 - [1 - e^{-\lambda 3^p}]^n.$$

9. (a) Since $f'(x) = -xe^{-x^2/2}/\sqrt{2\pi}$, $f'(x) < 0$ for $x > 0$ and $f'(x) > 0$ for $x < 0$.

   (b) Since $f''(x) = (x^2 - 1)e^{-x^2/2}/\sqrt{2\pi}$, we see that $f''(x) > 0$ for $|x| > 1$ and $f''(x) < 0$ for $|x| < 1$.

   (c) Rearrange $e^{x^2/2} \ge x^2/2$ to get $e^{-x^2/2} \le 2/x^2 \to 0$ as $|x| \to \infty$.

10. Following the hint, write $f(x) = \varphi((x-m)/\sigma)/\sigma$, where φ is the standard normal density. Observe that $f'(x) = \varphi'((x-m)/\sigma)/\sigma^2$ and $f''(x) = \varphi''((x-m)/\sigma)/\sigma^3$.

   (a) Since the argument of φ' is positive for $x > m$ and negative for $x < m$, $f(x)$ is decreasing for $x > m$ and increasing for $x < m$. Hence, f has a global maximum at $x = m$.

   (b) Since the absolute value of the argument of φ'' is greater than one if and only if $|x-m| > \sigma$, $f(x)$ is concave for $|x-m| < \sigma$ and convex for $|x-m| > \sigma$.

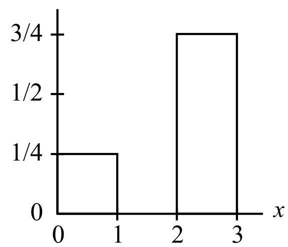
11. Since φ is bounded, $\lim_{\sigma\to\infty} \varphi((x-m)/\sigma)/\sigma = 0$. Hence, $\lim_{\sigma\to\infty} f(x) = 0$. For $x \neq m$, we have

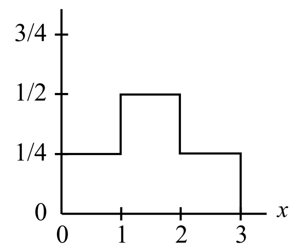
$$f(x) \;=\; \frac{\exp\Bigl[-\bigl(\frac{x-m}{\sigma}\bigr)^2\big/2\Bigr]}{\sqrt{2\pi}\,\sigma} \;\leq\; \frac{2}{\sqrt{2\pi}\,\sigma\bigl(\frac{x-m}{\sigma}\bigr)^2} \;=\; \frac{2\sigma}{\sqrt{2\pi}(x-m)^2} \;\to\; 0$$

as $\sigma \to 0$. Otherwise, since $f(m) = [\sqrt{2\pi}\,\sigma]^{-1}$, $\lim_{\sigma\to 0} f(m) = \infty$.

12. (a) $f(x) = \sum_n p_n f_n(x)$ is obviously nonnegative. Also,

$$\int_{-\infty}^{\infty} f(x)\,dx \;=\; \int_{-\infty}^{\infty} \sum_n p_n f_n(x)\,dx \;=\; \sum_n p_n \int_{-\infty}^{\infty} f_n(x)\,dx \;=\; \sum_n p_n \cdot 1 \;=\; \sum_n p_n \;=\; 1.$$

(b) 

(c) 

13. Clearly, $(g*h)(x) = \int_{-\infty}^{\infty} g(y)h(x-y)\,dy \geq 0$ since g and h are nonnegative. Next,

$$\begin{aligned}
\int_{-\infty}^{\infty} (g*h)(x)\,dx &= \int_{-\infty}^{\infty}\biggl(\int_{-\infty}^{\infty} g(y)h(x-y)\,dy\biggr)dx \\
&= \int_{-\infty}^{\infty} g(y)\biggl(\int_{-\infty}^{\infty} h(x-y)\,dx\biggr)dy \\
&= \int_{-\infty}^{\infty} g(y)\underbrace{\biggl(\int_{-\infty}^{\infty} h(\theta)\,d\theta\biggr)}_{=1}dy \;=\; \int_{-\infty}^{\infty} g(y)\,dy \;=\; 1.
\end{aligned}$$

14. (a) Let $p > 1$. On $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx$, use integration by parts with $u = x^{p-1}$ and $dv = e^{-x}\,dx$. Then $du = (p-1)x^{p-2}dx$, $v = -e^{-x}$, and

$$\Gamma(p) \;=\; \underbrace{-x^{p-1}e^{-x}\Big|_0^\infty}_{=0} + (p-1)\int_0^\infty x^{(p-1)-1}e^{-x}\,dx \;=\; (p-1)\Gamma(p-1).$$

(b) On $\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x}\,dx$, make the change of variable $x = y^2/2$ or $y = \sqrt{2x}$. Then $dx = y\,dy$ and $x^{-1/2} = \sqrt{2}/y$. Hence,

$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty \frac{\sqrt{2}}{y} e^{-y^2/2} y\,dy \;=\; \sqrt{2}\int_0^\infty e^{-y^2/2}\,dy \;=\; \sqrt{2}\sqrt{2\pi}\int_0^\infty \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy \\ &= \sqrt{2}\sqrt{2\pi}\cdot\frac{1}{2} \;=\; \sqrt{\pi}.\end{aligned}$$

(c) By repeatedly using the recursion formula in part (a), we have

$$\begin{aligned}\Gamma\biggl(\frac{2n+1}{2}\biggr) &= \frac{2n-1}{2}\Gamma\biggl(\frac{2n-1}{2}\biggr) \;=\; \frac{2n-1}{2}\cdot\frac{2n-3}{2}\Gamma\biggl(\frac{2n-3}{2}\biggr) \\ &\;\;\vdots \\ &= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2) \\ &= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} \\ &= \frac{(2n-1)!!}{2^n}\sqrt{\pi}.\end{aligned}$$

(d) First note that $g_p(y) = 0$ for $y \le 0$, and similarly for $g_q(y)$. Hence, in order to have $g_p(y)g_q(x-y) > 0$, we need $y > 0$ and $x - y > 0$, or equivalently, $x > y > 0$. Of course, if $x \le 0$ this does not happen. Thus, $(g_p * g_q)(x) = 0$ for $x \le 0$. For $x > 0$, we follow the hint and write

$$\begin{aligned}(g_p * g_q)(x) &= \int_{-\infty}^\infty g_p(y)g_q(x-y)\,dy \\ &= \int_0^x g_p(y)g_q(x-y)\,dy \\ &= \frac{1}{\Gamma(p)\Gamma(q)}\int_0^x y^{p-1}e^{-y}\cdot(x-y)^{q-1}e^{-(x-y)}\,dy \\ &= \frac{x^{q-1}e^{-x}}{\Gamma(p)\Gamma(q)}\int_0^x y^{p-1}(1-y/x)^{q-1}\,dy \\ &= \frac{x^q e^{-x}}{\Gamma(p)\Gamma(q)}\int_0^1 (x\theta)^{p-1}(1-\theta)^{q-1}\,d\theta, \quad \text{ch. of var. } \theta = y/x, \\ &= \frac{x^{p+q-1}e^{-x}}{\Gamma(p)\Gamma(q)}\int_0^1 \theta^{p-1}(1-\theta)^{q-1}\,d\theta. \qquad (*)\end{aligned}$$

Now, the left-hand side is a convolution of densities, and is therefore a density by Problem 13. In particular, this means that the left-hand side integrates to one. On the right-hand side, note that $\int_0^\infty x^{p+q-1}e^{-x}\,dx = \Gamma(p+q)$. Hence, integrating the above equation with respect to x from zero to infinity yields

$$1 \;=\; \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_0^1 \theta^{p-1}(1-\theta)^{q-1}\,d\theta.$$

Solving for the above integral and substituting the result into $(*)$, we find that $(g_p * g_q)(x) = g_{p+q}(x)$.

15. (a) In $\int_{-\infty}^{\infty} f_\lambda(x)\,dx = \int_{-\infty}^{\infty} \lambda f(\lambda x)\,dx$, make the change of variable $y = \lambda x$, $dy = \lambda\,dx$ to get

$$\int_{-\infty}^{\infty} f_\lambda(x)\,dx \;=\; \int_{-\infty}^{\infty} f(y)\,dy \;=\; 1.$$

(b) Observe that

$$g_{1,\lambda}(x) \;=\; \lambda\,\frac{(\lambda x)^0\,e^{-\lambda x}}{0!} \;=\; \lambda e^{-\lambda x},$$

which we recognize as the $\exp(\lambda)$ density.

(c) The desired probability is

$$P_m(t) \;:=\; \int_t^{\infty} \lambda\,\frac{(\lambda x)^{m-1}\,e^{-\lambda x}}{(m-1)!}\,dx.$$

Note that $P_1(t) = \int_t^\infty \lambda e^{-\lambda x}\,dx = e^{-\lambda t}$. For $m > 1$, apply integration by parts with $u = (\lambda x)^{m-1}/(m-1)!$ and $dv = \lambda e^{-\lambda x}\,dx$. Then

$$P_m(t) \;=\; \frac{(\lambda t)^{m-1}\,e^{-\lambda t}}{(m-1)!} + P_{m-1}(t).$$

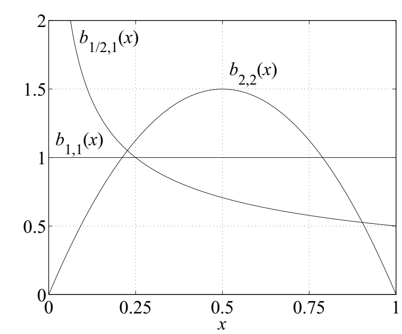
Applying this result recursively, we find that

$$P_m(t) \;=\; \frac{(\lambda t)^{m-1}\,e^{-\lambda t}}{(m-1)!} + \frac{(\lambda t)^{m-2}\,e^{-\lambda t}}{(m-2)!} + \cdots + e^{-\lambda t}.$$

(d) We have

$$\begin{aligned} g_{\frac{2m+1}{2},\frac{1}{2}}(x) \;&=\; \frac{1}{2}\,\frac{(\frac{1}{2}x)^{m-1/2}\,e^{-x/2}}{\Gamma((2m+1)/2)} \;=\; \frac{(1/2)^m(1/2)^{1/2}x^{m-1/2}e^{-x/2}}{\frac{(2m-1)!!}{2^m}\sqrt{\pi}} \\ &=\; \frac{x^{m-1/2}e^{-x/2}}{(2m-1)\cdots 5\cdot 3\cdot 1\cdot\sqrt{2\pi}}. \end{aligned}$$

16. (a) We see that $b_{1,1}(x) = 1$ is the uniform$(0,1)$ density, $b_{2,2}(x) = 6x(1-x)$, and $b_{1/2,1}(x) = 1/(2\sqrt{x})$.



(b) From Problem 14(d) and its hint, we have

$$g_{p+q}(x) \;=\; (g_p * g_q)(x) \;=\; \frac{x^{p+q-1}e^{-x}}{\Gamma(p)\Gamma(q)} \int_0^1 \theta^{p-1}(1-\theta)^{q-1}d\theta.$$

Integrating the left and right-hand sides with respect to x from zero to infinity yields

$$1 \;=\; \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 \theta^{p-1}(1-\theta)^{q-1}d\theta,$$

which says that the beta density integrates to one.

17. Starting with

$$\Gamma(p)\,\Gamma(q) \;=\; \Gamma(p+q)\int_0^1 u^{p-1}(1-u)^{q-1}\,du,$$

make the change of variable $u = \sin^2\theta$, $du = 2\sin\theta\cos\theta\,d\theta$. We obtain

$$\begin{aligned}
\Gamma(p)\,\Gamma(q) &= \Gamma(p+q)\int_0^1 u^{p-1}(1-u)^{q-1}\,du \\
&= \Gamma(p+q)\int_0^{\pi/2} (\sin^2\theta)^{p-1}(1-\sin^2\theta)^{q-1}\cdot 2\sin\theta\cos\theta\,d\theta \\
&= 2\Gamma(p+q)\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta.
\end{aligned}$$

Setting $p = q = 1/2$ on both sides yields

$$\Gamma(1/2)^2 \;=\; 2\int_0^{\pi/2} 1\,d\theta \;=\; \pi,$$

and it follows that $\Gamma(1/2) = \sqrt{\pi}$.

18. Starting with

$$\Gamma(p)\,\Gamma(q) \;=\; \Gamma(p+q)\int_0^1 u^{p-1}(1-u)^{q-1}\,du,$$

make the change of variable $u = \sin^2\theta$, $du = 2\sin\theta\cos\theta\,d\theta$. We obtain

$$\begin{aligned}
\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} &= \int_0^1 u^{p-1}(1-u)^{q-1}\,du \\
&= \int_0^{\pi/2} (\sin^2\theta)^{p-1}(1-\sin^2\theta)^{q-1}\cdot 2\sin\theta\cos\theta\,d\theta \\
&= 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta.
\end{aligned}$$

Setting $p = (n+1)/2$ and $q = 1/2$ on both sides yields

$$\frac{\Gamma\!\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\!\left(\frac{n+2}{2}\right)} \;=\; 2\int_0^{\pi/2} \sin^n\theta\,d\theta,$$

and the desired result follows.

19. Starting with the integral definition of $B(p,q)$, make the change of variable $u = 1 - e^{-\theta}$, which implies both $du = e^{-\theta}\,d\theta$ and $1-u = e^{-\theta}$. Hence,

$$\begin{aligned} B(p,q) &= \int_0^1 u^{p-1}(1-u)^{q-1}\,du = \int_0^\infty (1-e^{-\theta})^{p-1}(e^{-\theta})^{q-1}e^{-\theta}\,d\theta \\ &= \int_0^\infty (1-e^{-\theta})^{p-1}e^{-q\theta}\,d\theta. \end{aligned}$$

20. We first use the fact that the density is even and *then* make the change of variable $e^\theta = 1 + x^2/\nu$, which implies both $e^\theta\,d\theta = 2x/\nu\,dx$ and $x = \sqrt{\nu(e^\theta - 1)}$. Thus,

$$\begin{aligned} \int_{-\infty}^\infty \Big(1+\frac{x^2}{\nu}\Big)^{-(\nu+1)/2}dx &= 2\int_0^\infty \Big(1+\frac{x^2}{\nu}\Big)^{-(\nu+1)/2}dx \\ &= 2\int_0^\infty (e^\theta)^{-(\nu+1)/2}\cdot\tfrac{\nu}{2}e^\theta\,\frac{1}{\sqrt{\nu(e^\theta-1)}}\,d\theta \\ &= \sqrt{\nu}\int_0^\infty (e^\theta)^{-\nu/2}(e^\theta)^{1/2}/\sqrt{e^\theta-1}\,d\theta \\ &= \sqrt{\nu}\int_0^\infty (e^\theta)^{-\nu/2}(1-e^{-\theta})^{-1/2}\,d\theta \\ &= \sqrt{\nu}\int_0^\infty (1-e^{-\theta})^{1/2-1}e^{-\theta\nu/2}\,d\theta. \end{aligned}$$

By the preceding problem, this is equal to $\sqrt{\nu}\,B(1/2,\nu/2)$, and we see that Student's t density integrates to one.

21. (a) Using Stirling's formula,

$$\begin{aligned} \frac{\Gamma\Big(\dfrac{1+\nu}{2}\Big)}{\sqrt{\nu}\,\Gamma\Big(\dfrac{\nu}{2}\Big)} &\approx \frac{\sqrt{2\pi}\Big(\dfrac{1+\nu}{2}\Big)^{(1+\nu)/2-1/2}e^{-(1+\nu)/2}}{\sqrt{\nu}\,\sqrt{2\pi}\Big(\dfrac{\nu}{2}\Big)^{\nu/2-1/2}e^{-\nu/2}} = \frac{\Big(\dfrac{1+\nu}{2}\Big)^{\nu/2}e^{-1/2}}{\sqrt{\nu}\Big(\dfrac{\nu}{2}\Big)^{\nu/2-1/2}} \\ &= \Big(\frac{1+\nu}{\nu}\Big)^{\nu/2}\frac{(\nu/2)^{1/2}}{\sqrt{\nu}}e^{-1/2} = [(1+1/\nu)^\nu]^{1/2}\frac{1}{\sqrt{2}\,e^{1/2}} \\ &\to (e^1)^{1/2}\frac{1}{\sqrt{2}\,e^{1/2}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

(b) First write

$$\Big(1+\frac{x^2}{\nu}\Big)^{(\nu+1)/2} = \Big[\Big(1+\frac{x^2}{\nu}\Big)^\nu\Big]^{1/2}\Big(1+\frac{x^2}{\nu}\Big)^{1/2} \to [e^{x^2}]^{1/2}1^{1/2} = e^{x^2/2}.$$

It then follows that

$$\begin{aligned} f_\nu(x) &= \frac{\Big(1+\frac{x^2}{\nu}\Big)^{-(\nu+1)/2}}{\sqrt{\nu}B(\frac{1}{2},\frac{\nu}{2})} = \frac{\Gamma\Big(\dfrac{1+\nu}{2}\Big)}{\sqrt{\nu}\,\Gamma\Big(\dfrac{\nu}{2}\Big)}\cdot\frac{1}{\sqrt{\pi}\Big(1+\frac{x^2}{\nu}\Big)^{(\nu+1)/2}} \\ &\to \frac{1}{\sqrt{2\pi}\,e^{x^2/2}} = \frac{e^{-x^2/2}}{\sqrt{2\pi}}. \end{aligned}$$

22. Making the change of variable $t = 1/(1+z)$ as suggested in the hint, note that it is equivalent to $1+z = t^{-1}$, which implies $dz = -t^{-2}dt$. Thus,

$$\int_0^\infty \frac{z^{p-1}}{(1+z)^{p+q}}\,dz = \int_0^1 \left(\frac{1}{t}-1\right)^{p-1} t^{p+q}\frac{dt}{t^2} = \int_0^1 \left(\frac{1-t}{t}\right)^{p-1} t^{p+q-2}\,dt$$
$$= \int_0^1 (1-t)^{p-1}t^{q-1}\,dt = B(q,p) = B(p,q).$$

Hence, $f_Z(z)$ integrates to one.

23. $\mathsf{E}[X] = \displaystyle\int_1^\infty x\cdot\frac{2}{x^3}\,dx = \int_1^\infty 2x^{-2}\,dx = \left.\frac{-2}{x}\right|_1^\infty = 2.$

24. If the input-output relation has n levels, then the distance from $-V_{\max}$ to $+V_{\max}$ should be $n\Delta$; i.e., $n\Delta = 2V_{\max}$, or $\Delta = 2V_{\max}/n$. Next, we have from the example in the text that the performance is $\Delta^2/12$, and we need $\Delta^2/12 < \varepsilon$, or

$$\frac{1}{12}\left(\frac{2V_{\max}}{n}\right)^2 < \varepsilon.$$

Solving this for n yields $V_{\max}/\sqrt{3\varepsilon} < n = 2^b$. Taking natural logarithms, we have

$$b > \ln\bigl(V_{\max}/\sqrt{3\varepsilon}\bigr)\Big/\ln 2.$$

25. We use the change of variable $x = z-m$ as follows:

$$\mathsf{E}[Z] = \int_{-\infty}^\infty z f_Z(z)\,dz = \int_{-\infty}^\infty z f(z-m)\,dz = \int_{-\infty}^\infty (x+m)f(x)\,dx$$
$$= \int_{-\infty}^\infty x f(x)\,dx + m\int_{-\infty}^\infty f(x)\,dx = \mathsf{E}[X]+m = 0+m = m.$$

26. $\mathsf{E}[X^2] = \displaystyle\int_1^\infty x^2\cdot\frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x}\,dx = 2\ln x\Big|_1^\infty = 2(\infty-0) = \infty.$

27. First note that since Student's t density is even, $\mathsf{E}[|X|^k] = \int_{-\infty}^\infty |x|^k f_\nu(x)\,dx$ is proportional to

$$\int_0^\infty \frac{x^k}{(1+x^2/\nu)^{(\nu+1)/2}}\,dx = \int_0^1 \frac{x^k}{(1+x^2/\nu)^{(\nu+1)/2}}\,dx + \int_1^\infty \frac{x^k}{(1+x^2/\nu)^{(\nu+1)/2}}\,dx$$

With regard to this last integral, observe that

$$\int_1^\infty \frac{x^k}{(1+x^2/\nu)^{(\nu+1)/2}}\,dx \le \int_1^\infty \frac{x^k}{(x^2/\nu)^{(\nu+1)/2}}\,dx = \nu^{(\nu+1)/2}\int_1^\infty \frac{dx}{x^{\nu+1-k}},$$

which is finite if $\nu+1-k > 1$, or $k < \nu$. Next, instead of breaking the range of integration at one, we break it at the solution of $x^2/\nu = 1$, or $x = \sqrt{\nu}$. Then

$$\int_{\sqrt{\nu}}^\infty \frac{x^k}{(1+x^2/\nu)^{(\nu+1)/2}}\,dx \ge \int_{\sqrt{\nu}}^\infty \frac{x^k}{(x^2/\nu+x^2/\nu)^{(\nu+1)/2}}\,dx = \left(\tfrac{\nu}{2}\right)^{(\nu+1)/2}\int_{\sqrt{\nu}}^\infty \frac{dx}{x^{\nu+1-k}},$$

which is infinite if $\nu+1-k \le 1$, or $k \ge \nu$.

28. Begin with $\mathsf{E}[Y^4] = \mathsf{E}[(Z+n)^4] = \mathsf{E}[Z^4 + 4Z^3n + 6Z^2n^2 + 4Zn^3 + n^4]$. The moments of the standard normal were computed in an example in this chapter. Hence $\mathsf{E}[Y^4] = 3 + 4\cdot 0\cdot n + 6\cdot 1\cdot n^2 + 4\cdot 0\cdot n^3 + n^4 = 3 + 6n^2 + n^4$.

29. $\mathsf{E}[X^n] \;=\; \displaystyle\int_0^\infty x^n \frac{x^{p-1}e^{-x}}{\Gamma(p)}\,dx \;=\; \frac{1}{\Gamma(p)}\int_0^\infty x^{(n+p)-1}e^{-x}\,dx \;=\; \frac{\Gamma(n+p)}{\Gamma(p)}.$

30. (a) First write

$$\mathsf{E}[X] \;=\; \int_0^\infty x\cdot xe^{-x^2/2}\,dx \;=\; \frac{1}{2}\int_{-\infty}^\infty x^2e^{-x^2/2}\,dx \;=\; \frac{\sqrt{2\pi}}{2}\int_{-\infty}^\infty x^2\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx,$$

where the last integral is the second moment of a standard normal density, which is one. Hence, $\mathsf{E}[X] = \dfrac{\sqrt{2\pi}}{2} = \sqrt{\pi/2}$.

(b) For higher-order moments, first write

$$\mathsf{E}[X^n] \;=\; \int_0^\infty x^n\cdot xe^{-x^2/2}\,dx \;=\; \int_0^\infty x^{n+1}e^{-x^2/2}\,dx.$$

Now make the change of variable $t = x^2/2$, which implies $x = \sqrt{2t}$, or $dx = dt/\sqrt{2t}$. Hence,

$$\begin{aligned}\mathsf{E}[X^n] \;&=\; \int_0^\infty [(2t)^{1/2}]^{n+1}e^{-t}\,\frac{dt}{2^{1/2}t^{1/2}}\\ &=\; 2^{n/2}\int_0^\infty t^{[(n/2)+1]-1}e^{-t}\,dt \;=\; 2^{n/2}\Gamma(1+n/2).\end{aligned}$$

31. Let X_i denote the flow on link i, and put $Y_i := I_{(\beta,\infty)}(X_i)$ so that $Y_i = 1$ if the flow on link i is greater than β. Put $Z := \sum_{i=1}^n Y_i$ so that Z counts the number of links with flows greater than β. The buffer overflows if $Z > 2$. Since the X_i are i.i.d., so are the Y_i. Furthermore, the Y_i are Bernoulli(p), where $p = \mathsf{P}(X_i > \beta)$. Hence, $Z \sim \text{binomial}(n,p)$. Thus,

$$\begin{aligned}\mathsf{P}(Z>2) \;&=\; 1-\mathsf{P}(Z\le 2)\\ &=\; 1-\left[\binom{n}{0}(1-p)^n + \binom{n}{1}p(1-p)^{n-1} + \binom{n}{2}p^2(1-p)^{n-2}\right]\\ &=\; 1-(1-p)^{n-2}[(1-p)^2 + np(1-p) + \tfrac{1}{2}n(n-1)p^2].\end{aligned}$$

In remains to compute

$$p \;=\; \mathsf{P}(X_i>\beta) \;=\; \int_\beta^\infty xe^{-x^2/2}\,dx \;=\; -e^{x^2/2}\Big|_\beta^\infty \;=\; e^{-\beta^2/2}.$$

32. The key is to use the change of variable $\theta = \lambda x^p$, which implies both $d\theta = \lambda px^{p-1}\,dx$ and $x = (\theta/\lambda)^{1/p}$. Hence,

$$\begin{aligned}\mathsf{E}[X^n] \;&=\; \int_0^\infty x^n\cdot\lambda px^{p-1}e^{-\lambda x^p}\,dx \;=\; \int_0^\infty [(\theta/\lambda)^{1/p}]^n e^{-\theta}\,d\theta\\ &=\; (1/\lambda)^{n/p}\int_0^\infty \theta^{[(n/p)+1]-1}e^{-\theta}\,d\theta \;=\; \Gamma(1+n/p)/\lambda^{n/p}.\end{aligned}$$

33. Write

$$\begin{aligned}\mathsf{E}[Y] &= \mathsf{E}[(X^{1/4})^2] = \mathsf{E}[X^{1/2}] = \int_0^\infty x^{1/2}e^{-x}dx = \int_0^\infty x^{3/2-1}e^{-x}dx \\ &= \Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2.\end{aligned}$$

34. We have

$$\begin{aligned}\mathsf{P}\biggl(\bigcup_{i=1}^n \{X_i < \mu/2\}\biggr) &= 1 - \mathsf{P}\biggl(\bigcap_{i=1}^n \{X_i \geq \mu/2\}\biggr) \\ &= 1 - \prod_{i=1}^n \mathsf{P}(X_i \geq \mu/2) \\ &= 1 - \biggl(\int_{\mu/2}^\infty \lambda e^{-\lambda x}\,dx\biggr)^n, \quad \text{with } \lambda := 1/\mu, \\ &= 1 - (e^{-\lambda\mu/2})^n = 1 - e^{-n/2}.\end{aligned}$$

35. Let $X_i \sim \exp(\lambda)$ be i.i.d., where $\lambda = 1/20$. We must compute

$$\begin{aligned}\mathsf{P}\biggl(\bigcup_{i=1}^5 \{X_i > 25\}\biggr) &= 1 - \mathsf{P}\biggl(\bigcap_{i=1}^5 \{X_i \leq 25\}\biggr) \\ &= 1 - \prod_{i=1}^5 \mathsf{P}(X_i \leq 25) \\ &= 1 - \biggl(\int_0^{25} \lambda e^{-\lambda x}\,dx\biggr)^5 \\ &= 1 - (1 - e^{-25\lambda})^5 = 1 - (1 - e^{-5/4})^5 = 0.815.\end{aligned}$$

36. The first two calculations are

$$h(X) = \int_0^2 (1/2)\log 2\,dx = \log 2 \quad \text{and} \quad h(X) = \int_0^{1/2} 2\log(1/2)\,dx = \log(1/2).$$

For the third calculation, note that $-\ln f(x) = \frac{1}{2}[(x-m)/\sigma]^2 + \frac{1}{2}\ln 2\pi\sigma^2$. Then

$$\begin{aligned}h(X) &= \frac{1}{2}\int_{-\infty}^\infty f(x)\Bigl([(x-m)/\sigma]^2 + \ln 2\pi\sigma^2\Bigr)dx \\ &= \frac{1}{2}\biggl\{\mathsf{E}\biggl[\biggl(\frac{X-m}{\sigma}\biggr)^2\biggr] + \ln 2\pi\sigma^2\biggr\} = \tfrac{1}{2}\{1 + \ln 2\pi\sigma^2\} = \tfrac{1}{2}\ln 2\pi\sigma^2 e.\end{aligned}$$

37. The main difficulty is to compute

$$\int_{-\infty}^\infty x^{2n}(1 + x^2/\nu)^{-(\nu+1)/2}\,dx.$$

First use the fact that the integrand is even and *then* make the change of variable $e^\theta = 1 + x^2/\nu$, which implies both $e^\theta\,d\theta = 2x/\nu\,dx$ and $x = \sqrt{\nu(e^\theta - 1)}$. Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} x^{2n}(1+x^2/\nu)^{-(\nu+1)/2}\,dx &= \nu \int_0^\infty x^{2n-1}(1+x^2/\nu)^{-(\nu+1)/2}\,\frac{2x}{\nu}\,dx \\
&= \nu \int_0^\infty \big(\sqrt{\nu(e^\theta-1)}\,\big)^{2n-1}(e^\theta)^{-(\nu+1)/2}e^\theta\,d\theta \\
&= \nu^{n+1/2}\int_0^\infty (e^\theta-1)^{n-1/2}e^{-\theta(\nu+1)/2}e^\theta\,d\theta \\
&= \nu^{n+1/2}\int_0^\infty (1-e^{-\theta})^{n-1/2}e^{-\theta(\nu-2n)/2}\,d\theta \\
&= \nu^{n+1/2}\int_0^\infty (1-e^{-\theta})^{(n+1/2)-1}e^{-\theta(\nu-2n)/2}\,d\theta \\
&= \nu^{n+1/2}B\big(n+1/2,(\nu-2n)/2\big), \quad \text{by Problem 19.}
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathsf{E}[X^{2n}] &= \nu^{n+1/2}B\big(n+1/2,(\nu-2n)/2\big)\cdot\frac{1}{\sqrt{\nu}B(\frac{1}{2},\frac{\nu}{2})} \\
&= \nu^{n+1/2}\frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{\nu-2n}{2})}{\Gamma(\frac{\nu+1}{2})}\cdot\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu}\,\Gamma(\frac{1}{2})\Gamma(\frac{\nu}{2})} = \nu^n\frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{\nu-2n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{\nu}{2})}.
\end{aligned}$$

38. From $M_X(s) = e^{\sigma^2 s^2/2}$, we have $M_X'(s) = M_X(s)\sigma^2 s$ and then

$$M_X''(s) = M_X(s)\sigma^4 s^2 + M_X(s)\sigma^2.$$

Since $M_X(0) = 1$, we have $M_X''(1) = \sigma^2$.

39. Let $M(s) := e^{s^2/2}$ denote the moment generating function of the standard normal random variable. For the $N(m,\sigma^2)$ moment generating function, we use the change of variable $y = (x-m)/\sigma$, $dy = dx/\sigma$ to write

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{sx}\frac{\exp[-\frac{1}{2}(\frac{x-m}{\sigma})^2]}{\sqrt{2\pi}\,\sigma}\,dx &= \int_{-\infty}^{\infty} e^{s(\sigma y+m)}\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy = e^{sm}\int_{-\infty}^{\infty} e^{s\sigma y}\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy \\
&= e^{sm}M(s\sigma) = e^{sm+\sigma^2 s^2/2}.
\end{aligned}$$

40. $\mathsf{E}[e^{sY}] = \mathsf{E}[e^{s\ln(1/X)}] = \mathsf{E}[e^{\ln X^{-s}}] = \mathsf{E}[X^{-s}] = \displaystyle\int_0^1 x^{-s}\,dx = \frac{x^{1-s}}{1-s}\bigg|_0^1 = \frac{1}{1-s}.$

41. First note that

$$|x| = \begin{cases} x, & x \ge 0, \\ -x, & x < 0. \end{cases}$$

Then the Laplace(λ) mgf is

$$\mathsf{E}[e^{sX}] = \int_{-\infty}^{\infty} e^{sx}\cdot\tfrac{\lambda}{2}e^{-\lambda|x|}\,dx$$

$$\begin{aligned} &= \frac{\lambda}{2}\left[\int_0^\infty e^{sx}e^{-\lambda x}\,dx + \int_{-\infty}^0 e^{sx}e^{\lambda x}\,dx\right] \\ &= \frac{\lambda}{2}\left[\int_0^\infty e^{-x(\lambda-s)}\,dx + \int_{-\infty}^0 e^{x(\lambda+s)}\,dx\right]. \end{aligned}$$

Of these last two integrals, the one on the left is finite if $\lambda > \operatorname{Re} s$, while the second is finite if $\operatorname{Re} s > -\lambda$. For both of them to be finite, we need $-\lambda < \operatorname{Re} s < \lambda$. For such s both integrals are easy to evaluate. We get

$$M_X(s) := \mathsf{E}[e^{sX}] = \frac{\lambda}{2}\left[\frac{1}{\lambda-s}+\frac{1}{\lambda+s}\right] = \frac{\lambda}{2}\cdot\frac{2\lambda}{\lambda^2-s^2} = \frac{\lambda^2}{\lambda^2-s^2}.$$

Now, $M_X'(s) = 2s\lambda^2/(\lambda^2-s^2)^2$, and so the mean is $M_X'(0) = 0$. We continue with

$$M_X''(s) = 2\lambda^2\frac{(\lambda^2-s^2)^2+4s^2(\lambda^2-s^2)}{(\lambda^2-s^2)^4}.$$

Hence, the second moment is $M_X''(0) = 2/\lambda^2$. Since the mean is zero, the second moment is also the variance.

42. Since X is a nonnegative random variable, for $s \leq 0$, $sX \leq 0$ and $e^{sX} \leq 1$. Hence, for $s \leq 0$, $M_X(s) = \mathsf{E}[e^{sX}] \leq \mathsf{E}[1] = 1 < \infty$. For $s > 0$, we show that $M_X(s) = \infty$. We use the fact that for $z > 0$,

$$e^z = \sum_{n=0}^\infty \frac{z^n}{n!} \geq \frac{z^3}{3!}.$$

Then for $s > 0$, $sX > 0$, and we can write

$$M_X(s) = \mathsf{E}[e^{sX}] \geq \mathsf{E}\left[\frac{(sX)^3}{3!}\right] = \frac{s^3}{3!}\mathsf{E}[X^3] = \frac{2s^3}{3!}\int_1^\infty \frac{x^3}{x^3}\,dx = \infty.$$

43. We apply integration by parts with $u = x^{p-1}/\Gamma(p)$ and $dv = e^{-x(1-s)}\,dx$. Then $du = x^{p-2}/\Gamma(p-1)\,dx$ and $v = -e^{-x(1-s)}/(1-s)$. Hence,

$$\begin{aligned} M_p(s) &= \int_0^\infty e^{sx}\frac{x^{p-1}e^{-x}}{\Gamma(p)}\,dx = \int_0^\infty \frac{x^{p-1}}{\Gamma(p)}e^{-x(1-s)}\,dx \\ &= -\frac{x^{p-1}}{\Gamma(p)}\cdot\frac{e^{-x(1-s)}}{1-s}\bigg|_0^\infty + \frac{1}{1-s}\int_0^\infty \frac{x^{p-2}}{\Gamma(p-1)}e^{-x(1-s)}\,dx. \end{aligned}$$

The last term is $M_{p-1}(s)/(1-s)$. The other term is zero if $p > 1$ and $\operatorname{Re} s < 1$.

44. (a) In this case, we use the change of variable $t = x(1-s)$, which implies $x = t/(1-s)$ and $dx = dt/(1-s)$. Hence,

$$\begin{aligned} M_p(s) &= \int_0^\infty e^{sx}\frac{x^{p-1}e^{-x}}{\Gamma(p)}\,dx = \frac{1}{\Gamma(p)}\int_0^\infty x^{p-1}e^{-x(1-s)}\,dx \\ &= \frac{1}{\Gamma(p)}\int_0^\infty \left(\frac{t}{1-s}\right)^{p-1}e^{-t}\,\frac{dt}{1-s} \\ &= \left(\frac{1}{1-s}\right)^p\cdot\underbrace{\frac{1}{\Gamma(p)}\int_0^\infty t^{p-1}e^{-t}\,dt}_{=1} = \left(\frac{1}{1-s}\right)^p. \end{aligned}$$

(b) From $M_X(s) = (1-s)^{-p}$, we find $M_X'(s) = p(1-s)^{-p-1}$, $M_X''(s) = p(p+1)(1-s)^{-p-2}$, and so on. The general result is

$$M_X^{(n)}(s) \;=\; p(p+1)\cdots(p+[n-1])(1-s)^{-p-n} \;=\; \frac{\Gamma(n+p)}{\Gamma(p)}(1-s)^{-p-n}.$$

Hence, the Taylor series is

$$M_X(s) \;=\; \sum_{n=0}^{\infty}\frac{s^n}{n!}M_X^{(n)}(0) \;=\; \sum_{n=0}^{\infty}\frac{s^n}{n!}\cdot\frac{\Gamma(n+p)}{\Gamma(p)}.$$

45. (a) Make the change of variable $t = \lambda x$ or $x = t/\lambda$, $dx = dt/\lambda$. Thus,

$$\mathsf{E}[e^{sX}] \;=\; \int_0^\infty e^{sx}\,\frac{\lambda(\lambda x)^{p-1}e^{-\lambda x}}{\Gamma(p)}\,dx \;=\; \int_0^\infty e^{(s/\lambda)t}\,\frac{t^{p-1}e^{-t}}{\Gamma(p)}\,dt,$$

which is the moment generating function of g_p evaluated at s/λ. Hence,

$$\mathsf{E}[e^{sX}] \;=\; \Big(\frac{1}{1-s/\lambda}\Big)^p \;=\; \Big(\frac{\lambda}{\lambda-s}\Big)^p,$$

and the characteristic function is

$$\mathsf{E}[e^{j\nu X}] \;=\; \Big(\frac{1}{1-j\nu/\lambda}\Big)^p \;=\; \Big(\frac{\lambda}{\lambda-j\nu}\Big)^p.$$

(b) The Erlang(m,λ) mgf is $\Big(\dfrac{\lambda}{\lambda-s}\Big)^m$, and the chf is $\Big(\dfrac{\lambda}{\lambda-j\nu}\Big)^m$.

(c) The chi-squared with k degrees of freedom mgf is $\Big(\dfrac{1}{1-2s}\Big)^{k/2}$, and the chf is $\Big(\dfrac{1}{1-2j\nu}\Big)^{k/2}$.

46. First write

$$M_Y(s) \;=\; \mathsf{E}[e^{sY}] \;=\; \mathsf{E}[e^{sX^2}] \;=\; \int_{-\infty}^{\infty} e^{sx^2}\,\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \;=\; \int_{-\infty}^{\infty}\frac{e^{-x^2(1-2s)/2}}{\sqrt{2\pi}}\,dx.$$

If we let $(1-2s) = 1/\sigma^2$; i.e., $\sigma = 1/\sqrt{1-2s}$, then

$$M_Y(s) \;=\; \sigma\int_{-\infty}^{\infty}\frac{e^{-(x/\sigma)^2/2}}{\sqrt{2\pi}\,\sigma}\,dx \;=\; \sigma \;=\; \frac{1}{\sqrt{1-2s}}.$$

47. First observe that

$$\begin{aligned}
e^{sx^2}e^{-(x-m)^2/2} &= e^{-(x^2-2xm+m^2-2sx^2)/2} \;=\; e^{-[x^2(1-2s)-2xm]/2}e^{-m^2/2}\\
&= e^{-(1-2s)\{x^2-2xm/(1-2s)+[m/(1-2s)]^2-[m/(1-2s)]^2\}/2}e^{-m^2/2}\\
&= e^{-(1-2s)\{x-[m/(1-2s)]\}^2/2}e^{m^2/[2(1-2s)]}e^{-m^2/2}\\
&= e^{-(1-2s)\{x-[m/(1-2s)]\}^2/2}e^{sm^2/(1-2s)}.
\end{aligned}$$

If we now let $1-2s = 1/\sigma^2$, or $\sigma = 1/\sqrt{1-2s}$, and $\mu = m/(1-2s)$, then

$$\begin{aligned}
\mathsf{E}[e^{sY}] \;=\; \mathsf{E}[e^{sX^2}] \;&=\; \int_{-\infty}^{\infty} e^{sx^2}\,\frac{e^{-(x-m)^2/2}}{\sqrt{2\pi}}\,dx \;=\; e^{sm^2/(1-2s)}\sigma\int_{-\infty}^{\infty}\frac{e^{-[(x-\mu)/\sigma]^2/2}}{\sqrt{2\pi}\,\sigma}\,dx \\
&=\; e^{sm^2/(1-2s)}\sigma \;=\; \frac{e^{sm^2/(1-2s)}}{\sqrt{1-2s}}.
\end{aligned}$$

48. $\varphi_Y(\nu) \;=\; \mathsf{E}[e^{j\nu Y}] \;=\; \mathsf{E}[e^{j\nu(aX+b)}] \;=\; \mathsf{E}[e^{j(\nu a)X}]e^{j\nu b} \;=\; \varphi_X(a\nu)e^{j\nu b}$.

49. The key observation is that

$$|\nu| \;=\; \begin{cases} \nu, & \nu \geq 0, \\ -\nu, & \nu < 0. \end{cases}$$

It then follows that

$$\begin{aligned}
f_X(x) \;&=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda|\nu|}e^{-j\nu x}\,d\nu \\
&=\; \frac{1}{2\pi}\left[\int_0^{\infty} e^{-\lambda\nu}e^{-j\nu x}\,d\nu + \frac{1}{2\pi}\int_{-\infty}^{0} e^{\lambda\nu}e^{-j\nu x}\,d\nu\right] \\
&=\; \frac{1}{2\pi}\left[\int_0^{\infty} e^{-\nu(\lambda+jx)}\,d\nu + \int_{-\infty}^{0} e^{\nu(\lambda-jx)}\,d\nu\right] \\
&=\; \frac{1}{2\pi}\left[\frac{-1}{\lambda+jx}e^{-\nu(\lambda+jx)}\Big|_0^{\infty} + \frac{1}{\lambda-jx}e^{\nu(\lambda-jx)}\Big|_{-\infty}^{0}\right] \\
&=\; \frac{1}{2\pi}\left[\frac{1}{\lambda+jx}+\frac{1}{\lambda-jx}\right] \;=\; \frac{1}{2\pi}\left[\frac{2\lambda}{\lambda^2+x^2}\right] \;=\; \frac{\lambda/\pi}{\lambda^2+x^2}.
\end{aligned}$$

50. (a) $\displaystyle \frac{d}{dx}\left(\frac{e^{-x^2/2}}{\sqrt{2\pi}}\right) \;=\; -x\frac{e^{-x^2/2}}{\sqrt{2\pi}} \;=\; -xf(x).$

(b) $\displaystyle \varphi_X'(\nu) \;=\; \frac{d}{d\nu}\int_{-\infty}^{\infty} e^{j\nu x}f(x)\,dx \;=\; j\int_{-\infty}^{\infty} e^{j\nu x}xf(x)\,dx \;=\; -j\int_{-\infty}^{\infty} e^{j\nu x}f'(x)\,dx.$

(c) In this last integral, let $u = e^{j\nu x}$ and $dv = f'(x)\,dx$. Then $du = j\nu e^{j\nu x}\,dx$, $v = f(x)$, and the last integral is equal to

$$\underbrace{e^{j\nu x}f(x)\Big|_{-\infty}^{\infty}}_{=\,0} - j\nu\int_{-\infty}^{\infty} e^{j\nu x}f(x)\,dx \;=\; -j\nu\varphi_X(\nu).$$

(d) Combining (b) and (c), we have $\varphi_X'(\nu) = -j[-j\nu\varphi_X(\nu)] = -\nu\varphi_X(\nu)$.

(e) If $K(\nu) := \varphi_X(\nu)e^{\nu^2/2}$, then

$$K'(\nu) = \varphi_X'(\nu)e^{\nu^2/2} + \varphi_X(\nu)\cdot\nu e^{\nu^2/2} = -\nu\varphi_X(\nu)e^{\nu^2/2} + \varphi_X(\nu)\cdot\nu e^{\nu^2/2} = 0.$$

(f) By the mean-value theorem of calculus, for every ν, there is a ν_0 between 0 and ν such that $K(\nu) - K(0) = K'(\nu_0)(\nu - 0)$. Since the derivative is zero, we have $K(\nu) = K(0) = \varphi_X(0) = 1$. It then follows that $\varphi_X(\nu) = e^{-\nu^2/2}$.

51. Following the hints, we first write

$$\frac{d}{dx} x g_p(x) = \frac{d}{dx} \frac{x^p e^{-x}}{\Gamma(p)} = \frac{p x^{p-1} e^{-x} - x^p e^{-x}}{\Gamma(p)} = p g_p(x) - x g_p(x) = (p-x) g_p(x).$$

In

$$\varphi_X'(\nu) = \frac{d}{d\nu} \int_0^\infty e^{j\nu x} g_p(x)\,dx = j \int_0^\infty e^{j\nu x} x g_p(x)\,dx,$$

apply integration by parts with $u = x g_p(x)$ and $dv = e^{j\nu x}\,dx$. Then du is given above, $v = e^{j\nu x}/(j\nu)$, and

$$\begin{aligned}
\varphi_X'(\nu) &= j\Bigg[\underbrace{\frac{x g_p(x) e^{j\nu x}}{j\nu}\bigg|_0^\infty}_{=\,0} - \frac{1}{j\nu}\int_0^\infty e^{j\nu x}(p-x) g_p(x)\,dx\Bigg] \\
&= -\frac{1}{\nu}\left[p\int_0^\infty e^{j\nu x} g_p(x)\,dx - \frac{1}{j}\int_0^\infty e^{j\nu x}(jx) g_p(x)\,dx\right] \\
&= -\frac{1}{\nu}\left[p\varphi_X(\nu) - \frac{1}{j}\varphi_X'(\nu)\right] = -(p/\nu)\varphi_X(\nu) + (1/j\nu)\varphi_X'(\nu).
\end{aligned}$$

Rearrange this to get

$$\varphi_X'(\nu)(1 - 1/j\nu) = -(p/\nu)\varphi_X(\nu),$$

and multiply through by $-j\nu$ to get

$$\varphi_X'(\nu)(-j\nu + 1) = jp\varphi_X(\nu).$$

Armed with this, the derivative of $K(\nu) := \varphi_X(\nu)(1 - j\nu)^p$ is

$$\begin{aligned}
K'(\nu) &= \varphi_X'(\nu)(1-j\nu)^p + \varphi_X(\nu)p(1-j\nu)^{p-1}(-j) \\
&= (1-j\nu)^{p-1}[\varphi_X'(\nu)(1-j\nu) - jp\varphi_X(\nu)] = 0.
\end{aligned}$$

By the mean-value theorem of calculus, for every ν, there is a ν_0 between 0 and ν such that $K(\nu) - K(0) = K'(\nu_0)(\nu - 0)$. Since the derivative is zero, we have $K(\nu) = K(0) = \varphi_X(0) = 1$. It then follows that $\varphi_X(\nu) = 1/(1 - j\nu)^p$.

52. We use the formula $\mathsf{cov}(X,Z) = \mathsf{E}[XZ] - \mathsf{E}[X]\mathsf{E}[Z]$. The mean of an $\exp(\lambda)$ random variable is $1/\lambda$. Hence, $\mathsf{E}[X] = 1$. Since $Z := X + Y$, $\mathsf{E}[Z] = \mathsf{E}[X] + \mathsf{E}[Y]$. Since the Laplace random variable has zero mean, $\mathsf{E}[Y] = 0$. Hence, $\mathsf{E}[Z] = \mathsf{E}[X] = 1$. Next, $\mathsf{E}[XZ] = \mathsf{E}[X(X+Y)] = \mathsf{E}[X^2] + \mathsf{E}[XY] = \mathsf{E}[X^2] + \mathsf{E}[X]\mathsf{E}[Y]$ by independence. Since $\mathsf{E}[Y] = 0$, $\mathsf{E}[XZ] = \mathsf{E}[X^2] = \mathsf{var}(X) + (\mathsf{E}[X])^2 = 1 + 1^2 = 2$, where we have used the fact that the variance of an $\exp(\lambda)$ random variable is $1/\lambda^2$. We can now write $\mathsf{cov}(X,Z) = 2 - 1 = 1$. Since Z is the sum of independent, and therefore uncorrelated, random variables, $\mathsf{var}(Z) = \mathsf{var}(X+Y) = \mathsf{var}(X) + \mathsf{var}(Y) = 1 + 2 = 3$, where we have used the fact that the variance of a Laplace(λ) random variable is $2/\lambda^2$.

53. Since $Z = X + Y$, where $X \sim N(0,1)$ and $Y \sim \text{Laplace}(1)$ are independent, we have

$$\mathsf{var}(Z) \;=\; \mathsf{var}(X+Y) \;=\; \mathsf{var}(X)+\mathsf{var}(Y) \;=\; 1+2 \;=\; 3.$$

54. Write

$$M_Z(s) \;=\; \mathsf{E}[e^{sZ}] \;=\; \mathsf{E}[e^{s(X-Y)}] \;=\; \mathsf{E}[e^{sX}e^{-sY}] \;=\; \mathsf{E}[e^{sX}]\mathsf{E}[e^{-sY}] \;=\; M_X(s)M_Y(-s).$$

If $M_X(s) = M_Y(s) = \lambda/(\lambda - s)$, then

$$M_Z(s) \;=\; \frac{\lambda}{\lambda-s}\cdot\frac{\lambda}{\lambda-(-s)} \;=\; \frac{\lambda}{\lambda-s}\cdot\frac{\lambda}{\lambda+s} \;=\; \frac{\lambda^2}{\lambda^2-s^2},$$

which is the Laplace(λ) mgf.

55. Because the X_i are independent, we can write

$$M_{Y_n}(s) := \mathsf{E}[e^{sY_n}] = \mathsf{E}[e^{s(X_1+\cdots+X_n)}] = \mathsf{E}\left[\prod_{i=1}^n e^{sX_i}\right] = \prod_{i=1}^n \mathsf{E}[e^{sX_i}] = \prod_{i=1}^n M_{X_i}(s). \qquad (*)$$

(a) For $X_i \sim N(m_i, \sigma_i^2)$, $M_{X_i}(s) = e^{sm_i+\sigma_i^2 s^2/2}$. Hence,

$$M_{Y_n}(s) \;=\; \prod_{i=1}^n e^{sm_i+\sigma_i^2 s^2/2} \;=\; \exp\left[s\sum_{i=1}^n m_i + \left(\sum_{i=1}^n \sigma_i^2\right)s^2/2\right] \;=\; \underbrace{e^{sm+\sigma^2 s^2/2}}_{N(m,\sigma^2)\text{ mgf}},$$

provided we put $m := m_1+\cdots+m_n$ and $\sigma^2 := \sigma_1^2+\cdots+\sigma_n^2$.

(b) For Cauchy random variables, we must observe that the moment generating function exists only for $s = j\nu$. Equivalently, we must use characteristic functions. In this case, $(*)$ becomes

$$\varphi_{Y_n}(\nu) \;:=\; \mathsf{E}[e^{j\nu Y_n}] \;=\; \prod_{i=1}^n \varphi_{X_i}(\nu).$$

Now, the Cauchy(λ_i) chf is $\varphi_{X_i}(\nu) = e^{-\lambda_i|\nu|}$. Hence,

$$\varphi_{Y_n}(\nu) \;=\; \prod_{i=1}^n e^{-\lambda_i|\nu|} \;=\; \exp\left[-\left(\sum_{i=1}^n \lambda_i\right)|\nu|\right] \;=\; \underbrace{e^{-\lambda|\nu|}}_{\text{Cauchy}(\lambda)\text{ chf}},$$

provided we put $\lambda := \lambda_1+\cdots+\lambda_n$.

(c) For $X_i \sim \text{gamma}(p_i,\lambda)$, the mgf is $M_{X_i}(s) = [\lambda/(\lambda-s)]^{p_i}$. Hence,

$$M_{Y_n}(s) \;=\; \prod_{i=1}^n \left(\frac{\lambda}{\lambda-s}\right)^{p_i} \;=\; \left(\frac{\lambda}{\lambda-s}\right)^{p_1+\cdots+p_n} \;=\; \underbrace{\left(\frac{\lambda}{\lambda-s}\right)^{p}}_{\text{gamma}(p,\lambda)\text{ mgf}},$$

provided we put $p := p_1+\cdots+p_n$.

56. From part (c) of the preceding problem, $Y \sim \text{gamma}(rp,\lambda)$. The table inside the back cover of the text gives the nth moment of a gamma random variable. Hence,

$$\mathsf{E}[Y^n] \;=\; \frac{\Gamma(n+rp)}{\lambda^n\Gamma(rp)}.$$

57. Let T_i denote the time to transmit packet i. Then the time to transmit n packets is $T := T_1+\cdots+T_n$. We need to find the density of T. Since the T_i are exponential, we can apply the remark in the statement of Problem 55(c) to conclude that $T \sim \text{Erlang}(n,\lambda)$. Hence,

$$f_T(t) \;=\; \frac{\lambda(\lambda t)^{n-1}e^{-\lambda t}}{(n-1)!}, \quad t \geq 0.$$

58. Observe that $Y \;=\; \ln\left(\prod_{i=1}^{n}\frac{1}{X_i}\right) \;=\; \sum_{i=1}^{n}\ln\frac{1}{X_i}$. By Problem 40, each term is an $\exp(1)$ random variable. Hence, by the remark in the statement of Problem 55(c), $Y \sim \text{Erlang}(n,1)$; i.e.,

$$f_Y(y) \;=\; \frac{y^{n-1}e^{-y}}{(n-1)!}, \quad y \geq 0.$$

59. Consider the characteristic function,

$$\begin{aligned}\varphi_Y(\nu) \;&=\; \mathsf{E}[e^{j\nu Y}] \;=\; \mathsf{E}[e^{j\nu(\beta_1X_1+\cdots+\beta_nX_n)}] \;=\; \mathsf{E}\left[\prod_{i=1}^{n}e^{j(\nu\beta_i)X_i}\right] \;=\; \prod_{i=1}^{n}\mathsf{E}[e^{j(\nu\beta_i)X_i}]\\ &=\; \prod_{i=1}^{n}e^{-\lambda|\nu\beta_i|} \;=\; \prod_{i=1}^{n}e^{-\lambda\beta_i|\nu|} \;=\; \exp\left[-\lambda\left(\sum_{i=1}^{n}\beta_i\right)|\nu|\right].\end{aligned}$$

This is the chf of a Cauchy random variable with parameter $\lambda\sum_{i=1}^{n}\beta_i$. Hence,

$$f_Y(y) \;=\; \frac{\frac{\lambda}{\pi}\sum_{i=1}^{n}\beta_i}{\left(\lambda\sum_{i=1}^{n}\beta_i\right)^2+y^2}.$$

60. We need to compute $\mathsf{P}(|X-Y|\leq 2)$. If we put $Z := X-Y$, then we need to compute $\mathsf{P}(|Z|\leq 2)$. We first find the density of Z using characteristic functions. Write

$$\varphi_Z(\nu) = \mathsf{E}[e^{j\nu(X-Y)}] = \mathsf{E}[e^{j\nu X}e^{-j\nu Y}] = \mathsf{E}[e^{j\nu X}]\mathsf{E}[e^{j(-\nu)Y}] = e^{-|\nu|}e^{-|-\nu|} = e^{-2|\nu|},$$

which is the chf of a Cauchy(2) random variable. Since the Cauchy density is even,

$$\mathsf{P}(|Z|\leq 2) = 2\int_0^2 f_Z(z)\,dz = 2\left[\frac{1}{\pi}\tan^{-1}\left(\frac{z}{2}\right)+\frac{1}{2}\right]\Bigg|_0^2 = \frac{2}{\pi}\tan^{-1}(1) = \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2}.$$

61. Let $X := U+V+W$ be the sum of the three voltages. The alarm sounds if $X > x$. To find $\mathsf{P}(X > x)$, we need the density of X. Since U, V, and W are i.i.d. $\exp(\lambda)$ random variables, by the remark in the statement of Problem 55(c), $X \sim \text{Erlang}(3,\lambda)$. By Problem 15(c),

$$\mathsf{P}(X > x) \;=\; \sum_{k=0}^{2}\frac{(\lambda x)^k e^{-\lambda x}}{k!}.$$

62. Let $X_i \sim \text{Cauchy}(\lambda)$ be the i.i.d. line loads. Let $Y := X_1 + \cdots + X_n$ be the total load. The substation shuts down if $Y > \ell$. To find $\mathsf{P}(Y > \ell)$, we need to find the density of Y. By Problem 55(b), $Y \sim \text{Cauchy}(n\lambda)$, and so

$$\begin{aligned} \mathsf{P}(Y > \ell) &= \int_\ell^\infty f_Y(y)\,dy = \left[\frac{1}{\pi}\tan^{-1}\left(\frac{y}{n\lambda}\right) + \frac{1}{2}\right]\bigg|_\ell^\infty \\ &= 1 - \frac{1}{\pi}\left[\tan^{-1}\left(\frac{\ell}{n\lambda}\right) + \frac{1}{2}\right] = \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\frac{\ell}{n\lambda}\right). \end{aligned}$$

63. Let the $U_i \sim \text{uniform}[0,1]$ be the i.i.d. efficiencies of the extractors. Let $X_i = 1$ if extractor i operates with efficiency less than 0.25; in symbols, $X_i = I_{[0,0.25)}(U_i)$, which is Bernoulli(p) with $p = 0.25$. Then $Y := X_1 + \cdots + X_{13}$ is the number of extractors operating at less than 0.25 efficiency. The outpost operates normally if $Y < 3$. We must compute $\mathsf{P}(Y < 3)$. Since Y is the sum of i.i.d. Bernoulli(p) random variables, $Y \sim \text{binomial}(13, p)$. Thus,

$$\begin{aligned} \mathsf{P}(Y < 3) &= \mathsf{P}(Y = 0) + \mathsf{P}(Y = 1) + \mathsf{P}(Y = 2) \\ &= \binom{13}{0} p^0(1-p)^{13} + \binom{13}{1} p(1-p)^{12} + \binom{13}{2} p^2(1-p)^{11} \\ &= (1-p)^{11}[(1-p)^2 + 13p(1-p) + 78p^2] = 0.3326. \end{aligned}$$

64. By the remark in the statement of Problem 55(c), $R = T + A$ is chi-squared with $k = 2$ degrees of freedom. Since the number of degrees of freedom is even, R is Erlang$(k/2, 1/2) = $ Erlang$(1, 1/2) = \exp(1/2)$. Hence,

$$\mathsf{P}(R > r) = \int_r^\infty (1/2)e^{-x/2}\,dx = e^{-r/2}.$$

65. (a) Since c_{2n+k} is a density, it integrates to one. So,

$$\begin{aligned} \int_0^\infty c_{k,\lambda^2}(x)\,dx &= \int_0^\infty \sum_{n=0}^\infty \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!} c_{2n+k}(x)\,dx \\ &= \sum_{n=0}^\infty \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!} \underbrace{\int_0^\infty c_{2n+k}(x)\,dx}_{=1} \\ &= \underbrace{\sum_{n=0}^\infty \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!}}_{\text{Poisson}(\lambda^2/2)\text{ pmf}} = 1. \end{aligned}$$

(b) The mgf is

$$\begin{aligned} M_{k,\lambda^2}(s) &= \int_0^\infty e^{sx} c_{k,\lambda^2}(x)\,dx \\ &= \int_0^\infty e^{sx}\left[\sum_{n=0}^\infty \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!} c_{2n+k}(x)\right] dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!} \int_0^\infty e^{sx} c_{2n+k}(x)\,dx \\
&= \sum_{n=0}^{\infty} \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!} \left(\frac{1}{1-2s}\right)^{(2n+k)/2} \\
&= \frac{e^{-\lambda^2/2}}{(1-2s)^{k/2}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda^2/2}{1-2s}\right)^n \\
&= \frac{e^{-\lambda^2/2}}{(1-2s)^{k/2}} e^{(\lambda^2/2)/(1-2s)} \;=\; \frac{\exp\left[(\lambda^2/2)\left(\frac{1}{1-2s}-1\right)\right]}{(1-2s)^{k/2}} \\
&= \frac{\exp\left[(\lambda^2/2)\left(\frac{2s}{1-2s}\right)\right]}{(1-2s)^{k/2}} \;=\; \frac{\exp[s\lambda^2/(1-2s)]}{(1-2s)^{k/2}}.
\end{aligned}
$$

(c) If we first note that

$$
\frac{d}{ds}\left(\frac{s\lambda^2}{1-2s}\right)\bigg|_{s=0} \;=\; \frac{(1-2s)\lambda^2 - s\lambda^2(-2)}{1-2s}\bigg|_{s=0} \;=\; \lambda^2,
$$

then it is easy to show that $M'_{k,\lambda^2}(s)$ has the general form $\dfrac{\alpha(s)+\beta(s)}{(1-2s)^k}$, where $\alpha(0)=\lambda^2$ and $\beta(0)=k$. Hence, $\mathsf{E}[X] = M'_{k,\lambda^2}(0) = \lambda^2 + k$.

(d) The usual mgf argument gives

$$
\begin{aligned}
M_Y(s) \;=\; \mathsf{E}[e^{sY}] \;=\; \mathsf{E}[e^{s(X_1+\cdots+X_n)}] \;&=\; \prod_{i=1}^n M_{k_i,\lambda_i^2}(s) \\
&= \prod_{i=1}^n \frac{\exp[s\lambda_i^2/(1-2s)]}{(1-2s)^{k_i/2}} \\
&= \frac{\exp[s(\lambda_1^2+\cdots+\lambda_n^2)/(1-2s)]}{(1-2s)^{(k_1+\cdots+k_n)/2}}.
\end{aligned}
$$

If we put $k := k_1+\cdots+k_n$ and $\lambda^2 := \lambda_1^2+\cdots+\lambda_n^2$, we see that Y is noncentral chi-squared with k degrees of freedom and noncentrality parameter λ^2.

(e) We first consider

$$
\begin{aligned}
\frac{e^{\lambda\sqrt{x}}+e^{-\lambda\sqrt{x}}}{2} \;&=\; \frac{1}{2}\sum_{n=0}^{\infty} \frac{(\lambda\sqrt{x})^n}{n!}[1+(-1)^n] \;=\; \sum_{n=0}^{\infty} \frac{\lambda^{2n}x^n}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{\lambda^{2n}x^n}{1\cdot 3\cdot 5\cdots(2n-1)\cdot 2^n\cdot n!} \\
&= \sum_{n=0}^{\infty} \frac{2^n(\lambda^2/2)^n(x/2)^n}{1\cdot 3\cdot 5\cdots(2n-1)\cdot n!} \\
&= \sum_{n=0}^{\infty} \frac{\sqrt{\pi}(\lambda^2/2)^n(x/2)^n}{\Gamma(\frac{2n+1}{2})\cdot n!}, \quad \text{by Problem 14(c).}
\end{aligned}
$$

We can now write

$$\begin{aligned}\frac{e^{-(x+\lambda^2)/2}}{\sqrt{2\pi x}}\cdot\frac{e^{\lambda\sqrt{x}}+e^{-\lambda\sqrt{x}}}{2} &= \frac{e^{-(x+\lambda^2)/2}}{\sqrt{2\pi x}}\sum_{n=0}^{\infty}\frac{\sqrt{\pi}(\lambda^2/2)^n(x/2)^n}{\Gamma(\frac{2n+1}{2})\cdot n!}\\ &= \sum_{n=0}^{\infty}\frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!}\cdot\frac{(1/2)(x/2)^{n-1/2}e^{-x/2}}{\Gamma(\frac{2n+1}{2})}\\ &= \sum_{n=0}^{\infty}\frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!}c_{2n+1}(x) \;=\; c_{1,\lambda^2}(x).\end{aligned}$$

66. First, $\mathsf{P}(X \geq a) = \int_a^\infty 2x^{-3}\,dx = 1/a^2$, while, using the result of Problem 23, the Markov bound is $\mathsf{E}[X]/a = 2/a$. Thus, the true probability is $1/a^2$, but the bound is $2/a$, which decays much more slowly for large a.

67. We begin by noting that $\mathsf{P}(X \geq a) = e^{-a}$, $\mathsf{E}[X] = 1$, and $\mathsf{E}[X^2] = 2$. Hence, the Markov bound is $1/a$, and the Chebyshev bound is $2/a^2$. To find the Chernoff bound, we must minimize $h(s) := e^{-sa}M_X(s) = e^{-sa}/(1-s)$ over $0 < s < 1$. Now,

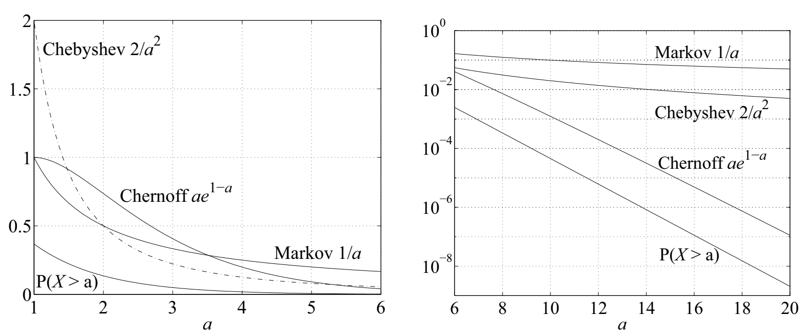
$$h'(s) \;=\; \frac{(1-s)(-a)e^{-sa}+e^{-sa}}{(1-s)^2}.$$

Solving $h'(s) = 0$, we find $s = (a-1)/a$, which is positive only for $a > 1$. Hence, the Chernoff bound is valid only for $a > 1$. For $a > 1$, the Chernoff bound is

$$h((a-1)/a) \;=\; \frac{e^{-a\cdot(a-1)/a}}{1-(a-1)/a} \;=\; ae^{1-a}.$$

(a) It is easy to see that the Markov bound is smaller than the Chebyshev bound for $0 < a < 2$. However, note that the Markov bound is greater than one for $0 < a < 1$, and the Chebyshev bound is greater than one for $0 < a < 2$.

(b) **MATLAB.**



The Markov bound is the smallest on $[1,2]$. The Chebyshev bound is the smallest from $a = 2$ to a bit more than $a = 5$. Beyond that, the Chernoff bound is the smallest.

CHAPTER 5
Problem Solutions

1. For $x \geq 0$,
$$F(x) = \int_0^x \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}.$$
For $x < 0$, $F(x) = 0$.

2. For $x \geq 0$,
$$F(x) = \int_0^x \frac{t}{\lambda^2} e^{-(t/\lambda)^2/2}\,dt = -e^{-(t/\lambda)^2/2}\Big|_0^x = 1 - e^{-(x/\lambda)^2/2}.$$
For $x < 0$, $F(x) = 0$.

3. For $x \geq 0$,
$$F(x) = \int_0^x \lambda p t^{p-1} e^{-\lambda t^p}\,dt = -e^{-\lambda t^p}\Big|_0^x = 1 - e^{-\lambda x^p}.$$
For $x < 0$, $F(x) = 0$.

4. For $x \geq 0$, first write
$$F(x) = \int_0^x \sqrt{\frac{2}{\pi}} \frac{t^2}{\lambda^3} e^{-(t/\lambda)^2/2}\,dt = \int_0^{x/\lambda} \sqrt{\frac{2}{\pi}} \theta^2 e^{-\theta^2/2}\,d\theta,$$
where we have used the change of variable $\theta = t/\lambda$. Next use integration by parts with $u = \theta$ and $dv = \theta e^{-\theta^2/2}\,d\theta$. Then
$$\begin{aligned} F(x) &= \sqrt{\frac{2}{\pi}} \left\{ -\theta e^{-\theta^2/2}\Big|_0^{x/\lambda} + \int_0^{x/\lambda} e^{-\theta^2/2}\,d\theta \right\} \\ &= -\sqrt{\frac{2}{\pi}} \frac{x}{\lambda} e^{-(x/\lambda)^2/2} + 2\int_0^{x/\lambda} \frac{e^{-\theta^2/2}}{\sqrt{2\pi}}\,d\theta \\ &= -\sqrt{\frac{2}{\pi}} \frac{x}{\lambda} e^{-(x/\lambda)^2/2} + 2[\Phi(x/\lambda) - 1/2] \\ &= 2\Phi(x/\lambda) - 1 - \sqrt{\frac{2}{\pi}} \frac{x}{\lambda} e^{-(x/\lambda)^2/2}. \end{aligned}$$
For $x < 0$, $F(x) = 0$.

5. For $y > 0$, $F(y) = \mathsf{P}(Y \leq y) = \mathsf{P}(e^Z \leq y) = \mathsf{P}(Z \leq \ln y) = F_Z(\ln y)$. Then $f_Y(y) = f_Z(\ln y)/y$ for $y > 0$. Since $Y := e^Z > 0$, $f_Y(y) = 0$ for $y \leq 0$.

6. To begin, write $F_Y(y) = \mathsf{P}(Y \le y) = \mathsf{P}(1-X \le y) = \mathsf{P}(1-y \le X) = 1 - F_X(1-y)$. Thus, $f_Y(y) = -f_X(1-y)\cdot(-1) = f_X(1-y)$. In the case of $X \sim \text{uniform}(0,1)$,

$$f_Y(y) \;=\; f_X(1-y) \;=\; I_{(0,1)}(1-y) \;=\; I_{(0,1)}(y),$$

since $0 < 1-y < 1$ if and only if $0 < y < 1$.

7. For $y > 0$,

$$F_Y(y) = \mathsf{P}(Y \le y) = \mathsf{P}(\ln(1/X) \le y) = \mathsf{P}(1/X \le e^y) = \mathsf{P}(X \ge e^{-y}) = 1 - F_X(e^{-y}).$$

Thus, $f_Y(y) = -f_X(e^{-y})\cdot(-e^{-y}) = e^{-y}$, since $f_X(e^{-y}) = I_{(0,1)}(e^{-y}) = 1$ for $y > 0$. Since $Y := \ln(1/X) > 0$, $f_Y(y) = 0$ for $y \le 0$.

8. For $y \ge 0$,

$$F_Y(y) = \mathsf{P}(\lambda X^P \le y) = \mathsf{P}(X^p \le y/\lambda) = \mathsf{P}(X \le (y/\lambda)^{1/p}) = F_X((y/\lambda)^{1/p}).$$

Thus, $f_Y(y) = f_X((y/\lambda)^{1/p}) \cdot \frac{1}{p}(y/\lambda)^{(1/p)-1}/\lambda$. Using the formula for the Weibull density, we find that $f_Y(y) = e^{-y}$ for $y \ge 0$. Since $Y := \lambda X^p \ge 0$, $f_Y(y) = 0$ for $y < 0$. Thus, $Y \sim \exp(1)$.

9. For $y \ge 0$, write $F_Y(y) = \mathsf{P}(\sqrt{X} \le y) = \mathsf{P}(X \le y^2) = F_X(y^2) = 1 - e^{-y^2}$. Thus,

$$f_Y(y) \;=\; -e^{-y^2}\cdot(-2y) \;=\; \frac{y}{(1/\sqrt{2})^2} e^{-(y/(1/\sqrt{2}))^2/2},$$

which is the Rayleigh$(1/\sqrt{2}\,)$ density.

10. Recall that the moment generating function of $X \sim N(m,\sigma^2)$ is $M_X(s) = \mathsf{E}[e^{sX}] = e^{sm+s^2\sigma^2/2}$. Thus,

$$\mathsf{E}[Y^n] \;=\; \mathsf{E}[(e^X)^n] \;=\; \mathsf{E}[e^{nX}] \;=\; M_X(n) \;=\; e^{nm+n^2\sigma^2/2}.$$

11. For $y > 0$, $F_Y(y) = \mathsf{P}(X^2 \le y) = \mathsf{P}(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$. Thus,

$$f_Y(y) \;=\; f_X(\sqrt{y})(\tfrac{1}{2}y^{-1/2}) - f_X(-\sqrt{y})(-\tfrac{1}{2}y^{-1/2}).$$

Since f_X is even,

$$f_Y(y) \;=\; y^{-1/2} f_X(\sqrt{y}) \;=\; \frac{e^{-y/2}}{\sqrt{2\pi y}}, \quad y > 0.$$

12. For $y > 0$,

$$\begin{aligned} F_Y(y) &= \mathsf{P}((X+m)^2 \le y) \\ &= \mathsf{P}(-\sqrt{y} \le X+m \le \sqrt{y}) \\ &= \mathsf{P}(-\sqrt{y}-m \le X \le \sqrt{y}-m) \\ &= F_X(\sqrt{y}-m) - F_X(-\sqrt{y}-m). \end{aligned}$$

Thus,

$$\begin{aligned} f_Y(y) &= f_X(\sqrt{y}-m)(\tfrac{1}{2}y^{-1/2}) - f_X(-\sqrt{y}-m)(-\tfrac{1}{2}y^{-1/2}) \\ &= \frac{1}{\sqrt{2\pi y}}\left[e^{-(\sqrt{y}-m)^2/2} + e^{-(-\sqrt{2}-m)^2/2}\right]\Big/2 \\ &= \frac{1}{\sqrt{2\pi y}}\left[e^{-(y-2\sqrt{y}m+m^2)/2} + e^{-(y+2\sqrt{y}m+m^2)/2}\right]\Big/2 \\ &= \frac{e^{-(y+m^2)/2}}{\sqrt{2\pi y}}\left[\frac{e^{m\sqrt{y}}+e^{-m\sqrt{y}}}{2}\right], \quad y > 0. \end{aligned}$$

13. Using the example mentioned in the hint, we have

$$F_{X_{\max}}(z) = \prod_{k=1}^{n} F_{X_k}(z) = F(z)^n \quad \text{and} \quad F_{X_{\min}}(z) = 1 - \prod_{k=1}^{n}[1-F_{X_k}(z)] = 1-[1-F(z)]^n.$$

14. Let $Z := \max(X,Y)$. Since X and Y are i.i.d., we have from the preceding problem that $F_Z(z) = F_X(z)^2$. Hence,

$$f_Z(z) = 2F_X(z)f_X(z) = 2(1-e^{-\lambda z})\cdot\lambda e^{-\lambda z}, \quad z \geq 0.$$

Next,

$$\begin{aligned} \mathsf{E}[Z] &= \int_0^\infty z f_Z(z)\,dz = 2\int_0^\infty \lambda z e^{-\lambda z}(1-e^{-\lambda z})\,dz \\ &= 2\int_0^\infty z\cdot\lambda e^{-\lambda z}\,dz - \int_0^\infty z\cdot(2\lambda)e^{-(2\lambda)z}\,dz \\ &= 2\frac{1}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}. \end{aligned}$$

15. Use the laws of total probability and substitution and the fact that conditioned on $X = m$, $Y \sim \text{Erlang}(m,\lambda)$. In particular, $\mathsf{E}[Y|X=m] = m/\lambda$. We can now write

$$\begin{aligned} \mathsf{E}[XY] &= \sum_{m=0}^{\infty}\mathsf{E}[XY|X=m]\mathsf{P}(X=m) = \sum_{m=0}^{\infty}\mathsf{E}[mY|X=m]\mathsf{P}(X=m) \\ &= \sum_{m=0}^{\infty} m\mathsf{E}[Y|X=m]\mathsf{P}(X=m) = \sum_{m=0}^{\infty} m(m/\lambda)\mathsf{P}(X=m) \\ &= \frac{1}{\lambda}\mathsf{E}[X^2] = \frac{\mu+\mu^2}{\lambda}, \quad \text{since } X \sim \text{Poisson}(\mu). \end{aligned}$$

16. The problem statement tells us that $\mathsf{P}(Y > y|X = n) = e^{-ny}$. Using the law of total probability and the pgf of $X \sim \text{Poisson}(\lambda)$, we have

$$\begin{aligned} \mathsf{P}(Y > y) &= \sum_{n=0}^{\infty}\mathsf{P}(Y > y|X=n)\mathsf{P}(X=n) = \sum_{n=0}^{\infty} e^{-ny}\mathsf{P}(X=n) \\ &= \mathsf{E}[e^{-yX}] = G_X(e^{-y}) = e^{\lambda(e^{-y}-1)}. \end{aligned}$$

17. (a) Using the law of substitution, independence, and the fact that $Y \sim N(0,1)$, write

$$F_{Z|X}(z|i) \;=\; \mathsf{P}(Z \le z|X=i) \;=\; \mathsf{P}(X+Y \le z|X=i) \;=\; \mathsf{P}(Y \le z-i|X=i)$$
$$=\; \mathsf{P}(Y \le z-i) \;=\; \Phi(z-i).$$

Next,

$$F_Z(z) \;=\; \sum_{i=0}^{1} F_{Z|X}(z|i)\mathsf{P}(X=i) \;=\; (1-p)\Phi(z)+p\Phi(z-1),$$

and so

$$f_Z(z) \;=\; \frac{(1-p)e^{-z^2/2}+pe^{-(z-1)^2/2}}{\sqrt{2\pi}}.$$

(b) From part (a), it is easy to see that $f_{Z|X}(z|i) = \exp[-(z-i)^2/2]/\sqrt{2\pi}$. Hence,

$$\frac{f_{Z|X}(z|1)}{f_{Z|X}(z|0)} \;\ge\; \frac{\mathsf{P}(X=0)}{\mathsf{P}(X=1)} \quad \text{becomes} \quad \frac{\exp[-(z-1)^2/2]}{\exp[-z^2/2]} \;\ge\; \frac{1-p}{p},$$

or $e^{z-1/2} \ge (1-p)/p$. Taking logarithms, we can further simplify this to

$$z \;\ge\; \frac{1}{2}+\ln\frac{1-p}{p}.$$

18. Use substitution and independence to write

$$F_{Z|A,X}(z|a,i) \;=\; \mathsf{P}(Z \le z|A=a,X=i) \;=\; \mathsf{P}(X/A+Y \le z|A=a,X=i)$$
$$=\; \mathsf{P}(Y \le z-i/a|A=a,X=i) \;=\; \mathsf{P}(Y \le z-i/a) \;=\; \Phi(z-i/a).$$

19. (a) Write

$$F_{Z_n}(z) \;=\; \mathsf{P}(Z_n \le z) \;=\; \mathsf{P}(\sqrt{Y_n} \le z) \;=\; \mathsf{P}(Y_n \le z^2) \;=\; F_{Y_n}(z^2).$$

For future reference, note that

$$f_{Z_n}(z) \;=\; f_{Y_n}(z^2)\cdot(2z).$$

Since Y_n is chi-squared with n degrees of freedom, i.e., gamma$(n/2,1/2)$,

$$f_{Y_n}(y) \;=\; \tfrac{1}{2}\frac{(y/2)^{n/2-1}e^{-y/2}}{\Gamma(n/2)}, \quad y>0,$$

and so

$$f_{Z_n}(z) \;=\; z\frac{(z^2/2)^{n/2-1}e^{-z^2/2}}{\Gamma(n/2)}, \quad z>0.$$

(b) When $n=1$, we obtain the **folded normal** density,

$$f_{Z_1}(z) \;=\; z\frac{(z^2/2)^{-1/2}e^{-z^2/2}}{\Gamma(1/2)} \;=\; 2\frac{e^{-z^2/2}}{\sqrt{2\pi}}, \quad z>0.$$

(c) When $n = 2$, we obtain the Rayleigh(1) density,

$$f_{Z_2}(z) = z\frac{(z^2/2)^0 e^{-z^2/2}}{\Gamma(1)} = ze^{-z^2/2}, \quad z > 0.$$

(d) When $n = 3$, we obtain the Maxwell density,

$$f_{Z_3}(z) = z\frac{(z^2/2)^{1/2} e^{-z^2/2}}{\Gamma(3/2)} = \frac{z^2}{\sqrt{2}} \cdot \frac{e^{-z^2/2}}{\frac{1}{2}\Gamma(\frac{1}{2})} = \sqrt{\frac{2}{\pi}}\, z^2 e^{-z^2/2}, \quad z > 0.$$

(e) When $n = 2m$, we obtain the Nakagami-m density,

$$f_{Z_{2m}}(z) = z\frac{(z^2/2)^{m-1} e^{-z^2/2}}{\Gamma(m)} = \frac{2}{2^m\Gamma(m)} z^{2m-1} e^{-z^2/2}, \quad z > 0.$$

20. Let $Z := X_1 + \cdots + X_n$. By Problem 55(a) in Chapter 4, $Z \sim N(0,n)$, and it follows that $V := Z/\sqrt{n} \sim N(0,1)$. We can now write $Y = Z^2 = nV^2$. By Problem 11, V^2 is chi-squared with one degree of freedom. Hence,

$$F_Y(y) = \mathsf{P}(nV^2 \le y) = \mathsf{P}(V^2 \le y/n),$$

and

$$f_Y(y) = f_{V^2}(y/n)/n = \frac{e^{-(y/n)/2}}{\sqrt{2\pi y/n}} \cdot \frac{1}{n} = \frac{e^{-(y/n)/2}}{\sqrt{2\pi n y}}, \quad y > 0.$$

21. (a) Since $F_Y(y) = \mathsf{P}(Y \le y) = \mathsf{P}(X^{1/q} \le y) = \mathsf{P}(X \le y^q) = F_X(y^q)$,

$$f_Y(y) = f_X(y^q) \cdot (qy^{q-1}) = qy^{q-1} \cdot \frac{(y^q)^{p-1} e^{-y^q}}{\Gamma(p)} = \frac{qy^{qp-1} e^{-y^q}}{\Gamma(p)}, \quad y > 0.$$

(b) Since $q > 0$, as $y \to 0$, $y^q \to 0$, and $e^{-y^q} \to 1$. Hence, the behavior of $f_Y(y)$ as $y \to 0$ is determined by the behavior of y^{p-1}. For $p > 1$, $p - 1 > 0$, and so $y^{p-1} \to 0$. For $p = 1$, $y^{p-1} = y^0 = 1$. For $0 < p < 1$, $p - 1 < 0$ and so $y^{p-1} = 1/y^{1-p} \to \infty$. Thus,

$$\lim_{y \to 0} f_Y(y) = \begin{cases} 0, & p > 1, \\ q/\Gamma(1/q), & p = 1, \\ \infty, & 0 < p < 1. \end{cases}$$

(c) We begin with the given formula

$$f_Y(y) = \frac{\lambda q(\lambda y)^{p-1} e^{-(\lambda y)^q}}{\Gamma(p/q)}, \quad y > 0.$$

(i) Taking $q = p$ and replacing λ with $\lambda^{1/p}$ yields

$$\begin{aligned} f_Y(y) &= \lambda^{1/p} p(\lambda^{1/p} y)^{p-1} e^{-(\lambda^{1/p} y)^p} = \lambda^{1/p} p \lambda^{1-1/p} y^{p-1} e^{-\lambda y^p} \\ &= \lambda p y^{p-1} e^{-\lambda y^p}, \end{aligned}$$

which is the Weibull(p,λ) density.

(ii) Taking $p = q = 2$ and replacing λ with $1/(\sqrt{2}\lambda)$ yields

$$f_Y(y) \;=\; 2/(\sqrt{2}\lambda)[y/(\sqrt{2}\lambda)]e^{-[y/(\sqrt{2}\lambda)]^2} \;=\; (y/\lambda^2)e^{-(y/\lambda)^2/2},$$

which is the required Rayleigh density.

(iii) Taking $p = 3$, $q = 2$, and replacing λ with $1/(\sqrt{2}\lambda)$ yields

$$\begin{aligned} f_Y(y) \;&=\; \frac{2/(\sqrt{2}\lambda)[y/(\sqrt{2}\lambda)]^2 e^{-[y/(\sqrt{2}\lambda)]^2}}{\Gamma(3/2)} \;=\; \frac{(y^2/\lambda^3)e^{-(y/\lambda)^2/2}}{\sqrt{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})} \\ &=\; \sqrt{\frac{2}{\pi}}\,\frac{y^2}{\lambda^3}e^{-(y/\lambda)^2}, \end{aligned}$$

which is the required Maxwell density.

