New York University, CIMS, CS, Course CSCI-GA.3140-001, Spring 2024

"Abstract Interpretation"

Ch. 15, Fixpoints

(Prerequisites to be self-studied)

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Class 3, Monday, February 5th, 2024, 4:55-6:55 PM, WWH, Room CIWW 202