(d) In

$$\mathsf{E}[Y^n] \;=\; \int_0^\infty y^n \frac{\lambda q(\lambda y)^{p-1}e^{-(\lambda y)^q}}{\Gamma(p/q)}\,dy,$$

make the change of variable $t = (\lambda y)^q$, $dt = q(\lambda y)^{q-1}\lambda\,dy$. Then

$$\begin{aligned} \mathsf{E}[Y^n] \;&=\; \frac{1}{\Gamma(p/q)\lambda^n}\int_0^\infty (\lambda y)^n(\lambda y)^{p-q}e^{-(\lambda y)^q}\lambda q(\lambda y)^{q-1}\,dy \\ &=\; \frac{1}{\Gamma(p/q)\lambda^n}\int_0^\infty (t^{1/q})^{n+p-q}e^{-t}\,dt \\ &=\; \frac{1}{\Gamma(p/q)\lambda^n}\int_0^\infty t^{(n+p)/q-1}e^{-t}\,dt \;=\; \frac{\Gamma((n+p)/q)}{\Gamma(p/q)\lambda^n}. \end{aligned}$$

(e) We use the same change of variable as in part (d) to write

$$\begin{aligned} F_Y(y) \;&=\; \int_0^y \frac{\lambda q(\lambda\theta)^{p-1}e^{-(\lambda\theta)^q}}{\Gamma(p/q)}\,d\theta \;=\; \frac{1}{\Gamma(p/q)}\int_0^{(\lambda y)^q}(t^{1/q})^{p-q}e^{-t}\,dt \\ &=\; \frac{1}{\Gamma(p/q)}\int_0^{(\lambda y)^q} t^{(p/q)-1}e^{-t}\,dt \;=\; G_{p/q}((\lambda y)^q). \end{aligned}$$

22. Following the hint, let $u = t^{-1}$ and $dv = te^{-t^2/2}\,dt$ so that $du = -1/t^2\,dt$ and $v = -e^{-t^2/2}$. Then

$$\begin{aligned} \int_x^\infty e^{-t^2/2}\,dt \;&=\; -\frac{e^{-t^2/2}}{t}\bigg|_x^\infty - \int_x^\infty \frac{e^{-t^2/2}}{t^2}\,dt \\ &=\; \frac{e^{-x^2/2}}{x} - \int_x^\infty \frac{e^{-t^2/2}}{t^2}\,dt \qquad (*) \\ &<\; \frac{e^{-x^2/2}}{x}. \end{aligned}$$

The next step is to write the integral in $(*)$ as

$$\int_x^\infty \frac{e^{-t^2/2}}{t^2}\,dt \;=\; \int_x^\infty \frac{1}{t^3}\cdot te^{-t^2/2}\,dt$$

and apply integration by parts with $u = t^{-3}$ and $dv = te^{-t^2/2}\,dt$ so that $du = -3/t^4\,dt$ and $v = -e^{-t^2/2}$. Then

$$\begin{aligned}\int_x^\infty \frac{e^{-t^2/2}}{t^2}\,dt &= -\frac{e^{-t^2/2}}{t^3}\bigg|_x^\infty - 3\int_x^\infty \frac{e^{-t^2/2}}{t^4}\,dt \\ &= \frac{e^{-x^2/2}}{x^3} - 3\int_x^\infty \frac{e^{-t^2/2}}{t^4}\,dt.\end{aligned}$$

Substituting this into $(*)$, we find that

$$\begin{aligned}\int_x^\infty e^{-t^2/2}\,dt &= \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + 3\int_x^\infty \frac{e^{-t^2/2}}{t^4}\,dt \\ &\geq \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3}.\end{aligned}$$

23. (a) Write

$$\begin{aligned}\mathsf{E}[F_X^c(Z)] &= \int_{-\infty}^\infty F_X^c(z) f_Z(z)\,dz \\ &= \int_{-\infty}^\infty \left[\int_z^\infty f_X(x)\,dx\right] f_Z(z)\,dz \\ &= \int_{-\infty}^\infty \left[\int_{-\infty}^\infty I_{(z,\infty)}(x) f_X(x)\,dx\right] f_Z(z)\,dz \\ &= \int_{-\infty}^\infty f_X(x)\left[\int_{-\infty}^\infty I_{(z,\infty)}(x) f_Z(z)\,dz\right] dx \\ &= \int_{-\infty}^\infty f_X(x)\left[\int_{-\infty}^\infty I_{(-\infty,x)}(z) f_Z(z)\,dz\right] dx \\ &= \int_{-\infty}^\infty f_X(x)\left[\int_{-\infty}^x f_Z(z)\,dz\right] dx \\ &= \int_{-\infty}^\infty f_X(x) F_Z(x)\,dx \;=\; \mathsf{E}[F_Z(X)].\end{aligned}$$

(b) From Problem 15(c) in Chapter 4, we have

$$F_Z(z) = 1 - \sum_{k=0}^{m-1} \frac{(\lambda z)^k e^{-\lambda z}}{k!}, \quad \underline{\underline{\text{for } z \geq 0,}}$$

and $F_Z(z) = 0$ for $z < 0$. Hence,

$$\begin{aligned}\mathsf{E}[F_Z(X)] &= \mathsf{E}\left[I_{[0,\infty)}(X)\left(1 - \sum_{k=0}^{m-1} \frac{(\lambda X)^k e^{-\lambda X}}{k!}\right)\right] \\ &= \mathsf{P}(X \geq 0) - \sum_{k=0}^{m-1} \frac{\lambda^k}{k!}\mathsf{E}[X^k e^{-\lambda X} I_{[0,\infty)}(X)].\end{aligned}$$

(c) Let $X \sim N(0,1)$ so that $\mathsf{E}[Q(Z)] = \mathsf{E}[F_X^c(Z)] = \mathsf{E}[F_Z(X)]$. Next, since $Z \sim \exp(\lambda) = \text{Erlang}(1,\lambda)$, we can use the result of part (b) to write

$$\begin{aligned}
\mathsf{E}[F_Z(X)] &= \mathsf{P}(X \ge 0) - \mathsf{E}[e^{-\lambda X} I_{[0,\infty)}(X)] \\
&= \frac{1}{2} - \int_0^\infty e^{-\lambda x} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \\
&= \frac{1}{2} - \frac{e^{\lambda^2/2}}{\sqrt{2\pi}} \int_0^\infty e^{-(x^2+2x\lambda+\lambda^2)/2}\,dx \\
&= \frac{1}{2} - \frac{e^{\lambda^2/2}}{\sqrt{2\pi}} \int_0^\infty e^{-(x+\lambda)^2/2}\,dx \\
&= \frac{1}{2} - \frac{e^{\lambda^2/2}}{\sqrt{2\pi}} \int_\lambda^\infty e^{-t^2/2}\,dt \;=\; \frac{1}{2} - e^{\lambda^2/2} Q(\lambda).
\end{aligned}$$

(d) Put $Z := \sigma\sqrt{Y}$ and make the following observations. First, for $z \ge 0$,

$$F_Z(z) \;=\; \mathsf{P}(\sigma\sqrt{Y} \le z) \;=\; \mathsf{P}(\sigma^2 Y \le z^2) \;=\; \mathsf{P}(Y \le (z/\sigma)^2) \;=\; F_Y((z/\sigma)^2),$$

and $F_Z(z) = 0$ for $z < 0$. Second, since Y is chi-squared with $2m$-degrees of freedom, $Y \sim \text{Erlang}(m, 1/2)$. Hence,

$$F_Z(z) \;=\; 1 - \sum_{k=0}^{m-1} \frac{((z/\sigma)^2/2)^k e^{-(z/\sigma)^2/2}}{k!}, \quad \underline{\underline{\text{for } z \ge 0.}}$$

Third, with $X \sim N(0,1)$,

$$\mathsf{E}[Q(\sigma\sqrt{Y})] \;=\; \mathsf{E}[Q(Z)] \;=\; \mathsf{E}[F_X^c(Z)] \;=\; \mathsf{E}[F_Z(X)]$$

is equal to

$$\mathsf{E}\left[I_{[0,\infty)}(X) \left(1 - \sum_{k=0}^{m-1} \frac{((X/\sigma)^2/2)^k e^{-(X/\sigma)^2/2}}{k!} \right) \right],$$

which simplifies to

$$\mathsf{P}(X \ge 0) - \sum_{k=0}^{m-1} \frac{1}{\sigma^{2k} 2^k k!} \mathsf{E}[X^{2k} e^{-(X/\sigma)^2/2} I_{[0,\infty)}(X)].$$

Now, $\mathsf{P}(X \ge 0) = 1/2$, and with $\widetilde{\sigma}^2 := (1 + 1/\sigma^2)^{-1}$, we have

$$\begin{aligned}
\mathsf{E}[X^{2k} e^{-(X/\sigma)^2/2} I_{[0,\infty)}(X)] &= \int_0^\infty x^{2k} e^{-(x/\sigma)^2/2} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \\
&= \frac{\widetilde{\sigma}}{\sqrt{2\pi}\,\widetilde{\sigma}} \int_0^\infty x^{2k} e^{-(x/\widetilde{\sigma})^2/2}\,dx \\
&= \frac{\widetilde{\sigma}}{2} \cdot \frac{1}{\sqrt{2\pi}\,\widetilde{\sigma}} \int_{-\infty}^\infty x^{2k} e^{-(x/\widetilde{\sigma})^2/2}\,dx \\
&= \frac{\widetilde{\sigma}}{2} \cdot 1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot \widetilde{\sigma}^{2k}.
\end{aligned}$$

Putting this all together, the desired result follows.

(e) If $m=1$ in part (d), then Z defined in solution of part (d) is Rayleigh(σ) by the same argument as in the solution of Problem 19(b). Hence, the desired result follows by taking $m=1$ in the result of part (d).

(f) Since the V_i are independent $\exp(\lambda_i)$, the mgf of $Y := V_1+\cdots+V_m$ is

$$M_Y(s) \;=\; \prod_{k=1}^m \frac{\lambda_k}{\lambda_k - s} \;=\; \sum_{k=1}^m c_k \frac{\lambda_k}{\lambda_k - s},$$

where the c_k are the result of expansion by partial fractions. Inverse transforming term by term, we find that

$$f_Y(y) \;=\; \sum_{k=1}^m c_k \cdot \lambda_k e^{-\lambda_k y}, \quad y \ge 0.$$

Using this, we can write

$$F_Y(y) \;=\; \sum_{k=1}^m c_k(1-e^{-\lambda_k y}), \quad y \ge 0.$$

Next, put $Z := \sqrt{Y}$, and note that for $z \ge 0$,

$$F_Z(z) \;=\; \mathsf{P}(\sqrt{Y} \le z) \;=\; \mathsf{P}(Y \le z^2) \;=\; F_Y(z^2).$$

Hence,

$$F_Z(z) \;=\; \sum_{k=1}^m c_k(1-e^{-\lambda_k z^2}), \quad z \ge 0.$$

We can now write that $\mathsf{E}[Q(Z)] = \mathsf{E}[F_X^c(Z)] = \mathsf{E}[F_Z(X)]$ is equal to

$$\mathsf{E}\biggl[I_{[0,\infty)}(X)\biggl(\sum_{k=1}^m c_k(1-e^{-\lambda_k X^2})\biggr)\biggr].$$

Now observe that with $\widetilde{\sigma}_k^2 := (1+2\lambda_k)^{-1}$,

$$\begin{aligned}
\mathsf{E}[I_{[0,\infty)}(X)e^{-\lambda_k X^2}] &= \int_0^\infty e^{-\lambda_k x^2}\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \;=\; \frac{1}{2}\int_{-\infty}^\infty e^{-\lambda_k x^2}\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{e^{-(1+2\lambda_k)x^2/2}}{\sqrt{2\pi}}\,dx \;=\; \frac{\widetilde{\sigma}_k}{2}\int_{-\infty}^\infty \frac{e^{-(x/\widetilde{\sigma}_k)^2/2}}{\sqrt{2\pi}\,\widetilde{\sigma}_k}\,dx \\
&= \frac{\widetilde{\sigma}_k}{2} \;=\; \frac{1}{2\sqrt{1+2\lambda_k}}.
\end{aligned}$$

Putting this all together, the desired result follows.

24. Recall that the noncentral chi-squared density with k degrees of freedom and noncentrality parameter λ^2 is given by

$$c_{k,\lambda^2}(t) \;:=\; \sum_{n=0}^\infty \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!}\,c_{2n+k}(t), \quad t > 0,$$

where c_{2n+k} denotes the central chi-squared density with $2n+k$ degrees of freedom. Hence,

$$\begin{aligned} C_{k,\lambda^2}(x) &= \int_0^x c_{k,\lambda^2}(t)\,dt = \int_0^x \sum_{n=0}^\infty \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!} c_{2n+k}(t)\,dt \\ &= \sum_{n=0}^\infty \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!} \int_0^x c_{2n+k}(t)\,dt = \sum_{n=0}^\infty \frac{(\lambda^2/2)^n e^{-\lambda^2/2}}{n!} C_{2n+k}(x). \end{aligned}$$

25. (a) Begin by writing

$$F_{Z_n}(z) = \mathsf{P}(\sqrt{Y_n} \le z) = \mathsf{P}(Y_n \le z^2) = F_{Y_n}(z^2).$$

Then $f_{Z_n}(z) = f_{Y_n}(z^2)\cdot 2z$. Next observe that

$$I_{n/2-1}(mz) = \sum_{\ell=0}^\infty \frac{(mz/2)^{2\ell+n/2-1}}{\ell!\Gamma(\ell+(n/2)-1+1)} = \sum_{\ell=0}^\infty \frac{(mz/2)^{2\ell+n/2-1}}{\ell!\Gamma(\ell+n/2)}.$$

From Problem 65 in Chapter 4,

$$\begin{aligned} f_{Y_n}(y) &:= \sum_{\ell=0}^\infty \frac{(m^2/2)^\ell e^{-m^2/2}}{\ell!} c_{2\ell+n}(y) \\ &= \sum_{\ell=0}^\infty \frac{(m^2/2)^\ell e^{-m^2/2}}{\ell!} \cdot \tfrac{1}{2} \frac{(y/2)^{(2\ell+n)/2-1} e^{-y/2}}{\Gamma((2\ell+n)/2)}. \end{aligned}$$

So,

$$\begin{aligned} f_{Z_n}(z) &= f_{Y_n}(z^2)\cdot 2z = 2ze^{-(m^2+z^2)/2} \cdot \tfrac{1}{2} \sum_{\ell=0}^\infty \frac{(m^2/2)^\ell (z^2/2)^{\ell+n/2-1}}{\ell!\Gamma(\ell+n/2)} \\ &= ze^{-(m^2+z^2)/2} \sum_{\ell=0}^\infty \frac{(mz)^{2\ell+n/2-1} m^{-n/2+1} (1/2)^{2\ell+n/2-1} z^{n/2-1}}{\ell!\Gamma(\ell+n/2)} \\ &= \frac{z^{n/2}}{m^{n/2-1}} e^{-(m^2+z^2)} I_{n/2-1}(mz). \end{aligned}$$

(b) Obvious.

(c) Begin by writing

$$F_{Y_n}(y) = \mathsf{P}(Z_n^2 \le y) = \mathsf{P}(Z_n \le \sqrt{y}) = F_{Z_n}(y^{1/2}).$$

Then

$$\begin{aligned} f_{Y_n}(y) &= f_{Z_n}(y^{1/2}) \cdot y^{-1/2}/2 \\ &= \tfrac{1}{2} \frac{(y^{1/2})^{n/2}}{m^{n/2-1}} e^{-(m^2+y)/2} I_{n/2-1}(m\sqrt{y}) y^{-1/2} \\ &= \tfrac{1}{2} \left(\frac{\sqrt{y}}{m}\right)^{n/2-1} e^{-(m^2+y)/2} I_{n/2-1}(m\sqrt{y}). \end{aligned}$$

(d) First write

$$\begin{aligned} F^c_{Z_n}(z) &= \int_z^\infty \frac{t^{n/2}}{m^{n/2-1}} e^{-(m^2+t^2)/2} I_{n/2-1}(mt)\,dt \\ &= \int_z^\infty \frac{(mt)^{n/2}}{m^{n-1}} e^{-(m^2+t^2)/2} I_{n/2-1}(mt)\,dt. \end{aligned}$$

Now apply integration by parts with

$$u = (mt)^{n/2-1} I_{n/2-1}(mt) \quad \text{and} \quad dv = mte^{-t^2/2}e^{-m^2/2}/m^{n-1}\,dt.$$

Then

$$v = -me^{-m^2/2}e^{-t^2/2}/m^{n-1}, \text{ and by the hint, } du = (mt)^{n/2-1} I_{n/2-2}(mt)\cdot m\,dt.$$

Thus,

$$\begin{aligned} F^c_{Z_n}(z) &= -(mt)^{n/2-1} I_{n/2-1}(mt) \frac{e^{-(m^2+t^2)/2}}{m^{n-2}}\bigg|_z^\infty \\ &\quad + \int_z^\infty \frac{(mt)^{n/2-1}}{m^{n-3}} e^{-(m^2+t^2)/2} I_{n/2-2}(mt)\,dt \\ &= \left(\frac{z}{m}\right)^{n/2-1} e^{-(m^2+z^2)/2} I_{n/2-1}(mz) + F_{Z_{n-2}}(z). \end{aligned}$$

(e) Using induction, this is immediate from part (d).

(f) It is easy to see that $Q(0,z) = \widetilde{Q}(0,z) = e^{-z^2/2}$. We then turn to

$$\begin{aligned} \frac{\partial}{\partial m}\widetilde{Q}(m,z) &= \frac{\partial}{\partial m} e^{-(m^2+z^2)/2} \sum_{k=0}^\infty (m/z)^k I_k(mz) \\ &= e^{-(m^2+z^2)/2} \sum_{k=0}^\infty \left\{ -m(m/z)^k I_k(mz) + z^{-2k}(mz)^k I_{k-1}(mz)z \right\} \\ &= e^{-(m^2+z^2)/2} \sum_{k=0}^\infty \left\{ -m(m/z)^k I_k(mz) + (m/z)^k I_{k-1}(mz)z \right\} \\ &= ze^{-(m^2+z^2)/2} \sum_{k=0}^\infty (m/z)^k \left\{ I_{k-1}(mz) - (m/z) I_k(mz) \right\} \\ &= ze^{-(m^2+z^2)/2} \sum_{k=0}^\infty (m/z)^k I_{k-1}(mz) - (m/z)^{k+1} I_k(mz) \\ &= ze^{-(m^2+z^2)/2} I_{-1}(mz) \;=\; ze^{-(m^2+z^2)/2} I_1(mz). \end{aligned}$$

To conclude, we compute

$$\begin{aligned} \frac{\partial Q}{\partial m} &= \frac{\partial}{\partial m} \int_z^\infty te^{-(m^2+t^2)/2} I_0(mt)\,dt \\ &= \int_z^\infty -mte^{-(m^2+t^2)/2} I_0(mt)\,dt + \int_z^\infty te^{-(m^2+t^2)/2} I_{-1}(mt)\cdot t\,dt. \end{aligned}$$

Write this last integral as

$$\int_z^\infty (mt)I_1(mt)\cdot(t/m)e^{-(m^2+t^2)/2}\,dt.$$

Now apply integration by parts with $u=(mt)I_1(mt)$ and $dv=te^{-(t^2+m^2)/2}/m\,dt$. Then $du=(mt)I_0(mt)m\,dt$ and $v=-e^{-(m^2+t^2)/2}/m$, and the above integral is equal to

$$ze^{-(m^2+z^2)/2}I_1(mz)+\int_z^\infty mte^{-(m^2+t^2)/2}I_0(mt)\,dt.$$

Putting this all together, we find that $\partial Q/\partial m = ze^{-(m^2+z^2)/2}I_1(mz)$.

26. Recall that $I_\nu(x) := \displaystyle\sum_{\ell=0}^\infty \frac{(x/2)^{2\ell+\nu}}{\ell!\Gamma(\ell+\nu+1)}$.

(a) Write

$$\frac{I_\nu(x)}{(x/2)^\nu} = \frac{1}{\Gamma(\nu+1)}+\sum_{\ell=1}^\infty \frac{(x/2)^{2\ell}}{\ell!\Gamma(\ell+\nu+1)} \to \frac{1}{\Gamma(\nu+1)} \quad \text{as } x\to 0.$$

Now write

$$\begin{aligned} f_{Z_n}(z) &= \frac{z^{n/2}}{m^{n/2-1}}e^{-(m^2+z^2)/2}I_{n/2-1}(mz) \\ &= \frac{z^{n/2}}{m^{n/2-1}}e^{-(m^2+z^2)/2}\frac{I_{n/2-1}(mz)}{(mz/2)^{n/2-1}}(mz/2)^{n/2-1} \\ &= \frac{z^{n-1}}{2^{n/2-1}}e^{-(m^2+z^2)/2}\frac{I_{n/2-1}(mz)}{(mz/2)^{n/2-1}}. \end{aligned}$$

Thus,

$$\lim_{z\to 0} f_{Z_n}(z) = \frac{e^{-m^2/2}}{2^{n/2-1}\Gamma(n/2)}\lim_{z\to 0} z^{n-1} = \begin{cases} 0, & n>1, \\ e^{-m^2/2}\sqrt{2/\pi}, & n=1, \\ \infty, & 0<n<1. \end{cases}$$

(b) First note that

$$I_{\nu-1}(x) = \sum_{\ell=0}^\infty \frac{(x/2)^{2\ell+\nu-1}}{\ell!\Gamma(\ell+\nu)} = \sum_{\ell=0}^\infty \frac{(\ell+\nu)(x/2)^{2\ell+\nu-1}}{\ell!\Gamma(\ell+\nu+1)}.$$

Second, with the change of index $\ell=k+1$,

$$\begin{aligned} I_{\nu+1}(x) &= \sum_{k=0}^\infty \frac{(x/2)^{2k+\nu+1}}{k!\Gamma(k+\nu+2)} = \sum_{\ell=1}^\infty \frac{(x/2)^{2\ell+\nu-1}}{(\ell-1)!\Gamma(\ell+\nu+1)} \\ &= \sum_{\ell=1}^\infty \frac{\ell(x/2)^{2\ell+\nu-1}}{\ell!\Gamma(\ell+\nu+1)} = \sum_{\ell=0}^\infty \frac{\ell(x/2)^{2\ell+\nu-1}}{\ell!\Gamma(\ell+\nu+1)}. \end{aligned}$$

It is now easy to see that

$$I_{\nu-1}(x)+I_{\nu+1}(x) \;=\; \sum_{\ell=0}^{\infty}\frac{(2\ell+\nu)(x/2)^{2\ell+\nu-1}}{\ell!\Gamma(\ell+\nu+1)} \;=\; 2I_\nu'(x).$$

It is similarly easy to see that

$$I_{\nu-1}(x)-I_{\nu+1}(x) \;=\; \sum_{\ell=0}^{\infty}\frac{\nu(x/2)^{2\ell+\nu-1}}{\ell!\Gamma(\ell+\nu+1)} \;=\; 2(\nu/x)I_\nu(x).$$

(c) To the integral $\tilde{I}_n(x) := (2\pi)^{-1}\int_{-\pi}^{\pi} e^{x\cos\theta}\cos(n\theta)\,d\theta$, apply integration by parts with $u = e^{x\cos\theta}$ and $dv = \cos n\theta\,d\theta$. Then $du = e^{x\cos\theta}(-x\sin\theta)\,d\theta$ and $v = \sin(n\theta)/n$. We find that

$$\tilde{I}_n(x) \;=\; \frac{1}{2\pi}\bigg[\underbrace{e^{x\cos\theta}\frac{\sin n\theta}{n}\bigg|_{-\pi}^{\pi}}_{=\,0} + \frac{x}{n}\int_{-\pi}^{\pi} e^{x\cos\theta}\sin n\theta\sin\theta\,d\theta\bigg].$$

We next use the identity $\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$ to get

$$\tilde{I}_n(x) \;=\; \frac{x}{2n}[\tilde{I}_{n-1}(x)-\tilde{I}_{n+1}(x)].$$

(d) Since $\tilde{I}_0(x) := (1/\pi)\int_0^{\pi} e^{x\cos\theta}\,d\theta$, make the change of variable $t = \theta-\pi/2$. Then

$$\begin{aligned}
\tilde{I}_0(x) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{x\cos(t+\pi/2)}\,dt \;=\; \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{-x\sin t}\,dt \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sum_{k=0}^{\infty}\frac{(-x\sin t)^k}{k!}\,dt \;=\; \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\underbrace{\int_{-\pi/2}^{\pi/2}\sin^k t\,dt}_{=\,0\text{ for }k\text{ odd}} \\
&= \frac{1}{\pi}\sum_{\ell=0}^{\infty}\frac{x^{2\ell}}{(2\ell)!}\int_{-\pi/2}^{\pi/2}\sin^{2\ell} t\,dt \;=\; \frac{2}{\pi}\sum_{\ell=0}^{\infty}\frac{x^{2\ell}}{(2\ell)!}\int_0^{\pi/2}\sin^{2\ell} t\,dt \\
&= \frac{2}{\pi}\sum_{\ell=0}^{\infty}\frac{x^{2\ell}}{(2\ell)!}\cdot\frac{\Gamma\Big(\frac{2\ell+1}{2}\Big)\sqrt{\pi}}{2\,\Gamma\Big(\frac{2\ell+2}{2}\Big)}, \quad \text{by Problem 18 in Ch. 4.}
\end{aligned}$$

Now,

$$(2\ell)! \;=\; 1\cdot3\cdot5\cdots(2\ell-1)\cdot2\cdot4\cdot6\cdots2\ell \;=\; 1\cdot3\cdot5\cdots(2\ell-1)\cdot2^\ell\ell!,$$

and from Problem 14(c) in Chapter 4,

$$\Gamma\Big(\frac{2\ell+1}{2}\Big) \;=\; \frac{1\cdot3\cdot5\cdots(2\ell-1)}{2^\ell}\sqrt{\pi}.$$

Hence,

$$\tilde{I}_0(x) \;=\; \sum_{\ell=0}^{\infty}\frac{(x/2)^{2\ell}}{\ell!\Gamma(\ell+1)} \;=:\; I_0(x).$$

(e) Begin by writing

$$\begin{aligned} I_{n\pm 1}(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}\cos([n\pm 1]\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}[\cos n\theta\cos\theta \mp \sin n\theta\sin\theta]\,d\theta. \end{aligned}$$

Then

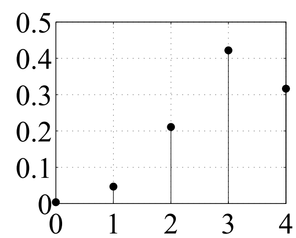
$$\tfrac{1}{2}[I_{n-1}(x)+I_{n+1}(x)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}\cos n\theta\cos\theta\,d\theta.$$

Since

$$I_n'(x) = \frac{\partial}{\partial x}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}\cos(n\theta)\,d\theta\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}\cos(n\theta)\cos\theta\,d\theta,$$

we see that $\frac{1}{2}[I_{n-1}(x)+I_{n+1}(x)] = I_n'(x)$.

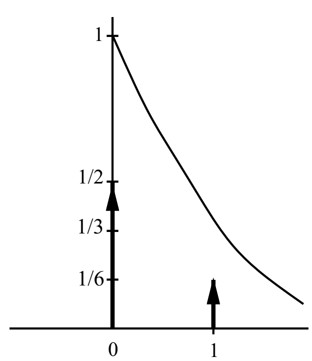
27. **MATLAB.**



28. See previous problem solution for graph. The probabilities are:

k	$\mathsf{P}(X=k)$
0	0.0039
1	0.0469
2	0.2109
3	0.4219
4	0.3164

29. (a) Sketch of $f(t)$:



(b) $\mathsf{P}(X=0)=\int_{\{0\}} f(t)\,dt = 1/2$, and $\mathsf{P}(X=1)=\int_{\{1\}} f(t)\,dt = 1/6$.

(c) We have

$$\begin{aligned}\mathsf{P}(0<X<1) &= \int_{0+}^{1-} f(t)\,dt = \int_{0+}^{1-} \tfrac{1}{3}e^{-t}u(t)+\tfrac{1}{2}\delta(t)+\tfrac{1}{6}\delta(t-1)\,dt \\ &= \int_{0+}^{1-} \tfrac{1}{3}e^{-t}\,dt = \int_0^1 \tfrac{1}{3}e^{-t}\,dt = -\tfrac{1}{3}e^{-t}\Big|_0^1 = \frac{1-e^{-1}}{3}\end{aligned}$$

and

$$\mathsf{P}(X>1) = \int_{1+}^{\infty} f(t)\,dt = \int_1^\infty \tfrac{1}{3}e^{-t}\,dt = -\tfrac{1}{3}e^{-t}\Big|_1^\infty = e^{-1}/3.$$

(d) Write

$$\begin{aligned}\mathsf{P}(0\le X\le 1) &= \mathsf{P}(X=0)+\mathsf{P}(0<X<1)+\mathsf{P}(X=1) \\ &= \frac{1}{2}+\frac{1-e^{-1}}{3}+\frac{1}{6} = 1-e^{-1}/3\end{aligned}$$

and

$$\mathsf{P}(X>1)=\mathsf{P}(X=1)+\mathsf{P}(X>1) = 1/6+e^{-1}/3 = \frac{1+2e^{-1}}{6}.$$

(e) Write

$$\begin{aligned}\mathsf{E}[X] &= \int_{-\infty}^{\infty} t f(t)\,dt = \int_0^\infty \tfrac{t}{3}e^{-t}\,dt+0\cdot\mathsf{P}(X=0)+1\cdot\mathsf{P}(X=1) \\ &= \tfrac{1}{3}\mathsf{E}[\text{exp RV w/}\lambda=1]+1/6 = 1/3+1/6 = 1/2.\end{aligned}$$

30. For the first part of the problem, we have $\mathsf{E}[e^X]=\int_{-\infty}^{\infty} e^x\cdot\frac{1}{2}[\delta(x)+I_{(0,1]}(x)]\,dx = \frac{1}{2}e^0+\frac{1}{2}\int_0^1 e^x\,dx = 1/2+e^x/2\Big|_0^1 = 1/2+(e-1)/2 = e/2$. For the second part, first write

$$\mathsf{P}(X=0|X\le 1/2) = \frac{\mathsf{P}(\{X=0\}\cap\{X\le 1/2\})}{\mathsf{P}(X\le 1/2)} = \frac{\mathsf{P}(X=0)}{\mathsf{P}(X\le 1/2)}.$$

Since $\mathsf{P}(X=0)=1/2$ and

$$\mathsf{P}(X\le 1/2)=\int_{-\infty}^{1/2}\tfrac{1}{2}[\delta(x)+I_{(0,1]}(x)]\,dx = \tfrac{1}{2}+\tfrac{1}{2}\int_0^{1/2} dx = \tfrac{1}{2}+\tfrac{1}{4}=\tfrac{3}{4},$$

we have $\mathsf{P}(X=0|X\le 1/2) = \frac{1/2}{3/4} = 2/3$.

31. The approach is to find the density and then compute $\mathsf{E}[X]=\int_{-\infty}^{\infty} x f_X(x)\,dx$. The catch is that the cdf has a jump at $x=1/2$, and so the density has an impulse there. Put

$$\tilde{f}_X(x) := \begin{cases} 2x, & 0<x<1/2, \\ 1, & 1/2<x<1, \\ 0, & \text{otherwise}.\end{cases}$$

Then the density is $f_X(x) = \tilde{f}_X(x) + \frac{1}{4}\delta(x-1/2)$. Hence,

$$\begin{aligned}
\mathsf{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{1/2} x\cdot 2x\,dx + \int_{1/2}^{1} x\cdot 1\,dx + \tfrac{1}{4}\cdot\tfrac{1}{2} \\
&= \tfrac{2}{3}\big(\tfrac{1}{2}\big)^3 + \tfrac{1}{2}[1-\big(\tfrac{1}{2}\big)^2] + \tfrac{1}{8} \\
&= \tfrac{1}{12} + \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{11}{24} + \tfrac{1}{8} = 7/12.
\end{aligned}$$

32. The approach is to find the density and then compute $\mathsf{E}[\sqrt{X}] = \int_{-\infty}^{\infty}\sqrt{x}\,f_X(x)\,dx$. The catch is that the cdf has a jump at $x=4$, and so the density has an impulse there. Put

$$\tilde{f}_X(x) := \begin{cases} x^{-1/2}/8, & 0<x<4, \\ 1/20, & 4<x<9, \\ 0, & \text{otherwise.}\end{cases}$$

Then the density is $f_X(x) = \tilde{f}_X(x) + \frac{1}{4}\delta(x-4)$. To begin, we compute

$$\begin{aligned}
\int_{-\infty}^{\infty} x^{1/2}\tilde{f}_X(x)\,dx &= \int_0^4 1/8\,dx + \int_4^9 x^{1/2}/20\,dx = 1/2 + x^{3/2}/30\Big|_4^9 \\
&= 1/2 + (27-8)/30 = 1/2 + 19/30 = 17/15.
\end{aligned}$$

The complete answer is

$$\mathsf{E}[\sqrt{X}] = 17/15 + \sqrt{4}/4 = 17/15 + 1/2 = 34/30 + 15/30 = 49/30.$$

33. First note that for $y<0$,

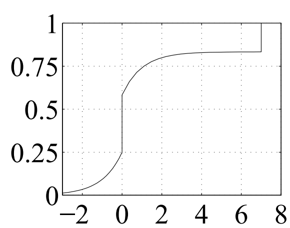
$$\int_{-\infty}^{y} e^{-|t|}\,dt = \int_{-\infty}^{y} e^t\,dt = e^y,$$

and for $y \ge 0$,

$$\int_{-\infty}^{y} e^{-|t|}\,dt = \int_{-\infty}^{0} e^t\,dt + \int_0^y e^{-t}\,dt = 1 + (-e^{-t})\Big|_0^y = 1 + 1 - e^{-y} = 2 - e^{-y}.$$

Hence,

$$\begin{aligned}
F_Y(y) &= \begin{cases} e^y/4, & y<0, \\ (2-e^{-y})/4 + 1/3, & 0\le y<7, \\ (2-e^{-y})/4 + 1/3 + 1/6, & y\ge 7,\end{cases} \\
&= \begin{cases} e^y/4, & y<0, \\ 5/6 - e^{-y}/4, & 0\le y<7, \\ 1-e^{-y}/4, & y \ge 7.\end{cases}
\end{aligned}$$



34. Let $\{Y=0\} = \{\text{loose connection}\}$, and $\{Y=1\} = \{Y=0\}^c$. Then $\mathsf{P}(Y=0) = \mathsf{P}(Y=1) = 1/2$. Using the law of total probability,

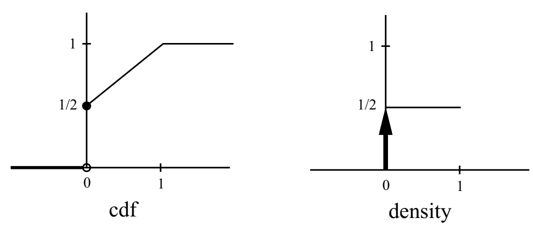
$$\begin{aligned} F_X(x) &= \mathsf{P}(X \le x) = \sum_{i=0}^{1} \mathsf{P}(X \le x | Y = i)\mathsf{P}(Y=i) \\ &= \tfrac{1}{2}\bigg[\mathsf{P}(X \le x|Y=0) + \int_{-\infty}^{x} I_{(0,1]}(t)\,dt\bigg]. \end{aligned}$$

Since $\mathsf{P}(X=0|Y=0)=1$, we see that

$$F_X(x) = \begin{cases} 1, & x \ge 1, \\ \frac{1}{2}(1+x), & 0 \le x < 1, \\ 0, & x < 0. \end{cases}$$

Since there is a jump at $x=0$, we must be careful in computing the density. It is

$$f_X(x) = \tfrac{1}{2}[I_{(0,1)}(x) + \delta(x)].$$



35. (a) Recall that as x varies from $+1$ to -1, $\cos^{-1}x$ varies from 0 to π. Hence, $F_X(x) = \mathsf{P}(\cos\Theta \le x) = \mathsf{P}\big(\Theta \in [-\pi, -\theta_x] \cup [\theta_x, \pi]\big)$, where $\theta_x := \cos^{-1}x$. Since $\Theta \sim \text{uniform}[-\pi,\pi]$,

$$F_X(x) = \frac{\pi - \theta_x}{2\pi} + \frac{-\theta_x - (-\pi)}{2\pi} = 2\frac{\pi-\theta_x}{2\pi} = 1 - \frac{\cos^{-1}x}{\pi}.$$

(b) Recall that as y varies from $+1$ to -1, $\sin^{-1}y$ varies from $\pi/2$ to $-\pi/2$. For $y \ge 0$,

$$\begin{aligned} F_Y(y) &= \mathsf{P}(\sin\Theta \le y) = \mathsf{P}\big(\Theta \in [-\pi, \theta_y] \cup [\pi - \theta_y, \pi]\big) \\ &= \frac{\theta_y}{2\pi} + \frac{\theta_y - (-\pi)}{2\pi} = \frac{\pi + 2\theta_y}{2\pi} = \frac{1}{2} + \frac{\sin^{-1}y}{\pi}, \end{aligned}$$

and for $y < 0$,

$$\begin{aligned} F_Y(y) &= \mathsf{P}(\sin\Theta \le y) = \mathsf{P}(\Theta \in [-\pi - \theta_y, \theta_y]) \\ &= \frac{2\theta_y + \pi}{2\pi} = \frac{1}{2} + \frac{\sin^{-1}y}{\pi}. \end{aligned}$$

(c) Write

$$f_X(x) = -\frac{1}{\pi} \cdot \frac{-1}{\sqrt{1-x^2}} = \frac{1/\pi}{\sqrt{1-x^2}},$$

and

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{1-y^2}} = \frac{1/\pi}{\sqrt{1-y^2}}.$$

(d) First write

$$F_Z(z) = \mathsf{P}(Z \le z) = \mathsf{P}\Big(\frac{Y+1}{2} \le z\Big) = \mathsf{P}(Y \le 2z-1) = F_Y(2z-1).$$

Then differentiate to get

$$\begin{aligned} f_Z(z) &= f_Y(2z-1)\cdot 2 = \frac{2}{\pi\sqrt{1-(2z-1)^2}} = \frac{1}{\sqrt{\pi}\sqrt{\pi}\sqrt{z(1-z)}} \\ &= \frac{\Gamma(\frac{1}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} z^{1/2-1}(1-z)^{1/2-1}, \end{aligned}$$

which is the beta density with $p = q = 1/2$.

36. The cdf is

$$\begin{aligned} F_Y(y) &= \begin{cases} 1, & y \ge 3, \\ \displaystyle\int_{-1-\sqrt{1+y}}^{-1+\sqrt{1+y}} 1/4\,dx, & -1 \le y < 3, \\ 0, & y < -1, \end{cases} \\ &= \begin{cases} 1, & y \ge 3, \\ \frac{1}{2}\sqrt{1+y}, & -1 \le y < 3, \\ 0, & y < -1, \end{cases} \end{aligned}$$

and the density is

$$f_Y(y) = \begin{cases} \dfrac{1}{4\sqrt{1+y}}, & -1 < y < 3, \\ 0, & \text{otherwise.} \end{cases}$$

37. The cdf is

$$F_Y(y) = \begin{cases} 1, & y \ge 2, \\ \dfrac{2(3-\sqrt{2/y})}{6}, & 1/2 \le y < 2, \\ 1/3, & 0 \le y < 1/2, \\ 0, & y < 0, \end{cases}$$

and the density is

$$f_Y(y) = \frac{\sqrt{2}}{6} y^{-3/2} I_{(1/2,2)}(y) + \frac{1}{3}\delta(y) + \frac{1}{3}\delta(y-2).$$

38. For $0 \le y \le 1$, we first compute the cdf

$$\begin{aligned} F_Y(y) &= \mathsf{P}(Y \le y) \;=\; \mathsf{P}(\sqrt{1-R^2} \le y) \;=\; \mathsf{P}(1-R^2 \le y^2) \;=\; \mathsf{P}(1-y^2 \le R^2) \\ &= \mathsf{P}(\sqrt{1-y^2} \le R) \;=\; \frac{\sqrt{2}-\sqrt{1-y^2}}{\sqrt{2}} \;=\; 1-\frac{1}{\sqrt{2}}(1-y^2)^{1/2}. \end{aligned}$$

We then differentiate to get the density

$$f_Y(y) = \begin{cases} \dfrac{y}{\sqrt{2(1-y^2)}}, & 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

39. The first thing to note is that for $0 \le R \le \sqrt{2}$, $0 \le R^2 \le 2$. It is then easy to see that the minimum value of $Z = [R^2(1-R^2/4)]^{-1}$ occurs when $R^2 = 2$ or $R = \sqrt{2}$. Hence, the random variable Z takes values in the range $[1,\infty)$. So, for $z \ge 1$, we write

$$\begin{aligned} F_Z(z) &= \mathsf{P}\biggl(\frac{1}{R^2(1-R^2/4)} \le z\biggr) \;=\; \mathsf{P}\Bigl(R^2(1-R^2/4) \ge 1/z\Bigr) \\ &= \mathsf{P}\Bigl((R^2/4)(1-R^2/4) \ge 1/(4z)\Bigr). \end{aligned}$$

Put $y := 1/(4z)$ and observe that $x(1-x) \ge y$ if and only if $0 \ge x^2 - x + y$, with equality if and only if

$$x = \frac{1 \pm \sqrt{1-4y}}{2}.$$

Since we will have $x = R^2/4 \le 1/2$, we need the negative root. Thus,

$$\begin{aligned} F_Z(z) &= \mathsf{P}\biggl((R^2/4) \ge \frac{1-\sqrt{1-4y}}{2}\biggr) \;=\; \mathsf{P}\Bigl(R^2 \ge 2[1-\sqrt{1-1/z}]\Bigr) \\ &= \mathsf{P}\Bigl(R \ge \sqrt{2[1-\sqrt{1-1/z}]}\Bigr) \;=\; \frac{\sqrt{2}-\sqrt{2[1-\sqrt{1-1/z}]}}{\sqrt{2}} \\ &= 1-\sqrt{1-\sqrt{1-1/z}}. \end{aligned}$$

Differentiating, we obtain

$$\begin{aligned} f_Z(z) &= -\frac{1}{2}\bigl(1-\sqrt{1-1/z}\bigr)^{-1/2}\frac{d}{dz}[1-\sqrt{1-1/z}] \\ &= -\frac{1}{2}\bigl(1-\sqrt{1-1/z}\bigr)^{-1/2}\cdot -\frac{1}{2}(1-1/z)^{-1/2}\cdot\frac{1}{z^2} \\ &= \frac{1}{4z^2}[(1-\sqrt{1-1/z})(1-1/z)]^{-1/2}. \end{aligned}$$

40. First note that as R varies from 0 to $\sqrt{2}$, T varies from π to $\sqrt{2}\,\pi$. For $\pi \le t \le \sqrt{2}\,\pi$, write

$$F_T(t) = \mathsf{P}(T \le t) \;=\; \mathsf{P}\biggl(\frac{\pi}{\sqrt{1-R^2/4}} \le t\biggr) \;=\; \mathsf{P}\biggl(\frac{\pi^2}{t^2} \le 1-R^2/4\biggr)$$

$$= \mathsf{P}(R^2/4 \le 1-\pi^2/t^2) \;=\; \mathsf{P}(R^2 \le 4[1-\pi^2/t^2])$$
$$= \mathsf{P}\Big(R \le 2\sqrt{1-\pi^2/t^2}\,\Big) \;=\; \sqrt{2}(1-\pi^2/t^2)^{1/2}.$$

Differentiating, we find that

$$f_T(t) \;=\; \frac{\pi^2\sqrt{2}}{t^3}(1-\pi^2/t^2)^{-1/2} \;=\; \frac{\pi^2\sqrt{2}}{t^2\sqrt{t^2-\pi^2}}.$$

For the second part of the problem, observe that as R varies between 0 and $\sqrt{2}$, M varies between 1 and $e^{-\pi}$. For m in this range, note that $\ln m < 0$, and write

$$\begin{aligned}
F_M(m) &= \mathsf{P}(M \le m) \;=\; \mathsf{P}(e^{-\pi(R/2)/\sqrt{1-R^2/4}} \le m) \\
&= \mathsf{P}\Big(\pi(R/2)\Big/\sqrt{1-R^2/4} \ge -\ln m\Big) \\
&= \mathsf{P}\Big(\pi^2(R^2/4)\Big/(1-R^2/4) \ge (-\ln m)^2\Big) \\
&= \mathsf{P}\Big((R^2/4) \ge 1/[1+\{\pi/(-\ln m)\}^2]\Big) \\
&= \mathsf{P}\Big(R \ge 2[1+\{\pi/(-\ln m)\}^2]^{-1/2}\Big) \\
&= 1-\sqrt{2}[1+\{\pi/(-\ln m)\}^2]^{-1/2},
\end{aligned}$$

Differentiating, we find that

$$f_M(m) \;=\; \frac{\sqrt{2}\pi^2}{m(-\ln m)^3}[1+\{\pi/(-\ln m)\}^2]^{-3/2} \;=\; \frac{\sqrt{2}\pi^2}{m[(-\ln m)^2+\pi^2]^{3/2}}.$$

41. (a) For $X \sim \text{uniform}[-1,1]$,

$$F_Y(y) \;=\; \begin{cases} 1, & y \ge 1, \\ (y+\sqrt{y})/2, & 0 \le y < 1, \\ 0, & y < 0, \end{cases}$$

and

$$f_Y(y) \;=\; \begin{cases} \frac{1}{2}[1+1/(2\sqrt{y})], & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) For $X \sim \text{uniform}[-1,2]$,

$$F_Y(y) \;=\; \begin{cases} 1, & y \ge 2, \\ (y+1)/3, & 1 \le y < 2, \\ (y+\sqrt{y})/3, & 0 \le y < 1, \\ 0, & y < 0, \end{cases}$$

and

$$f_Y(y) \;=\; \begin{cases} 1/3, & 1 < y < 2 \\ \frac{1}{3}[1+1/(2\sqrt{y})], & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) For $X \sim \text{uniform}[-2,3]$,

$$F_Y(y) = \begin{cases} 1, & y \geq 2, \\ (y+2)/5, & 1 \leq y < 2, \\ (y+\sqrt{y})/5, & 0 \leq y < 1, \\ 0, & y < 0, \end{cases}$$

and

$$f_Y(y) = \frac{1}{5}[I_{(1,2)}(y) + \{1 + 1/(2\sqrt{y})\}I_{(0,1)}(y) + \delta(y-1) + \delta(y-2)].$$

(d) For $X \sim \exp(\lambda)$,

$$F_Y(y) = \begin{cases} 1, & y \geq 2, \\ \mathsf{P}(X \leq y) + \mathsf{P}(X \geq 3), & 0 \leq y < 2, \\ 0, & y < 0, \end{cases}$$
$$= \begin{cases} 1, & y \geq 2, \\ (1 - e^{-\lambda y}) + e^{-3\lambda}, & 0 \leq y < 2, \\ 0, & y < 0, \end{cases}$$

and

$$f_Y(y) = \lambda e^{-\lambda y} I_{(0,2)}(y) + e^{-3\lambda}\delta(y) + (e^{-2\lambda} - e^{-3\lambda})\delta(y-2).$$

42. (a) If $X \sim \text{uniform}[-1,1]$, then $Y = g(X) = 0$, and so

$$F_Y(y) = u(y) \text{ (the unit step function)}, \quad \text{and} \quad f_Y(y) = \delta(y).$$

(b) If $X \sim \text{uniform}[-2,2]$,

$$F_Y(y) = \begin{cases} 1, & y \geq 1, \\ \frac{1}{2}(y+1), & 0 \leq y < 1, \\ 0, & y < 0, \end{cases}$$

and

$$f_Y(y) = \frac{1}{2}[I_{(0,1)}(y) + \delta(y)].$$

(c) We have

$$F_Y(y) = \begin{cases} 1, & y \geq 1, \\ (y+1)/3, & 0 \leq y < 1, \\ 0, & y < 0, \end{cases}$$

and

$$f_Y(y) = \tfrac{1}{3}[I_{(0,1)}(y) + \delta(y) + \delta(y-1)].$$

(d) If $X \sim \text{Laplace}(\lambda)$,

$$F_Y(y) = \begin{cases} 1, & y \geq 1, \\ 2\int_0^{y+1} \frac{\lambda}{2} e^{-\lambda x}\,dx, & 0 \leq y < 1, \\ 0, & y < 0, \end{cases}$$
$$= \begin{cases} 1, & y \geq 1, \\ 1 - e^{-\lambda(y+1)}, & 0 \leq y < 1, \\ 0, & y < 0, \end{cases}$$

and
$$f_Y(y) = \lambda e^{-\lambda(y+1)} I_{(0,1)}(y) + (1-e^{-\lambda})\delta(y) + e^{-2\lambda}\delta(y-1).$$

43. (a) If $X \sim \text{uniform}[-3,2]$,
$$F_Y(y) = \begin{cases} 1, & y \geq 1, \\ \frac{1}{5}(y^{1/3}+y+2), & 0 \leq y < 1, \\ \frac{1}{5}[y+2-(-y)^{1/2}], & -1 \leq y < 0, \\ 0, & y < -1, \end{cases}$$
and
$$f_Y(y) = \tfrac{1}{5}[1+1/(3y^{2/3})]I_{(0,1)}(y) + \tfrac{1}{5}[1+1/(2\sqrt{-y})]I_{(-1,0)}(y) + \tfrac{1}{5}\delta(y-1).$$

(b) If $X \sim \text{uniform}[-3,1]$,
$$F_Y(y) = \begin{cases} 1, & y \geq 1, \\ \frac{1}{4}(y^{1/3}+y+2), & 0 \leq y < 1, \\ \frac{1}{4}[y+2-(-y)^{1/2}], & -1 \leq y < 0, \\ 0, & y < -1, \end{cases}$$
and
$$f_Y(y) = \tfrac{1}{4}[1+1/(3y^{2/3})]I_{(0,1)}(y) + \tfrac{1}{4}[1+1/(2\sqrt{-y})]I_{(-1,0)}(y).$$

(c) If $X \sim \text{uniform}[-1,1]$,
$$F_Y(y) = \begin{cases} 1, & y \geq 1, \\ \frac{1}{2}(y^{1/3}+1), & 0 \leq y < 1, \\ \frac{1}{2}[1-(-y)^{1/2}], & -1 \leq y < 0, \\ 0, & y < -1, \end{cases}$$
and
$$f_Y(y) = \frac{1}{6y^{2/3}} I_{(0,1)}(y) + \frac{1}{4\sqrt{-y}} I_{(-1,0)}(y).$$

44. We have
$$F_Y(y) = \begin{cases} 0, & y < -1, \\ \frac{1}{6}[y+1+\sqrt{y+1}], & -1 \leq y < 1, \\ \frac{1}{6}[3+\sqrt{y+1}\,], & 1 \leq y < 8, \\ 1, & y \geq 8, \end{cases}$$
and
$$f_Y(y) = \tfrac{1}{6}\delta(y-1) + [\tfrac{1}{6} + \tfrac{1}{12}(y+1)^{-1/2}]I_{(-1,1)}(y) + \tfrac{1}{12}(y+1)^{-1/2}I_{(1,8)}(y).$$

45. We have
$$F_Y(y) = \begin{cases} 0, & y < 0, \\ (y^2+\sqrt{y}+y+1)/4, & 0 \leq y < 1, \\ 1, & y \geq 1, \end{cases}$$
and
$$f_Y(y) = \tfrac{1}{4}\delta(y) + \tfrac{1}{4}[2y+1/(2\sqrt{y})+1]I_{(0,1)}(y).$$

46. We have

$$F_Y(y) \;=\; \begin{cases} 1, & y \ge 1, \\ \frac{1}{6}[4+3y-y^2], & 0 \le y < 1, \\ \frac{1}{6}[3+2y-y^2], & -1 \le y < 0, \\ 0, & y < -1, \end{cases}$$

and

$$f_Y(y) \;=\; [\tfrac{1}{2} - \tfrac{1}{3}y]I_{(0,1)}(y) + \tfrac{1}{3}[1-y]I_{(-1,0)}(y) + \tfrac{1}{6}\delta(y).$$

47. We have

$$F_Y(y) \;=\; \begin{cases} 1, & y \ge 1, \\ \frac{1}{3}[1+y+\sqrt{y/(2-y)}], & 0 \le y < 1, \\ 0, & y < 0, \end{cases}$$

and

$$f_Y(y) \;=\; \tfrac{1}{3}\left[\delta(y) + 1 + \left(\frac{y}{2-y}\right)^{-1/2}\frac{1}{(2-y)^2}\right].$$

48. Observe that g is a periodic sawtooth function with period one. Also note that since $0 \le g(x) < 1$, $Y = g(X)$ takes values in $[0,1)$.

(a) We begin by writing

$$F_Y(y) = \mathsf{P}(Y \le y) = \mathsf{P}(g(X) \le y) = \sum_{k=0}^{\infty} \mathsf{P}(k \le X \le k+y) = \sum_{k=0}^{\infty} F_X(k+y) - F_X(k).$$

When $X \sim \exp(1)$, we obtain, for $0 \le y < 1$,

$$F_Y(y) \;=\; \sum_{k=0}^{\infty}(1-e^{-k}e^{-y}) - (1-e^{-k}) \;=\; \sum_{k=0}^{\infty} e^{-k}(1-e^{-y}) \;=\; \frac{1-e^{-y}}{1-e^{-1}}.$$

Differentiating, we get

$$f_Y(y) \;=\; \frac{e^{-y}}{1-e^{-1}}, \quad 0 \le y < 1.$$

We say that Y has a "truncated" exponential density.

(b) When $X \sim \text{uniform}[0,1)$, we obtain $Y = X \sim \text{uniform}[0,1)$.

(c) Suppose $X \sim \text{uniform}[\nu, \nu+\delta)$, where $\nu = m+\delta$ for some integer $m \ge 0$ and some $0 < \delta < 1$. For $0 \le y < \delta$

$$F_Y(y) \;=\; (m+1+y) - (m+1) \;=\; y,$$

and for $\delta \le y < 1$,

$$F_Y(y) \;=\; [(m+1+\delta) - (m+1)] + [(m+y) - (m+\delta)] \;=\; y.$$

Since $F_Y(y) = y$ for $0 \le y < 1$, $Y \sim \text{uniform}[0,1)$.

49. In the derivation of Property (vii) of cdfs, we started with the formula

$$(-\infty, x_0) \;=\; \bigcup_{n=1}^{\infty} (-\infty, x_0 - \tfrac{1}{n}].$$

However, we can also write

$$(-\infty, x_0) \;=\; \bigcup_{n=1}^{\infty} (-\infty, x_0 - \tfrac{1}{n}).$$

Hence,

$$G(x_0) = \mathsf{P}(X < x_0) \;=\; \mathsf{P}\bigg(\bigcup_{n=1}^{\infty} \{X < x_0 - \tfrac{1}{n}\}\bigg) = \lim_{N\to\infty} \mathsf{P}(X < x_0 - \tfrac{1}{N}) = \lim_{N\to\infty} G(x_0 - \tfrac{1}{N}).$$

Thus, G is left continuous. In a similar way, we can adapt the derivation of Property (vi) of cdfs to write

$$\mathsf{P}(X \le x_0) = \mathsf{P}\bigg(\bigcap_{n=1}^{\infty} \{X < x_0 + \tfrac{1}{n}\}\bigg) = \lim_{N\to\infty} \mathsf{P}(X < x_0 + \tfrac{1}{n}) = \lim_{N\to\infty} G(x_0 + \tfrac{1}{n}) = G(x_0+).$$

To conclude, write

$$\mathsf{P}(X = x_0) \;=\; \mathsf{P}(X \le x_0) - \mathsf{P}(X < x_0) \;=\; G(x_0+) - G(x_0).$$

50. First note that since $F_Y(t)$ is right continuous, so is $1 - F_Y(t) = \mathsf{P}(Y > t)$. Next, we use the assumption $\mathsf{P}(Y > t + \Delta t | T > t) = \mathsf{P}(Y > \Delta t)$ to show that with $h(t) := \mathsf{P}(Y > t)$, $h(t + \Delta t) = h(t) + h(\Delta t)$. To this end, write

$$\begin{aligned} h(\Delta t) \;&=\; \ln \mathsf{P}(Y > \Delta t) \;=\; \ln \mathsf{P}(Y > t + \Delta t | Y > t) \;=\; \ln \frac{\mathsf{P}(\{Y > t + \Delta t\} \cap \{Y > t\})}{\mathsf{P}(Y > t)} \\ &=\; \ln \frac{\mathsf{P}(Y > t + \Delta t)}{\mathsf{P}(Y > t)} \;=\; \ln \mathsf{P}(Y > t + \Delta t) - \ln \mathsf{P}(Y > t) \;=\; h(t + \Delta t) - h(t). \end{aligned}$$

Rewrite this result as $h(t + \Delta t) = h(t) + h(\Delta t)$. Then with $\Delta t = t$, we have $h(2t) = 2h(t)$. With $\Delta t = 2t$, we have $h(3t) = h(t) + h(2t) = h(t) + 2h(t) = 3h(t)$. In general, $h(nt) = nh(t)$. In a similar manner we can show that

$$h(t) \;=\; h\Big(\frac{t}{m} + \cdots + \frac{t}{m}\Big) \;=\; mh(t/m),$$

and so $h(t/m) = h(t)/m$. We now have that for rational $a = n/m$, $h(at) = h(n(t/m)) = nh(t/m) = (n/m)h(t) = ah(t)$. For general $a \ge 0$, let $a_k \downarrow a$ with a_k rational. Then by the right continuity of h,

$$h(at) \;=\; \lim_{k\to\infty} h(a_k t) \;=\; \lim_{k\to\infty} a_k h(t) \;=\; ah(t).$$

We can now write

$$h(t) \;=\; h(t \cdot 1) \;=\; th(1).$$

Thus,
$$t \cdot h(1) \;=\; h(t) \;=\; \ln \mathsf{P}(Y > t) \;=\; \ln(1 - F_Y(t)),$$
and we have $1 - F_Y(t) = e^{h(1)t}$, which implies $Y \sim \exp(-h(1))$. Of couse, $-h(1) = -\ln \mathsf{P}(Y > 1) = -\ln[1 - F_Y(1)]$.

51. We begin with

$$\mathsf{E}[Y_n] \;=\; \mathsf{E}\biggl[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\biggl(\frac{X_i - m}{\sigma}\biggr)\biggr] \;=\; \frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n}\bigl(\mathsf{E}[X_i] - m\bigr) \;=\; 0.$$

For the variance, we use the fact that since independent random variables are uncorrelated, the variance of the sum is the sum of the variances. Thus,

$$\mathsf{var}(Y_n) \;=\; \sum_{i=1}^{n}\mathsf{var}\biggl(\frac{X_i - m}{\sigma\sqrt{n}}\biggr) \;=\; \sum_{i=1}^{n}\frac{\mathsf{var}(X_i)}{n\sigma^2} \;=\; \sum_{i=1}^{n}\frac{\sigma^2}{n\sigma^2} \;=\; 1.$$

52. Let X_i denote the time to transmit the ith packet, where X_i has mean m and variance σ^2. The total time to transmit n packets is $T_n := X_1 + \cdots + X_n$. The expected total time is $\mathsf{E}[T_n] = nm$. Since we do not know the distribution of the X_i, we cannot know the distribution of T_n. However, we use the central limit theorem to approximate $\mathsf{P}(T_n > 2nm)$. Note that the sample mean $M_n = T_n/n$. Write

$$\begin{aligned}
\mathsf{P}(T_n > 2nm) \;&=\; \mathsf{P}(\tfrac{1}{n}T_n > 2m) \;=\; \mathsf{P}(M_n > 2m) \;=\; \mathsf{P}(M_n - m > m) \\
&=\; \mathsf{P}\biggl(\frac{M_n - m}{\sigma/\sqrt{n}} > \frac{m}{\sigma/\sqrt{n}}\biggr) \;=\; \mathsf{P}\biggl(Y_n > \frac{m}{\sigma/\sqrt{n}}\biggr) \\
&=\; 1 - F_{Y_n}(m\sqrt{n}/\sigma) \;\approx\; 1 - \Phi(m\sqrt{n}/\sigma),
\end{aligned}$$

by the central limit theorem.

53. Let $X_i = 1$ if bit i is in error, and $X_i = 0$ otherwise. Then $\mathsf{P}(X_i = 1) = p$. Although the problem does not say so, let us assume that the X_i are independent. Then $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the fraction of bits in error. We cannot reliably decode of $M_n > t$. To approximate the probability that we cannot reliably decode, write

$$\begin{aligned}
\mathsf{P}(M_n > t) \;&=\; \mathsf{P}\biggl(\frac{M_n - m}{\sigma/\sqrt{n}} > \frac{t - m}{\sigma/\sqrt{n}}\biggr) \;=\; \mathsf{P}\biggl(Y_n > \frac{t - m}{\sigma/\sqrt{n}}\biggr) \;=\; 1 - F_{Y_n}\biggl(\frac{t - m}{\sigma/\sqrt{n}}\biggr) \\
&\approx\; 1 - \Phi\biggl(\frac{t - m}{\sigma/\sqrt{n}}\biggr) \;=\; 1 - \Phi\biggl(\frac{t - p}{\sqrt{p(1-p)/n}}\biggr),
\end{aligned}$$

since $m = \mathsf{E}[X_i] = p$ and $\sigma^2 = \mathsf{var}(X_i) = p(1-p)$.

54. If the X_i are i.i.d. Poisson(1), then $T_n := X_1 + \cdots + X_n$ is Poisson(n). Thus,

$$\mathsf{P}(T_n = k) \;=\; \frac{n^k e^{-n}}{k!} \;\approx\; \frac{1}{\sqrt{2\pi}}\exp\biggl[-\frac{1}{2}\biggl(\frac{k - n\cdot 1}{1\cdot\sqrt{n}}\biggr)^2\biggr]\frac{1}{1\cdot\sqrt{n}}.$$

Taking $k = n$, we obtain $n^n e^{-n} \approx n!/\sqrt{2\pi n}$ or $n! \approx \sqrt{2\pi}\, n^{n+1/2}e^{-n}$.

55. Recall that g_n is the density of $X_1 + \cdots + X_n$. If the X_i are i.i.d. uniform$[-1,1]$, then g_n is the convolution of $(1/2)I_{[-1,1]}(x)$ with itself n times. From graphical considerations, it is clear that $g_n(x) = 0$ for $|x| > n$; i.e., $x_{\max} = n$.

56. To begin, write

$$\varphi_{Y_n}(\nu) \;=\; \mathsf{E}\big[e^{j\nu(X_1+\cdots+X_n)/\sqrt{n}}\big] \;=\; \mathsf{E}\bigg[\prod_{i=1}^{n} e^{j(\nu/\sqrt{n}\,)X_i}\bigg] \;=\; \prod_{i=1}^{n}\Big[\tfrac{1}{2}e^{-j\nu/\sqrt{n}} + \tfrac{1}{2}e^{j\nu/\sqrt{n}}\Big]$$
$$=\; \cos^n(\nu/\sqrt{n}\,) \;\approx\; \Big(1 - \frac{\nu^2/2}{n}\Big)^n \;\to\; e^{-\nu^2/2}.$$

57. (a) MTTF $= \mathsf{E}[T] = n$ (from Erlang in table).

(b) The desired probability is

$$R(t) \;:=\; \mathsf{P}(T > t) \;=\; \int_t^\infty \frac{\tau^{n-1}\,e^{-\tau}}{(n-1)!}\,d\tau.$$

Let $P_n(t)$ denote the above integral. Then $P_1(t) = \int_t^\infty e^{-\tau}\,d\tau = e^{-t}$. For $n > 1$, apply integration by parts with $u = \tau^{n-1}/(n-1)!$ and $dv = e^{-\tau}\,d\tau$. Then

$$P_n(t) \;=\; \frac{t^{n-1}\,e^{-t}}{(n-1)!} + P_{n-1}(t).$$

Applying this result recursively, we find that

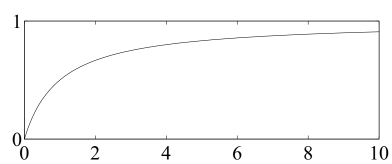
$$P_n(t) \;=\; \frac{t^{n-1}\,e^{-t}}{(n-1)!} + \frac{t^{n-2}\,e^{-t}}{(n-2)!} + \cdots + e^{-t},$$

which is the desired result.

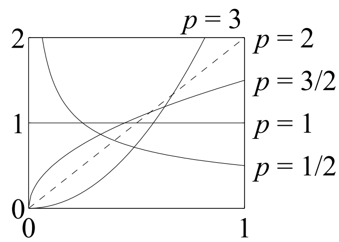
(c) The failure rate is

$$r(t) \;=\; \frac{f_T(t)}{R(t)} \;=\; \frac{t^{n-1}e^{-t}/(n-1)!}{\displaystyle\sum_{k=0}^{n-1}\frac{t^k}{k!}e^{-t}} \;=\; \frac{t^{n-1}}{\displaystyle(n-1)!\sum_{k=0}^{n-1}\frac{t^k}{k!}}.$$

For $n = 2$, $r(t) = t/(1+t)$:



58. (a) Let $\lambda = 1$. For $p = 1/2$, $r(t) = 1/(2\sqrt{2})$. For $p = 1$, $r(t) = 1$. For $p = 3/2$, $r(t) = 3\sqrt{t}/2$. For $p = 2$, $r(t) = 2t$. For $p = 3$, $r(t) = 3t^2$.



(b) We have from the text that

$$R(t) = \exp\left[-\int_0^t r(\tau)\,d\tau\right] = \exp\left[-\int_0^t \lambda p \tau^{p-1}\,d\tau\right] = e^{-\lambda t^p}.$$

(c) The MTTF is

$$\mathsf{E}[T] = \int_0^\infty R(t)\,dt = \int_0^\infty e^{-\lambda t^p}\,dt.$$

Now make the change of variable $\theta = \lambda t^p$, or $t = (\theta/\lambda)^{1/p}$. Then

$$\begin{aligned}\mathsf{E}[T] &= \int_0^\infty e^{-\theta}\left(\frac{\theta}{\lambda}\right)^{1/p-1}\frac{d\theta}{\lambda p} = \frac{1}{p\lambda^{1/p}}\int_0^\infty \theta^{1/p-1}e^{-\theta}\,d\theta \\ &= \frac{1}{p\lambda^{1/p}}\Gamma(1/p) = \frac{\Gamma(1/p+1)}{\lambda^{1/p}}.\end{aligned}$$

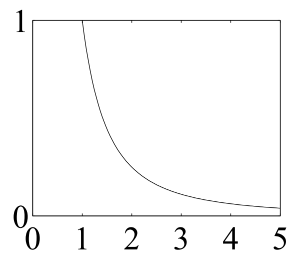
(d) Using the result of part (b),

$$f_T(t) = -R'(t) = \lambda t^{p-1}e^{-\lambda t^p}, \quad t > 0.$$

59. (a) Write

$$\begin{aligned}R(t) &= \exp\left[-\int_0^\infty r(\tau)\,d\tau\right] = \exp\left[-\int_{t_0}^t p\tau^{-1}\,d\tau\right] \\ &= \exp[-p(\ln t - \ln t_0)] = \exp[\ln(t_0/t)^p] = (t_0/t)^p, \quad t \ge t_0.\end{aligned}$$

(b) If $t_0 = 1$ and $p = 2$, $R(t) = 1/t^2$ has the form



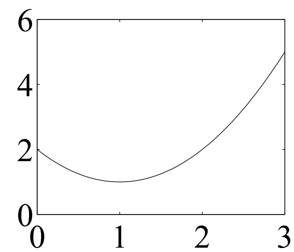
(c) For $p > 1$, the MTTF is

$$\mathsf{E}[T] = \int_0^\infty R(t)\,dt = \int_{t_0}^\infty (t_0/t)^p\,dt = t_0^p\left(\frac{t^{1-p}}{1-p}\right)\Big|_{t_0}^\infty = \frac{t_0}{p-1}.$$

(d) For the density, write

$$f_T(t) \;=\; r(t)R(t) \;=\; \frac{p}{t}\cdot\left(\frac{t_0}{t}\right)^p \;=\; \frac{pt_0^p}{t^{p+1}}, \quad t \ge t_0.$$

60. (a) A sketch of $r(t) = t^2 - 2t + 2$ for $t \ge 0$ is:



(b) We first compute

$$\int_0^t r(\tau)\,d\tau \;=\; \int_0^t \tau^2 - 2\tau + 2\,d\tau \;=\; \tfrac{1}{3}t^3 - t^2 + 2t.$$

Then

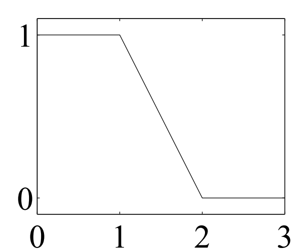
$$f_T(t) \;=\; r(t)e^{-\int_0^t r(\tau)\,d\tau} \;=\; [t^2 - 2t + 2]e^{-(\frac{1}{3}t^3 - t^2 + 2t)}, \quad t \ge 0.$$

61. (a) If $T \sim$ uniform$[1,2]$, then for $0 \le t < 1$, $R(t) = \mathsf{P}(T > t) = 1$, and for $t \ge 2$, $R(t) = \mathsf{P}(T > t) = 0$. For $1 \le t < 2$,

$$R(t) \;=\; \mathsf{P}(T > t) \;=\; \int_t^2 1\,d\tau \;=\; 2 - t.$$

The complete formula and sketch are

$$R(t) \;=\; \begin{cases} 1, & 0 \le t < 1, \\ 2-t, & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}$$



(b) The failure rate is

$$r(t) \;=\; -\frac{d}{dt}\ln R(t) \;=\; -\frac{d}{dt}\ln(2-t) \;=\; \frac{1}{2-t}, \quad 1 < t < 2.$$

(c) Since $T \sim \text{uniform}[1,2]$, the MTTF is $\mathsf{E}[T] = 1.5$.

62. Write

$$R(t) \;:=\; \mathsf{P}(T > t) \;=\; \mathsf{P}(T_1 > t, T_2 > t) \;=\; \mathsf{P}(T_1 > t)\mathsf{P}(T_2 > t) \;=\; R_1(t)R_2(t).$$

63. Write

$$\begin{aligned} R(t) \;&:=\; \mathsf{P}(T > t) \;=\; \mathsf{P}(\{T_1 > t\} \cup \{T_2 > t\}) \;=\; 1 - \mathsf{P}(T_1 \le t, T_2 \le t) \\ &=\; 1 - \mathsf{P}(T_1 \le t)\mathsf{P}(T_2 \le t) \;=\; 1 - [1 - R_1(t)][1 - R_2(t)] \\ &=\; 1 - [1 - R_1(t) - R_2(t) + R_1(t)R_2(t)] \\ &=\; R_1(t) + R_2(t) - R_1(t)R_2(t). \end{aligned}$$

64. We follow the hint and write

$$\mathsf{E}[Y^n] \;=\; \mathsf{E}[T] \;=\; \int_0^\infty \mathsf{P}(T > t)\,dt \;=\; \int_0^\infty \mathsf{P}(Y^n > t)\,dt \;=\; \int_0^\infty \mathsf{P}(Y > t^{1/n})\,dt.$$

We then make the change of variable $y = t^{1/n}$, or $t = y^n$, $dt = ny^{n-1}\,dy$, to get

$$\mathsf{E}[Y^n] \;=\; \int_0^\infty \mathsf{P}(Y > y) \cdot ny^{n-1}\,dy.$$

CHAPTER 6
Problem Solutions

1. Since the X_i are uncorrelated with common mean m and common variance σ^2,

$$\begin{aligned}
\mathsf{E}[S_n^2] &= \frac{1}{n-1}\left(\mathsf{E}\left[\sum_{i=1}^n X_i^2\right] - n\mathsf{E}[M_n]\right) \\
&= \frac{1}{n-1}\left[n(\sigma^2+m^2) - n\Big\{\mathsf{var}(M_n) + (\mathsf{E}[M_n])^2\Big\}\right] \\
&= \frac{1}{n-1}\left[n(\sigma^2+m^2) - n\{\sigma^2/n+m^2\}\right] \\
&= \frac{1}{n-1}\left[(n-1)\sigma^2+nm^2-nm^2\right] \;=\; \sigma^2.
\end{aligned}$$

2. (a) The mean of a Rayleigh(λ) random variable is $\lambda\sqrt{\pi/2}$. Consider

$$\lambda_n := M_n/\sqrt{\pi/2}.$$

Then

$$\mathsf{E}[\lambda_n] = \mathsf{E}[M_n/\sqrt{\pi/2}\,] = \mathsf{E}[M_n]/\sqrt{\pi/2} = \lambda\sqrt{\pi/2}/\sqrt{\pi/2} = \lambda.$$

Thus, λ_n is unbiased. Next,

$$\lambda_n = M_n/\sqrt{\pi/2} \to \mathsf{E}[M_n]/\sqrt{\pi/2} = \lambda\sqrt{\pi/2}/\sqrt{\pi/2} = \lambda,$$

and we see that λ_n is strongly consistent.

(b) **MATLAB.** Add the line of code `lambdan=mean(X)/sqrt(pi/2)`.

(c) **MATLAB.** Since $M_n \approx \lambda\sqrt{\pi/2}$, we solve for π and put

$$\pi_n := 2(M_n/\lambda)^2.$$

Since $\lambda = 3$, add the line of code `pin=2*(mean(X)/3)^2`.

3. (a) The mean of a gamma(p,λ) random variable is p/λ. We put

$$p_n := \lambda M_n.$$

Then $\mathsf{E}[p_n] = \lambda\mathsf{E}[M_n] = \lambda\cdot p/\lambda = p$. Also $p_n = \lambda M_n \to \lambda(p/\lambda) = p$. Thus, p_n is unbiased and strongly consistent.

(b) **MATLAB.** In this problem $\lambda = 1/2$ and $p = k/2$, or $k = 2p$. We use $k_n := 2p_n = 2(\lambda M_n) = 2((1/2)M_n) = M_n$. We therefore add the line of code `kn=mean(X)`.

4. (a) Since the mean of a noncentral chi-squared random variable with k degrees of freedom and noncentrality parameter λ^2 is $k+\lambda^2$, we put

$$\lambda_n^2 := M_n - k.$$

Then $\mathsf{E}[\lambda_n^2] = \mathsf{E}[M_n - k] = \mathsf{E}[M_n] - k = (k+\lambda^2) - k = \lambda^2$, and we see that λ_n^2 is an unbiased estimator of λ^2. Next, since $\lambda_n^2 = M_n - k \to \mathsf{E}[M_n] - k = (k+\lambda^2) - k = \lambda^2$, the estimator is strongly consistent.

(b) **MATLAB.** Since $k = 5$, add the line of code `lambda2n=mean(X)-5`.

5. (a) Since the mean of a gamma(p,λ) random variable is p/λ, we put $\lambda_n := p/M_n$. Then $\lambda_n = p/M_n \to p/\mathsf{E}[M_n] = p/(p/\lambda) = \lambda$, and we see that λ_n is a strongly consistent estimator of λ.

(b) **MATLAB.** Since $p = 3$, add the line of code `lambdan=3/mean(X)`.

6. (a) Since the variance of a Laplace(λ) random variable is $2/\lambda^2$, we put

$$\lambda_n := \sqrt{2/S_n^2}.$$

Since S_n^2 converges to the variance, we have $\lambda_n \to \sqrt{2/(2/\lambda^2)} = \lambda$, and we see that λ_n is a strongly consistent estimator of λ.

(b) **MATLAB.** Add the line of code `lambdan=sqrt(2/var(X))`.

7. (a) The mean of a gamma(p,λ) random variable is p/λ. The second moment is $p(p+1)/\lambda^2$. Hence, the variance is

$$\frac{p(p+1)}{\lambda^2} - \frac{p^2}{\lambda^2} = \frac{p}{\lambda^2}.$$

Thus, $M_n \approx p/\lambda$ and $S_n^2 \approx p/\lambda^2$. Solving for p and λ suggests that we put

$$\lambda_n := \frac{M_n}{S_n^2} \quad \text{and} \quad p_n := \lambda_n M_n.$$

Now, $M_n \to p/\lambda$ and $S_n^2 \to p/\lambda^2$. It follows that $\lambda_n \to (p/\lambda)\big/(p/\lambda^2) = \lambda$ and then $p_n \to \lambda \cdot (p/\lambda) = p$. Hence, λ_n is a strongly consistent estimator of λ, and p_n is a strongly consistent estimator of p.

(b) **MATLAB.** Add the code

```
Mn = mean(X)
lambdan = Mn/var(X)
pn = lambdan*Mn
```

8. Using results from the problem referenced in the hint, we have

$$\mathsf{E}[X^q] = \frac{\Gamma\big((q+p)/q\big)}{\Gamma(p/q)\lambda^q} = \frac{\Gamma(1+p/q)}{\Gamma(p/q)\lambda^q} = \frac{p/q}{\lambda^q}.$$

This suggests that we put

$$\lambda_n := \left(\frac{p/q}{\frac{1}{n}\sum_{i=1}^n X_i^q} \right)^{1/q}.$$

Then

$$\lambda_n \to \left(\frac{p/q}{(p/q)/\lambda^q} \right)^{1/q} = \lambda,$$

and we see that λ_n is a strongly consistent estimator of λ.

9. In the preceding problem $\mathsf{E}[X^q] = (p/q)/\lambda^q$. Now consider

$$\mathsf{E}[X^{2q}] = \frac{\Gamma\big((2q+p)/q\big)}{\Gamma(p/q)\lambda^{2q}} = \frac{\Gamma(2+p/q)}{\Gamma(p/q)\lambda^{2q}} = \frac{(1+p/q)(p/q)}{\lambda^{2q}}.$$

In this equation, replace λ^q by $(p/q)/\mathsf{E}[X^q]$ and solve for (p/q). Thus,

$$p/q = \frac{(\mathsf{E}[X^q])^2}{\mathsf{var}(X^q)}.$$

This suggests that we first put

$$\overline{X_n^q} := \frac{1}{n}\sum_{i=1}^n X_i^q$$

and then

$$p_n := q\frac{(\overline{X_n^q})^2}{\frac{1}{n-1}\Big[\Big(\sum_{i=1}^n (X_i^q)^2\Big) - n(\overline{X_n^q})^2\Big]}$$

and

$$\lambda_n := \left(\frac{p_n/q}{\overline{X_n^q}}\right)^{1/q} = \left(\frac{\overline{X_n^q}}{\frac{1}{n-1}\Big[\Big(\sum_{i=1}^n (X_i^q)^2\Big) - n(\overline{X_n^q})^2\Big]}\right)^{1/q}.$$

10. **MATLAB.** OMITTED.

11. **MATLAB.** The required script can be created using the code from Problem 2 followed by the lines

```
global lambdan
lambdan = mean(X)/sqrt(pi/2)
```

followed by the script from Problem 10 modifed as follows: The chi-squared statistic Z can be computed by inserting the lines

```
p = CDF(b) - CDF(a);
Z = sum((H-n*p).^2./(n*p))
```

after the creation of the right edge sequence in the script given in Problem 10, where `CDF` is the function

```
function y = CDF(t) % Rayleigh CDF
global lambdan
y = zeros(size(t));
i = find(t>0);
y(i) = 1-exp(-(t(i)/lambdan).^2/2);
```

In addition, the line defining `y` in the script from Problem 10 should be changed to `y=PDF(t)`, where `PDF` is the function

```
function y = PDF(t) % Rayleigh density
global lambdan
y = zeros(size(t));
i = find(t>0);
y(i) = (t(i)/lambdan^2).*exp(-(t(i)/lambdan).^2/2);
```

Finally, the chi-squared statistic Z should be compared with $z_\alpha = 22.362$, since $\alpha = 0.05$ and since there are $m = 15$ bins and $r = 1$ estimated parameter, the degrees of freedom parameter is $k = m-1-r = 15-1-1 = 13$ in the chi-squared table in the text.

12. **MATLAB.** Similar to the solution of Problem 11 except that it is easier to use the MATLAB function `chi2cdf` or `gamcdf` to compute the required cdfs for evaluating the chi-squared statistic Z. For the same reasons as in Problem 11, $z_\alpha = 22.362$.

13. **MATLAB.** Similar to the solution of Problem 11 except that it is easier to use the MATLAB function `ncx2cdf` to compute the required cdfs for evaluating the chi-squared statistic Z. For the same reasons as in Problem 11, $z_\alpha = 22.362$.

14. **MATLAB.** Similar to the solution of Problem 11 except that it is easier to use the MATLAB function `gamcdf` to compute the required cdfs for evaluating the chi-squared statistic Z. For the same reasons as in Problem 11, $z_\alpha = 22.362$.

15. **MATLAB.** Similar to the solution of Problem 11. For the same reasons as in Problem 11, $z_\alpha = 22.362$.

16. Since

$$\mathsf{E}[H_j] = \mathsf{E}\left[\sum_{i=1}^n I_{[e_j,e_{j+1})}(X_i)\right] = \sum_{i=1}^n \mathsf{P}(e_j \le X_i < e_{j+1}) = \sum_{i=1}^n p_j = np_j,$$

$$\mathsf{E}\left[\frac{H_j - np_j}{\sqrt{np_j}}\right] = 0.$$

Since the X_i are i.i.d., the $I_{[e_j,e_{j+1})}(X_i)$ are i.i.d. Bernoulli(p_j). Hence,

$$\begin{aligned}\mathsf{E}[(H_j - np_j)^2] = \mathsf{var}(H_j) &= \mathsf{var}\left(\sum_{i=1}^n I_{[e_j,e_{j+1})}(X_i)\right)\\ &= \sum_{i=1}^n \mathsf{var}\Big(I_{[e_j,e_{j+1})}(X_i)\Big) = n\cdot p_j(1-p_j),\end{aligned}$$

and so

$$\mathsf{E}\left[\left(\frac{H_j - np_j}{\sqrt{np_j}}\right)^2\right] = 1 - p_j.$$

17. If f is an even density, then

$$F(-x) = \int_{-\infty}^{-x} f(t)\,dt = -\int_{\infty}^{x} f(-\theta)\,d\theta = \int_{x}^{\infty} f(\theta)\,d\theta = 1 - F(x).$$

18. The width of any confidence interval is $w = 2\sigma y/\sqrt{n}$. If $\sigma = 2$ and $n = 100$,

$$w_{99\%} = \frac{2 \cdot 2 \cdot 2.576}{10} = 1.03.$$

To make $w_{99\%} < 1/4$ requires

$$\frac{2\sigma y}{\sqrt{n}} < 1/4 \quad \text{or} \quad n > (8\sigma y)^2 = (16 \cdot 2.576)^2 = 1699.$$

19. First observe that with $X_i = m + W_i$, $\mathsf{E}[X_i] = m$, and $\mathsf{var}(X_i) = \mathsf{var}(W_i) = 4$. So, $\sigma^2 = 4$. For 95% confidence interval, $\sigma y_{\alpha/2}/\sqrt{n} = 2 \cdot 1.960/10 = 0.392$, and so

$$m = 14.846 \pm 0.392 \quad \text{with 95\% probability.}$$

The corresponding confidence interval is $[14.454, 15.238]$.

20. Write

$$\begin{aligned}
\mathsf{P}(|M_n - m| \le \delta) &= \mathsf{P}(-\delta \le M_n - m \le \delta) = \mathsf{P}\left(-\delta \le \frac{1}{n}\sum_{i=1}^{n}(m + W_i) - m \le \delta\right) \\
&= \mathsf{P}\left(-\delta \le \frac{1}{n}\sum_{i=1}^{n} W_i \le \delta\right) = \mathsf{P}\Bigg(-n\delta \le \underbrace{\sum_{i=1}^{n} W_i}_{\text{Cauchy}(n)} \le n\delta\Bigg) \\
&= \frac{2}{\pi}\tan^{-1}(n\delta/n),
\end{aligned}$$

which is equal to $2/3$ if and only if $\tan^{-1}(\delta) = \pi/3$, or $\delta = \sqrt{3}$.

21. **Matlab.** OMITTED.

22. We use the formula $m = M_n \pm y_{\alpha/2} S_n/\sqrt{n} = 10.083 \pm (1.960)(0.568)/10$ to get

$$m = 10.083 \pm 0.111 \quad \text{with 95\% probability,}$$

and the confidence intervale is $[9.972, 10.194]$.

23. We use the formula $m = M_n \pm y_{\alpha/2} S_n/\sqrt{n} = 4.422 \pm (1.812)(0.957)/10$ to get

$$m = 4.422 \pm 0.173 \quad \text{with 93\% probability,}$$

and the confidence interval is $[4.249, 4.595]$.

24. We have $M_n = \text{number defective}/n = 10/100 = 0.1$. We use the formula $m = M_n \pm y_{\alpha/2}S_n/\sqrt{n} = 0.1 \pm (1.645)(.302)/10$ to get

$$m = 0.1 \pm 0.0497 \quad \text{with 90\% probability.}$$

The number of defectives is $10\,000m$, or

$$\text{number of defectives} = 1000 \pm 497 \quad \text{with 90\% probability.}$$

Thus, we are 90% sure that the number of defectives is between 503 and 1497 out of a total of 10 000 units.

25. We have $M_n = 1559/3000$. We use the formula $m = M_n \pm y_{\alpha/2}S_n/\sqrt{n} = 0.520 \pm (1.645)(.5)/\sqrt{3000}$ to get

$$m = 0.520 \pm 0.015 \quad \text{with 90\% probability,}$$

and the confidence interval is $[0.505, 0.535]$. Hence, the probability is at least 90% that more than 50.5% of the voters will vote for candiate A. So we are 90% sure that candidate A will win. The 99% confidence interval is given by

$$m = 0.520 \pm 0.024 \quad \text{with 99\% probability,}$$

and the confidence interval is $[0.496, 0.544]$. Hence, we are *not* 99% sure that candidate A will win.

26. We have $M_n = \text{number defective}/n = 48/500 = 0.0960$. We use the formula $m = M_n \pm y_{\alpha/2}S_n/\sqrt{n} = 0.0960 \pm (1.881)(.295)/\sqrt{500}$ to get

$$m = 0.0960 \pm 0.02482 \quad \text{with 94\% probability.}$$

The number of defectives is $100\,000m$, or

$$\text{number of defectives} = 9600 \pm 2482 \quad \text{with 94\% probability.}$$

Thus, we are 94% sure that the number of defectives is between 7118 and 12 082 out of a total of 100 000 units.

27. (a) We have $M_n = 6/100$. We use the formula $m = M_n \pm y_{\alpha/2}S_n/\sqrt{n} = 0.06 \pm (2.170)(.239)/10$ to get

$$m = 0.06 \pm 0.0519 \quad \text{with 97\% probability.}$$

We are thus 97% sure that $p = m$ lies in the interval $[0.0081, 0.1119]$. Thus, we are *not* 97% sure that $p < 0.1$.

(b) We have $M_n = 71/1000$. We use the formula $m = M_n \pm y_{\alpha/2}S_n/\sqrt{n} = 0.071 \pm (2.170)(.257)/\sqrt{1000}$ to get

$$m = 0.071 \pm 0.018 \quad \text{with 97\% probability.}$$

We are thus 97% sure that $p = m$ lies in the interval $[0.053, 0.089]$. Thus, we are 97% sure that $p < 0.089 < 0.1$.

28. (a) Let T_i denote the time to transmit the ith packet. Then we need to compute

$$\mathsf{P}\bigg(\bigcup_{i=1}^{n}\{T_i > t\}\bigg) = 1 - \mathsf{P}\bigg(\bigcap_{i=1}^{n}\{T_i \leq t\}\bigg) = 1 - F_{T_1}(t)^n = 1 - (1 - e^{-t/\mu})^n.$$

(b) Using the notation from part (a), $T = T_1 + \cdots + T_n$. Since the T_i are i.i.d. $\exp(1/\mu)$, T is Erlang$(n, 1/\mu)$ by Problem 55(c) in Chapter 4 and the remark following it. Hence,

$$f_T(t) = (1/\mu)\frac{(t/\mu)^{n-1}e^{-t/\mu}}{(n-1)!}, \quad t \geq 0.$$

(c) We have

$$\mu = M_n \pm \frac{y_{\alpha/2}S_n}{\sqrt{n}} = 1.994 \pm \frac{1.960(1.798)}{10} = 1.994 \pm 0.352$$

and confidence interval $[1.642, 2.346]$ with 95% probability.

29. **MATLAB.** OMITTED.

30. By the hint, $\sum_{i=1}^{n} X_i$ is Gaussian with mean nm and variance $n\sigma^2$. Since $M_n = \big(\sum_{i=1}^{n} X_i\big)/n$, it is easy to see that M_n is still Gaussian, and its mean is $(nm)/n = m$. Its variance is $(n\sigma^2)/n^2 = \sigma^2/n$. Next, $M_n - m$ remains Gaussian but with mean zero and the same variance σ^2/n. Finally, $(M_n - m)/\sqrt{\sigma/n}$ remains Gaussian and with mean zero, but its variance is $(\sigma^2/n)/(\sqrt{\sigma^2/n}\,)^2 = 1$.

31. We use the fomula $m = M_n \pm y_{\alpha/2}S_n/\sqrt{n}$, where in this Gaussian case, $y_{\alpha/2}$ is taken from the tables using Student's t distribution with $n = 10$. Thus,

$$m = M_n \pm \frac{y_{\alpha/2}S_n}{\sqrt{n}} = 14.832 \pm \frac{2.262 \cdot 1.904}{\sqrt{10}} = 14.832 \pm 1.362,$$

and the confidence interval is $[13.470, 16.194]$ with 95% probability.

32. We use $[nV_n^2/u, nV_n^2/\ell]$, where u and ℓ are chosen from the appropriate table. For a 95% confidence interval, $\ell = 74.222$ and $u = 129.561$. Thus,

$$\left[\frac{nV_n^2}{u}, \frac{nV_n^2}{\ell}\right] = \left[\frac{100(4.413)}{129.561}, \frac{100(4.413)}{74.222}\right] = [3.406, 5.946].$$

33. We use $[(n-1)S_n^2/u, (n-1)S_n^2/\ell]$, where u and ℓ are chosen from the appropriate table. For a 95% confidence interval, $\ell = 73.361$ and $u = 128.422$. Thus,

$$\left[\frac{(n-1)S_n^2}{u}, \frac{(n-1)S_n^2}{\ell}\right] = \left[\frac{99(4.736)}{128.422}, \frac{99(4.736)}{73.361}\right] = [3.651, 6.391].$$

34. For the two-sided test at the 0.05 significance level, we compare $|Z_n|$ with $y_{\alpha/2} = 1.960$. Since $|Z_n| = 1.8 \leq 1.960 = y_{\alpha/2}$, we *accept* the null hypothesis. For the one-sided test of $m > m_0$ at the 0.05 significance level, we compare Z_n with $-y_\alpha = -1.645$. Since it is not the case that $Z_n = -1.80 > -1.645 = -y_\alpha$, we do *not* accept the null hypothesis.

35. Suppose $\Phi(-y) = \alpha$. Then by Problem 17, $\Phi(-y) = 1 - \Phi(y)$, and so $1 - \Phi(y) = \alpha$, or $\Phi(y) = 1 - \alpha$.

36. (a) Since $Z_n = 1.50 \leq y_\alpha = 1.555$, the Internet service provider accepts the null hypothesis.

    (b) Since $Z_n = 1.50 > -1.555 = -y_\alpha$, we accept the null hypothesis; i.e., we *reject* the claim of the Internet service provider.

37. The computer vendor would take the null hypothesis to be $m \leq m_0$. To give the vendor the benefit of the doubt, the consumer group uses $m \leq m_0$ as the null hypothesis. To accept the null hypothesis would require $Z_n \leq y_\alpha$. Only by using the sigificance level of 0.10, which has $y_\alpha = 1.282$, can the consumer group give the benefit of the doubt to the vendor and still reject the vendor's claim.

38. Giving itself the benfit of the doubt, the company uses the null hypothesis $m > m_0$ and uses a 0.05 significance level. The null hypothesis will be accepted if $Z_n > -y_\alpha = -1.645$. Since $Z_n = -1.6 > -1.645$, the company believes it has justified its claim.

39. Write

$$\begin{aligned}
e(\widehat{g}) &= \sum_{k=1}^n |Y_k - (\widehat{a}x_k + \widehat{b})|^2 = \sum_{k=1}^n |Y_k - (\widehat{a}x_k + [\overline{Y} - \widehat{a}\overline{x}])|^2 \\
&= \sum_{k=1}^n |(Y_k - \overline{Y}) + \widehat{a}(x_k - \overline{x})|^2 \\
&= \sum_{k=1}^n \Big[(Y_k - \overline{Y})^2 - 2\widehat{a}(x_k - \overline{x})(Y_k - \overline{Y}) + \widehat{a}^2(x_k - \overline{x})^2\Big] \\
&= S_{YY} - 2\widehat{a}S_{xY} + \widehat{a}^2 S_{xx} \\
&= S_{YY} - 2\widehat{a}S_{xY} + \widehat{a}\Big(\frac{S_{xY}}{S_{xx}}\Big)S_{xx} = S_{YY} - \widehat{a}S_{xY} = S_{YY} - S_{xY}^2/S_{xx}.
\end{aligned}$$

40. Write

$$\begin{aligned}
\mathsf{E}[Y|X=x] &= \mathsf{E}[g(X) + W|X=x] = \mathsf{E}[g(x) + W|X=x] \\
&= g(x) + \mathsf{E}[W|X=x] = g(x) + \mathsf{E}[W] = g(x).
\end{aligned}$$

41. **MATLAB.** OMITTED.

42. **MATLAB.** OMITTED.

43. **MATLAB.** If $z = c/t^q$, then $\ln z = \ln c - q \ln t$. If $y = \ln z$ and $x = \ln t$, then $y = (-q)x + \ln c$. If $y \approx a(1)x + a(2)$, then $q = -a(1)$ and $c = \exp(a(2))$. Hence, the two lines of code that we need are

```
qhat = -a(1)
chat = exp(a(2))
```

44. Obvious.

45. Write

$$f_{\widetilde{Z}}(z) = e^{sz} f_Z(z)/M_Z(s) = e^{sz}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\Big/ e^{s^2/2} = \frac{e^{-(z-s)^2/2}}{\sqrt{2\pi}}.$$

To make $\mathsf{E}[\widetilde{Z}] = t$, put $s = t$. Then $\widetilde{Z} \sim (t,1)$.

46. Write

$$f_{\widetilde{Z}}(z) = e^{sz} f_Z(z)/M_Z(s) = e^{sz}\frac{\lambda(\lambda z)^{p-1}e^{-\lambda z}}{\Gamma(p)}\Big/\Big(\frac{\lambda}{\lambda-s}\Big)^p, \quad z > 0.$$

Then

$$\mathsf{E}[\widetilde{Z}] = \Big(\frac{\lambda}{\lambda-s}\Big)^{-p}\int_0^\infty z\frac{\lambda(\lambda z)^{p-1}e^{-z(\lambda-s)}}{\Gamma(p)}\,dz = \int_0^\infty z\frac{(\lambda-s)^p z^{p-1}e^{-z(\lambda-s)}}{\Gamma(p)}\,dz,$$

which is the mean of a gamma$(p,\lambda-s)$ density. Hence, $\mathsf{E}[\widetilde{Z}] = p/(\lambda-s)$. To make $\mathsf{E}[\widetilde{Z}] = t$, we need $p/(\lambda-s) = t$ or $s = \lambda - p/t$.

47. **MATLAB.** OMITTED.

48. First,

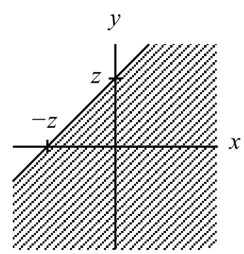
$$p_{\widetilde{Z}}(z_i) = e^{sz_i}p_Z(z_i)\Big/[(1-p)+pe^s].$$

Then

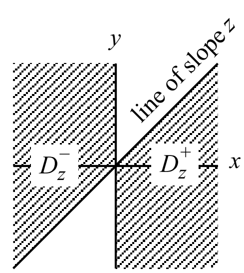
$$p_{\widetilde{Z}}(1) = e^s p/[(1-p)+pe^s] \quad \text{and} \quad p_{\widetilde{Z}}(0) = (1-p)/[(1-p)+pe^s].$$

CHAPTER 7
Problem Solutions

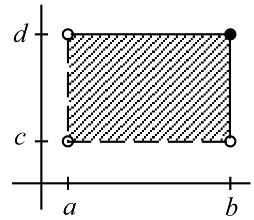
1. We have
$$F_Z(z) \;=\; \mathsf{P}(Z \le z) \;=\; \mathsf{P}(Y - X \le z) \;=\; \mathsf{P}((X,Y) \in A_z),$$
where
$$A_z \;:=\; \{(x,y) : y - x \le z\} \;=\; \{(x,y) : y \le x + z\}.$$



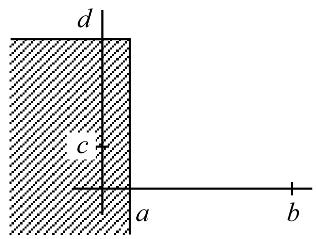
2. We have
$$\begin{aligned} F_Z(z) \;&=\; \mathsf{P}(Z \le z) \;=\; \mathsf{P}(Y/X \le z) \;=\; \mathsf{P}\Big(\{Y/X \le z\} \cap \Big[\{X < 0\} \cup \{X > 0\}\Big]\Big) \\ &=\; \mathsf{P}((X,Y) \in D_z^- \cup D_z^+) \;=\; \mathsf{P}((X,Y) \in A_z), \end{aligned}$$
where
$$A_z \;:=\; D_z^- \cup D_z^+,$$
and
$$D_z^- \;:=\; \{(x,y) : y/x \le z \text{ and } x < 0\} \;=\; \{(x,y) : y \ge zx \text{ and } x < 0\},$$
and
$$D_z^+ \;:=\; \{(x,y) : y/x \le z \text{ and } x > 0\} \;=\; \{(x,y) : y \le zx \text{ and } x > 0\}.$$



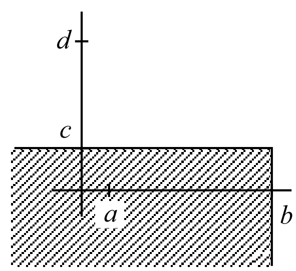
3. (a) $R := (a,b] \times (c,d]$ is



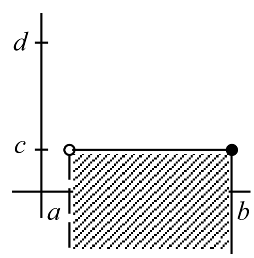
(b) $A := (-\infty, a] \times (-\infty, d]$ is



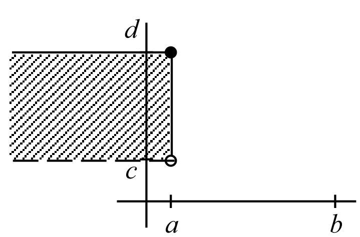
(c) $B := (-\infty, b] \times (-\infty, c]$ is



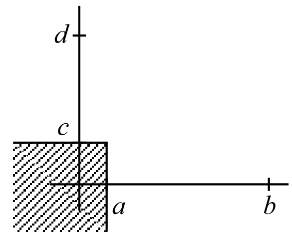
(d) $C := (a, b] \times (-\infty, c]$ is



(e) $D := (-\infty, a] \times (c, d]$ is



(f) $A \cap B$ is



4. Following the hint and then observing that R and $A \cup B$ are disjoint, we have

$$\mathsf{P}\big((X,Y) \in (-\infty,b] \times (-\infty,d]\big) \;=\; \mathsf{P}\big((X,Y) \in R\big) + \mathsf{P}\big((X,Y) \in A \cup B\big). \qquad (*)$$

Next, by the inclusion–exclusion formula,

$$\begin{aligned}\mathsf{P}\big((X,Y) \in A \cup B\big) \;&=\; \mathsf{P}\big((X,Y) \in A\big) + \mathsf{P}\big((X,Y) \in B\big) - \mathsf{P}\big((X,Y) \in A \cap B\big) \\ &=\; F_{XY}(a,d) + F_{XY}(b,c) - F_{XY}(a,c).\end{aligned}$$

Hence, $(*)$ becomes

$$F_{XY}(b,d) \;=\; \mathsf{P}\big((X,Y) \in R\big) + F_{XY}(a,d) + F_{XY}(b,c) - F_{XY}(a,c),$$

which is easily rearranged to get the rectangle formula,

$$\mathsf{P}\big((X,Y) \in R\big) \;=\; F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,c) + F_{XY}(a,c).$$

5. (a) $\{(x,y) : |x| \le y \le 1\}$ is NOT a product set.
   (b) $\{(x,y) : 2 < x \le 4,\, 1 \le y < 2\} = (2,4] \times [1,2)$.
   (c) $\{(x,y) : 2 < x \le 4,\, y = 1\} = (2,4] \times \{1\}$.
   (d) $\{(x,y) : 2 < x \le 4\} = (2,4] \times \mathbb{R}$.
   (e) $\{(x,y) : y = 1\} = \mathbb{R} \times \{1\}$.
   (f) $\{(1,1),(2,1),(3,1)\} = \{1,2,3\} \times \{1\}$.
   (g) The union of $\{(1,3),(2,3),(3,3)\}$ and the set in (f) is equal to $\{1,2,3\} \times \{1,3\}$.
   (h) $\{(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(3,1)\}$ is NOT a product set.

6. We have

$$F_X(x) \;=\; \begin{cases} 1, & x \ge 2, \\ x-1, & 1 \le x < 2, \\ 0, & x < 1, \end{cases} \qquad \text{and} \qquad F_Y(y) \;=\; \begin{cases} 1 - \frac{e^{-y} - e^{-2y}}{y}, & y \ge 0, \\ 0, & y < 0, \end{cases}$$

where the quotient involving division by y is understood as taking its limiting value of one when $y = 0$. Since $F_X(x)F_Y(y) \ne F_{XY}(x,y)$ when $1 \le x \le 2$ and $y > 0$, X and Y are NOT independent.

7. We have

$$F_X(x) = \begin{cases} 1, & x \geq 3, \\ 2/7, & 2 \leq x < 3, \\ 0, & x < 2, \end{cases} \quad \text{and} \quad F_Y(y) = \begin{cases} \frac{1}{7}[7 - 2e^{-2y} - 5e^{-3y}], & y \geq 0, \\ 0, & y < 0. \end{cases}$$

8. Let $y > 0$. First compute

$$\begin{aligned} \frac{\partial}{\partial y}\left(\frac{y + e^{-x(y+1)}}{y+1}\right) &= \frac{(y+1)\{1 + e^{-x(y+1)}(-x)\} - \{y + e^{-x(y+1)}\}(1)}{(y+1)^2} \\ &= \frac{(y+1) - x(y+1)e^{-x(y+1)} - y - e^{-x(y+1)}}{(y+1)^2} \\ &= \frac{1 - e^{-x(y+1)}\{1 + x(y+1)\}}{(y+1)^2}. \end{aligned}$$

Then compute

$$\begin{aligned} \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{y + e^{-x(y+1)}}{y+1}\right)\right] &= \frac{\{1 + x(y+1)\}e^{-x(y+1)}(y+1) - e^{-x(y+1)}(y+1)}{(y+1)^2} \\ &= \frac{xe^{-x(y+1)}(y+1)^2}{(y+1)^2} = xe^{-x(y+1)}, \quad x, y > 0. \end{aligned}$$

9. The first step is to recognize that $f_{XY}(x,y)$ factors into

$$f_{XY}(x,y) = \frac{\exp[-|y-x| - x^2/2]}{2\sqrt{2\pi}} = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot \frac{e^{-|y-x|}}{2}.$$

When integrating this last factor with respect to y, make the change of variable $\theta = y - x$ to get

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tfrac{1}{2}e^{-|y-x|}\,dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tfrac{1}{2}e^{-|\theta|}\,d\theta \\ &= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\theta}\,d\theta = \frac{e^{-x^2/2}}{\sqrt{2\pi}}. \end{aligned}$$

Thus, $X \sim N(0,1)$.

10. The first step is to factor $f_{XY}(x,y)$ as

$$f_{XY}(x,y) = \frac{4e^{-(x-y)^2/2}}{y^5\sqrt{2\pi}} = \frac{4}{y^5} \cdot \frac{e^{-(x-y)^2/2}}{\sqrt{2\pi}}.$$

Regarding this last factor a function of x, it is an $N(y,1)$ density. In other words, when integrated with respect to x, the result is one. In symbols,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{4}{y^5} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2}}{\sqrt{2\pi}}\,dx = \frac{4}{y^5}, \quad y \geq 1.$$

11. We first analyze $U := \max(X,Y)$. Then

$$F_U(u) \;=\; \mathsf{P}(\max(X,Y) \le u) \;=\; \mathsf{P}(X \le u \text{ and } Y \le u) \;=\; F_{XY}(u,u),$$

and the density is

$$f_U(u) \;=\; \frac{\partial F_{XY}(x,y)}{\partial x}\bigg|_{x=u,\,y=u} + \frac{\partial F_{XY}(x,y)}{\partial y}\bigg|_{x=u,\,y=u}.$$

If X and Y are independent, then

$$F_U(u) \;=\; F_X(u)F_Y(u), \quad \text{and} \quad f_U(u) \;=\; f_X(u)F_Y(u) + F_X(u)f_Y(u).$$

If in addition X and Y have the same density, say f, (and therefore the same cdf, say F), then

$$F_U(u) \;=\; F(u)^2, \quad \text{and} \quad f_U(u) \;=\; 2F(u)f(u).$$

We next analyze $V := \min(X,Y)$. Using the inclusion–exclusion formula,

$$\begin{aligned} F_V(v) &= \mathsf{P}(\min(X,Y) \le v) \;=\; \mathsf{P}(X \le v \text{ or } Y \le v) \\ &= \mathsf{P}(X \le v) + \mathsf{P}(Y \le v) - \mathsf{P}(X \le v \text{ and } Y \le v) \\ &= F_X(v) + F_Y(v) - F_{XY}(v,v). \end{aligned}$$

The density is

$$f_V(v) \;=\; f_X(v) + f_Y(v) - \frac{\partial F_{XY}(x,y)}{\partial x}\bigg|_{x=v,\,y=v} - \frac{\partial F_{XY}(x,y)}{\partial y}\bigg|_{x=v,\,y=v}.$$

If X and Y are independent, then

$$F_V(v) \;=\; F_X(v) + F_Y(v) - F_X(v)F_Y(v),$$

and

$$f_V(v) \;=\; f_X(v) + f_Y(v) - f_X(v)F_Y(v) - F_X(v)f_Y(v).$$

If in addition X and Y have the same density f and cdf F, then

$$F_V(v) \;=\; 2F(v) - F(v)^2, \quad \text{and} \quad f_V(v) \;=\; 2[f(v) - F(v)f(v)].$$

12. Since $X \sim \text{gamma}(p,1)$ and $Y \sim \text{gamma}(q,1)$ are independent, we have from Problem 55(c) in Chapter 4 that $Z \sim \text{gamma}(p+q,1)$. Since $p = q = 1/2$, we further have $Z \sim \text{gamma}(1,1) = \exp(1)$. Hence, $\mathsf{P}(Z > 1) = e^{-1}$.

13. We have from Problem 55(b) in Chapter 4 that $Z \sim \text{Cauchy}(\lambda + \mu)$. Since $\lambda = \mu = 1/2$, $Z \sim \text{Cauchy}(1)$. Thus,

$$\mathsf{P}(Z \le 1) \;=\; \frac{1}{\pi}\tan^{-1}(1) + \frac{1}{2} \;=\; \frac{1}{\pi}\frac{\pi}{4} + \frac{1}{2} \;=\; \frac{3}{4}.$$

14. First write

$$F_Z(z) \;=\; \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z-y} f_{XY}(x,y)\,dx\right]dy.$$

It then follows that

$$f_Z(z) \;=\; \frac{\partial}{\partial z}F_Z(z) \;=\; \int_{-\infty}^{\infty}\frac{\partial}{\partial z}\int_{-\infty}^{z-y} f_{XY}(x,y)\,dx\,dy \;=\; \int_{-\infty}^{\infty} f_{XY}(z-y,y)\,dy.$$

15. First write

$$\begin{aligned}
F_Z(z) &= \mathsf{P}((X,Y)\in A_z) \;=\; \mathsf{P}((X,Y)\in B_z^+\cup B_z^-)\\
&= \iint\limits_{B_z^+} f_{XY}(x,y)\,dx\,dy + \iint\limits_{B_z^-} f_{XY}(x,y)\,dx\,dy\\
&= \int_0^{\infty}\left[\int_{-\infty}^{z/y} f_{XY}(x,y)\,dx\right]dy + \int_{-\infty}^{0}\left[\int_{z/y}^{\infty} f_{XY}(x,y)\,dx\right]dy.
\end{aligned}$$

Then

$$\begin{aligned}
f_Z(z) \;=\; \frac{\partial}{\partial z}F_Z(z) &= \int_0^{\infty} f_{XY}(z/y,y)/y\,dy - \int_{-\infty}^{0} f_{XY}(z/y,y)/y\,dy\\
&= \int_0^{\infty} f_{XY}(z/y,y)/|y|\,dy + \int_{-\infty}^{0} f_{XY}(z/y,y)/|y|\,dy\\
&= \int_{-\infty}^{\infty} f_{XY}(z/y,y)/|y|\,dy.
\end{aligned}$$

16. For the cdf, write

$$F_Z(z) \;=\; \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z+x} f_{XY}(x,y)\,dy\right]dx.$$

Then

$$f_Z(z) \;=\; \frac{\partial}{\partial z}F_Z(z) \;=\; \int_{-\infty}^{\infty}\frac{\partial}{\partial z}\int_{-\infty}^{z+x} f_{XY}(x,y)\,dy\,dx \;=\; \int_{-\infty}^{\infty} f_{XY}(x,z+x)\,dx.$$

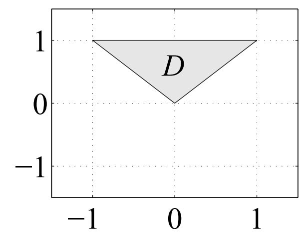
17. For the cdf, write

$$\begin{aligned}
F_Z(z) &= \mathsf{P}(Z\le z) \;=\; \mathsf{P}(Y/X\le z)\\
&= \mathsf{P}(Y\le Xz, X>0) + \mathsf{P}(Y\ge Xz, X<0)\\
&= \int_0^{\infty}\left[\int_{-\infty}^{xz} f_{XY}(x,y)\,dy\right]dx + \int_{-\infty}^{0}\left[\int_{xz}^{\infty} f_{XY}(x,y)\,dy\right]dx.
\end{aligned}$$

Then

$$f_Z(z) = \frac{\partial}{\partial z}F_Z(z) = \int_0^{\infty} f_{XY}(x,xz)x\,dx - \int_{-\infty}^{0} f_{XY}(x,xz)x\,dx = \int_{-\infty}^{\infty} f_{XY}(x,xz)|x|\,dx.$$

18. (a) The region D:

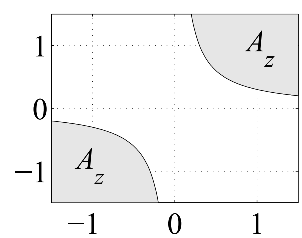


(b) Since $f_{XY}(x,y) = Kx^n y^m$ for $(x,y) \in D$ and since D contains negative values of x, **we must have *n* even** in order that the density be nonnegative. In this case, the integral of the density over the region D must be one. Hence, K must be such that

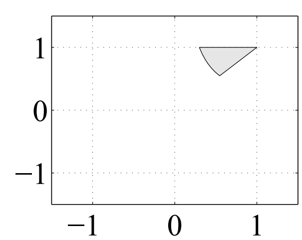
$$\begin{aligned}
1 &= \iint_D f_{XY}(x,y)\,dy\,dx \;=\; \int_{-1}^{1}\left[\int_{|x|}^{1} Kx^n y^m\,dy\right]dx \;=\; \int_{-1}^{1} Kx^n\left[\frac{y^{m+1}}{m+1}\right]_{|x|}^{1} dx \\
&= \int_{-1}^{1} \frac{Kx^n}{m+1}[1-|x|^{m+1}]\,dx \;=\; \frac{K}{m+1}\left[\int_{-1}^{1} x^n - |x|^{n+m+1}\,dx\right] \\
&= \frac{2K}{m+1}\int_0^1 x^n - x^{n+m+1}\,dx \;=\; \frac{2K}{m+1}\left[\frac{1}{n+1} - \frac{1}{n+m+2}\right] \\
&= \frac{2K}{(n+1)(n+m+2)}.
\end{aligned}$$

Hence, $K = (n+1)(n+m+2)/2$.

(c) A sketch of A_z with $z = 0.3$:



(d) A sketch of $A_z \cap D$ with $z = 0.3$:



(e)

$$\mathsf{P}((X,Y) \in A_z) \;=\; \int_z^{\sqrt{z}}\left[\int_{z/x}^{1} Kx^n y^m\,dy\right]dx + \int_{\sqrt{z}}^{1}\left[\int_x^1 Kx^n y^m\,dy\right]dx$$

$$\begin{aligned}
&= \int_z^{\sqrt{z}} Kx^n \left[\int_{z/x}^1 y^m\,dy\right]dx + \int_{\sqrt{z}}^1 Kx^n\left[\int_x^1 y^m\,dy\right]dx \\
&= \frac{K}{m+1}\left[\int_z^{\sqrt{z}} x^n[1-(z/x)^{m+1}]\,dx + \int_{\sqrt{z}}^1 x^n[1-x^{m+1}]\,dx\right] \\
&= \frac{K}{m+1}\left[\int_z^{\sqrt{z}} x^n - z^{m+1}x^{n-m-1}\,dx + \int_{\sqrt{z}}^1 x^n - x^{n+m+1}\,dx\right] \\
&= \frac{K}{m+1}\left[\int_z^1 x^n\,dx - \int_z^{\sqrt{z}} z^{m+1}x^{n-m-1}\,dx - \int_{\sqrt{z}}^1 x^{n+m+1}\,dx\right] \\
&= \frac{K}{m+1}\left[\frac{1-z^n}{n+1} - \int_z^{\sqrt{z}} z^{m+1}x^{n-m-1}\,dx - \frac{1-z^{(n+m+2)/2}}{n+m+2}\right].
\end{aligned}$$

If $n \neq m$, the remaining integral is equal to

$$\frac{(\sqrt{z}\,)^{n+m+2} - z^{n+1}}{n-m}.$$

Otherwise, the integral is equal to

$$-\frac{z^{m+1}}{2}\ln z.$$

19. Let $X \sim \text{uniform}[0,w]$ and $Y \sim \text{uniform}[0,h]$. We need to compute $\mathsf{P}(XY \geq \lambda wh)$. Before proceeding, we make a few observations. First, since $X \geq 0$, we can write for $z > 0$,

$$\mathsf{P}(XY \geq z) = \int_0^\infty \int_{z/x}^\infty f_{XY}(x,y)\,dy\,dx = \int_0^\infty f_X(x)\int_{z/x}^\infty f_Y(y)\,dy\,dx.$$

Since $Y \sim \text{uniform}[0,h]$, the inner integral will be zero if $z/x > h$. Since $z/x \leq h$ if and only if $x \geq z/h$,

$$\mathsf{P}(XY \geq z) = \int_{z/h}^\infty f_X(x)\int_{z/x}^h \frac{1}{h}\,dy\,dx = \int_{z/h}^\infty f_X(x)\Big(1-\frac{z}{xh}\Big)\,dx.$$

We can now write

$$\begin{aligned}
\mathsf{P}(XY \geq \lambda wh) &= \int_{\lambda wh/h}^\infty f_X(x)\Big(1-\frac{\lambda wh}{xh}\Big)\,dx = \int_{\lambda w}^\infty f_X(x)\Big(1-\frac{\lambda w}{x}\Big)\,dx \\
&= \int_{\lambda w}^w \frac{1}{w}\Big(1-\frac{\lambda w}{x}\Big)\,dx = (1-\lambda) - \lambda\ln\frac{w}{\lambda w} = (1-\lambda)+\lambda\ln\lambda.
\end{aligned}$$

20. We first compute

$$\begin{aligned}
\mathsf{E}[XY] &= \mathsf{E}[\cos\Theta\sin\Theta] = \tfrac{1}{2}\mathsf{E}[\sin 2\Theta] = \frac{1}{4\pi}\int_{-\pi}^{\pi}\sin 2\theta\,d\theta \\
&= \frac{\cos(-2\pi)-\cos(2\pi)}{8\pi} = 0.
\end{aligned}$$

Similarly,

$$\mathsf{E}[X] \;=\; \mathsf{E}[\cos\Theta] \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\theta\,d\theta \;=\; \frac{\sin(\pi)-\sin(-\pi)}{2\pi} \;=\; \frac{0-0}{2\pi} \;=\; 0,$$

and

$$\mathsf{E}[Y] \;=\; \mathsf{E}[\sin\Theta] \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\theta\,d\theta \;=\; \frac{\cos(-\pi)-\cos(\pi)}{2\pi} \;=\; \frac{(-1)-(-1)}{2\pi} \;=\; 0.$$

To prove that X and Y are not independent, we argue by contradiction. However, before we begin, observe that since (X,Y) satisfies

$$X^2+Y^2 \;=\; \cos^2\Theta+\sin^2\Theta \;=\; 1,$$

(X,Y) always lies on the unit circle. Now consider the square of side one centered at the origin,

$$S \;:=\; \{(x,y) : |x| \leq 1/2, \text{ and } |y| \leq 1/2\}.$$

Since this region lies strictly inside the unit circle, $\mathsf{P}((X,Y)\in S)=0$. Now, to obtain a contradiction suppose that X and Y are independent. Then $f_{XY}(x,y)=f_X(x)f_Y(y)$, where f_X and f_Y are both arcsine densities by Problem 35 in Chapter 5. Hence, for $|x|<1$ and $|y|<1$, $f_{XY}(x,y)\geq f_{XY}(0,0)=1/\pi^2$. We can now write

$$\mathsf{P}((X,Y)\in S) \;=\; \iint_S f_{XY}(x,y)\,dx\,dy \;\geq\; \iint_S 1/\pi^2\,dx\,dy \;=\; 1/\pi^2 \;>\; 0,$$

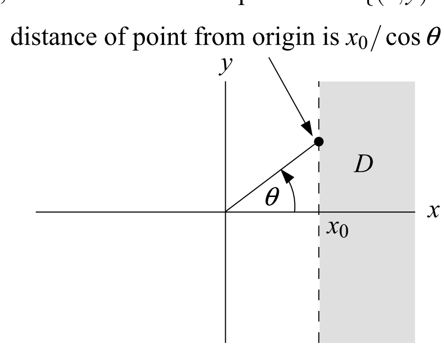
which is a contradiction.

21. If $\mathsf{E}[h(X)k(Y)]=\mathsf{E}[h(X)]\mathsf{E}[k(Y)]$ for all bounded continuous functions h and k, then we may specialize this equation to the functions $h(x)=e^{j\nu_1 x}$ and $k(y)=e^{j\nu_2 y}$ to show that the joint characteristic function satisfies

$$\varphi_{XY}(\nu_1,\nu_2) := \mathsf{E}[e^{j(\nu_1 X+\nu_2 Y)}] = \mathsf{E}[e^{j\nu_1 X}e^{j\nu_2 Y}] = \mathsf{E}[e^{j\nu_1 X}]\mathsf{E}[e^{j\nu_2 Y}] = \varphi_X(\nu_1)\varphi_Y(\nu_2).$$

Since the joint characteristic function is the product of the marginal characteristic functions, X and Y are independent.

22. (a) Following the hint, let D denote the half-plane $D:=\{(x,y): x>x_0\}$,



and write

$$\mathsf{P}(X > x_0) \;=\; \mathsf{P}(X > x_0, Y \in \mathbb{R}) \;=\; \mathsf{P}((X,Y) \in D),$$

where X and Y are both $N(0,1)$. Then

$$\mathsf{P}((X,Y) \in D) \;=\; \iint_D \frac{e^{-(x^2+y^2)/2}}{2\pi}\,dx\,dy.$$

Now convert to polar coordinates and write

$$\begin{aligned}
\mathsf{P}((X,Y) \in D) \;&=\; \int_{-\pi/2}^{\pi/2} \int_{x_0/\cos\theta}^{\infty} \frac{e^{-r^2/2}}{2\pi}\, r\,dr\,d\theta \\
&=\; \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} -e^{-r^2/2}\Big|_{x_0/\cos\theta}^{\infty}\,d\theta \\
&=\; \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \exp[-(x_0/\cos\theta)^2/2]\,d\theta \\
&=\; \frac{1}{\pi} \int_0^{\pi/2} \exp\biggl(\frac{-x_0^2}{2\cos^2\theta}\biggr)\,d\theta.
\end{aligned}$$

(b) In the preceding integral, make the change of variable $\theta = \pi/2 - t$, $d\theta = -dt$. Then $\cos\theta$ becomes $\cos(\pi/2 - t) = \sin t$, and the preceding integral becomes

$$\frac{1}{\pi} \int_0^{\pi/2} \exp\biggl(\frac{-x_0^2}{2\sin^2 t}\biggr)\,dt.$$

23. Write

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x)\,dy = \int_{-\infty}^{\infty} \frac{f_{XY}(x,y)}{f_X(x)}\,dy = \frac{1}{f_X(x)} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{f_X(x)}{f_X(x)} = 1.$$

24. First write

$$\begin{aligned}
\mathsf{P}((X,Y) \in A | x < X \le x + \Delta x) \;&=\; \frac{\mathsf{P}\Bigl(\bigl\{(X,Y) \in A\bigr\} \cap \bigl\{x < X \le x + \Delta x\bigr\}\Bigr)}{\mathsf{P}(x < X \le x + \Delta x)} \\
&=\; \frac{\mathsf{P}\Bigl(\bigl\{(X,Y) \in A\bigr\} \cap \bigl\{(X,Y) \in (x, x+\Delta x] \times \mathbb{R}\bigr\}\Bigr)}{\mathsf{P}(x < X \le x + \Delta x)} \\
&=\; \frac{\mathsf{P}\Biggl(\Bigl\{(X,Y) \in A \cap \Bigl[(x, x+\Delta x] \times \mathbb{R}\Bigr]\Bigr\}\Biggr)}{\mathsf{P}(x < X \le x + \Delta x)} \\
&=\; \frac{\displaystyle\iint_{A \cap \bigl[(x,x+\Delta x] \times \mathbb{R}\bigr]} f_{XY}(t,y)\,dy\,dt}{\displaystyle\int_x^{x+\Delta x} f_X(\tau)\,d\tau}
\end{aligned}$$

$$= \frac{\displaystyle\iint I_A(t,y) I_{(x,x+\Delta x]\times \mathbb{R}}(t,y) f_{XY}(t,y)\,dy\,dt}{\displaystyle\int_x^{x+\Delta x} f_X(\tau)\,d\tau}$$

$$= \frac{\displaystyle\int_x^{x+\Delta x}\int_{-\infty}^{\infty} I_A(t,y) f_{XY}(t,y)\,dy\,dt}{\displaystyle\int_x^{x+\Delta x} f_X(\tau)\,d\tau}$$

$$= \frac{\displaystyle\frac{1}{\Delta x}\int_x^{x+\Delta x}\int_{-\infty}^{\infty} I_A(t,y) f_{XY}(t,y)\,dy\,dt}{\displaystyle\frac{1}{\Delta x}\int_x^{x+\Delta x} f_X(\tau)\,d\tau.}$$

It now follows that

$$\lim_{\Delta t\to\infty} \mathsf{P}((X,Y)\in A|x<X\le x+\Delta x) = \frac{\displaystyle\int_{-\infty}^{\infty} I_A(x,y) f_{XY}(x,y)\,dy}{f_X(x)} = \int_{-\infty}^{\infty} I_A(x,y) f_{Y|X}(y|x)\,dy.$$

25. We first compute, for $x>0$,

$$f_{Y|X}(y|x) = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy}, \quad y>0.$$

As a function of y, this is an exponential density with parameter x. This is very different from $f_Y(y) = 1/(y+1)^2$. We next compute, for $y>0$,

$$f_{X|Y}(x|y) = \frac{xe^{-x(y+1)}}{1/(y+1)^2} = (y+1)^2 xe^{-(y+1)x}, \quad x>0.$$

As a function of x this is an Erlang$(2,y+1)$ density, which is not the same as $f_X \sim \exp(1)$.

26. We first compute, for $x>0$,

$$f_{Y|X}(y|x) = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy}, \quad y>0.$$

As a function of y, this is an exponential density with parameter x. Hence,

$$\mathsf{E}[Y|X=x] = \int_0^\infty y f_{Y|X}(y|x)\,dy = 1/x.$$

We next compute, for $y>0$,

$$f_{X|Y}(x|y) = \frac{xe^{-x(y+1)}}{1/(y+1)^2} = (y+1)^2 xe^{-(y+1)x}, \quad x>0.$$

As a function of x this is an Erlang$(2,y+1)$ density. Hence,

$$\mathsf{E}[X|Y=y] = \int_0^\infty x f_{X|Y}(x|y)\,dx = 2/(y+1).$$

27. Write

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathsf{P}(X \in B|Y = y) f_Y(y)\,dy &= \int_{-\infty}^{\infty} \left[\int_B f_{X|Y}(x|y)\,dx \right] f_Y(y)\,dy \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} I_B(x) f_{X|Y}(x|y)\,dx \right] f_Y(y)\,dy \\
&= \int_{-\infty}^{\infty} I_B(x) \left[\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \right] dx \\
&= \int_B f_X(x)dx \;=\; \mathsf{P}(X \in B).
\end{aligned}$$

28. For $z \geq 0$,

$$\begin{aligned}
f_Z(z) &= \int_{-\sqrt{z}}^{\sqrt{z}} \frac{\frac{e^{-(z-y^2)/(2\sigma^2)}}{\sqrt{2\pi}\sigma}}{\sqrt{z-y^2}} \cdot \frac{e^{-(y/\sigma)^2/2}}{\sqrt{2\pi}\sigma}\,dy \;=\; \frac{e^{-z/(2\sigma^2)}}{2\pi\sigma^2} \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}}\,dy \\
&= \frac{e^{-z/(2\sigma^2)}}{\pi\sigma^2} \int_0^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}}\,dy \;=\; \frac{e^{-z/(2\sigma^2)}}{\pi\sigma^2} \int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt \\
&= \frac{e^{-z/(2\sigma^2)}}{\pi\sigma^2} [\sin^{-1}(1) - \sin^{-1}(0)] \;=\; \frac{e^{-z/(2\sigma^2)}}{2\sigma^2},
\end{aligned}$$

which is an exponential density with parameter $1/(2\sigma^2)$.

29. For $z \geq 0$,

$$\begin{aligned}
f_Z(z) &= \int_0^{\infty} x \cdot \lambda e^{-\lambda x z} \cdot \lambda e^{-\lambda x}\,dx + \int_0^{\infty} y \cdot \lambda e^{-\lambda y z} \cdot \lambda e^{-\lambda y}\,dy \\
&= 2 \int_0^{\infty} x \cdot \lambda e^{-\lambda x z} \cdot \lambda e^{-\lambda x}\,dx \\
&= \frac{2\lambda^2}{\lambda z + \lambda} \int_0^{\infty} x \cdot [\lambda z + \lambda] e^{-x[\lambda z + \lambda]}\,dx.
\end{aligned}$$

Now, this last integral is the expectation of an exponential density with parameter $\lambda z + \lambda$. Hence,

$$f_Z(z) \;=\; \frac{2\lambda^2}{\lambda z + \lambda} \cdot \frac{1}{\lambda z + \lambda} \;=\; \frac{2}{(z+1)^2}, \quad z \geq 0.$$

30. Using the law of total probability, substitution, and independence, we have

$$\begin{aligned}
\mathsf{P}(X \leq Y) &= \int_0^{\infty} \mathsf{P}(X \leq Y|X = x) f_X(x)\,dx \;=\; \int_0^{\infty} \mathsf{P}(x \leq Y|X = x) f_X(x)\,dx \\
&= \int_0^{\infty} \mathsf{P}(Y \geq x) f_X(x)\,dx \;=\; \int_0^{\infty} e^{-\mu x} \cdot \lambda e^{-\lambda x}\,dx \\
&= \lambda \int_0^{\infty} e^{-x(\lambda+\mu)}\,dx \;=\; \frac{\lambda}{\lambda+\mu} \left(-e^{-(\lambda+\mu)} \right) \Big|_0^{\infty} \;=\; \frac{\lambda}{\lambda+\mu}.
\end{aligned}$$

31. Using the law of total probability, substitution, and independence, we have

$$\begin{aligned}
\mathsf{P}(Y/\ln(1+X^2)>1) &= \int_{-\infty}^{\infty} \mathsf{P}(Y/\ln(1+X^2)>1|X=x)f_X(x)\,dx \\
&= \int_1^2 \mathsf{P}(Y>\ln(1+x^2)|X=x)\cdot 1\,dx \\
&= \int_1^2 \mathsf{P}(Y>\ln(1+x^2))\,dx \;=\; \int_1^2 e^{-\ln(1+x^2)}\,dx \\
&= \int_1^2 e^{\ln(1+x^2)^{-1}}\,dx \;=\; \int_1^2 \frac{1}{1+x^2}\,dx \\
&= \tan^{-1}(2)-\tan^{-1}(1).
\end{aligned}$$

32. First find the cdf using the law of total probability and substitution. Then differentiate to obtain the density.

(a) For $Z = e^X Y$,

$$\begin{aligned}
F_Z(z) &= \mathsf{P}(Z\le z) \;=\; \mathsf{P}(e^XY\le z) \;=\; \int_{-\infty}^{\infty}\mathsf{P}(e^XY\le z|X=x)f_X(x)\,dx \\
&= \int_{-\infty}^{\infty}\mathsf{P}(Y\le ze^{-x}|X=x)f_X(x)\,dx \;=\; \int_{-\infty}^{\infty}F_{Y|X}(ze^{-x}|x)f_X(x)\,dx.
\end{aligned}$$

Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Y|X}(ze^{-x}|x)e^{-x}f_X(x)\,dx,$$

and so

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x,ze^{-x})e^{-x}\,dx.$$

(b) Since $Z=|X+Y|\ge 0$, we know that $F_Z(z)$ and $f_Z(z)$ are zero for $z<0$. For $z\ge 0$, write

$$\begin{aligned}
F_Z(z) &= \mathsf{P}(Z\le z) \;=\; \mathsf{P}(|X+Y|\le z) \;=\; \int_{-\infty}^{\infty}\mathsf{P}(|X+Y|\le z|X=x)f_X(x)\,dx \\
&= \int_{-\infty}^{\infty}\mathsf{P}(|x+Y|\le z|X=x)f_X(x)\,dx \\
&= \int_{-\infty}^{\infty}\mathsf{P}(-z\le x+Y\le z|X=x)f_X(x)\,dx \\
&= \int_{-\infty}^{\infty}\mathsf{P}(-z-x\le Y\le z-x|X=x)f_X(x)\,dx \\
&= \int_{-\infty}^{\infty}\big[F_{Y|X}(z-x|x)-F_{Y|X}(-z-x|x)\big]f_X(x)\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty}\big[f_{Y|X}(z-x|x)-f_{Y|X}(-z-x|x)(-1)\big]f_X(x)\,dx \\
&= \int_{-\infty}^{\infty}\big[f_{Y|X}(z-x|x)+f_{Y|X}(-z-x|x)\big]f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} f_{XY}(x,z-x)+f_{XY}(x,-z-x)\,dx.
\end{aligned}$$

33. (a) First find the cdf of Z using the law of total probability, substitution, and independence. Then differentiate to obtain the density. Write

$$\begin{aligned}
F_Z(z) &= \mathsf{P}(Z \le z) \;=\; \mathsf{P}(Y/X \le z) \;=\; \int_{-\infty}^{\infty} \mathsf{P}(Y/X \le z \mid X = x) f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} \mathsf{P}(Y/x \le z \mid X = x) f_X(x)\,dx \;=\; \int_{-\infty}^{\infty} \mathsf{P}(Y/x \le z) f_X(x)\,dx \\
&= \int_{-\infty}^{0} \mathsf{P}(Y/x \le z) f_X(x)\,dx + \int_{0}^{\infty} \mathsf{P}(Y/x \le z) f_X(x)\,dx \\
&= \int_{-\infty}^{0} \mathsf{P}(Y \ge zx) f_X(x)\,dx + \int_{0}^{\infty} \mathsf{P}(Y \le zx) f_X(x)\,dx \\
&= \int_{-\infty}^{0} [1 - F_Y(zx)] f_X(x)\,dx + \int_{0}^{\infty} F_Y(zx) f_X(x)\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{0} -f_Y(zx) x f_X(x)\,dx + \int_{0}^{\infty} f_Y(zx) x f_X(x)\,dx \\
&= \int_{-\infty}^{0} f_Y(zx)|x| f_X(x)\,dx + \int_{0}^{\infty} f_Y(zx)|x| f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} f_Y(zx) f_X(x)|x|\,dx.
\end{aligned}$$

(b) Using the result of part (a) and the fact that the integrand is even, write

$$f_Z(z) \;=\; \int_{-\infty}^{\infty} \frac{e^{-|zx|^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma} \frac{e^{-(x/\sigma)^2/2}}{\sqrt{2\pi}\sigma} |x|\,dx \;=\; \frac{2}{2\pi} \int_0^{\infty} \frac{x}{\sigma^2} e^{-(x/\sigma)^2[1+z^2]/2}\,dx.$$

Now make the change of variable $\theta = (x/\sigma)\sqrt{1+z^2}$ to get

$$f_Z(z) \;=\; \frac{1/\pi}{1+z^2} \int_0^{\infty} \theta e^{-\theta^2/2}\,d\theta \;=\; \frac{1/\pi}{1+z^2} \left(-e^{-\theta^2/2}\right)\Big|_0^{\infty} \;=\; \frac{1/\pi}{1+z^2},$$

which is the Cauchy(1) density.

(c) Using the result of part (a) and the fact that the integrand is even, write

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \tfrac{\lambda}{2} e^{-\lambda|zx|} \tfrac{\lambda}{2} e^{-\lambda|x|} |x|\,dx \;=\; \frac{\lambda^2}{2} \int_0^{\infty} x e^{-x\lambda(|z|+1)}\,dx \\
&= \frac{\lambda^2/2}{\lambda(|z|+1)} \int_0^{\infty} x \cdot \lambda(|z|+1) e^{-\lambda(|z|+1)x}\,dx,
\end{aligned}$$

where this last integral is the mean of an exponential density with parameter $\lambda(|z|+1)$. Hence,

$$f_Z(z) \;=\; \frac{\lambda^2/2}{\lambda(|z|+1)} \cdot \frac{1}{\lambda(|z|+1)} \;=\; \frac{1}{2(|z|+1)^2}.$$

(d) For $z > 0$, use the result of part (a) and the fact that the integrand is even, write

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \tfrac{1}{2} I_{[-1,1]}(zx) \frac{e^{-x^2/2}}{\sqrt{2\pi}} |x| \, dx \;=\; \int_0^{1/z} x \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx \\ &= \frac{1}{\sqrt{2\pi}} (-e^{-x^2/2}) \Big|_0^{1/z} \;=\; \frac{1 - e^{-1/(2z^2)}}{\sqrt{2\pi}}. \end{aligned}$$

The same formula holds for $z < 0$, and it is easy to check that $f_Z(0) = 1/\sqrt{2\pi}$.

(e) Since $Z \geq 0$, for $z \geq 0$, we use the result from part (a) to write

$$\begin{aligned} f_Z(z) &= \int_0^{\infty} \tfrac{zx}{\lambda^2} e^{-(zx/\lambda)^2/2} \tfrac{x}{\lambda^2} e^{-(x/\lambda)^2/2} x \, dx \;=\; z \int_0^{\infty} \left(\tfrac{x}{\lambda}\right)^3 e^{-(x/\lambda)^2(z^2+1)/2} \frac{dx}{\lambda} \\ &= z \int_0^{\infty} \theta^3 e^{-\theta^2(z^2+1)/2} \, d\theta \;=\; z \int_0^{\infty} \left(\frac{t}{\sqrt{z^2+1}}\right)^3 e^{-t^2/2} \frac{dt}{\sqrt{z^2+1}} \\ &= \frac{z}{(z^2+1)^2} \int_0^{\infty} t^3 e^{-t^2/2} \, dt \;=\; \frac{2z}{(z^2+1)^2} \int_0^{\infty} s e^{-s} \, ds. \end{aligned}$$

This last integral is the mean of an $\exp(1)$ density, which is one. Hence $f_Z(z) = 2z/(z^2+1)^2$.

34. For the cdf, use the law of total probability, substitution, and independence to write

$$\begin{aligned} F_Z(z) &= \mathsf{P}(Z \leq z) \;=\; \mathsf{P}(Y/\ln X \leq z) \;=\; \int_0^{\infty} \mathsf{P}(Y/\ln X \leq z | X = x) f_X(x) \, dx \\ &= \int_0^{\infty} \mathsf{P}(Y/\ln x \leq z | X = x) f_X(x) \, dx \;=\; \int_0^{\infty} \mathsf{P}(Y/\ln x \leq z) f_X(x) \, dx \\ &= \int_0^{1} \mathsf{P}(Y \geq z \ln x) f_X(x) \, dx + \int_1^{\infty} \mathsf{P}(Y \leq z \ln x) f_X(x) \, dx. \end{aligned}$$

Then

$$\begin{aligned} f_Z(z) &= \int_0^1 -f_Y(z \ln x)(\ln x) f_X(x) \, dx + \int_1^{\infty} f_Y(z \ln x)(\ln x) f_X(x) \, dx \\ &= \int_0^{\infty} f_X(x) f_Y(z \ln x) |\ln x| \, dx. \end{aligned}$$

35. Use the law of total probability, substitution, and independence to write

$$\begin{aligned} \mathsf{E}[e^{(X+Z)U}] &= \int_{-1/2}^{1/2} \mathsf{E}[e^{(X+Z)U} | U = u] \, du \;=\; \int_{-1/2}^{1/2} \mathsf{E}[e^{(X+Z)u} | U = u] \, du \\ &= \int_{-1/2}^{1/2} \mathsf{E}[e^{(X+Z)u}] \, du \;=\; \int_{-1/2}^{1/2} \mathsf{E}[e^{Xu}] \mathsf{E}[e^{Zu}] \, du \;=\; \int_{-1/2}^{1/2} \frac{1}{(1-u)^2} \, du \\ &= \frac{1}{1-u} \bigg|_{-1/2}^{1/2} \;=\; \frac{1}{1-1/2} - \frac{1}{1+1/2} \;=\; 2 - \frac{2}{3} \;=\; \frac{4}{3}. \end{aligned}$$

36. Use the law of total probability and substitution to write

$$\begin{aligned}\mathsf{E}[X^2Y] &= \int_1^2 \mathsf{E}[X^2Y|Y=y]\,dy = \int_1^2 \mathsf{E}[X^2y|Y=y]\,dy = \int_1^2 y\mathsf{E}[X^2|Y=y]\,dy \\ &= \int_1^2 y\cdot(2/y^2)\,dy = 2\int_1^2 1/y\,dy = 2\ln 2.\end{aligned}$$

37. Use the law of total probability and substitution to write

$$\begin{aligned}\mathsf{E}[X^nY^r] &= \int_0^\infty \mathsf{E}[X^nY^r|Y=y]f_Y(y)\,dy = \int_0^\infty \mathsf{E}[X^ny^r|Y=y]f_Y(y)\,dy \\ &= \int_0^\infty y^r\mathsf{E}[X^n|Y=y]f_Y(y)\,dy = \int_0^\infty y^r\frac{\Gamma(n+p)}{y^n\Gamma(p)}f_Y(y)\,dy \\ &= \frac{\Gamma(n+p)}{\Gamma(p)}\int_0^\infty y^{r-n}f_Y(y)\,dy = \frac{\Gamma(n+p)}{\Gamma(p)}\mathsf{E}[Y^{r-n}] = \frac{\Gamma(n+p)}{\Gamma(p)}\cdot\frac{(r-n)!}{\lambda^{r-n}}.\end{aligned}$$

38. (a) Use the law of total probability, substitution, and independence to find the cdf. Write

$$\begin{aligned}F_Y(y) &= \mathsf{P}(Y\le y) = \mathsf{P}(e^{VU}\le y) = \int_0^\infty \mathsf{P}(e^{VU}\le y|V=v)f_V(v)\,dv \\ &= \int_0^\infty \mathsf{P}(e^{vU}\le y|V=v)f_V(v)\,dv = \int_0^\infty \mathsf{P}(e^{vU}\le y)f_V(v)\,dv \\ &= \int_0^\infty \mathsf{P}(U\le \tfrac{1}{v}\ln y)f_V(v)\,dv.\end{aligned}$$

Then

$$f_Y(y) = \int_0^\infty \tfrac{1}{vy}f_U(\tfrac{1}{v}\ln y)f_V(v)\,dv.$$

To determine when $f_U(\frac{1}{v}\ln y)$ is nonzero, we consider the cases $y>1$ and $y<1$ separately. For $y>1$, $\frac{1}{v}\ln y\ge 0$ for all $v\ge 0$, and $\frac{1}{v}\ln y\le 1/2$ for $v\ge 2\ln y$. Thus, $f_U(\frac{1}{v}\ln y)=1$ for $v\ge 2\ln y$, and we can write

$$f_Y(y) = \int_{2\ln y}^\infty \tfrac{1}{yv}\cdot ve^{-v}\,dv = \frac{e^{-2\ln y}}{y} = \frac{1}{y^3}.$$

For $y<1$, $f_U(\frac{1}{v}\ln y)=1$ for $-1/2\le\frac{1}{v}\ln y$, or $v\ge -2\ln y$. Thus,

$$f_Y(y) = \int_{-2\ln y}^\infty \tfrac{1}{yv}\cdot ve^{-v}\,dv = \frac{1}{y}e^{2\ln y} = y.$$

Putting this all together, we have

$$f_Y(y) = \begin{cases} 1/y^3, & y\ge 1, \\ y, & 0\le y<1, \\ 0, & y<0.\end{cases}$$

(b) Using the density of part (a),

$$\mathsf{E}[Y] = \int_0^1 y^2\,dy + \int_1^\infty \frac{1}{y^2}\,dy = \frac{1}{3} + 1 = \frac{4}{3}.$$

(c) Using the law of total probability, substitution, and independence, we have

$$\begin{aligned}
\mathsf{E}[e^{VU}] &= \int_{-1/2}^{1/2} \mathsf{E}[e^{VU}|U=u]\,du = \int_{-1/2}^{1/2} \mathsf{E}[e^{Vu}|U=u]\,du \\
&= \int_{-1/2}^{1/2} \mathsf{E}[e^{Vu}]\,du = \int_{-1/2}^{1/2} \frac{1}{(1-u)^2}\,du = \frac{1}{1-u}\bigg|_{-1/2}^{1/2} = \frac{4}{3}.
\end{aligned}$$

39. (a) This problem is interesting because the answer does not depend on the random variable X. Assuming X has a density $f_X(x)$, first write

$$\begin{aligned}
\mathsf{E}[\cos(X+Y)] &= \int_{-\infty}^{\infty} \mathsf{E}[\cos(X+Y)|X=x]\,f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} \mathsf{E}[\cos(x+Y)|X=x]\,f_X(x)\,dx.
\end{aligned}$$

Now use the conditional density of Y given $X = x$ to write

$$\mathsf{E}[\cos(x+Y)|X=x] = \int_{x-\pi}^{x+\pi} \cos(x+y)\,\frac{dy}{2\pi} = \int_{2x-\pi}^{2x+\pi} \cos\theta\,\frac{d\theta}{2\pi} = 0,$$

since we are integrating $\cos\theta$ over an interval of length 2π. Thus, $\mathsf{E}[\cos(X+Y)] = 0$ as well.

(b) Write

$$\begin{aligned}
\mathsf{P}(Y>y) &= \int_{-\infty}^{\infty} \mathsf{P}(Y>y|X=x)\,f_X(x)\,dx = \int_1^2 \mathsf{P}(Y>y|X=x)\,dx \\
&= \int_1^2 e^{-xy}\,dx = \frac{e^{-y}-e^{-2y}}{y}.
\end{aligned}$$

(c) Begin in the usual way by writing

$$\begin{aligned}
\mathsf{E}[Xe^Y] &= \int_{-\infty}^{\infty} \mathsf{E}[Xe^Y|X=x]\,f_X(x)\,dx = \int_{-\infty}^{\infty} \mathsf{E}[xe^Y|X=x]\,f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} x\mathsf{E}[e^Y|X=x]\,f_X(x)\,dx.
\end{aligned}$$

Now observe that

$$\mathsf{E}[e^Y|X=x] = \mathsf{E}[e^{sY}|X=x]\Big|_{s=1} = e^{s^2x^2/2}\Big|_{s=1} = e^{x^2/2}.$$

Then continue with

$$\begin{aligned}
\mathsf{E}[Xe^Y] &= \int_{-\infty}^{\infty} xe^{x^2/2} f_X(x)\,dx = \frac{1}{4}\int_3^7 xe^{x^2/2}\,dx = \frac{1}{4}\big(e^{x^2/2}\big)\Big|_3^7 \\
&= \frac{e^{49/2}-e^{9/2}}{4}.
\end{aligned}$$

(d) Write

$$\begin{aligned}
\mathsf{E}[\cos(XY)] &= \int_{-\infty}^{\infty} \mathsf{E}[\cos(XY)|X=x] f_X(x)\,dx = \int_1^2 \mathsf{E}[\cos(xY)|X=x]\,dx \\
&= \int_1^2 \mathsf{E}[\mathrm{Re}(e^{jxY})|X=x]\,dx = \mathrm{Re}\int_1^2 \mathsf{E}[e^{jxY}|X=x]\,dx \\
&= \mathrm{Re}\int_1^2 e^{-x^2(1/x)/2}\,dx = \int_1^2 e^{-x/2}\,dx = 2(e^{-1/2}-e^{-1}).
\end{aligned}$$

40. Using the law of total probability, substitution, and independence,

$$\begin{aligned}
M_Y(s) &= \mathsf{E}[e^{sY}] = \mathsf{E}[e^{sZX}] = \int_0^{\infty} \mathsf{E}[e^{sZX}|Z=z] f_Z(z)\,dz \\
&= \int_0^{\infty} \mathsf{E}[e^{szX}|Z=z] f_Z(z)\,dz = \int_0^{\infty} \mathsf{E}[e^{szX}] f_Z(z)\,dz \\
&= \int_0^{\infty} e^{(sz)^2\sigma^2/2} f_Z(z)\,dz = \int_0^{\infty} e^{(sz)^2\sigma^2/2} z e^{-z^2/2}\,dz \\
&= \int_0^{\infty} z e^{-(1-s^2\sigma^2)z^2/2}\,dz.
\end{aligned}$$

Now make the change of variable $t = z\sqrt{1-s^2\sigma^2}$ to get

$$M_Y(s) = \int_0^{\infty} \frac{t}{\sqrt{1-s^2\sigma^2}} e^{-t^2/2} \frac{dt}{\sqrt{1-s^2\sigma^2}} = \frac{1}{1-s^2\sigma^2} = \frac{1/\sigma^2}{1/\sigma^2 - s^2}.$$

Hence, $Y \sim \text{Laplace}(1/\sigma)$.

41. Using the law of total probability and substitution,

$$\begin{aligned}
\mathsf{E}[X^n Y^m] &= \int_0^{\infty} \mathsf{E}[X^n Y^m|Y=y] f_Y(y)\,dy = \int_0^{\infty} \mathsf{E}[X^n y^m|Y=y] f_Y(y)\,dy \\
&= \int_0^{\infty} y^m \mathsf{E}[X^n|Y=y] f_Y(y)\,dy = \int_0^{\infty} y^m 2^{n/2} y^n \Gamma(1+n/2) f_Y(y)\,dy \\
&= 2^{n/2}\Gamma(1+n/2)\mathsf{E}[Y^{n+m}] = 2^{n/2}\Gamma(1+n/2)\frac{(n+m)!}{\beta^{n+m}}.
\end{aligned}$$

42. (a) We use the law of total probability, substitution, and independence to write

$$\begin{aligned}
F_Z(z) &= \mathsf{P}(Z \le z) = \mathsf{P}(X/Y \le z) = \int_0^{\infty} \mathsf{P}(X/Y \le z|Y=y) f_Y(y)\,dy \\
&= \int_0^{\infty} \mathsf{P}(X/y \le z|Y=y) f_Y(y)\,dy = \int_0^{\infty} \mathsf{P}(X \le zy|Y=y) f_Y(y)\,dy \\
&= \int_0^{\infty} \mathsf{P}(X \le zy) f_Y(y)\,dy = \int_0^{\infty} F_X(zy) f_Y(y)\,dy.
\end{aligned}$$

Differentiating, we have

$$f_Z(z) = \int_0^{\infty} f_X(zy) y \cdot f_Y(y)\,dy = \int_0^{\infty} \lambda \frac{(\lambda zy)^{p-1} e^{-\lambda zy}}{\Gamma(p)} \cdot \lambda \frac{(\lambda y)^{q-1} e^{-\lambda y}}{\Gamma(q)} \cdot y\,dy.$$

Making the change of variable $w = \lambda y$, we obtain

$$\begin{aligned} f_Z(z) &= \int_0^\infty \frac{(zw)^{p-1}e^{-zw}}{\Gamma(p)} \cdot \frac{w^{q-1}e^{-w}}{\Gamma(q)} \cdot w\,dw \\ &= \frac{z^{p-1}}{\Gamma(p)\Gamma(q)} \int_0^\infty w^{p+q-1}e^{-w(1+z)}\,dw. \end{aligned}$$

Now make the change of variable $\theta = w(1+z)$ so that $d\theta = (1+z)\,dw$ and $w = \theta/(1+z)$. Then

$$\begin{aligned} f_Z(z) &= \frac{z^{p-1}}{\Gamma(p)\Gamma(q)} \int_0^\infty \left(\frac{\theta}{1+z}\right)^{p+q-1} e^{-\theta}\,\frac{d\theta}{(1+z)} \\ &= \frac{z^{p-1}}{\Gamma(p)\Gamma(q)(1+z)^{p+q}} \underbrace{\int_0^\infty \theta^{p+q-1}e^{-\theta}\,d\theta}_{=\,\Gamma(p+q)} \\ &= \frac{z^{p-1}}{B(p,q)(1+z)^{p+q}}. \end{aligned}$$

(b) Starting with $V := Z/(1+Z)$, we first write

$$\begin{aligned} F_V(v) &= \mathsf{P}\left(\frac{Z}{1+Z} \leq v\right) = \mathsf{P}(Z \leq v + vZ) = \mathsf{P}(Z(1-v) \leq v) \\ &= \mathsf{P}(Z \leq v/(1-v)). \end{aligned}$$

Differentiating, we have

$$f_V(v) = f_Z\left(\frac{v}{1-v}\right)\frac{(1-v)+v}{(1-v)^2} = f_Z\left(\frac{v}{1-v}\right)\frac{1}{(1-v)^2}.$$

Now apply the formula derived in part (a) and use the fact that

$$1 + \frac{v}{1-v} = \frac{1}{1-v}$$

to get

$$f_V(v) = \frac{[v/(1-v)]^{p-1}}{B(p,q)(\frac{1}{1-v})^{p+q}} \cdot \frac{1}{(1-v)^2} = \frac{v^{p-1}(1-v)^{q-1}}{B(p,q)},$$

which is the beta density with parameters p and q.

43. Put $q := (n-1)p$ and $Z_i := \sum_{j \neq i} X_j$, which is gamma$(q,\lambda)$ by Problem 55(c) in Chapter 4. Now observe that $Y_i = X_i/(X_i + Z_i)$, which has a beta density with parameters p and $q := (n-1)p$ by Problem 42(b).

44. Using the law of total probability, substitution, and independence,

$$\begin{aligned} F_Z(z) &= \mathsf{P}(Z \leq z) = \mathsf{P}(X/\sqrt{Y/k} \leq z) = \mathsf{P}(X \leq z\sqrt{Y/k}\,) \\ &= \int_0^\infty \mathsf{P}(X \leq z\sqrt{Y/k}\,|Y=y) f_Y(y)\,dy = \int_0^\infty \mathsf{P}(X \leq z\sqrt{y/k}\,|Y=y) f_Y(y)\,dy \\ &= \int_0^\infty \mathsf{P}(X \leq z\sqrt{y/k}\,) f_Y(y)\,dy. \end{aligned}$$

Then

$$\begin{aligned}
f_Z(z) &= \int_0^\infty f_X(z\sqrt{y/k}\,)\sqrt{y/k}\,f_Y(y)\,dy \\
&= \int_0^\infty \frac{e^{-(z^2y/k)/2}}{\sqrt{2\pi}}\sqrt{y/k}\,\frac{\frac{1}{2}(y/2)^{k/2-1}e^{-y/2}}{\Gamma(k/2)}\,dy \\
&= \frac{1}{2\sqrt{\pi}\sqrt{k}\,\Gamma(k/2)}\int_0^\infty (y/2)^{k/2-1/2}e^{-y(1+z^2/k)/2}\,dy.
\end{aligned}$$

Now make the change of variable $\theta = y(1+z^2/k)/2$, $d\theta = (1+z^2/k)/2\,dy$ to get

$$\begin{aligned}
f_Z(z) &= \frac{1}{\sqrt{\pi}\sqrt{k}\,\Gamma(k/2)}\int_0^\infty \left(\frac{\theta}{1+z^2/k}\right)^{k/2-1/2} e^{-\theta}\,\frac{d\theta}{1+z^2/k} \\
&= \frac{1}{\Gamma(1/2)\sqrt{k}\,\Gamma(k/2)(1+z^2/k)^{k/2+1/2}}\int_0^\infty \theta^{k/2+1/2-1}e^{-\theta}\,d\theta \\
&= \frac{1}{\Gamma(1/2)\sqrt{k}\,\Gamma(k/2)(1+z^2/k)^{k/2+1/2}}\Gamma(k/2+1/2) \;=\; \frac{(1+z^2/k)^{-(k+1)/2}}{\sqrt{k}\,B(1/2,k/2)},
\end{aligned}$$

which is the required Student's t density with k degrees of freedom.

45. We use the law of total probability, substitution, and independence to write

$$\begin{aligned}
F_Z(z) &= \mathsf{P}(Z \le z) \;=\; \mathsf{P}(X/Y \le z) \;=\; \int_0^\infty \mathsf{P}(X/Y \le z | Y = y) f_Y(y)\,dy \\
&= \int_0^\infty \mathsf{P}(X/y \le z | Y = y) f_Y(y)\,dy \;=\; \int_0^\infty \mathsf{P}(X \le zy | Y = y) f_Y(y)\,dy \\
&= \int_0^\infty \mathsf{P}(X \le zy) f_Y(y)\,dy \;=\; \int_0^\infty F_X(zy) f_Y(y)\,dy.
\end{aligned}$$

Differentiating, we have

$$f_Z(z) = \int_0^\infty f_X(zy)y\cdot f_Y(y)\,dy = \int_0^\infty \lambda r\frac{(\lambda zy)^{p-1}e^{-(\lambda zy)^r}}{\Gamma(p/r)}\cdot\lambda r\frac{(\lambda y)^{q-1}e^{-(\lambda y)^r}}{\Gamma(q/r)}\cdot y\,dy.$$

Making the change of variable $w = \lambda y$, we obtain

$$\begin{aligned}
f_Z(z) &= \int_0^\infty r\frac{(zw)^{p-1}e^{-(zw)^r}}{\Gamma(p/r)}\cdot r\frac{w^{q-1}e^{-w^r}}{\Gamma(q/r)}\cdot w\,dw \\
&= \frac{r^2z^{p-1}}{\Gamma(p/r)\Gamma(q/r)}\int_0^\infty w^{p+q-1}e^{-w^r(1+z^r)}\,dw.
\end{aligned}$$

Now make the change of variable $\theta = w^r(1+z^r)$ so that $d\theta = rw^{r-1}(1+z^r)\,dw$ and $w = (\theta/[1+z^r])^{1/r}$. Then

$$f_Z(z) = \frac{r^2z^{p-1}}{\Gamma(p/r)\Gamma(q/r)}\int_0^\infty\left(\frac{\theta}{1+z^r}\right)^{(p+q-1)/r} e^{-\theta}\,\frac{d\theta}{r(1+z^r)\left(\frac{\theta}{1+z^r}\right)^{(r-1)/r}}$$

$$= \frac{rz^{p-1}}{\Gamma(p/r)\Gamma(q/r)(1+z^r)^{(p+q)/r}} \underbrace{\int_0^\infty \theta^{(p+q)/r-1} e^{-\theta}\, d\theta}_{=\Gamma((p+q)/r)}$$

$$= \frac{rz^{p-1}}{B(p/r,q/r)(1+z^r)^{(p+q)/r}}.$$

46. For $0 < z \le 1$,

$$f_Z(z) = \frac{1}{4}\int_0^z y^{-1/2}(z-y)^{-1/2}\, dy = \frac{1}{4z}\int_0^z \frac{1}{\sqrt{(y/z)(1-(y/z))}}\, dy.$$

Now make the change of variable $t^2 = y/z$, $2t\,dt = dy/z$ to get

$$f_Z(z) = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt = \frac{1}{2}\sin^{-1} t\Big|_0^1 = \pi/4.$$

Next, for $1 < z \le 2$,

$$\begin{aligned} f_Z(z) &= \frac{1}{4z}\int_{z-1}^1 \frac{1}{\sqrt{(y/z)(1-(y/z))}}\, dy = \frac{1}{2}\int_{\sqrt{1-1/z}}^{1/\sqrt{z}} \frac{1}{\sqrt{1-t^2}}\, dt \\ &= \frac{1}{2}\sin^{-1} t\Big|_{\sqrt{1-1/z}}^{1/\sqrt{z}} = \frac{1}{2}\big[\sin^{-1}(1/\sqrt{z}\,) - \sin^{-1}(\sqrt{1-1/z}\,)\big]. \end{aligned}$$

Putting this all together yields

$$f_Z(z) = \begin{cases} \pi/4, & 0 < z \le 1, \\ \frac{1}{2}\big[\sin^{-1}(1/\sqrt{z}\,) - \sin^{-1}(\sqrt{1-1/z}\,)\big], & 1 < z \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

47. Let ψ denote the $N(0,1)$ density. Using

$$f_{UV}(u,v) = \psi_\rho(u,v) = \psi(u)\cdot \underbrace{\frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{v-\rho u}{\sqrt{1-\rho^2}}\right)}_{N(\rho u,1-\rho^2)\text{ density in } v},$$

we see that

$$\begin{aligned} f_U(u) &= \int_{-\infty}^\infty f_{UV}(u,v)\, dv = \int_{-\infty}^\infty \psi(u)\cdot\frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{v-\rho u}{\sqrt{1-\rho^2}}\right) dv \\ &= \psi(u)\underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{v-\rho u}{\sqrt{1-\rho^2}}\right) dv}_{\text{density in } v \text{ integrates to one}} = \psi(u). \end{aligned}$$

Similarly writing

$$f_{UV}(u,v) = \psi_\rho(u,v) = \frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{u-\rho v}{\sqrt{1-\rho^2}}\right)\cdot\psi(v),$$

we have

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} f_{UV}(u,v)\,du = \int_{-\infty}^{\infty} \frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{u-\rho v}{\sqrt{1-\rho^2}}\right)\cdot\psi(v)\,du \\ &= \psi(v)\int_{-\infty}^{\infty} \frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{v-\rho u}{\sqrt{1-\rho^2}}\right)du = \psi(v). \end{aligned}$$

48. Using

$$\psi_\rho(u,v) = \psi(u)\cdot\frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{v-\rho u}{\sqrt{1-\rho^2}}\right),$$

we can write

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{\sigma_X\sigma_Y}\psi_\rho\left(\frac{x-m_X}{\sigma_X},\frac{y-m_Y}{\sigma_Y}\right) \\ &= \frac{1}{\sigma_X\sigma_Y}\psi\left(\frac{x-m_X}{\sigma_X}\right)\cdot\frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{\frac{y-m_Y}{\sigma_Y}-\rho\frac{x-m_X}{\sigma_X}}{\sqrt{1-\rho^2}}\right). \qquad (*) \end{aligned}$$

Then in $\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$, make the change of variable $v = (y-m_Y)/\sigma_Y$, $dv = dy/\sigma_Y$ to get

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{1}{\sigma_X}\psi\left(\frac{x-m_X}{\sigma_X}\right)\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{v-\rho\frac{x-m_X}{\sigma_X}}{\sqrt{1-\rho^2}}\right)dv}_{\text{density in } v \text{ integrates to one}} \\ &= \frac{1}{\sigma_X}\psi\left(\frac{x-m_X}{\sigma_X}\right). \end{aligned}$$

Thus, $f_X \sim N(m_X,\sigma_X^2)$. Using this along with $(*)$, we obtain

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1}{\sigma_Y\sqrt{1-\rho^2}}\psi\left(\frac{\frac{y-m_Y}{\sigma_Y}-\rho\frac{x-m_X}{\sigma_X}}{\sqrt{1-\rho^2}}\right) \\ &= \frac{1}{\sigma_Y\sqrt{1-\rho^2}}\psi\left(\frac{y-[m_Y+\frac{\sigma_Y}{\sigma_X}\rho(x-m_X)]}{\sigma_Y\sqrt{1-\rho^2}}\right). \end{aligned}$$

Thus, $f_{Y|X}(\cdot|x) \sim N\Big(m_Y+\frac{\sigma_Y}{\sigma_X}\rho(x-m_X),\sigma_Y^2(1-\rho^2)\Big)$. Proceeding in an analogous way, using

$$\psi_\rho(u,v) = \frac{1}{\sqrt{1-\rho^2}}\psi\left(\frac{u-\rho v}{\sqrt{1-\rho^2}}\right)\cdot\psi(v),$$

we can write

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{\sigma_X\sigma_Y}\psi_\rho\left(\frac{x-m_X}{\sigma_X},\frac{y-m_Y}{\sigma_Y}\right) \\ &= \frac{1}{\sigma_X\sigma_Y\sqrt{1-\rho^2}}\psi\left(\frac{\frac{x-m_X}{\sigma_X}-\rho\frac{y-m_Y}{\sigma_Y}}{\sqrt{1-\rho^2}}\right)\cdot\psi\left(\frac{y-m_Y}{\sigma_Y}\right). \qquad (**) \end{aligned}$$

Then in $\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$, make the change of variable $u = (x-m_X)/\sigma_X$, $du = dx/\sigma_X$ to get

$$f_Y(y) \;=\; \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \;=\; \frac{1}{\sigma_Y}\psi\biggl(\frac{y-m_Y}{\sigma_Y}\biggr)\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{1-\rho^2}}\psi\Biggl(\frac{u-\rho\frac{y-m_Y}{\sigma_Y}}{\sqrt{1-\rho^2}}\Biggr)du}_{\text{density in } u \text{ integrates to one}}$$

$$= \frac{1}{\sigma_Y}\psi\biggl(\frac{y-m_Y}{\sigma_Y}\biggr).$$

Thus, $f_Y \sim N(m_Y, \sigma_Y^2)$. Using this along with $(**)$, we obtain

$$\begin{aligned} f_{X|Y}(x|y) \;&=\; \frac{f_{XY}(x,y)}{f_Y(y)} \;=\; \frac{1}{\sigma_X\sqrt{1-\rho^2}}\psi\Biggl(\frac{\frac{x-m_X}{\sigma_X}-\rho\frac{y-m_Y}{\sigma_Y}}{\sqrt{1-\rho^2}}\Biggr) \\ &=\; \frac{1}{\sigma_X\sqrt{1-\rho^2}}\psi\Biggl(\frac{x-[m_X+\frac{\sigma_X}{\sigma_Y}\rho(y-m_Y)]}{\sigma_X\sqrt{1-\rho^2}}\Biggr). \end{aligned}$$

Thus, $f_{X|Y}(\cdot|y) \sim N\Bigl(m_X + \frac{\sigma_X}{\sigma_Y}\rho(y-m_Y), \sigma_X^2(1-\rho^2)\Bigr)$.

49. From the solution of Problem 48, we have that

$$f_{Y|X}(\cdot|x) \sim N\Bigl(m_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-m_X), \sigma_Y^2(1-\rho^2)\Bigr)$$

and

$$f_{X|Y}(\cdot|y) \sim N\Bigl(m_X + \frac{\sigma_X}{\sigma_Y}\rho(y-m_Y), \sigma_X^2(1-\rho^2)\Bigr).$$

Hence,

$$\mathsf{E}[Y|X=x] \;=\; \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dx \;=\; m_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-m_X),$$

and

$$\mathsf{E}[X|Y=y] \;=\; \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx \;=\; m_X + \frac{\sigma_X}{\sigma_Y}\rho(y-m_Y).$$

50. From Problem 48, we know that $f_X \sim N(m_X, \sigma_X^2)$. Hence, $\mathsf{E}[X] = m_X$ and $\mathsf{E}[X^2] = \mathsf{var}(X) + m_X^2 = \sigma_X^2 + m_X^2$. To compute $\mathsf{cov}(X,Y)$, we use the law of total probability and substitution to write

$$\begin{aligned} \mathsf{cov}(X,Y) \;&=\; \mathsf{E}[(X-m_X)(Y-m_Y)] \;=\; \int_{-\infty}^{\infty} \mathsf{E}[(X-m_X)(Y-m_Y)|Y=y]f_Y(y)\,dy \\ &=\; \int_{-\infty}^{\infty} \mathsf{E}[(X-m_X)(y-m_Y)|Y=y]f_Y(y)\,dy \\ &=\; \int_{-\infty}^{\infty} (y-m_Y)\mathsf{E}[(X-m_X)|Y=y]f_Y(y)\,dy \\ &=\; \int_{-\infty}^{\infty} (y-m_Y)\bigl\{\mathsf{E}[X|Y=y]-m_X\bigr\}f_Y(y)\,dy \\ &=\; \int_{-\infty}^{\infty} (y-m_Y)\Bigl\{\frac{\sigma_X}{\sigma_Y}\rho(y-m_Y)\Bigr\}f_Y(y)\,dy \end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma_X}{\sigma_Y}\rho \int_{-\infty}^{\infty} (y-m_Y)^2 f_Y(y)\,dy \;=\; \frac{\sigma_X}{\sigma_Y}\rho\cdot \mathsf{E}[(Y-m_Y)^2] \\
&= \frac{\sigma_X}{\sigma_Y}\rho\cdot\sigma_Y^2 \;=\; \sigma_X\sigma_Y\rho.
\end{aligned}$$

It then follows that

$$\frac{\mathsf{cov}(X,Y)}{\sigma_X\sigma_Y} \;=\; \rho.$$

51. (a) Using the results of Problem 47, we have

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv \;=\; \frac{1}{2}\int_{-\infty}^{\infty} \psi_{\rho_1}(u,v)+\psi_{\rho_2}(u,v)\,dv \\
&= \frac{1}{2}[\psi(u)+\psi(u)] \;=\; \psi(u),
\end{aligned}$$

and

$$\begin{aligned}
f_V(v) &= \int_{-\infty}^{\infty} f_{UV}(u,v)\,du \;=\; \frac{1}{2}\int_{-\infty}^{\infty} \psi_{\rho_1}(u,v)+\psi_{\rho_2}(u,v)\,du \\
&= \frac{1}{2}[\psi(v)+\psi(v)] \;=\; \psi(v).
\end{aligned}$$

Thus, f_U and f_V are $N(0,1)$ densities.

(b) Write

$$\begin{aligned}
\overline{\rho} &:= \mathsf{E}[UV] \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv\,f_{UV}(u,v)\,du\,dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv\cdot\frac{1}{2}[\psi_{\rho_1}(u,v)+\psi_{\rho_2}(u,v)]\,du\,dv \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv\psi_{\rho_1}(u,v)\,du\,dv + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv\psi_{\rho_2}(u,v)\,du\,dv\right] \\
&= \frac{\rho_1+\rho_2}{2}.
\end{aligned}$$

(c) If indeed

$$f_{UV}(u,v) \;=\; \frac{1}{2}[\psi_{\rho_1}(u,v)+\psi_{\rho_2}(u,v)]$$

is a bivariate normal density, then

$$f_{UV}(u,v) \;=\; \frac{\exp\Bigl(\frac{-1}{2(1-\overline{\rho}^{\,2})}[u^2-2\overline{\rho}uv+v^2]\Bigr)}{2\pi\sqrt{1-\overline{\rho}^{\,2}}}.$$

In particular then,

$$f_{UV}(u,u) \;=\; \frac{1}{2}[\psi_{\rho_1}(u,u)+\psi_{\rho_2}(u,u)],$$

or

$$\frac{e^{-u^2/(1+\overline{\rho})}}{2\pi\sqrt{1-\overline{\rho}^{\,2}}} \;=\; \frac{1}{2}\left[\frac{e^{-u^2/(1+\rho_1)}}{2\pi\sqrt{1-\rho_1^2}}+\frac{e^{-u^2/(1+\rho_2)}}{2\pi\sqrt{1-\rho_2^2}}\right].$$

Since $t := u^2 \geq 0$ is arbitrary, part (iii) of the hint tells us that

$$\frac{1}{2\pi\sqrt{1-\overline{\rho}^{\,2}}} \;=\; \frac{-1}{4\pi\sqrt{1-\overline{\rho}_1^2}} \;=\; \frac{-1}{4\pi\sqrt{1-\overline{\rho}_2^2}} \;=\; 0,$$

which is false.

(d) First observe that

$$f_{V|U}(v|u) \;=\; \frac{f_{UV}(u,v)}{f_U(u)} \;=\; \frac{1}{2}\Bigg(\underbrace{\frac{\psi_{\rho_1}(u,v)}{\psi(u)}}_{N(\rho_1 u,1-\rho_1^2)}+\underbrace{\frac{\psi_{\rho_2}(u,v)}{\psi(u)}}_{N(\rho_2 u,1-\rho_2^2)}\Bigg).$$

Hence,

$$\begin{aligned}\int_{-\infty}^{\infty} v^2 f_{V|U}(v|u)\,dv \;&=\; \frac{1}{2}[(1-\rho_1^2)+(\rho_1 u)^2+(1-\rho_2^2)+(\rho_2 u)^2] \\ &=\; \frac{1}{2}[2-\rho_1^2-\rho_2^2+(\rho_1^2+\rho_2^2)u^2],\end{aligned}$$

which depends on u unless $\rho_1=\rho_2=0$.

52. For $u_0, v_0 \geq 0$, let $D := \{(u,v) : u \geq u_0, v \geq v_0\}$. Then

$$\begin{aligned}\mathsf{P}(U>u_0, V>v_0) \;&=\; \iint\limits_D \psi_\rho(u,v)\,du\,dv \\ &=\; \int_0^{\tan^{-1}(v_0/u_0)}\int_{v_0/\sin\theta}^{\infty} \psi_\rho(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \qquad (\#) \\ &\quad+\int_{\tan^{-1}(v_0/u_0)}^{\pi/2}\int_{u_0/\cos\theta}^{\infty} \psi_\rho(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta. \quad (\#\#)\end{aligned}$$

Now, since

$$\psi_\rho(r\cos\theta, r\sin\theta) \;=\; \frac{\exp\Big(\frac{-r^2}{2(1-\rho^2)}[1-\rho\sin 2\theta]\Big)}{2\pi\sqrt{1-\rho^2}},$$

we can express the anti-derivative of $r\psi_\rho(r\cos\theta, r\sin\theta)$ with respect to r in closed form as

$$\frac{-\sqrt{1-\rho^2}}{2\pi(1-\rho\sin 2\theta)}\exp\bigg(\frac{-r^2}{2(1-\rho^2)}[1-\rho\sin 2\theta]\bigg).$$

Hence, the double integral in (#) reduces to

$$\int_0^{\tan^{-1}(v_0/u_0)} h(v_0^2,\theta)\,d\theta.$$

The double integral in (##) reduces to

$$\int_{\tan^{-1}(v_0/u_0)}^{\pi/2} \frac{\sqrt{1-\rho^2}}{2\pi(1-\rho\sin 2\theta)}\exp\bigg(\frac{-u_0^2}{2(1-\rho^2)\cos^2\theta}[1-\rho\sin 2\theta]\bigg)\,d\theta.$$

Applying the change of variable $t = \pi/2 - \theta$, $dt = -d\theta$, we obtain

$$\int_0^{\pi/2-\tan^{-1}(v_0/u_0)} h_\rho(u_0^2, t)\,dt.$$

53. If $\rho = 0$ in Problem 52, then U and V are independent. Taking $u_0 = v_0 = x_0$,

$$\begin{aligned} Q(x_0)^2 &= \int_0^{\pi/4} h_0(x_0^2, \theta)\,d\theta + \int_0^{\pi/4} h_0(x_0^2, \theta)\,d\theta \\ &= 2\int_0^{\pi/4} \frac{\exp[-x_0^2/(2\sin^2\theta)]}{2\pi}\,d\theta \;=\; \frac{1}{\pi}\int_0^{\pi/4} \exp\biggl(\frac{-x_0^2}{2\sin^2\theta}\biggr)d\theta. \end{aligned}$$

54. Factor

$$f_{XYZ}(x,y,z) \;=\; \frac{2\exp[-|x-y| - (y-z)^2/2]}{z^5\sqrt{2\pi}}, \quad z \geq 1,$$

as

$$f_{XYZ}(x,y,z) \;=\; \frac{4}{z^5} \cdot \frac{e^{-(y-z)^2/2}}{\sqrt{2\pi}} \cdot \tfrac{1}{2}e^{-|x-y|}.$$

Now, the second factor on the right is an $N(z,1)$ density in the variable y, and the third factor is a Laplace(1) density that has been shifted to have mean y. Hence, the integral of the third factor with respect to x yields one, and we have by inspection that

$$f_{YZ}(y,z) \;=\; \frac{4}{z^5} \cdot \frac{e^{-(y-z)^2/2}}{\sqrt{2\pi}}, \quad z \geq 1.$$

We then easily see that

$$f_{X|YZ}(x|y,z) \;:=\; \frac{f_{XYZ}(x,y,z)}{f_{YZ}(y,z)} \;=\; \tfrac{1}{2}e^{-|x-y|}, \quad z \geq 1.$$

Next, since the right-hand factor in the formula for $f_{YZ}(y,z)$ is an $N(z,1)$ density in y, if we integrate this factor with respect to y, we get one. Thus,

$$f_Z(z) \;=\; \frac{4}{z^5}, \quad z \geq 1.$$

We can now see that

$$f_{Y|Z}(y|z) \;:=\; \frac{f_{YZ}(y,z)}{f_Z(z)} \;=\; \frac{e^{-(y-z)^2/2}}{\sqrt{2\pi}}, \quad z \geq 1.$$

55. To find $f_{XY}(x,y)$, first write

$$\begin{aligned} f_{XYZ}(x,y,z) &:= \frac{e^{-(x-y)^2/2}e^{-(y-z)^2/2}e^{-z^2/2}}{(2\pi)^{3/2}} \;=\; \frac{e^{-(x-y)^2/2}e^{-(z-y/2)^2}e^{-y^2/4}}{(2\pi)^{3/2}} \\ &= \frac{e^{-(x-y)^2/2}e^{-(y/\sqrt{2}\,)^2/2}}{2\pi} \cdot \frac{e^{-(z-y/2)^2}}{\sqrt{2\pi}} \\ &= \frac{e^{-(x-y)^2/2}e^{-(y/\sqrt{2}\,)^2/2}}{2\pi\sqrt{2}} \cdot \frac{e^{-(z-y/2)^2}}{\sqrt{2\pi}/\sqrt{2}} \\ &= \frac{e^{-(x-y)^2/2}e^{-(y/\sqrt{2}\,)^2/2}}{2\pi\sqrt{2}} \cdot \frac{e^{-[(z-y/2)/(1/\sqrt{2}\,)]^2/2}}{\sqrt{2\pi}/\sqrt{2}}. \end{aligned}$$

Now the right-hand factor is an $N(y/2,1/2)$ density in the variable z. Hence, its integral with respect to z is one. We thus have

$$f_{XY}(x,y) = \frac{e^{-(x-y)^2/2}e^{-(y/\sqrt{2}\,)^2/2}}{2\pi\sqrt{2}} = \frac{e^{-(y/\sqrt{2}\,)^2/2}}{\sqrt{2\pi}\sqrt{2}} \cdot \frac{e^{-(x-y)^2/2}}{\sqrt{2\pi}},$$

which shows that $Y \sim N(0,2)$, and given $Y = y$, X is conditionally $N(y,1)$. Thus,

$$\mathsf{E}[Y] = 0 \quad \text{and} \quad \mathsf{var}(Y) = 2.$$

Next,

$$\mathsf{E}[X] = \int_{-\infty}^{\infty} \mathsf{E}[X|Y=y] f_Y(y)\,dy = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \mathsf{E}[Y] = 0,$$

and

$$\begin{aligned}\mathsf{var}(X) = \mathsf{E}[X^2] &= \int_{-\infty}^{\infty} \mathsf{E}[X^2|Y=y] f_Y(y)\,dy = \int_{-\infty}^{\infty} (1+y^2)\cdot f_Y(y)\,dy \\ &= 1+\mathsf{E}[Y^2] = 1+\mathsf{var}(Y) = 1+2 = 3.\end{aligned}$$

Finally,

$$\begin{aligned}\mathsf{E}[XY] &= \int_{-\infty}^{\infty} \mathsf{E}[XY|Y=y] f_Y(y)\,dy = \int_{-\infty}^{\infty} \mathsf{E}[Xy|Y=y] f_Y(y)\,dy \\ &= \int_{-\infty}^{\infty} y\mathsf{E}[X|Y=y] f_Y(y)\,dy = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \mathsf{E}[Y^2] = \mathsf{var}(Y) = 2.\end{aligned}$$

56. First write

$$\begin{aligned}\mathsf{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[XY|Y=y,Z=z] f_{YZ}(y,z)\,dy\,dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[Xy|Y=y,Z=z] f_{YZ}(y,z)\,dy\,dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\mathsf{E}[X|Y=y,Z=z] f_{YZ}(y,z)\,dy\,dz.\end{aligned}$$

Since $f_{X|YZ}(\cdot\,|y,z) \sim N(y,z^2)$, the preceding conditional expectation is just y. Hence,

$$\mathsf{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{YZ}(y,z)\,dy\,dz = \mathsf{E}[Y^2] = \int_{-\infty}^{\infty} \mathsf{E}[Y^2|Z=z] f_Z(z)\,dz.$$

Since $f_{Y|Z}(\cdot\,|z) \sim \exp(z)$, the preceding conditional expectation is just $2/z^2$. Thus,

$$\mathsf{E}[XY] = \int_{-\infty}^{\infty} \frac{2}{z^2} f_Z(z)\,dz = \int_1^2 \frac{2}{z^2}\cdot\frac{3}{7}z^2\,dz = \frac{6}{7}.$$

A similar analysis yields

$$\begin{aligned}\mathsf{E}[YZ] &= \int_{-\infty}^{\infty} \mathsf{E}[YZ|Z=z] f_Z(z)\,dz = \int_{-\infty}^{\infty} \mathsf{E}[Yz|Z=z] f_Z(z)\,dz \\ &= \int_{-\infty}^{\infty} z\mathsf{E}[Y|Z=z] f_Z(z)\,dz = \int_{-\infty}^{\infty} z(1/z) f_Z(z)\,dz = \int_{-\infty}^{\infty} f_Z(z)\,dz = 1.\end{aligned}$$

57. Write

$$\begin{aligned}
\mathsf{E}[XYZ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[XYZ|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[Xyz|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yz\mathsf{E}[X|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz.
\end{aligned}$$

Since $f_{X|YZ}(\cdot|y,z)$ is a shifted Laplace density with mean y, the preceding conditional expectation is just y. Hence,

$$\begin{aligned}
\mathsf{E}[XYZ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 z f_{YZ}(y,z)\,dy\,dz \;=\; \mathsf{E}[Y^2Z] \;=\; \int_{-\infty}^{\infty} \mathsf{E}[Y^2Z|Z=z]\,f_Z(z)\,dy \\
&= \int_{-\infty}^{\infty} \mathsf{E}[Y^2z|Z=z]\,f_Z(z)\,dy \;=\; \int_{-\infty}^{\infty} z\mathsf{E}[Y^2|Z=z]\,f_Z(z)\,dy.
\end{aligned}$$

Since $f_{Y|Z}(\cdot|z) \sim N(z,1)$, the preceding conditional expectation is just $1+z^2$. Thus,

$$\begin{aligned}
\mathsf{E}[XYZ] &= \int_{-\infty}^{\infty} z(1+z^2)f_Z(z)\,dz \;=\; \int_1^{\infty}[z+z^3]\cdot 4/z^5\,dz \;=\; 4\int_1^{\infty} z^{-4}+z^{-2}\,dz \\
&= 4(1/3+1) \;=\; 16/3.
\end{aligned}$$

58. Write

$$\begin{aligned}
\mathsf{E}[XYZ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[XYZ|X=x,Y=y]\,f_{XY}(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[xyZ|X=x,Y=y]\,f_{XY}(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\mathsf{E}[Z|X=x,Y=y]\,f_{XY}(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2y f_{XY}(x,y)\,dx\,dy \;=\; \mathsf{E}[X^2Y] \;=\; \int_{-\infty}^{\infty} \mathsf{E}[X^2Y|X=x]\,f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} \mathsf{E}[x^2Y|X=x]\,f_X(x)\,dx \;=\; \int_{-\infty}^{\infty} x^2\mathsf{E}[Y|X=x]\,f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} x^3 f_X(x)\,dx \;=\; \int_1^2 x^3\,dx \;=\; 15/4.
\end{aligned}$$

59. We use the law of total probability, substitution, and independence to write

$$\begin{aligned}
\varphi_Y(\nu) &= \mathsf{E}[e^{j\nu Y}] \;=\; \mathsf{E}[e^{j\nu\sum_{i=1}^N X_i}] \;=\; \sum_{n=1}^{\infty} \mathsf{E}[e^{j\nu\sum_{i=1}^N X_i}|N=n]\mathsf{P}(N=n) \\
&= \sum_{n=1}^{\infty} \mathsf{E}[e^{j\nu\sum_{i=1}^n X_i}|N=n]\mathsf{P}(N=n) \\
&= \sum_{n=1}^{\infty} \mathsf{E}[e^{j\nu\sum_{i=1}^n X_i}]\mathsf{P}(N=n)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \mathsf{E}\left[\prod_{i=1}^{n} e^{j\nu X_i}\right] \mathsf{P}(N=n) \\
&= \sum_{n=1}^{\infty} \left[\prod_{i=1}^{n} \mathsf{E}[e^{j\nu X_i}]\right] \mathsf{P}(N=n) \\
&= \sum_{n=1}^{\infty} \varphi_X(\nu)^n \mathsf{P}(N=n) \;=\; G_N(\varphi_X(\nu)).
\end{aligned}$$

Now, if $N \sim \text{geometric}_1(p)$, $G_N(z) = [(1-p)z]/[1-pz]$, and if $X \sim \exp(\lambda)$, $\varphi_X(\nu) = \lambda/(\lambda - j\nu)$. Then

$$\varphi_Y(\nu) = \frac{(1-p)\varphi_X(\nu)}{1-p\varphi_X(\nu)} = \frac{(1-p)\lambda/(\lambda-j\nu)}{1-p\lambda/(\lambda-j\nu)} = \frac{(1-p)\lambda}{(\lambda-j\nu)-p\lambda} = \frac{(1-p)\lambda}{(1-p)\lambda-j\nu},$$

which is the $\exp((1-p)\lambda)$ characteristic function. Thus, $Y \sim \exp((1-p)\lambda)$.

CHAPTER 8
Problem Solutions

1. We have

$$\begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 230 & 280 & 330 \\ 340 & 410 & 480 \\ 450 & 540 & 630 \end{bmatrix}$$

and

$$\operatorname{tr} \begin{bmatrix} 230 & 280 & 330 \\ 340 & 410 & 480 \\ 450 & 540 & 630 \end{bmatrix} = 1270.$$

2. **MATLAB.** See the answer to the previous problem.

3. **MATLAB.** We have

$$A' = \begin{bmatrix} 7 & 4 \\ 8 & 5 \\ 9 & 6 \end{bmatrix}.$$

4. Write

$$\operatorname{tr}(AB) = \sum_{i=1}^{r} (AB)_{ii} = \sum_{i=1}^{r} \left(\sum_{k=1}^{n} A_{ik} B_{ki} \right) = \sum_{k=1}^{n} \left(\sum_{i=1}^{r} B_{ki} A_{ik} \right) = \sum_{k=1}^{n} (BA)_{kk} = \operatorname{tr}(BA).$$

5. (a) Write

$$\operatorname{tr}(AB') = \sum_{i=1}^{r} (AB')_{ii} = \sum_{i=1}^{r} \left(\sum_{k=1}^{n} A_{ik} (B')_{ki} \right) = \sum_{i=1}^{r} \sum_{k=1}^{n} A_{ik} B_{ik}.$$

(b) If $\operatorname{tr}(AB') = 0$ for all B, then in particular, it is true for $B = A$; i.e.,

$$0 = \operatorname{tr}(AA') = \sum_{i=1}^{r} \sum_{k=1}^{n} A_{ik}^2,$$

which implies $A_{ik} = 0$ for all i and k. In other words, A is the zero matrix of size $r \times n$.

6. Following the hint, we first write

$$0 \le \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y \rangle = \|x\|^2 - 2\lambda \langle x, y \rangle + \lambda^2 \|y\|^2.$$

Taking $\lambda = \langle x, y \rangle / \|y\|^2$ yields

$$0 \le \|x\|^2 - 2 \frac{|\langle x, y \rangle|^2}{\|y\|^2} + \frac{|\langle x, y \rangle|^2}{\|y\|^4} \|y\|^2 = \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2},$$

which can be rearranged to get $|\langle x,y\rangle|^2 \leq \|x\|^2\,\|y\|^2$. Conversely, suppose $|\langle x,y\rangle|^2 = \|x\|^2\,\|y\|^2$. There are two cases to consider. If $y \neq 0$, then reversing the above sequence of observations implies $0 = \|x-\lambda y\|^2$, which implies $x = \lambda y$. On the other hand, if $y = 0$ and if

$$|\langle x,y\rangle| \;=\; \|x\|\,\|y\|,$$

then we must have $\|x\| = 0$; i.e., $x = 0$ and $y = 0$, and in this case $x = \lambda y$ for all λ.

7. Consider the ij component of $\mathsf{E}[XB]$. Since

$$\begin{aligned}(\mathsf{E}[XB])_{ij} &= \mathsf{E}[(XB)_{ij}] \;=\; \mathsf{E}\Big[\sum_k X_{ik}B_{kj}\Big] \;=\; \sum_k \mathsf{E}[X_{ik}]B_{kj} \;=\; \sum_k (\mathsf{E}[X])_{ik}B_{kj} \\ &= (\mathsf{E}[X]B)_{ij}\end{aligned}$$

holds for all ij, $\mathsf{E}[XB] = \mathsf{E}[X]B$.

8. $\operatorname{tr}(\mathsf{E}[X]) \;=\; \sum_i (\mathsf{E}[X])_{ii} \;=\; \sum_i \mathsf{E}[X_{ii}] \;=\; \mathsf{E}\Big[\sum_i X_{ii}\Big] \;=\; \mathsf{E}[\operatorname{tr}(X)]$.

9. Write

$$\begin{aligned}\mathsf{E}[\|X-\mathsf{E}[X]\|^2] &= \mathsf{E}[(X-\mathsf{E}[X])'(X-\mathsf{E}[X])], \quad \text{which is a scalar,} \\ &= \operatorname{tr}\Big\{\mathsf{E}[(X-\mathsf{E}[X])'(X-\mathsf{E}[X])]\Big\} \\ &= \mathsf{E}\Big[\operatorname{tr}\Big\{(X-\mathsf{E}[X])'(X-\mathsf{E}[X])\Big\}\Big], \quad \text{by Problem 8,} \\ &= \mathsf{E}\Big[\operatorname{tr}\Big\{(X-\mathsf{E}[X])(X-\mathsf{E}[X])'\Big\}\Big], \quad \text{by Problem 4,} \\ &= \operatorname{tr}\Big\{\mathsf{E}[(X-\mathsf{E}[X])(X-\mathsf{E}[X])']\Big\}, \quad \text{by Problem 8,} \\ &= \operatorname{tr}(C) \;=\; \sum_{i=1}^n C_{ii} \;=\; \sum_{i=1}^n \operatorname{var}(X_i).\end{aligned}$$

10. Since $\mathsf{E}\big[[X,Y,Z]'\big] = \big[\mathsf{E}[X],\mathsf{E}[Y],\mathsf{E}[Z]\big]'$, and

$$\operatorname{cov}([X,Y,Z]') \;=\; \begin{bmatrix} \mathsf{E}[X^2] & \mathsf{E}[XY] & \mathsf{E}[XZ] \\ \mathsf{E}[YX] & \mathsf{E}[Y^2] & \mathsf{E}[YZ] \\ \mathsf{E}[ZX] & \mathsf{E}[ZY] & \mathsf{E}[Z^2] \end{bmatrix} - \begin{bmatrix} \mathsf{E}[X]^2 & \mathsf{E}[X]\mathsf{E}[Y] & \mathsf{E}[X]\mathsf{E}[Z] \\ \mathsf{E}[Y]\mathsf{E}[X] & \mathsf{E}[Y]^2 & \mathsf{E}[Y]\mathsf{E}[Z] \\ \mathsf{E}[Z]\mathsf{E}[X] & \mathsf{E}[Z]\mathsf{E}[Y] & \mathsf{E}[Z]^2 \end{bmatrix},$$

we begin by computing all the entries of these matrices. For the mean vector,

$$\mathsf{E}[Z] \;=\; \int_1^2 z\cdot\tfrac{3}{7}z^2\,dz \;=\; \tfrac{3}{7}\int_1^2 z^3\,dz \;=\; \tfrac{3}{7}\cdot\tfrac{1}{4}z^4\Big|_1^2 \;=\; 3\cdot 15/28 \;=\; 45/28.$$

Next,

$$\mathsf{E}[Y] \;=\; \int_1^2 \mathsf{E}[Y|Z=z]f_Z(z)\,dz \;=\; \int_1^2 \tfrac{1}{z}\cdot\tfrac{3}{7}z^2\,dz \;=\; \tfrac{3}{7}\int_1^2 z\,dz \;=\; \tfrac{3}{7}\cdot\tfrac{1}{2}z^2\Big|_1^2 \;=\; 9/14.$$

Since $\mathsf{E}[U] = 0$ and since U and Z are independent,

$$\mathsf{E}[X] \;=\; \mathsf{E}[ZU+Y] \;=\; \mathsf{E}[Z]\mathsf{E}[U]+\mathsf{E}[Y] \;=\; \mathsf{E}[Y] \;=\; 9/14.$$

Thus, the desired mean vector is

$$\mathsf{E}\big[[X,Y,Z]'\big] \;=\; [9/14, 9/14, 45/28]'.$$

We next compute the correlations. First,

$$\begin{aligned}\mathsf{E}[YZ] \;&=\; \int_1^2 \mathsf{E}[YZ|Z=z]f_Z(z)\,dz \;=\; \int_1^2 \mathsf{E}[Yz|Z=z]f_Z(z)\,dz \\ &=\; \int_1^2 z\mathsf{E}[Y|Z=z]f_Z(z)\,dz \;=\; \int_1^2 z(1/z)f_Z(z)\,dz \;=\; \int_1^2 f_Z(z)\,dz \;=\; 1.\end{aligned}$$

Next,

$$\mathsf{E}[Z^2] \;=\; \int_1^2 z^2\cdot\tfrac{3}{7}z^2\,dz \;=\; \tfrac{3}{7}\int_1^2 z^4\,dz \;=\; \tfrac{3}{7}\cdot\tfrac{1}{5}z^5\Big|_1^2 \;=\; 93/35.$$

Again using the fact that $\mathsf{E}[U]=0$ and independence,

$$\mathsf{E}[XZ] \;=\; \mathsf{E}[(ZU+Y)Z] \;=\; \mathsf{E}[Z^2]\mathsf{E}[U]+\mathsf{E}[YZ] \;=\; \mathsf{E}[YZ] \;=\; 1.$$

Now,

$$\mathsf{E}[Y^2] \;=\; \int_1^2 \mathsf{E}[Y^2|Z=z]f_Z(z)\,dz \;=\; \int_1^2 (2/z^2)\cdot\tfrac{3}{7}z^2\,dz \;=\; 6/7.$$

We can now compute

$$\mathsf{E}[XY] \;=\; \mathsf{E}[(ZU+Y)Y] \;=\; \mathsf{E}[ZY]\mathsf{E}[U]+\mathsf{E}[Y^2] \;=\; 6/7,$$

and

$$\begin{aligned}\mathsf{E}[X^2] \;&=\; \mathsf{E}[(ZU+Y)^2] \;=\; \mathsf{E}[Z^2]\mathsf{E}[U^2]+2\mathsf{E}[U]\mathsf{E}[ZY]+\mathsf{E}[Y^2] \\ &=\; \mathsf{E}[Z^2]+\mathsf{E}[Y^2] \;=\; 93/35+6/7 \;=\; 123/35.\end{aligned}$$

We now have that

$$\begin{aligned}\mathsf{cov}([X,Y,Z]') \;&=\; \begin{bmatrix} 123/35 & 6/7 & 1 \\ 6/7 & 6/7 & 1 \\ 1 & 1 & 93/35 \end{bmatrix} - \begin{bmatrix} 81/196 & 81/196 & 405/392 \\ 81/196 & 81/196 & 405/392 \\ 405/392 & 405/392 & 2025/784 \end{bmatrix} \\ &=\; \begin{bmatrix} 3.1010 & 0.4439 & -0.0332 \\ 0.4439 & 0.4439 & -0.0332 \\ -0.0332 & -0.0332 & 0.0742 \end{bmatrix}.\end{aligned}$$

11. Since $\mathsf{E}\big[[X,Y,Z]'\big] = \big[\mathsf{E}[X],\mathsf{E}[Y],\mathsf{E}[Z]\big]'$, and

$$\mathsf{cov}([X,Y,Z]') \;=\; \begin{bmatrix} \mathsf{E}[X^2] & \mathsf{E}[XY] & \mathsf{E}[XZ] \\ \mathsf{E}[YX] & \mathsf{E}[Y^2] & \mathsf{E}[YZ] \\ \mathsf{E}[ZX] & \mathsf{E}[ZY] & \mathsf{E}[Z^2] \end{bmatrix} - \begin{bmatrix} \mathsf{E}[X]^2 & \mathsf{E}[X]\mathsf{E}[Y] & \mathsf{E}[X]\mathsf{E}[Z] \\ \mathsf{E}[Y]\mathsf{E}[X] & \mathsf{E}[Y]^2 & \mathsf{E}[Y]\mathsf{E}[Z] \\ \mathsf{E}[Z]\mathsf{E}[X] & \mathsf{E}[Z]\mathsf{E}[Y] & \mathsf{E}[Z]^2 \end{bmatrix},$$

we compute all the entries of these matrices. To make this job easier, we first factor

$$f_{XYZ}(x,y,z) \;=\; \frac{2\exp[-|x-y|-(y-z)^2/2]}{z^5\sqrt{2\pi}}, \quad z\geq 1,$$

as $f_{X|YZ}(x|y,z)f_{Y|Z}(y|z)f_Z(z)$ by writing

$$f_{XYZ}(x,y,z) \;=\; \tfrac{1}{2}e^{-|x-y|}\cdot\frac{e^{-(y-z)^2/2}}{\sqrt{2\pi}}\cdot\frac{4}{z^5}, \quad z\geq 1.$$

We then see that as a function of x, $f_{X|YZ}(x|y,z)$ is a shifted Laplace(1) density. Similarly, as a function of y, $f_{Y|Z}(y|z)$ is an $N(z,1)$ density. Thus,

$$\begin{aligned}
\mathsf{E}[X] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{E}[X|Y=y,Z=z]f_{YZ}(y,z)\,dy\,dz \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{YZ}(y,z)\,dy\,dz \\
&= \mathsf{E}[Y] \;=\; \int_{-\infty}^{\infty}\mathsf{E}[Y|Z=z]f_Z(z)\,dz \;=\; \int_{-\infty}^{\infty} z f_Z(z)\,dz \;=\; \mathsf{E}[Z] \\
&= \int_1^{\infty} z\cdot\frac{4}{z^5}\,dz \;=\; \int_1^{\infty}\frac{4}{z^4}\,dz \;=\; 4/3.
\end{aligned}$$

Thus, $\mathsf{E}\big[[X,Y,Z]'\big]=[4/3,4/3,4/3]'$. We next compute

$$\begin{aligned}
\mathsf{E}[X^2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{E}[X^2|Y=y,Z=z]f_{YZ}(y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(2+y^2)f_{YZ}(y,z)\,dy\,dz \;=\; 2+\mathsf{E}[Y^2],
\end{aligned}$$

where

$$\mathsf{E}[Y^2] \;=\; \int_{-\infty}^{\infty}\mathsf{E}[Y^2|Z=z]f_Z(z)\,dz \;=\; \int_{-\infty}^{\infty}(1+z^2)f_Z(z)\,dz \;=\; 1+\mathsf{E}[Z^2].$$

Now,

$$\mathsf{E}[Z^2] \;=\; \int_1^{\infty} z^2\frac{4}{z^5}\,dz \;=\; \int_1^{\infty}\frac{4}{z^3}\,dz \;=\; 2.$$

Thus, $\mathsf{E}[Y^2]=3$ and $\mathsf{E}[X^2]=5$. We next turn to

$$\begin{aligned}
\mathsf{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{E}[XY|Y=y,Z=z]f_{YZ}(y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{E}[Xy|Y=y,Z=z]f_{YZ}(y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\mathsf{E}[X|Y=y,Z=z]f_{YZ}(y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{YZ}(y,z)\,dy\,dz \;=\; \mathsf{E}[Y^2] \;=\; 3.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathsf{E}[XZ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{E}[XZ|Y=y,Z=z]f_{YZ}(y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{E}[Xz|Y=y,Z=z]f_{YZ}(y,z)\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yz f_{YZ}(y,z)\,dy\,dz \;=\; \mathsf{E}[YZ] \\
&= \int_{-\infty}^{\infty}\mathsf{E}[YZ|Z=z]f_Z(z)\,dz \;=\; \int_{-\infty}^{\infty} z\mathsf{E}[Y|Z=z]f_Z(z)\,dz \\
&= \int_{-\infty}^{\infty} z^2 f_Z(z)\,dz \;=\; \mathsf{E}[Z^2] \;=\; 2.
\end{aligned}$$

We now have that

$$\begin{aligned}\mathsf{cov}([X,Y,Z]') &= \begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \frac{16}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix} 29 & 11 & 2 \\ 11 & 11 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3.2222 & 1.2222 & 0.2222 \\ 1.2222 & 1.2222 & 0.2222 \\ 0.2222 & 0.2222 & 0.2222 \end{bmatrix}.\end{aligned}$$

12. Since $\mathsf{E}\big[[X,Y,Z]'\big] = \big[\mathsf{E}[X],\mathsf{E}[Y],\mathsf{E}[Z]\big]'$, and

$$\mathsf{cov}([X,Y,Z]') = \begin{bmatrix} \mathsf{E}[X^2] & \mathsf{E}[XY] & \mathsf{E}[XZ] \\ \mathsf{E}[YX] & \mathsf{E}[Y^2] & \mathsf{E}[YZ] \\ \mathsf{E}[ZX] & \mathsf{E}[ZY] & \mathsf{E}[Z^2] \end{bmatrix} - \begin{bmatrix} \mathsf{E}[X]^2 & \mathsf{E}[X]\mathsf{E}[Y] & \mathsf{E}[X]\mathsf{E}[Z] \\ \mathsf{E}[Y]\mathsf{E}[X] & \mathsf{E}[Y]^2 & \mathsf{E}[Y]\mathsf{E}[Z] \\ \mathsf{E}[Z]\mathsf{E}[X] & \mathsf{E}[Z]\mathsf{E}[Y] & \mathsf{E}[Z]^2 \end{bmatrix},$$

we compute all the entries of these matrices. We begin with

$$\begin{aligned}\mathsf{E}[Z] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[Z|Y=y,X=x]\,f_{XY}(x,y)\,dy\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,f_{XY}(x,y)\,dy\,dx = \mathsf{E}[X] = 3/2.\end{aligned}$$

Next,

$$\mathsf{E}[Y] = \int_{-\infty}^{\infty} \mathsf{E}[Y|X=x]\,f_X(x)\,dx = \int_{-\infty}^{\infty} x\,f_X(x)\,dx = \mathsf{E}[X] = 3/2.$$

We now compute

$$\begin{aligned}\mathsf{E}[XY] &= \int_{-\infty}^{\infty} \mathsf{E}[XY|X=x]\,f_X(x)\,dx = \int_{-\infty}^{\infty} \mathsf{E}[xY|X=x]\,f_X(x)\,dx \\ &= \int_{-\infty}^{\infty} x\mathsf{E}[Y|X=x]\,f_X(x)\,dx = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \mathsf{E}[X^2] \\ &= \mathsf{var}(X)+\mathsf{E}[X]^2 = 1/12+(3/2)^2 = 7/3.\end{aligned}$$

Then

$$\begin{aligned}\mathsf{E}[XZ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[XZ|Y=y,X=x]\,f_{XY}(x,y)\,dy\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\mathsf{E}[Z|Y=y,X=x]\,f_{XY}(x,y)\,dy\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{XY}(x,y)\,dy\,dx = \mathsf{E}[X^2] = 7/3,\end{aligned}$$

and

$$\begin{aligned}\mathsf{E}[YZ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[YZ|Y=y,X=x]\,f_{XY}(x,y)\,dy\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\mathsf{E}[Z|Y=y,X=x]\,f_{XY}(x,y)\,dy\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,f_{XY}(x,y)\,dy\,dx = \mathsf{E}[XY] = 7/3.\end{aligned}$$

Next,

$$\mathsf{E}[Y^2] \;=\; \int_{-\infty}^{\infty} \mathsf{E}[Y^2|X=x]\,f_X(x)\,dx \;=\; \int_{-\infty}^{\infty} 2x^2 f_X(x)\,dx \;=\; 2\mathsf{E}[X^2] \;=\; 14/3,$$

and

$$\begin{aligned}
\mathsf{E}[Z^2] \;&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[Z^2|Y=y,X=x]\,f_{XY}(x,y)\,dy\,dx \\
&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+x^2)\,f_{XY}(x,y)\,dy\,dx \;=\; 1+\mathsf{E}[X^2] \;=\; 1+7/3 \;=\; 10/3.
\end{aligned}$$

We now have that

$$\begin{aligned}
\mathsf{cov}([X,Y,Z]') \;&=\; \frac{1}{3}\begin{bmatrix} 7 & 7 & 7 \\ 7 & 14 & 7 \\ 7 & 7 & 10 \end{bmatrix} - \Big(\frac{3}{2}\Big)^2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\
&=\; \frac{1}{12}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 29 & 1 \\ 1 & 1 & 13 \end{bmatrix} \;=\; \begin{bmatrix} 0.0833 & 0.0833 & 0.0833 \\ 0.0833 & 2.4167 & 0.0833 \\ 0.0833 & 0.0833 & 1.0833 \end{bmatrix}.
\end{aligned}$$

13. Since $\mathsf{E}\big[[X,Y,Z]'\big] = \big[\mathsf{E}[X],\mathsf{E}[Y],\mathsf{E}[Z]\big]'$, and

$$\mathsf{cov}([X,Y,Z]') \;=\; \begin{bmatrix} \mathsf{E}[X^2] & \mathsf{E}[XY] & \mathsf{E}[XZ] \\ \mathsf{E}[YX] & \mathsf{E}[Y^2] & \mathsf{E}[YZ] \\ \mathsf{E}[ZX] & \mathsf{E}[ZY] & \mathsf{E}[Z^2] \end{bmatrix} - \begin{bmatrix} \mathsf{E}[X]^2 & \mathsf{E}[X]\mathsf{E}[Y] & \mathsf{E}[X]\mathsf{E}[Z] \\ \mathsf{E}[Y]\mathsf{E}[X] & \mathsf{E}[Y]^2 & \mathsf{E}[Y]\mathsf{E}[Z] \\ \mathsf{E}[Z]\mathsf{E}[X] & \mathsf{E}[Z]\mathsf{E}[Y] & \mathsf{E}[Z]^2 \end{bmatrix},$$

we compute all the entries of these matrices. In order to do this, we first note that $Z \sim N(0,1)$. Next, as a function of y, $f_{Y|Z}(y|z)$ is an $N(z,1)$ density. Similarly, as a function of x, $f_{X|YZ}(x|y,z)$ is an $N(y,1)$ density. Hence, $\mathsf{E}[Z]=0$,

$$\mathsf{E}[Y] \;=\; \int_{-\infty}^{\infty} \mathsf{E}[Y|Z=z]\,f_Z(z)\,dz \;=\; \int_{-\infty}^{\infty} z f_Z(z)\,dz \;=\; \mathsf{E}[Z] \;=\; 0,$$

and

$$\begin{aligned}
\mathsf{E}[X] \;&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[X|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\
&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{YZ}(y,z)\,dy\,dz \;=\; \mathsf{E}[Y] \;=\; 0.
\end{aligned}$$

We next compute

$$\begin{aligned}
\mathsf{E}[XY] \;&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[XY|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\
&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[Xy|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\
&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\mathsf{E}[X|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{YZ}(y,z)\,dy\,dz \\
&=\; \mathsf{E}[Y^2] \;=\; \int_{-\infty}^{\infty} \mathsf{E}[Y^2|Z=z]\,f_Z(z)\,dz \;=\; \int_{-\infty}^{\infty} (1+z^2)\,f_Z(z)\,dz \\
&=\; 1+\mathsf{E}[Z^2] \;=\; 2.
\end{aligned}$$

Then

$$\begin{aligned}\mathsf{E}[YZ] &= \int_{-\infty}^{\infty} \mathsf{E}[YZ|Z=z]\,f_Z(z)\,dz \;=\; \int_{-\infty}^{\infty} z\mathsf{E}[Y|Z=z]\,f_Z(z)\,dz \\ &= \int_{-\infty}^{\infty} z^2 f_Z(z)\,dz \;=\; \mathsf{E}[Z^2] \;=\; 1,\end{aligned}$$

and

$$\begin{aligned}\mathsf{E}[XZ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[XZ|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\mathsf{E}[X|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yz f_{YZ}(y,z)\,dy\,dz \;=\; \mathsf{E}[YZ] \;=\; 1.\end{aligned}$$

Next,

$$\mathsf{E}[Y^2] \;=\; \int_{-\infty}^{\infty} \mathsf{E}[Y^2|Z=z]\,f_Z(z)\,dz \;=\; \int_{-\infty}^{\infty} (1+z^2)\,f_Z(z)\,dz \;=\; 1+\mathsf{E}[Z^2] \;=\; 2,$$

and

$$\begin{aligned}\mathsf{E}[X^2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[X^2|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+y^2)\,f_{YZ}(y,z)\,dy\,dz \;=\; 1+\mathsf{E}[Y^2] \;=\; 3.\end{aligned}$$

Since $\mathsf{E}[Z^2]=1$, we now have that

$$\mathsf{cov}([X,Y,Z]') \;=\; \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

14. We first note that $Z \sim N(0,1)$. Next, as a function of y, $f_{Y|Z}(y|z)$ is an $N(z,1)$ density. Similarly, as a function of x, $f_{X|YZ}(x|y,z)$ is an $N(y,1)$ density. Hence,

$$\begin{aligned}\mathsf{E}[e^{j(\nu_1 X+\nu_2 Y+\nu_3 Z)}] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[e^{j(\nu_1 X+\nu_2 Y+\nu_3 Z)}|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[e^{j(\nu_1 X+\nu_2 y+\nu_3 z)}|Y=y,Z=z]\,f_{YZ}(y,z)\,dy\,dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathsf{E}[e^{j\nu_1 X}|Y=y,Z=z]\,e^{j\nu_2 y}e^{j\nu_3 z}f_{YZ}(y,z)\,dy\,dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\nu_1 y-\nu_1^2/2}e^{j\nu_2 y}e^{j\nu_3 z}f_{YZ}(y,z)\,dy\,dz \\ &= e^{-\nu_1^2/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j(\nu_1+\nu_2)y}e^{j\nu_3 z}f_{YZ}(y,z)\,dy\,dz \\ &= e^{-\nu_1^2/2}\mathsf{E}[e^{j(\nu_1+\nu_2)Y}e^{j\nu_3 Z}]\end{aligned}$$

$$\begin{aligned}
&= e^{-\nu_1^2/2}\int_{-\infty}^{\infty}\mathsf{E}[e^{j(\nu_1+\nu_2)Y}e^{j\nu_3 Z}|Z=z]f_Z(z)\,dz\\
&= e^{-\nu_1^2/2}\int_{-\infty}^{\infty}e^{j\nu_3 z}\mathsf{E}[e^{j(\nu_1+\nu_2)Y}|Z=z]f_Z(z)\,dz\\
&= e^{-\nu_1^2/2}\int_{-\infty}^{\infty}e^{j\nu_3 z}e^{j(\nu_1+\nu_2)z-(\nu_1+\nu_2)^2/2}f_Z(z)\,dz\\
&= e^{-\nu_1^2/2}e^{-(\nu_1+\nu_2)^2/2}\int_{-\infty}^{\infty}e^{j(\nu_1+\nu_2+\nu_3)z}f_Z(z)\,dz\\
&= e^{-\nu_1^2/2}e^{-(\nu_1+\nu_2)^2/2}\mathsf{E}[e^{j(\nu_1+\nu_2+\nu_3)z}]\\
&= e^{-\nu_1^2/2}e^{-(\nu_1+\nu_2)^2/2}e^{-(\nu_1+\nu_2+\nu_3)^2/2}\\
&= e^{-[\nu_1^2+(\nu_1+\nu_2)^2+(\nu_1+\nu_2+\nu_3)^2]/2}.
\end{aligned}$$

15. $R_Y := \mathsf{E}[YY'] = \mathsf{E}[(AX)(AX)'] = \mathsf{E}[AXX'A'] = A\mathsf{E}[XX']A' = AR_XA'$.

16. First, since $R = \mathsf{E}[XX']$, we see that $R' = \mathsf{E}[XX']' = \mathsf{E}[(XX')'] = \mathsf{E}[XX'] = R$. Thus, R is symmetric. Next, define the scalar $Y := c'X$. Then

$$0 \le \mathsf{E}[Y^2] = \mathsf{E}[YY'] = \mathsf{E}[(c'X)(c'X)'] = \mathsf{E}[c'XX'c] = c'\mathsf{E}[XX']c = c'Rc,$$

and we see that R is positive semidefinite.

17. Use the Cauchy–Schwarz inequality to write

$$\begin{aligned}
\big|(C_{XY})_{ij}\big| &= \big|\mathsf{E}[(X_i-m_{X,i})(Y_j-m_{Y,j})]\big|\\
&\le \sqrt{\mathsf{E}[(X_i-m_{X,i})^2]\mathsf{E}[(Y_j-m_{Y,j})^2]} = \sqrt{(C_X)_{ii}(C_Y)_{jj}}.
\end{aligned}$$

18. If

$$P = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix},$$

then

$$\begin{bmatrix}U\\V\end{bmatrix} := P'\begin{bmatrix}X\\Y\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix} = \begin{bmatrix}X\cos\theta+Y\sin\theta\\ -X\sin\theta+Y\cos\theta\end{bmatrix}.$$

We now have to find θ such that $\mathsf{E}[UV]=0$. Write

$$\begin{aligned}
\mathsf{E}[UV] &= \mathsf{E}[(X\cos\theta+Y\sin\theta)(-X\sin\theta+Y\cos\theta)]\\
&= \mathsf{E}[(Y^2-X^2)\sin\theta\cos\theta+XY(\cos^2\theta-\sin^2\theta)]\\
&= \frac{\sigma_Y^2-\sigma_X^2}{2}\sin 2\theta+\mathsf{E}[XY]\cos 2\theta,
\end{aligned}$$

which is zero if and only if

$$\tan 2\theta = \frac{2\mathsf{E}[XY]}{\sigma_X^2-\sigma_Y^2}.$$

Hence,

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\mathsf{E}[XY]}{\sigma_X^2-\sigma_Y^2}\right).$$

19. (a) Write $(e_i e_i')_{mn} = (e_i)_m (e_i')_n$, which equals one if and only if $m = i$ and $n = i$. Hence, $e_i e_i'$ must be all zeros except at position $i\,i$ where it is one.

(b) Write

$$\begin{aligned} E'E &= \begin{bmatrix} e_1 & e_4 & e_5 \end{bmatrix} \begin{bmatrix} e_1' \\ e_4' \\ e_5 \end{bmatrix} \\ &= e_1 e_1' + e_4 e_4' + e_5 e_5' \\ &= \operatorname{diag}(1,0,0,0,0) + \operatorname{diag}(0,0,0,1,0) + \operatorname{diag}(0,0,0,0,1) \\ &= \operatorname{diag}(1,0,0,1,1). \end{aligned}$$

20. (a) We must solve $P'C_X P = \text{diagonal}$. With $X := U+V$, we have

$$\begin{aligned} C_X &= \mathsf{E}[XX'] = \mathsf{E}[(U+V)(U+V)'] = C_U + \mathsf{E}[UV'] + \mathsf{E}[VU'] + C_V \\ &= QMQ' + I = Q(M+I)Q'. \end{aligned}$$

Hence, $Q'C_X Q = M+I$, which is diagonal. The point here is that we may take $P = Q$.

(b) We now put $Y := P'X = QX = Q(U+V)$. Then

$$\begin{aligned} C_Y &= \mathsf{E}[YY'] = \mathsf{E}[Q(U+V)(U'+V')Q'] \\ &= Q\{C_U + \mathsf{E}[UV'] + \mathsf{E}[VU'] + C_V\}Q' = QC_U Q' + QIQ' = M+I. \end{aligned}$$

21. Starting with $u = x+y$ and $v = x-y$, we have

$$x = \frac{u+v}{2} \quad \text{and} \quad y = \frac{u-v}{2}.$$

We can now write

$$dH = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}, \quad \det dH = -1/2,$$

and

$$|\det dH| = 1/2,$$

and so

$$f_{UV}(u,v) = f_{XY}\Big(\frac{u+v}{2}, \frac{u-v}{2}\Big) \cdot \frac{1}{2}.$$

22. Starting with $u = xy$ and $v = y/x$, write $y = xv$. Then $u = x^2 v$, and $x = (u/v)^{1/2}$. We also have $y = (u/v)^{1/2} v = (uv)^{1/2}$. Then

$$\begin{aligned} &\tfrac{\partial x}{\partial u} = (1/2)/\sqrt{uv}, \qquad \tfrac{\partial x}{\partial v} = (-1/2)\sqrt{u}/v^{3/2}, \\ &\tfrac{\partial y}{\partial u} = (1/2)\sqrt{v/u}, \qquad \tfrac{\partial y}{\partial v} = (1/2)\sqrt{u/v}. \end{aligned}$$

In other words,

$$dH = \begin{bmatrix} (1/2)/\sqrt{uv} & (-1/2)\sqrt{u}/v^{3/2} \\ (1/2)\sqrt{v/u} & (1/2)\sqrt{u/v} \end{bmatrix},$$

and so

$$|\det dH| = \left|\frac{1}{4v} + \frac{1}{4v}\right| = \frac{1}{2|v|}.$$

Thus,

$$f_{UV}(u,v) = f_{XY}\big(\sqrt{u/v}, \sqrt{uv}\big)\frac{1}{2v}, \quad u,v > 0.$$

For f_U, write

$$f_U(u) = \int_0^\infty f_{UV}(u,v)\,dv = \int_0^\infty f_{XY}\big(\sqrt{u/v}, \sqrt{uv}\big)\frac{1}{2v}\,dv$$

Now make the change of variable $y = \sqrt{uv}$, or $y^2 = uv$. Then $2y\,dy = u\,dv$, and

$$f_U(u) = \int_0^\infty f_{XY}\Big(\frac{u}{y}, y\Big)\frac{1}{y}\,dy.$$

For f_V, write

$$f_V(v) = \int_0^\infty f_{UV}(u,v)\,du = \int_0^\infty f_{XY}\big(\sqrt{u/v}, \sqrt{uv}\big)\frac{1}{2v}\,du.$$

This time make the change of variable $x = \sqrt{u/v}$ or $x^2 = u/v$. Then $2x\,dx = du/v$, and

$$f_V(v) = \int_0^\infty f_{XY}(x, vx)x\,dx.$$

23. Starting with $u = x$ and $v = y/x$, we find that $y = xv = uv$. Hence,

$$dH = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & u \end{bmatrix}, \quad \det dH = u, \quad \text{and} \quad |\det dH| = |u|.$$

Then

$$f_{UV}(u,v) = f_{XY}(u, uv)|u| = \tfrac{\lambda}{2}e^{-\lambda|u|} \cdot \tfrac{\lambda}{2}e^{-\lambda|uv|} \cdot |u|,$$

and

$$\begin{aligned} f_V(v) &= \int_{-\infty}^\infty \tfrac{\lambda^2}{4}|u|e^{-\lambda(1+|v|)|u|}\,du = \tfrac{\lambda^2}{2}\int_0^\infty ue^{-\lambda(1+|v|)u}\,du \\ &= \frac{\lambda}{2(1+|v|)}\int_0^\infty u \cdot \lambda(1+|v|)e^{-\lambda(1+|v|)u}\,du. \end{aligned}$$

Now, this last integral is the mean of an exponential density with parameter $\lambda(1+|v|)$. Hence,

$$f_V(v) = \frac{\lambda}{2(1+|v|)} \cdot \frac{1}{\lambda(1+|v|)} = \frac{1}{2(1+|v|)^2}.$$

24. Starting with $u = \sqrt{-2\ln x}\cos(2\pi y)$ and $v = \sqrt{-2\ln x}\sin(2\pi y)$, we have

$$u^2 + v^2 \;=\; (-2\ln x)[\cos^2(2\pi y) + \sin^2(2\pi y)] \;=\; -2\ln x.$$

Hence, $x = e^{-(u^2+v^2)/2}$. We also have

$$\frac{v}{u} \;=\; \tan(2\pi y) \quad \text{or} \quad y \;=\; \frac{1}{2\pi}\tan^{-1}(v/u).$$

We can now write

$$\begin{array}{ll} \frac{\partial x}{\partial u} = -ue^{-(u^2+v^2)/2}, & \frac{\partial x}{\partial v} = -ve^{-(u^2+v^2)/2}, \\ \frac{\partial y}{\partial u} = \frac{1}{2\pi}\frac{1}{1+(u/v)^2}\cdot\frac{-v}{u^2}, & \frac{\partial y}{\partial v} = \frac{1}{2\pi}\frac{1}{1+(u/v)^2}\cdot\frac{1}{u}. \end{array}$$

In other words,

$$dH \;=\; \left[\begin{array}{cc} -ue^{-(u^2+v^2)/2} & -ve^{-(u^2+v^2)/2} \\ \frac{1}{2\pi}\frac{1}{1+(u/v)^2}\cdot\frac{-v}{u^2} & \frac{1}{2\pi}\frac{1}{1+(u/v)^2}\cdot\frac{1}{u} \end{array}\right],$$

and so

$$\begin{aligned} |\det dH| \;&=\; \left|\frac{-1}{2\pi}\frac{e^{-(u^2+v^2)/2}}{1+(v/u)^2} - \frac{1}{2\pi}\frac{e^{-(u^2+v^2)/2}}{1+(v/u)^2}\cdot\frac{v^2}{u^2}\right| \\ &=\; \frac{e^{-(u^2+v^2)/2}}{2\pi}\left|\frac{1}{1+(v/u)^2} + \frac{(v/u)^2}{1+(v/u)^2}\right| \\ &=\; \frac{e^{-u^2/2}}{\sqrt{2\pi}}\frac{e^{-v^2/2}}{\sqrt{2\pi}}. \end{aligned}$$

We next use the formula $f_{UV}(u,v) = f_{XY}(x,y)\cdot|\det dH|$, where $x = e^{-(u^2+v^2)/2}$ and $y = \tan^{-1}(v/u)/(2\pi)$. Fortunately, since these formulas for x and y lie in $(0,1]$, $f_{XY}(x,y) = I_{(0,1]}(x)I_{(0,1]}(y) = 1$, and we see that

$$f_{UV}(u,v) \;=\; 1\cdot|\det dH| \;=\; \frac{e^{-u^2/2}}{\sqrt{2\pi}}\frac{e^{-v^2/2}}{\sqrt{2\pi}}.$$

Integrating out v shows that $f_U(u) = e^{-u^2/2}/\sqrt{2\pi}$, and integrating out v shows that $f_V(v) = e^{-v^2/2}/\sqrt{2\pi}$. It now follows that $f_{UV}(u,v) = f_U(u)f_V(v)$, and we see that U and V are independent.

25. Starting with $u = x+y$ and $v = x/(x+y) = x/u$, we see that $v = x/u$, or $x = uv$. Next, from $y = u - x = u - uv$, we get $y = u(1-v)$. We can now write

$$\begin{array}{ll} \frac{\partial x}{\partial u} = v, & \frac{\partial x}{\partial v} = u, \\ \frac{\partial y}{\partial u} = 1-v, & \frac{\partial y}{\partial v} = -u. \end{array}$$

In other words,

$$dH \;=\; \left[\begin{array}{cc} v & u \\ 1-v & -u \end{array}\right],$$

and so

$$|\det dH| \;=\; |-uv - u(1-v)| \;=\; |u|.$$

We next write

$$f_{UV}(u,v) \;=\; f_{XY}(uv, u(1-v))|u|.$$

If X and Y are independent gamma RVs, then for $u > 0$ and $0 < v < 1$,

$$\begin{aligned} f_{UV}(u,v) &= \frac{\lambda(\lambda uv)^{p-1}e^{-\lambda uv}}{\Gamma(p)} \cdot \frac{\lambda(\lambda u(1-v))^{q-1}e^{-\lambda u(1-v)}}{\Gamma(q)} \cdot u \\ &= \frac{\lambda(\lambda u)^{p+q-1}e^{-\lambda u}}{\Gamma(p+q)} \cdot \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} v^{p-1}(1-v)^{q-1}, \end{aligned}$$

which we recognize as the product of a gamma$(p+q,\lambda)$ and a beta(p,q) density. Hence, it is easy to integrate out either u or v and show that $f_{UV}(u,v) = f_U(u)f_V(v)$, where $f_U \sim$ gamma$(p+q,\lambda)$ and $f_V \sim$ beta(p,q). Thus, U and V are independent.

26. Starting with $u = x+y$ and $v = x/y$, write $x = yv$ and then $u = yv+y = y(v+1)$. Solve for $y = u/(v+1)$ and $x = u-y = u-u/(v+1) = uv/(v+1)$. Next,

$$\begin{aligned} &\tfrac{\partial x}{\partial u} = v/(v+1), && \tfrac{\partial x}{\partial v} = u/(v+1)^2, \\ &\tfrac{\partial y}{\partial u} = 1/(v+1), && \tfrac{\partial y}{\partial v} = -u/(v+1)^2. \end{aligned}$$

In other words,

$$dH \;=\; \begin{bmatrix} v/(v+1) & u/(v+1)^2 \\ 1/(v+1) & -u/(v+1)^2 \end{bmatrix},$$

and so

$$|\det dH| \;=\; \left|\frac{-uv}{(v+1)^3} - \frac{u}{(v+1)^3}\right| \;=\; \left|\frac{-u(v+1)}{(v+1)^3}\right| \;=\; \frac{|u|}{(v+1)^2}.$$

We next write

$$f_{UV}(u,v) \;=\; f_{XY}\left(\frac{uv}{v+1}, \frac{u}{v+1}\right)\frac{|u|}{(v+1)^2}.$$

When $X \sim$ gamma(p,λ) and $Y \sim$ gamma(q,λ) are independent, then U and V are nonnegative, and

$$\begin{aligned} f_{UV}(u,v) &= \lambda\frac{[\lambda uv/(v+1)]^{p-1}e^{-\lambda uv/(v+1)}}{\Gamma(p)} \cdot \lambda\frac{[\lambda u/(v+1)]^{q-1}e^{-\lambda u/(v+1)}}{\Gamma(q)} \cdot \frac{u}{(v+1)^2} \\ &= \lambda\frac{(\lambda u)^{p+q-1}e^{-\lambda u}}{\Gamma(p+q)} \cdot \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \cdot \frac{v^{p-1}}{(v+1)^{p+q}} \\ &= \lambda\frac{(\lambda u)^{p+q-1}e^{-\lambda u}}{\Gamma(p+q)} \cdot \frac{v^{p-1}}{B(p,q)(v+1)^{p+q}}, \end{aligned}$$

which shows that U and V are independent with the required marginal densities.

27. From solution of the Example at the end of the section,

$$f_{R,\Theta}(r,\theta) = f_{XY}(r\cos\theta, r\sin\theta)r.$$

What is different in this problem is that X and Y are *correlated* Gaussian random variables; i.e.,

$$f_{XY}(x,y) = \frac{e^{-(x^2-2\rho xy+y^2)/[2(1-\rho^2)]}}{2\pi\sqrt{1-\rho^2}}.$$

Hence,

$$\begin{aligned} f_{R,\Theta}(r,\theta) &= \frac{re^{-(r^2\cos^2\theta-2\rho(r\cos\theta)(r\sin\theta)+r^2\sin^2\theta)/[2(1-\rho^2)]}}{2\pi\sqrt{1-\rho^2}} \\ &= \frac{re^{-r^2(1-\rho\sin 2\theta)/[2(1-\rho^2)]}}{2\pi\sqrt{1-\rho^2}}. \end{aligned}$$

To find the density of Θ, we must integrate this with respect to r. Notice that the integrand is proportional to $re^{-\lambda r^2/2}$, whose anti-derivative is $-e^{-\lambda r^2/2}/\lambda$. Here we have $\lambda = (1-\rho\sin 2\theta)/(1-\rho^2)$. We can now write

$$\begin{aligned} f_\Theta(\theta) &= \int_0^\infty f_{R,\Theta}(r,\theta)\,dr = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_0^\infty re^{-\lambda r^2/2}\,dr = \frac{1}{2\pi\sqrt{1-\rho^2}}\cdot\frac{1}{\lambda} \\ &= \frac{1-\rho^2}{2\pi\sqrt{1-\rho^2}(1-\rho\sin 2\theta)} = \frac{\sqrt{1-\rho^2}}{2\pi(1-\rho\sin 2\theta)}. \end{aligned}$$

28. Since $X \sim N(0,1)$, we have $\mathsf{E}[X] = \mathsf{E}[X^3] = 0$, $\mathsf{E}[X^4] = 3$, and $\mathsf{E}[X^6] = 15$. Since $W \sim N(0,1)$, $\mathsf{E}[W] = 0$ too. Hence, $\mathsf{E}[Y] = \mathsf{E}[X^3+W] = 0$. It then follows that m_Y, m_X and $b = m_X - Am_y$ are zero. We next compute

$$C_{XY} = \mathsf{E}[XY] = \mathsf{E}[X(X^3+W)] = \mathsf{E}[X^4] + \mathsf{E}[X]\mathsf{E}[W] = 3+0 = 3,$$

and

$$C_Y = \mathsf{E}[Y^2] = \mathsf{E}[X^6+2X^3W+W^2] = 15+0+1 = 16.$$

Hence, $AC_Y = C_{XY}$ implies $A = 3/16$, and then $\widehat{X} = A(Y-m_Y)+m_X = (3/16)Y$.

29. First note that since X and W are zero mean, so is Y. Next,

$$C_{XY} = \mathsf{E}[XY] = \mathsf{E}[X(X+W)] = \mathsf{E}[X^2] + \mathsf{E}[XW] = \mathsf{E}[X^2] + \mathsf{E}[X]\mathsf{E}[W] = \mathsf{E}[X^2] = 1,$$

and

$$\begin{aligned} C_Y &= \mathsf{E}[Y^2] = \mathsf{E}[(X+W)^2] = \mathsf{E}[X^2+2XW+W^2] \\ &= \mathsf{E}[X^2] + 2\mathsf{E}[X]\mathsf{E}[W] + \mathsf{E}[W^2] = 1+0+2/\lambda^2 = 1+2/\lambda^2. \end{aligned}$$

Then $A = C_{XY}/C_Y = 1/[1+2/\lambda^2]$, and

$$\widehat{X} = \frac{\lambda^2}{2+\lambda^2}Y.$$

30. We first have $\mathsf{E}[Y] = \mathsf{E}[GX+W] = Gm_X + 0 = Gm_X$. Next,

$$\begin{aligned} C_{XY} &= \mathsf{E}[(X-m_X)(Y-m_Y)'] \;=\; \mathsf{E}[(X-m_X)(GX+W-Gm_X)'] \\ &= \mathsf{E}[(X-m_X)(G\{X-m_X\}+W)'] \;=\; C_X G' + C_{XW} \;=\; C_X G', \end{aligned}$$

and

$$\begin{aligned} C_Y &= \mathsf{E}[(Y-m_Y)(Y-m_Y)'] \;=\; \mathsf{E}[(G\{X-m_X\}+W)(G\{X-m_X\}+W)'] \\ &= GC_XG' + GC_{XW} + C_{WX}G' + C_W \;=\; GC_XG' + C_W, \end{aligned}$$

since X and W are uncorrelated. Solving $AC_Y = C_{XY}$ implies

$$A = C_XG'(GC_XG'+C_W)^{-1} \quad \text{and} \quad \widehat{X} = C_XG'(GC_XG'+C_W)^{-1}(Y-Gm_X)+m_X.$$

31. We begin with the result of Problem 30 that

$$A \;=\; C_XG'(GC_XG'+C_W)^{-1}.$$

Following the hint, we make the identifications $\alpha = C_W$, $\gamma = C_X$, $\beta = G$, and $\delta = G'$. Then

$$\begin{aligned} A &= C_XG'(\alpha+\beta\gamma\delta)^{-1} \\ &= C_XG'[C_W^{-1} - C_W^{-1}G(C_X^{-1}+G'C_W^{-1}G)^{-1}G'C_W^{-1}] \\ &= C_XG'C_W^{-1} - C_XG'C_W^{-1}G(C_X^{-1}+G'C_W^{-1}G)^{-1}G'C_W^{-1} \\ &= [C_X - C_XG'C_W^{-1}G(C_X^{-1}+G'C_W^{-1}G)^{-1}]G'C_W^{-1} \\ &= [C_X(C_X^{-1}+G'C_W^{-1}G) - C_XG'C_W^{-1}G](C_X^{-1}+G'C_W^{-1}G)^{-1}G'C_W^{-1} \\ &= [I + C_XG'C_W^{-1}G - C_XG'C_W^{-1}G](C_X^{-1}+G'C_W^{-1}G)^{-1}G'C_W^{-1} \\ &= (C_X^{-1}+G'C_W^{-1}G)^{-1}G'C_W^{-1}. \end{aligned}$$

32. We begin with

$$\widehat{X} = A(Y-m_Y)+m_X, \quad \text{where} \quad AC_Y = C_{XY}.$$

Next, with $Z := BX$, we have $m_Z = Bm_X$ and

$$C_{ZY} = \mathsf{E}[(Z-m_Z)(Y-m_Y)'] \;=\; \mathsf{E}[B(X-m_X)(Y-m_Y)'] \;=\; BC_{XY}.$$

We must solve $\widetilde{A}C_Y = C_{ZY}$. Starting with $AC_Y = C_{XY}$, multiply this equation by B to get $(BA)C_Y = BC_{XY} = C_{ZY}$. We see that $\widetilde{A} := BA$ solves the required equation. Hence, the linear MMSE estimate of X based on Z is

$$(BA)(Y-m_Y)+m_Z \;=\; (BA)(Y-m_Y)+Bm_X \;=\; B\{A(Y-m_Y)+m_X\} \;=\; B\widehat{X}.$$

33. We first show that the orthogonality condition

$$\mathsf{E}[(CY)'(X-AY)] \;=\; 0, \quad \text{for all } C,$$

implies A is optimal. Write

$$\begin{aligned}
\mathsf{E}[\|X-BY\|^2] &= \mathsf{E}[\|(X-AY)+(AY-BY)\|^2] \\
&= \mathsf{E}[\|(X-AY)+(A-B)Y)\|^2] \\
&= \mathsf{E}[\|X-AY\|^2]+2\mathsf{E}[\{(A-B)Y\}'(X-AY)]+\mathsf{E}[\|(A-B)Y\|^2] \\
&= \mathsf{E}[\|X-AY\|^2]+\mathsf{E}[\|(A-B)Y\|^2] \\
&\geq \mathsf{E}[\|X-AY\|^2],
\end{aligned}$$

where the cross terms vanish by taking $C = A-B$ in the orthogonality condition. Next, rewrite the orthogonality condition as

$$\begin{aligned}
\mathsf{E}[(CY)'(X-AY)] &= \operatorname{tr}\{\mathsf{E}[(CY)'(X-AY)]\} = \mathsf{E}[\operatorname{tr}\{(CY)'(X-AY)\}] \\
&= \mathsf{E}[\operatorname{tr}\{(X-AY)(CY)'\}] = \operatorname{tr}\{\mathsf{E}[(X-AY)(CY)']\} \\
&= \operatorname{tr}\{(R_{XY}-AR_Y)C'\}.
\end{aligned}$$

Now, this expression must be zero for all C, including $C = R_{XY}-AR_Y$. However, since $\operatorname{tr}(DD') = 0$ implies $D = 0$, we conclude that the optimal A solves $AR_Y = R_{XY}$. Next, the best constant estimator is easily found by writing

$$\mathsf{E}[\|X-b\|^2] = \mathsf{E}[\|(X-m_X)+(m_X-b)\|^2] = \mathsf{E}[\|X-m_X\|^2]+\|m_X-b\|^2.$$

Hence, the optimal value of b is $b = m_X$.

34. To begin, write

$$\begin{aligned}
\mathsf{E}[(X-\widehat{X})(X-\widehat{X})'] &= \mathsf{E}[\{(X-m_X)-A(Y-m_Y)\}\{(X-m_X)-A(Y-m_Y)\}'] \\
&= C_X - AC_{YX} - C_{XY}A' + AC_YA'. \qquad (*)
\end{aligned}$$

We now use the fact that $AC_Y = C_{XY}$. If we multiply $AC_Y = C_{XY}$ on the right by A', we obtain

$$AC_YA' = C_{XY}A'.$$

Furthermore, since AC_YA' is symmetric, we can take the transpose of the above expression and obtain

$$AC_YA' = AC_{YX}.$$

By making appropriate substitutions in $(*)$, we find that the error covariance is also given by

$$C_X - AC_{YX}, \quad C_X - C_{XY}A', \quad \text{and} \quad C_X - AC_YA'.$$

35. Write

$$\begin{aligned}
\mathsf{E}[\|X-\widehat{X}\|^2] &= \mathsf{E}[(X-\widehat{X})'(X-\widehat{X})] = \operatorname{tr}\Big\{\mathsf{E}[(X-\widehat{X})'(X-\widehat{X})]\Big\} \\
&= \mathsf{E}[\operatorname{tr}\{(X-\widehat{X})'(X-\widehat{X})\}] = \mathsf{E}[\operatorname{tr}\{(X-\widehat{X})(X-\widehat{X})'\}] \\
&= \operatorname{tr}\{\mathsf{E}[(X-\widehat{X})(X-\widehat{X})']\} = \operatorname{tr}\{C_X - AC_{YX}\}.
\end{aligned}$$

36. **MATLAB.** We found

$$A = \begin{bmatrix} -0.0622 & 0.0467 & -0.0136 & -0.1007 \\ 0.0489 & -0.0908 & -0.0359 & -0.1812 \\ -0.0269 & 0.0070 & -0.0166 & 0.0921 \\ 0.0619 & 0.0205 & -0.0067 & 0.0403 \end{bmatrix}$$

and MSE $= 0.0806$.

37. Write X in the form $X = [Y', Z']'$, where $Y := [X_1, \ldots, X_m]'$. Then

$$C_X = \mathsf{E}\left[\begin{bmatrix} Y \\ Z \end{bmatrix} \begin{bmatrix} Y' & Z' \end{bmatrix}\right] = \begin{bmatrix} C_Y & C_{YZ} \\ C_{ZY} & C_Z \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_2' & C_3 \end{bmatrix},$$

and

$$C_{XY} = \mathsf{E}\left[\begin{bmatrix} Y \\ Z \end{bmatrix} Y'\right] = \begin{bmatrix} C_Y \\ C_{ZY} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2' \end{bmatrix}.$$

Solving $AC_Y = C_{XY}$ becomes

$$AC_1 = \begin{bmatrix} C_1 \\ C_2' \end{bmatrix}, \quad \text{or} \quad A = \begin{bmatrix} I \\ C_2'C_1^{-1} \end{bmatrix}.$$

The linear MMSE estimate of $X = [Y', Z']'$ is

$$AY = \begin{bmatrix} I \\ C_2'C_1^{-1} \end{bmatrix} Y = \begin{bmatrix} Y \\ C_2'C_1^{-1}Y \end{bmatrix}.$$

In other words, $\widehat{Y} = Y$ and $\widehat{Z} = C_2'C_1^{-1}Y$. Note that the matrix required for the linear MMSE estimate of Z based on Y is the solution of $BC_Y = C_{ZY}$ or $BC_1 = C_2'$; i.e., $B = C_2'C_1^{-1}$. Next, the error covariance for estimating X based on Y is

$$\begin{aligned} \mathsf{E}[(X-\widehat{X})(X-\widehat{X})'] = C_X - AC_{YX} &= \begin{bmatrix} C_1 & C_2 \\ C_2' & C_3 \end{bmatrix} - \begin{bmatrix} I \\ C_2'C_1^{-1} \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} \\ &= \begin{bmatrix} C_1 & C_2 \\ C_2' & C_3 \end{bmatrix} - \begin{bmatrix} C_1 & C_2 \\ C_2' & C_2'C_1^{-1}C_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & C_3 - C_2'C_1^{-1}C_2 \end{bmatrix}, \end{aligned}$$

and the MSE is

$$\mathsf{E}[\|X-\widehat{X}\|^2] = \operatorname{tr}(C_X - AC_{YX}) = \operatorname{tr}(C_3 - C_2'C_1^{-1}C_2).$$

38. Since P' decorrelates Y, the covariance matrix C_Z of $Z := P'Y$ is diagonal. Writing $\tilde{A}C_Z = C_{XZ}$ in component form, and using the fact that C_Z is diagonal,

$$\sum_k \tilde{A}_{ik}(C_Z)_{kj} = (C_{XZ})_{ij}$$

becomes

$$\tilde{A}_{ij}(C_Z)_{jj} = (C_{XZ})_{ij}$$

If $(C_Z)_{jj} \neq 0$, then $\tilde{A}_{ij} = (C_{XZ})_{ij}/(C_Z)_{jj}$. If $(C_Z)_{jj} = 0$, then $\tilde{A}_{ij}(C_Z)_{jj} = (C_{XZ})_{ij}$ can be solved only if $(C_{XZ})_{ij} = 0$, which we now show to be the case by using the Cauchy–Schwarz inequality. Write

$$\begin{aligned} |(C_{XZ})_{ij}| &= \big|\mathsf{E}[(X_i - (m_X)_i)(Z_j - (m_Z)_j)]\big| \\ &\leq \sqrt{\mathsf{E}[(X_i - (m_X)_i)^2]\mathsf{E}[(Z_j - (m_Z)_j)^2]} \\ &= \sqrt{(C_X)_{ii}(C_Z)_{jj}}. \end{aligned}$$

Hence, if $(C_Z)_{jj} = 0$ then $(C_{XZ})_{ij} = 0$, and any value of $\tilde{A}_{ij}$ solves $\tilde{A}_{ij}(C_Z)_{jj} = (C_{XZ})_{ij}$. Now that we have shown that we can always solve $\tilde{A}C_Z = C_{XZ}$, observe that this equation is equivalent to

$$\tilde{A}(P'C_YP) = C_{XY}P \quad \text{or} \quad (\tilde{A}P')C_Y = C_{XY}.$$

Thus, $A = \tilde{A}P'$ solves the original problem.

39. Since X has the form $X = [Y', Z']'$, if we take $G = [I, 0]$ and $W \equiv 0$, then

$$GX + W = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = Y.$$

40. Write

$$\mathsf{E}\left[\frac{1}{n}\sum_{k=1}^n X_k^2\right] = \frac{1}{n}\sum_{k=1}^n \mathsf{E}[X_k^2] = \frac{1}{n}\sum_{k=1}^n \sigma^2 = \sigma^2.$$

41. Write

$$\mathsf{E}\left[\frac{1}{n}\sum_{k=1}^n X_kX_k'\right] = \frac{1}{n}\sum_{k=1}^n \mathsf{E}[X_kX_k'] = \frac{1}{n}\sum_{k=1}^n C = C.$$

42. **MATLAB.** Additional code:

```
Mn = mean(X,2)
MnMAT = kron(ones(1,n),Mn);
Chat = (X-MnMAT)*(X-MnMAT)'/(n-1)
```

43. We must first find $f_{Y|X}(y|x)$. To this end, use substitution and independence to write

$$\mathsf{P}(Y \leq y|X = x) = \mathsf{P}(X + W \leq y|X = x) = \mathsf{P}(x + W \leq y|X = x) = \mathsf{P}(W \leq y - x).$$

Then $f_{Y|X}(y|x) = f_W(y - x) = (\lambda/2)e^{-\lambda|y-x|}$. For fixed y, the maximizing value of x is $x = y$. Hence, $g_{\mathrm{ML}}(y) = y$.

44. By the same argument as in the solution of Problem 43, $f_{Y|X}(y|x) = (\lambda/2)e^{-\lambda|y-x|}$. When $X \sim \exp(\mu)$ and we maximize over x, we must impose the constraint $x \geq 0$. Hence,

$$g_{\mathrm{ML}}(y) = \underset{x\geq 0}{\operatorname{argmax}}\, \tfrac{\lambda}{2}e^{-\lambda|y-x|} = \begin{cases} y, & y \geq 0, \\ 0, & y < 0. \end{cases}$$

When $X \sim \text{uniform}[0,1]$,

$$g_{\text{ML}}(y) = \underset{0 \le x \le 1}{\operatorname{argmax}} \tfrac{\lambda}{2} e^{-\lambda|y-x|} = \begin{cases} y, & 0 \le y \le 1, \\ 1, & y > 1, \\ 0, & y < 0. \end{cases}$$

45. By the same argument as in the solution of Problem 43, $f_{Y|X}(y|x) = (\lambda/2)e^{-\lambda|y-x|}$. For the MAP estimator, we must maximize

$$f_{Y|X}(y|x)f_X(x) = \tfrac{\lambda}{2} e^{-\lambda|y-x|} \cdot \mu e^{-\mu x}, \quad x \ge 0.$$

By considering separately the cases $x \le y$ and $x > y$,

$$f_{Y|X}(y|x)f_X(x) = \begin{cases} \frac{\mu\lambda}{2} e^{-\lambda y} e^{(\lambda-\mu)x}, & 0 \le x \le y, \\ \frac{\mu\lambda}{2} e^{\lambda y} e^{-(\lambda+\mu)x}, & x > y, x \ge 0. \end{cases}$$

When $y \ge 0$, observe that the two formulas agree at $x = y$ and have the common value $(\mu\lambda/2)e^{-\mu y}$; in fact, if $\lambda > \mu$, the first formula is maximized at $x = y$, while the second formula is always maximized at $x = y$. If $y < 0$, then only the second formula is valid, and its region of validity is $x \ge 0$. This formula is maximized at $x = 0$. Hence, for $\lambda > \mu$,

$$g_{\text{MAP}}(y) = \begin{cases} y, & y \ge 0, \\ 0, & y < 0. \end{cases}$$

We now consider the case $\lambda \le \mu$. As before, if $y < 0$, the maximizing value of x is zero. If $y \ge 0$, then the maximum value of $f_{Y|X}(y|x)f_X(x)$ for $0 \le x \le y$ occurs at $x = 0$ with a maximum value of $(\mu\lambda/2)e^{-\lambda y}$. The maximum value of $f_{Y|X}(y|x)f_X(x)$ for $x \ge y$ occurs at $x = y$ with a maximum value of $(\mu\lambda/2)e^{-\mu y}$. For $\lambda < \mu$,

$$\max\{(\mu\lambda/2)e^{-\lambda y}, (\mu\lambda/2)e^{-\mu y}\} = (\mu\lambda/2)e^{-\lambda y},$$

which corresponds to $x = 0$. Hence, for $\lambda < \mu$,

$$g_{\text{MAP}}(y) = 0, \quad -\infty < y < \infty.$$

46. From the formula

$$f_{XY}(x,y) = (x/y^2)e^{-(x/y)^2/2} \cdot \lambda e^{-\lambda y}, \quad x, y > 0,$$

we see that $Y \sim \exp(\lambda)$, and that given $Y = y$, $X \sim \text{Rayleigh}(y)$. Hence, the MMSE estimator is $\mathsf{E}[X|Y = y] = \sqrt{\pi/2}\,y$. To compute the MAP estimator, we must solve

$$\underset{x \ge 0}{\operatorname{argmax}} (x/y^2)e^{-(x/y)^2/2}.$$

We do this by differentiating with respect to x and setting the derivative equal to zero. Write

$$\frac{\partial}{\partial x}(x/y^2)e^{-(x/y)^2/2} = \frac{e^{-(x/y)^2/2}}{y^2}\left[1 - \frac{x^2}{y^2}\right].$$

Hence,

$$g_{\text{MAP}}(y) = y.$$

47. Suppose that

$$\mathsf{E}[(X-g_1(Y))h(Y)] \;=\; 0 \quad \text{and} \quad \mathsf{E}[(X-g_2(Y))h(Y)] \;=\; 0.$$

Subtracting the second equation from the first yields

$$\mathsf{E}[\{g_2(Y)-g_1(Y)\}h(Y)] \;=\; 0. \tag{$*$}$$

Since h is an arbitrary bounded function, put $h(y) := \operatorname{sgn}[g_2(y)-g_1(y)]$, where

$$\operatorname{sgn}(x) \;:=\; \begin{cases} 1, & x>0, \\ 0, & x=0, \\ -1, & x<0. \end{cases}$$

Note also that $x \cdot \operatorname{sgn}(x) = |x|$. Then $(*)$ becomes $\mathsf{E}[|g_2(Y)-g_1(Y)|] = 0$.

CHAPTER 9
Problem Solutions

1. We first compute

$$\det C = \det \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix} = \sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\rho)^2 = \sigma_1^2\sigma_2^2(1-\rho^2),$$

and $\sqrt{\det C} = \sigma_1\sigma_2\sqrt{1-\rho^2}$. Next,

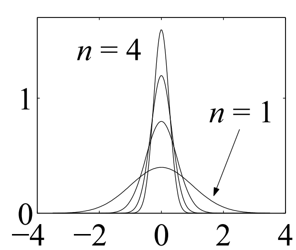
$$C^{-1} = \frac{1}{\det C}\begin{bmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_1\sigma_2\rho & \sigma_1^2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sigma_1^2(1-\rho^2)} & \dfrac{-\rho}{\sigma_1\sigma_2(1-\rho^2)} \\ \dfrac{-\rho}{\sigma_1\sigma_2(1-\rho^2)} & \dfrac{1}{\sigma_2^2(1-\rho^2)} \end{bmatrix},$$

and

$$\begin{aligned} [\,x\;\;y\,]\,C^{-1}\begin{bmatrix} x \\ y \end{bmatrix} &= [\,x\;\;y\,]\begin{bmatrix} \dfrac{x}{\sigma_1^2(1-\rho^2)} - \dfrac{\rho y}{\sigma_1\sigma_2(1-\rho^2)} \\ \dfrac{-\rho x}{\sigma_1\sigma_2(1-\rho^2)} + \dfrac{y}{\sigma_1^2(1-\rho^2)} \end{bmatrix} \\ &= \frac{x^2}{\sigma_1^2(1-\rho^2)} - \frac{\rho xy}{\sigma_1\sigma_2(1-\rho^2)} - \frac{\rho xy}{\sigma_1\sigma_2(1-\rho^2)} + \frac{y^2}{\sigma_1^2(1-\rho^2)} \\ &= \frac{1}{1-\rho^2}\left[\left(\frac{x}{\sigma_1}\right)^2 - 2\rho\frac{xy}{\sigma_1\sigma_2} + \left(\frac{y}{\sigma_2}\right)^2\right], \end{aligned}$$

and the result follows.

2. Here is the plot:



3. (a) First write $c_1X + c_2Y = c_1X + c_2(3X) = (c_1+3c_2)X$, which is easily seen to be $N(0,(c_1+3c_2)^2)$. Thus, X and Y are jointly Gaussian.

   (b) Observe that $\mathsf{E}[XY] = \mathsf{E}[X(3X)] = 3\mathsf{E}[X^2] = 3$ and $\mathsf{E}[Y^2] = \mathsf{E}[(3X)^2] = 9\mathsf{E}[X^2] = 9$. Since X and Y have zero means,

$$\mathsf{cov}([X,Y]') = \begin{bmatrix} \mathsf{E}[X^2] & \mathsf{E}[XY] \\ \mathsf{E}[YX] & \mathsf{E}[Y^2] \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

(c) The conditional cdf of Y given $X=x$ is

$$F_{Y|X}(y|x) \;=\; \mathsf{P}(Y \le y|X=x) \;=\; \mathsf{P}(3X \le y|X=x).$$

By substitution, this last conditional probability is $\mathsf{P}(3x \le y|X=x)$. The event $\{3x \le y\}$ is deterministic and therefore independent of X. Hence, we can drop the conditioning and get

$$F_{Y|X}(y|x) \;=\; \mathsf{P}(3x \le y).$$

If $3x \le y$, then $\{3x \le y\} = \Omega$, and $\{3x \le y\} = \varnothing$ otherwise. Hence, the above probability is just $u(y-3x)$.

4. If $Y = \sum_{i=1}^n c_i X_i$, its characteristic function is

$$\begin{aligned}\varphi_Y(\nu) \;&=\; \mathsf{E}[e^{j\nu(\sum_{i=1}^n c_i X_i)}] \;=\; \mathsf{E}\biggl[\prod_{i=1}^n e^{j\nu c_i X_i}\biggr] \;=\; \prod_{i=1}^n \mathsf{E}[e^{j\nu c_i X_i}] \;=\; \prod_{i=1}^n \varphi_{X_i}(\nu c_i) \\ &=\; \prod_{i=1}^n e^{j(\nu c_i)m_i - (\nu c_i)^2\sigma_i^2/2} \;=\; e^{j\nu(\sum_{i=1}^n c_i m_i) - \nu^2(\sum_{i=1}^n c_i^2\sigma_i^2)/2},\end{aligned}$$

which is the characteristic function of an $N(\sum_{i=1}^n c_i m_i, \sum_{i=1}^n c_i^2\sigma_i^2)$ random variable.

5. First, $\mathsf{E}[Y] = \mathsf{E}[AX+b] = A\mathsf{E}[X]+b = Am+b$. Second,

$$\mathsf{E}[\{Y-(Am+b)\}\{Y-(Am+b)\}'] \;=\; \mathsf{E}[A(X-m)(X-m)'A'] \;=\; ACA'.$$

6. Write out

$$\begin{aligned}Y_1 \;&=\; X_1 \\ Y_2 \;&=\; X_1 + X_2 \\ Y_3 \;&=\; X_1 + X_2 + X_3 \\ &\;\;\vdots\end{aligned}$$

In general, $Y_n = Y_{n-1} + X_n$, or $X_n = Y_n - Y_{n-1}$, which we can write in matrix-vector notation as

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ -1 & 1 & 0 & & & 0 \\ 0 & -1 & 1 & 0 & & 0 \\ & & & \ddots & & \\ 0 & & 0 & -1 & 1 & 0 \\ 0 & & \cdots & 0 & -1 & 1 \end{bmatrix}}_{=:\,A} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_{n-1} \\ Y_n \end{bmatrix}.$$

Since $X = AY$ and Y is Gaussian, so is X.

7. Since $X \sim N(0,C)$, the scalar $Y := \nu' X$ is also Gaussian and has zero mean. Hence, $\mathsf{E}[(\nu' X X' \nu)^k] = \mathsf{E}[(Y^2)^k] = \mathsf{E}[Y^{2k}] = (2k-1)\cdots 5\cdot 3\cdot 1\cdot(\mathsf{E}[Y^2])^k$. Now observe that $\mathsf{E}[Y^2] = \mathsf{E}[\nu' X X' \nu] = \nu' C \nu$ and the result follows.

8. We first have

$$\mathsf{E}[Y_j] \;=\; \mathsf{E}[X_j - \overline{X}] \;=\; m - \mathsf{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \;=\; m - \frac{1}{n}\sum_{i=1}^{n} m \;=\; 0.$$

Next, since $\mathsf{E}[\overline{X}Y_j] = \mathsf{E}[\overline{X}(X_j - \overline{X})]$, we first compute

$$\begin{aligned}\mathsf{E}[\overline{X}\,\overline{X}] &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathsf{E}[X_iX_j] \;=\; \frac{1}{n^2}\left\{\sum_{i=1}^{n}(\sigma^2+m^2) + \sum_{i\neq j} m^2\right\}\\ &= \frac{1}{n^2}\left\{n(\sigma^2+m^2)+n(n-1)m^2\right\} \;=\; \frac{1}{n^2}\left\{n\sigma^2+n^2m^2\right\} \;=\; \frac{\sigma^2}{n}+m^2,\end{aligned}$$

and

$$\mathsf{E}[\overline{X}X_j] \;=\; \frac{1}{n}\sum_{i=1}^{n}\mathsf{E}[X_iX_j] \;=\; \frac{1}{n}\left\{(\sigma^2+m^2)+(n-1)m^2\right\} \;=\; \frac{\sigma^2}{n}+m^2.$$

It now follows that $\mathsf{E}[\overline{X}Y_j] = 0$.

9. Following the hint, in the expansion of $\mathsf{E}[(\nu' X)^{2k}]$, the sum of all the coefficients of $\nu_{i_1}\cdots\nu_{i_{2k}}$ is $(2k)!\mathsf{E}[X_{i_1}\cdots X_{i_{2k}}]$. The corresponding sum of coeficients in the expansion of

$$(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1\cdot(\nu' C\nu)^k$$

is

$$(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1\cdot 2^k k!\sum_{j_1,\ldots,j_{2k}} C_{j_1j_2}\cdots C_{j_{2k-1}j_{2k}},$$

where the sum is over all $j_1,\ldots,j_{2k}$ that are permutations of $i_1,\ldots,i_{2k}$ *and* such that the product $C_{j_1j_2}\cdots C_{j_{2k-1}j_{2k}}$ is distinct. Since

$$\begin{aligned}(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1\cdot 2^k k! &= (2k-1)(2k-3)\cdots 5\cdot 3\cdot 1\cdot(2k)(2k-2)\cdots 4\cdot 2\\ &= (2k)!,\end{aligned}$$

Wick's Theorem follows.

10. Write

$$\begin{aligned}\mathsf{E}[X_1X_2X_3X_4] &= \sum_{j_1,j_2,j_3,j_4} C_{j_1j_2}C_{j_3j_4} \;=\; C_{12}C_{34}+C_{13}C_{24}+C_{14}C_{23}.\\ \mathsf{E}[X_1X_3^2X_4] &= C_{13}C_{34}+C_{14}C_{33}.\\ \mathsf{E}[X_1^2X_2^2] &= C_{11}C_{22}+C_{12}^2.\end{aligned}$$

11. Put $a := [a_1,\ldots,a_n]'$. Then $Y = a'X$, and

$$\begin{aligned}\varphi_Y(\eta) &= \mathsf{E}[e^{j\eta Y}] \;=\; \mathsf{E}[e^{j\eta(a'X)}] \;=\; \mathsf{E}[e^{j(\eta a)'X}] \;=\; \varphi_X(\eta a)\\ &= e^{j(\eta a)'m-(\eta a)'C(\eta a)/2} \;=\; e^{j\eta(a'm)-(a'Ca)\eta^2/2},\end{aligned}$$

which is the characteristic function of a scalar $N(a'm, a'Ca)$ random variable.

12. With $X = [U', W']'$ and $\nu = [\alpha', \beta']'$, we have $\varphi_X(\nu) = e^{j\nu' m - \nu' C \nu/2}$, where

$$\nu' m = \begin{bmatrix} \alpha' & \beta' \end{bmatrix} \begin{bmatrix} m_U \\ m_W \end{bmatrix} = \alpha' m_U + \beta' m_W,$$

and

$$\nu' C \nu = \begin{bmatrix} \alpha' & \beta' \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha' & \beta' \end{bmatrix} \begin{bmatrix} S\alpha \\ T\beta \end{bmatrix} = \alpha' S \alpha + \beta' T \beta.$$

Thus,

$$\varphi_X(\nu) = e^{j(\alpha' m_U + \beta' m_W) - (\alpha' S\alpha + \beta' T\beta)/2} = e^{j\alpha' m_U - \alpha' S\alpha/2} e^{j\beta' m_W - \beta' T\beta/2},$$

which has the required form $\varphi_U(\alpha)\varphi_W(\beta)$ of a product of Gaussian characteristic functions.

13. (a) Since X is $N(0,C)$ and $Y := C^{-1/2}X$, Y is also normal. It remains to find the mean and covariance of Y. We have $\mathsf{E}[Y] = \mathsf{E}[C^{-1/2}X] = C^{-1/2}\mathsf{E}[X] = 0$ and $\mathsf{E}[YY'] = \mathsf{E}[C^{-1/2}XX'C^{-1/2}] = C^{-1/2}\mathsf{E}[XX']C^{-1/2} = C^{-1/2}CC^{-1/2} = I$. Hence, $Y \sim N(0,I)$.

(b) Since the covariance matrix of Y is diagonal, the components of Y are uncorrelated. Since Y is also Gaussian, the components of Y are independent. Since the covariance matrix of Y is the indentity, each $Y_k \sim N(0,1)$. Hence, each Y_k^2 is chi-squared with one degree of freedom by Problem 46 in Chapter 4 or Problem 11 in Chapter 5.

(c) By the Remark in Problem 55(c) in Chapter 4, V is chi-squared with n degrees of freedom.

14. Since

$$Z := \det \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} = X^2 + Y^2,$$

where X and Y are independent $N(0,1)$, observe that X^2 and Y^2 are chi-squared with one degree of freedom. Hence, Z is chi-squared with two degrees of freedom, which is the same as $\exp(1/2)$.

15. Begin with

$$f_X(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-j\nu' x} e^{j\nu' m - \nu' C\nu/2}\, d\nu = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-j(x-m)'\nu} e^{-\nu' C\nu/2}\, d\nu.$$

Now make the multivariate change of variable $\zeta = C^{1/2}\nu$, $d\zeta = \det C^{1/2}\, d\nu$. Then

$$\begin{aligned} f_X(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-j(x-m)'C^{-1/2}\zeta} e^{-\zeta'\zeta/2} \frac{d\zeta}{\det C^{1/2}} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-j\{C^{-1/2}(x-m)\}'\zeta} e^{-\zeta'\zeta/2} \frac{d\zeta}{\sqrt{\det C}}. \end{aligned}$$

Put $t = C^{-1/2}(x-m)$ so that

$$f_X(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-jt'\zeta}e^{-\zeta'\zeta/2}\,\frac{d\zeta}{\sqrt{\det C}} = \frac{1}{\sqrt{\det C}}\prod_{i=1}^n\biggl(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jt_i\zeta_i}e^{-\zeta_i^2/2}\,d\zeta_i\biggr).$$

Observe that $e^{-\zeta_i^2/2}$ is the characteristic function of a scalar $N(0,1)$ random variable. Hence,

$$f_X(x) \;=\; \frac{1}{\sqrt{\det C}}\prod_{i=1}^n\frac{e^{-t_i^2/2}}{\sqrt{2\pi}} \;=\; \frac{1}{(2\pi)^{n/2}\sqrt{\det C}}\exp[-t't/2].$$

Recalling that $t = C^{-1/2}(x-m)$ yields

$$f_X(x) \;=\; \frac{\exp[-\frac{1}{2}(x-m)'C^{-1}(x-m)]}{(2\pi)^{n/2}\sqrt{\det C}}.$$

16. First observe that

$$Z \;:=\; \det\begin{bmatrix} X & Y \\ U & V\end{bmatrix} \;=\; XV-YU.$$

Then consider the conditional cumulative distribution function,

$$\begin{aligned} F_{Z|UV}(z|u,v) \;&=\; \mathsf{P}(Z\le z|U=u,V=v) \;=\; \mathsf{P}(XV-YU\le z|U=u,V=v) \\ &=\; \mathsf{P}(Xv-Yu\le z|U=u,V=v). \end{aligned}$$

Since $[X,Y]'$ and $[U,V]'$ are jointly Gaussian and uncorrelated, they are independent. Hence, we can drop the conditioning and get

$$F_{Z|UV}(z|u,v) \;=\; \mathsf{P}(Xv-Yu\le z).$$

Next, since X and Y are independent and $N(0,1)$, $Xv-Yu \sim N(0,u^2+v^2)$. Hence,

$$f_{Z|UV}(\cdot\,|u,v)\sim N(0,u^2+v^2).$$

17. We first use the fact that A solves $AC_Y = C_{XY}$ to show that $(X-m_X)-A(Y-m_Y)$ and Y are uncorrelated. Write

$$\mathsf{E}[\{(X-m_X)-A(Y-m_Y)\}Y'] \;=\; C_{XY}-AC_Y \;=\; 0.$$

We next show that $(X-m_X)-A(Y-m_Y)$ and Y are jointly Gaussian by writing them as an affine transformation of the Gaussian vector $[X',Y']'$; i.e.,

$$\begin{bmatrix} (X-m_X)-A(Y-m_Y) \\ Y\end{bmatrix} \;=\; \begin{bmatrix} I & -A \\ 0 & I\end{bmatrix}\begin{bmatrix} X \\ Y\end{bmatrix} + \begin{bmatrix} Am_Y-m_X \\ 0\end{bmatrix}.$$

It now follows that $(X-m_X)-A(Y-m_Y)$ and Y are independent. Using the hints on substitution and independence, we compute the conditional characteristic function,

$$\begin{aligned} \mathsf{E}[e^{\nu'X}|Y=y] \;&=\; \mathsf{E}\bigl[e^{j\nu'[(X-m_X)-A(Y-m_Y)]}\cdot e^{j\nu'[m_X+A(Y-m_Y)]}\big|Y=y\bigr] \\ &=\; e^{j\nu'[m_X+A(y-m_Y)]}\,\mathsf{E}\bigl[e^{j\nu'[(X-m_X)-A(Y-m_Y)]}\big|Y=y\bigr] \\ &=\; e^{j\nu'[m_X+A(y-m_Y)]}\,\mathsf{E}\bigl[e^{j\nu'[(X-m_X)-A(Y-m_Y)]}\bigr]. \end{aligned}$$

This last expectation is the characteristic function of the zero-mean Gaussian random vector $(X-m_X)-A(Y-m_Y)$. To compute its covariance matrix first observe that since $AC_Y = C_{XY}$, we have $AC_YA' = C_{XY}A'$. Then

$$\begin{aligned}
&\mathsf{E}[\{(X-m_X)-A(Y-m_Y)\}\{(X-m_X)-A(Y-m_Y)\}'] \\
&\qquad\qquad = C_X - C_{XY}A' - AC_{YX} + AC_YA' \\
&\qquad\qquad = C_X - AC_{YX}.
\end{aligned}$$

We now have

$$\mathsf{E}[e^{\nu' X}|Y=y] \;=\; e^{j\nu'[m_X+A(y-m_Y)]}e^{-\nu'[C_X-AC_{YX}]\nu/2}.$$

Thus, given $Y=y$, X is conditionally $N\big(m_X+A(y-m_Y), C_X-AC_{YX}\big)$.

18. First observe that

$$Z \;:=\; \det\begin{bmatrix} X & Y \\ U & V \end{bmatrix} \;=\; XV-YU.$$

Then consider the conditional cumulative distribution function,

$$\begin{aligned}
F_{Z|UV}(z|u,v) \;&=\; \mathsf{P}(Z\le z|U=u,V=v) \;=\; \mathsf{P}(XV-YU\le z|U=u,V=v) \\
&=\; \mathsf{P}(Xv-Yu\le z|U=u,V=v).
\end{aligned}$$

Since $[X,Y,U,V]'$ is Gaussian, given $U=u$ and $V=v$, $[X,Y]'$ is conditionally

$$N\!\left(A\begin{bmatrix} u \\ v \end{bmatrix}, C_{[X,Y]'} - AC_{[U,V]',[X,Y]'}\right),$$

where A solves $AC_{[U,V]'} = C_{[X,Y]',[U,V]'}$. We now turn to the conditional distribution of $Xv-Yu$. Since the conditional distribution of $[X,Y]'$ is Gaussian, so is the conditional distribution of the linear combination $Xv-Yu$. Hence, all we need to find are the conditional mean and the conditional variance of $Xv-Yu$; i.e.,

$$\begin{aligned}
\mathsf{E}[Xv-Yu|U=u,V=v] \;&=\; \mathsf{E}\left[\begin{bmatrix} v & u \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}\middle| U=u,V=v\right] \\
&=\; \begin{bmatrix} v & u \end{bmatrix}\mathsf{E}\left[\begin{bmatrix} X \\ Y \end{bmatrix}\middle| U=u,V=v\right] \\
&=\; \begin{bmatrix} v & u \end{bmatrix} A\begin{bmatrix} u \\ v \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{E}\left[\left\{\begin{bmatrix} v & u \end{bmatrix}\left(\begin{bmatrix} X \\ Y \end{bmatrix} - A\begin{bmatrix} u \\ v \end{bmatrix}\right)\right\}^2\middle| U=u,V=v\right] \\
&\quad=\; \mathsf{E}\left[\left\{\begin{bmatrix} v & u \end{bmatrix}\left(\begin{bmatrix} X \\ Y \end{bmatrix} - A\begin{bmatrix} U \\ V \end{bmatrix}\right)\right\}^2\middle| U=u,V=v\right] \\
&\quad=\; \begin{bmatrix} v & u \end{bmatrix}\mathsf{E}\left[\left(\begin{bmatrix} X \\ Y \end{bmatrix} - A\begin{bmatrix} U \\ V \end{bmatrix}\right)\left(\begin{bmatrix} X \\ Y \end{bmatrix} - A\begin{bmatrix} U \\ V \end{bmatrix}\right)'\middle| U=u,V=v\right]\begin{bmatrix} v \\ u \end{bmatrix} \\
&\quad=\; \begin{bmatrix} v & u \end{bmatrix}\Big(C_{[X,Y]'} - AC_{[U,V]',[X,Y]'}\Big)\begin{bmatrix} v \\ u \end{bmatrix},
\end{aligned}$$

where the last step uses the fact that $[X,Y]' - A[U,V]'$ is independent of $[U,V]'$. If $[X,Y]'$ and $[U,V]'$ are uncorrelated, i.e., $C_{[X,Y]',[U,V]'} = 0$, then $A = 0$ solves $AC_{[U,V]'} = 0$; in this case, the conditional mean is zero, and the conditional variance simplifies to

$$\begin{bmatrix} v & u \end{bmatrix} C_{[X,Y]'} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} v & u \end{bmatrix} I \begin{bmatrix} v \\ u \end{bmatrix} = v^2 + u^2.$$

19. First write

$$Z - \mathsf{E}[Z] = (X + jY) - (m_X + jm_Y) = (X - m_X) + j(Y - m_Y).$$

Then

$$\begin{aligned} \mathsf{cov}(Z) &:= \mathsf{E}[(Z - \mathsf{E}[Z])(Z - \mathsf{E}[Z])^*] \\ &= \mathsf{E}[\{(X - m_X) + j(Y - m_Y)\}\{(X - m_X) + j(Y - m_Y)\}^*] \\ &= \mathsf{E}[\{(X - m_X) + j(Y - m_Y)\}\{(X - m_X) - j(Y - m_Y)\}] \\ &= \mathsf{var}(X) - j\,\mathsf{cov}(X,Y) + j\,\mathsf{cov}(Y,X) + \mathsf{var}(Y) = \mathsf{var}(X) + \mathsf{var}(Y). \end{aligned}$$

20. (a) Write

$$\begin{aligned} K &:= \mathsf{E}[(Z - \mathsf{E}[Z])(Z - \mathsf{E}[Z])^H] \\ &= \mathsf{E}[\{(X - m_X) + j(Y - m_Y)\}\{(X - m_X) + j(Y - m_Y)\}^H] \\ &= \mathsf{E}[\{(X - m_X) + j(Y - m_Y)\}\{(X - m_X)^H - j(Y - m_Y)^H\}] \\ &= C_X - jC_{XY} + jC_{YX} + C_Y = (C_X + C_Y) + j(C_{YX} - C_{XY}). \end{aligned}$$

(b) If $C_{XY} = -C_{YX}$, the $(C_{XY})_{ii} = -(C_{YX})_{ii}$ implies

$$\begin{aligned} \mathsf{E}[(X_i - (m_X)_i)(Y_i - (m_Y)_i)] &= -\mathsf{E}[(Y_i - (m_Y)_i)(X_i - (m_X)_i)] \\ &= -\mathsf{E}[(X_i - (m_X)_i)(Y_i - (m_Y)_i)]. \end{aligned}$$

Hence, $\mathsf{E}[(X_i - (m_X)_i)(Y_i - (m_Y)_i)] = 0$, and we see that X_i and Y_i are uncorrelated.

(c) By part (a), if K is real, then $C_{YX} = C_{XY}$. Circular symmetry implies $C_{YX} = -C_{XY}$. It follows that $C_{XY} = -C_{XY}$, and then $C_{XY} = 0$; i.e., X and Y are uncorrelated.

21. First,

$$f_X(x) = \frac{e^{-x'(2I)x/2}}{(2\pi)^{n/2}(1/2)^{n/2}} \quad \text{and} \quad f_Y(y) = \frac{e^{-y'(2I)y/2}}{(2\pi)^{n/2}(1/2)^{n/2}}.$$

Then

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{e^{-(x'x + y'y)}}{\pi^n} = \frac{e^{-(x+jy)^H(x+jy)}}{\pi^n}.$$

22. (a) Immediate from Problem 20(a).

(b) Since $\nu' Q \nu$ is a scalar, $(\nu' Q \nu)' = \nu' Q \nu$. Since $Q' = -Q$, $(\nu' Q \nu)' = \nu' Q' \nu = -\nu' Q \nu$. Thus, $\nu' Q \nu = -\nu' Q \nu$, and it follows that $\nu' Q \nu = 0$.

(c) Begin by observing that if $K = R + jQ$ and $w = \nu + j\theta$, then

$$Kw = (R + jQ)(\nu + j\theta) = R\nu + jR\theta + jQ\nu - Q\theta.$$

Next,

$$\begin{aligned} w^H(Kw) &= (\nu' - j\theta')(R\nu + jR\theta + jQ\nu - Q\theta) \\ &= \nu'R\nu + j\nu'R\theta + j\nu'Q\nu - \nu'Q\theta - j\theta'R\nu + \theta'R\theta + \theta'Q\nu - j\theta'Q\theta \\ &= \nu'R\nu + j\nu'R\theta + j\nu'Q\nu + \theta'Q\nu - j\nu'R\theta + \theta'R\theta + \theta'Q\nu - j\theta'Q\theta \\ &= \nu'R\nu + \theta'R\theta + 2\theta'Q\nu, \end{aligned}$$

where we have used the result of parts (a) and (b).

23. First,

$$AZ = (\alpha + j\beta)(X + jY) = (\alpha X - \beta Y) + j(\beta X + \alpha Y).$$

Second,

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \alpha X - \beta Y \\ \beta X + \alpha Y \end{bmatrix}.$$

Now assume that circular symmetry holds; i.e., $C_X = C_Y$ and $C_{XY} = -C_{YX}$. Put $U := \alpha X - \beta Y$ and $V := \beta X + \alpha Y$. Assuming zero means to simplify the notation,

$$\begin{aligned} C_U = \mathsf{E}[(\alpha X - \beta Y)(\alpha X - \beta Y)'] &= \alpha C_X \alpha' - \beta C_{YX}\alpha' - \alpha C_{XY}\beta' + \beta' C_Y \beta \\ &= \alpha C_X \alpha' - \beta C_{YX}\alpha' + \alpha C_{YX}\beta' + \beta' C_X \beta. \end{aligned}$$

Similarly,

$$\begin{aligned} C_V = \mathsf{E}[(\beta X + \alpha Y)(\beta X + \alpha Y)'] &= \beta C_X \beta' + \alpha C_{YX}\beta' + \beta C_{XY}\alpha' + \alpha C_Y \alpha' \\ &= \beta C_X \beta' + \alpha C_{YX}\beta' - \beta C_{YX}\alpha' + \alpha C_X \alpha'. \end{aligned}$$

Hence, $C_U = C_V$. It remains to compute

$$\begin{aligned} C_{UV} = \mathsf{E}[(\alpha X - \beta Y)(\beta X + \alpha Y)'] &= \alpha C_X \beta' + \alpha C_{XY}\alpha' - \beta C_{YX}\beta' - \beta C_Y \alpha' \\ &= \alpha C_X \beta' - \alpha C_{YX}\alpha' - \beta C_{YX}\beta' - \beta C_X \alpha' \end{aligned}$$

and

$$\begin{aligned} C_{VU} = \mathsf{E}[(\beta X + \alpha Y)(\alpha X - \beta Y)'] &= \beta C_X \alpha' - \beta C_{XY}\beta' + \alpha C_{YX}\alpha' - \alpha C_Y \beta' \\ &= \beta C_X \alpha' + \beta C_{YX}\beta' + \alpha C_{YX}\alpha' - \alpha C_X \beta', \end{aligned}$$

which shows that $C_U = -C_V$. Thus, if Z is circularly symmetric, so is AZ.

24. To begin, note that with $R = [X', U']'$ and $I = [Y', V']'$,

$$C_R = \begin{bmatrix} C_X & C_{XU} \\ C_{UX} & C_U \end{bmatrix}, \quad C_I = \begin{bmatrix} C_Y & C_{YV} \\ C_{VY} & C_V \end{bmatrix},$$

and

$$C_{RI} = \begin{bmatrix} C_{XY} & C_{XV} \\ C_{UY} & C_{UV} \end{bmatrix}, \quad C_{IR} = \begin{bmatrix} C_{YX} & C_{YU} \\ C_{VX} & C_{VU} \end{bmatrix}.$$

Also, Θ is circularly symmetric means $C_R = C_I$ and $C_{RI} = -C_{IR}$.

(a) We assume zero means to simplify the notation. First,

$$\begin{aligned} K_{ZW} &= \mathsf{E}[ZW^H] \;=\; \mathsf{E}[(X+jY)(U+jV)^H] \;=\; \mathsf{E}[(X+jY)(U^H-jV^H)] \\ &= C_{XU} - jC_{XV} + jC_{YU} + C_{YV} \\ &= 2(C_{XU} - jC_{XV}), \quad \text{since } \Theta \text{ is circularly symmetric.} \end{aligned}$$

Second,

$$C_{\widetilde{Z}\widetilde{W}} \;=\; \begin{bmatrix} C_{XU} & C_{XV} \\ C_{YU} & C_{YV} \end{bmatrix} \;=\; \begin{bmatrix} C_{XU} & C_{XV} \\ -C_{XV} & C_{XU} \end{bmatrix}, \quad \text{since } \Theta \text{ is circularly symmetric.}$$

It is now clear that $K_{ZW} = 0$ if and only if $C_{\widetilde{Z}\widetilde{W}} = 0$.

(b) Assuming zero means again, we compute

$$\begin{aligned} K_W &= \mathsf{E}[WW^H] \;=\; \mathsf{E}[(U+jV)(U+jV)^H] \;=\; \mathsf{E}[(U+jV)(U^H-jV^H)] \\ &= C_U - jC_{UV} + jC_{VU} + C_V \;=\; 2(C_U - jC_{UV}). \end{aligned}$$

We now see that $AK_W = K_{ZW}$ becomes

$$2(\alpha + j\beta)(C_U - jC_{UV}) \;=\; 2(C_{XU} - jC_{XV})$$

or

$$(\alpha C_U + \beta C_{UV}) + j(\beta C_U - \alpha C_{UV}) \;=\; C_{XU} - jC_{XV}. \qquad (*)$$

We also have

$$C_{\widetilde{W}} \;=\; \begin{bmatrix} C_U & C_{UV} \\ C_{VU} & C_V \end{bmatrix}$$

so that $\widetilde{A}C_{\widetilde{W}} = C_{\widetilde{Z}\widetilde{W}}$ becomes

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} C_U & C_{UV} \\ C_{VU} & C_V \end{bmatrix} \;=\; C_{\widetilde{Z}\widetilde{W}}$$

or

$$\begin{bmatrix} \alpha C_U - \beta C_{VU} & \alpha C_{UV} - \beta C_V \\ \beta C_U + \alpha C_{VU} & \beta C_{UV} + \alpha C_V \end{bmatrix} \;=\; C_{\widetilde{Z}\widetilde{W}}$$

or

$$\begin{bmatrix} \alpha C_U + \beta C_{UV} & \alpha C_{UV} - \beta C_U \\ \beta C_U - \alpha C_{UV} & \beta C_{UV} + \alpha C_U \end{bmatrix} \;=\; \begin{bmatrix} C_{XU} & C_{XV} \\ -C_{XV} & C_{XU} \end{bmatrix},$$

which is equivalent to $(*)$.

(c) If A solves $AK_W = K_{ZW}$, then by part (b), $\widetilde{A}$ solves $\widetilde{A}C_{\widetilde{W}} = C_{\widetilde{Z}\widetilde{W}}$. Hence, by Problem 17, given $\widetilde{W} = \widetilde{w}$,

$$\widetilde{Z} \sim N\Big(m_{\widetilde{Z}} + \widetilde{A}(\widetilde{w} - m_{\widetilde{W}}), C_{\widetilde{Z}} - \widetilde{A}C_{\widetilde{W}\widetilde{Z}}\Big).$$

Next, $C_{\widetilde{Z}} - \widetilde{A}C_{\widetilde{W}\widetilde{Z}}$ is equivalent to

$$\begin{bmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{bmatrix} - \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} C_{UX} & C_{UY} \\ C_{VX} & C_{VY} \end{bmatrix},$$

which, by the circular symmetry of Θ, becomes

$$\begin{bmatrix} C_X & C_{XY} \\ -C_{XY} & C_X \end{bmatrix} - \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} C_{UX} & C_{UY} \\ -C_{UY} & C_{UX} \end{bmatrix},$$

or

$$\begin{bmatrix} C_X & C_{XY} \\ -C_{XY} & C_X \end{bmatrix} - \begin{bmatrix} \alpha C_{UX} + \beta C_{UY} & \alpha C_{UY} - \beta C_{UX} \\ \beta C_{UX} - \alpha C_{UY} & \beta C_{UY} + \alpha C_{UX} \end{bmatrix},$$

which is equivalent to

$$2(C_X - jC_{XY}) - (\alpha + j\beta) \cdot 2(C_{UX} - jC_{UY}),$$

which is exactly $K_Z - AK_{WZ}$. Thus, given $W = w$,

$$Z \sim N\big(m_Z + A(w - m_W), K_Z - AK_{WZ}\big).$$

25. Let $Z = X + jY$ with X and Y independent $N(0,1/2)$ as in the text.

(a) Since X and Y are zero mean,

$$\mathsf{cov}(Z) \;=\; \mathsf{E}[ZZ^*] \;=\; \mathsf{E}[X^2 + Y^2] \;=\; \tfrac{1}{2} + \tfrac{1}{2} \;=\; 1.$$

(b) First write $2|Z|^2 = 2(X^2 + Y^2) = (\sqrt{2}X)^2 + (\sqrt{2}Y)^2$. Now, $\sqrt{2}X$ and $\sqrt{2}Y$ are both $N(0,1)$. Hence, their squares are chi-squared with one degree of freedom by Problem 46 in Chapter 4 or Problem 11 in Chapter 5. Hence, by Problem 55(c) in Chapter 4 and the remark following it, $2|Z|^2$ is chi-squared with two degrees of freedom.

26. With $X \sim N(m_r,1)$ and $Y \sim N(m_i,1)$, it follows either from Problem 47 in Chapter 4 or from Problem 12 in Chapter 5 that X^2 and Y^2 are noncentral chi-squared with one degree of freedom and respective noncentrality parameters m_r^2 and m_i^2. Since X and Y are independent, it follows from Problem 65 in Chapter 4 that $X^2 + Y^2$ is noncentral chi-squared with two degrees of freedom and noncentrality parameter $m_r^2 + m_i^2$. It is now immediate from Problem 26 in Chapter 5 that $|Z| = \sqrt{X^2 + Y^2}$ has the orginal Rice density.

27. (a) The covariance matrix of W is

$$\mathsf{E}[WW^H] = \mathsf{E}[K^{-1/2}ZZ^H K^{-1/2}] = K^{-1/2}\mathsf{E}[ZZ^H]K^{-1/2} = K^{-1/2}KK^{-1/2} = I.$$

Hence,

$$f_W(w) \;=\; \frac{e^{-w^H w}}{\pi^n} \;=\; \prod_{k=1}^{n} \frac{e^{-|w_k|^2}}{\pi}.$$

(b) By part (a), the $W_k = U_k + jV_k$ are i.i.d. $N(0,1)$ with

$$f_{W_k}(w) \;=\; \frac{e^{-|w_k|^2}}{\pi} \;=\; \frac{e^{-(u_k^2+v_k^2)}}{\big(\sqrt{2\pi/2}\,\big)^2} \;=\; \frac{e^{-[(u_k/(1/\sqrt{2}\,))^2+(v_k/(1/\sqrt{2}\,))^2]/2}}{\big(\sqrt{2\pi/2}\,\big)^2}$$

$$=\; \frac{e^{-[u_k/(1/\sqrt{2}\,)]^2/2}}{\sqrt{2\pi/2}} \cdot \frac{e^{-[v_k/(1/\sqrt{2}\,)]^2/2}}{\sqrt{2\pi/2}} \;=\; f_{U_k V_k}(u_k, v_k).$$

Hence, U_k and V_k are independent $N(0,1/2)$.

(c) Write

$$2\|W\|^2 \;=\; \sum_{k=1}^{n} (\sqrt{2}\,U_k)^2 + (\sqrt{2}\,V_k)^2.$$

Since $\sqrt{2}\,U_k$ and $\sqrt{2}\,V_k$ are independent $N(0,1)$, their squares are chi-squared with one degree of freedom by Problem 46 in Chapter 4 or Problem 11 in Chapter 5. Next, $(\sqrt{2}\,U_k)^2+(\sqrt{2}\,V_k)^2$ is chi-squared with two degrees of freedom by Problem 55(c) in Chapter 4 and the remark following it. Similarly, since the W_k are indepdendent, $2\|W\|^2$ is chi-squared with $2n$ degrees of freedom.

28. (a) Write

$$0 \;=\; (u+v)'M(u+v) \;=\; u'Mu+v'Mu+u'Mv+v'Mv \;=\; 2v'Mu,$$

since $M'=M$. Hence, $v'Mu=0$.

(b) By part (a) with $v=Mu$ we have

$$0 \;=\; v'Mu \;=\; (Mu)'Mu \;=\; \|Mu\|^2.$$

Hence, $Mu=0$ for all u, and it follows that M must be the zero matrix.

29. We have from the text that

$$\begin{bmatrix} \nu' & \theta' \end{bmatrix} \begin{bmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{bmatrix} \begin{bmatrix} \nu \\ \theta \end{bmatrix}$$

is equal to

$$\nu'C_X\nu + \nu'C_{XY}\theta + \theta'C_{YX}\nu + \theta'C_Y\theta,$$

which, upon noting that $\nu'C_{XY}\theta$ is a scalar and therefore equal to its transpose, simplifies to

$$\nu'C_X\nu + 2\theta'C_{YX}\nu + \theta'C_Y\theta. \qquad (*)$$

We also have from the text (via Problem 22) that

$$w^HKw = \nu'(C_X+C_Y)\nu + \theta'(C_X+C_Y)\theta + 2\theta'(C_{YX}-C_{XY})\nu.$$

If $(*)$ is equal to $w^HKw/2$ for all ν and all θ, then in particular, this must hold for all ν when $\theta=0$. This implies

$$\nu'C_X\nu \;=\; \nu'\frac{C_X+C_Y}{2}\nu \quad \text{or} \quad \nu'\frac{C_X-C_Y}{2}\nu \;=\; 0.$$

Since ν is arbitrary, $(C_X-C_Y)/2=0$, or $C_X=C_Y$. This means that we can now write

$$w^HKw/2 \;=\; \nu'C_X\nu + \theta'C_Y\theta + \theta'(C_{YX}-C_{XY})\nu.$$

Comparing this with $(*)$ shows that

$$2\theta'C_{YX}\nu \;=\; \theta'(C_{YX}-C_{XY})\nu \quad \text{or} \quad \theta'(C_{YX}+C_{XY})\nu \;=\; 0.$$

Taking $\theta=\nu$ arbitrary and noting that $C_{YX}+C_{XY}$ is symmetric, it follows that $C_{YX}+C_{XY}=0$, and so $C_{XY}=-C_{YX}$.

30. (a) Since Γ is $2n \times 2n$, $\det(2\Gamma) = 2^{2n}\det\Gamma$. From the hint it follows that $\det\Gamma = (\det K)^2/2^{2n}$.

(b) Write

$$\begin{aligned}
VV^{-1} &= (A+BCD)[A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}] \\
&= (A+BCD)A^{-1}[I - B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}] \\
&= (I+BCDA^{-1})[I - B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}] \\
&= I + BCDA^{-1} - B(C^{-1}+DA^{-1}B)^{-1}DA^{-1} \\
&\quad - BCDA^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1} \\
&= I + BCDA^{-1} \\
&\quad - B[I+CDA^{-1}B](C^{-1}+DA^{-1}B)^{-1}DA^{-1} \\
&= I + BCDA^{-1} \\
&\quad - BC[C^{-1}+DA^{-1}B](C^{-1}+DA^{-1}B)^{-1}DA^{-1} \\
&= I + BCDA^{-1} - BCDA^{-1} = I.
\end{aligned}$$

(c) To begin, write

$$\begin{aligned}
\Gamma\Gamma^{-1} &= \begin{bmatrix} C_X & -C_{YX} \\ C_{YX} & C_X \end{bmatrix} \begin{bmatrix} \Delta^{-1} & C_X^{-1}C_{YX}\Delta^{-1} \\ -\Delta^{-1}C_{YX}C_X^{-1} & \Delta^{-1} \end{bmatrix} \\
&= \begin{bmatrix} C_X\Delta^{-1} + C_{YX}\Delta^{-1}C_{YX}C_X^{-1} & C_{YX}\Delta^{-1} - C_{YX}\Delta^{-1} \\ C_{YX}\Delta^{-1} - C_X\Delta^{-1}C_{YX}C_X^{-1} & C_{YX}C_X^{-1}C_{YX}\Delta^{-1} + C_X\Delta^{-1} \end{bmatrix} \\
&= \begin{bmatrix} C_X\Delta^{-1} + C_{YX}\Delta^{-1}C_{YX}C_X^{-1} & 0 \\ C_{YX}\Delta^{-1} - C_X\Delta^{-1}C_{YX}C_X^{-1} & (C_{YX}C_X^{-1}C_{YX} + C_X)\Delta^{-1} \end{bmatrix} \\
&= \begin{bmatrix} C_X\Delta^{-1} + C_{YX}\Delta^{-1}C_{YX}C_X^{-1} & 0 \\ C_{YX}\Delta^{-1} - C_X\Delta^{-1}C_{YX}C_X^{-1} & I \end{bmatrix}.
\end{aligned}$$

Using the hint that

$$\Delta^{-1} = C_X^{-1} - C_X^{-1}C_{YX}\Delta^{-1}C_{YX}C_X^{-1},$$

we easily obtain

$$C_X\Delta^{-1} = I - C_{YX}\Delta^{-1}C_{YX}C_X^{-1},$$

from which it follows that

$$\Gamma\Gamma^{-1} = \begin{bmatrix} I & 0 \\ C_{YX}\Delta^{-1} - C_X\Delta^{-1}C_{YX}C_X^{-1} & I \end{bmatrix}.$$

To show that the lower-left block is also zero, use the hint to write

$$\begin{aligned}
&C_{YX}\Delta^{-1} - C_X\Delta^{-1}C_{YX}C_X^{-1} \\
&\quad = C_{YX}[C_X^{-1} - C_X^{-1}C_{YX}\Delta^{-1}C_{YX}C_X^{-1}] - C_X\Delta^{-1}C_{YX}C_X^{-1} \\
&\quad = C_{YX}C_X^{-1} - C_{YX}C_X^{-1}C_{YX}\Delta^{-1}C_{YX}C_X^{-1} - C_X\Delta^{-1}C_{YX}C_X^{-1} \\
&\quad = C_{YX}C_X^{-1} - [C_{YX}C_X^{-1}C_{YX} + C_X]\Delta^{-1}C_{YX}C_X^{-1} \\
&\quad = C_{YX}C_X^{-1} - \Delta\Delta^{-1}C_{YX}C_X^{-1} = 0.
\end{aligned}$$

(d) Write

$$\begin{aligned}
KK^{-1} &= 2(C_X + jC_{YX})(\Delta^{-1} - jC_X^{-1}C_{YX}\Delta^{-1})/2 \\
&= C_X\Delta^{-1} + C_{YX}C_X^{-1}C_{YX}\Delta^{-1} + j(C_{YX}\Delta^{-1} - C_{YX}\Delta^{-1}) \\
&= \Delta\Delta^{-1} = I.
\end{aligned}$$

(e) We begin with

$$\begin{aligned}
\begin{bmatrix} x' & y' \end{bmatrix} \Gamma^{-1} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \Delta^{-1} & C_X^{-1}C_{YX}\Delta^{-1} \\ -\Delta^{-1}C_{YX}C_X^{-1} & \Delta^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \Delta^{-1}x + C_X^{-1}C_{YX}\Delta^{-1}y \\ -\Delta^{-1}C_{YX}C_X^{-1}x + \Delta^{-1}y \end{bmatrix} \\
&= x'\Delta^{-1}x + x'C_X^{-1}C_{YX}\Delta^{-1}y - y'\Delta^{-1}C_{YX}C_X^{-1}x + y'\Delta^{-1}y.
\end{aligned}$$

Now, since each of the above terms on the third line is a scalar, each term is equal to its transpose. In particular,

$$y'\Delta^{-1}C_{YX}C_X^{-1}x = x'C_X^{-1}C_{XY}\Delta^{-1}y = -x'C_X^{-1}C_{YX}\Delta^{-1}y.$$

Hence,

$$\tfrac{1}{2}\begin{bmatrix} x' & y' \end{bmatrix} \Gamma^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \tfrac{1}{2}(x'\Delta^{-1}x + 2x'C_X^{-1}C_{YX}\Delta^{-1}y + y'\Delta^{-1}y). \qquad (*)$$

We next compute

$$\begin{aligned}
z^H K^{-1} z &= \tfrac{1}{2}(x' - jy')(\Delta^{-1} - jC_X^{-1}C_{YX}\Delta^{-1})(x + jy) \\
&= \tfrac{1}{2}(x' - jy')[(\Delta^{-1}x + C_X^{-1}C_{YX}\Delta^{-1}y) + j(\Delta^{-1}y - C_X^{-1}C_{YX}\Delta^{-1}x)] \\
&= \tfrac{1}{2}(x' - jy')[(\Delta^{-1}x + C_X^{-1}C_{YX}\Delta^{-1}y) + j(\Delta^{-1}y - \Delta^{-1}C_{YX}C_X^{-1}x)]
\end{aligned}$$

by the hint that $C_X^{-1}C_{YX}\Delta^{-1} = \Delta^{-1}C_{YX}C_X^{-1}$. We continue with

$$\begin{aligned}
z^H K^{-1} z &= \tfrac{1}{2}[\{x'(\Delta^{-1}x + C_X^{-1}C_{YX}\Delta^{-1}y) + y'(\Delta^{-1}y - \Delta^{-1}C_{YX}C_X^{-1}x)\} \\
&\quad + j\{x'(\Delta^{-1}y - \Delta^{-1}C_{YX}C_X^{-1}x) - y'(\Delta^{-1}x + C_X^{-1}C_{YX}\Delta^{-1}y)\}] \\
&= \tfrac{1}{2}[\{x'\Delta^{-1}x + x'C_X^{-1}C_{YX}\Delta^{-1}y + y'\Delta^{-1}y - y'\Delta^{-1}C_{YX}C_X^{-1}x\} \\
&\quad + j\{x'\Delta^{-1}y - x'\Delta^{-1}C_{YX}C_X^{-1}x - y'\Delta^{-1}x - y'C_X^{-1}C_{YX}\Delta^{-1}y\}].
\end{aligned}$$

We now use the fact that since each of the terms in the last line is a scalar, it is equal to its transpose. Also $C_{XY} = -C_{YX}$. Hence,

$$\begin{aligned}
z^H K^{-1} z &= \tfrac{1}{2}[\{x'\Delta^{-1}x + 2x'C_X^{-1}C_{YX}\Delta^{-1}y + y'\Delta^{-1}y\} \\
&\quad - j\{x'\Delta^{-1}C_{YX}C_X^{-1}x + y'C_X^{-1}C_{YX}\Delta^{-1}y\}].
\end{aligned}$$

Since

$$x'\Delta^{-1}C_{YX}C_X^{-1}x = (x'\Delta^{-1}C_{YX}C_X^{-1}x)' = -x'C_X^{-1}C_{YX}\Delta^{-1}x = -x'\Delta^{-1}C_{YX}C_X^{-1}x,$$

and similarly for $y'C_X^{-1}C_{YX}\Delta^{-1}y$, the two imaginary terms above are zero.

CHAPTER 10
Problem Solutions

1. Write

$$
\begin{aligned}
m_X(t) &:= \mathsf{E}[X_t] \;=\; \mathsf{E}[g(t,Z)] \\
&= g(t,1)\mathsf{P}(Z=1)+g(t,2)\mathsf{P}(Z=2)+g(t,3)\mathsf{P}(Z=3) \\
&= p_1 a(t)+p_2 b(t)+p_3 c(t),
\end{aligned}
$$

and

$$
\begin{aligned}
R_X(t,s) &:= \mathsf{E}[X_t X_s] \;=\; \mathsf{E}[g(t,Z)g(s,Z)] \\
&= g(t,1)g(s,1)p_1+g(t,2)g(s,2)p_2+g(t,3)g(s,3)p_3 \\
&= a(t)a(s)p_1+b(t)b(s)p_2+c(t)c(s)p_3.
\end{aligned}
$$

2. Imitating the derivation of the Cauchy–Schwarz inequality for random variables in Chapter 2 of the text, write

$$
\begin{aligned}
0 &\le \int_{-\infty}^{\infty} |g(\theta)-\lambda h(\theta)|^2\,d\theta \\
&= \int_{-\infty}^{\infty} |g(\theta)|^2\,d\theta-\lambda\int_{-\infty}^{\infty} h(\theta)g(\theta)^*\,d\theta \\
&\qquad -\lambda^*\int_{-\infty}^{\infty} g(\theta)h(\theta)^*\,d\theta+|\lambda|^2\int_{-\infty}^{\infty} |h(\theta)|^2\,d\theta.
\end{aligned}
$$

Then put

$$
\lambda \;=\; \frac{\int_{-\infty}^{\infty} g(\theta)h(\theta)^*\,d\theta}{\int_{-\infty}^{\infty} |h(\theta)|^2\,d\theta}
$$

to get

$$
\begin{aligned}
0 &\le \int_{-\infty}^{\infty} |g(\theta)|^2\,d\theta-\frac{\left|\int_{-\infty}^{\infty} g(\theta)h(\theta)^*\,d\theta\right|^2}{\int_{-\infty}^{\infty} |h(\theta)|^2\,d\theta} \\
&\qquad -\frac{\left|\int_{-\infty}^{\infty} g(\theta)h(\theta)^*\,d\theta\right|^2}{\int_{-\infty}^{\infty} |h(\theta)|^2\,d\theta}+\frac{\left|\int_{-\infty}^{\infty} g(\theta)h(\theta)^*\,d\theta\right|^2}{\left(\int_{-\infty}^{\infty} |h(\theta)|^2\,d\theta\right)^2}\int_{-\infty}^{\infty} |h(\theta)|^2\,d\theta \\
&= \int_{-\infty}^{\infty} |g(\theta)|^2\,d\theta-\frac{\left|\int_{-\infty}^{\infty} g(\theta)h(\theta)^*\,d\theta\right|^2}{\int_{-\infty}^{\infty} |h(\theta)|^2\,d\theta}.
\end{aligned}
$$

Rearranging yields the desired result.

3. Write

$$\begin{aligned} C_X(t_1,t_2) &= \mathsf{E}[(X_{t_1}-m_X(t_1))(X_{t_2}-m_X(t_2))] \\ &= \mathsf{E}[X_{t_1}X_{t_2}]-m_X(t_1)\mathsf{E}[X_{t_2}]-\mathsf{E}[X_{t_1}]m_X(t_2)+m_X(t_1)m_X(t_2) \\ &= \mathsf{E}[X_{t_1}X_{t_2}]-m_X(t_1)m_X(t_2)-m_X(t_1)m_X(t_2)+m_X(t_1)m_X(t_2) \\ &= R_X(t_1,t_2)-m_X(t_1)m_X(t_2). \end{aligned}$$

Similarly,

$$\begin{aligned} C_{XY}(t_1,t_2) &= \mathsf{E}[(X_{t_1}-m_X(t_1))(Y_{t_2}-m_Y(t_2))] \\ &= \mathsf{E}[X_{t_1}Y_{t_2}]-m_X(t_1)\mathsf{E}[Y_{t_2}]-\mathsf{E}[X_{t_1}]m_Y(t_2)+m_X(t_1)m_Y(t_2) \\ &= \mathsf{E}[X_{t_1}X_{t_2}]-m_X(t_1)m_Y(t_2)-m_X(t_1)m_Y(t_2)+m_X(t_1)m_Y(t_2) \\ &= R_{XY}(t_1,t_2)-m_X(t_1)m_Y(t_2). \end{aligned}$$

4. Write

$$\begin{aligned} 0 \;\le\; \mathsf{E}\bigg[\bigg|\sum_{i=1}^n c_i X_{t_i}\bigg|^2\bigg] \;=\; \mathsf{E}\bigg[\bigg(\sum_{i=1}^n c_i X_{t_i}\bigg)\bigg(\sum_{k=1}^n c_k X_{t_k}\bigg)^*\bigg] \;=\; \sum_{i=1}^n\sum_{k=1}^n c_i \mathsf{E}[X_{t_i}X_{t_k}]c_k^* \\ = \sum_{i=1}^n\sum_{k=1}^n c_i R_X(t_i,t_k)c_k^*. \end{aligned}$$

5. Since X_t has zero mean, $\mathsf{var}(X_t) = R_X(t,t) = t$. Thus, $X_t \sim N(0,t)$, and

$$f_{X_t}(x) \;=\; \frac{e^{-x^2/(2t)}}{\sqrt{2\pi t}}.$$

6. First note that by making the change of variable $k = n-i$, we have

$$Y_n \;=\; \sum_{k=-\infty}^{\infty} h(k)X_{n-k}.$$

(a) Write

$$\begin{aligned} m_Y(n) \;=\; \mathsf{E}[Y_n] \;=\; \mathsf{E}\bigg[\sum_{k=-\infty}^{\infty} h(k)X_{n-k}\bigg] &= \sum_{k=-\infty}^{\infty} h(k)\mathsf{E}[X_{n-k}] \\ &= \sum_{k=-\infty}^{\infty} h(k)m_X(n-k). \end{aligned}$$

(b) Write

$$\begin{aligned} \mathsf{E}[X_nY_m] \;=\; \mathsf{E}\bigg[X_n\bigg(\sum_{k=-\infty}^{\infty} h(k)X_{m-k}\bigg)\bigg] &= \sum_{k=-\infty}^{\infty} h(k)\mathsf{E}[X_nX_{m-k}] \\ &= \sum_{k=-\infty}^{\infty} h(k)R_X(n,m-k). \end{aligned}$$

(c) Write

$$\begin{aligned}
\mathsf{E}[Y_n Y_m] &= \mathsf{E}\bigg[\bigg(\sum_{l=-\infty}^{\infty} h(l) X_{n-l}\bigg) Y_m\bigg] = \sum_{l=-\infty}^{\infty} h(l)\mathsf{E}[X_{n-l}Y_m] \\
&= \sum_{l=-\infty}^{\infty} h(l) R_{XY}(n-l,m) = \sum_{l=-\infty}^{\infty} h(l)\bigg[\sum_{k=-\infty}^{\infty} h(k) R_X(n-l,m-k)\bigg].
\end{aligned}$$

7. Let $X_t = \cos(2\pi f t + \Theta)$, where $\Theta \sim \text{uniform}[-\pi,\pi]$.

(a) Consider choices $t_1 = 0$ and $t_2 = -(\pi/2)/(2\pi f)$. Then $X_{t_1} = \cos(\Theta)$ and $X_{t_2} = \sin(\Theta)$, which are not jointly continuous.

(b) Write

$$\begin{aligned}
\mathsf{E}[g(X_t)] &= \mathsf{E}[g(\cos(2\pi f t + \Theta))] = \int_{-\pi}^{\pi} g(\cos(2\pi f t + \theta))\,\frac{d\theta}{2\pi} \\
&= \int_{-\pi+2\pi f t}^{\pi+2\pi f t} g(\cos(\tau))\,\frac{d\tau}{2\pi} = \int_{-\pi}^{\pi} g(\cos(\tau))\,\frac{d\tau}{2\pi},
\end{aligned}$$

since the integrand has period 2π and the range of integration has length 2π. Thus, $\mathsf{E}[g(X_t)]$ does not depend on t.

8. (a) Using independence and a trigonometric identity, write

$$\begin{aligned}
\mathsf{E}[Y_{t_1}Y_{t_2}] &= \mathsf{E}[X_{t_1}X_{t_2}\cos(2\pi f t_1 + \Theta)\cos(2\pi f t_2 + \Theta)] \\
&= \mathsf{E}[X_{t_1}X_{t_2}]\mathsf{E}[\cos(2\pi f t_1 + \Theta)\cos(2\pi f t_2 + \Theta)] \\
&= \tfrac{1}{2}R_X(t_1 - t_2)\mathsf{E}\big[\cos(2\pi f[t_1 - t_2]) + \cos(2\pi f[t_1 + t_2] + 2\Theta)\big] \\
&= \tfrac{1}{2}R_X(t_1 - t_2)\big\{\cos(2\pi f[t_1 - t_2]) + \underbrace{\mathsf{E}\big[\cos(2\pi f[t_1 + t_2] + 2\Theta)\big]\big\}}_{=0} \\
&= \tfrac{1}{2}R_X(t_1 - t_2)\cos(2\pi f[t_1 - t_2]).
\end{aligned}$$

(b) A similar argument yields

$$\mathsf{E}[X_{t_1}Y_{t_2}] = \mathsf{E}[X_{t_1}X_{t_2}\cos(2\pi f t_2 + \Theta)] = \mathsf{E}[X_{t_1}X_{t_2}]\underbrace{\mathsf{E}[\cos(2\pi f t_2 + \Theta)]}_{=0} = 0.$$

(c) It is clear that Y_t is zero mean. Together with part (a) it follows that Y_t is WSS.

9. By Problem 7(b), $F_{X_t}(x) = \mathsf{P}(X_t \le x) = \mathsf{E}[I_{(-\infty,x]}(X_t)]$ does not depend on t, and so we can restrict attention to the case $t = 0$. Since $X_0 = \cos(\Theta)$ has the arcsine density of Problem 35 in Chapter 5, $f(x) = (1/\pi)/\sqrt{1-x^2}$ for $|x| < 1$.

10. Second-order strict stationarity means that for every two-dimensional set B, for every t_1, t_2, and Δt, $\mathsf{P}((X_{t_1+\Delta t}, X_{t_2+\Delta t}) \in B)$ does not depend on Δt. In particular, this is true whenever B has the form $B = A \times \mathbb{R}$ for any one-dimensional set A; i.e.,

$$\begin{aligned}
\mathsf{P}((X_{t_1+\Delta t}, X_{t_2+\Delta t}) \in B) = \mathsf{P}((X_{t_1+\Delta t}, X_{t_2+\Delta t}) \in A \times \mathbb{R}) &= \mathsf{P}(X_{t_1} \in A, X_{t_2} \in \mathbb{R}) \\
&= \mathsf{P}(X_{t_1} \in A).
\end{aligned}$$

does not depend on Δt. Hence X_t is first-order strictly stationary.

11. (a) If $p_1 = p_2 = 0$ and $p_3 = 1$, then $X_t = c(t) = -1$ with probability one. Then $\mathsf{E}[X_t] = -1$ does not depend on t, and $\mathsf{E}[X_{t_1}X_{t_2}] = (-1)^2 = 1$ depends on t_1 and t_2 only through their difference $t_1 - t_2$; in fact the correlation function is a constant function of $t_1 - t_2$. Thus, X_t is WSS.

(b) If $p_1 = 1$ and $p_2 = p_3 = 0$, then $X_t = e^{-|t|}$ with probability one. Then $\mathsf{E}[X_t] = e^{-|t|}$ depends on t. Hence, X_t is not WSS.

(c) First, the only way to have $X_0 = 1$ is to have $X_t = a(t) = e^{-|t|}$, which requires $Z = 1$. Hence,

$$\mathsf{P}(X_0 = 1) \;=\; \mathsf{P}(Z = 1) \;=\; p_1.$$

Second, the only way to have $X_t \leq 0$ for $0 \leq t \leq 0.5$ is to have $X_t = c(t) = -1$, which requires $Z = 3$. Hence,

$$\mathsf{P}(X_t \leq 0, 0 \leq t \leq 0.5) \;=\; \mathsf{P}(Z = 3) \;=\; p_3.$$

Third, the only way to have $X_t \leq 0$ for $0.5 \leq t \leq 1$ is to have $X_t = b(t) = \sin(2\pi t)$ or $X_t = c(t) = -1$. Hence,

$$\mathsf{P}(X_t \leq 0, 0.5 \leq t \leq 1) \;=\; \mathsf{P}(Z = 2 \text{ or } Z = 3) \;=\; p_2 + p_3.$$

12. With $Y_k := q(X_k, X_{k+1}, \ldots, X_{k+L-1})$, write

$$\mathsf{E}[e^{j(\nu_1 Y_{1+m} + \cdots + \nu_n Y_{n+m})}] \;=\; \mathsf{E}[e^{j\{\nu_1 q(X_{1+m},\ldots,X_{m+L}) + \cdots + \nu_n q(X_{n+m},\ldots,X_{n+m+L-1})\}}].$$

The exponential on the right is just a function of $X_{1+m}, \ldots, X_{n+L-1+m}$. Since the X_k process is strictly stationary, the expectation on the right is unchanged if we replace $X_{1+m}, \ldots, X_{n+L-1+m}$ by $X_1, \ldots, X_{n+L-1}$; i.e., the above right-hand side is equal to

$$\mathsf{E}[e^{j\{\nu_1 q(X_1,\ldots,X_L) + \cdots + \nu_n q(X_n,\ldots,X_{n+L-1})\}}] \;=\; \mathsf{E}[e^{j(\nu_1 Y_1 + \cdots + \nu_n Y_n)}].$$

Hence, the Y_k process is strictly stationary.

13. We begin with

$$\mathsf{E}[g(X_0)] \;=\; \mathsf{E}[X_0 I_{[0,\infty)}(X_0)] \;=\; \int_{-\infty}^{\infty} x I_{[0,\infty)}(x) \cdot \tfrac{\lambda}{2} e^{-\lambda|x|}\,dx \;=\; \tfrac{1}{2}\int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx,$$

which is just $1/2$ times the expectation of an $\exp(\lambda)$ random variable. We thus see that $\mathsf{E}[g(X_0)] = 1/(2\lambda)$. Next, for $n \neq 0$, we compute

$$\begin{aligned}\mathsf{E}[g(X_n)] \;&=\; \int_{-\infty}^{\infty} x I_{[0,\infty)}(x) \cdot \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \;=\; \frac{1}{\sqrt{2\pi}}\int_0^{\infty} x e^{-x^2/2}\,dx \\ &=\; \frac{1}{\sqrt{2\pi}}\big(-e^{-x^2/2}\big)\Big|_0^{\infty} \;=\; \frac{1}{\sqrt{2\pi}}.\end{aligned}$$

Hence, $\mathsf{E}[g(X_0)] \neq \mathsf{E}[g(X_n)]$ for $n \neq 0$, and it follows that X_k is not strictly stationary.

14. First consider the mean function,

$$\mathsf{E}[q(t+T)] \;=\; \frac{1}{T_0}\int_0^{T_0} q(t+\theta)\,d\theta \;=\; \frac{1}{T_0}\int_t^{t+T_0} q(\tau)\,d\tau \;=\; \frac{1}{T_0}\int_0^{T_0} q(\tau)\,d\tau,$$

where we have used the fact that since q has period T_0, the integral of q over any interval of length T_0 yields the same result. The second thing to consider is the correlation function. Write

$$\begin{aligned}\mathsf{E}[q(t_1+T)q(t_2+T)] &= \frac{1}{T_0}\int_0^{T_0} q(t_1+\theta)q(t_2+\theta)\,d\theta \\ &= \frac{1}{T_0}\int_{t_2}^{t_2+T_0} q(t_1+\tau-t_2)q(\tau)\,d\tau \\ &= \frac{1}{T_0}\int_0^{T_0} q([t_1-t_2]+\tau)q(\tau)\,d\tau,\end{aligned}$$

where we have used the fact that as a function of τ, the product $q([t_1-t_2]+\tau)q(\tau)$ has period T_0. Since the mean function does not depend on t, and since the correlation function depends on t_1 and t_2 only through their difference, X_t is WSS.

15. For arbitrary functions h, write

$$\begin{aligned}\mathsf{E}[h(X_{t_1+\Delta t},\ldots,X_{t_n+\Delta t})] &= \mathsf{E}\big[h\big(q(t_1+\Delta t+T),\ldots,q(t_n+\Delta t+T)\big)\big] \\ &= \frac{1}{T_0}\int_0^{T_0} h\big(q(t_1+\Delta t+\theta),\ldots,q(t_n+\Delta t+\theta)\big)\,d\theta \\ &= \frac{1}{T_0}\int_{\Delta t}^{\Delta t+T_0} h\big(q(t_1+\tau),\ldots,q(t_n+\tau)\big)\,d\tau \\ &= \frac{1}{T_0}\int_0^{T_0} h\big(q(t_1+\tau),\ldots,q(t_n+\tau)\big)\,d\tau,\end{aligned}$$

where the last step follows because we are integrating a function of period T_0 over an interval of length T_0. Hence,

$$\mathsf{E}[h(X_{t_1+\Delta t},\ldots,X_{t_n+\Delta t})] \;=\; \mathsf{E}[h(X_{t_1},\ldots,X_{t_n})],$$

and we see that X_t is strictly stationary.

16. First write $\mathsf{E}[Y_n] = \mathsf{E}[X_n - X_{n-1}] = \mathsf{E}[X_n] - \mathsf{E}[X_{n-1}] = 0$ since $\mathsf{E}[X_n]$ does not depend on n. Next, write

$$\begin{aligned}\mathsf{E}[Y_nY_m] &= \mathsf{E}[(X_n-X_{n-1})(X_m-X_{m-1})] \\ &= R_X(n-m) - R_X([n-1]-m) - R_X(n-[m-1]) + R_X(n-m) \\ &= 2R_X(n-m) - R_X(n-m-1) - R_X(n-m+1),\end{aligned}$$

which depends on n and m only through their difference. Hence, Y_n is WSS.

17. From the Fourier transform table, $S_X(f) = \sqrt{2\pi}\,e^{-(2\pi f)^2/2}$.

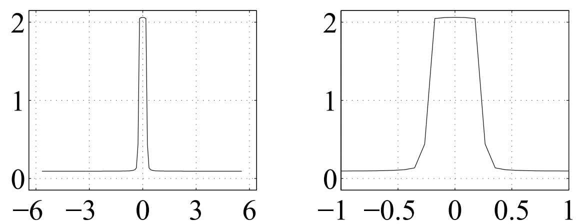
18. From the Fourier transform table, $S_X(f) = \pi e^{-2\pi|f|}$.

19. (a) Since correlation functions are real and even, we can write

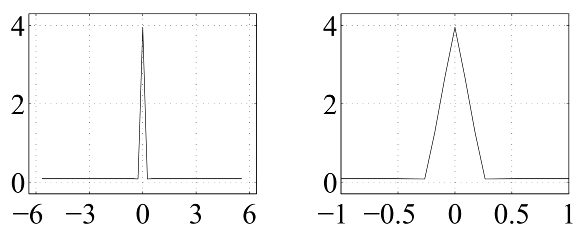
$$\begin{aligned} S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau}\, d\tau \;=\; \int_{-\infty}^{\infty} R_X(\tau)\cos(2\pi f\tau)\, d\tau \\ &= 2\int_0^{\infty} R_X(\tau)\cos(2\pi f\tau)\, d\tau \;=\; 2\,\mathrm{Re}\int_0^{\infty} R_X(\tau) e^{-j2\pi f\tau}\, d\tau. \end{aligned}$$

(b) OMITTED.

(c) The requested plot is at the left; at the right the plot is focused closer to $f = 0$.



(d) The requested plot is at the left; at the right the plot is focused closer to $f = 0$.



20. (a) $R_X(n) = \mathsf{E}[X_{k+n}X_k] = \mathsf{E}[X_kX_{k+n}] = R_X(-n)$.

(b) Since $R_X(n)$ is real and even, we can write

$$\begin{aligned} S_X(f) &= \sum_{n=-\infty}^{\infty} R_X(n) e^{-j2\pi fn} \;=\; \sum_{n=-\infty}^{\infty} R_X(n)[\cos(2\pi fn) - j\sin(2\pi fn)] \\ &= \sum_{n=-\infty}^{\infty} R_X(n)\cos(2\pi fn) - j\underbrace{\sum_{n=-\infty}^{\infty} \overbrace{R_X(n)\sin(2\pi fn)}^{\text{odd function of } n}}_{=\,0} \\ &= \sum_{n=-\infty}^{\infty} R_X(n)\cos(2\pi fn), \end{aligned}$$

which is a real and even function of f.

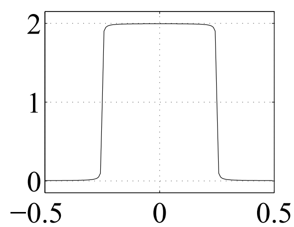
21. (a) Since correlation functions are real and even, we can write

$$S_X(f) = \sum_{n=-\infty}^{\infty} R_X(n) e^{-j2\pi fn}$$

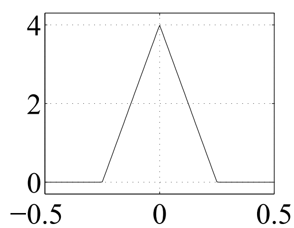
$$
\begin{aligned}
&= R_X(0) + \sum_{n=1}^{\infty} R_X(n) e^{-j2\pi f n} + \sum_{n=-\infty}^{-1} R_X(n) e^{-j2\pi f n} \\
&= R_X(0) + \sum_{n=1}^{\infty} R_X(n) e^{-j2\pi f n} + \sum_{k=1}^{\infty} R_X(-k) e^{j2\pi f k} \\
&= R_X(0) + 2\sum_{n=1}^{\infty} R_X(n) \cos(2\pi f n) \\
&= R_X(0) + 2\,\mathrm{Re} \sum_{n=1}^{\infty} R_X(n) e^{-j2\pi f n}.
\end{aligned}
$$

(b) OMITTED.

(c) Here is the plot:



(d) Here is the plot:



22. Write

$$
\begin{aligned}
\int_{-\infty}^{\infty} h(-t) e^{-j2\pi f t}\, dt &= \int_{-\infty}^{\infty} h(\theta) e^{-j2\pi f(-\theta)}\, d\theta \\
&= \int_{-\infty}^{\infty} h(\theta) e^{j2\pi f\theta}\, d\theta \\
&= \left(\int_{-\infty}^{\infty} h(\theta)^* e^{-j2\pi f\theta}\, d\theta \right)^* \\
&= \left(\int_{-\infty}^{\infty} h(\theta) e^{-j2\pi f\theta}\, d\theta \right)^*, \quad \text{since } h \text{ is real,} \\
&= H(f)^*.
\end{aligned}
$$

23. We begin with $S_{XY}(f) = H(f)^* S_X(f) = (j2\pi f)^* S_X(f) = -j2\pi f S_X(f)$. It then follows that

$$
R_{XY}(\tau) = -\frac{d}{d\tau} R_X(\tau) = -\frac{d}{d\tau} e^{-\tau^2/2} = \tau e^{-\tau^2/2}.
$$

Similarly, since $S_Y(f) = |H(f)|^2 S_X(f) = -(j2\pi f)(j2\pi f)S_X(f)$,

$$R_Y(\tau) \;=\; -\frac{d^2}{d\tau^2}R_X(\tau) \;=\; \frac{d}{d\tau}R_{XY}(\tau) \;=\; \frac{d}{d\tau}\tau e^{-\tau^2/2} \;=\; e^{-\tau^2/2}(1-\tau^2).$$

24. Since $R_X(\tau) = 1/(1+\tau^2)$, we have from the transform table that $S_X(f) = \pi e^{-2\pi|f|}$. Similarly, since $h(t) = 3\sin(\pi t)/(\pi t)$, we have from the transform table that $H(f) = 3I_{[-1/2,1/2]}(f)$. We can now write

$$S_Y(f) \;=\; |H(f)|^2 S_X(f) \;=\; 9I_{[-1/2,1/2]}(f)\cdot\pi e^{-2\pi|f|} \;=\; 9\pi e^{-2\pi|f|}I_{[-1/2,1/2]}(f).$$

25. First note that since $R_X(\tau) = e^{-\tau^2/2}$, $S_X(f) = \sqrt{2\pi}e^{-(2\pi f)^2/2}$.

(a) $S_{XY}(f) \;=\; H(f)^* S_X(f) \;=\; \big[e^{-(2\pi f)^2/2}\big]^*\sqrt{2\pi}e^{-(2\pi f)^2/2} \;=\; \sqrt{2\pi}\,e^{-(2\pi f)^2}$.

(b) Writing

$$S_{XY}(f) \;=\; \frac{1}{\sqrt{2}}\sqrt{2\pi}\sqrt{2}e^{-(\sqrt{2})^2(2\pi f)^2/2},$$

we have from the transform table that

$$R_{XY}(\tau) \;=\; \frac{1}{\sqrt{2}}e^{-(\tau/\sqrt{2})^2/2} \;=\; \frac{1}{\sqrt{2}}e^{-\tau^2/4}.$$

(c) Write

$$\mathsf{E}[X_{t_1}Y_{t_2}] \;=\; R_{XY}(t_1-t_2) \;=\; \frac{1}{\sqrt{2}}e^{-(t_1-t_2)^2/4}.$$

(d) $S_Y(f) \;=\; |H(f)|^2 S_X(f) \;=\; e^{-(2\pi f)^2}\cdot\sqrt{2\pi}e^{-(2\pi f)^2/2} \;=\; \sqrt{2\pi}e^{-3(2\pi f)^2/2}$.

(e) Writing

$$S_Y(f) \;=\; \frac{1}{\sqrt{3}}\sqrt{2\pi}\sqrt{3}e^{-(\sqrt{3})^2(2\pi f)^2/2},$$

we have from the transform table that

$$R_Y(\tau) \;=\; \frac{1}{\sqrt{3}}e^{-(\tau/\sqrt{3})^2/2} \;=\; \frac{1}{\sqrt{3}}e^{-\tau^2/6}.$$

26. We have from the transform table that $S_X(f) = [\sin(\pi f)/(\pi f)]^2$. The goal is to choose a filter $H(f)$ so that $R_Y(\tau) = \sin(\pi\tau)/(\pi\tau)$; i.e., so that $S_Y(f) = I_{[-1/2,1/2]}(f)$. Thus, the formula $S_Y(f) = |H(f)|^2 S_X(f)$ becomes

$$I_{[-1/2,1/2]}(f) \;=\; |H(f)|^2\left[\frac{\sin(\pi f)}{\pi f}\right]^2.$$

We therefore take

$$H(f) \;=\; \begin{cases} \dfrac{\pi f}{\sin(\pi f)}, & |f| \le 1/2, \\ 0, & |f| > 1/2. \end{cases}$$

27. Since Y_t and Z_t are responses of LTI systems to a WSS input, Y_t and Z_t are individually WSS. If we can show that $\mathsf{E}[Y_{t_1}Z_{t_2}]$ depends on t_1 and t_2 only throught their difference, then Y_t and Z_t will be J-WSS. We show this to be the case. Write

$$\begin{aligned}
\mathsf{E}[Y_{t_1}Z_{t_2}] &= \mathsf{E}\biggl[\int_{-\infty}^{\infty} h(\theta)X_{t_1-\theta}\,d\theta \int_{-\infty}^{\infty} g(\tau)X_{t_2-\tau}\,d\tau\biggr] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\theta)g(\tau)\mathsf{E}[X_{t_1-\theta}X_{t_2-\tau}]\,d\tau\,d\theta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\theta)g(\tau)R_X([t_1-\theta]-[t_2-\tau])\,d\tau\,d\theta,
\end{aligned}$$

which depends on $t_1 - t_2$ as claimed.

28. Observe that if $h(t) := \delta(t) - \delta(t-1)$, then $\int_{-\infty}^{\infty} h(\tau)X_{t-\tau}\,d\tau = X_t - X_{t-1}$.

(a) Since $Y_t := X_t - X_{t-1}$ is the response of an LTI system to a WSS input, X_t and Y_t is J-WSS.

(b) Since $H(f) = 1 - e^{-j2\pi f}$,

$$\begin{aligned}
|H(f)|^2 &= (1-e^{-j2\pi f})(1-e^{j2\pi f}) \;=\; 2 - e^{j2\pi f} - e^{-j2\pi f} \\
&= 2 - 2\frac{e^{j2\pi f}+e^{-j2\pi f}}{2} \;=\; 2[1-\cos(2\pi f)],
\end{aligned}$$

we have

$$S_Y(f) = |H(f)|^2 S_X(f) = 2[1-\cos(2\pi f)]\cdot\frac{2}{1+(2\pi f)^2} = \frac{4[1-\cos(2\pi f)]}{1+(2\pi f)^2}.$$

29. In $Y_t = \int_{t-3}^{t} X_\tau\,d\tau$, make the change of variable $\theta = t - \tau$, $d\theta = -d\tau$ to get

$$Y_t \;=\; \int_0^3 X_{t-\theta}\,d\theta \;=\; \int_{-\infty}^{\infty} I_{[0,3]}(\theta)X_{t-\theta}\,d\theta.$$

This shows that Y_t is the response to X_t of the LTI system with impulse $h(t) = I_{[0,3]}(t)$. Hence, Y_t is WSS.

30. Apply the formula

$$H(0) \;=\; \biggl(\int_{-\infty}^{\infty} h(t)e^{-j2\pi f t}\,dt\biggr)\bigg|_{f=0}$$

with $h(t) = (1/\pi)[(\sin t)/t]^2$. Then from the table, $H(f) = (1-\pi|f|)I_{[-1/\pi,1/\pi]}(f)$, and we find that

$$1 \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\Bigl(\frac{\sin t}{t}\Bigr)^2 dt, \quad \text{which is equivalent to} \quad \pi \;=\; \int_{-\infty}^{\infty}\Bigl(\frac{\sin t}{t}\Bigr)^2 dt.$$

31. (a) Following the hint, we have

$$\mathsf{E}[X_nY_m] \;=\; \sum_{k=-\infty}^{\infty} h(k)R_X(n,m-k) \;=\; \sum_{k=-\infty}^{\infty} h(k)R_X(n-m+k).$$

(b) Similarly,

$$\begin{aligned}
\mathsf{E}[Y_n Y_m] &= \sum_{l=-\infty}^{\infty} h(l)\left[\sum_{k=-\infty}^{\infty} h(k) R_X(n-l, m-k)\right] \\
&= \sum_{l=-\infty}^{\infty} h(l)\left[\sum_{k=-\infty}^{\infty} h(k) R_X(n-l-[m-k])\right] \\
&= \sum_{l=-\infty}^{\infty} h(l)\left[\sum_{k=-\infty}^{\infty} h(k) R_X([n-m]-[l-k])\right].
\end{aligned}$$

(c) By part (a),

$$R_X(n) = \sum_{k=-\infty}^{\infty} h(k) R_X(n+k),$$

and so

$$\begin{aligned}
S_{XY}(f) &= \sum_{n=-\infty}^{\infty} R_X(n) e^{-j2\pi f n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} h(k) R_X(n+k)\right] e^{-j2\pi f n} \\
&= \sum_{k=-\infty}^{\infty} h(k)\left[\sum_{n=-\infty}^{\infty} R_X(n+k) e^{-j2\pi f n}\right] \\
&= \sum_{k=-\infty}^{\infty} h(k)\left[\sum_{m=-\infty}^{\infty} R_X(m) e^{-j2\pi f(m-k)}\right] \\
&= \left[\sum_{k=-\infty}^{\infty} h(k) e^{j2\pi f k}\right]\left[\sum_{m=-\infty}^{\infty} R_X(m) e^{-j2\pi f m}\right] \\
&= \left[\sum_{k=-\infty}^{\infty} h(k) e^{-j2\pi f k}\right]^* S_X(f), \quad \text{since } h(k) \text{ is real}, \\
&= H(f)^* S_X(f).
\end{aligned}$$

(d) By part (b),

$$R_Y(n) = \sum_{l=-\infty}^{\infty} h(l)\left[\sum_{k=-\infty}^{\infty} h(k) R_X(n-[l-k])\right],$$

and so

$$\begin{aligned}
S_Y(f) &= \sum_{n=-\infty}^{\infty}\left\{\sum_{l=-\infty}^{\infty} h(l)\left[\sum_{k=-\infty}^{\infty} h(k) R_X(n-[l-k])\right]\right\} e^{-j2\pi f n} \\
&= \sum_{l=-\infty}^{\infty} h(l) \sum_{k=-\infty}^{\infty} h(k) \sum_{n=-\infty}^{\infty} R_X(n-[l-k]) e^{-j2\pi f n} \\
&= \sum_{l=-\infty}^{\infty} h(l) \sum_{k=-\infty}^{\infty} h(k) \sum_{m=-\infty}^{\infty} R_X(m) e^{-j2\pi f(m+[l-k])} \\
&= \sum_{l=-\infty}^{\infty} h(l) e^{-j2\pi f l} \sum_{k=-\infty}^{\infty} h(k) e^{j2\pi f k} \sum_{m=-\infty}^{\infty} R_X(m) e^{-j2\pi f m} \\
&= H(f) H(f)^* S_X(f) = |H(f)|^2 S_X(f).
\end{aligned}$$

32. By the hint,

$$\frac{1}{2T}\int_0^T |x(t)|^2\,dt \;=\; \frac{nE_0}{2T}+\frac{1}{2T}\int_0^\tau |x(t)|^2\,dt.$$

We first observe that since

$$T_0 \;\le\; T_0+\frac{\tau}{n} \;=\; \frac{T}{n} \;\le\; T_0+\frac{T_0}{n} \;\to\; T_0,$$

$T/n \to T_0$. It follows that

$$\frac{nE_0}{2T} \;=\; \frac{E_0}{2(T/n)} \;\to\; \frac{E_0}{2T_0}.$$

We next show that the integral on the right goes to zero. Write

$$\frac{1}{2T}\int_0^\tau |x(t)|^2\,dt \;\le\; \frac{1}{2T}\int_0^{T_0} |x(t)|^2\,dt \;=\; \frac{E_0}{2T} \;\to\; 0.$$

A similar argument shows that

$$\frac{1}{2T}\int_{-T}^0 |x(t)|^2\,dt \;\to\; E_0/(2T_0).$$

Putting this all together shows that $(1/2T)\int_{-T}^T |x(t)|^2\,dt \to E_0/T_0$.

33. Write

$$\mathsf{E}\bigg[\int_{-\infty}^\infty X_t^2\,dt\bigg] \;=\; \int_{-\infty}^\infty \mathsf{E}[X_t^2]\,dt \;=\; \int_{-\infty}^\infty R_X(0)\,d\tau \;=\; \infty.$$

34. If the function $q(W) := \int_0^W S_1(f)-S_2(f)\,df$ is identically zero, then so is its derivative, $q'(W) = S_1(f)-S_2(f)$. But then $S_1(f) = S_2(f)$ for all $f \ge 0$.

35. If $h(t) = I_{[-T,T]}(t)$, and white noise is applied to the corresponding system, the cross power spectral density of the input and output is

$$S_{XY}(f) \;=\; H(f)^*N_0/2 \;=\; 2T\frac{\sin(2\pi Tf)}{2\pi Tf}\cdot\frac{N_0}{2},$$

which is real and even, but not nonnegative. Similarly, if $h(t) = e^{-t}I_{[0,\infty)}(t)$,

$$S_{XY}(f) \;=\; \frac{N_0/2}{1+j2\pi f},$$

which is complex valued.

36. First write

$$S_Y(f) \;=\; |H(f)|^2N_0/2 \;=\; \begin{cases} (1-f^2)^2N_0/2, & |f|\le 1,\\ 0, & |f|>1.\end{cases}$$

Then

$$\begin{aligned} P_Y \;&=\; \int_{-\infty}^\infty S_Y(f)\,df \;=\; \frac{N_0}{2}2\int_0^1 (1-2f^2+f^4)\,df \\ &=\; N_0\Big[f-\tfrac{2}{3}f^3+\tfrac{1}{5}f^5\Big]\Big|_0^1 \;=\; N_0[1-\tfrac{2}{3}+\tfrac{1}{5}] \;=\; 8N_0/15.\end{aligned}$$

37. First note that $R_X(\tau) = e^{-(2\pi\tau)^2/2}$ has power spectral density $S_X(f) = e^{-f^2/2}/\sqrt{2\pi}$. Using the definition of $H(f) = \sqrt{|f|}\,I_{[-1,1]}(f)$,

$$\begin{aligned}
\mathsf{E}[Y_t^2] &= R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\,df = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\,df = \int_{-1}^{1} |f| S_X(f)\,df \\
&= 2\int_0^1 f S_X(f)\,df = \sqrt{\frac{2}{\pi}}\int_0^1 f e^{-f^2/2}\,df = \sqrt{\frac{2}{\pi}}\Big(-e^{-f^2/2}\Big)\Big|_0^1 \\
&= (1-e^{-1/2})\sqrt{\frac{2}{\pi}}.
\end{aligned}$$

38. First observe that

$$S_Y(f) = |H(f)|^2 S_X(f) = \left(\frac{\sin \pi f}{\pi f}\right)^2 N_0/2.$$

This is the Fourier transform of

$$R_Y(\tau) = (1-|\tau|)I_{[-1,1]}(\tau)\tfrac{N_0}{2}.$$

Then $P_Y = R_Y(0) = N_0/2$.

39. (a) First note that

$$H(f) = \frac{1/(RC)}{1/(RC) + j2\pi f} = \frac{1}{1 + j(2\pi f)RC}.$$

Then

$$S_{XY}(f) = H(f)^* S_X(f) = \frac{N_0/2}{1 - j(2\pi f)RC}.$$

(b) The inverse Fourier transform of $S_{XY}(f) = H(f)^* N_0/2$ is

$$R_{XY}(\tau) = h(-\tau)^* N_0/2 = h(-\tau)N_0/2,$$

where the last step follows because h is real. Hence,

$$R_{XY}(\tau) = \frac{N_0}{2RC} e^{\tau/(RC)} u(-\tau).$$

(c) $\mathsf{E}[X_{t_1} Y_{t_2}] = R_{XY}(t_1 - t_2) = (N_0/(2RC))e^{(t_1-t_2)/(RC)} u(t_2 - t_1)$.

(d) $S_Y(f) = |H(f)|^2 S_X(f) = \dfrac{N_0/2}{1 + (2\pi f RC)^2}$.

(e) Since

$$S_Y(f) = \frac{N_0/2}{1+(2\pi f RC)^2} = \frac{N_0}{2(RC)^2}\cdot\frac{2(1/(RC))}{\left(\frac{1}{RC}\right)^2 + (2\pi f)^2}\left(\frac{RC}{2}\right),$$

we have that

$$R_Y(\tau) = \frac{N_0}{4RC} e^{-|\tau|/(RC)}.$$

(f) $P_Y = R_Y(0) = N_0/(4RC)$.

40. To begin, write $\mathsf{E}[Y_{t+1/2}Y_t] = R_Y(1/2)$. Next, since the input has power spectral density $N_0/2$ and since $h(t) = 1/(1+t^2)$ has transform $H(f) = \pi e^{-2\pi|f|}$, we can write

$$S_Y(f) = |H(f)|^2 S_X(f) = |\pi e^{-2\pi|f|}|^2 \tfrac{N_0}{2} = \pi^2 e^{-4\pi|f|}\tfrac{N_0}{2} = \tfrac{\pi N_0}{2}\cdot \pi e^{-2\pi(2)|f|}.$$

From the transform table, we conclude that

$$R_Y(\tau) = \frac{\pi N_0}{2}\cdot\frac{2}{4+\tau^2} = \frac{\pi N_0}{4+\tau^2},$$

and so $\mathsf{E}[Y_{t+1/2}Y_t] = R_Y(1/2) = 4\pi N_0/17$.

41. Since $H(f) = \sin(\pi T f)/(\pi T f)$, we can write

$$S_Y(f) = |H(f)|^2\frac{N_0}{2} = T\left(\frac{\sin \pi T f}{\pi T f}\right)^2 \frac{N_0}{2T}.$$

Then

$$R_Y(\tau) = \frac{N_0}{2T}(1-|\tau|/T)I_{[-T,T]}(\tau).$$

42. To begin, write

$$Y_t = e^{-t}\int_{-\infty}^{t} e^{\theta}X_\theta\,d\theta = \int_{-\infty}^{t} e^{-(t-\theta)}X_\theta\,d\theta = \int_{-\infty}^{\infty} e^{-(t-\theta)}u(t-\theta)X_\theta\,d\theta,$$

where u is the unit-step function. We then see that Y_t is the response to X_t of the LTI system with impulse response $h(t) := e^{-t}u(t)$. Hence, we know from the text that X_t and Y_t are jointly wide-sense stationary. Next, since $S_X(f) = N_0/2$, $R_X(\tau) = (N_0/2)\delta(\tau)$. We then compute in the time domain,

$$\begin{aligned} R_{XY}(\tau) &= \int_{-\infty}^{\infty} h(-\alpha)R_X(\tau-\alpha)\,d\alpha = \frac{N_0}{2}\int_{-\infty}^{\infty} h(-\alpha)\delta(\tau-\alpha)\,d\alpha = \frac{N_0}{2}h(-\tau) \\ &= \frac{N_0}{2}e^{\tau}u(-\tau). \end{aligned}$$

Next,

$$S_{XY}(f) = H(f)^* S_X(f) = \frac{N_0/2}{1-j2\pi f},$$

and

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{N_0/2}{1+(2\pi f)^2}.$$

It then follows that $R_Y(\tau) = (N_0/4)e^{-|\tau|}$.

43. Consider the impulse response

$$h(\tau) := \sum_{n=-\infty}^{\infty} h_n\delta(\tau-n).$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} h(\tau)X_{t-\tau}\,d\tau &= \int_{-\infty}^{\infty}\biggl[\sum_{n=-\infty}^{\infty} h_n\delta(\tau-n)\biggr]X_{t-\tau}\,d\tau \\ &= \sum_{n=-\infty}^{\infty} h_n \int_{-\infty}^{\infty} X_{t-\tau}\delta(\tau-n)\,d\tau \;=\; \sum_{n=-\infty}^{\infty} h_n X_{t-n} \;=:\; Y_t.\end{aligned}$$

(a) Since Y_t is the response of the LTI system with impulse response $h(t)$ to the WSS input X_t, X_t and Y_t are J-WSS.

(b) Since Y_t is the response of the LTI system with impulse response $h(t)$ to the WSS input X_t, $S_Y(f) = |H(f)|^2 S_X(f)$, where

$$\begin{aligned}H(f) &= \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty}\biggl[\sum_{n=-\infty}^{\infty} h_n\delta(\tau-n)\biggr]e^{-j2\pi f\tau}\,d\tau \\ &= \sum_{n=-\infty}^{\infty} h_n \int_{-\infty}^{\infty} \delta(\tau-n)e^{-j2\pi f\tau}\,d\tau \;=\; \sum_{n=-\infty}^{\infty} h_n e^{-j2\pi fn}\end{aligned}$$

has period one. Hence, $P(f) = |H(f)|^2$ is real, nonnegative, and has period one.

44. When the input power spectral density is $S_W(f) = 3$, the output power spectral density is $|H(f)|^2 \cdot 3$. We are also told that this output power spectral density is equal to e^{-f^2}. Hence, $|H(f)|^2 \cdot 3 = e^{-f^2}$, or $|H(f)|^2 = e^{-f^2}/3$. Next, if $S_X(f) = e^{f^2} I_{[-1,1]}(f)$, then $S_Y(f) = |H(f)|^2 S_X(f) = (e^{-f^2}/3)\cdot e^{f^2} I_{[-1,1]}(f) = (1/3)I_{[-1,1]}(f)$. It then follows that

$$R_Y(\tau) = \frac{1}{3}\cdot 2\frac{\sin(2\pi\tau)}{2\pi\tau} = \frac{2}{3}\cdot\frac{\sin(2\pi\tau)}{2\pi\tau}.$$

45. Since $H(f) = GI_{[-B,B]}(f)$ and Y_t is the response to white noise, the output power spectral density is $S_Y(f) = G^2 I_{[-B,B]}(f)\cdot N_0/2$, and so

$$R_Y(\tau) = \frac{G^2N_0}{2}\cdot 2B\frac{\sin(2\pi B\tau)}{2\pi B\tau} = G^2BN_0\cdot\frac{\sin(2\pi B\tau)}{2\pi B\tau}.$$

Note that

$$R_Y(k\Delta t) = R_Y(k/(2B)) = G^2BN_0\cdot\frac{\sin(2\pi Bk/(2B))}{2\pi Bk/(2B)} = G^2BN_0\cdot\frac{\sin(\pi k)}{\pi k},$$

which is G^2BN_0 for $k=0$ and zero otherwise. It is obvious that the X_i are zero mean. Since $\mathsf{E}[X_iX_j] = R_Y(i-j)$, and the X_i are uncorrelated with variance $\mathsf{E}[X_i^2] = R_Y(0) = G^2BN_0$.

46. (a) First write $R_X(\tau) = \mathsf{E}[X_{t+\tau}X_t^*]$. Then $R_X(-\tau) = \mathsf{E}[X_{t-\tau}X_t^*] = (\mathsf{E}[X_tX_{t-\tau}^*])^* = R_X(\tau)^*$.

(b) Since

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}\,d\tau,$$

we can write

$$\begin{aligned} S_X(f)^* &= \int_{-\infty}^{\infty} R_X(\tau)^* e^{j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty} R_X(-t)^* e^{-j2\pi ft}\,dt \\ &= \int_{-\infty}^{\infty} R_X(t) e^{-j2\pi ft}\,dt, \quad \text{by part (a)}, \\ &= S_X(f). \end{aligned}$$

Since $S_X(f)$ is equal to its complex conjugate, $S_X(f)$ is real.

(c) Write

$$\begin{aligned} \mathsf{E}[X_{t_1}Y_{t_2}^*] &= \mathsf{E}\bigg[X_{t_1}\bigg(\int_{-\infty}^{\infty} h(\theta)X_{t_2-\theta}\,d\theta\bigg)^*\bigg] = \int_{-\infty}^{\infty} h(\theta)^*\mathsf{E}[X_{t_1}X_{t_2-\theta}^*]\,d\theta \\ &= \int_{-\infty}^{\infty} h(\theta)^* R_X([t_1-t_2]+\theta)\,d\theta = \int_{-\infty}^{\infty} h(-\beta)^* R_X([t_1-t_2]-\beta)\,d\beta. \end{aligned}$$

(d) By part (c),

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(-\beta)^* R_X(\tau-\beta)\,d\beta,$$

which is the convolution of $h(-\cdot)^*$ and R_X. Hence, the transform of this equation is the product of the transform of $h(-\cdot)^*$ and S_X. We just have to observe that

$$\int_{-\infty}^{\infty} h(-\beta)^* e^{-j2\pi f\beta}\,d\beta = \int_{-\infty}^{\infty} h(t)^* e^{j2\pi ft}\,dt = \bigg[\int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt\bigg]^* = H(f)^*.$$

Hence, $S_{XY}(f) = H(f)^* S_X(f)$. Next, since

$$\begin{aligned} R_Y(\tau) &= \mathsf{E}[Y_{t+\tau}Y_t^*] = \mathsf{E}\bigg[\bigg(\int_{-\infty}^{\infty} h(\theta)X_{t+\tau-\theta}\,d\theta\bigg)Y_t^*\bigg] \\ &= \int_{-\infty}^{\infty} h(\theta)\mathsf{E}[X_{t+\tau-\theta}Y_t^*]\,d\theta = \int_{-\infty}^{\infty} h(\theta)R_{XY}(\tau-\theta)\,d\theta, \end{aligned}$$

is a convolution, its transform is

$$S_Y(f) = H(f)S_{XY}(f) = H(f)H(f)^*S_X(f) = |H(f)|^2 S_X(f).$$

47. (a) $R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}\,df = \int_{-\infty}^{\infty} \delta(f)e^{j2\pi f\tau}\,df = e^{j2\pi 0\tau} = 1.$

(b) Write

$$\begin{aligned} R_X(\tau) &= \int_{-\infty}^{\infty} [\delta(f-f_0)+\delta(f+f_0)]e^{j2\pi f\tau}\,df = e^{j2\pi f_0\tau} + e^{-j2\pi f_0\tau} \\ &= 2\cos(2\pi f_0\tau). \end{aligned}$$

(c) First write

$$\begin{aligned} S_X(f) = e^{-f^2/2} &= \Big[e^{-(\frac{1}{2\pi})^2(2\pi f)^2/2}\cdot\frac{1}{2\pi}\cdot\sqrt{2\pi}\Big]\frac{2\pi}{\sqrt{2\pi}} \\ &= \Big[e^{-(\frac{1}{2\pi})^2(2\pi f)^2/2}\cdot\frac{1}{2\pi}\cdot\sqrt{2\pi}\Big]\sqrt{2\pi}. \end{aligned}$$

From the Fourier transform table with $\sigma = 1/(2\pi)$,

$$R_X(\tau) = \sqrt{2\pi}e^{-(2\pi\tau)^2/2}.$$

(d) From the transform table with $\lambda = 1/(2\pi)$,

$$R_X(\tau) = \frac{1}{\pi} \cdot \frac{1/(2\pi)}{(1/(2\pi))^2 + \tau^2} = \frac{2}{1 + (2\pi\tau)^2}.$$

48. Write

$$\mathsf{E}[X_t^2] = R_X(0) = \left[\int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}\,df\right]\bigg|_{\tau=0} = \int_{-\infty}^{\infty} S_X(f)\,df = \int_{-W}^{W} 1\,df = 2W.$$

49. (a) $e^{-f}u(f)$ is not even.

(b) $e^{-f^2}\cos(f)$ is not nonnegative.

(c) $(1-f^2)/(1+f^4)$ is not nonnegative.

(d) $1/(1+jf^2)$ is not real valued.

50. (a) Since $\sin\tau$ is odd, it is NOT a valid correlation function.

(b) Since the Fourier transform of $\cos\tau$ is $[\delta(f-1)+\delta(f+1)]/2$, which is real, even, and nonnegative, $\cos\tau$ IS a valid correlation function.

(c) Since the Fourier transform of $e^{-\tau^2/2}$ is $\sqrt{2\pi}\,e^{-(2\pi f)^2/2}$, which is real, even, and nonnegative, $e^{-\tau^2/2}$ IS a valid correlation function.

(d) Since the Fourier transform of $e^{-|\tau|}$ is $2/[1+(2\pi f)^2]$, which is real, even, and nonnegative, $e^{-|\tau|}$ IS a valid correlation function.

(e) Since the value of $\tau^2 e^{-|\tau|}$ at $\tau = 0$ is less than the value for other values of τ, $\tau^2 e^{-|\tau|}$ is NOT a valid correlation function.

(f) Since the Fourier transform of $I_{[-T,T]}(\tau)$ is $(2T)\sin(2\pi Tf)/(2\pi Tf)$ is not nonnegative, $I_{[-T,T]}(\tau)$ is NOT a valid correlation function.

51. Since $R_0(\tau)$ is a correlation function, $S_0(f)$ is real, even, and nonnegative. Since $R(\tau) = R_0(\tau)\cos(2\pi f_0\tau)$,

$$S(f) = \tfrac{1}{2}[S_0(f-f_0) + S_0(f+f_0)],$$

which is obviously real and nonnegative. It is also even since

$$\begin{aligned} S(-f) &= \tfrac{1}{2}[S_0(-f-f_0) + S_0(-f+f_0)] \\ &= \tfrac{1}{2}[S_0(f+f_0) + S_0(f-f_0)], \quad \text{since } S_0 \text{ is even,} \\ &= S(f). \end{aligned}$$

Since $S(f)$ is real, even, and nonnegative, $R(\tau)$ is a valid correlation function.

52. First observe that the Fourier transform of $\overline{R}(\tau) = R(\tau-\tau_0) + R(\tau+\tau_0)$ is $\overline{S}(f) = 2S(f)\cos(2\pi f\tau_0)$. Hence, the answer cannot be (a) because it is possible to have $S(f) > 0$ and $\cos(2\pi f\tau_0) < 0$ for some values of f. Let $S(f) = I_{[-1/(4\tau_0),1/(4\tau_0)]}(f)$, which is real, even, and nonnegative. Hence, its inverse transform, which we denote by $R(\tau)$, is a correlation function. In this case, $\overline{S}(f) = 2S(f)\cos(2\pi f\tau_0) \geq 0$ for all f, and is real and even too. Hence, for this choice of $R(\tau)$, $\overline{R}(\tau)$ is a correlation function. Therefore, the answer is (b).

53. To begin, write

$$R(\tau) = \int_{-\infty}^{\infty} S(f)e^{j2\pi f\tau}\,df = \int_{-\infty}^{\infty} S(f)[\cos(2\pi f\tau) - j\sin(2\pi f\tau)]\,df.$$

Since S is real and even, the integral of $S(f)\sin(2\pi f\tau)$ is zero, and we have

$$R(\tau) = \int_{-\infty}^{\infty} S(f)\cos(2\pi f\tau)\,df,$$

which is a real and even function of τ. Finally,

$$\begin{aligned}|R(\tau)| &= \left|\int_{-\infty}^{\infty} S(f)e^{j2\pi f\tau}\,df\right| \leq \int_{-\infty}^{\infty} \left|S(f)e^{j2\pi f\tau}\right|\,df \\ &= \int_{-\infty}^{\infty} |S(f)|\,df = \int_{-\infty}^{\infty} S(f)df = R(0).\end{aligned}$$

54. Let $S_0(f)$ denote the Fourier transform of $R_0(\tau)$, and let $S(f)$ denote the Fourier transform of $R(\tau)$.

(a) The derivation in the text showing that the transform of a correlation function is real and even uses only the fact that correlation functions are real and even. Hence, $S_0(f)$ is real and even. Furthermore, since R is the convolution of R_0 with itself, $S(f) = S_0(f)^2$, which is real, even, and nonnegative. Hence, $R(\tau)$ is a correlation function.

(b) If $R_0(\tau) = I_{[-T,T]}(\tau)$, then

$$S_0(f) = 2T\frac{\sin(2\pi Tf)}{2\pi Tf} \quad \text{and} \quad S(f) = 2T\cdot 2T\left[\frac{\sin(2\pi Tf)}{2\pi Tf}\right]^2.$$

Hence, $R(\tau) = 2T\cdot\big(1-|\tau|/(2T)\big)I_{[-2T,2T]}(\tau)$.

55. Taking $\alpha = N_0/2$ as in the text,

$$h(t) = v(t_0 - t) = \sin(t_0 - t)I_{[0,\pi]}(t_0 - t).$$

Then h is causal for $t_0 \geq \pi$.

56. Since $v(t) = e^{(t/\sqrt{2})^2/2}$, $V(f) = \sqrt{2\pi}\sqrt{2}e^{-2(2\pi f)^2/2} = 2\sqrt{\pi}e^{-(2\pi f)^2}$. Then

$$\begin{aligned}H(f) &= \alpha\frac{V(f)^* e^{-j2\pi f t_0}}{S_X(f)} = \alpha\frac{2\sqrt{\pi}e^{-(2\pi f)^2}e^{-j2\pi f t_0}}{e^{-(2\pi f)^2/2}} \\ &= 2\alpha\sqrt{\pi}e^{-(2\pi f)^2/2}e^{-j2\pi f t_0} = \sqrt{2}\,\alpha\cdot\sqrt{2\pi}e^{-(2\pi f)^2/2}e^{-j2\pi f t_0},\end{aligned}$$

and it follows that $h(t) = \sqrt{2}\,\alpha e^{-(t-t_0)^2/2}$.

57. Let $v_0(n) := \sum_k h(n-k)v(k)$ and $Y_n := \sum_k h(n-k)X_k$. The SNR is $v_0(n_0)^2/\mathsf{E}[Y_{n_0}^2]$. We have

$$\mathsf{E}[Y_{n_0}^2] \;=\; \int_{-1/2}^{1/2} S_Y(f)\,df \;=\; \int_{-1/2}^{1/2} |H(f)|^2 S_X(f)\,df.$$

Let $V(f) := \sum_k v(k)e^{-j2\pi fk}$. Then

$$\begin{aligned}
|v_0(n_0)|^2 &= \left|\int_{-1/2}^{1/2} H(f)V(f)e^{j2\pi f n_0}\,df\right|^2 \\
&= \left|\int_{-1/2}^{1/2} H(f)\sqrt{S_X(f)}\cdot\frac{V(f)e^{j2\pi f n_0}}{\sqrt{S_X(f)}}\,df\right|^2 \\
&\le \int_{-1/2}^{1/2} |H(f)|^2 S_X(f)\,df \int_{-1/2}^{1/2} \frac{|V(f)^* e^{-j2\pi f n_0}|^2}{S_X(f)}\,df,
\end{aligned}$$

with equality if and only if

$$H(f)\sqrt{S_X(f)} \;=\; \alpha\frac{V(f)^* e^{-j2\pi f n_0}}{\sqrt{S_X(f)}} \qquad (\#)$$

for some constant α. It is now clear that the SNR is upper bounded by

$$\int_{-1/2}^{1/2} \frac{|V(f)^* e^{-j2\pi f n_0}|^2}{S_X(f)}\,df$$

and that the SNR equals the bound if and only if (#) holds with equality for some constant α. Hence, the matched filter transfer function is

$$H(f) \;=\; \alpha\frac{V(f)^* e^{-j2\pi f n_0}}{S_X(f)}.$$

58. We begin by writing

$$\mathsf{E}[U_t] \;=\; \mathsf{E}[V_t + X_t] \;=\; \mathsf{E}[V_t] + \mathsf{E}[X_t],$$

which does not depend on t since V_t and X_t are each individually WSS. Next write

$$\mathsf{E}[U_{t_1}V_{t_2}] \;=\; \mathsf{E}[(V_{t_1}+X_{t_1})V_{t_2}] \;=\; R_V(t_1-t_2) + R_{XV}(t_1-t_2). \qquad (*)$$

Now write

$$\mathsf{E}[U_{t_1}U_{t_2}] \;=\; \mathsf{E}[U_{t_1}(V_{t_2}+X_{t_2}] \;=\; \mathsf{E}[U_{t_1}V_{t_2}] + \mathsf{E}[U_{t_1}X_{t_2}].$$

By $(*)$, the term $\mathsf{E}[U_{t_1}V_{t_2}]$ depends on t_1 and t_2 only through their difference. Since

$$\mathsf{E}[U_{t_1}X_{t_2}] \;=\; \mathsf{E}[(V_{t_1}+X_{t_1})X_{t_2}] \;=\; R_{VX}(t_1-t_2) + R_X(t_1-t_2),$$

it follows that $\mathsf{E}[U_{t_1}U_{t_2}]$ depends on t_1 and t_2 only through their difference. Hence, U_t and V_t are J-WSS.

59. The assumptions in the problem imply that V_t and X_t are J-WSS, and by the preceding problem, it follows that U_t and V_t are J-WSS. We can therefore apply the formulas for the Wiener filter derived in the text. It just remains to compute the quantities used in the formulas. First,

$$R_{VU}(\tau) = \mathsf{E}[V_{t+\tau}U_t] = \mathsf{E}[V_{t+\tau}(V_t+X_t)] = R_V(\tau)+R_{XV}(\tau) = R_V(\tau),$$

which implies $S_{VU}(f) = S_V(f)$. Similarly,

$$\begin{aligned} R_U(\tau) &= \mathsf{E}[U_{t+\tau}U_t] = \mathsf{E}[(V_{t+\tau}+X_{t+\tau})U_t] \\ &= R_{VU}(\tau)+\mathsf{E}[X_{t+\tau}(V_t+X_t)] \\ &= R_V(\tau)+R_{XV}(\tau)+R_X(\tau) = R_V(\tau)+R_X(\tau), \end{aligned}$$

and so $S_U(f) = S_V(f)+S_X(f)$. We then have

$$H(f) = \frac{S_{VU}(f)}{S_U(f)} = \frac{S_V(f)}{S_V(f)+S_X(f)}.$$

60. The formula for $R_V(\tau)$ implies $S_V(f) = (1-|f|)I_{[-1,1]}(f)$. We then have

$$\begin{aligned} H(f) &= \frac{S_V(f)}{S_V(f)+S_X(f)} = \frac{(1-|f|)I_{[-1,1]}(f)}{(1-|f|)I_{[-1,1]}(f)+1-I_{[-1,1]}(f)} \\ &= \frac{(1-|f|)I_{[-1,1]}(f)}{1-|f|I_{[-1,1]}(f)} = I_{[-1,1]}(f), \end{aligned}$$

and so

$$h(t) = 2\frac{\sin(2\pi t)}{2\pi t}.$$

61. To begin, write

$$\begin{aligned} \mathsf{E}[|V_t-\widehat{V}_t|^2] &= \mathsf{E}[(V_t-\widehat{V}_t)(V_t-\widehat{V}_t)] = \mathsf{E}[(V_t-\widehat{V}_t)V_t]-\mathsf{E}[(V_t-\widehat{V}_t)\widehat{V}_t] \\ &= \mathsf{E}[(V_t-\widehat{V}_t)V_t], \quad \text{by the orthogonality principle,} \\ &= \mathsf{E}[V_t^2]-\mathsf{E}[\widehat{V}_tV_t] = R_V(0)-\mathsf{E}[\widehat{V}_tV_t] = \int_{-\infty}^{\infty} S_V(f)\,df-\mathsf{E}[\widehat{V}_tV_t]. \end{aligned}$$

Next observe that

$$\begin{aligned} \mathsf{E}[\widehat{V}_tV_t] &= \mathsf{E}\bigg[\bigg(\int_{-\infty}^{\infty} h(\theta)U_{t-\theta}\,d\theta\bigg)V_t\bigg] = \int_{-\infty}^{\infty} h(\theta)\mathsf{E}[V_tU_{t-\theta}]\,d\theta \\ &= \int_{-\infty}^{\infty} h(\theta)R_{VU}(\theta)\,d\theta = \int_{-\infty}^{\infty} h(\theta)R_{VU}(\theta)^*\,d\theta, \quad \text{since } R_{VU}(\theta) \text{ is real,} \\ &= \int_{-\infty}^{\infty} H(f)S_{VU}(f)^*\,df, \quad \text{by Parseval's formula,} \\ &= \int_{-\infty}^{\infty} \frac{S_{VU}(f)}{S_U(f)}S_{VU}(f)^*\,df = \int_{-\infty}^{\infty} \frac{|S_{VU}(f)|^2}{S_U(f)}\,df. \end{aligned}$$

Putting these two observations together yields

$$\mathsf{E}[|V_t-\widehat{V}_t|^2] = \int_{-\infty}^{\infty} S_V(f)\,df-\int_{-\infty}^{\infty} \frac{|S_{VU}(f)|^2}{S_U(f)}\,df = \int_{-\infty}^{\infty} \bigg[S_V(f)-\frac{|S_{VU}(f)|^2}{S_U(f)}\bigg]df.$$

62. Denote the optimal estimator by $\widehat{V}_n = \sum_{k=-\infty}^{\infty} h(k)U_{n-k}$, and denote any other estimator by $\widetilde{V}_n = \sum_{k=-\infty}^{\infty} \tilde{h}(k)U_{n-k}$. The discrete-time orthogonality principle says that if

$$\mathsf{E}\bigg[(V_n - \widehat{V}_n)\sum_{k=-\infty}^{\infty} \tilde{h}(k)U_{n-k}\bigg] \;=\; 0 \qquad (*)$$

for *every* $\tilde{h}$, then h is optimal in that $\mathsf{E}[|V_n - \widehat{V}_n|^2] \le \mathsf{E}[|V_n - \widetilde{V}_n|^2]$ for every $\tilde{h}$. To establish the orthogonality principle, assume the above equation holds for every choice of $\tilde{h}$. Then we can write

$$\begin{aligned}
\mathsf{E}[|V_n - \widetilde{V}_n|^2] &= \mathsf{E}[|(V_n - \widehat{V}_n) + (\widehat{V}_n - \widetilde{V}_n)|^2] \\
&= \mathsf{E}[|V_n - \widehat{V}_n|^2 + 2(V_n - \widehat{V}_n)(\widehat{V}_n - \widetilde{V}_n) + |\widehat{V}_n - \widetilde{V}_n|^2] \\
&= \mathsf{E}[|V_n - \widehat{V}_n|^2] + 2\mathsf{E}[(V_n - \widehat{V}_n)(\widehat{V}_n - \widetilde{V}_n)] + \mathsf{E}[|\widehat{V}_n - \widetilde{V}_n|^2]. \quad (**)
\end{aligned}$$

Now observe that

$$\begin{aligned}
\mathsf{E}[(V_n - \widehat{V}_n)(\widehat{V}_n - \widetilde{V}_n)] &= \mathsf{E}\bigg[(V_n - \widehat{V}_n)\bigg(\sum_{k=-\infty}^{\infty} h(k)U_{n-k} - \sum_{k=-\infty}^{\infty} \tilde{h}(k)U_{n-k}\bigg)\bigg] \\
&= \mathsf{E}\bigg[(V_n - \widehat{V}_n)\sum_{k=-\infty}^{\infty} \big[h(k) - \tilde{h}(k)\big]U_{n-k}\bigg] \;=\; 0, \quad \text{by } (*).
\end{aligned}$$

We can now continue $(**)$ writing

$$\mathsf{E}[|V_n - \widetilde{V}_n|^2] \;=\; \mathsf{E}[|V_n - \widehat{V}_n|^2] + \mathsf{E}[|\widehat{V}_n - \widetilde{V}_n|^2] \;\ge\; \mathsf{E}[|V_n - \widehat{V}_n|^2],$$

and thus h is the filter that minimizes the mean-squared error.

The next task is to characterize the filter h that satisfies the orthogonality condition for every choice of $\tilde{h}$. Write the orthogonality condition as

$$\begin{aligned}
0 &= \mathsf{E}\bigg[(V_n - \widehat{V}_n)\sum_{k=-\infty}^{\infty} \tilde{h}(k)U_{t-k}\bigg] \;=\; \mathsf{E}\bigg[\sum_{k=-\infty}^{\infty} \tilde{h}(k)(V_n - \widehat{V}_n)U_{t-k}\bigg] \\
&= \sum_{k=-\infty}^{\infty} \mathsf{E}[\tilde{h}(k)(V_n - \widehat{V}_n)U_{t-k}] \;=\; \sum_{k=-\infty}^{\infty} \tilde{h}(k)\mathsf{E}[(V_n - \widehat{V}_n)U_{t-k}] \\
&= \sum_{k=-\infty}^{\infty} \tilde{h}(k)[R_{VU}(k) - R_{\widehat{V}U}(k)].
\end{aligned}$$

Since this must hold for all $\tilde{h}$, take $\tilde{h}(k) = R_{VU}(k) - R_{\widehat{V}U}(k)$ to get

$$\sum_{k=-\infty}^{\infty} \big|R_{VU}(k) - R_{\widehat{V}U}(k)\big|^2 \;=\; 0.$$

Thus, the orthogonality condition holds for all $\tilde{h}$ if and only if $R_{VU}(k) = R_{\widehat{V}U}(k)$ for all k.

The next task is to analyze $R_{\widehat{V}U}$. Recall that $\widehat{V}_n$ is the response of an LTI system to input U_n. Applying the result of Problem 31(a) with X replaced by U and Y replaced by $\widehat{V}$, we have, also using the fact that R_U is even,

$$R_{\widehat{V}U}(m) = R_{U\widehat{V}}(-m) = \sum_{k=-\infty}^{\infty} h(k)R_U(m-k).$$

Taking discrete-time Fourier transforms of

$$R_{VU}(m) = R_{\widehat{V}U}(m) = \sum_{k=-\infty}^{\infty} h(k)R_U(m-k)$$

yields

$$S_{VU}(f) = H(f)S_U(f), \quad \text{and so} \quad H(f) = \frac{S_{VU}(f)}{S_U(f)}.$$

63. We have

$$\begin{aligned} H(f) &= \frac{S_V(f)}{S_V(f)+S_X(f)} = \frac{2\lambda/[\lambda^2+(2\pi f)^2]}{2\lambda/[\lambda^2+(2\pi f)^2]+1} = \frac{2\lambda}{2\lambda+\lambda^2+(2\pi f)^2} \\ &= \frac{\lambda}{A}\cdot\frac{2A}{A^2+(2\pi f)^2}, \end{aligned}$$

where $A := \sqrt{2\lambda+\lambda^2}$. Hence, $h(t) = (\lambda/A)e^{-A|t|}$.

64. To begin, write

$$K(f) = \frac{\lambda+j2\pi f}{A+j2\pi f} = \frac{\lambda}{A+j2\pi f} + j2\pi f\frac{1}{1+j2\pi f}.$$

Then

$$\begin{aligned} k(t) &= \lambda e^{-At}u(t) + \frac{d}{dt}e^{-At}u(t) \\ &= \lambda e^{-At}u(t) - Ae^{-At}u(t) + e^{-At}\delta(t) \\ &= (\lambda - A)e^{-At}u(t) + \delta(t), \end{aligned}$$

since $e^{-At}\delta(t) = \delta(t)$ for both $t = 0$ and for $t \neq 0$. This is a causal impulse response.

65. Let $Z_t := V_{t+\Delta t}$. Then the causal Wiener filter for Z_t yields the prediction or smoothing filter for $V_{t+\Delta t}$. The Wiener–Hopf equation for Z_t is

$$R_{ZU}(\tau) = \int_0^\infty h_{\Delta t}(\theta)R_U(\tau-\theta)\,d\theta, \quad \tau \geq 0.$$

Now, $R_{ZU}(\tau) = \mathsf{E}[Z_{t+\tau}U_t] = \mathsf{E}[V_{t+\tau+\Delta t}U_t] = R_{VU}(\tau+\Delta t)$, and so we must solve

$$R_{VU}(\tau+\Delta t) = \int_0^\infty h_{\Delta t}(\theta)R_U(\tau-\theta)\,d\theta, \quad \tau \geq 0.$$

For white noise with $R_U(\tau) = \delta(\tau)$, this reduces to

$$R_{VU}(\tau+\Delta t) \;=\; \int_0^\infty h_{\Delta t}(\theta)\delta(\tau-\theta)\,d\theta \;=\; h_{\Delta t}(\tau), \quad \tau \geq 0.$$

If $h(t) = R_{VU}(t)u(t)$ denotes the causal Wiener filter, then for $\Delta t \geq 0$ (prediction), we can write

$$h_{\Delta t}(\tau) \;=\; R_{VU}(\tau+\Delta t) \;=\; h(\tau+\Delta t), \quad \tau \geq 0.$$

If $\Delta t < 0$ (smoothing), we can write $h_{\Delta t}(\tau) = h(\tau+\Delta t)$ only for $\tau \geq -\Delta t$. For $0 \leq \tau < -\Delta t$, $h(\tau+\Delta t) = 0$ while $h_{\Delta t}(\tau) = R_{VU}(\tau+\Delta t)$.

66. By the hint, the limit of the double sums is the desired double integral. If we can show that each of these double sums is nonnegative, then the limit will also be nonnegative. To this end put $Z_i := X_{t_i} e^{-j2\pi f t_i}\Delta t_i$. Then

$$\begin{aligned} 0 \;\leq\; \mathsf{E}\Bigg[\bigg|\sum_{i=1}^n Z_i\bigg|^2\Bigg] \;&=\; \mathsf{E}\Bigg[\bigg(\sum_{i=1}^n Z_i\bigg)\bigg(\sum_{k=1}^n Z_k\bigg)^*\Bigg] \;=\; \sum_{i=1}^n\sum_{k=1}^n \mathsf{E}[Z_i Z_k^*] \\ &=\; \sum_{i=1}^n\sum_{k=1}^n \mathsf{E}[X_{t_i}X_{t_k}]e^{-j2\pi f t_i}e^{j2\pi f t_k}\Delta t_i \Delta t_k \\ &=\; \sum_{i=1}^n\sum_{k=1}^n R_X(t_i - t_k)e^{-j2\pi f t_i}e^{j2\pi f t_k}\Delta t_i \Delta t_k. \end{aligned}$$

67. (a) The Fourier transform of $C_Y(\tau) = e^{-|\tau|}$ is $2/[1+(2\pi f)^2]$, which is continuous at $f=0$. Hence, we have convergence in mean square of $\frac{1}{2T}\int_{-T}^T Y_t\,dt$ to $\mathsf{E}[Y_t]$.

(b) The Fourier transform of $C_Y(\tau) = \sin(\pi\tau)/(\pi\tau)$ is $I_{[-1/2,1/2]}(f)$, which is continuous at $f=0$. Hence, we have convergence in mean square of $\frac{1}{2T}\int_{-T}^T Y_t\,dt$ to $\mathsf{E}[Y_t]$.

68. We first point out that this is *not* a question about mean-square convergence. Write

$$\frac{1}{2T}\int_{-T}^T \cos(2\pi t + \Theta)\,dt \;=\; \frac{\sin(2\pi T + \Theta) - \sin(2\pi(-T)+\Theta)}{2T\cdot 2\pi}.$$

Since $|\sin x| \leq 1$, we can write

$$\left|\frac{1}{2T}\int_{-T}^T \cos(2\pi t + \Theta)\,dt\right| \;\leq\; \frac{2}{4\pi T} \;\to\; 0,$$

and so the limit in question exists and is equal to zero.

69. As suggested by the hint, put $Y_t := X_{t+\tau}X_t$. It will be sufficient if Y_t is WSS and if the Fourier transform of the covariance function of Y_t is continuous at the origin. First, since X_t is WSS, the mean of Y_t is

$$\mathsf{E}[Y_t] \;=\; \mathsf{E}[X_{t+\tau}X_t] \;=\; R_X(\tau),$$

which does not depend on t. Before examining the correlation function of Y_t, we assume that X_t is fourth-order strictly stationary so that

$$\mathsf{E}[Y_{t_1}Y_{t_2}] \;=\; \mathsf{E}[X_{t_1+\tau}X_{t_1}X_{t_2+\tau}X_{t_2}]$$

must be unchanged if on the right-hand side we subtract t_2 from every subscript expression to get

$$\mathsf{E}[X_{t_1+\tau-t_2}X_{t_1-t_2}X_\tau X_0].$$

Since this depends on t_1 and t_2 only through their difference, we see that Y_t is WSS if X_t is fourth-order strictly stationary. Now, the covariance function of Y_t is

$$C(\theta) \;=\; \mathsf{E}[X_{\theta+\tau}X_\theta X_\tau X_0] - R_X(\tau)^2.$$

If the Fourier transform of this function of θ is continuous at the origin, then

$$\frac{1}{2T}\int_{-T}^{T} X_{t+\tau}X_t\,dt \to R_X(\tau).$$

70. As suggested by the hint, put $Y_t := I_B(X_t)$. It will be sufficient if Y_t is WSS and if the Fourier transform of the covariance function of Y_t is continuous at the origin. We assume at the outset that X_t is second-order strictly stationary. Then the mean of Y_t is

$$\mathsf{E}[Y_t] \;=\; \mathsf{E}[I_B(X_t)] \;=\; \mathsf{P}(X_t \in B),$$

which does not depend on t. Similarly,

$$\mathsf{E}[Y_{t_1}Y_{t_2}] \;=\; \mathsf{E}[I_B(X_{t_1})I_B(X_{t_2})] \;=\; \mathsf{P}(X_{t_1} \in B, X_{t_2} \in B)$$

must be unchanged if on the right-hand side we subtract t_2 from every subscript to get

$$\mathsf{P}(X_{t_1-t_2} \in B, X_0 \in B).$$

Since this depends on t_1 and t_2 only through their difference, we see that Y_t is WSS if X_t is second-order strictly stationary. Now, the covariance function of Y_t is

$$C(\theta) \;=\; \mathsf{P}(X_\theta \in B, X_0 \in B) - \mathsf{P}(X_t \in B)^2.$$

If the Fourier transform of this function of θ is continuous at the origin, then

$$\frac{1}{2T}\int_{-T}^{T} I_B(X_t)\,dt \to \mathsf{P}(X_t \in B).$$

71. We make the following definition and apply the hints:

$$\begin{aligned}
\overline{R}_{XY}(\tau) &:= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(\tau+\theta,\theta)\,d\theta \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty} h(\alpha)R_X(\tau+\theta,\theta-\alpha)\,d\alpha\right]d\theta \\
&= \int_{-\infty}^{\infty} h(\alpha)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_X(\tau+\theta,\theta-\alpha)\,d\theta\right]d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T-\alpha}^{T-\alpha} R_X(\tau+\alpha+\beta,\beta)\,d\beta\right]d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\underbrace{\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_X(\tau+\alpha+\beta,\beta)\,d\beta\right]}_{=\,\overline{R}_X(\tau+\alpha)}d\alpha.
\end{aligned}$$

72. Write

$$\begin{aligned}
\overline{R}_Y(\tau) &:= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_Y(\tau+\theta,\theta)\,d\theta \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty} h(\beta)R_{XY}(\tau+\theta-\beta,\theta)\,d\beta\right]d\theta \\
&= \int_{-\infty}^{\infty} h(\beta)\underbrace{\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}([\tau-\beta]+\theta,\theta)\,d\theta\right]}_{=\overline{R}_{XY}(\tau-\beta)}d\beta.
\end{aligned}$$

73. Let $S_{XY}(f)$ denote the Fourier transform of $\overline{R}_{XY}(\tau)$, and let $S_Y(f)$ denote the Fourier transform of $\overline{R}_Y(\tau)$. Then by the preceding two problems,

$$S_Y(f) \;=\; H(f)S_{XY}(f) \;=\; H(f)H(-f)S_X(f) \;=\; |H(f)|^2 S_X(f),$$

where, since h is real, $H(-f) = H(f)^*$.

74. This is an instance of Problem 32.

CHAPTER 11
Problem Solutions

1. With $\lambda = 3$ and $t = 10$,

$$\mathsf{P}(N_t = 0) \;=\; e^{-\lambda t} \;=\; e^{-3 \cdot 10} \;=\; e^{-30} \;=\; 9.358 \times 10^{-14}.$$

2. With $\lambda = 12$ per *minute* and $t = 20$ *seconds*, $\lambda t = 4$. Thus,

$$\begin{aligned}
\mathsf{P}(N_t > 3) \;&=\; 1 - \mathsf{P}(N_t \le 3) \;=\; 1 - e^{-\lambda t}\left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!}\right) \\
&=\; 1 - e^{-4}\left(1 + 4 + \frac{4^2}{2} + \frac{4^3}{6}\right) \;=\; 1 - e^{-4}(5 + 8 + 32/3) \\
&=\; 1 - e^{-4}(39/3 + 32/3) \;=\; 1 - e^{-4}(71/3) \;=\; 0.5665.
\end{aligned}$$

3. (a) $\mathsf{P}(N_5 = 10) \;=\; \dfrac{(2 \cdot 5)^{10} e^{-2 \cdot 5}}{10!} \;=\; 0.125.$

(b) We have

$$\begin{aligned}
\mathsf{P}\left(\bigcap_{i=1}^{5} \{N_i - N_{i-1} = 2\}\right) \;&=\; \prod_{i=1}^{5} \mathsf{P}(N_i - N_{i-1} = 2) \;=\; \prod_{i=1}^{5} \left(\frac{2^2 e^{-2}}{2!}\right) \\
&=\; (2e^{-2})^5 \;=\; 32e^{-10} \;=\; 1.453 \times 10^{-3}.
\end{aligned}$$

4. Let N_t denote the number of crates sold through time t (in days). Then $N_i - N_{i-1}$ is the number of crates sold on day i, and so

$$\begin{aligned}
\mathsf{P}\left(\bigcap_{i=1}^{5} \{N_i - N_{i-1} \ge 3\}\right) \;&=\; \prod_{i=1}^{5} \mathsf{P}(N_i - N_{i-1} \ge 3) \;=\; \prod_{i=1}^{5} \Big[1 - \mathsf{P}(N_i - N_{i-1} \le 2)\Big] \\
&=\; \prod_{i=1}^{5} \Big[1 - e^{-\lambda}\big(1 + \lambda + \lambda^2/2!\big)\Big] \\
&=\; \Big[1 - e^{-3}\big(1 + 3 + 9/2\big)\Big]^5 \;=\; 0.06385.
\end{aligned}$$

5. Let N_t denote the number of fishing rods sold through time t (in days). Then $N_i - N_{i-1}$ is the number of crates sold on day i, and so

$$\begin{aligned}
\mathsf{P}\left(\bigcup_{i=1}^{5} \{N_i - N_{i-1} \ge 3\}\right) \;&=\; 1 - \mathsf{P}\left(\bigcap_{i=1}^{5} \{N_i - N_{i-1} \le 2\}\right) \\
&=\; 1 - \prod_{i=1}^{5} \mathsf{P}(N_i - N_{i-1} \le 2)
\end{aligned}$$

$$
\begin{aligned}
&= 1-\prod_{i=1}^{5}\big[e^{-\lambda}(1+\lambda+\lambda^2/2!)\big] \\
&= 1-\big[e^{-2}(1+2+4/2!)\big]^5 \;=\; 1-e^{-10}\cdot 5^5 \;=\; 0.858.
\end{aligned}
$$

6. Since the average time between hit songs is 7 months, the rate is $\lambda = 1/7$ per month.

   (a) Since a year is 12 months, we write

$$
\begin{aligned}
\mathsf{P}(N_{12} > 2) &= 1-\mathsf{P}(N_{12} \leq 2) \;=\; 1-e^{-12\lambda}[1+12\lambda+(12\lambda)^2/2!] \\
&= 1-e^{-12/7}[1+12/7+(12/7)^2/2] \;=\; 0.247.
\end{aligned}
$$

   (b) Let T_n denote the time of the nth hit song. Since $T_n = X_1+\cdots+X_n$, $\mathsf{E}[T_n] = n\mathsf{E}[X_1] = 7n$. For $n=10$, $\mathsf{E}[T_{10}] = 70$ months.

7. (a) Since $N_0 \equiv 0$, $N_t = N_t - N_0$. Since $(0,t]\cap(t,t+\Delta t] = \varnothing$, $N_t - N_0$ and $N_{t+\Delta t} - N_t$ are independent.

   (b) Write

$$
\begin{aligned}
\mathsf{P}(N_{t+\Delta t} = k+\ell | N_t = k) &= \mathsf{P}(N_{t+\Delta t} - N_t = \ell | N_t = k), \quad \text{by substitution,} \\
&= \mathsf{P}(N_{t+\Delta t} - N_t = \ell | N_t - N_0 = k), \quad \text{since } N_0 \equiv 0, \\
&= \mathsf{P}(N_{t+\Delta t} - N_t = \ell), \quad \text{by independent increments.}
\end{aligned}
$$

   (c) Write

$$
\begin{aligned}
\mathsf{P}(N_t = k | N_{t+\Delta t} = k+\ell) &= \frac{\mathsf{P}(N_{t+\Delta t} = k+\ell | N_t = k)\mathsf{P}(N_t = k)}{\mathsf{P}(N_{t+\Delta t} = k+\ell)} \\
&= \frac{\mathsf{P}(N_{t+\Delta t} - N_t = \ell)\mathsf{P}(N_t = k)}{\mathsf{P}(N_{t+\Delta t} = k+\ell)}, \quad \text{by part (b),} \\
&= \frac{\frac{(\lambda\Delta t)^\ell e^{-\lambda\Delta t}}{\ell!}\cdot\frac{(\lambda t)^k e^{-\lambda t}}{k!}}{\frac{[\lambda(t+\Delta t)]^{k+\ell}e^{-\lambda(t+\Delta t)}}{(k+\ell)!}} \\
&= \binom{k+\ell}{k}\left(\frac{t}{t+\Delta t}\right)^k\left(\frac{\Delta t}{t+\Delta t}\right)^\ell.
\end{aligned}
$$

   (d) In part (c), put $\ell = n-k$ and put $p = t/(t+\Delta t)$. Then

$$
\mathsf{P}(N_t = k | N_{t+\Delta t} = n) \;=\; \binom{n}{k}p^k(1-p)^{n-k}, \quad k = 0,\ldots,n.
$$

8. The nth customer arrives at time $T_n \sim \text{Erlang}(n,\lambda)$. Hence,

$$
\mathsf{E}[T_n] \;=\; \frac{\Gamma(1+n)}{\lambda\Gamma(n)} \;=\; \frac{n\Gamma(n)}{\lambda\Gamma(n)} \;=\; \frac{n}{\lambda}.
$$

   Alternatively, since $T_n = X_1+\cdots+X_n$, where the X_i are i.i.d. $\exp(\lambda)$, $\mathsf{E}[T_n] = n\mathsf{E}[X_i] = n/\lambda$.

9. (a) $\mathsf{E}[X_i] = 1/\lambda = 0.5$ weeks.

(b) $\mathsf{P}(N_2 = 0) = e^{-\lambda\cdot 2} = e^{-4} = 0.0183$.

(c) $\mathsf{E}[N_{12}] = \lambda\cdot 12 = 24$ snowstorms.

(d) Write

$$\begin{aligned}
\mathsf{P}\bigg(\bigcup_{i=1}^{12}\{N_i - N_{i-1} \geq 5\}\bigg) &= 1 - \mathsf{P}\bigg(\bigcap_{i=1}^{12}\{N_i - N_{i-1} \leq 4\}\bigg) \\
&= 1 - [e^{-\lambda}(1+\lambda+\lambda^2/2+\lambda^3/6+\lambda^4/24)]^{12} \\
&= 1 - [e^{-2}(1+2+2+4/3+2/3)]^{12} \\
&= 1 - [7e^{-2}]^{12} = 1 - 0.523 = 0.477.
\end{aligned}$$

10. First observe that since $1/\lambda = 2$ months, $\lambda = 1/2$ per month.

(a) $\mathsf{P}(N_4 = 0) = e^{-\lambda\cdot 4} = e^{-4/2} = e^{-2} = 0.135$.

(b) Write

$$\begin{aligned}
\mathsf{P}\bigg(\bigcup_{i=1}^{4}\{N_i - N_{i-1} \geq 2\}\bigg) &= 1 - \mathsf{P}\bigg(\bigcap_{i=1}^{4}\{N_i - N_{i-1} \leq 1\}\bigg) \\
&= 1 - \prod_{i=1}^{4}\mathsf{P}(N_i - N_{i-1} \leq 1) \\
&= 1 - \prod_{i=1}^{4}[e^{-\lambda\cdot 1} + e^{-\lambda\cdot 1}(\lambda\cdot 1)] \\
&= 1 - [e^{-\lambda}(1+\lambda)]^4 = 1 - [\tfrac{3}{2}e^{-1/2}]^4 \\
&= 1 - \tfrac{81}{16}e^{-2} = 0.315.
\end{aligned}$$

11. We need to find $\mathsf{var}(T_n) = \mathsf{var}(X_1 + \cdots + X_n)$. Since the X_i are independent, they are uncorrelated, and so the variance of the sum is the sum of the variances. Since $X_i \sim \exp(\lambda)$, $\mathsf{var}(T_n) = n\,\mathsf{var}(X_i) = n/\lambda^2$. An alternative approach is to use the fact that $T_n \sim \text{Erlang}(n,\lambda)$. Since the moments of T_n are available,

$$\mathsf{var}(T_n) = \mathsf{E}[T_n^2] - (\mathsf{E}[T_n])^2 = \frac{(1+n)n}{\lambda^2} - \Big(\frac{n}{\lambda}\Big)^2 = \frac{n}{\lambda^2}.$$

12. To begin, use the law of total probability, substitution, and independence to write

$$\begin{aligned}
\mathsf{E}[z^{Y_t}] &= \mathsf{E}[z^{N_{\ln(1+tU)}}] = \int_0^1 \mathsf{E}[z^{N_{\ln(1+tU)}}|U=u]\,du = \int_0^1 \mathsf{E}[z^{N_{\ln(1+tu)}}|U=u]\,du \\
&= \int_0^1 \mathsf{E}[z^{N_{\ln(1+tu)}}]\,du = \int_0^1 \exp[(z-1)\ln(1+tu)]\,du = \int_0^1 e^{\ln(1+tu)^{z-1}}\,du \\
&= \int_0^1 (1+tu)^{z-1}\,du.
\end{aligned}$$

To compute this integral, we need to treat the cases $z = 0$ and $z \neq 0$ separately. We find that

$$\mathsf{E}[z^{Y_t}] = \begin{cases} \dfrac{\ln(1+t)}{t}, & z = 0, \\ \dfrac{(1+t)^z - 1}{tz}, & z \neq 0. \end{cases}$$

13. Denote the arrival times of N_t by $T_1, T_2, \ldots$, and let $X_k := T_k - T_{k-1}$ denote the interarrival times. Similarly, denote the arrival times of M_t by $S_1, S_2, \ldots$. (As it turns out, we do not need the interarrival times of M_t.) Then for arbitrary $k > 1$, we use the law of total probability, substitution, and independence to compute

$$\begin{aligned}
&\mathsf{P}(M_{T_k} - M_{T_{k-1}} = m) \\
&\quad = \int_0^\infty \int_\theta^\infty \mathsf{P}(M_{T_k} - M_{T_{k-1}} = m | T_k = t, T_{k-1} = \theta) f_{T_k T_{k-1}}(t,\theta)\,dt\,d\theta \\
&\quad = \int_0^\infty \int_\theta^\infty \mathsf{P}(M_t - M_\theta = m | T_k = t, T_{k-1} = \theta) f_{T_k T_{k-1}}(t,\theta)\,dt\,d\theta \\
&\quad = \int_0^\infty \int_\theta^\infty \mathsf{P}(M_t - M_\theta = m) f_{T_k T_{k-1}}(t,\theta)\,dt\,d\theta \\
&\quad = \int_0^\infty \int_\theta^\infty \frac{[\mu(t-\theta)]^m e^{-\mu(t-\theta)}}{m!} f_{T_k T_{k-1}}(t,\theta)\,dt\,d\theta \\
&\quad = \mathsf{E}\bigg[\frac{[\mu(T_k - T_{k-1})]^m e^{-\mu(T_k - T_{k-1})}}{m!}\bigg] \\
&\quad = \mathsf{E}\bigg[\frac{(\mu X_k)^m e^{-\mu X_k}}{m!}\bigg] = \int_0^\infty \frac{(\mu x)^m e^{-\mu x}}{m!} \cdot \lambda e^{-\lambda x}\,dx \\
&\quad = \frac{\mu^m \lambda}{(\lambda+\mu)m!} \underbrace{\int_0^\infty x^m \cdot (\lambda+\mu) e^{-(\lambda+\mu)x}\,dx}_{m\text{th moment of } \exp(\lambda+\mu) \text{ density}} \\
&\quad = \frac{\mu^m \lambda}{(\lambda+\mu)m!} \cdot \frac{m!}{(\lambda+\mu)^m} = \frac{\lambda}{\lambda+\mu}\bigg(\frac{\mu}{\lambda+\mu}\bigg)^m,
\end{aligned}$$

which is a geometric$_0(\mu/(\lambda+\mu))$ pmf in m.

14. It suffices to show that the probability generating function $G_{M_t}(z)$ has the form $e^{\mu(z-1)}$ for some $\mu > 0$. We use the law of total probability, substitution, and independence to write

$$\begin{aligned}
G_{M_t}(z) &= \mathsf{E}[z^{M_t}] = \mathsf{E}[z^{\sum_{i=1}^{N_t} Y_i}] = \sum_{n=0}^\infty \mathsf{E}[z^{\sum_{i=1}^{N_t} Y_i} | N_t = n]\mathsf{P}(N_t = n) \\
&= \sum_{n=0}^\infty \mathsf{E}[z^{\sum_{i=1}^{n} Y_i} | N_t = n]\mathsf{P}(N_t = n) = \sum_{n=0}^\infty \mathsf{E}[z^{\sum_{i=1}^{n} Y_i}]\mathsf{P}(N_t = n) \\
&= \sum_{n=0}^\infty \bigg(\prod_{i=1}^n \mathsf{E}[z^{Y_i}]\bigg)\mathsf{P}(N_t = n) = \sum_{n=0}^\infty [pz + (1-p)]^n \mathsf{P}(N_t = n) \\
&= G_{N_t}\big(pz + (1-p)\big) = e^{\lambda t([pz+(1-p)]-1)} = e^{p\lambda t(z-1)}.
\end{aligned}$$

Thus, $M_t \sim \text{Poisson}(p\lambda t)$.

15. Let $M_t = \sum_{i=1}^{N_t} V_i$ denote the total energy throught time t, with $M_t = 0$ for $N_t = 0$. We use the law of total probability, substitution, and independence to write

$$\begin{aligned}
\mathsf{E}[M_t] &= \sum_{n=0}^{\infty} \mathsf{E}[M_t|N_t = n]\mathsf{P}(N_t = n) = \sum_{n=1}^{\infty} \mathsf{E}\bigg[\sum_{i=1}^{N_t} V_i \bigg| N_t = n\bigg]\mathsf{P}(N_t = n) \\
&= \sum_{n=1}^{\infty} \mathsf{E}\bigg[\sum_{i=1}^{n} V_i \bigg| N_t = n\bigg]\mathsf{P}(N_t = n) = \sum_{n=1}^{\infty} \bigg(\sum_{i=1}^{n} \mathsf{E}[V_i]\bigg)\mathsf{P}(N_t = n) \\
&= \sum_{n=1}^{\infty} n\mathsf{E}[V_1]\mathsf{P}(N_t = n) = \mathsf{E}[V_1]\sum_{n=1}^{\infty} n\mathsf{P}(N_t = n) = \mathsf{E}[V_1]\mathsf{E}[N_t] = \mathsf{E}[V_1](\lambda t).
\end{aligned}$$

The average time between lightning strikes is $1/\lambda$ minutes.

16. We have

$$\lambda = 5.170 \pm \frac{2.132(1.96)}{10} = 5.170 \pm 0.418 \text{ with 95\% probability.}$$

In other words, $\lambda \in [4.752, 5.588]$ with 95% probability.

17. To begin, observe that

$$Y = \sum_{i=1}^{\infty} g(T_i) = \sum_{i=1}^{\infty}\sum_{k=1}^{n} g_k I_{(t_{k-1},t_k]}(T_i) = \sum_{k=1}^{n} g_k \sum_{i=1}^{\infty} I_{(t_{k-1},t_k]}(T_i) = \sum_{k=1}^{n} g_k(N_{t_k} - N_{t_{k-1}})$$

is a sum of independent random variables. Then

$$\mathsf{E}[Y] = \sum_{k=1}^{n} g_k \mathsf{E}[N_{t_k} - N_{t_{k-1}}] = \sum_{k=1}^{n} g_k \lambda(t_k - t_{k-1}) = \int_0^{\infty} g(\tau)\lambda\, d\tau,$$

since g is piecewise constant. Next,

$$\begin{aligned}
\varphi_Y(\nu) &= \mathsf{E}[e^{j\nu Y}] = \mathsf{E}[e^{j\nu \sum_{k=1}^{n} g_k(N_{t_k} - N_{t_{k-1}})}] = \prod_{k=1}^{n} \mathsf{E}[e^{j\nu g_k(N_{t_k} - N_{t_{k-1}})}] \\
&= \prod_{k=1}^{n} \exp\Big[\lambda(t_k - t_{k-1})(e^{j\nu g_k} - 1)\Big] = \exp\bigg[\sum_{k=1}^{n} \lambda(t_k - t_{k-1})(e^{j\nu g_k} - 1)\bigg] \\
&= \exp\bigg[\int_0^{\infty} \big(e^{j\nu g(\tau)} - 1\big)\lambda\, d\tau\bigg],
\end{aligned}$$

since $e^{j\nu g(\tau)} - 1$ is piecewise constant with values $e^{j\nu g_k} - 1$ (or the value zero if τ does not lie in any $(t_{k-1}, t_k]$). We now compute the correlation,

$$\begin{aligned}
\mathsf{E}[YZ] &= \mathsf{E}\bigg[\bigg(\sum_{k=1}^{n} g_k(N_{t_k} - N_{t_{k-1}})\bigg)\bigg(\sum_{l=1}^{n} h_l(N_{t_l} - N_{t_{l-1}})\bigg)\bigg] \\
&= \sum_{k \neq l} g_k h_l \mathsf{E}[(N_{t_k} - N_{t_{k-1}})(N_{t_l} - N_{t_{l-1}})] + \sum_k g_k h_k \mathsf{E}[(N_{t_k} - N_{t_{k-1}})^2] \\
&= \sum_{k \neq l} g_k h_l \lambda^2 (t_k - t_{k-1})(t_l - t_{l-1}) + \sum_k g_k h_k [\lambda(t_k - t_{k-1}) + \lambda^2 (t_k - t_{k-1})^2]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k,l} g_k h_l \lambda^2 (t_k - t_{k-1})(t_l - t_{l-1}) + \sum_k g_k h_k \lambda (t_k - t_{k-1}) \\
&= \left[\sum_k g_k \lambda (t_k - t_{k-1})\right]\left[\sum_l h_l \lambda (t_l - t_{l-1})\right] + \sum_k g_k h_k \lambda (t_k - t_{k-1}) \\
&= \int_0^\infty g(\tau)\lambda\, d\tau \int_0^\infty h(\tau)\lambda\, d\tau + \int_0^\infty g(\tau)h(\tau)\lambda\, d\tau.
\end{aligned}$$

Hence,

$$\mathsf{cov}(Y,Z) \;=\; \mathsf{E}[YZ] - \mathsf{E}[Y]\mathsf{E}[Z] \;=\; \int_0^\infty g(\tau)h(\tau)\lambda\, d\tau.$$

18. The key is to use $g(\tau) = h(t-\tau)$ in the preceding problem. It then immediately follows that

$$\mathsf{E}[Y_t] \;=\; \int_0^\infty h(t-\tau)\lambda\, d\tau, \quad \varphi_{Y_t}(\nu) \;=\; \exp\left[\int_0^\infty \big(e^{j\nu h(t-\tau)} - 1\big)\lambda\, d\tau\right],$$

and

$$\mathsf{cov}(Y_t, Y_s) \;=\; \int_0^\infty h(t-\tau)h(s-\tau)\lambda\, d\tau.$$

19. **MATLAB.** Replace the line

```
X = -log(rand(1))/lambda; % Generate exp(lambda) RV
```

with

```
X = randn(1)^2; % Generate chi-squared RV
```

20. If N_t is a Poisson process of rate λ, then F_X is the exp(λ) cdf. Hence,

$$\mathsf{P}(X_1 < Y_1) \;=\; \int_0^\infty F_X(y) f_Y(y)\, dy \;=\; \int_0^\infty [1 - e^{-\lambda y}] f_Y(y)\, dy \;=\; 1 - M_Y(-\lambda).$$

If M_t is a Poisson process of rate μ, then M_Y is the exp(μ) mgf, and

$$\mathsf{P}(X_1 < Y_1) \;=\; 1 - \frac{\mu}{\mu - (-\lambda)} \;=\; 1 - \frac{\mu}{\mu + \lambda} \;=\; \frac{\lambda}{\lambda + \mu}.$$

21. Since $T_n := X_1 + \cdots + X_n$ is the sum of i.i.d. random variables, $\mathsf{var}(T_n) = n\,\mathsf{var}(X_1)$. Since $X_1 \sim$ uniform$[0,1]$, $\mathsf{var}(X_1) = 1/12$, and $\mathsf{var}(T_n) = n/12$.

22. In the case of a Poisson process, T_k is Erlang(k,λ). Hence,

$$\begin{aligned}
\sum_{k=1}^\infty F_k(t) &= \sum_{k=1}^\infty \left[1 - \sum_{l=0}^{k-1} \frac{(\lambda t)^l e^{-\lambda t}}{l!}\right] \;=\; \sum_{k=1}^\infty \left[\sum_{l=k}^\infty \frac{(\lambda t)^l e^{-\lambda t}}{l!}\right] \\
&= \sum_{k=1}^\infty \left[\sum_{l=0}^\infty \frac{(\lambda t)^l e^{-\lambda t}}{l!} I_{[k,\infty)}(l)\right] \;=\; \sum_{l=0}^\infty \frac{(\lambda t)^l e^{-\lambda t}}{l!} \left[\sum_{k=1}^\infty I_{[k,\infty)}(l)\right] \\
&= \sum_{l=0}^\infty l \frac{(\lambda t)^l e^{-\lambda t}}{l!} \;=\; \mathsf{E}[N_t] \;=\; \lambda t.
\end{aligned}$$

23. To begin, write

$$\begin{aligned} \mathsf{E}[N_t|X_1=x] &= \mathsf{E}\bigg[\sum_{n=1}^{\infty} I_{[0,t]}(T_n)\bigg|X_1=x\bigg] = \sum_{n=1}^{\infty} \mathsf{E}\big[I_{[0,t]}(X_1+\cdots+X_n)\big|X_1=x\big] \\ &= \sum_{n=1}^{\infty} \mathsf{E}\big[I_{[0,t]}(x+X_2+\cdots+X_n)\big|X_1=x\big]. \end{aligned}$$

(a) If $x>t$, then $x+X_2+\cdots+X_n>t$, and so $I_{[0,t]}(x+X_2+\cdots+X_n)=0$. Thus, $\mathsf{E}[N_t|X_1=x]=0$ for $x>t$.

(b) First, for $n=1$ and $x\le t$, $I_{[0,t]}(x)=1$. Next, for $n\ge 2$ and $x\le t$, $I_{[0,t]}(x+X_2+\cdots+X_n)=I_{[0,t-x]}(X_2+\cdots+X_n)$. So,

$$\begin{aligned} \mathsf{E}[N_t|X_1=x] &= 1+\sum_{n=2}^{\infty} \mathsf{E}\big[I_{[0,t-x]}(X_2+\cdots+X_n)\big] \\ &= 1+\sum_{n=1}^{\infty} \mathsf{E}\big[I_{[0,t-x]}(X_1+\cdots+X_n)\big] \\ &= 1+\mathsf{E}[N_{t-x}], \end{aligned}$$

where we have used fact that the X_i are i.i.d.

(c) By the law of total probability,

$$\begin{aligned} \mathsf{E}[N_t] &= \int_0^{\infty} \mathsf{E}[N_t|X_1=x]f(x)\,dx \\ &= \int_0^t \mathsf{E}[N_t|X_1=x]f(x)\,dx+\int_t^{\infty} \underbrace{\mathsf{E}[N_t|X_1=x]}_{=\,0\ \text{for}\ x>t}f(x)\,dx \\ &= \int_0^t \Big(1+\mathsf{E}[N_{t-x}]\Big)f(x)\,dx = F(t)+\int_0^t \mathsf{E}[N_{t-x}]f(x)\,dx. \end{aligned}$$

24. With the understanding that $m(t)=0$ for $t<0$, we can write the renewal equation as

$$m(t) = F(t)+\int_0^{\infty} m(t-x)f(x)\,dx,$$

where the last term is a convolution. Hence, taking the Laplace transform of the renewal equation yields

$$M(s) := \int_0^{\infty} m(t)e^{st}\,dt = \int_0^{\infty} F(t)e^{st}\,dt+M(s)M_X(s).$$

Using integration by parts, we have

$$\int_0^{\infty} F(t)e^{st}\,dt = \frac{F(t)e^{st}}{s}\bigg|_0^{\infty}-\frac{1}{s}\int_0^{\infty} f(t)e^{st}\,dt.$$

Thus,

$$M(s) = -\frac{1}{s}M_X(s)+M(s)M_X(s),$$

which we can rearrange as

$$M(s)[1-M_X(s)] \;=\; -\frac{1}{s}M_X(s),$$

or

$$M(s) \;=\; -\frac{1}{s}\cdot\frac{M_X(s)}{1-M_X(s)} \;=\; -\frac{1}{s}\cdot\frac{\lambda/(\lambda-s)}{1-\lambda/(\lambda-s)} \;=\; -\frac{1}{s}\cdot\frac{\lambda}{-s} \;=\; \frac{\lambda}{s^2}.$$

It follows that $m(t) = \lambda t u(t)$.

25. For $0 \le s < t < \infty$, write

$$\begin{aligned}
\mathsf{E}[V_tV_s] &= \mathsf{E}\bigg[\int_0^t X_\tau\,d\tau\int_0^s X_\theta\,d\theta\bigg] \;=\; \int_0^s\bigg(\int_0^t \mathsf{E}[X_\tau X_\theta]\,d\tau\bigg)d\theta \\
&= \int_0^s\bigg(\int_0^t R_X(\tau-\theta)\,d\tau\bigg)d\theta \;=\; \sigma^2\int_0^s\bigg(\int_0^t \delta(\tau-\theta)\,d\tau\bigg)d\theta \\
&= \sigma^2\int_0^s d\theta \;=\; \sigma^2 s.
\end{aligned}$$

26. For $0 \le s < t < \infty$, write

$$\begin{aligned}
\mathsf{E}[W_tW_s] &= \mathsf{E}[(W_t-W_s)W_s]+\mathsf{E}[W_s^2] \\
&= \mathsf{E}[(W_t-W_s)(W_s-W_0)]+\sigma^2 s \\
&= \mathsf{E}[W_t-W_s]\mathsf{E}[W_s-W_0]+\sigma^2 s \;=\; 0\cdot 0+\sigma^2 s \;=\; \sigma^2 s.
\end{aligned}$$

27. For $t_2 > t_1$, write

$$\begin{aligned}
\mathsf{E}[Y_{t_1}Y_{t_2}] &= \mathsf{E}[e^{W_{t_1}}e^{W_{t_2}}] \;=\; \mathsf{E}[e^{W_{t_2}-W_{t_1}}e^{2W_{t_1}}] \;=\; \mathsf{E}[e^{W_{t_2}-W_{t_1}}e^{2(W_{t_1}-W_0)}] \\
&= \mathsf{E}[e^{W_{t_2}-W_{t_1}}]\mathsf{E}[e^{2(W_{t_1}-W_0)}] \;=\; e^{\sigma^2(t_2-t_1)/2}e^{4\sigma^2 t_1/2} \;=\; e^{\sigma^2(t_2+3t_1)/2}.
\end{aligned}$$

28. Since $\mathsf{cov}(W_{t_i},W_{t_j}) = \sigma^2\min(t_i,t_j)$,

$$\mathsf{cov}(X) \;=\; \sigma^2\begin{bmatrix} t_1 & t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & t_2 & \cdots & t_2 \\ t_1 & t_2 & t_3 & \cdots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & & t_n \end{bmatrix}.$$

29. Let $0 \le t_1 < \cdots < t_{n+1} < \infty$ and $0 \le s_1 < \cdots < s_{m+1} < \infty$, and suppose that

$$g(\tau) \;=\; \sum_{i=1}^n g_i I_{(t_i,t_{i+1}]}(\tau) \quad\text{and}\quad h(\tau) \;=\; \sum_{j=1}^m h_j I_{(s_j,s_{j+1}]}(\tau).$$

Denote the distinct points of $\{t_i\}\cup\{s_j\}$ in increasing order by $\theta_1 < \cdots < \theta_p$. Then

$$g(\tau) \;=\; \sum_{k=1}^p \tilde{g}_k I_{(\theta_k,\theta_{k+1}]}(\tau) \quad\text{and}\quad h(\tau) \;=\; \sum_{k=1}^p \tilde{h}_k I_{(\theta_k,\theta_{k+1}]}(\tau),$$

where the $\tilde{g}_k$ are taken from the g_i, and the $\tilde{h}_k$ are taken from the h_j. We can now write

$$\begin{aligned}\int_0^\infty g(\tau)\,dW_\tau + \int_0^\infty h(\tau)\,dW_\tau &= \sum_{k=1}^p \tilde{g}_k(W_{\theta_{k+1}} - W_{\theta_k}) + \sum_{k=1}^p \tilde{h}_k(W_{\theta_{k+1}} - W_{\theta_k}) \\ &= \sum_{k=1}^p [\tilde{g}_k + \tilde{h}_k](W_{\theta_{k+1}} - W_{\theta_k}) \\ &= \int_0^\infty [g(\tau)+h(\tau)]\,dW_\tau.\end{aligned}$$

30. Following the hint, we first write

$$\begin{aligned}\mathsf{E}\bigg[\bigg(\int_0^\infty g(\tau)\,dW_\tau - \int_0^\infty h(\tau)\,dW_\tau\bigg)^2\bigg] &= \mathsf{E}\bigg[\bigg(\int_0^\infty g(\tau)\,dW_\tau\bigg)^2\bigg] \\ &\quad - 2\mathsf{E}\bigg[\int_0^\infty g(\tau)\,dW_\tau \int_0^\infty h(\tau)\,dW_\tau\bigg] \\ &\quad + \mathsf{E}\bigg[\bigg(\int_0^\infty h(\tau)\,dW_\tau\bigg)^2\bigg] \\ &= \sigma^2\int_0^\infty g(\tau)^2\,d\tau + \sigma^2\int_0^\infty h(\tau)^2\,d\tau \\ &\quad - 2\mathsf{E}\bigg[\int_0^\infty g(\tau)\,dW_\tau \int_0^\infty h(\tau)\,dW_\tau\bigg].\end{aligned}$$

Second, we write

$$\begin{aligned}\mathsf{E}\bigg[\bigg(\int_0^\infty g(\tau)\,dW_\tau - \int_0^\infty h(\tau)\,dW_\tau\bigg)^2\bigg] &= \mathsf{E}\bigg[\bigg(\int_0^\infty [g(\tau)-h(\tau)]\,dW_\tau\bigg)^2\bigg] \\ &= \sigma^2\int_0^\infty [g(\tau)-h(\tau)]^2\,d\tau \\ &= \sigma^2\int_0^\infty g(\tau)^2\,d\tau + \sigma^2\int_0^\infty h(\tau)^2\,d\tau \\ &\quad - 2\sigma^2\int_0^\infty g(\tau)h(\tau)\,d\tau.\end{aligned}$$

Comparing these two formulas, we see that

$$\mathsf{E}\bigg[\int_0^\infty g(\tau)\,dW_\tau \int_0^\infty h(\tau)\,dW_\tau\bigg] = \sigma^2\int_0^\infty g(\tau)h(\tau)\,d\tau.$$

31. (a) Following the hint,

$$Y_t := \int_0^t g(\tau)\,dW_\tau = \int_0^\infty g(\tau)I_{[0,t]}(\tau)\,dW_\tau,$$

it follows that

$$\mathsf{E}[Y_t^2] = \sigma^2\int_0^\infty [g(\tau)I_{[0,t]}(\tau)]^2\,d\tau = \sigma^2\int_0^t g(\tau)^2\,d\tau.$$

(b) For $t_1, t_2 \geq 0$, write

$$\begin{aligned}
\mathsf{E}[Y_{t_1} Y_{t_2}] &= \mathsf{E}\biggl[\int_0^\infty g(\tau) I_{[0,t_1]}(\tau)\, dW_\tau \int_0^\infty g(\tau) I_{[0,t_2]}(\tau)\, dW_\tau\biggr] \\
&= \sigma^2 \int_0^\infty \bigl\{g(\tau) I_{[0,t_1]}(\tau)\bigr\}\bigl\{g(\tau) I_{[0,t_2]}(\tau)\bigr\}\, d\tau \\
&= \sigma^2 \int_0^\infty g(\tau)^2 I_{[0,t_1]}(\tau) I_{[0,t_2]}(\tau)\, d\tau \;=\; \sigma^2 \int_0^{\min(t_1,t_2)} g(\tau)^2\, d\tau.
\end{aligned}$$

32. By independence of V and the Wiener process along with the result of the previous problem,

$$R_Y(t_1,t_2) = e^{-\lambda(t_1+t_2)}\mathsf{E}[V^2] + \sigma^2 e^{-\lambda(t_1+t_2)} \int_0^{\min(t_1,t_2)} e^{2\lambda\tau}\, d\tau.$$

If $t_1 \leq t_2$, this last integral is equal to

$$\int_0^{t_1} e^{2\lambda\tau}\, d\tau = \frac{1}{2\lambda}(e^{2\lambda t_1} - 1),$$

and so

$$\mathsf{E}[Y_{t_1} Y_{t_2}] = e^{-\lambda(t_1+t_2)}\Bigl(q^2 - \frac{\sigma^2}{2\lambda}\Bigr) + \frac{\sigma^2}{2\lambda} e^{-\lambda(t_2-t_1)}.$$

Similarly, if $t_2 < t_1$,

$$\mathsf{E}[Y_{t_1} Y_{t_2}] = e^{-\lambda(t_1+t_2)}\Bigl(q^2 - \frac{\sigma^2}{2\lambda}\Bigr) + \frac{\sigma^2}{2\lambda} e^{-\lambda(t_1-t_2)}.$$

In either case, we can write

$$\mathsf{E}[Y_{t_1} Y_{t_2}] = e^{-\lambda(t_1+t_2)}\Bigl(q^2 - \frac{\sigma^2}{2\lambda}\Bigr) + \frac{\sigma^2}{2\lambda} e^{-\lambda|t_1-t_2|}.$$

33. Write

$$\mathsf{E}[Y_{t_1} Y_{t_2}] = \frac{e^{-\lambda(t_1+t_2)}}{2\lambda}\mathsf{E}[W_{e^{2\lambda t_1}} W_{e^{2\lambda t_2}}] = \frac{e^{-\lambda(t_1+t_2)}}{2\lambda} \cdot \sigma^2 \min(e^{2\lambda t_1}, e^{2\lambda t_2}).$$

For $t_1 \leq t_2$, this reduces to

$$\frac{e^{-\lambda(t_1+t_2)}}{2\lambda} \cdot \sigma^2 e^{2\lambda t_1} = \frac{\sigma^2}{2\lambda} e^{-\lambda(t_2-t_1)}.$$

If $t_2 < t_1$, we have

$$\frac{e^{-\lambda(t_1+t_2)}}{2\lambda} \cdot \sigma^2 e^{2\lambda t_2} = \frac{\sigma^2}{2\lambda} e^{-\lambda(t_1-t_2)}.$$

We conclude that

$$\mathsf{E}[Y_{t_1} Y_{t_2}] = \frac{\sigma^2}{2\lambda} e^{-\lambda|t_1-t_2|}.$$

34. (a) $P(t) := \mathsf{E}[Y_t^2] = \mathsf{E}\biggl[\biggl(\int_0^t g(\tau)\,dW_\tau\biggr)^2\biggr] = \int_0^t g(\tau)^2\,d\tau.$

(b) If $g(t)$ is never zero, then for $0 \le t_1 < t_2 < \infty$,

$$P(t_2) - P(t_1) = \int_{t_1}^{t_2} g(\tau)^2\,d\tau > 0.$$

Thus, $P(t_1) < P(t_2)$.

(c) First,

$$\mathsf{E}[X_t] = \mathsf{E}[Y_{P^{-1}(t)}] = \mathsf{E}\biggl[\int_0^{P^{-1}(t)} g(\tau)\,dW_\tau\biggr] = 0$$

since Wiener integrals have zero mean. Second,

$$\mathsf{E}[X_t^2] = \mathsf{E}\biggl[\biggl(\int_0^{P^{-1}(t)} g(\tau)\,dW_\tau\biggr)^2\biggr] = \int_0^{P^{-1}(t)} g(\tau)^2\,d\tau = P(P^{-1}(t)) = t.$$

35. (a) For $t > 0$, $\mathsf{E}[W_t^2] = \mathsf{E}[(W_t - W_0)^2] = t$.

(b) For $s < 0$, $\mathsf{E}[W_s^2] = \mathsf{E}[(W_0 - W_s)^2] = -s$.

(c) From parts (a) and (b) we see that no matter what the sign of t, $\mathsf{E}[W_t^2] = |t|$. Whether $t > s$ or $s < t$, we can write

$$|t-s| = \mathsf{E}[(W_t - W_s)^2] = \mathsf{E}[W_t^2] - 2\mathsf{E}[W_t W_s] + \mathsf{E}[W_s^2],$$

and so

$$\mathsf{E}[W_t W_s] = \frac{\mathsf{E}[W_t^2] + \mathsf{E}[W_s^2] - |t-s|}{2} = \frac{|t| + |s| - |t-s|}{2}.$$

36. (a) Write

$$\begin{aligned}\mathsf{P}(X = x_k) &= \mathsf{P}((X,Y) \in \{x_k\} \times \mathbb{R}) = \sum_i \int_{-\infty}^{\infty} I_{\{x_k\}}(x_i)\overbrace{I_{\mathbb{R}}(y)}^{=1} f_{XY}(x_i,y)\,dy \\ &= \int_{-\infty}^{\infty} \biggl[\sum_i I_{\{x_k\}}(x_i) f_{XY}(x_i,y)\biggr] dy = \int_{-\infty}^{\infty} f_{XY}(x_k,y)\,dy.\end{aligned}$$

(b) Write

$$\begin{aligned}\mathsf{P}(Y \in C) &= \mathsf{P}\bigl((X,Y) \in \mathbb{R} \times C\bigr) = \sum_i \int_{-\infty}^{\infty} I_{\mathbb{R}}(x_i) I_C(y) f_{XY}(x_i,y)\,dy \\ &= \int_{-\infty}^{\infty} I_C(y)\biggl[\sum_i f_{XY}(x_i,y)\biggr] dy = \int_C \biggl[\sum_i f_{XY}(x_i,y)\biggr] dy.\end{aligned}$$

(c) First write

$$\mathsf{P}(Y \in C | X = x_k) = \frac{\mathsf{P}(X = x_k, Y \in C)}{\mathsf{P}(X = x_k)} = \frac{\mathsf{P}((X,Y) \in \{x_k\} \times C)}{\mathsf{P}(X = x_k)}.$$

Then since

$$\mathsf{P}((X,Y)\in\{x_k\}\times C) \;=\; \sum_i \int_{-\infty}^{\infty} I_{\{x_k\}}(x_i) I_C(y) f_{XY}(x_i,y)\,dy \;=\; \int_C f_{XY}(x_k,y)\,dy,$$

we have

$$\mathsf{P}(Y\in C|X=x_k) \;=\; \frac{\displaystyle\int_C f_{XY}(x_k,y)\,dy}{p_X(x_k)} \;=\; \int_C \left[\frac{f_{XY}(x_k,y)}{p_X(x_k)}\right] dy.$$

(d) Write

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathsf{P}(X\in B|Y=y) f_Y(y)\,dy \;&=\; \int_{-\infty}^{\infty} \left[\sum_i I_B(x_i) p_{X|Y}(x_i|y)\right] f_Y(y)\,dy \\
&=\; \sum_i I_B(x_i) \int_{-\infty}^{\infty} \frac{f_{XY}(x_i,y)}{f_Y(y)} f_Y(y)\,dy \\
&=\; \sum_i I_B(x_i) \int_{-\infty}^{\infty} f_{XY}(x_i,y)\,dy \\
&=\; \sum_i I_B(x_i) \int_{-\infty}^{\infty} I_{\mathbb{R}}(y) f_{XY}(x_i,y)\,dy \\
&=\; \sum_i \int_{-\infty}^{\infty} I_{B\times\mathbb{R}}(x_i,y) f_{XY}(x_i,y)\,dy \\
&=\; \mathsf{P}((X,Y)\in B\times\mathbb{R}) \\
&=\; \mathsf{P}(X\in B, Y\in\mathbb{R}) \;=\; \mathsf{P}(X\in B).
\end{aligned}$$

37. The key observations are that since F is nondecreasing,

$$F^{-1}(U)\le x \;\Rightarrow\; U\le F(x),$$

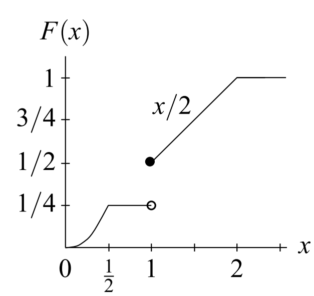
and since F^{-1} is nondecreasing,

$$U\le F(x) \;\Rightarrow\; F^{-1}(U)\le x.$$

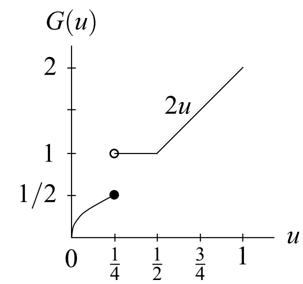
Hence, $\{F^{-1}(U)\le x\}=\{U\le F(x)\}$, and with $X:=F^{-1}(U)$ we can write

$$\mathsf{P}(X\le x) \;=\; \mathsf{P}(F^{-1}(U)\le x) \;=\; \mathsf{P}(U\le F(x)) \;=\; \int_0^{F(x)} 1\,du \;=\; F(x).$$

38. (a) The cdf is



(b) For $1/2 \le u < 1$, $B_u = [2u, \infty)$, and $G(u) = 2u$. For $1/4 < u < 1/2$, $B_u = [1, \infty)$, and $G(u) = 1$. For $u = 1/4$, $B_u = [1/2, \infty)$, and $G(u) = 1/2$. For $0 \le u < 1/4$, $B_u = [\sqrt{u}, \infty)$, and $G(u) = \sqrt{u}$. Hence,



39. Suppose $G(u) \le x$. Since F is nondecreasing, $F(G(u)) \le F(x)$. By the definition of $G(u)$, $F(G(u)) \ge u$. Thus, $F(x) \ge F(G(u)) \ge u$. Now suppose $u \le F(x)$. Then by the definition of $G(u)$, $G(u) \le x$.

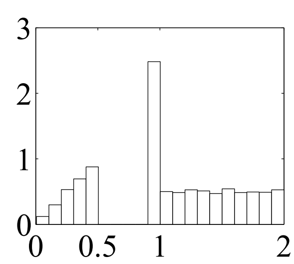
40. With $X := G(U)$, write

$$\mathsf{P}(X \le x) \;=\; \mathsf{P}(G(U) \le x) \;=\; \mathsf{P}(U \le F(x)) \;=\; \int_0^{F(x)} 1\,du \;=\; F(x).$$

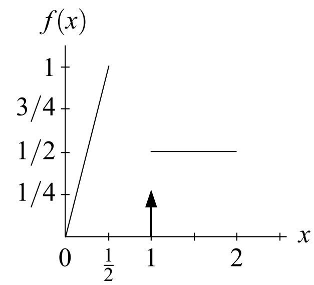
41. To compute $G(u)$, we used the MATLAB function

```
function x = G(u)
i1 = find(u <= .25);
i2 = find(.25 < u & u <= .5);
i3 = find(.5 < u & u <= 1);
x(i1) = sqrt(u(i1));
x(i2) = 1;
x(i3) = 2*u(i3);
```

We obtained the histogram



This is explained by noting that the density corresponding to $F(x)$ is impulsive:



42. For an integer-valued process,

$$\mu_{m,n}(B) = \sum_{j_m,\ldots,j_n} I_B(j_m,\ldots,j_n)p_{m,n}(j_m,\ldots,j_n).$$

If $B = \{i_m\} \times \cdots \times \{i_n\}$, then

$$\begin{aligned}
\mu_{m,n+1}(B \times \mathbb{R}) &= \sum_{j_m,\ldots,j_n,j} I_{B\times\mathbb{R}}(j_m,\ldots,j_n,j)p_{m,n+1}(j_m,\ldots,j_n,j) \\
&= \sum_j p_{m,n+1}(i_m,\ldots,i_n,j), \\
\mu_{m-1,n}(\mathbb{R} \times B) &= \sum_{j,j_m,\ldots,j_n} I_{\mathbb{R}\times B}(j,j_m,\ldots,j_n)p_{m-1,n}(j,j_m,\ldots,j_n) \\
&= \sum_j p_{m-1,n}(j,i_m,\ldots,i_n),
\end{aligned}$$

and $\mu_{m,n}(B) = p_{m,n}(i_m,\ldots,i_n)$. Hence, if the conditions

$$\mu_{m,n+1}(B \times \mathbb{R}) = \mu_{m,n}(B) \quad \text{and} \quad \mu_{m-1,n}(\mathbb{R} \times B) = \mu_{m,n}(B)$$

hold for all B and we take $B = \{i_m\} \times \cdots \times \{i_n\}$, then we obtain

$$\sum_j p_{m,n+1}(i_m,\ldots,i_n,j) = p_{m,n}(i_m,\ldots,i_n)$$

and

$$\sum_j p_{m-1,n}(j,i_m,\ldots,i_n) = p_{m,n}(i_m,\ldots,i_n).$$

Conversely, suppose these two formulas hold. Fix an arbitrary $B \subset \mathbb{R}^{n-m+1}$. If we multiply both formulass by $I_B(i_m,\ldots,i_n)$ and sum over all $i_m,\ldots,i_n$, we obtain the original consistency conditions.

43. For the first condition, write

$$\begin{aligned}
\sum_j p_{m,n+1}(i_m,\ldots,i_n,j) &= \sum_j q(i_m)r(i_{m+1}|i_m)\cdots r(i_n|i_{n-1})r(j|i_n) \\
&= \underbrace{q(i_m)r(i_{m+1}|i_m)\cdots r(i_n|i_{n-1})}_{=\,p_{m,n}(i_m,\ldots,i_n)} \underbrace{\sum_j r(j|i_n)}_{=\,1}.
\end{aligned}$$

For the second condition, write

$$\begin{aligned}\sum_j p_{m-1,n}(j,i_m,\ldots,i_n) &= \sum_j q(j)r(i_m|j)r(i_{m+1}|i_m)\cdots r(i_n|i_{n-1}) \\ &= \underbrace{\underbrace{\Bigg(\sum_j q(j)r(i_m|j)\Bigg)}_{=\,q(i_m)} r(i_{m+1}|i_m)\cdots r(i_n|i_{n-1})}_{=\,p_{m,n}(i_m,\ldots,i_n)}.\end{aligned}$$

44. If

$$\begin{aligned}\mu_n(B_n) &= \mu_{n+1}(B_n\times\mathbb{R}) \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} I_{B_n}(x_1,\ldots,x_n)I_{\mathbb{R}}(y)f_{n+1}(x_1,\ldots,x_n,y)\,dy\,dx_n\cdots dx_1 \\ &= \underset{B_n}{\int\cdots\int}\bigg[\int_{-\infty}^{\infty} f_{n+1}(x_1,\ldots,x_n,y)\,dy\bigg]dx_n\cdots dx_1,\end{aligned}$$

then necessarily the quantity in square brackets is the joint density $f_n(x_1,\ldots,x_n)$. Conversely, if the quantity in square brackets is equal to $f_n(x_1,\ldots,x_n)$, we can write

$$\begin{aligned}\mu_{n+1}(B_n\times\mathbb{R}) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} I_{B_n}(x_1,\ldots,x_n)I_{\mathbb{R}}(y)f_{n+1}(x_1,\ldots,x_n,y)\,dy\,dx_n\cdots dx_1 \\ &= \underset{B_n}{\int\cdots\int}\bigg[\int_{-\infty}^{\infty} f_{n+1}(x_1,\ldots,x_n,y)\,dy\bigg]dx_n\cdots dx_1 \\ &= \underset{B_n}{\int\cdots\int} f_n(x_1,\ldots,x_n)dx_n\cdots dx_1 \;=\; \mu_n(B_n).\end{aligned}$$

45. The key is to write

$$\begin{aligned}&\mu_{t_1,\ldots,t_{n+1}}(B_{n,k}) \\ &\quad= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} I_{B_n}(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)I_{\mathbb{R}}(x_k)f_{t_1,\ldots,t_{n+1}}(x_1,\ldots,x_{n+1})\,dx_{n+1}\cdots dx_1 \\ &\quad= \underset{B_n}{\int\cdots\int}\bigg[\int_{-\infty}^{\infty} f_{t_1,\ldots,t_{n+1}}(x_1,\ldots,x_{n+1})\,dx_k\bigg]dx_{n+1}\cdots dx_{k+1}dx_{k-1}\cdots dx_1.\end{aligned}$$

Then if $\mu_{t_1,\ldots,t_{n+1}}(B_{n,k}) = \mu_n(B_n)$, we must have

$$f_{t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n}(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n) \;=\; \int_{-\infty}^{\infty} f_{t_1,\ldots,t_{n+1}}(x_1,\ldots,x_{n+1})\,dx_k$$

and conversely.

46. By the hint, $[W_{t_1},\ldots,W_{t_n}]'$ is a linear transformation of a vector of independent Gaussian increments, which is a Gaussian vector. Hence, $[W_{t_1},\ldots,W_{t_n}]'$ is a Gaussian vector. Since n and the times $t_1<\cdots<t_n$ are arbitrary, W_t is a Gaussian process.

47. Let $X = [W_{t_1} - W_0, \ldots, W_{t_n} - W_{t_{n-1}}]'$, and let $Y = [W_{t_1}, \ldots, W_{t_n}]'$. Then $Y = AX$, where A denotes the matrix given in the statement of Problem 46. Since the components of X are independent,

$$f_X(x) \;=\; \prod_{i=1}^n \frac{e^{-x_i^2/[2(t_i-t_{i-1})]}}{\sqrt{2\pi(t_i-t_{i-1})}},$$

where it is understood that $t_0 := 0$. Using the example suggested in the hint, we have

$$f_Y(y) \;=\; \frac{f_X(x)}{|\det A|}\bigg|_{x=A^{-1}y}.$$

Fortunately, $\det A = 1$, and by definition, $X_i = W_{t_i} - W_{t_{i-1}}$. Hence,

$$f_{t_1,\ldots,t_n}(w_1,\ldots,w_n) \;=\; \prod_{i=1}^n \frac{e^{-(w_i-w_{i-1})^2/[2(t_i-t_{i-1})]}}{\sqrt{2\pi(t_i-t_{i-1})}},$$

where it is understood that $w_0 := 0$.

48. By the hint, it suffices to show that C is positive semidefinite. Write

$$\begin{aligned}
a'Ca \;=\; \sum_{i=1}^n\sum_{k=1}^n a_i a_k C_{ik} \;&=\; \sum_{i=1}^n\sum_{k=1}^n a_i a_k R(t_i-t_k) \\
&=\; \sum_{i=1}^n\sum_{k=1}^n a_i a_k \int_{-\infty}^{\infty} S(f)e^{j2\pi f(t_i-t_k)}\,df \\
&=\; \int_{-\infty}^{\infty} S(f)\bigg|\sum_{i=1}^n a_i e^{j2\pi f t_i}\bigg|^2 df \;\ge\; 0.
\end{aligned}$$

CHAPTER 12
Problem Solutions

1. If $\mathsf{P}(A|X=i,Y=j,Z=k)=h(i)$, then

$$\begin{aligned}\mathsf{P}(A,X=i,Y=j,Z=k) &= \mathsf{P}(A|X=i,Y=j,Z=k)\mathsf{P}(X=i,Y=j,Z=k)\\ &= h(i)\mathsf{P}(X=i,Y=j,Z=k).\end{aligned}$$

Summing over j yields

$$\mathsf{P}(A,X=i,Z=k) = h(i)\mathsf{P}(X=i,Z=k), \qquad (*)$$

and thus $h(i)=\mathsf{P}(A|X=i,Z=k)$. Next, summing $(*)$ over k yields

$$\mathsf{P}(A,X=i) = h(i)\mathsf{P}(X=i),$$

and then $h(i)=\mathsf{P}(A|X=i)$ as well.

2. Write

$$\begin{aligned}X_1 &= g(X_0,Z_1)\\ X_2 &= g(X_1,Z_2) = g(g(X_0,Z_1),Z_2)\\ X_3 &= g(X_2,Z_3) = g(g(g(X_0,Z_1),Z_2),Z_3)\\ &\vdots\end{aligned}$$

In general, X_n is a function of $X_0,Z_1,\ldots,Z_n$, and thus $(X_0,\ldots,X_n)$ is a function of $(X_0,Z_1,\ldots,Z_n)$, which is independent of Z_{n+1}. Now observe that

$$\begin{aligned}\mathsf{P}(X_{n+1}=i_{n+1}|X_n=i_n,\ldots,X_0=i_0) &= \mathsf{P}(g(X_n,Z_{n+1})=i_{n+1}|X_n=i_n,\ldots,X_0=i_0)\\ &= \mathsf{P}(g(i_n,Z_{n+1})=i_{n+1}|X_n=i_n,\ldots,X_0=i_0)\\ &= \mathsf{P}(g(i_n,Z_{n+1})=i_{n+1}),\end{aligned}$$

where we have used substitution and independence. Since

$$\mathsf{P}(X_{n+1}=i_{n+1}|X_n=i_n,\ldots,X_0=i_0)$$

does not depend on $i_{n-1},\ldots,i_0$, X_n is a Markov chain.

3. Write

$$\mathsf{P}(A\cap B|C) = \frac{\mathsf{P}(A\cap B\cap C)}{\mathsf{P}(C)} = \frac{\mathsf{P}(A\cap B\cap C)}{\mathsf{P}(B\cap C)}\cdot\frac{\mathsf{P}(B\cap C)}{\mathsf{P}(C)} = \mathsf{P}(A|B\cap C)\mathsf{P}(B|C).$$

4. Write

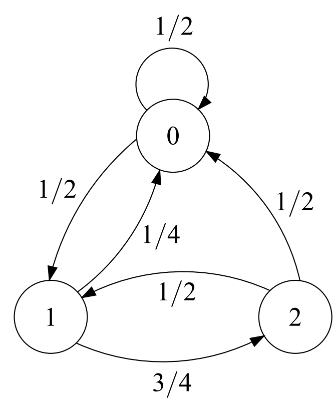
$$\begin{aligned}
&\mathsf{P}(X_0 = i, X_1 = j, X_2 = k, X_3 = l) \\
&\quad = \mathsf{P}(X_3 = l | X_2 = k, X_1 = j, X_0 = i) \\
&\qquad \cdot \mathsf{P}(X_2 = k | X_1 = j, X_0 = i)\mathsf{P}(X_1 = j | X_0 = i)\mathsf{P}(X_0 = i) \\
&\quad = p_{kl}\, p_{jk}\, p_{ij}\, \nu_i.
\end{aligned}$$

5. The first equation we write is

$$\pi_0 = \sum_k \pi_k p_{k0} \quad \text{as} \quad \pi_0 = \pi_0 p_{00} + \pi_1 p_{10} = \pi_0(1-a) + \pi_1 b.$$

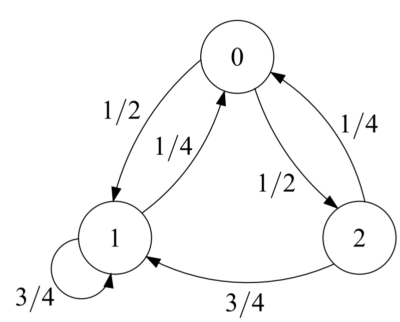
This tells us that $\pi_1 = (a/b)\pi_0$. Then we use the fact that $\pi_0 + \pi_1 = 1$ to write $\pi_0 + (a/b)\pi_0 = 1$, or $\pi_0 = 1/(1+a/b) = b/(a+b)$. We also have $\pi_1 = (a/b)\pi_0 = a/(a+b)$.

6. The state transition diagram is



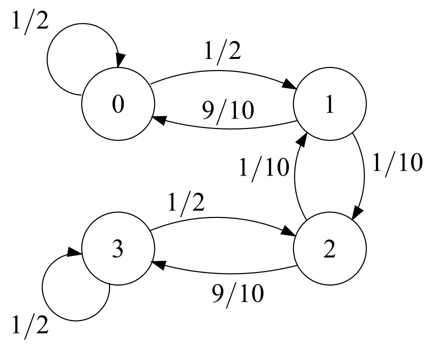
The stationary distribution is $\pi_0 = 5/12$, $\pi_1 = 1/3$, $\pi_2 = 1/4$.

7. The state transition diagram is



The stationary distribution is $\pi_0 = 1/5$, $\pi_1 = 7/10$, $\pi_2 = 1/10$.

8. The state transition diagram is



The stationary distribution is $\pi_0 = 9/28$, $\pi_1 = 5/28$, $\pi_2 = 5/28$, $\pi_3 = 9/28$.

9. The first equation is

$$\pi_0 \;=\; \pi_0 p_{00} + \pi_1 p_{10} \;=\; \pi_0(1-a) + \pi_1 b,$$

which tells us that $\pi_1 = (a/b)\pi_0$. The next equation is

$$\pi_1 \;=\; \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} \;=\; \pi_0 a + \pi_1(1-[a+b]) + \pi_2 b,$$

which tells us that

$$(a+b)\pi_1 \;=\; \pi_0 a + \pi_2 b \quad \text{or} \quad (a+b)(a/b)\pi_0 \;=\; \pi_0 a + \pi_2 b,$$

from which it follows that $\pi_2 = (a/b)^2\pi_0$. Now suppose that $\pi_i = (a/b)^i\pi_0$ holds for $i = 0,\ldots,j < N$. Then from

$$\pi_j \;=\; \pi_{j-1}p_{j-1,j} + \pi_j p_{jj} + \pi_{j+1}p_{j+1,j} \;=\; \pi_{j-1}a + \pi_j(1-[a+b]) + \pi_{j+1}b,$$

we obtain

$$(a+b)\pi_j \;=\; \pi_{j-1}a + \pi_{j+1}b \quad \text{or} \quad (a+b)(a/b)^j\pi_0 \;=\; (a/b)^{j-1}a + \pi_{j+1}b,$$

from which it follows that $\pi_{j+1} = (a/b)^{j+1}\pi_0$. To find π_0, we use the fact that $\pi_0 + \cdots + \pi_N = 1$. Using the finite geometric series formula, we have

$$1 \;=\; \sum_{j=0}^{N}(a/b)^j\pi_0 \;=\; \pi_0\frac{1-(a/b)^{N+1}}{1-a/b}.$$

Hence,

$$\pi_j \;=\; \frac{1-a/b}{1-(a/b)^{N+1}}\left(\frac{a}{b}\right)^j, \quad j = 0,\ldots,N, \quad a \neq b.$$

If $a = b$, then $\pi_j = \pi_0$ for $j = 0,\ldots,N$ implies $\pi_j = 1/(N+1)$.

10. Let π and $\widehat{\pi}$ denote the two solutions in the example. For $0 \le \lambda \le 1$, put

$$\widetilde{\pi}_j \;:=\; \lambda\pi_j + (1-\lambda)\widehat{\pi}_j \;\ge\; 0.$$

Then

$$\sum_j \widetilde{\pi}_j = \sum_j \bigl[\lambda\pi_j + (1-\lambda)\widehat{\pi}_j\bigr] = \lambda\sum_j \pi_j + (1-\lambda)\sum_j \widehat{\pi}_j = \lambda\cdot 1 + (1-\lambda)\cdot 1 = 1.$$

Hence, $\widetilde{\pi}$ is a valid pmf. Finally, note that

$$\begin{aligned}\sum_i \widetilde{\pi}_i p_{ij} \;&=\; \sum_i [\lambda\pi_i + (1-\lambda)\widehat{\pi}_i]p_{ij} \;=\; \lambda\sum_i \pi_i p_{ij} + (1-\lambda)\sum_i \widehat{\pi}_i p_{ij} \\ &=\; \lambda\pi_j + (1-\lambda)\widehat{\pi}_j \;=\; \widetilde{\pi}_j.\end{aligned}$$

11. **MATLAB.** OMITTED.

12. Write

$$\begin{aligned}\mathsf{E}[T_1(j)|X_0 = i] \;&=\; \sum_{k=1}^{\infty} k\mathsf{P}(T_1(j) = k|X_0 = i) + \infty\cdot\mathsf{P}(T_1(j) = \infty|X_0 = i) \\ &=\; \sum_{k=1}^{\infty} k f_{ij}^{(k)} + \infty(1 - f_{ij}).\end{aligned}$$

13. (a) First note that $\{T_1 = 2\} = \{X_2 = j, X_1 \ne j\}$ and that

$$\begin{aligned}\{T_2 = 5\} \;=\; \{X_5 = j\} \cap \Bigl[\; &\{X_4 = j, X_3 \ne j, X_2 \ne j, X_1 \ne j\} \\ \cup &\{X_4 \ne j, X_3 = j, X_2 \ne j, X_1 \ne j\} \\ \cup &\{X_4 \ne j, X_3 \ne j, X_2 = j, X_1 \ne j\} \\ \cup &\{X_4 \ne j, X_3 \ne j, X_2 \ne j, X_1 = j\} \;\Bigr].\end{aligned}$$

Then

$$\{T_2 = 5\} \cap \{T_1 = 2\} \;=\; \{X_5 = j, X_4 \ne j, X_3 \ne j, X_2 = j, X_1 \ne j\}.$$

It now follows that

$$\mathsf{P}(T_2 = 5|T_1 = 2, X_0 = i) \;=\; \mathsf{P}(X_5 = j, X_4 \ne j, X_3 \ne j | X_2 = j, X_1 \ne j, X_0 = i).$$

(b) Write

$$\mathsf{P}(X_5 = j, X_4 \ne j, X_3 \ne j, X_2 = j, X_1 \ne j, X_0 = i)$$

as

$$\mathsf{P}\biggl(\bigcup_{l \ne j}\{X_5 = j, X_4 \ne j, X_3 \ne j, X_2 = j, X_1 = l, X_0 = i\}\biggr),$$

which is equal to

$$\sum_{l \ne j} \mathsf{P}(X_5 = j, X_4 \ne j, X_3 \ne j | X_2 = j, X_1 = l, X_0 = i)\mathsf{P}(X_2 = j, X_1 = l, X_0 = i).$$

By the Markov property, this simplifies to

$$\sum_{l\neq j} \mathsf{P}(X_5 = j, X_4 \neq j, X_3 \neq j | X_2 = j)\mathsf{P}(X_2 = j, X_1 = l, X_0 = i),$$

which is just

$$\mathsf{P}(X_5 = j, X_4 \neq j, X_3 \neq j | X_2 = j)\mathsf{P}(X_2 = j, X_1 \neq j, X_0 = i).$$

It now follows that $\mathsf{P}(X_5 = j, X_4 \neq j, X_3 \neq j | X_2 = j, X_1 \neq j, X_0 = i)$ is equal to

$$\mathsf{P}(X_5 = j, X_4 \neq j, X_3 \neq j | X_2 = j).$$

(c) Write

$$\begin{aligned} \mathsf{P}(X_5 = j, X_4 \neq j, X_3 \neq j | X_2 = j) &= \sum_{k\neq j, l\neq j} \mathsf{P}(X_5 = j, X_4 = k, X_3 = l | X_2 = j) \\ &= \sum_{k\neq j, l\neq j} \mathsf{P}(X_3 = j, X_2 = k, X_1 = l | X_0 = j) \\ &= \mathsf{P}(X_3 = j, X_2 \neq j, X_1 \neq j | X_0 = j). \end{aligned}$$

14. Write

$$\begin{aligned} \mathsf{P}(V(j) = \infty | X_0 = i) &= \mathsf{P}\bigg(\bigcap_{L=1}^{\infty} \{V(j) \geq L\} \bigg| X_0 = i\bigg) \\ &= \lim_{M\to\infty} \mathsf{P}\bigg(\bigcap_{L=1}^{M} \{V(j) \geq L\} \bigg| X_0 = i\bigg), \quad \text{limit property of } \mathsf{P}, \\ &= \lim_{M\to\infty} \mathsf{P}(V(j) \geq M | X_0 = i), \quad \text{decreasing events.} \end{aligned}$$

15. First write

$$\begin{aligned} \mathsf{E}[V(j)|X_0 = i] &= \mathsf{E}\bigg[\sum_{n=1}^{\infty} I_{\{j\}}(X_n) \bigg| X_0 = i\bigg] = \sum_{n=1}^{\infty} \mathsf{E}[I_{\{j\}}(X_n)|X_0 = i] \\ &= \sum_{n=1}^{\infty} \mathsf{P}(X_n = j | X_0 = i) = \sum_{n=1}^{\infty} p_{ij}^{(n)}. \qquad (*) \end{aligned}$$

Next,

$$\begin{aligned} \mathsf{P}(V(j) = \infty | X_0 = i) &= \lim_{L\to\infty} \mathsf{P}(V(j) \geq L | X_0 = i), \quad \text{from preceding problem,} \\ &= \lim_{L\to\infty} f_{ij}(f_{jj})^{L-1}, \quad \text{from the text.} \end{aligned}$$

Now, if j is recurrent ($f_{jj} = 1$), the above limit is f_{ij}. If $f_{ij} > 0$, then $V(j) = \infty$ with positive probability, and so the expectation on the left in $(*)$ must be infinite, and hence, so is the sum on the far right in $(*)$. If $f_{ij} = 0$, then for any $n = 1, 2, \ldots$, we can write

$$\begin{aligned} 0 = f_{ij} := \mathsf{P}(T_1(j) < \infty | X_i = 0) &\geq \mathsf{P}\bigg(\bigcup_{m=1}^{\infty} \{X_m = j\} \bigg| X_i = 0\bigg) \\ &\geq \mathsf{P}(X_n = j | X_0 = i) = p_{ij}^{(n)}, \end{aligned}$$

and it follows that $\sum_{n=1}^{\infty} p_{ij}^{(n)} = 0$.

In the transient case ($f_{jj} < 1$), we have from the text that

$$\mathsf{P}(V(j) \geq L | X_0 = i) \;=\; f_{ij}(f_{jj})^{L-1}.$$

Using the preceding problem along with $f_{jj} < 1$, we have

$$\mathsf{P}(V(j) = \infty | X_0 = i) \;=\; \lim_{L\to\infty} f_{ij}(f_{jj})^{L-1} \;=\; 0.$$

Now, we also have from the text that

$$\mathsf{P}(V(j) = L | X_0 = i) \;=\; \begin{cases} f_{ij}(f_{jj})^{L-1}(1-f_{jj}), & L \geq 1, \\ 1 - f_{ij}, & L = 0, \end{cases}$$

where we use the fact that the number of visits to state j is zero if and only if we never visit state j, and this happens if and only if $T_1(j) = \infty$. We can therefore write

$$\mathsf{E}[V(j)|X_0 = i] \;=\; \sum_{L=0}^{\infty} L \cdot \mathsf{P}(V(j) = L | X_0 = i) \;=\; \frac{f_{ij}}{1 - f_{jj}} \;<\; \infty,$$

and it follows that the last sum in $(*)$ is finite too.

16. From $m\lambda - 1 \leq \lfloor m\lambda \rfloor \leq m\lambda$,

$$\frac{1}{m\lambda} \;\leq\; \frac{1}{\lfloor m\lambda \rfloor} \;\leq\; \frac{1}{m\lambda - 1}.$$

Then

$$\frac{1}{\lambda} \;\leq\; \frac{m}{\lfloor m\lambda \rfloor} \;\leq\; \frac{1}{\lambda - 1/m} \;\to\; \frac{1}{\lambda}.$$

17. (a) Write $h(j) = I_S(j) = \sum_{l=1}^{n} I_{\{s_l\}}(j)$. Then

$$\begin{aligned} \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} h(X_k) \;&=\; \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m}\left[\sum_{l=1}^{n} I_{\{s_l\}}(X_k)\right] \;=\; \sum_{l=1}^{n} \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} I_{\{s_l\}}(X_k) \\ &=\; \sum_{l=1}^{n} \lim_{m\to\infty} V_m(s_l)/m \;=\; \sum_{l=1}^{n} \pi_{s_l}, \quad \text{by Theorems 3 and 4,} \\ &=\; \sum_j I_S(j)\pi_j \;=\; \sum_j h(j)\pi_j. \end{aligned}$$

(b) With $h(j) = \sum_{l=1}^{n} c_l I_{S_l}(j)$, where each S_l is a finite subset of states, we can write

$$\begin{aligned} \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} h(X_k) \;&=\; \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m}\left[\sum_{l=1}^{n} c_l I_{S_l}(X_k)\right] \;=\; \sum_{l=1}^{n} c_l \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} I_{S_l}(X_k) \\ &=\; \sum_{l=1}^{n} c_l \left[\sum_j I_{S_l}(j)\pi_j\right], \quad \text{by part (a)} \\ &=\; \sum_j \left[\sum_{l=1}^{n} c_l I_{S_l}(j)\right]\pi_j \;=\; \sum_j h(j)\pi_j. \end{aligned}$$

(c) If $h(j) = 0$ for all but finitely many states, then there are at most finitely many distinct nonzero values that $h(j)$ can take, say $c_1, \ldots, c_n$. Put $S_l := \{j : h(j) = c_l\}$. By the assumption about h, each S_l is a finite set. Furthermore, we can write

$$h(j) \;=\; \sum_{l=1}^{n} c_l I_{S_l}(j).$$

Hence, the desired result is immediate by part (b).

18. **Matlab.** OMITTED.

19. **Matlab.** OMITTED.

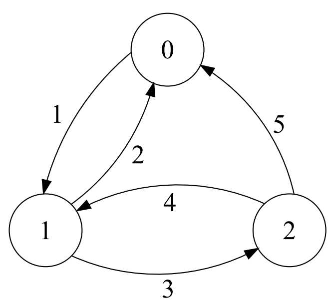
20. Suppose $k \in A_i \cap A_j$. Since $k \in A_i$, $i \leftrightarrow k$. Since $k \in A_j$, $j \leftrightarrow k$. Hence, $i \leftrightarrow j$. Now, if $l \in A_i$, $l \leftrightarrow i \leftrightarrow j$ and we see that $l \in A_j$. Similarly, if $l \in A_j$, $l \leftrightarrow j \leftrightarrow i$ and we see that $l \in A_i$. Thus, $A_i = A_j$.

21. Yes. As pointed out in the discussion at the end of the section, by combining Theorem 8 and Theorem 6, the chain has a unique stationary distribution. Now, by Theorem 4, all states are positive recurrent, which by definition means that the expected time to return to a state is finite.

22. Write

$$g_{i,i+n} \;=\; \lim_{\Delta t \downarrow 0} \frac{(\lambda \Delta t)^n e^{-\lambda \Delta t}}{\Delta t \cdot n!} \;=\; \frac{\lambda}{n!} \lim_{\Delta t \downarrow 0} (\lambda \Delta t)^{n-1} e^{-\lambda \Delta t} \;=\; 0, \quad \text{for } n \geq 2.$$

23. The state transition diagram is



The stationary distribution is $\pi_0 = 11/15$, $\pi_1 = 1/5$, $\pi_2 = 1/15$.

24. **Matlab.** Change the line `A=P-In;` to `A=P;`.

25. The forward equation is

$$\begin{aligned} p'_{ij}(t) &= \sum_k p_{ik}(t) g_{kj} \;=\; \sum_{k=j-1}^{j+1} p_{ik} g_{kj} \\ &= p_{i,j-1}(t)\lambda_{j-1} - p_{ij}(t)[\lambda_j + \mu_j] + p_{i,j+1}(t)\mu_{j+1}. \end{aligned}$$

The backward equation is

$$p'_{ij}(t) \;=\; \sum_k g_{ik}p_{kj}(t) \;=\; \sum_{k=i-1}^{i+1} g_{ik}p_{ij}(t)$$
$$= \; \mu_i p_{i-1,j}(t) - (\lambda_i+\mu_i)p_{ii}(t) + \lambda_i p_{i+1,j}(t).$$

The chain is conservative because $g_{i,i-1}+g_{i,i+1} = \mu_i+\lambda_i = -[-(\lambda_i+\mu_i)] = -g_{ii} < \infty$.

26. We use the formula $0 = \sum_k \pi_k g_{kj}$. For $j=0$, we have

$$0 \;=\; \sum_k \pi_k g_{k0} \;=\; \pi_0(-\lambda_0)+\pi_1\mu_1,$$

which implies $\pi_1 = (\lambda_0/\mu_1)\pi_0$. For $j=1$, we have

$$0 \;=\; \sum_k \pi_k g_{k1} \;=\; \pi_0\lambda_0 + \pi_1[-(\lambda_1+\mu_1)] + \pi_2\mu_2$$
$$= \; \pi_0\lambda_0 + (\lambda_0/\mu_1)\pi_0[-(\lambda_1+\mu_1)] + \pi_2\mu_2 \;=\; \pi_0(\lambda_0\lambda_1/\mu_1)+\pi_2\mu_2,$$

which implies $\pi_2 = \pi_0\lambda_0\lambda_1/(\mu_1\mu_2)$. Now suppose that $\pi_i = \pi_0\lambda_0\cdots\lambda_{i-1}/(\mu_1\cdots\mu_i)$ for $i=1,\ldots,j$. Then from

$$0 \;=\; \sum_k \pi_k g_{kj} \;=\; \pi_{j-1}\lambda_{j-1} - \pi_j(\lambda_j+\mu_j) + \pi_{j+1}\mu_{j+1}$$
$$= \; \frac{\lambda_0\cdots\lambda_{j-2}}{\mu_1\cdots\mu_{j-1}}\pi_0\lambda_{j-1} - \frac{\lambda_0\cdots\lambda_{j-1}}{\mu_1\cdots\mu_j}\pi_0(\lambda_j+\mu_j) + \pi_{j+1}\mu_{j+1},$$

and it follows that

$$\pi_{j+1} \;=\; \frac{\lambda_0\cdots\lambda_j}{\mu_1\cdots\mu_{j+1}}\pi_0.$$

Now, let

$$B \;:=\; \sum_{j=1}^{\infty} \frac{\lambda_0\cdots\lambda_{j-1}}{\mu_1\cdots\mu_j} \;<\; \infty.$$

From the condition $\sum_{j=0}^{\infty}\pi_j = 1$, we have

$$1 \;=\; \pi_0+\pi_1+\cdots \;=\; \pi_0 + \pi_0\sum_{j=1}^{\infty}\frac{\lambda_0\cdots\lambda_{j-1}}{\mu_1\cdots\mu_j} \;=\; \pi_0(1+B).$$

It then follows that $\pi_0 = 1/(1+B)$, and

$$\pi_j \;=\; \frac{1}{1+B}\left(\frac{\lambda_0\cdots\lambda_{j-1}}{\mu_1\cdots\mu_j}\right), \quad j\geq 1.$$

If $\lambda_i = \lambda$ and $\mu_i = \mu$, then $B = \sum_{j=1}^{\infty}(\lambda/\mu)^j$, which is finite if and only if $\lambda<\mu$. In in this case,

$$1+B \;=\; \sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^k \;=\; \frac{1}{1+\lambda/\mu},$$

and

$$\pi_j \;=\; (1-\lambda/\mu)(\lambda/\mu)^j \;\sim\; \text{geometric}_0(\lambda/\mu).$$

27. The solution is very similar the that of the preceding problem. In this case, put

$$B_N := \sum_{j=1}^{N} \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} < \infty,$$

and then $\pi_0 = 1/(1+B_N)$, and

$$\pi_j = \frac{1}{1+B_N}\left(\frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j}\right), \quad j = 1,\ldots,N.$$

If $\lambda_i = \lambda$ and $\mu_i = \mu$, then

$$1+B_N = \sum_{k=0}^{N} (\lambda/\mu)^k = \frac{1-(\lambda/\mu)^{N+1}}{1-\lambda/\mu},$$

and

$$\pi_j = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}}(\lambda/\mu)^j, \quad j = 0,\ldots,N.$$

If $\lambda = \mu$, then $\pi_j = 1/(N+1)$.

28. To begin, write

$$m_i(t) := \mathsf{E}[X_t|X_0 = i] = \sum_j j\mathsf{P}(X_t = j|X_0 = i) = \sum_j j p_{ij}(t).$$

Then

$$\begin{aligned}
m_i'(t) &= \sum_j j p_{ij}'(t) = \sum_j j\left[\sum_k p_{ik}(t) g_{kj}\right] = \sum_k p_{ik}(t)\left[\sum_j j g_{kj}\right] \\
&= \sum_k p_{ik}(t)\Big[(k-1)g_{k,k-1} + k g_{kk} + (k+1)g_{k,k+1}\Big] \\
&= \sum_k p_{ik}(t)\Big[(k-1)(k\mu) - k(k\lambda+\alpha+k\mu) + (k+1)(k\lambda+\alpha)\Big] \\
&= \sum_k p_{ik}(t)[-k\mu + k\lambda + \alpha] = -\mu m_i(t) + \lambda m_i(t) + \alpha.
\end{aligned}$$

We must solve

$$m_i'(t) + (\mu-\lambda)m_i(t) = \alpha, \quad m_i(0) = i.$$

If $\mu \neq \lambda$, it is readily verified that

$$m_i(t) = \left(i - \frac{\alpha}{\mu-\lambda}\right)e^{-(\mu-\lambda)t} + \frac{\alpha}{\mu-\lambda}$$

solves the equation. If $\mu = \lambda$, then $m_i(t) = \alpha t + i$ solves $m_i'(t) = \alpha$ with $m_i(0) = i$.

29. To begin, write

$$p_{ij}'(t) = \sum_k p_{ik}(t) g_{kj} = p_{i,j-1}(t) g_{j-1,j} + p_{ij}(t) g_{jj} = \lambda p_{i,j-1}(t) - \lambda p_{ij}(t). \quad (*)$$

Observe that for $j = i$, $p_{i,i-1}(t) = 0$ since the chain can not go from state i to a lower state. Thus,

$$p'_{ii}(t) \;=\; -\lambda p_{ii}(t).$$

Also, $p_{ii}(0) = \mathsf{P}(X_t = i|X_0 = i) = 1$. Thus, $p_{ii}(t) = e^{-\lambda t}$, and it is easily verified that

$$p_{i,i+n}(t) \;=\; \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0,1,2,\ldots,$$

solves $(*)$. Thus, X_t is a Poisson process of rate λ with $X_0 = i$ instead of $X_0 = 0$.

30. (a) The assumption that $\hat{D} > -\infty$ implies that the π_k are not all zero, in which case, the π_k would not sum to one and would not be a pmf. Aside from this, it is clear that $\pi_k \geq 0$ for all k and that $\sum_k \pi_k = \hat{D}/\hat{D} = 1$. Next, since $p_{jj} = 0$, write

$$\begin{aligned}\sum_k \pi_k p_{kj} \;&=\; \sum_{k\neq j}[\hat{\pi}_k g_{kk}/\hat{D}]\frac{g_{kj}}{-g_{kk}} \;=\; \frac{-1}{\hat{D}}\sum_{k\neq j}\hat{\pi}_k g_{kj} \\ &=\; \frac{-1}{\hat{D}}\bigg[\underbrace{\bigg(\sum_k \hat{\pi}_k g_{kj}\bigg)}_{=\,0} - \hat{\pi}_j g_{jj}\bigg] \;=\; \hat{\pi}_j g_{jj}/\hat{D} \;=\; \pi_j.\end{aligned}$$

(b) The assumption that $\check{D} > -\infty$ implies that the π_k are not all zero. Aside from this, it is clear that $\pi_k \geq 0$ for all k and that $\sum_k \pi_k = \check{D}/\check{D} = 1$. Next, write

$$\begin{aligned}\sum_k \pi_k g_{kj} \;&=\; \sum_k [(\check{\pi}_k/g_{kk})/\check{D}]g_{kj} \;=\; \frac{-1}{\check{D}}\sum_k \check{\pi}_k \frac{g_{kj}}{-g_{kk}} \\ &=\; \frac{-1}{\check{D}}\bigg[\bigg(\sum_{k\neq j}\check{\pi}_k\frac{g_{kj}}{-g_{kk}}\bigg) - \check{\pi}_j\bigg] \\ &=\; \frac{-1}{\check{D}}\bigg[\underbrace{\bigg(\sum_k \check{\pi}_k p_{kj}\bigg)}_{=\,\check{\pi}_j} - \check{\pi}_j\bigg], \quad \text{since } p_{jj} = 0, \\ &=\; 0.\end{aligned}$$

(c) Since $\hat{\pi}_k$ and $\check{\pi}_k$ are pmfs, they sum to one. Hence, if $g_{ii} = g$ for all i, $\hat{D} = g$ and $\check{D} = 1/g$. In (a), $\pi_k = \hat{\pi}_k g_{kk}/\hat{D} = \hat{\pi}_k g/g = \hat{\pi}_k$. In (b), $\pi_k = (\check{\pi}_k/g_{kk})/\check{D} = (\check{\pi}_k/g)/(1/g) = \check{\pi}_k$.

31. Following the hints, write

$$\begin{aligned}\mathsf{P}(T > t+\Delta t|T > t, X_0 = i) \;&=\; \mathsf{P}(X_s = i, 0 \leq s \leq t+\Delta t|X_s = i, 0 \leq s \leq t) \\ &=\; \mathsf{P}(X_s = i, t \leq s \leq t+\Delta t|X_s = i, 0 \leq s \leq t) \\ &=\; \mathsf{P}(X_s = i, t \leq s \leq t+\Delta t|X_t = i) \\ &=\; \mathsf{P}(X_s = i, 0 \leq s \leq \Delta t|X_0 = i) \\ &=\; \mathsf{P}(T > \Delta t|X_0 = i).\end{aligned}$$

The exponential parameter is

$$\begin{aligned}\lim_{\Delta t\downarrow 0}\frac{1-\mathsf{P}(T>\Delta t|X_0=i)}{\Delta t} &= \lim_{\Delta t\downarrow 0}\frac{1-\mathsf{P}(X_s=i,0\le s\le \Delta t|X_0=i)}{\Delta t}\\ &= \lim_{\Delta t\downarrow 0}\frac{1-\mathsf{P}(X_{\Delta t}=i|X_0=i)}{\Delta t} \;=:\; -g_{ii}.\end{aligned}$$

32. Using substitution in reverse, write

$$\begin{aligned}&\mathsf{P}(W_t\le y|W_s=x,W_{s_{n-1}}=x_{n-1},\ldots,W_{s_0}=x_0)\\ &\qquad= \mathsf{P}(W_t-x\le y-x|W_s=x,W_{s_{n-1}}=x_{n-1},\ldots,W_{s_0}=x_0)\\ &\qquad= \mathsf{P}(W_t-W_s\le y-x|W_s=x,W_{s_{n-1}}=x_{n-1},\ldots,W_{s_0}=x_0).\end{aligned}$$

Now, using the fact that $W_0\equiv 0$, this last conditional probability is equal to

$$\mathsf{P}(W_t-W_s\le y-x|W_s-W_{s_{n-1}}=x-x_{n-1},\ldots,W_{s_1}-W_{s_0}=x_1-x_0,W_{s_0}-W_0=x_0).$$

Since the Wiener process has independent increments that are Gaussian, this last expression reduces to

$$\mathsf{P}(W_t-W_s\le y-x) \;=\; \int_{-\infty}^{y-x}\frac{\exp[-\{\theta/[\sigma\sqrt{t-s}\,]\}^2/2]}{\sqrt{2\pi\sigma^2(t-s)}}\,d\theta.$$

Since this depends on x but not on $x_{n-1},\ldots,x_0$,

$$\mathsf{P}(W_t\le y|W_s=x,W_{s_{n-1}}=x_{n-1},\ldots,W_{s_0}=x_0) \;=\; \mathsf{P}(W_t\le y|W_s=x).$$

Hence, W_t is a Markov process.

33. Write

$$\begin{aligned}&\sum_y \mathsf{P}(X=x|Y=y,Z=z)\mathsf{P}(Y=y|Z=z)\\ &\qquad= \sum_y \frac{\mathsf{P}(X=x,Y=y,Z=z)}{\mathsf{P}(Y=y,Z=z)}\cdot\frac{\mathsf{P}(Y=y,Z=z)}{\mathsf{P}(Z=z)}\\ &\qquad= \frac{1}{\mathsf{P}(Z=z)}\sum_y \mathsf{P}(X=x,Y=y,Z=z)\\ &\qquad= \frac{\mathsf{P}(X=x,Z=z)}{\mathsf{P}(Z=z)} \;=\; \mathsf{P}(X=x|Z=z).\end{aligned}$$

34. Using the law of total conditional probability, write

$$\begin{aligned}P_{t+s}(B) &= \mathsf{P}(X_{t+s}\in B|X_0=x)\\ &= \int_{-\infty}^{\infty}\mathsf{P}(X_{t+s}\in B|X_s=z,X_0=x)f_{X_s|X_0}(z|x)\,dz\\ &= \int_{-\infty}^{\infty}\mathsf{P}(X_{t+s}\in B|X_s=z)f_s(x,z)\,dz\\ &= \int_{-\infty}^{\infty}\mathsf{P}(X_t\in B|X_0=z)f_s(x,z)\,dz\\ &= \int_{-\infty}^{\infty}P_t(z,B)f_s(x,z)\,dz.\end{aligned}$$

CHAPTER 13
Problem Solutions

1. First observe that $\mathsf{E}[|X_n|^p] = n^{p/2}\mathsf{P}(U \le 1/n) = n^{p/2}\cdot(1/n) = n^{p/2-1}$, which goes to zero if and only if $p < 2$. Thus, $1 \le p < 2$.

2. $\mathsf{E}\bigg[\bigg|\dfrac{N_t}{t} - \lambda\bigg|^2\bigg] \;=\; \dfrac{1}{t^2}\mathsf{E}[|N_t - \lambda t|^2] \;=\; \dfrac{\mathsf{var}(N_t)}{t^2} \;=\; \dfrac{\lambda t}{t^2} \;=\; \dfrac{\lambda}{t} \;\to\; 0.$

3. Given $\varepsilon > 0$, let $n \ge N$ imply $|C(k)| \le \varepsilon/2$. Then for such n,
$$\frac{1}{n}\sum_{k=0}^{n-1} C(k) \;=\; \frac{1}{n}\sum_{k=0}^{N-1} C(k) + \frac{1}{n}\sum_{k=N}^{n} C(k)$$
and
$$\bigg|\frac{1}{n}\sum_{k=0}^{n-1} C(k)\bigg| \;\le\; \frac{1}{n}\sum_{k=0}^{N-1}|C(k)| + \frac{1}{n}\sum_{k=N}^{n}\varepsilon/2 \;\le\; \frac{1}{n}\sum_{k=0}^{N-1}|C(k)| + \varepsilon/2.$$
For large enough n, the right-hand side will be less that ε.

4. Applying the hint followed by the Cauchy–Schwarz inequality shows that
$$\bigg|\frac{1}{n}\sum_{k=0}^{n-1} C(k)\bigg| \;\le\; \sqrt{\mathsf{E}[(X_1-m)^2]\mathsf{E}[(M_n-m)^2]}.$$
Hence, if M_n converges in mean square to m, the left-hand side goes to zero as $n \to \infty$.

5. Starting with the hint, we write
$$\begin{aligned}
\mathsf{E}[ZI_A] &= \mathsf{E}[(Z-Z_n)I_A] + \mathsf{E}[Z_nI_A] \\
&\le \mathsf{E}[Z-Z_n] + n\mathsf{E}[I_A], \quad \text{since } Z_n \le Z \text{ and } Z_n \le n, \\
&= \mathsf{E}[|Z-Z_n|] + n\mathsf{P}(A).
\end{aligned}$$
Since Z_n converges in mean to Z, given $\varepsilon > 0$, we can let $n \ge N$ imply $\mathsf{E}[|Z-Z_n|] \le \varepsilon/2$. Then in particular we have
$$\mathsf{E}[ZI_A] \;<\; \varepsilon/2 + N\mathsf{P}(A).$$
Hence, if $0 < \delta < \varepsilon/(2N)$,
$$\mathsf{P}(A) < \delta \quad \text{implies} \quad \mathsf{E}[ZI_A] < \varepsilon/2 + N\varepsilon/(2N) = \varepsilon.$$

6. Following the hint, we find that given $\varepsilon > 0$, there exists a $\delta > 0$ such that
$$\mathsf{P}(U \le \Delta x) = \Delta x < \delta \quad \text{implies} \quad \mathsf{E}[f(x+U)I_{\{U \le \Delta x\}}] = \int_0^{\Delta x} f(x+t)\,dt < \varepsilon.$$

Now make the change of variable $\theta = x+t$ to get

$$\int_0^{\Delta x} f(x+t)\,dt \;=\; \int_x^{x+\Delta x} f(\theta)\,d\theta \;=\; F(x+\Delta x) - F(x).$$

For $0 < -\Delta x < \delta$, take $A = \{U > 1+\Delta x\}$ and $Z = f(x-1+U)$. Then $\mathsf{P}(U > 1+\Delta x) = -\Delta x$ implies

$$\mathsf{E}[f(x+1+U)I_{\{U>1+Dx\}}] = \int_{1+\Delta x}^{1} f(x-1+t)\,dt < \varepsilon.$$

In this last integral make the change of variable $\theta = x-1+t$ to get

$$\int_{x+\Delta x}^{x} f(\theta)\,d\theta \;=\; F(x) - F(x+\Delta x).$$

Thus, $F(x+\Delta x) - F(x) > -\varepsilon$. We can now write that given $\varepsilon > 0$, there exists a δ such that $|\Delta x| < \delta$ implies $|F(x+\Delta x) - F(x)| < \varepsilon$.

7. Following the hint, we can write

$$\exp\Big[\frac{1}{p}\ln(|X|/\alpha)^p + \frac{1}{q}\ln(|Y|/\beta)^q\Big] \;\le\; \frac{1}{p}e^{\ln(|X|/\alpha)^p} + \frac{1}{q}e^{\ln(|Y|/\beta)^q}$$

or

$$\exp\Big[\ln(|X|/\alpha) + \ln(|Y|/\beta)\Big] \;\le\; \frac{1}{p}\left(\frac{|X|}{\alpha}\right)^p + \frac{1}{q}\left(\frac{|Y|}{\beta}\right)^q$$

or

$$\frac{|XY|}{\alpha\beta} \;\le\; \frac{1}{p}\frac{|X|^p}{\alpha^p} + \frac{1}{q}\frac{|Y|^q}{\beta^q}.$$

Hence,

$$\frac{\mathsf{E}[|XY|]}{\alpha\beta} \;\le\; \frac{1}{p}\frac{\mathsf{E}[|X|^p]}{\alpha^p} + \frac{1}{q}\frac{\mathsf{E}[|Y|^q]}{\beta^q} \;=\; \frac{1}{p}\frac{\alpha^p}{\alpha^p} + \frac{1}{q}\frac{\beta^q}{\beta^q} \;=\; 1.$$

The hint assumes neither α nor β is zero or infinity. However, if either α or β is zero, then both sides of the inequality are zero. If neither is zero and one of them is infinity, then the right-hand side is infinity and the inequality is trivial.

8. Following the hint, with $X = |Z|^\alpha$, $Y = 1$, and $p = \beta/\alpha$, we have

$$\mathsf{E}[|Z|^\alpha] \;=\; \mathsf{E}[|XY|] \;\le\; \mathsf{E}[|X|^p]^{1/p}\mathsf{E}[|Y|^q]^{1/q} \;=\; \mathsf{E}[(|Z|^\alpha)^{\beta/\alpha}]^{\alpha/\beta}\cdot 1 \;=\; \mathsf{E}[|Z|^\beta]^{\alpha/\beta}.$$

Raising both sides to the $1/\alpha$ yields $\mathsf{E}[|Z|^\alpha]^{1/\alpha} \le \mathsf{E}[|Z|^\beta]^{1/\beta}$.

9. By Lyapunov's inequality, $\mathsf{E}[|X_n - X|^\alpha]^{1/\alpha} \le \mathsf{E}[|X_n - X|^\beta]^{1/\beta}$. Raising both sides to the α power yields $\mathsf{E}[|X_n - X|^\alpha] \le \mathsf{E}[|X_n - X|^\beta]^{\alpha/\beta}$. Hence, if $\mathsf{E}[|X_n - X|^\beta] \to 0$, then $\mathsf{E}[|X_n - X|^\alpha] \to 0$ too.

10. Following the hint, write

$$\begin{aligned}
\mathsf{E}[|X+Y|^p] &= \mathsf{E}[|X+Y|\,|X+Y|^{p-1}] \\
&\le \mathsf{E}[|X|\,|X+Y|^{p-1}] + \mathsf{E}[|Y|\,|X+Y|^{p-1}] \\
&\le \mathsf{E}[|X|^p]^{1/p}\mathsf{E}[(|X+Y|^{p-1})^q]^{1/q} + \mathsf{E}[|Y|^p]^{1/p}\mathsf{E}[(|X+Y|^{p-1})^q]^{1/q},
\end{aligned}$$

where $1/q := 1 - 1/p$. Hence, $1/q = (p-1)/p$ and $q = p/(p-1)$. Now divide the above inequality by $\mathsf{E}[|X+Y|^p]^{(p-1)/p}$ to get

$$\mathsf{E}[|X+Y|^p]^{1/p} \;\leq\; \mathsf{E}[|X|^p]^{1/p} + \mathsf{E}[|Y|^p]^{1/p}.$$

11. For a Wiener process, $W_t - W_{t_0} \sim N(0, \sigma^2|t-t_0|)$. To simplify the notation, put $\tilde{\sigma}^2 := \sigma^2|t-t_0|$. Then

$$\begin{aligned}
\mathsf{E}[|W_t - W_{t_0}|] &= \int_{-\infty}^{\infty} |x| \frac{e^{-(x/\tilde{\sigma})^2/2}}{\sqrt{2\pi}\,\tilde{\sigma}}\,dx \;=\; 2\int_0^{\infty} x \frac{e^{-(x/\tilde{\sigma})^2/2}}{\sqrt{2\pi}\,\tilde{\sigma}}\,dx \\
&= \frac{2\tilde{\sigma}}{\sqrt{2\pi}} \int_0^{\infty} te^{-t^2/2}\,dt \;=\; \frac{2\tilde{\sigma}}{\sqrt{2\pi}}\Big(-e^{-t^2/2}\Big)\Big|_0^{\infty} \;=\; 2\sigma\sqrt{\frac{|t-t_0|}{2\pi}},
\end{aligned}$$

which goes to zero as $|t-t_0| \to 0$.

12. Let $t_0 > 0$ be arbitrary. Since $\mathsf{E}[|N_t - N_{t_0}|^2] = \lambda|t-t_0| + \lambda^2|t-t_0|^2$, it is clear that as $t \to t_0$, $\mathsf{E}[|N_t - N_{t_0}|^2] \to 0$. Hence, N_t is continuous in mean square.

13. First write

$$\begin{aligned}
R(t,s) - R(\tau,\theta) &= R(t,s) - R(t,\theta) + R(t,\theta) - R(\tau,\theta) \\
&= \mathsf{E}[X_t(X_s - X_\theta)] + \mathsf{E}[(X_t - X_\tau)X_\theta].
\end{aligned}$$

Then by the Cauchy–Schwarz inequality,

$$|R(t,s) - R(\tau,\theta)| \;\leq\; \sqrt{\mathsf{E}[X_t^2]\mathsf{E}[(X_s - X_\theta)^2]} + \sqrt{\mathsf{E}[(X_t - X_\tau)^2]\mathsf{E}[X_\theta^2]},$$

which goes to zero as $(t,s) \to (\tau,\theta)$. Note that we need the boundedness of $\mathsf{E}[X_t^2]$ for t near τ.

14. (a) $\mathsf{E}[|X_{t+T} - X_t|^2] = R(0) - 2R(T) + R(0) = 0$.

    (b) Write

$$|R(t+T) - R(t)| \;=\; \big|\mathsf{E}[(X_{t+T} - X_t)X_0]\big| \;\leq\; \sqrt{\mathsf{E}[|X_{t+T} - X_t|^2]\mathsf{E}[X_0^2]},$$

which is zero by part (a). Hence, $R(t+T) = R(t)$ for all t.

15. Write

$$\|(X_n + Y_n) - (X+Y)\|_p = \|(X_n - X) + (Y_n - Y)\|_p \leq \|X_n - X\|_p + \|Y_n - Y\|_p \to 0.$$

16. Let t_0 be arbitrary Since

$$\begin{aligned}
\|Z_t - Z_{t_0}\|_p &= \|(X_t + Y_t) - (X_{t_0} + Y_{t_0})\|_p \\
&= \|(X_t - X_{t_0}) + (Y_t - Y_{t_0})\|_p \\
&\leq \|X_t - X_{t_0}\|_p + \|Y_t - Y_{t_0}\|_p,
\end{aligned}$$

it is clear that if X_t and Y_t are continuous in mean of order p, then so is $Z_t := X_t + Y_t$.

17. Let $\varepsilon > 0$ be given. Since $\|X_n - X\|_p \to 0$, there is an N such that for $n \geq N$, $\|X_n - X\|_p < \varepsilon/2$. Thus, for $n, m \geq N$,

$$\|X_n - X_m\|_p = \|(X_n - X) + (X - X_m)\|_p \leq \|X_n - X\|_p + \|X - X_m\|_p < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

18. Suppose X_n is Cauchy in L^p. With $\varepsilon = 1$, there is an N such that for all $n, m \geq N$, $\|X_n - X_m\|_p < 1$. In particular, with $m = N$ we have from

$$\big|\|X_n\|_p - \|X_N\|_p\big| \;\leq\; \|X_n - X_N\|_p$$

that

$$\|X_n\|_p \;\leq \|X_n - X_N\|_p + \|X_N\|_p \;<\; 1 + \|X_N\|_p, \quad \text{for } n \geq N.$$

To get a bound that also holds for $n = 1, \ldots, N-1$, write

$$\|X_n\|_p \;\leq\; \max(\|X_1\|_p, \ldots, \|X_{N-1}\|_p, 1 + \|X_N\|_p).$$

19. Since X_n converges, it is Cauchy and therefore bounded by the preceding two problems. Hence, we can write $\|X_n\|_p \leq B < \infty$ for some constant B. Given $\varepsilon > 0$, let $n \geq N$ imply $\|X_n - X\|_p < \varepsilon/(2\|Y\|_q)$ *and* $\|Y_n - Y\|_q < \varepsilon/(2B)$. Then

$$\begin{aligned}
\|X_nY_n - XY\|_1 &= \mathsf{E}[|X_nY_n - X_nY + X_nY - XY|] \\
&\leq \mathsf{E}[|X_n(Y_n - Y)|] + \mathsf{E}[|(X_n - X)Y|] \\
&\leq \|X_n\|_p\|Y_n - Y\|_q + \|X_n - X\|_p\|Y\|_q, \quad \text{by Hölder's inequality,} \\
&< B \cdot \frac{\varepsilon}{2B} + \frac{\varepsilon}{2\|Y\|_q}\|Y\|_q \;=\; \varepsilon.
\end{aligned}$$

20. If X_n converges in mean of order p to both X and Y, write

$$\|X - Y\|_p \;=\; \|(X - X_n) + (X_n - Y)\|_p \;\leq\; \|X - X_n\|_p + \|X_n - Y\|_p \;\to\; 0.$$

Since $\|X - Y\|_p = 0$, $\mathsf{E}[|X - Y|^p] = 0$.

21. For the rightmost inequality, observe that

$$\|X - Y\|_p \;=\; \|X + (-Y)\|_p \;\leq\; \|X\|_p + \|-Y\|_p \;=\; \|X\|_p + \|Y\|_p.$$

For the remaining inequality, first write

$$\|X\|_p \;=\; \|(X - Y) + Y\|_p \;\leq\; \|X - Y\|_p + \|Y\|_p,$$

from which it follows that

$$\|X\|_p - \|Y\|_p \;\leq\; \|X - Y\|_p. \tag{$*$}$$

Similarly, from

$$\|Y\|_p \;=\; \|(Y - X) + X\|_p \;\leq\; \|Y - X\|_p + \|X\|_p,$$

it follows that

$$\|Y\|_p - \|X\|_p \;\leq\; \|Y - X\|_p \;=\; \|X - Y\|_p. \tag{$**$}$$

From $(*)$ and $(**)$ it follows that

$$\big|\|X\|_p - \|Y\|_p\big| \;\leq\; \|X - Y\|_p.$$

22. By taking pth roots, we see that

$$\lim_{n\to\infty} \mathsf{E}[|X_n|^p] \;=\; \mathsf{E}[|X|^p] \qquad (\#)$$

is equivalent to $\|X_n\|_p \to \|X\|_p$. Since

$$\big|\|X\|_p - \|Y\|_p\big| \;\le\; \|X-Y\|_p,$$

we see that convergence in mean of order p implies (#).

23. Write

$$\begin{aligned}\|X+Y\|_p^2 + \|X-Y\|_p^2 &= \langle X+Y, X+Y\rangle + \langle X-Y, X-Y\rangle \\ &= \langle X,X\rangle + 2\langle X,Y\rangle + \langle Y,Y\rangle + \langle X,X\rangle - 2\langle X,Y\rangle + \langle Y,Y\rangle \\ &= 2(\|X\|_p^2 + \|Y\|_p^2).\end{aligned}$$

24. Write

$$\begin{aligned}|\langle X_n,Y_n\rangle - \langle X,Y\rangle| &= |\langle X_n,Y_n\rangle - \langle X,Y_n\rangle + \langle X,Y_n\rangle - \langle X,Y\rangle| \\ &\le |\langle X_n - X, Y\rangle| + |\langle X, Y_n - Y\rangle| \\ &\le \|X_n - X\|_2\|Y_n\|_2 + \|X\|_2\|Y_n - Y\|_2 \;\to\; 0.\end{aligned}$$

Here we have used the Cauchy–Schwarz inequality and the fact that since Y_n converges, it is bounded.

25. As in the example, for $n > m$, we can write

$$\|Y_n - Y_m\|_2^2 \;\le\; \sum_{k=m+1}^{n}\sum_{l=m+1}^{n} |h_k|\,|h_l|\,|\langle X_k, X_l\rangle|.$$

In this problem, $\langle X_k, X_l\rangle = 0$ for $k \ne l$. Hence,

$$\|Y_n - Y_m\|_2^2 \;\le\; \sum_{k=m+1}^{n} B|h_k|^2 \;\le\; B\sum_{k=m+1}^{n}|h_k|^2 \;\to\; 0$$

as $n > m \to \infty$ on account of the assumption that $\sum_{k=1}^\infty |h_k|^2 < \infty$.

26. Put $Y_n := \sum_{k=1}^n h_k X_k$. It suffices to show that Y_n is Cauchy in L^p. Write, for $n > m$,

$$\|Y_n - Y_m\|_p \;=\; \bigg\|\sum_{k=m+1}^{n} h_k X_k\bigg\|_p \;\le\; \sum_{k=m+1}^{n} |h_k|\,\|X_k\|_p \;=\; B^{1/p}\sum_{k=m+1}^{n}|h_k| \;\to\; 0$$

as $n > m \to \infty$ on account of the assumption that $\sum_{k=1}^\infty |h_k| < \infty$.

27. Write

$$\begin{aligned}\mathsf{E}[YZ] &= \mathsf{E}\bigg[\bigg(\sum_{i=1}^{n} X_{\tau_i}(t_i - t_{i-1})\bigg)\bigg(\sum_{j=1}^{\nu} X_{\theta_j}(s_j - s_{j-1})\bigg)\bigg] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{\nu} R(\tau_i,\theta_j)(t_i - t_{i-1})(s_j - s_{j-1}).\end{aligned}$$

28. First note that if

$$Y := \sum_{i=1}^{n} g(\tau_i) X_{\tau_i}(t_i - t_{i-1}) \quad \text{and} \quad Z := \sum_{j=1}^{\nu} g(\theta_j) X_{\theta_j}(s_j - s_{j-1}),$$

then

$$\mathsf{E}[YZ] = \sum_{i=1}^{n} \sum_{j=1}^{\nu} g(\tau_i) R(\tau_i, \theta_j) g(\theta_j)(t_i - t_{i-1})(s_j - s_{j-1}).$$

Hence, given finer and finer partitions, with Y_m defined analogously to Y above, we see that

$$\begin{aligned}\mathsf{E}[|Y_m - Y_k|^2] &= \mathsf{E}[Y_m^2] - 2\mathsf{E}[Y_m Y_k] + \mathsf{E}[Y_k^2] \\ &\to \int_a^b \int_a^b g(t)R(t,s)g(s)\,dt\,ds - 2\int_a^b \int_a^b g(t)R(t,s)g(s)\,dt\,ds \\ &\quad + \int_a^b \int_a^b g(t)R(t,s)g(s)\,dt\,ds = 0.\end{aligned}$$

Thus, Y_m is Cauchy in L^2, and there exists a $Y \in L^2$ with $\|Y_m - Y\|_2 \to 0$. Furthermore, since Y_m converges in mean square to Y, $\mathsf{E}[Y_m^2] \to \mathsf{E}[Y^2]$, and it is clear that

$$\mathsf{E}[Y_m^2] \to \int_a^b \int_a^b g(t)R(t,s)g(s)\,dt\,ds.$$

29. Consider the formula

$$\int_0^T R(t-s)\varphi(s)\,ds = \lambda\varphi(t), \quad 0 \le t \le T. \qquad (*)$$

Since R is defined for all t, we can extend the definition of φ on the right-hand side in the obvious way. Furthermore, since R has period T, so will the extended definition of φ. Hence, both R and φ have Fourier series representations, say

$$R(t) = \sum_n r_n e^{j2\pi nt/T} \quad \text{and} \quad \varphi(s) = \sum_n \varphi_n e^{j2\pi ns/T}.$$

Substituting these into $(*)$ yields

$$\sum_n r_n e^{j2\pi nt/T} \underbrace{\int_0^T e^{-j2\pi ns/T} \varphi(s)\,ds}_{= T\varphi_n} = \sum_n \lambda \varphi_n e^{j2\pi nt/T}.$$

It follows that $Tr_n\varphi_n = \lambda\varphi_n$. Now, if φ is an eigenfunction, φ cannot be the zero function. Hence, there is at least one value of n with $\varphi_n \neq 0$. For all n with $\varphi_n \neq 0$, $\lambda = Tr_n$. Thus,

$$\varphi(t) = \sum_{n: r_n = \lambda/T} \varphi_n e^{j2\pi nt/T}.$$

30. If $\int_a^b R(t,s)\varphi(s)\,ds = \lambda\varphi(t)$ then

$$\begin{aligned} 0 \;\le\; \int_a^b\int_a^b R(t,s)\varphi(t)\varphi(s)\,dt\,ds \;&=\; \int_a^b \varphi(t)\biggl[\int_a^b R(t,s)\varphi(s)\,ds\biggr]dt \\ &= \int_a^b \varphi(t)[\lambda\varphi(t)]\,dt \;=\; \lambda\int_a^b \varphi(t)^2\,dt. \end{aligned}$$

Hence, $\lambda \ge 0$.

31. To begin, write

$$\begin{aligned} \lambda_k\int_a^b \varphi_k(t)\varphi_m(t)\,dt \;&=\; \int_a^b \lambda_k\varphi_k(t)\varphi_m(t)\,dt \;=\; \int_a^b\biggl[\int_a^b R(t,s)\varphi_k(s)\,ds\biggr]\varphi_m(t)\,dt \\ &= \int_a^b \varphi_k(s)\biggl[\int_a^b R(s,t)\varphi_m(t)\,dt\biggr]ds, \quad \text{since } R(t,s)=R(s,t), \\ &= \int_a^b \varphi_k(s)\cdot\lambda_m\varphi_m(s)\,ds \;=\; \lambda_m\int_a^b \varphi_k(s)\varphi_m(s)\,ds. \end{aligned}$$

We can now write

$$(\lambda_k-\lambda_m)\int_a^b \varphi_k(t)\varphi_m(t)\,dt \;=\; 0.$$

If $\lambda_k \ne \lambda_m$, we must have $\int_a^b \varphi_k(t)\varphi_m(t)\,dt = 0$.

32. (a) Write

$$\begin{aligned} \int_0^T R(t,s)g(s)\,ds \;&=\; \int_0^T\biggl[\sum_{k=1}^\infty \lambda_k\varphi_k(t)\varphi_k(s)\biggr]g(s)\,ds \\ &= \sum_{k=1}^\infty \lambda_k\varphi_k(t)\int_0^T g(s)\varphi_k(s)\,ds \;=\; \sum_{k=1}^\infty \lambda_k g_k\varphi_k(t). \end{aligned}$$

(b) Write

$$\begin{aligned} \int_0^T R(t,s)\varphi(s)\,ds \;&=\; \int_0^T R(t,s)\biggl[g(s)-\sum_{k=1}^\infty g_k\varphi_k(s)\biggr]ds \\ &= \int_0^T R(t,s)g(s)\,ds - \sum_{k=1}^\infty g_k \underbrace{\int_0^T R(t,s)\varphi_k(s)\,ds}_{=\,\lambda_k\varphi_k(t)} \\ &= \sum_{k=1}^\infty \lambda_k g_k\varphi_k(t) - \sum_{k=1}^\infty \lambda_k g_k\varphi_k(t) \;=\; 0. \end{aligned}$$

(c) Write

$$\mathsf{E}[Z^2] = \int_0^T\int_0^T \varphi(t)R(t,s)\varphi(s)\,dt\,ds = \int_0^T \varphi(t)\biggl[\int_0^T R(t,s)\varphi(s)\,ds\biggr]dt = 0.$$

33. Consider the equation

$$\begin{aligned}
\lambda\varphi(t) &= \int_0^T e^{-|t-s|}\varphi(s)\,ds && (\#) \\
&= \int_0^t e^{-(t-s)}\varphi(s)\,ds + \int_t^T e^{-(s-t)}\varphi(s)\,ds \\
&= e^{-t}\int_0^t e^{s}\varphi(s)\,ds + e^{t}\int_t^T e^{-s}\varphi(s)\,ds.
\end{aligned}$$

Differentiating yields

$$\begin{aligned}
\lambda\varphi'(t) &= -e^{-t}\int_0^t e^{s}\varphi(s)\,ds + e^{-t}e^{t}\varphi(t) + e^{t}\int_t^T e^{-s}\varphi(s)\,ds + e^{t}(-e^{-t}\varphi(t)) \\
&= -e^{-t}\int_0^t e^{s}\varphi(s)\,ds + e^{t}\int_t^T e^{-s}\varphi(s)\,ds. && (\#\#)
\end{aligned}$$

Differentiating again yields

$$\begin{aligned}
\lambda\varphi''(t) &= e^{-t}\int_0^t e^{s}\varphi(s)\,ds - e^{-t}e^{t}\varphi(t) + e^{t}\int_t^T e^{-s}\varphi(s)\,ds + e^{t}(-e^{-t}\varphi(t)) \\
&= \int_0^T e^{-|t-s|}\varphi(s)\,ds - 2\varphi(t) \\
&= \lambda\varphi(t) - 2\varphi(t), \quad \text{by } (\#), \\
&= (\lambda-2)\varphi(t).
\end{aligned}$$

We can now write

$$\varphi''(t) = (1-2/\lambda)\varphi(t) = -(2/\lambda-1)\varphi(t).$$

For $0<\lambda<2$, put $\mu := \sqrt{2/\lambda-1}$. Then $\varphi''(t) = -\mu^2\varphi(t)$. Hence, we must have

$$\varphi(t) = A\cos\mu t + B\sin\mu t$$

for some constants A and B.

34. The required orthogonality principle is that

$$\mathsf{E}\bigg[(X_t-\widehat{X}_t)\sum_{i=1}^L c_iA_i\bigg] = 0$$

for all constants c_i, where

$$\widehat{X}_t = \sum_{j=1}^L \widehat{c}_jA_j.$$

In particular, we must have $\mathsf{E}[(X_t-\widehat{X}_t)A_i]=0$. Now, we know from the text that $\mathsf{E}[X_tA_i] = \lambda_i\varphi_i(t)$. We also have

$$\mathsf{E}[\widehat{X}_tA_i] = \sum_{j=1}^L \widehat{c}_j\mathsf{E}[A_jA_i] = \widehat{c}_i\lambda_i.$$

Hence, $\widehat{c}_i = \varphi_i(t)$, and we have

$$\widehat{X}_t = \sum_{i=1}^{L} A_i \varphi_i(t).$$

35. Since $\|Y_n - Y\|_2 \to 0$, by the hint, $\mathsf{E}[Y_n] \to \mathsf{E}[Y]$ and $\mathsf{E}[Y_n^2] \to \mathsf{E}[Y^2]$. Since g_n is piecewise constant, we know that $\mathsf{E}[Y_n] = 0$, and so $\mathsf{E}[Y] = 0$ too. Next, an argument analogous to the one in Problem 21 tells us that if $\|g_n - g\| \to 0$, then $\|g_n\| \to \|g\|$. Hence,

$$\mathsf{E}[Y^2] = \lim_{n\to\infty} \mathsf{E}[Y_n^2] = \lim_{n\to\infty} \sigma^2 \int_0^\infty g_n(t)^2\,dt = \sigma^2 \int_0^\infty g(t)^2\,dt.$$

36. First write

$$\|Y - \widetilde{Y}\|_2 = \|Y - Y_n\|_2 + \|Y_n - \widetilde{Y}_n\|_2 + \|\widetilde{Y}_n - \widetilde{Y}\|_2,$$

where the first and last terms on the right go to zero. As for the middle term, write

$$\begin{aligned}\|Y_n - \widetilde{Y}_n\|_2^2 &= \mathsf{E}\bigg[\bigg(\int_0^\infty g_n(t)\,dW_t - \int_0^\infty \widetilde{g}_n(t)\,dW_t\bigg)^2\bigg] \\ &= \mathsf{E}\bigg[\bigg(\int_0^\infty [g_n(t) - \widetilde{g}_n(t)]\,dW_t\bigg)^2\bigg] = \sigma^2 \int_0^\infty [g_n(t) - \widetilde{g}_n(t)]^2\,dt.\end{aligned}$$

We can now write

$$\|Y_n - \widetilde{Y}_n\|_2 = \sigma\|g_n - \widetilde{g}_n\| \leq \sigma\|g_n - g\| + \sigma\|g - \widetilde{g}_n\| \to 0.$$

37. We know from our earlier work that the Wiener integral is linear on piecewise-constant functions. To analyze the general case, let $Y = \int_0^\infty g(t)\,dW_t$ and $Z = \int_0^\infty h(t)\,dW_t$. We must show that $aY + bZ = \int_0^\infty ag(t) + bh(t)\,dW_t$. Let $g_n(t)$ and $h_n(t)$ be piecewise-constant functions such that $\|g_n - g\| \to 0$ and $\|h_n - h\| \to 0$ and such that $Y_n := \int_0^\infty g_n(t)\,dW_t$ and $Z_n := \int_0^\infty h_n(t)\,dW_t$ converge in mean square to Y and Z, respectively. Now observe that

$$aY_n + bZ_n = a\int_0^\infty g_n(t)\,dW_t + b\int_0^\infty h_n(t)\,dW_t = \int_0^\infty ag_n(t) + bh_n(t)\,dW_t, \qquad (*)$$

since g_n and h_n are piecewise constant. Next, since

$$\|(ag_n + bh_n) - (ag + bh)\| \leq |a|\,\|g_n - g\| + |b|\,\|h_n - h\| \to 0,$$

it follows that the right-hand side of $(*)$ converges in mean square to $\int_0^\infty ag(t) + bh(t)\,dW_t$. Since the left-hand side of $(*)$ converges in mean square to $aY + bZ$, the desired result follows.

38. We must find all values of β for which $\mathsf{E}[|X_t/t|^2] \to 0$. First compute

$$\mathsf{E}[X_t^2] = \mathsf{E}\bigg[\bigg(\int_0^t \tau^\beta\,dW_\tau\bigg)^2\bigg] = \int_0^t \tau^{2\beta}\,d\tau = \frac{t^{2\beta+1}}{2\beta+1}.$$

Then $t^{2\beta+1}/t^2 = t^{2\beta-1} \to 0$ if and only if $2\beta - 1 < 0$; i.e., $0 < \beta < 1/2$.

39. Using the law of total probability, substitution, and independence, write

$$\begin{aligned} \mathsf{E}[Y_T^2] &= \mathsf{E}\bigg[\bigg(\int_0^T \tau^n\,dW_\tau\bigg)^2\bigg] = \int_0^\infty \mathsf{E}\bigg[\bigg(\int_0^T \tau^n\,dW_\tau\bigg)^2\bigg|T=t\bigg] f_T(t)\,dt \\ &= \int_0^\infty \mathsf{E}\bigg[\bigg(\int_0^t \tau^n\,dW_\tau\bigg)^2\bigg|T=t\bigg] f_T(t)\,dt = \int_0^\infty \mathsf{E}\bigg[\bigg(\int_0^t \tau^n\,dW_\tau\bigg)^2\bigg] f_T(t)\,dt. \end{aligned}$$

Now use properties of the Wiener process to write

$$\begin{aligned} \mathsf{E}[Y_T^2] &= \int_0^\infty \bigg(\int_0^t \tau^{2n}\,d\tau\bigg) f_T(t)\,dt = \int_0^\infty \frac{t^{2n+1}}{2n+1} f_T(t)\,dt = \frac{\mathsf{E}[T^{2n+1}]}{2n+1} \\ &= \frac{(2n+1)!/\lambda^{2n+1}}{2n+1} = \frac{(2n)!}{\lambda^{2n+1}}. \end{aligned}$$

40. (a) Write

$$\begin{aligned} g(t+\Delta t)-g(t) &= \mathsf{E}[f(W_{t+\Delta t})]-\mathsf{E}[f(W_t)] \\ &= \mathsf{E}[f(W_{t+\Delta t})-f(W_t)] \\ &\approx \mathsf{E}[f'(W_t)(W_{t+\Delta t}-W_t)]+\tfrac{1}{2}\mathsf{E}[f''(W_t)(W_{t+\Delta t}-W_t)^2] \\ &= \mathsf{E}[f'(W_t-W_0)(W_{t+\Delta t}-W_t)] \\ &\quad +\tfrac{1}{2}\mathsf{E}[f''(W_t-W_0)(W_{t+\Delta t}-W_t)^2] \\ &= \mathsf{E}[f'(W_t-W_0)]\cdot 0+\tfrac{1}{2}\mathsf{E}[f''(W_t-W_0)]\cdot\Delta t. \end{aligned}$$

It then follows that $g'(t) = \frac{1}{2}\mathsf{E}[f''(W_t)]$.

(b) If $f(x) = e^x$, then $g'(t) = \frac{1}{2}\mathsf{E}[e^{W_t}] = \frac{1}{2}g(t)$. In this case, $g(t) = e^{t/2}$, since $g(0) = \mathsf{E}[e^{W_0}] = 1$.

(c) We have by direct calculation that $g(t) = \mathsf{E}[e^{W_t}] = e^{s^2t/2}\big|_{s=1} = e^{t/2}$.

41. Let C be the ball of radius r, $C := \{Y \in L^p : \|Y\|_p \le r\}$. For $X \notin C$, i.e., $\|X\|_p > r$, we show that

$$\widehat{X} = \frac{r}{\|X\|_p}X.$$

To begin, note that the proposed formula for $\widehat{X}$ satisfies $\|\widehat{X}\|_p = r$ so that $\widehat{X} \in C$ as required. Now observe that

$$\|X-\widehat{X}\|_p = \bigg\|X-\frac{r}{\|X\|_p}X\bigg\|_p = \bigg|1-\frac{r}{\|X\|_p}\bigg|\,\|X\|_p = \|X\|_p - r.$$

Next, for any $Y \in C$,

$$\begin{aligned} \|X-Y\|_p &\ge \big|\|X\|_p-\|Y\|_p\big| \\ &= \|X\|_p-\|Y\|_p \\ &\ge \|X\|_p - r \\ &= \|X-\widehat{X}\|_p. \end{aligned}$$

Thus, no $Y \in C$ is closer to X than $\widehat{X}$.

42. Suppose that $\widehat{X}$ and $\widetilde{X}$ are both elements of a subspace M and that $\langle X-\widehat{X},Y\rangle = 0$ for all $Y \in M$ and $\langle X-\widetilde{X},Y\rangle = 0$ for all $Y \in M$. Then write

$$\|\widehat{X}-\widetilde{X}\|_2^2 \;=\; \langle \widehat{X}-\widetilde{X},\widehat{X}-\widetilde{X}\rangle \;=\; \langle (\widehat{X}-X)+(X-\widetilde{X}),\underbrace{\widehat{X}-\widetilde{X}}_{\in M}\rangle \;=\; 0+0 \;=\; 0.$$

43. For $\widehat{X}_N$ is to be the projection of $\widehat{X}_M$ onto N, it is sufficient that the orthogonality principle be satisfied. In other words, it suffices to show that

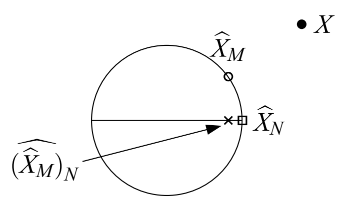
$$\langle \widehat{X}_M-\widehat{X}_N,Y\rangle \;=\; 0, \quad \text{for all } Y \in N.$$

Observe that

$$\langle \widehat{X}_M-\widehat{X}_N,Y\rangle \;=\; \langle (\widehat{X}_M-X)+(X-\widehat{X}_N),Y\rangle \;=\; -\langle X-\widehat{X}_M,Y\rangle + \langle X-\widehat{X}_N,Y\rangle.$$

Now, the last term on the right is zero by the orthogonality principle for the projection of X onto N, since $Y \in N$. To show that $\langle X-\widehat{X}_M,Y\rangle = 0$, observe that since $N \subset M$, $Y \subset N$ implies $Y \subset M$. By the orthogonality principle for the projection of X onto M, $\langle X-\widehat{X}_M,Y\rangle = 0$ for $Y \in M$.

44. In the diagram, M is the disk and N is the horizontal line segment. The $\circ$ is $\widehat{X}_M$, the projection of X onto the disk M. The $\square$ is $\widehat{X}_N$, the projection of X onto line segment N. The $\times$ is $\widehat{(\widehat{X}_M)_N}$, the projection of the circle $\widehat{X}_M$ onto the line segment N. We see that $\widehat{(\widehat{X}_M)_N} \neq \widehat{X}_N$.



45. Suppose that $g_n(Y) \in M$ and $g_n(Y)$ converges in mean square to some X. We must show that $X \in M$. Since $g_n(Y)$ converges, it is Cauchy. Writing

$$\begin{aligned}\|g_n(Y)-g_m(Y)\|_2^2 \;&=\; \mathsf{E}[|g_n(Y)-g_m(Y)|^2] \\ &=\; \int_{-\infty}^{\infty} |g_n(y)-g_m(y)|^2 f_Y(y)\,dy \;=\; \|g_n-g_m\|_Y,\end{aligned}$$

we see that g_n is Cauchy in G, which is complete. Hence, there exists a $g \in G$ with $\|g_n-g\|_Y \to 0$. We claim $X = g(Y)$. Write

$$\begin{aligned}\|g(Y)-X\|_2 \;&=\; \|g(Y)-g_n(Y)+g_n(Y)-X\|_2 \\ &\leq\; \|g(Y)-g_n(Y)\|_2 + \|g_n(Y)-X\|_2,\end{aligned}$$

where the last term goes to zero by assumption. Now observe that

$$\begin{aligned}\|g(Y)-g_n(Y)\|_2^2 \;=\; \mathsf{E}[|g(Y)-g_n(Y)|^2] \;&=\; \int_{-\infty}^{\infty} |g(y)-g_n(y)|^2 f_Y(y)\,dy \\ &=\; \|g-g_n\|_Y^2 \;\to\; 0.\end{aligned}$$

Thus, $X = g(Y) \in M$ as required.

46. We claim that the required projection is $\int_0^1 f(t)\,dW_t$. Note that this is an element of M since

$$\int_0^1 f(t)^2\,dt \;\le\; \int_0^\infty f(t)^2\,dt \;<\; \infty.$$

Consider the orthogonality condition

$$\mathsf{E}\bigg[\bigg(\int_0^\infty f(t)\,dW_t - \int_0^1 f(t)\,dW_t\bigg)\int_0^1 g(t)\,dW_t\bigg] = \mathsf{E}\bigg[\bigg(\int_1^\infty f(t)\,dW_t\bigg)\bigg(\int_0^1 g(t)\,dW_t\bigg)\bigg].$$

Now put

$$\tilde{f}(t) \;:=\; \begin{cases} f(t), & t>1, \\ 0, & 0\le t\le 1, \end{cases} \qquad\text{and}\quad \tilde{g}(t) \;:=\; \begin{cases} 0, & t>1, \\ g(t), & 0\le t\le 1, \end{cases}$$

so that the orthgonality condition becomes

$$\mathsf{E}\bigg[\bigg(\int_0^\infty \tilde{f}(t)\,dW_t\bigg)\bigg(\int_0^\infty \tilde{g}(t)\,dW_t\bigg)\bigg] \;=\; \int_0^\infty \tilde{f}(t)\tilde{g}(t)\,dt,$$

which is zero since $\tilde{f}(t)\tilde{g}(t)=0$ for all $t\ge 0$.

47. The function $g(t)$ will be optimal if the orthogonality condition

$$\mathsf{E}\bigg[\bigg(X-\int_0^\infty g(\tau)\,dW_\tau\bigg)\int_0^\infty \tilde{g}(\tau)\,dW_\tau\bigg] \;=\; 0$$

holds for all $\tilde{g}$ with $\int_0^\infty \tilde{g}(\tau)^2\,d\tau$. In particular, this must be true for $\tilde{g}(\tau)=I_{[0,t]}(\tau)$. In this case, the above expectation reduces to

$$\mathsf{E}\bigg[X\int_0^\infty I_{[0,t]}(\tau)\,dW_\tau\bigg] - \mathsf{E}\bigg[\int_0^\infty g(\tau)\,dW_\tau\int_0^\infty I_{[0,t]}(\tau)\,dW_\tau\bigg].$$

Now, since

$$\int_0^\infty I_{[0,t]}(\tau)\,dW_\tau \;=\; \int_0^t dW_\tau \;=\; W_t,$$

we have the further simplification

$$\mathsf{E}[XW_t]-\sigma^2\int_0^\infty g(\tau)I_{[0,t]}(\tau)\,d\tau \;=\; \mathsf{E}[XW_t]-\sigma^2\int_0^t g(\tau)\,d\tau.$$

Since this must be equal to zero for all $t\ge 0$, we can differentiate and obtain

$$g(t) \;=\; \frac{1}{\sigma^2}\cdot\frac{d}{dt}\mathsf{E}[XW_t].$$

48. (a) Using the methods of Chapter 5, it is not too hard to show that

$$F_Y(y) \;=\; \begin{cases} 1, & y\ge 1/4, \\ [2-\sqrt{1-4y}\,]/2, & 0\le y<1/4, \\ [3/2-(1/2)\sqrt{1-4y}\,]/2, & -2\le y<0, \\ 0, & y<-2. \end{cases}$$

It then follows that

$$f_Y(y) = \begin{cases} 1/\sqrt{1-4y}, & 0 < y < 1/4, \\ 1/(2\sqrt{1-4y}\,), & -2 < y < 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) We must find a $\widehat{g}(y)$ such that

$$\mathsf{E}[v(X)g(Y)] = \mathsf{E}[\widehat{g}(Y)g(Y)], \quad \text{for all bounded } g.$$

For future reference, note that

$$\mathsf{E}[v(X)g(Y)] = \mathsf{E}[v(X)g(X(1-X))] = \frac{1}{2}\int_{-1}^{1} v(x)g(x(1-x))\,dx.$$

Now, by considering the problem of solving $g(x) = y$ for x in the two cases $0 \le y \le 1/4$ and $-2 \le y < 0$ suggests that we try

$$\widehat{g}(y) = \begin{cases} \frac{1}{2}v\Big(\frac{1+\sqrt{1-4y}}{2}\Big) + \frac{1}{2}v\Big(\frac{1-\sqrt{1-4y}}{2}\Big), & 0 \le y \le 1/4, \\ v\Big(\frac{1-\sqrt{1-4y}}{2}\Big), & -2 \le y < 0. \end{cases}$$

To check, we compute

$$\begin{aligned} \mathsf{E}[\widehat{g}(Y)g(Y)] &= \mathsf{E}[\widehat{g}(X(1-X))g(X(1-X))] \\ &= \frac{1}{2}\bigg[\int_{-1}^{0} + \int_{0}^{1/2} + \int_{1/2}^{1}\bigg]\widehat{g}(x(1-x))g(x(1-x))\,dx \\ &= \frac{1}{2}\bigg[\int_{-1}^{0} v(x)g(x(1-x))\,dx \\ &\qquad + \int_{0}^{1/2} \frac{v(1-x)+v(x)}{2} g(x(1-x))\,dx \\ &\qquad + \int_{1/2}^{1} \frac{v(x)+v(1-x)}{2} g(x(1-x))\,dx\bigg] \\ &= \frac{1}{2}\bigg[\int_{-1}^{0} v(x)g(x(1-x))\,dx + \int_{0}^{1} v(x)g(x(1-x))\,dx\bigg] \\ &= \int_{-1}^{1} v(x)g(x(1-x))\,dx. \end{aligned}$$

49. (a) Using the methods of Chapter 5, it is not too hard to show that

$$F_Y(y) = \begin{cases} F_\Theta(\sin^{-1}(y)) + 1 - F_\Theta(\pi - \sin^{-1}(y)), & 0 \le y \le 1, \\ F_\Theta(\sin^{-1}(y)) - F_\Theta(-\pi - \sin^{-1}(y)), & -1 \le y < 0. \end{cases}$$

Hence,

$$f_Y(y) = \begin{cases} \dfrac{f_\Theta(\sin^{-1}(y)) + f_\Theta(\pi - \sin^{-1}(y))}{\sqrt{1-y^2}}, & 0 \le y < 1, \\ \dfrac{f_\Theta(\sin^{-1}(y)) + f_\Theta(-\pi - \sin^{-1}(y))}{\sqrt{1-y^2}}, & -1 < y < 0. \end{cases}$$

(b) Consider

$$\mathsf{E}[v(X)g(Y)] \;=\; \mathsf{E}[v(\cos\Theta)g(\sin\Theta)] \;=\; \int_{-\pi}^{\pi} v(\cos\theta)g(\sin\theta)f_\Theta(\theta)\,d\theta.$$

Next, write

$$\int_0^{\pi/2} v(\cos\theta)g(\sin\theta)f_\Theta(\theta)\,d\theta \;=\; \int_0^1 v(\sqrt{1-y^2}\,)g(y)\frac{f_\Theta(\sin^{-1}(y))}{\sqrt{1-y^2}}\,dy$$

and

$$\int_{-\pi/2}^{0} v(\cos\theta)g(\sin\theta)f_\Theta(\theta)\,d\theta \;=\; \int_{-1}^0 v(\sqrt{1-y^2}\,)g(y)\frac{f_\Theta(\sin^{-1}(y))}{\sqrt{1-y^2}}\,dy.$$

Similarly, we can write with a bit more work

$$\begin{aligned}
\int_{\pi/2}^{\pi} v(\cos\theta)g(\sin\theta)f_\Theta(\theta)\,d\theta \;&=\; \int_0^{\pi/2} v(\cos(\pi-t))g(\sin(\pi-t))f_\Theta(\pi-t)\,dt \\
&=\; \int_0^{\pi/2} v(-\cos t)g(\sin t)f_\Theta(\pi-t)\,dt \\
&=\; \int_0^1 v(-\sqrt{1-y^2}\,)g(y)\frac{f_\Theta(\pi-\sin^{-1}(y))}{\sqrt{1-y^2}}\,dy,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\pi}^{-\pi/2} v(\cos\theta)g(\sin\theta)f_\Theta(\theta)\,d\theta \\
&\qquad=\; \int_{-\pi/2}^{0} v(\cos(-\pi-t))g(\sin(-\pi-t))f_\Theta(-\pi-t)\,dt \\
&\qquad=\; \int_{-\pi/2}^{0} v(-\cos t)g(\sin t)f_\Theta(-\pi-t)\,dt \\
&\qquad=\; \int_{-1}^{0} v(-\sqrt{1-y^2}\,)g(y)\frac{f_\Theta(-\pi-\sin^{-1}(y))}{\sqrt{1-y^2}}\,dy.
\end{aligned}$$

Putting all this together, we see that

$$\begin{aligned}
&\mathsf{E}[v(X)g(Y)] \\
&\quad=\; \int_0^1 \frac{v(\sqrt{1-y^2}\,)f_\Theta(\sin^{-1}(y)) + v(-\sqrt{1-y^2}\,)f_\Theta(\pi-\sin^{-1}(y))}{f_\Theta(\sin^{-1}(y)) + f_\Theta(\pi-\sin^{-1}(y))} \\
&\qquad\cdot g(y)f_Y(y)\,dy \\
&\qquad+ \int_{-1}^0 \frac{v(\sqrt{1-y^2}\,)f_\Theta(\sin^{-1}(y)) + v(-\sqrt{1-y^2}\,)f_\Theta(-\pi-\sin^{-1}(y))}{f_\Theta(\sin^{-1}(y)) + f_\Theta(-\pi-\sin^{-1}(y))} \\
&\qquad\cdot g(y)f_Y(y)\,dy.
\end{aligned}$$

We conclude that

$$\mathsf{E}[v(X)|Y=y]$$
$$= \begin{cases} \frac{v(\sqrt{1-y^2}\,)f_\Theta(\sin^{-1}(y))+v(-\sqrt{1-y^2}\,)f_\Theta(\pi-\sin^{-1}(y))}{f_\Theta(\sin^{-1}(y))+f_\Theta(\pi-\sin^{-1}(y))}, & 0<y<1, \\ \frac{v(\sqrt{1-y^2}\,)f_\Theta(\sin^{-1}(y))+v(-\sqrt{1-y^2}\,)f_\Theta(-\pi-\sin^{-1}(y))}{f_\Theta(\sin^{-1}(y))+f_\Theta(-\pi-\sin^{-1}(y))}, & -1<y<0. \end{cases}$$

50. Since $X \geq 0$ and $g(y) = I_B(\mathsf{E}[X|Y]) \geq 0$, $\mathsf{E}[Xg(Y)] \geq 0$. On the other hand,

$$\mathsf{E}[X|Y]I_{(-\infty,-1/n)}(\mathsf{E}[X|Y]) \;\leq\; \frac{-1}{n}I_{(-\infty,-1/n)}(\mathsf{E}[X|Y]),$$

and so

$$0 \;\leq\; \mathsf{E}[Xg(Y)] \;=\; \mathsf{E}[\mathsf{E}[X|Y]g(Y)] \;\leq\; \frac{-1}{n}\mathsf{P}(\mathsf{E}[X|Y] < -1/n) \;<\; 0,$$

which is a contradiction. Hence, $\mathsf{P}(\mathsf{E}[X|Y] < 0) = 0$.

51. Write

$$\begin{aligned} \mathsf{E}[Xg(Y)] &= \mathsf{E}[(X^+ - X^-)g(Y)] \\ &= \mathsf{E}[X^+g(Y)] - \mathsf{E}[X^-g(Y)] \\ &= \mathsf{E}\big[\mathsf{E}[X^+|Y]g(Y)\big] - \mathsf{E}\big[\mathsf{E}[X^-|Y]g(Y)\big] \\ &= \mathsf{E}\big[\big(\mathsf{E}[X^+|Y] - \mathsf{E}[X^-|Y]\big)g(Y)\big]. \end{aligned}$$

By uniqueness, we conclude that $\mathsf{E}[X|Y] = \mathsf{E}[X^+|Y] - \mathsf{E}[X^-|Y]$.

52. Following the hint, we begin with

$$\big|\mathsf{E}[X|Y]\big| = \big|\mathsf{E}[X^+|Y] - \mathsf{E}[X^-|Y]\big| \leq \big|\mathsf{E}[X^+|Y]\big| + \big|\mathsf{E}[X^-|Y]\big| = \mathsf{E}[X^+|Y] + \mathsf{E}[X^-|Y].$$

Then

$$\mathsf{E}\big[\big|\mathsf{E}[X|Y]\big|\big] \;\leq\; \mathsf{E}\big[\mathsf{E}[X^+|Y]\big] + \mathsf{E}\big[\mathsf{E}[X^-|Y]\big] \;=\; \mathsf{E}[X^+] + \mathsf{E}[X^-] \;<\; \infty.$$

53. To show that $\mathsf{E}[h(Y)X|Y] = h(Y)\mathsf{E}[X|Y]$, we have to show that the right-hand side satisfies the characterizing equation of the left-hand side. Since the characterizing equation for the left-hand side is

$$\mathsf{E}[\{h(Y)X\}g(Y)] \;=\; \mathsf{E}\big[\mathsf{E}[h(Y)X|Y]g(Y)\big], \quad \text{for all bounded } g,$$

we must show that

$$\mathsf{E}[\{h(Y)X\}g(Y)] \;=\; \mathsf{E}\big[\{h(Y)\mathsf{E}[X|Y]\}g(Y)\big], \quad \text{for all bounded } g. \qquad (*)$$

The only other thing we know is the characterizing equation for $\mathsf{E}[X|Y]$, which is

$$\mathsf{E}[Xg(Y)] \;=\; \mathsf{E}\big[\mathsf{E}[X|Y]g(Y)\big], \quad \text{for all bounded } g.$$

Since g in the above formula is an arbitrary and bounded function, and since h is also bounded, we can rewrite the above formula by replacing $g(Y)$ with $g(Y)h(Y)$ for arbitrary bounded g. We thus have

$$\mathsf{E}[X\{g(Y)h(Y)\}] \;=\; \mathsf{E}\big[\mathsf{E}[X|Y]\{g(Y)h(Y)\}\big], \quad \text{for all bounded } g,$$

which is equivalent to $(*)$.

54. We must show that $\mathsf{E}[X|q(Y)]$ satisfies the characterizing equation of $\mathsf{E}\big[\mathsf{E}[X|Y]\big|q(Y)\big]$. To write down the characterizing equation for $\mathsf{E}\big[\mathsf{E}[X|Y]\big|q(Y)\big]$, it is convenient to use the notation $Z := \mathsf{E}[X|Y]$. Then the characterizing equation for $\mathsf{E}[Z|q(Y)]$ is

$$\mathsf{E}[Zg(q(Y))] \;=\; \mathsf{E}\big[\mathsf{E}[Z|q(Y)]g(q(Y))\big], \quad \text{for all bounded } g.$$

We must show that this equation holds when $\mathsf{E}[Z|q(Y)]$ is replaced by $\mathsf{E}[X|q(Y)]$; i.e., we must show that

$$\mathsf{E}[Zg(q(Y))] \;=\; \mathsf{E}\big[\mathsf{E}[X|q(Y)]g(q(Y))\big], \quad \text{for all bounded } g.$$

Replacing Z by its definition, we must show that

$$\mathsf{E}[\mathsf{E}[X|Y]g(q(Y))] \;=\; \mathsf{E}\big[\mathsf{E}[X|q(Y)]g(q(Y))\big], \quad \text{for all bounded } g. \qquad (**)$$

We begin with the characterizing equation for $\mathsf{E}[X|Y]$, which is

$$\mathsf{E}[Xh(Y)] \;=\; \mathsf{E}\big[\mathsf{E}[X|Y]h(Y)\big], \quad \text{for all bounded } h.$$

Since h is bounded and arbitrary, we can replace $h(Y)$ by $g(q(Y))$ for arbitrary bounded g. Thus,

$$\mathsf{E}[Xg(q(Y))] \;=\; \mathsf{E}\big[\mathsf{E}[X|Y]g(q(Y))\big], \quad \text{for all bounded } g.$$

We next consider the characterizing equation of $\mathsf{E}[X|q(Y)]$, which is

$$\mathsf{E}[Xg(q(Y))] \;=\; \mathsf{E}\big[\mathsf{E}[X|q(Y)]g(q(Y))\big], \quad \text{for all bounded } g.$$

Combining these last two equations yields $(**)$.

55. The desired result,

$$\mathsf{E}[\{h(Y)X\}g(Y)] \;=\; \mathsf{E}[\{h(Y)\mathsf{E}[X|Y]\}g(Y)],$$

can be rewritten as

$$\mathsf{E}[(X-\mathsf{E}[X|Y])g(Y)h(Y)] \;=\; 0,$$

where g is a bounded function and $h(Y) \in L^2$. But then $g(Y)h(Y) \in L^2$, and therefore this last equation must hold by the orthogonality principle since $\mathsf{E}[X|Y]$ is the projection of X onto $M = \{v(Y) : \mathsf{E}[v(Y)^2] < \infty\}$.

56. Let $h_n(Y)$ be bounded and converge to $h(Y)$. Then for bounded g,

$$\begin{aligned}
\mathsf{E}[\{h(Y)X\}g(Y)] &= \mathsf{E}\Big[\lim_{n\to\infty} h_n(Y)Xg(Y)\Big] \\
&= \lim_{n\to\infty} \mathsf{E}[X\{h_n(Y)g(Y)\}] \\
&= \lim_{n\to\infty} \mathsf{E}\big[\mathsf{E}[X|Y]\{h_n(Y)g(Y)\}\big] \\
&= \mathsf{E}\Big[\lim_{n\to\infty} h_n(Y)\mathsf{E}[X|Y]g(Y)\Big] \\
&= \mathsf{E}\big[\{h(Y)\mathsf{E}[X|Y]\}g(Y)\big].
\end{aligned}$$

By uniqueness, $\mathsf{E}[h(Y)X|Y] = h(Y)\mathsf{E}[X|Y]$.

57. First write

$$Y = \mathsf{E}[Y|Y] = \mathsf{E}[X_1+\cdots+X_n|Y] = \sum_{i=1}^n \mathsf{E}[X_i|Y].$$

By symmetry, we must have $\mathsf{E}[X_i|Y] = \mathsf{E}[X_1|Y]$ for all i. Then $Y = n\mathsf{E}[X_1|Y]$, or $\mathsf{E}[X_1|Y] = Y/n$.

58. Write

$$\begin{aligned}
\mathsf{E}[X_{n+1}|Y_1,\ldots,Y_n] &= \mathsf{E}[Y_{n+1}+Y_n+\cdots+Y_1|Y_1,\ldots,Y_n] \\
&= \mathsf{E}[Y_{n+1}|Y_1,\ldots,Y_n]+Y_n+\cdots+Y_1 \\
&= \mathsf{E}[Y_{n+1}]+X_n, \quad \text{by indep. \& def. of } X_n, \\
&= X_n, \quad \text{since } \mathsf{E}[Y_{n+1}] = 0.
\end{aligned}$$

59. For $n \geq 1$,

$$\mathsf{E}[X_{n+1}] = \mathsf{E}\big[\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1]\big] = \mathsf{E}[X_n],$$

where the second equality uses the definition of a martingale. Hence, $\mathsf{E}[X_n] = \mathsf{E}[X_1]$ for $n \geq 1$.

60. For $n \geq 1$,

$$\mathsf{E}[X_{n+1}] = \mathsf{E}\big[\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1]\big] \leq \mathsf{E}[X_n],$$

where the inequality uses the definition of a supermartingale. Since $X_n \geq 0$, $\mathsf{E}[X_n] \geq 0$. Hence, $0 \leq \mathsf{E}[X_n] \leq \mathsf{E}[X_1]$ for $n \geq 1$.

61. Since $X_{n+1} := \mathsf{E}[Z|Y_{n+1},\ldots,Y_1]$,

$$\begin{aligned}
\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1] &= \mathsf{E}\big[\mathsf{E}[Z|Y_{n+1},\ldots,Y_1]\big|Y_n,\ldots,Y_1\big] \\
&= \mathsf{E}[Z|Y_n,\ldots,Y_1], \quad \text{by the smoothing property,} \\
&=: X_n.
\end{aligned}$$

62. Since $X_n := w(Y_n)\cdots w(Y_1)$, observe that X_n is a function of $Y_1,\ldots,Y_n$ and that $X_{n+1} = w(Y_{n+1})X_n$. Then

$$\begin{aligned}
\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1] &= \mathsf{E}[w(Y_{n+1})X_n|Y_n,\ldots,Y_1] = X_n\mathsf{E}[w(Y_{n+1})|Y_n,\ldots,Y_1] \\
&= X_n\mathsf{E}[w(Y_{n+1})], \quad \text{by independence.}
\end{aligned}$$

It remains to compute

$$\mathsf{E}[w(Y_{n+1})] \;=\; \int_{-\infty}^{\infty} w(y)f(y)\,dy \;=\; \int_{-\infty}^{\infty} \frac{\tilde{f}(y)}{f(y)} f(y)\,dy \;=\; \int_{-\infty}^{\infty} \tilde{f}(y)\,dy \;=\; 1.$$

Hence, $\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1] = X_n$; i.e., X_n is a martingale with respect to Y_n.

63. To begin, write

$$\begin{aligned} w_{n+1}(y_1,\ldots,y_{n+1}) \;&=\; \frac{\tilde{f}_{Y_{n+1}\cdots Y_1}(y_{n+1},\ldots,y_1)}{f_{Y_{n+1}\cdots Y_1}(y_{n+1},\ldots,y_1)} \\ &=\; \frac{\tilde{f}_{Y_{n+1}|Y_n\cdots Y_1}(y_{n+1}|y_n,\ldots,y_1)}{f_{Y_{n+1}|Y_n\cdots Y_1}(y_{n+1}|y_n,\ldots,y_1)} \cdot \frac{\tilde{f}_{Y_n\cdots Y_1}(y_n,\ldots,y_1)}{f_{Y_n\cdots Y_1}(y_n,\ldots,y_1)}. \end{aligned}$$

If we put

$$\hat{w}_{n+1}(y_{n+1},\ldots,y_1) \;:=\; \frac{\tilde{f}_{Y_{n+1}|Y_n\cdots Y_1}(y_{n+1}|y_n,\ldots,y_1)}{f_{Y_{n+1}|Y_n\cdots Y_1}(y_{n+1}|y_n,\ldots,y_1)},$$

then $X_{n+1} := w_{n+1}(Y_1,\ldots,Y_{n+1}) = \hat{w}_{n+1}(Y_{n+1},\ldots,Y_1)X_n$, where X_n is a function of $Y_1,\ldots,Y_n$. We can now write

$$\begin{aligned} \mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1] \;&=\; \mathsf{E}[\hat{w}_{n+1}(Y_{n+1},\ldots,Y_1)X_n|Y_n,\ldots,Y_1] \\ &=\; X_n\mathsf{E}[\hat{w}_{n+1}(Y_{n+1},\ldots,Y_1)|Y_n,\ldots,Y_1]. \end{aligned}$$

We now show that this last factor is equal to one. Write

$$\begin{aligned} &\mathsf{E}[\hat{w}_{n+1}(Y_{n+1},Y_n,\ldots,Y_1)|Y_n = y_n,\ldots,Y_1 = y_1] \\ &\qquad=\; \mathsf{E}[\hat{w}_{n+1}(Y_{n+1},y_n\ldots,y_1)|Y_n = y_n,\ldots,Y_1 = y_1] \\ &\qquad=\; \int_{-\infty}^{\infty} \hat{w}_{n+1}(y,y_n,\ldots,y_1) f_{Y_{n+1}|Y_n\cdots Y_1}(y|y_n,\ldots,y_1)\,dy \\ &\qquad=\; \int_{-\infty}^{\infty} \tilde{f}_{Y_{n+1}|Y_n\cdots Y_1}(y|y_n,\ldots,y_1)\,dy \;=\; 1. \end{aligned}$$

64. Since W_k depends on Y_k and Y_{k-1}, and since $X_n = W_1 + \cdots + W_n$, X_n depends on $Y_0,\ldots,Y_n$. Note also that $X_{n+1} = X_n + W_{n+1}$. Now write

$$\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_0] \;=\; X_n + \mathsf{E}[W_{n+1}|Y_n,\ldots,Y_0].$$

Next,

$$\begin{aligned} \mathsf{E}[W_{n+1}|Y_n = y_n,\ldots,Y_0 = y_0] \;&=\; \mathsf{E}\left[Y_{n+1} - \int_{-\infty}^{\infty} zp(z|Y_n)\,dz \,\middle|\, Y_n = y_n,\ldots,Y_0 = y_0\right] \\ &=\; \mathsf{E}\left[Y_{n+1} - \int_{-\infty}^{\infty} zp(z|y_n)\,dz \,\middle|\, Y_n = y_n,\ldots,Y_0 = y_0\right] \\ &=\; \mathsf{E}[Y_{n+1}|Y_n = y_n,\ldots,Y_0 = y_0] - \int_{-\infty}^{\infty} zp(z|y_n)\,dz \\ &=\; \int_{-\infty}^{\infty} zf_{Y_{n+1}|Y_n\cdots Y_0}(z|y_n,\ldots,y_0)\,dz - \int_{-\infty}^{\infty} zp(z|y_n)\,dz \end{aligned}$$

$$= \int_{-\infty}^{\infty} z f_{Y_{n+1}|Y_n}(z|y_n)\,dz - \int_{-\infty}^{\infty} zp(z|y_n)\,dz$$
$$= \int_{-\infty}^{\infty} zp(z|y_n)\,dz - \int_{-\infty}^{\infty} zp(z|y_n)\,dz \;=\; 0.$$

Hence, $\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_0] = X_n$; i.e., X_n is a martingale with respect to Y_n.

65. First write

$$\begin{aligned}
\mathsf{E}[Y_{n+1}|X_n = i_n,\ldots,X_0 = i_0] &= \mathsf{E}\big[\rho^{X_{n+1}}\big|X_n = i_n,\ldots,X_0 = i_0\big] \\
&= \sum_j \rho^j \mathsf{P}(X_{n+1} = j|X_n = i_n,\ldots,X_0 = i_0) \\
&= \sum_j \rho^j \mathsf{P}(X_{n+1} = j|X_n = i_n).
\end{aligned}$$

Next, for $0 < i < N$,

$$\begin{aligned}
\sum_j \rho^j \mathsf{P}(X_{n+1} = j|X_n = i) &= \rho^{i-1}(1-a) + \rho^{i+1}a \\
&= \Big(\frac{1-a}{a}\Big)^{i-1}(1-a) + \Big(\frac{1-a}{a}\Big)^{i+1}a \\
&= \Big(\frac{1-a}{a}\Big)^{i-1}\Big[1-a+\Big(\frac{1-a}{a}\Big)^2 a\Big] \\
&= \Big(\frac{1-a}{a}\Big)^{i-1}\Big[1-a+\frac{(1-a)^2}{a}\Big] \\
&= \frac{(1-a)^i}{a^{i-1}}\Big[1+\frac{1-a}{a}\Big] \;=\; \rho^i.
\end{aligned}$$

If $X_n = 0$ or $X_n = 1$, then $X_{n+1} = X_n$, and so

$$\mathsf{E}[\rho^{X_{n+1}}|X_n,\ldots,X_0] \;=\; \mathsf{E}[\rho^{X_n}|X_n,\ldots,X_0] \;=\; \rho^{X_n} \;=\; Y_n.$$

Hence, in all cases, $\mathsf{E}[Y_{n+1}|X_n,\ldots,X_0] = Y_n$, and so Y_n is a martingale with respect to X_n.

66. First, since X_n is a submartingale, it is clear that

$$A_{n+1} \;:=\; A_n + (\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1] - X_n) \;\geq\; 0$$

and is a function of $Y_1,\ldots,Y_n$. To show that M_n is a martingale, write

$$\begin{aligned}
\mathsf{E}[M_{n+1}|Y_n,\ldots,Y_1] &= \mathsf{E}[X_{n+1} - A_{n+1}|Y_n,\ldots,Y_1] \\
&= \mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1] - A_{n+1} \\
&= \mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1] - \Big[A_n + (\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1] - X_n)\Big] \\
&= X_n - A_n \;=\; M_n.
\end{aligned}$$

67. Without loss of generality, assume $f_1 \leq f_2$. Then

$$\mathsf{E}[Z_{f_1}Z_{f_2}^*] \;=\; \mathsf{E}[Z_{f_1}(Z_{f_2} - Z_{f_1})^*] + \mathsf{E}[|Z_{f_1}|^2]$$

$$
\begin{aligned}
&= \mathsf{E}[(Z_{f_1} - Z_{-1/2})(Z_{f_2} - Z_{f_1})^*] + \mathsf{E}[|Z_{f_1}|^2] \\
&= \mathsf{E}[Z_{f_1} - Z_{-1/2}]\mathsf{E}[(Z_{f_2} - Z_{f_1})^*] + \mathsf{E}[|Z_{f_1}|^2] \\
&= 0 \cdot 0 + \mathsf{E}[|Z_{f_1}|^2] \;=\; \int_{-1/2}^{f_1} S(\nu)\,d\nu.
\end{aligned}
$$

68. Using the geometric series formula,

$$
\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} e^{j2\pi fn} &= \frac{1}{n}\cdot\frac{e^{j2\pi f} - \left(e^{j2\pi f}\right)^{n+1}}{1 - e^{j2\pi f}} \;=\; \frac{e^{j2\pi f}}{n}\cdot\frac{1 - e^{j2\pi fn}}{1 - e^{j2\pi f}} \\
&= \frac{e^{j2\pi f}}{n}\cdot\frac{e^{j\pi fn}}{e^{j\pi f}}\cdot\frac{e^{-j\pi fn} - e^{j\pi fn}}{e^{-j\pi f} - e^{j\pi f}} \;=\; \frac{e^{j2\pi f}}{n}\cdot\frac{e^{j\pi fn}}{e^{j\pi f}}\cdot\frac{\sin(\pi fn)}{\sin(\pi f)}.
\end{aligned}
$$

69. Following the hint, write

$$
\begin{aligned}
\mathsf{E}[|Y|^2] \;=\; \mathsf{E}[YY^*] \;&=\; \mathsf{E}\left[\left(\sum_{n=-N}^{N} d_n X_n\right)\left(\sum_{k=-N}^{N} d_k X_k\right)^*\right] \\
&= \sum_{n=-N}^{N}\sum_{k=-N}^{N} d_n d_k^* \mathsf{E}[X_n X_k] \;=\; \sum_{n=-N}^{N}\sum_{k=-N}^{N} d_n d_k^* R(n-k) \\
&= \sum_{n=-N}^{N}\sum_{k=-N}^{N} d_n d_k^* \int_{-1/2}^{1/2} S(f) e^{j2\pi f(n-k)}\,df \\
&= \int_{-1/2}^{1/2} S(f)\left|\sum_{n=-N}^{N} d_n e^{j2\pi fn}\right|^2 df.
\end{aligned}
$$

Hence, if $\sum_{n=-N}^{N} d_n e^{j2\pi fn} = 0$, then $\mathsf{E}[|Y|^2] = 0$.

70. Write

$$
\begin{aligned}
\mathsf{E}[T(G_0)T(H_0)] \;&=\; \mathsf{E}\left[\left(\sum_{n=-N}^{N} g_n X_n\right)\left(\sum_{k=-N}^{N} h_k X_k\right)^*\right] \;=\; \sum_{n=-N}^{N}\sum_{k=-N}^{N} g_n h_n^* R(n-k) \\
&= \sum_{n=-N}^{N}\sum_{k=-N}^{N} g_n h_n^* \int_{-1/2}^{1/2} S(f) e^{j2\pi f(n-k)}\,df \\
&= \int_{-1/2}^{1/2} S(f)\left(\sum_{n=-N}^{N} g_n e^{j2\pi fn}\right)\left(\sum_{k=-N}^{N} h_k e^{j2\pi fk}\right)^* df \\
&= \int_{-1/2}^{1/2} S(f) G_0(f) H_0(f)^*\,df.
\end{aligned}
$$

71. (a) Write

$$
\begin{aligned}
\|T(G) - Y\|_2 \;&=\; \|T(G) - T(G_n) + T(G_n) - T(\widetilde{G}_n) + T(\widetilde{G}_n) - Y\|_2 \\
&\le\; \|T(G) - T(G_n)\|_2 + \|T(G_n) - T(\widetilde{G}_n)\|_2 + \|T(\widetilde{G}_n) - Y\|_2.
\end{aligned}
$$

On the right-hand side, the first and third terms go to zero. To analyze the middle term, write

$$\begin{aligned}\|T(G_n)-T(\widetilde{G}_n)\|_2 &= \|T(G_n-\widetilde{G}_n)\|_2 = \|G_n-\widetilde{G}_n\| \\ &\leq \|G_n-G\|+\|G-\widetilde{G}_n\| \to 0.\end{aligned}$$

(b) To show T is norm preserving on $L^2(S)$, let $G_n \to G$ with $T(G_n)\to T(G)$. Then by Problem 21, $\|T(G_n)\|_2 \to \|T(G)\|_2$, and similarly, $\|G_n\|\to\|G\|$. Now write

$$\begin{aligned}\|T(G)\|_2 = \lim_{n\to\infty}\|T(G_n)\|_2 &= \lim_{n\to\infty}\|G_n\|_2, \quad \text{since } G_n \text{ is a trig. polynomial,} \\ &= \|G\|.\end{aligned}$$

(c) To show T is linear on $L^2(S)$, fix $G,H\in L^2(S)$, and let G_n and H_n be trigonometric polynomials converging to G and H, respectively. Then

$$\alpha G_n+\beta H_n \to \alpha G+\beta H, \tag{$*$}$$

and we can write

$$\begin{aligned}&\|T(\alpha G+\beta H)-\{\alpha T(G)+\beta T(H)\}\|_2 \\ &\quad= \|T(\alpha G+\beta H)-T(\alpha G_n+\beta H_n)\|_2 \\ &\qquad+\|T(\alpha G_n+\beta H_n)-\{\alpha T(G_n)+\beta T(H_n)\}\|_2 \\ &\qquad+\|\{\alpha T(G_n)+\beta T(H_n)\}-\{\alpha T(G)+\beta T(H)\}\|_2.\end{aligned}$$

Now, the first term on the right goes to zero on account of $(*)$ and the defintion of T. The second term on the right is equal to zero because T is linear on trigonometric polynomials. The third term goes to zero upon observing that

$$\begin{aligned}\|\{\alpha T(G_n)+\beta T(H_n)\}-\{\alpha T(G)+\beta T(H)\}\|_2 &\leq |\alpha|\,\|T(G_n)-T(G)\|_2 \\ &\quad+|\beta|\,\|T(H_n)-T(H)\|_2.\end{aligned}$$

(d) Using parts (b) and (c), $\|T(G)-T(H)\|_2=\|T(G-H)\|_2=\|G-H\|$ implies that T is actually uniformly continuous.

72. It suffices to show that $I_{[-1/2,f]}\in L^2(S)$. Write

$$\int_{-1/2}^{1/2}\big|I_{[-1/2,f]}(\nu)\big|^2 S(\nu)\,d\nu = \int_{-1/2}^{f}S(\nu)\,d\nu \leq \int_{-1/2}^{1/2}S(\nu)\,d\nu = R(0) = \mathsf{E}[X_n^2]<\infty.$$

73. (a) We know that for trigonometric polynomials, $\mathsf{E}[T(G)]=0$. Hence, if G_n is a sequence of trigonometric polynomials converging to G in $L^2(S)$, then $T(G_n)\to T(G)$ in L^2, and then $\mathsf{E}[T(G_n)]\to\mathsf{E}[T(G)]$.

(b) For trigonometric polynomials G and H, we have

$$\langle T(G),T(H)\rangle := \mathsf{E}[T(G)T(H)^*] = \int_{-1/2}^{1/2}G(f)H(f)^*S(f)\,df.$$

If G_n and H_n are sequences of trigonometric polynomials converging to G and H in $L^2(S)$, then we can use the result of Problem 24 to write

$$\begin{aligned}\langle T(G), T(H)\rangle &= \lim_{n\to\infty}\langle T(G_n), T(H_n)\rangle = \lim_{n\to\infty}\int_{-1/2}^{1/2} G_n(f)H_n(f)^* S(f)\,df \\ &= \int_{-1/2}^{1/2} G(f)H(f)^* S(f)\,df.\end{aligned}$$

(c) For $-1/2 \le f_1 < f_2 \le f_3 < f_4 \le 1/2$, write

$$\begin{aligned}\mathsf{E}[(Z_{f_2}-Z_{f_1})(Z_{f_4}-Z_{f_3})^*] &= \mathsf{E}[T(I_{(f_1,f_2]})T(I_{(f_3,f_4]})^*] \\ &= \int_{-1/2}^{1/2} I_{(f_1,f_2]}(f) I_{(f_3,f_4]}(f) S(f)\,df = 0.\end{aligned}$$

74. (a) We take

$$L^2(S_Y) := \left\{G : \int_{-1/2}^{1/2} |G(f)|^2 S_Y(f)\,df < \infty\right\}.$$

(b) Write

$$\begin{aligned}T_Y(G_0) &= \sum_{n=-N}^{N} g_n Y_n = \sum_{n=-N}^{N} g_n \sum_{k=-\infty}^{\infty} h_k X_{n-k} \\ &= \sum_{n=-N}^{N} g_n \sum_{k=-\infty}^{\infty} h_k \int_{-1/2}^{1/2} e^{j2\pi f(n-k)}\,dZ_f \\ &= \int_{-1/2}^{1/2} G_0(f)H(f)\,dZ_f.\end{aligned}$$

(c) For $G \in L^2(S_Y)$, let G_n be a sequence of trigonometric polynomials converging to G in $L^2(S_Y)$. Since T_Y is continuous,

$$\begin{aligned}T_Y(G) &= \lim_{n\to\infty} T_Y(G_n) = \lim_{n\to\infty}\int_{-1/2}^{1/2} G_n(f)H(f)\,dZ_f \\ &= \lim_{n\to\infty} T(G_nH) = T(GH), \quad \text{since } T \text{ is continuous,} \\ &= \int_{-1/2}^{1/2} G(f)H(f)\,dZ_f.\end{aligned}$$

(d) Using part (c),

$$\begin{aligned}V_f := T_Y(I_{[-1/2,f]}) &= \int_{-1/2}^{1/2} I_{[-1/2,f]}(\nu)H(\nu)\,dZ_\nu \qquad (*) \\ &= \int_{-1/2}^{f} H(\nu)\,dZ_\nu.\end{aligned}$$

(e) A slight generalization of $(*)$ establishes the result for piecewise-constant functions. For general $G \in L^2(S_Y)$, approximate G by a sequence of piecewise-constant functions G_n and write

$$\begin{aligned}\int_{-1/2}^{1/2} G(f)\,dV_f &= \lim_{n\to\infty}\int_{-1/2}^{1/2} G_n(f)\,dV_f = \lim_{n\to\infty}\int_{-1/2}^{1/2} G_n(f)H(f)\,dZ_f \\ &= \lim_{n\to\infty} T(G_nH) = T(GH) = \int_{-1/2}^{1/2} G(f)H(f)\,dZ_f.\end{aligned}$$

CHAPTER 14
Problem Solutions

1. We must show that $\mathsf{P}(|X_n| \geq \varepsilon) \to 0$. Since $X_n \sim \text{Cauchy}(1/n)$ has an even density, we can write

$$\mathsf{P}(|X_n| \geq \varepsilon) \;=\; 2\int_{\varepsilon}^{\infty} \frac{1/(n\pi)}{(1/n)^2 + x^2}\,dx \;=\; 2\int_{n\varepsilon}^{\infty} \frac{1/\pi}{1+y^2}\,dy \;\to\; 0.$$

2. Since $|c_n - c| \to 0$, given $\varepsilon > 0$, there is an N such that for $n \geq N$, $|c_n - c| < \varepsilon$. For such n,

$$\{\omega \in \Omega : |c_n - c| \geq \varepsilon\} \;=\; \varnothing,$$

and so $\mathsf{P}(|c_n - c| \geq \varepsilon) = 0$.

3. We show that X_n converges in probability to zero. Observe that X_n takes only the values n and zero. Hence, for $0 < \varepsilon < 1$, $|X_n| \geq \varepsilon$ if and only if $X_n = n$, which happens if and only if $U \leq 1/\sqrt{n}$. We can now write

$$\mathsf{P}(|X_n| \geq \varepsilon) \;=\; \mathsf{P}(U \leq 1/\sqrt{n}\,) \;=\; 1/\sqrt{n} \;\to\; 0.$$

4. Write

$$\mathsf{P}(|X_n| \geq \varepsilon) \;=\; \mathsf{P}(|V| \geq \varepsilon c_n) \;=\; 2\int_{\varepsilon c_n}^{\infty} f_V(v)\,dv \;\to\; 0,$$

since $c_n \to \infty$.

5. Using the hint,

$$\mathsf{P}(X \neq Y) \;=\; \mathsf{P}\biggl(\bigcup_{k=1}^{\infty}\{|X-Y| \geq 1/k\}\biggr) \;=\; \lim_{K\to\infty} \mathsf{P}(|X-Y| \geq 1/K).$$

We now show that the above limit is zero. To begin, observe that

$$|X_n - X| < 1/(2K) \quad \text{and} \quad |X_n - Y| < 1/(2K)$$

imply

$$|X-Y| = |X - X_n + X_n - Y| \leq |X - X_n| + |X_n - Y| < 1/(2K) + 1/(2K) = 1/K.$$

Hence, $|X-Y| \geq 1/K$ implies $|X_n - X| \geq 1/(2K)$ or $|X_n - Y| \geq 1/(2K)$, and we can write

$$\begin{aligned}
\mathsf{P}(|X-Y| \geq 1/K) \;&\leq\; \mathsf{P}\bigl(\{|X_n - X| \geq 1/(2K)\} \cup \{|X_n - Y| \geq 1/(2K)\}\bigr) \\
&\leq\; \mathsf{P}\bigl(\{|X_n - X| \geq 1/(2K)\}\bigr) + \mathsf{P}\bigl(\{|X_n - Y| \geq 1/(2K)\}\bigr),
\end{aligned}$$

which goes to zero as $n \to \infty$

6. Let $\varepsilon > 0$ be given. We must show that for every $\eta > 0$, for all sufficiently large n, $\mathsf{P}(|X_n - X| \geq \varepsilon) < \eta$. Without loss of generality, assume $\eta < \varepsilon$. Let n be so large that $\mathsf{P}(|X_n - X| \geq \eta) < \eta$. Then since $\eta < \varepsilon$,

$$\mathsf{P}(|X_n - X| \geq \varepsilon) \;\leq\; \mathsf{P}(|X_n - X| \geq \eta) \;<\; \eta.$$

7. (a) Observe that

$$\begin{aligned} \mathsf{P}(|X| > \alpha) &= 1 - \mathsf{P}(|X| \leq \alpha) \;=\; 1 - \mathsf{P}(-\alpha \leq X \leq \alpha) \\ &= 1 - [F_X(\alpha) - F_X((-\alpha)-)] \;=\; 1 - F_X(\alpha) + F_X((-\alpha)-) \\ &\leq 1 - F_X(\alpha) + F_X(-\alpha). \end{aligned}$$

Now, since $F_X(x) \to 1$ as $x \to \infty$ and $F_X(x) \to 0$ as $x \to -\infty$, for large α, $1 - F_X(\alpha) < \varepsilon/8$ and $F_X(-\alpha) < \varepsilon/8$, and then $\mathsf{P}(|X| > \alpha) < \varepsilon/4$. Similarly, for large β, $\mathsf{P}(|Y| > \beta) < \varepsilon/4$.

(b) Observe that if the four conditions hold, then

$$|X_n| = |X_n - X + X| \leq |X_n - X| + |X| < \delta + \alpha < 2\alpha,$$

and similarly, $|Y_n| \leq 2\beta$. Now that both (X_n, Y_n) and (X, Y) lie in the rectangle,

$$|g(X_n, Y_n) - g(X, Y)| \;<\; \varepsilon.$$

(c) By part (b), observe that

$$\begin{aligned} \{|g(X_n, Y_n) - g(X, Y)| \geq \varepsilon\} \;\subset\; & \{|X_n - X| \geq \delta\} \cup \{|Y_n - Y| \geq \delta\} \\ & \cup \{|X| > \alpha\} \cup \{|Y| > \beta\}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathsf{P}(|g(X_n, Y_n) - g(X, Y)| \geq \varepsilon) &\leq \mathsf{P}(|X_n - X| \geq \delta) + \mathsf{P}(|Y_n - Y| \geq \delta) \\ &\quad + \mathsf{P}(|X| > \alpha) + \mathsf{P}(|Y| > \beta) \\ &< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 \;=\; \varepsilon. \end{aligned}$$

8. (a) Since the X_i are i.i.d., the X_i^2 are i.i.d. and therefore uncorrelated and have common mean $\mathsf{E}[X_i^2] = \sigma^2 + m^2$ and common variance

$$\mathsf{E}\big[\big(X_i^2 - \mathsf{E}[X_i^2]\big)^2\big] \;=\; \mathsf{E}[X_i^4] - \big(\mathsf{E}[X_i^2]\big)^2 \;=\; \gamma^4 - (\sigma^2 + m^2)^2.$$

By the weak law of large numbers, V_n converges in probability to $\mathsf{E}[X_i^2] = \sigma^2 + m^2$.

(b) Observe that $S_n^2 = g(M_n, V_n)n/(n-1)$, where $g(m, v) := v - m^2$ is continuous. Hence, by the preceding problem, $g(M_n, V_n)$ converges in probability to $g(m, v) = (\sigma^2 + m^2) - m^2 = \sigma^2$. Next, by Problem 2, $n/(n-1)$ converges in probability to 1, and then the product $[n/(n-1)]g(M_n, V_n)$ converges in probability to $1 \cdot g(m, v) = \sigma^2$.

9. (a) Since X_n converges in probability to X, with $\varepsilon = 1$ we have $\mathsf{P}(|X_n - X| \geq 1) \to 0$ as $n \to \infty$. Now, if $|X - X_n| < 1$, then $\big||X| - |X_n|\big| < 1$, and it follows that

$$|X| \;<\; 1 + |X_n| \;\leq\; 1 + |Y|.$$

Equivalently,

$$|X| \;\geq\; |Y| + 1 \quad \text{implies} \quad |X_n - X| \geq 1.$$

Hence,

$$\mathsf{P}(|X| \geq |Y| + 1) \;\leq\; \mathsf{P}(|X_n - X| \geq 1) \;\to\; 0.$$

(b) Following the hint, write

$$\begin{aligned}
\mathsf{E}[|X_n - X|] \;&=\; \mathsf{E}[|X_n - X| I_{A_n}] + \mathsf{E}[|X_n - X| I_{A_n^c}] \\
&\leq\; \mathsf{E}\big[\big(|X_n| + |X|\big) I_{A_n}\big] + \varepsilon \mathsf{P}(A_n^c) \\
&\leq\; \mathsf{E}\big[\big(Y + |X|\big) I_{A_n}\big] + \varepsilon \\
&\leq\; \varepsilon + \varepsilon,
\end{aligned}$$

since for large n, $\mathsf{P}(A_n) < \delta$ implies $\mathsf{E}\big[\big(Y + |X|\big) I_{A_n}\big] < \varepsilon$. Hence, $\mathsf{E}[|X_n - X|] < 2\varepsilon$.

10. (a) Suppose $g(x)$ is bounded, nonnegative, and $g(x) \to 0$ as $x \to 0$. Then given $\varepsilon > 0$, there exists a $\delta > 0$ such that $g(x) < \varepsilon/2$ for all $|x| < \delta$. For $|x| \geq \delta$, we use the fact that g is bounded to write $g(x) \leq G$ for some positive, finite G. Since X_n converges to zero in probability, for large n, $\mathsf{P}(|X_n| \geq \delta) < \varepsilon/(2G)$. Now write

$$\begin{aligned}
\mathsf{E}[g(X_n)] \;&=\; \mathsf{E}[g(X_n) I_{[0,\delta)}(|X_n|)] + \mathsf{E}[g(X_n) I_{[\delta,\infty)}(|X_n|)] \\
&\leq\; \mathsf{E}[(\varepsilon/2) I_{[0,\delta)}(|X_n|)] + \mathsf{E}[G I_{[\delta,\infty)}(|X_n|)] \\
&=\; \frac{\varepsilon}{2} \mathsf{P}(|X_n| < \delta) + G \mathsf{P}(|X_n| \geq \delta) \\
&<\; \frac{\varepsilon}{2} + G \frac{\varepsilon}{2G} \;=\; \varepsilon.
\end{aligned}$$

(b) By applying part (a) to the function $g(x) = x/(1+x)$, it follows that if X_n converges in probability to zero, then

$$\lim_{n \to \infty} \mathsf{E}\left[\frac{|X_n|}{1 + |X_n|}\right] \;=\; 0.$$

Now we show that if the above limit holds, then X_n must converge in probability to zero. Following the hint, we use the fact that $g(x) = 1/(1+x)$ is an increasing function for $x \geq 0$. Write

$$\begin{aligned}
\mathsf{E}[g(X_n)] \;&=\; \mathsf{E}[g(X_n) I_{[\varepsilon,\infty)}(|X_n|)] + \mathsf{E}[g(X_n) I_{[0,\varepsilon)}(|X_n|)] \\
&\geq\; \mathsf{E}[g(X_n) I_{[\varepsilon,\infty)}(|X_n|)] \\
&\geq\; \mathsf{E}[g(\varepsilon) I_{[\varepsilon,\infty)}(|X_n|)] \\
&=\; g(\varepsilon) \mathsf{P}(|X_n| \geq \varepsilon).
\end{aligned}$$

Thus, if $g(x)$ is nonnegative and nondecreasing, if $\mathsf{E}[g(X_n)] \to 0$, then X_n converges in distribution to zero.

11. First note that for the constant random variable $Y \equiv c$, $F_Y(y) = I_{[c,\infty)}(y)$. Similarly, for $Y_n \equiv c_n$, $F_{Y_n}(y) = I_{[c_n,\infty)}(y)$. Since the only point at which F_Y is not continuous is $y = c$, we must show that $I_{[c_n,\infty)}(y) \to I_{[c,\infty)}(y)$ for all $y \neq c$. Consider a y with $c < y$. For all sufficiently large n, c_n will be very close to c — so close that $c_n < y$, which implies $F_{Y_n}(y) = 1 = F_Y(y)$. Now consider $y < c$. For all sufficiently large n, c_n will be very close to c — so close that $y < c_n$, which implies $F_{Y_n}(y) = 0 = F_Y(y)$.

12. For $0 < c < \infty$, $F_Y(y) = \mathsf{P}(cX \leq y) = F_X(y/c)$, and $F_{Y_n}(y) = F_X(y/c_n)$. Now, y is a continuity point of F_Y if and only if y/c is a continuity point of F_X. For such y, since $y/c_n \to y/c$, $F_X(y/c_n) \to F_X(y/c)$. For $c = 0$, $Y \equiv 0$, and $F_Y(y) = I_{[0,\infty)}(y)$. For $y \neq 0$,

$$y/c_n \;\to \begin{cases} +\infty, & y > 0, \\ -\infty, & y < 0, \end{cases}$$

and so

$$F_{Y_n}(y) \;=\; F_X(y/c_n) \;\to\; \begin{cases} 1, & y > 0, \\ 0, & y < 0, \end{cases}$$

which is exactly $F_Y(y)$ for $y \neq 0$.

13. Since $X_t \leq Y_t \leq Z_t$,

$$\{Z_t \leq y\} \;\subset\; \{Y_t \leq y\} \;\subset\; \{X_t \leq y\},$$

we can write

$$\mathsf{P}(Z_t \leq y) \;\leq\; \mathsf{P}(Y_t \leq y) \;\leq\; \mathsf{P}(X_t \leq y),$$

or $F_{Z_t}(y) \leq F_{Y_t}(y) \leq F_{X_t}(y)$, and it follows that

$$\underbrace{\lim_{t\to\infty} F_{Z_t}(y)}_{=\,F(y)} \;\leq\; \lim_{t\to\infty} F_{Y_t}(y) \;\leq\; \underbrace{\lim_{t\to\infty} F_{X_t}(y)}_{=\,F(y)}.$$

Thus, $F_{Y_t}(y) \to F(y)$.

14. Since X_n converges in mean to X, the inequality

$$\big|\mathsf{E}[X_n] - \mathsf{E}[X]\big| \;=\; \big|\mathsf{E}[X_n - X]\big| \;\leq\; \mathsf{E}[|X_n - X|]$$

shows that $\mathsf{E}[X_n] \to \mathsf{E}[X]$. We now need the following implications:

conv. in mean $\Rightarrow$ conv. in probability $\Rightarrow$ conv. in distribution $\Rightarrow \varphi_{X_n}(\nu) \to \varphi_X(\nu)$.

Since X_n is exponential, we also have

$$\varphi_{X_n}(\nu) \;=\; \frac{1/\mathsf{E}[X_n]}{1/\mathsf{E}[X_n] - j\nu} \;\to\; \frac{1/\mathsf{E}[X]}{1/\mathsf{E}[X] - j\nu}.$$

Since limits are unique, the above right-hand side must be $\varphi_X(\nu)$, which implies X is an exponential random variable.

15. Since X_n and Y_n each converge in distribution to constants x and y, respectively, they also converge in probability. Hence, as noted in the text, $X_n + Y_n$ converges in probability to $x + y$. Since convergence in probability implies convergence in distribution, $X_n + Y_n$ converges in distribution to $x + y$.

16. Following the hint, we note that each Y_n is a finite linear combination of independent Gaussian increments. Hence, each Y_n is Gaussian. Since Y is the mean-square limit of the Gaussian Y_n, the distribution of Y is also Gaussian by the example cited in the hint. Furthermore, since each $Y_n = \int_0^\infty g_n(\tau)\,dW_\tau$, Y_n has zero mean and variance $\sigma^2 \int_0^\infty g_n(\tau)^2\,d\tau = \sigma^2\|g_n\|^2$. By the cited example, Y has zero mean. Also, since

$$\big|\|g_n\| - \|g\|\big| \;\le\; \|g_n - g\| \;\to\; 0,$$

we have

$$\begin{aligned}\mathsf{var}(Y) \;=\; \mathsf{E}[Y^2] \;=\; \lim_{n\to\infty}\mathsf{E}[Y_n^2] \;&=\; \lim_{n\to\infty}\mathsf{var}(Y_n)\\ &=\; \lim_{n\to\infty}\sigma^2\|g_n\|^2 \;=\; \sigma^2\|g\|^2 \;=\; \sigma^2\int_0^\infty g(\tau)^2\,d\tau.\end{aligned}$$

17. For any constants $c_1,\ldots,c_n$, write

$$\sum_{i=1}^n c_i X_{t_i} \;=\; \sum_{i=1}^n c_i\int_0^\infty g(t_i,\tau)\,dW_\tau \;=\; \int_0^\infty \underbrace{\bigg(\sum_{i=1}^n c_i g(t_i,\tau)\bigg)}_{=:\,g(\tau)}\,dW_\tau,$$

which is normal by the previous problem.

18. The plan is to show that the increments are Gaussian and uncorrelated. It will then follow that the increments are independent. For $0 \le u < v \le s < t < \infty$, write

$$\begin{bmatrix} X_v - X_u \\ X_t - X_s \end{bmatrix} \;=\; \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} X_u \\ X_v \\ X_s \\ X_t \end{bmatrix}.$$

By writing

$$X_t \;:=\; \int_0^t g(\tau)\,dW_\tau \;=\; \int_0^\infty g(\tau)I_{[0,t]}(\tau)\,dW_\tau \;=\; \int_0^\infty h(t,\tau)\,dW_\tau,$$

where $h(t,\tau) := g(\tau)I_{[0,t]}(\tau)$, we see that by the preceding problem, the vector on the right above is Gaussian, and hence, so are the increments in the vector on the left. Next,

$$\begin{aligned}\mathsf{E}[(X_t - X_s)(X_v - X_u)] \;&=\; \mathsf{E}\bigg[\int_s^t g(\tau)\,dW_\tau \int_u^v g(\tau)\,dW_\tau\bigg]\\ &=\; \mathsf{E}\bigg[\int_0^\infty g(\tau)I_{(s,t]}(\tau)\,dW_\tau \int_0^\infty g(\tau)I_{(u,v]}(\tau)\,dW_\tau\bigg]\\ &=\; \sigma^2\int_0^\infty g(\tau)^2 I_{(s,t]}(\tau)I_{(u,v]}(\tau)\,d\tau \;=\; 0.\end{aligned}$$

19. Given a linear combination $\sum_{k=1}^n c_k A_k$, put $g(\tau) := \sum_{k=1}^n c_k\varphi_k(\tau)$. Then

$$\sum_{k=1}^n c_k A_k = \sum_{k=1}^n c_k\bigg[\int_a^b X_\tau\varphi_k(\tau)\,d\tau\bigg] = \int_a^b X_\tau\bigg[\sum_{k=1}^n c_k\varphi_k(\tau)\bigg]d\tau = \int_a^b g(\tau)X_\tau\,d\tau,$$

which is a mean-square limit of sums of the form

$$\sum_i g(\tau_i) X_{\tau_i}(t_i - t_{i-1}).$$

Since X_τ is a Gaussian process, these sums are Gaussian, and hence, so is their mean-square limit.

20. If $M_{X_n}(s) \to M_X(s)$, then this holds when $s = j\nu$; i.e., $\varphi_{X_n}(\nu) \to \varphi_X(\nu)$. But this implies that X_n converges in distribution to X. Similarly, if $G_{X_n}(z) \to G_X(z)$, then this holds when $z = e^{j\nu}$; i.e., $\varphi_{X_n}(\nu) \to \varphi_X(\nu)$. But this implies X_n converges in distribution to X.

21. (a) Write

$$\begin{aligned} p_n(k) &= F_{X_n}(k+1/2) - F_{X_n}(k-1/2) \\ &\to F_X(k+1/2) - F_X(k-1/2) \;=\; p(k), \end{aligned}$$

where we have used the fact that since X is integer valued, $k \pm 1/2$ is a continuity point of F_X.

(b) The continuity points of F_X are the noninteger values of x. For such $x > 0$, suppose $k < x < k+1$. Then

$$F_{X_n}(x) = \mathsf{P}(X_n \le x) = \mathsf{P}(X_n \le k) = \sum_{i=0}^{k} p_n(i) \to \sum_{i=0}^{k} p(i) = \mathsf{P}(X \le k) = F_X(x).$$

22. Let

$$p_n(k) \;:=\; \mathsf{P}(X_n = k) \;=\; \binom{n}{k} p_n^k (1-p_n)^{n-k} \;=\; \frac{n!}{k!(n-k)!} p_n^k (1-p_n)^{n-k}$$

and $p(k) := \mathsf{P}(X = k) = \lambda^k e^{-\lambda}/k!$. Next, by Stirling's formula, as $n \to \infty$,

$$q_n \;:=\; \frac{\sqrt{2\pi}\, n^{n+1/2} e^{-n}}{n!} \to 1 \quad \text{and} \quad r_n(k) \;:=\; \frac{(n-k)!}{\sqrt{2\pi}(n-k)^{n-k+1/2} e^{-(n-k)}} \;\to\; 1,$$

and so $q_n r_n(k) \to 1$ as well. If we can show that $p_n(k) q_n r_n(k) \to p(k)$, then

$$\lim_{n\to\infty} p_n(k) \;=\; \lim_{n\to\infty} \frac{p_n(k) q_n r_n(k)}{q_n r_n(k)} \;=\; \frac{\lim_{n\to\infty} p_n(k) q_n r_n(k)}{\lim_{n\to\infty} q_n r_n(k)} \;=\; \frac{p(k)}{1} \;=\; p(k).$$

Now write

$$\begin{aligned} \lim_{n\to\infty} p_n(k) q_n r_n(k) &= \lim_{n\to\infty} p_n(k) \frac{\sqrt{2\pi}\, n^{n+1/2} e^{-n}}{n!} \cdot \frac{(n-k)!}{\sqrt{2\pi}(n-k)^{n-k+1/2} e^{-(n-k)}} \\ &= \frac{e^{-k}}{k!} \lim_{n\to\infty} p_n^k (1-p_n)^{n-k} \cdot \frac{n^{n+1/2}}{(n-k)^{n-k+1/2}} \end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-k}}{k!} \lim_{n\to\infty} p_n^k (1-p_n)^{n-k} \cdot \frac{n^{n+1/2}}{n^{n-k+1/2}(1-k/n)^{n-k+1/2}} \\
&= \frac{e^{-k}}{k!} \lim_{n\to\infty} \frac{(np_n)^k}{(1-k/n)^{-k+1/2}} \cdot \frac{(1-p_n)^{n-k}}{(1-k/n)^n} \\
&= \frac{e^{-k}}{k!} \lim_{n\to\infty} \frac{(np_n)^k}{(1-k/n)^{-k+1/2}} \cdot \frac{[1-(np_n)/n]^n (1-p_n)^{-k}}{(1-k/n)^n} \\
&= \frac{e^{-k}}{k!} \cdot \frac{\lambda^k}{1} \cdot \frac{e^{-\lambda}\cdot 1}{e^{-k}} \;=\; \frac{\lambda^k e^{-\lambda}}{k!} \;=\; p(k).
\end{aligned}$$

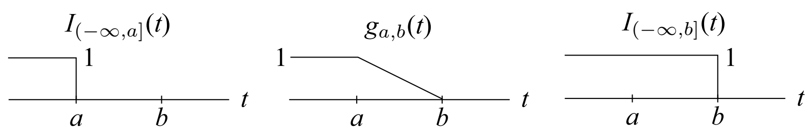
Note that since $np_n \to \lambda$, $p_n \to 0$, and so $(1-p_n)^{-k} \to 1$.

23. First write

$$G_{X_n}(z) \;=\; [(1-p_n)+p_n z]^n \;=\; [1+p_n(z-1)]^n \;=\; \left[1+\frac{np_n(z-1)}{n}\right]^n.$$

Since $np_n(z-1) \to \lambda(z-1)$, $G_{X_n}(z) \to e^{\lambda(z-1)} = G_X(z)$.

24. (a) Here are the sketches:



(b) From part (a) we can write

$$I_{(-\infty,a]}(Y) \;\le\; g_{a,b}(Y) \;\le\; I_{(-\infty,b]}(Y),$$

from which it follows that

$$\underbrace{\mathsf{E}[I_{(-\infty,a]}(Y)]}_{=\,\mathsf{P}(Y\le a)} \;\le\; \mathsf{E}[g_{a,b}(Y)] \;\le\; \underbrace{\mathsf{E}[I_{(-\infty,b]}(Y)]}_{=\,\mathsf{P}(Y\le b)},$$

or

$$F_Y(a) \;\le\; \mathsf{E}[g_{a,b}(Y)] \;\le\; F_Y(b).$$

(c) Since $F_{X_n}(x) \le \mathsf{E}[g_{x,x+\Delta x}(X_n)]$,

$$\overline{\lim_{n\to\infty}} F_{X_n}(x) \;\le\; \lim_{n\to\infty} \mathsf{E}[g_{x,x+\Delta x}(X_n)] \;=\; \mathsf{E}[g_{x,x+\Delta x}(X)] \;\le\; F_X(x+\Delta x).$$

(d) Similarly,

$$F_X(x-\Delta x) \;\le\; \mathsf{E}[g_{x-\Delta x,x}(X)] \;=\; \lim_{n\to\infty} \mathsf{E}[g_{x-\Delta x,x}(X_n)] \;\le\; \underline{\lim_{n\to\infty}} F_{X_n}(x).$$

(e) If x is a continuity point of F_X, then given any $\varepsilon > 0$, for sufficiently small Δx,

$$F_X(x)-\varepsilon \;<\; F_X(x-\Delta x) \quad \text{and} \quad F_X(x+\Delta x) \;<\; F_X(x)+\varepsilon.$$

Combining this with parts (c) and (d), we obtain

$$F_X(x) - \varepsilon \;\le\; \varliminf_{n\to\infty} F_{X_n}(x) \;\le\; \varlimsup_{n\to\infty} F_{X_n}(x) \;<\; F_X(x) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the liminf and the limsup are equal, $\lim_{n\to\infty} F_{X_n}(x)$ exists and is equal to $F_X(x)$.

25. If X_n converges in distribution to zero, then X_n converges in probability to zero, and by Problem 10,

$$\mathsf{E}\left[\frac{|X_n|}{1+|X_n|}\right] \;=\; 0.$$

Conversely, if the above limit holds, then by Problem 10, X_n converges in probability to zero, which implies convergence in distribution to zero.

26. Observe that $f_n(x) = nf(nx)$ implies

$$F_n(x) \;=\; F(nx) \;\to\; \begin{cases} 1, & x > 0, \\ F(0), & x = 0, \\ 0, & x < 0. \end{cases}$$

So, for $x \neq 0$, $F_n(x) \to I_{[0,\infty)}(x)$, which is the cdf of $X \equiv 0$. In other words, X_n converges in distribution to zero, which implies convergence in probability to zero.

27. Since X_n converges in mean square to X, X_n converges in distribution to X. Since $g(x) := x^2 e^{-x^2}$ is a bounded continuous function, $\mathsf{E}[g(X_n)] \to \mathsf{E}[g(X)]$.

28. Let $0 < \delta < 1$ be given. For large t, we have $|u(t) - 1| < \delta$, or

$$-\delta \;<\; u(t) - 1 \;<\; \delta \quad \text{or} \quad 1 - \delta \;<\; u(t) \;<\; 1 + \delta.$$

Hence, for $z > 0$,

$$\mathsf{P}(Z_t \le z(1-\delta)) \;\le\; \mathsf{P}(Z_t \le zu(t)) \;\le\; \mathsf{P}(Z_t \le z(1+\delta)).$$

Rewrite this as

$$F_{Z_t}(z(1-\delta)) \;\le\; \mathsf{P}(Z_t \le zu(t)) \;\le\; F_{Z_t}(z(1+\delta)).$$

Then

$$F(z(1-\delta)) \;=\; \lim_{t\to\infty} F_{Z_t}(z(1-\delta)) \;\le\; \varliminf_{t\to\infty} \mathsf{P}(Z_t \le zu(t))$$

and

$$\varlimsup_{t\to\infty} \mathsf{P}(Z_t \le zu(t)) \;\le\; \lim_{t\to\infty} F_{Z_t}(z(1+\delta)) \;=\; F_Z(z(1+\delta)).$$

Since $0 < \delta < 1$ is arbitrary, and since F is continuous, we must have

$$\lim_{t\to\infty} \mathsf{P}(Z_t \le zu(t)) \;=\; F(z).$$

The case for $z < 0$ is similar.

29. Rewrite $\mathsf{P}(c(t)Z_t \le z)$ as $\mathsf{P}(Z_t \le z/c(t)) = \mathsf{P}(Z_t \le (z/c)(c/c(t))$. Then if we put $u(t) := c/c(t)$, we have by the preceding problem that $\mathsf{P}(c(t)Z_t \le z) \to F(z/c)$.

30. First write $F_{X_t}(x) = \mathsf{P}(Z_t + s(t) \le x) = \mathsf{P}(Z_t \le x - s(t))$. Let $\varepsilon > 0$ be given. Then $s(t) \to 0$ implies that for large t, $|s(t)| < \varepsilon$, or

$$-\varepsilon < s(t) < \varepsilon \quad \text{or} \quad -\varepsilon < -s(t) < \varepsilon.$$

Then

$$F_{Z_t}(x-\varepsilon) = \mathsf{P}(Z_t \le x-\varepsilon) \le F_{X_t}(x) \le \mathsf{P}(Z_t \le x+\varepsilon) = F_{Z_t}(x+\varepsilon).$$

It follows that

$$F(x-\varepsilon) \le \varliminf_{t\to\infty} F_{X_t}(x) \le \varlimsup_{t\to\infty} F_{X_t}(x) \le F(x+\varepsilon).$$

Since F is continuous and $\varepsilon > 0$ is arbitrary,

$$\lim_{t\to\infty} F_{X_t}(x) = F(x).$$

31. Starting with $N_{\lfloor t \rfloor} \le N_t \le N_{\lceil t \rceil}$, it is easy to see that

$$X_t := \frac{\frac{N_{\lfloor t \rfloor}}{t} - \lambda}{\sqrt{\lambda/t}} \le Y_t \le \frac{\frac{N_{\lceil t \rceil}}{t} - \lambda}{\sqrt{\lambda/t}} =: Z_t.$$

According to Problem 13, it suffices to show that X_t and Z_t converge in distribution to $N(0,1)$ random variables. By the preceding two problems, the distribution limit of Z_t is the same as that of $c(t)Z_t + s(t)$ if $c(t) \to 1$ and $s(t) \to 0$. We take

$$c(t) := \frac{t/\lceil t \rceil}{\sqrt{t/\lceil t \rceil}} = \sqrt{t/\lceil t \rceil} \to 1$$

and

$$s(t) := \frac{\lambda t/\lceil t \rceil}{\sqrt{\lambda/\lceil t \rceil}} - \frac{\lambda}{\sqrt{\lambda/\lceil t \rceil}} = \lambda \frac{\big(t/\lceil t \rceil - 1\big)}{\sqrt{\lambda}} \sqrt{\lceil t \rceil} \to 0.$$

Finally, observe that

$$c(t)Z_t + s(t) = \frac{\frac{N_{\lceil t \rceil}}{\lceil t \rceil} - \lambda}{\sqrt{\lambda/\lceil t \rceil}},$$

goes through the values of Y_n and therefore converges in distribution to an $N(0,1)$ random variable. It is similar to show that the distribution limit of X_t is also $N(0,1)$.

32. (a) Let $G := \{X_n \to X\}$. For $\omega \in G$,

$$\frac{1}{1+X_n(\omega)^2} \to \frac{1}{1+X(\omega)^2}.$$

Since $\mathsf{P}(G^c) = 0$, $1/(1+X_n^2)$ converges almost surely to $1/(1+X^2)$.

(b) Since almost sure convergence implies convergence in probability, which implies convergence in distribution, we have $1/(1+X_n^2)$ converging in distribution to $1/(1+X^2)$. Since $g(x) = 1/(1+x^2)$ is bounded and continuous, $\mathsf{E}[g(X_n)] \to \mathsf{E}[g(X)]$; i.e.,

$$\lim_{n\to\infty} \mathsf{E}\left[\frac{1}{1+X_n^2}\right] = \mathsf{E}\left[\frac{1}{1+X^2}\right].$$

33. Let $G_X := \{X_n \to X\}$ and $G_Y := \{Y_n \to Y\}$, where $\mathsf{P}(G_X^c) = \mathsf{P}(G_Y^c) = 0$. Let $G := \{g(X_n, Y_n) \to g(X, Y)\}$. We must show that $\mathsf{P}(G^c) = 0$. Our plan is to show that $G_X \cap G_Y \subset G$. It follows that $G^c \subset G_X^c \cup G_Y^c$, and we can then write

$$\mathsf{P}(G^c) \le \mathsf{P}(G_X^c) + \mathsf{P}(G_Y^c) = 0.$$

For $\omega \in G_X \cap G_Y$, $(X_n(\omega), Y_n(\omega)) \to (X(\omega), Y(\omega))$. Since $g(x,y)$ is continuous, for such ω, $g(X_n(\omega), Y_n(\omega)) \to g(X(\omega), Y(\omega))$. Thus, $G_X \cap G_Y \subset G$.

34. Let $G_X := \{X_n \to X\}$ and $G_{XY} := \{X = Y\}$, where $\mathsf{P}(G_X^c) = \mathsf{P}(G_{XY}^c) = 0$. Put $G_Y := \{X_n \to Y\}$. We must show that $\mathsf{P}(G_Y^c) = 0$. Our plan is to show that $G_X \cap G_{XY} \subset G_Y$. It follows that $G_Y^c \subset G_X^c \cup G_{XY}^c$, and we can then write

$$\mathsf{P}(G_Y^c) \le \mathsf{P}(G_X^c) + \mathsf{P}(G_{XY}^c) = 0.$$

For $\omega \in G_X \cap G_{XY}$, $X_n(\omega) \to X(\omega)$ and $X(\omega) = Y(\omega)$. Hence, for such ω, $X_n(\omega) \to Y(\omega)$; i.e., $\omega \in G_Y$. Thus, $G_X \cap G_{XY} \subset G_Y$.

35. Let $G_X := \{X_n \to X\}$ and $G_Y := \{X_n \to Y\}$, where $\mathsf{P}(G_X^c) = \mathsf{P}(G_Y^c) = 0$. Put $G_{XY} := \{X = Y\}$. We must show that $\mathsf{P}(G_{XY}^c) = 0$. Our plan is to show that $G_X \cap G_Y \subset G_{XY}$. It follows that $G_{XY}^c \subset G_X^c \cup G_Y^c$, and we can then write

$$\mathsf{P}(G_{XY}^c) \le \mathsf{P}(G_X^c) + \mathsf{P}(G_Y^c) = 0.$$

For $\omega \in G_X \cap G_Y$, $X_n(\omega) \to X(\omega)$ and $X_n(\omega) \to Y(\omega)$. Since limits of sequences of numbers are unique, for such ω, $X(\omega) = Y(\omega)$; i.e., $\omega \in G_{XY}$. Thus, $G_X \cap G_Y \subset G_{XY}$.

36. Let $G_X := \{X_n \to X\}$, $G_Y := \{Y_n \to Y\}$, and $G_I := \bigcap_{n=1}^\infty \{X_n \le Y_n\}$, where $\mathsf{P}(G_X^c) = \mathsf{P}(G_Y^c) = \mathsf{P}(G_I^c) = 0$. This last equality follows because

$$\mathsf{P}(G_I^c) \le \sum_{n=1}^\infty \mathsf{P}(X_n > Y_n) = 0.$$

Let $G := \{X \le Y\}$. We must show that $\mathsf{P}(G^c) = 0$. Our plan is to show that $G_X \cap G_Y \cap G_I \subset G$. It follows that $G^c \subset G_X^c \cup G_Y^c \cup G_I^c$, and we can then write

$$\mathsf{P}(G^c) \le \mathsf{P}(G_X^c) + \mathsf{P}(G_Y^c) + \mathsf{P}(G_I^c) = 0.$$

For $\omega \in G_X \cap G_Y \cap G_I$, $X_n(\omega) \to X(\omega)$, $Y_n(\omega) \to Y(\omega)$, and for all n, $X_n(\omega) \le Y_n(\omega)$. By properties of sequences of real numbers, for such ω, we must have $X(\omega) \le Y(\omega)$. Thus, $G_X \cap G_Y \cap G_I \subset G$.

37. Suppose X_n converges almost surely to X, and X_n converges in mean to Y. Then X_n converges in probability to X and to Y. By Problem 5, $X = Y$ a.s.

38. If $X(\omega) > 0$, then $c_n X(\omega) \to \infty$. If $X(\omega) < 0$, then $c_n X(\omega) \to -\infty$. If $X(\omega) = 0$, then $c_n X(\omega) = 0 \to 0$. Hence,

$$Y(\omega) \;=\; \begin{cases} +\infty, & \text{if } X(\omega) > 0, \\ 0, & \text{if } X(\omega) = 0, \\ -\infty, & \text{if } X(\omega) < 0. \end{cases}$$

39. By a limit property of probability, we can write

$$\begin{aligned} \mathsf{P}\bigg(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\{X_n = j\}\bigg| X_0 = i\bigg) \;&=\; \lim_{M\to\infty} \mathsf{P}\bigg(\bigcup_{n=M}^{\infty}\{X_n = j\}\bigg| X_0 = i\bigg) \\ &\le\; \lim_{M\to\infty} \sum_{n=M}^{\infty} \mathsf{P}(X_n = j | X_0 = i) \;=\; \lim_{M\to\infty} \sum_{n=M}^{\infty} p_{ij}^{(n)}, \end{aligned}$$

which must be zero since the $p_{ij}^{(n)}$ are summable.

40. Following the hint, write

$$\mathsf{P}(S = \infty) \;=\; \mathsf{P}\bigg(\bigcap_{n=1}^{\infty}\{S > n\}\bigg) \;=\; \lim_{N\to\infty} \mathsf{P}(S > N) \;\le\; \lim_{N\to\infty} \frac{\mathsf{E}[S]}{N} \;=\; 0.$$

41. Since S_n converges in mean to S, by the problems referred to in the hint,

$$\mathsf{E}[S] = \lim_{n\to\infty} \mathsf{E}[S_n] = \lim_{n\to\infty} \sum_{k=1}^{n} \mathsf{E}[|h_k||X_k|] = \lim_{n\to\infty} \sum_{k=1}^{n} |h_k| \mathsf{E}[|X_k|] \le B^{1/p} \sum_{k=1}^{\infty} |h_k| < \infty.$$

42. From the inequality

$$\int_n^{n+1} \frac{1}{t^p}\,dt \;\ge\; \int_n^{n+1} \frac{1}{(n+1)^p}\,dt \;=\; \frac{1}{(n+1)^p},$$

we obtain

$$\infty \;>\; \int_1^{\infty} \frac{1}{t^p}\,dt \;=\; \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{t^p}\,dt \;\ge\; \sum_{n=1}^{\infty} \frac{1}{(n+1)^p} \;=\; \sum_{n=2}^{\infty} \frac{1}{n^p},$$

which implies $\sum_{n=1}^{\infty} 1/n^p < \infty$.

43. Write

$$\begin{aligned} \mathsf{E}[M_n^4] \;&=\; \frac{1}{n^4} \mathsf{E}\bigg[\bigg(\sum_{i=1}^{n} X_i\bigg)\bigg(\sum_{j=1}^{n} X_j\bigg)\bigg(\sum_{l=1}^{n} X_l\bigg)\bigg(\sum_{m=1}^{n} X_m\bigg)\bigg] \\ &=\; \frac{1}{n^4}\bigg\{\sum_{i=1}^{n} \mathsf{E}[X_i^4] + \sum_{i=1}^{n}\sum_{l=1, l\neq i}^{n} \mathsf{E}[X_i X_i X_l X_l] + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \mathsf{E}[X_i X_j X_i X_j] \\ &\qquad + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \mathsf{E}[X_i X_j X_j X_i] + 4\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \underbrace{\mathsf{E}[X_i^3 X_j]}_{=\,0}\bigg\} \\ &=\; \frac{1}{n^4}[n\gamma + 3n(n-1)\sigma^4]. \end{aligned}$$

44. It suffices to show that $\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) < \infty$. To this end, write

$$\mathsf{P}(|X_n| \geq \varepsilon) \;=\; 2\int_{\varepsilon}^{\infty} \tfrac{n}{2} e^{-nx}\,dx \;=\; -e^{-nx}\Big|_{\varepsilon}^{\infty} \;=\; e^{-n\varepsilon}.$$

Then

$$\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) \;=\; \sum_{n=1}^{\infty} (e^{-\varepsilon})^n \;=\; \frac{e^{-\varepsilon}}{1-e^{-\varepsilon}} \;<\; \infty.$$

45. First use the Rayleigh cdf to write

$$\mathsf{P}(|X_n| \geq \varepsilon) \;=\; \mathsf{P}(X_n \geq \varepsilon) \;=\; e^{-(n\varepsilon)^2/2}.$$

Then

$$\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) \;=\; \sum_{n=1}^{\infty} e^{-\varepsilon^2 n^2/2} \;\leq\; \sum_{n=1}^{\infty} (e^{-\varepsilon^2/2})^n \;=\; \frac{1}{1-e^{-\varepsilon^2/2}} \;<\; \infty.$$

Thus, X_n converges almost surely to zero.

46. For $n > 1/\varepsilon$, write

$$\mathsf{P}(|X_n| \geq \varepsilon) \;=\; \int_{\varepsilon}^{\infty} \frac{p-1}{n^{p-1}} x^{-p}\,dx \;=\; \frac{1}{n^{p-1}\varepsilon^{p-1}}.$$

Then

$$\sum_{n>1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) \;=\; \sum_{n<1/\varepsilon} \mathsf{P}(|X_n| \geq \varepsilon) + \frac{1}{\varepsilon^{p-1}} \sum_{n>1/\varepsilon}^{\infty} \frac{1}{n^{p-1}} \;<\; \infty$$

since $p-1 > 1$. Thus, X_n converges almost surely to zero.

47. To begin, first observe that

$$\mathsf{E}[|X_n - X|] \;=\; \mathsf{E}[|n^2(-1)^n Y_n|] \;=\; n^2\mathsf{E}[Y_n] \;=\; n^2 p_n.$$

Also observe that

$$\mathsf{P}(|X_n - X| \geq \varepsilon) \;=\; \mathsf{P}(n^2 Y_n \geq \varepsilon) \;=\; \mathsf{P}(Y_n \geq \varepsilon/n^2) \;=\; \mathsf{P}(Y_n = 1) \;=\; p_n.$$

In order to have X_n converge almost surely to X, it is sufficient to consider p_n such that $\sum_{n=1}^{\infty} p_n < \infty$.

(a) If $p_n = 1/n^{3/2}$, then $\sum_{n=1}^{\infty} p_n < \infty$, but $n^2 p_n = n^{1/2} \not\to 0$. For this choice of p_n, X_n converges almost surely but not in mean to X.

(b) If $p_n = 1/n^3$, then $\sum_{n=1}^{\infty} p_n < \infty$ and $n^2 p_n = 1/n \to 0$. For this choice of p_n, X_n converges almost surely and in mean to X.

48. To apply the weak law of large numbers of this section to $(1/n)\sum_{i=1}^{n} X_i^2$ requires only that the X_i^2 be i.i.d. and have finite mean; there is no second-moment requirement on X_i^2 (which would be a requirement on X_i^4).

49. By writing
$$\frac{N_n}{n} = \frac{1}{n}\sum_{k=1}^{n} N_k - N_{k-1},$$
which is a sum of i.i.d. Poisson(λ) random variables, we see that N_n/n converges almost surely to λ by the strong law of large numbers. We next observe that $N_{\lfloor t\rfloor} \leq N_t \leq N_{\lceil t\rceil}$. Then
$$\frac{N_{\lfloor t\rfloor}}{t} \leq \frac{N_t}{t} \leq \frac{N_{\lceil t\rceil}}{t},$$
and it follows that
$$\frac{N_{\lfloor t\rfloor}}{\lceil t\rceil} \leq \frac{N_t}{t} \leq \frac{N_{\lceil t\rceil}}{\lfloor t\rfloor},$$
and then
$$\underbrace{\frac{\lfloor t\rfloor}{\lceil t\rceil}}_{\to 1}\cdot\underbrace{\frac{N_{\lfloor t\rfloor}}{\lfloor t\rfloor}}_{\to\lambda} \leq \frac{N_t}{t} \leq \underbrace{\frac{N_{\lceil t\rceil}}{\lceil t\rceil}}_{\to\lambda}\cdot\underbrace{\frac{\lceil t\rceil}{\lfloor t\rfloor}}_{\to 1}.$$
Hence N_t/t converges almost surely to λ.

50. (a) By the strong law of large numbers, for ω not in a set of probability zero,
$$\frac{1}{n}\sum_{k=1}^{n} X_k(\omega) \to \mu.$$
Hence, for $\varepsilon > 0$, for all sufficiently large n,
$$\left|\frac{1}{n}\sum_{k=1}^{n} X_k(\omega) - \mu\right| < \varepsilon,$$
which implies
$$\frac{1}{n}\sum_{k=1}^{n} X_k(\omega) - \mu < \varepsilon,$$
from which it follows that
$$\sum_{k=1}^{n} X_k(\omega) < n(\mu+\varepsilon).$$

(b) Given $M > 0$, let $\varepsilon > 0$ and choose n in part (a) so that both $n(\mu+\varepsilon) > M$ and $n \geq M$ hold. Then
$$T_n(\omega) = \sum_{k=1}^{n} X_k(\omega) < n(\mu+\varepsilon).$$
Now, for $t > n(\mu+\varepsilon) \geq T_n(\omega)$,
$$N_t(\omega) = \sum_{k=1}^{\infty} I_{[0,t]}(T_k(\omega)) \geq n \geq M.$$

(c) As noted in the solution of part (a), the strong law of large numbers implies

$$\frac{T_n}{n} = \frac{1}{n}\sum_{k=1}^{n} X_k \to \mu \quad \text{a.s.}$$

Hence, $n/T_n \to 1/\mu$ a.s.

(d) First observe that

$$Y_{N_t} = \frac{N_t}{T_{N_t}} \geq \frac{N_t}{t},$$

and so

$$\frac{1}{\mu} = \overline{\lim_{t\to\infty}} Y_{N_t} \geq \overline{\lim_{t\to\infty}} \frac{N_t}{t}.$$

Next,

$$Y_{N_t+1} = \frac{N_t+1}{T_{N_t+1}} \leq \frac{N_t+1}{t} = \frac{N_t}{t} + \frac{1}{t},$$

and so

$$\frac{1}{\mu} = \underline{\lim_{t\to\infty}} Y_{N_t+1} \leq \underline{\lim_{t\to\infty}} \frac{N_t}{t} + 0.$$

Hence,

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu}.$$

51. Let $X_n := nI_{(0,1/n]}(U)$, where $U \sim \text{uniform}(0,1]$. Then for every ω, $U(\omega) \in (0,1]$. For $n > 1/U(\omega)$, or $U(\omega) > 1/n$, $X_n(\omega) = 0$. Thus, $X_n(\omega) \to 0$ for every ω. However,

$$\mathsf{E}[X_n] = n\mathsf{P}(U \leq 1/n) = 1.$$

Hence, X_n does not converge in mean to zero.

52. Suppose D_ε contains n points, say $a \leq x_1 < \cdots < x_n \leq b$. Then

$$\begin{aligned}
n\varepsilon &< \sum_{k=1}^{n} G(x_k+) - G(x_k-) \\
&= G(x_n+) + \sum_{k=1}^{n-1} G(x_k+) - \sum_{k=2}^{n} G(x_k-) - G(x_1-) \\
&\leq G(b) + \sum_{k=1}^{n-1} G(x_{k+1}) - \sum_{k=2}^{n} G(x_{k-1}) - G(a) \\
&= G(b) + \sum_{k=1}^{n-1} [G(x_{k+1}) - G(x_k)] - G(a) \\
&\leq G(b) + G(x_n) - G(x_1) - G(a) \leq 2[G(b) - G(a)].
\end{aligned}$$

53. Write

$$\mathsf{P}\left(U \in \bigcup_{n=1}^{\infty} K_n\right) = \mathsf{P}\left(\bigcup_{n=1}^{\infty}\{U \in K_n\}\right) \leq \sum_{n=1}^{\infty} \mathsf{P}(U \in K_n) \leq \sum_{n=1}^{\infty} \frac{2\varepsilon}{2^n} = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the probability in question must be zero.

CHAPTER 15
Problem Solutions

1. Using the formula $F_{W_t}(x) = F_{W_1}(t^{-H}x)$, we see that

$$\lim_{t \downarrow 0} F_{W_t}(x) = \begin{cases} F_{W_1}(\infty) = 1, & x > 0, \\ F_{W_1}(-\infty) = 0, & x < 0, \end{cases}$$

which is the cdf of the zero random variable for $x \neq 0$. Hence, W_t converges in distribution to the zero random variable.

Next,

$$X(\omega) := \lim_{t \to \infty} t^H W_1(\omega) = \begin{cases} \infty, & \text{if } W_1(\omega) > 0, \\ 0, & \text{if } W_1(\omega) = 0, \\ -\infty, & \text{if } W_1(\omega) < 0. \end{cases}$$

Thus,

$$\mathsf{P}(X = \infty) = \mathsf{P}(W_1 > 0) = 1 - F_{W_1}(0), \quad \mathsf{P}(X = -\infty) = \mathsf{P}(W_1 < 0) = F_{W_1}(0-),$$

and

$$\mathsf{P}(X = 0) = \mathsf{P}(W_1 = 0) = F_{W_1}(0) - F_{W_1}(0-).$$

2. Since the joint characteristic function of a zero-mean Gaussian process is completely determined by the covariance matrix, we simply observe that

$$\mathsf{E}[W_{\lambda t_1} W_{\lambda t_2}] = \sigma^2 \min(\lambda t_1, \lambda t_2) = \lambda \sigma^2 \min(t_1, t_2),$$

and

$$\mathsf{E}[(\lambda^{1/2} W_{t_1})(\lambda^{1/2} W_{t_2})] = \lambda \sigma^2 \min(t_1, t_2).$$

3. Fix $\tau > 0$, and consider the process $Z_t := W_t - W_{t-\tau}$. Since the Wiener process is Gaussian with zero mean, so is the process Z_t. Hence, it suffices to consider the covariance

$$\mathsf{E}[Z_{t_1} Z_{t_2}] = \mathsf{E}[(W_{t_1} - W_{t_1-\tau})(W_{t_2} - W_{t_2-\tau})].$$

The time intervals involved do not overlap if $t_1 < t_2 - \tau$ or if $t_2 < t_1 - \tau$. Hence,

$$\mathsf{E}[Z_{t_1} Z_{t_2}] = \begin{cases} 0, & |t_2 - t_1| > \tau, \\ \sigma^2(|t_2 - t_1| + \tau), & |t_2 - t_1| \leq \tau, \end{cases}$$

which depends on t_1 and t_2 only through their difference.

4. For $H = 1/2$, $q_H(\theta) = I_{(0,\infty)}(\theta)$, and $C_H = 1$. So,

$$B_H(t) - B_H(s) = \int_{-\infty}^{\infty} [I_{(-\infty,t)}(\tau) - I_{(-\infty,s)}(\tau)]\, dW_\tau = W_t - W_s.$$

5. It suffices to show that

$$\int_1^\infty [(1+\theta)^{H-1/2} - \theta^{H-1/2}]^2 \, d\theta \;<\; \infty.$$

Consider the function $f(t) := t^{H-1/2}$. By the mean-value theorem of calculus,

$$(1+\theta)^{H-1/2} - \theta^{H-1/2} \;=\; f'(\hat{t}), \quad \text{for some } \hat{t} \in (\theta, \theta+1).$$

Since $f'(t) = (H-1/2)t^{H-3/2}$,

$$\big|(1+\theta)^{H-1/2} - \theta^{H-1/2}\big| \;\leq\; |H-1/2|/\theta^{3/2-H}.$$

Then

$$\int_1^\infty [(1+\theta)^{H-1/2} - \theta^{H-1/2}]^2 \, d\theta \;\leq\; (H-1/2)^2 \int_1^\infty \frac{1}{\theta^{3-2H}} \, d\theta \;<\; \infty,$$

since $3-2H > 1$.

6. The expression

$$\mathsf{P}\bigg(\bigg|\frac{M_n - \mu}{\sigma/n^{1-H}}\bigg| \leq y\bigg) \;=\; 1-\alpha$$

says that

$$\mu \;=\; M_n \pm \frac{y\sigma}{n^{1-H}} \quad \text{with probability } 1-\alpha.$$

Hence, the width of the confidence interval is

$$\frac{2y\sigma}{n^{1-H}} \;=\; \frac{2y\sigma}{\sqrt{n}} \cdot n^{H-1/2}.$$

7. Suppose $\mathsf{E}[Y_k^2] = \sigma^2 k^{2H}$ for $k = 1, \ldots, n$. Substituting this into the required formula yields

$$\big(\mathsf{E}[Y_{n+1}^2] - \sigma^2 n^{2H}\big) - \big(\sigma^2 n^{2H} - \sigma^2 (n-1)^{2H}\big) \;=\; 2C(n),$$

which we are assuming is equal to

$$\sigma^2 \big[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\big].$$

It follows that $\mathsf{E}[Y_{n+1}^2] = \sigma^2 (n+1)^{2H}$.

8. (a) Clearly,

$$\widetilde{X}_\nu^{(m)} \;:=\; \sum_{k=(\nu-1)m+1}^{\nu m} (X_k - \mu)$$

is zero mean. Also,

$$\mathsf{E}\big[\widetilde{X}_\nu^{(m)} \widetilde{X}_n^{(m)}\big] \;=\; \sum_{k=(\nu-1)m+1}^{\nu m} \sum_{l=(n-1)m+1}^{nm} C(k-l)$$

$$
\begin{aligned}
&= \sum_{k=(\nu-1)m+1}^{\nu m} \sum_{i=1}^{m} C(k-[i+(n-1)m]) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} C([j+(\nu-1)m]-[i+(n-1)m]) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} C(j-i+m(\nu-n)).
\end{aligned}
$$

Thus, $\widetilde{X}_\nu^{(m)}$ is WSS.

(b) From the solution of part (a), we see that

$$
\begin{aligned}
\widetilde{C}^{(m)}(0) &= \sum_{i=1}^{m} \sum_{j=1}^{m} C(j-i) \\
&= mC(0)+2\sum_{k=1}^{m-1} C(k)(m-k) \\
&= \mathsf{E}[Y_m^2] \;=\; \sigma^2 m^{2H}, \quad \text{by Problem 7.}
\end{aligned}
$$

Thus, $C^{(m)}(0) = \sigma^2 m^{2H}/m^2 = \sigma^2 m^{2H-2}$.

9. Starting with

$$
\lim_{m\to\infty} \frac{C^{(m)}(n)}{m^{2H-2}} \;=\; \frac{\sigma_\infty^2}{2}\big[|n+1|^{2H} - 2|n|^{2H} + |n-1|^{2H}\big],
$$

put $n=0$ to get

$$
\lim_{m\to\infty} \frac{C^{(m)}(0)}{m^{2H-2}} \;=\; \sigma_\infty^2.
$$

Then observe that

$$
\frac{C^{(m)}(0)}{m^{2H-2}} \;=\; \frac{\mathsf{E}[(X_1^{(m)}-\mu)^2]}{m^{2H-2}} \;=\; \frac{\mathsf{E}\bigg[\bigg|\frac{1}{m}\displaystyle\sum_{k=1}^{m}(X_k-\mu)\bigg|^2\bigg]}{m^{2H-2}}.
$$

10. Following the hint, observe that

$$
\begin{aligned}
\frac{\widetilde{C}^{(m)}(n)}{m^{2H}} &= \frac{1}{2}\,\frac{\big(\mathsf{E}[Y_{(n+1)m}^2]-\mathsf{E}[Y_{nm}^2]\big) - \big(\mathsf{E}[Y_{nm}^2]-\mathsf{E}[Y_{(n-1)m}^2]\big)}{m^{2H}} \\
&= \frac{1}{2}\bigg[(n+1)^{2H}\frac{\mathsf{E}[Y_{(n+1)m}^2]}{[(n+1)m]^{2H}} - 2n^{2H}\frac{\mathsf{E}[Y_{nm}^2]}{(nm)^{2H}} + (n-1)^{2H}\frac{\mathsf{E}[Y_{(n-1)m}^2]}{[(n-1)m]^{2H}}\bigg] \\
&\to \frac{\sigma_\infty^2}{2}[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}].
\end{aligned}
$$

11. If the equation cited in the text holds, then in particular,

$$
\lim_{m\to\infty} \frac{\widetilde{C}^{(m)}(0)}{m^{2H-2}} \;=\; \sigma_\infty^2,
$$

and

$$\begin{aligned}\lim_{m\to\infty}\rho^{(m)}(n) &= \lim_{m\to\infty}\frac{\widetilde{C}^{(m)}(n)}{\widetilde{C}^{(m)}(0)} = \lim_{m\to\infty}\frac{\widetilde{C}^{(m)}(n)/m^{2H-2}}{\widetilde{C}^{(m)}(0)/m^{2H-2}}\\ &= \frac{1}{2}[|n+1|^{2H}-2|n|^{2H}+|n-1|^{2H}].\end{aligned}$$

Conversely, if both conditions hold,

$$\begin{aligned}\lim_{m\to\infty}\frac{\widetilde{C}^{(m)}(n)}{m^{2H-2}} &= \lim_{m\to\infty}\frac{\widetilde{C}^{(m)}(0)\rho^{(m)}(n)}{m^{2H-2}}\\ &= \lim_{m\to\infty}\frac{\widetilde{C}^{(m)}(0)}{m^{2H-2}}\lim_{m\to\infty}\rho^{(m)}(n)\\ &= \frac{\sigma_\infty^2}{2}[|n+1|^{2H}-2|n|^{2H}+|n-1|^{2H}].\end{aligned}$$

12. To begin, write

$$\begin{aligned}S(f) &= \left|1-e^{-j2\pi f}\right|^{-2d} = \left|e^{-j\pi f}[e^{j\pi f}-e^{-j\pi f}]\right|^{-2d} = \left|2j\frac{e^{j\pi f}-e^{-j\pi f}}{2j}\right|^{-2d}\\ &= \left|2\sin(\pi f)\right|^{-2d} = \left[4\sin^2(\pi f)\right]^{-d}.\end{aligned}$$

Then

$$\begin{aligned}C(n) &= \int_{-1/2}^{1/2}\left[4\sin^2(\pi f)\right]^{-d}e^{j2\pi fn}\,df\\ &= \int_{-1/2}^{1/2}\left[4\sin^2(\pi f)\right]^{-d}\cos(2\pi fn)\,df\\ &= 2\int_0^{1/2}\left[4\sin^2(\pi f)\right]^{-d}\cos(2\pi fn)\,df\\ &= 2\int_0^{\pi}\left[4\sin^2(\nu/2)\right]^{-d}\cos(n\nu)\,\frac{d\nu}{2\pi}\\ &= \frac{1}{\pi}\int_0^{\pi}\left[4\sin^2(\nu/2)\right]^{-d}\cos(n\nu)\,d\nu. \qquad (*)\end{aligned}$$

Next, as suggested in the hint, apply the change of variable $\theta = 2\pi-\nu$ to

$$\frac{1}{\pi}\int_{\pi}^{2\pi}\left[4\sin^2(\nu/2)\right]^{-d}\cos(n\nu)\,d\nu$$

to get

$$\frac{1}{\pi}\int_0^{\pi}\left[4\sin^2([2\pi-\theta]/2)\right]^{-d}\cos(n[2\pi-\theta])\,d\theta,$$

which, using a trigonometric identity, is equal to $(*)$. Thus,

$$\begin{aligned}C(n) &= \frac{1}{2\pi}\int_0^{2\pi}\left[4\sin^2(\nu/2)\right]^{-d}\cos(n\nu)\,d\nu\\ &= \frac{1}{\pi}\int_0^{\pi}\left[4\sin^2(t)\right]^{-d}\cos(2nt)\,dt.\end{aligned}$$

By the formula provided in the hint,

$$\begin{aligned} C(n) &= \frac{\cos(n\pi)\Gamma(2-2d)2^{p-1}2^{1-p}}{(1-2d)\Gamma((2-2d+2n)/2)\Gamma((2-2d-2n)/2)} \\ &= \frac{(-1)^n\Gamma(1-2d)}{\Gamma(1-d+n)\Gamma(1-d-n)}. \end{aligned}$$

13. Following hint (i), let $u = \sin^2\theta$ and $dv = \theta^{\alpha-3}\,d\theta$. Then $du = 2\sin\theta\cos\theta\,d\theta$ and $v = \theta^{\alpha-2}/(\alpha-2)$. Hence,

$$\int_\varepsilon^r \theta^{\alpha-3}\sin^2\theta\,d\theta = \frac{\theta^{\alpha-2}\sin^2\theta}{\alpha-2}\bigg|_\varepsilon^r - \frac{1}{\alpha-2}\int_\varepsilon^r \theta^{\alpha-2}\sin 2\theta\,d\theta.$$

Next,

$$\begin{aligned} \frac{1}{2-\alpha}\int_\varepsilon^r \theta^{\alpha-2}\sin 2\theta\,d\theta &= \frac{1}{2-\alpha}\int_{2\varepsilon}^{2r}(t/2)^{\alpha-2}\sin t\,\frac{dt}{2} \\ &= \frac{(1/2)^{\alpha-1}}{2-\alpha}\int_{2\varepsilon}^{2r} t^{\alpha-2}\sin t\,dt \\ &= \frac{2^{1-\alpha}}{2-\alpha}\left[\frac{t^{\alpha-1}}{\alpha-1}\sin t\bigg|_{2\varepsilon}^{2r} + \frac{1}{1-\alpha}\int_{2\varepsilon}^{2r} t^{\alpha-1}\cos t\,dt\right]. \end{aligned}$$

Now write

$$\int_{2\varepsilon}^{2r} t^{\alpha-1}\cos t\,dt = \operatorname{Re}\int_{2\varepsilon}^{2r} t^{\alpha-1}e^{-jt}\,dt \to \operatorname{Re} e^{-j\alpha\pi/2}\Gamma(\alpha) = \cos(\alpha\pi/2)\Gamma(\alpha)$$

as $\varepsilon \to 0$ and $r \to \infty$ by hint (iii). To obtain the complete result, observe that

$$\theta^{\alpha-2}\sin^2\theta = \theta^\alpha\left(\frac{\sin\theta}{\theta}\right)^2 \quad \text{and} \quad t^{\alpha-1}\sin t = t^\alpha\frac{\sin t}{t}$$

both tend to zero as their arguments tend to zero or to infinity.

14. (a) First observe that

$$Q(-f) = \sum_{i=1}^\infty \frac{1}{|i-f|^{2H+1}} = \sum_{l=-\infty}^{-1}\frac{1}{|-l-f|^{2H+1}} = \sum_{i=-\infty}^{-1}\frac{1}{|i+f|^{2H+1}}.$$

Hence,

$$S(f) = \left[Q(-f) + \frac{1}{|f|^{2H+1}} + Q(f)\right]\sin^2(\pi f),$$

and then

$$\frac{S(f)}{|f|^{1-2H}} = |f|^{2H-1}[Q(-f)+Q(f)]\sin^2(\pi f) + \frac{\sin^2(\pi f)}{f^2} \to \pi^2.$$

(b) We have

$$\int_{-1/2}^{1/2} S(f)\,df \;=\; \sigma^2 \;=\; \pi^2 \cdot \frac{4\cos([1-H]\pi)\Gamma(2-2H)}{(2\pi)^{2-2H}(2H-1)2H}$$
$$= \frac{(2\pi)^{2H}\cos(\pi H)\Gamma(2-2H)}{2H(1-2H)}.$$

15. Write

$$\begin{aligned} C(n) &= \frac{(-1)^n\Gamma(1-2d)}{\Gamma(n+1-d)\Gamma(1-d-n)} \\ &= \frac{(-1)^n\Gamma(1-2d)}{\Gamma(n+1-d)(-1)^n\Gamma(d)\Gamma(1-d)/\Gamma(n+d)} \\ &= \frac{\Gamma(1-2d)}{\Gamma(1-d)\Gamma(d)} \cdot \frac{\Gamma(n+d)}{\Gamma(n+1-d)}. \end{aligned}$$

Now, with $\varepsilon = 1-d$, observe that

$$\begin{aligned} \frac{\Gamma(n+1-\varepsilon)}{\Gamma(n+\varepsilon)} &\sim \frac{(n+1-\varepsilon)^{n+1-\varepsilon-1/2}e^{-(n+1-\varepsilon)}}{(n+\varepsilon)^{n+\varepsilon-1/2}e^{-(n+\varepsilon)}} \\ &= e^{1-2d}\frac{n^{n+1/2-\varepsilon}[1+(1-\varepsilon)/n)]^{n+1/2-\varepsilon}}{n^{n+\varepsilon-1/2}(1+\varepsilon/n)^{n+\varepsilon-1/2}} \\ &= e^{1-2d}n^{2d-1}\frac{[1+(1-\varepsilon)/n)]^{n+1/2-\varepsilon}}{(1+\varepsilon/n)^{n+\varepsilon-1/2}}. \end{aligned}$$

Thus,

$$n^{1-2d}\frac{\Gamma(n+1-\varepsilon)}{\Gamma(n+\varepsilon)} \to e^{1-2d}\frac{\left(e^{1-\varepsilon}\right)^{1/2-\varepsilon}}{\left(e^{\varepsilon}\right)^{\varepsilon-1/2}} \;=\; e^{1-2d}e^{1/2-\varepsilon} \;=\; e^{1/2-d}.$$

Thus, $\alpha = 1-2d$ and $c = e^{1/2-d}\Gamma(1-2d)/[\Gamma(1-d)\Gamma(d)]$.

16. Evaluating the integral, we have

$$\frac{I_n}{n^{1-\alpha}} \;=\; \frac{n^{1-\alpha}-k^{1-\alpha}}{(1-\alpha)n^{1-\alpha}} \;=\; \frac{1-(k/n)^{1-\alpha}}{1-\alpha} \;\to\; \frac{1}{1-\alpha}.$$

Now, given a small $\varepsilon > 0$, for large n, $I_n/n^{1-\alpha} > 1/(1-\alpha)-\varepsilon$, which implies $I_n/n^{-\alpha} > n(1/(1-\alpha)-\varepsilon) \to \infty$. With B_n as in the hint, we have from $B_n + n^{-\alpha} - k^{-\alpha} \le I_n \le B_n$ that

$$\frac{B_n}{n^{1-\alpha}} + \frac{1}{n} + \frac{k^{-\alpha}}{n^{1-\alpha}} \;\le\; \frac{I_n}{n^{1-\alpha}} \;\le\; \frac{B_n}{n^{1-\alpha}}$$

or

$$\frac{I_n}{n^{1-\alpha}} \;\le\; \frac{B_n}{n^{1-\alpha}} \;\le\; \frac{I_n}{n^{1-\alpha}} - \frac{1}{n} - \frac{k^{-\alpha}}{n^{1-\alpha}}.$$

Thus, $B_n/n^{1-\alpha} \to 1/(1-\alpha)$.

17. We begin with the inequality

$$\underbrace{\sum_{\nu=k}^{n-1} \nu^{1-\alpha}}_{=:\,B_n} \;\le\; \underbrace{\int_k^n t^{1-\alpha}\,dt}_{=:\,I_n} \;\le\; \underbrace{\sum_{\nu=k}^{n-1} (\nu+1)^{1-\alpha}}_{=\,B_n - k^{1-\alpha} + n^{1-\alpha}}.$$

Next, since $I_n = (n^{2-\alpha} - k^{2-\alpha})/(2-\alpha)$,

$$\frac{I_n}{n^{2-\alpha}} \;=\; \frac{1-(k/n)^{2-\alpha}}{2-\alpha} \;\to\; \frac{1}{2-\alpha}.$$

Also,

$$\frac{I_n}{n^{2-\alpha}} \;\le\; \frac{B_n}{n^{2-\alpha}} - \frac{k^{1-\alpha}}{n^{2-\alpha}} + \frac{1}{n}$$

and

$$\frac{B_n}{n^{2-\alpha}} \;\le\; \frac{I_n}{n^{2-\alpha}}$$

imply $B_n/n^{2-\alpha} \to 1/(2-\alpha)$ as required.

18. Observe that

$$\begin{aligned}
\sum_{n=q}^{\infty} |h_n| \;&=\; \sum_{n=q}^{\infty} \left| \sum_{k=n-q}^{n} \alpha_k b_{n-k} \right| \\
&\le\; \sum_{n=q}^{\infty} \sum_{k=0}^{\infty} |\alpha_k||b_{n-k}| I_{[n-q,n]}(k) \\
&=\; \sum_{k=0}^{\infty} |\alpha_k| \sum_{n=q}^{\infty} |b_{n-k}| I_{[0,q]}(n-k) \\
&\le\; M\biggl(\sum_{i=0}^{q} |b_i|\biggr) \sum_{k=0}^{\infty} (1+\delta/2)^{-k} \\
&=\; M\biggl(\sum_{i=0}^{q} |b_i|\biggr) \frac{1}{1-1/(1+\delta/2)} \;<\; \infty.
\end{aligned}$$

19. Following the hint, we first compute

$$\begin{aligned}
\sum_n \alpha_{m-n} Y_n \;&=\; \sum_n \alpha_{m-n} \biggl(\sum_k a_k X_{n-k} \biggr) \;=\; \sum_n \alpha_{m-n} \sum_l a_{n-l} X_l \\
&=\; \sum_l X_l \sum_n \alpha_{m-n} a_{n-l} \;=\; \sum_l X_l \underbrace{\sum_\nu \alpha_{m-(\nu+l)} a_\nu}_{=\,\delta(m-l)},
\end{aligned}$$

where the reduction to the impulse follows because the convolution of α_n and a_n corresponds in the z-transform domain to the product $[1/A(z)] \cdot A(z) = 1$, and 1 is the transform of the unit impulse. Thus, $\sum_n \alpha_{m-n} Y_n = X_m$.

Next,

$$\begin{aligned}\sum_n \alpha_{m-n} Y_n &= \sum_n \alpha_{m-n}\left(\sum_k b_k Z_{n-k}\right) = \sum_n \alpha_{m-n}\sum_l b_{n-l} Z_l \\ &= \sum_l Z_l \sum_n \alpha_{m-n} b_{n-l} = \sum_l Z_l \underbrace{\sum_k \alpha_{m-(l+k)} b_k}_{=\,h_{m-l}}\end{aligned}$$

since this last convolution corresponds in the z-transform domain to the product $[1/A(z)]\cdot B(z) =: H(z)$.

20. Since X_n is WSS, $\mathsf{E}[|X_n|^2]$ is a finite constant. Since $\sum_{k=0}^{\infty}|h_k| < \infty$, we have by an example in Chapter 13 or by Problem 26 in Chapter 13 that $\sum_{k=0}^{m} h_k X_{n-k}$ converges in mean square as $m \to \infty$. By another example in Chapter 13,

$$\mathsf{E}[Y_n] = \mathsf{E}\left[\sum_{k=0}^{\infty} h_k X_{n-k}\right] = \sum_{k=0}^{\infty} h_k \mathsf{E}[X_{n-k}].$$

If $\mathsf{E}[X_{n-k}] = \mu$, then

$$\mathsf{E}[Y_n] = \sum_{k=0}^{\infty} h_k \mu$$

is finite and does not depend on n. Next, by the continuity of the inner product (Problem 24 in Chapter 13),

$$\begin{aligned}\mathsf{E}\left[X_l\left(\sum_{k=0}^{\infty} h_k X_{n-k}\right)\right] &= \lim_{m\to\infty}\mathsf{E}\left[X_l\left(\sum_{k=0}^{m} h_k X_{n-k}\right)\right] = \lim_{m\to\infty}\sum_{k=0}^{m} h_k \mathsf{E}[X_l X_{n-k}] \\ &= \lim_{m\to\infty}\sum_{k=0}^{m} h_k R_X(l-n+k) = \sum_{k=0}^{\infty} h_k R_X([l-n]+k).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathsf{E}[Y_n Y_l] &= \mathsf{E}\left[\left(\sum_{k=0}^{\infty} h_k X_{n-k}\right) Y_l\right] = \lim_{m\to\infty}\mathsf{E}\left[\left(\sum_{k=0}^{m} h_k X_{n-k}\right) Y_l\right] \\ &= \lim_{m\to\infty}\sum_{k=0}^{m} h_k \mathsf{E}[X_{n-k} Y_l] = \lim_{m\to\infty}\sum_{k=0}^{m} h_k R_{XY}(n-k-l) \\ &= \sum_{k=0}^{\infty} h_k R_{XY}([n-l]-k).\end{aligned}$$

Thus, X_n and Y_n are J-WSS.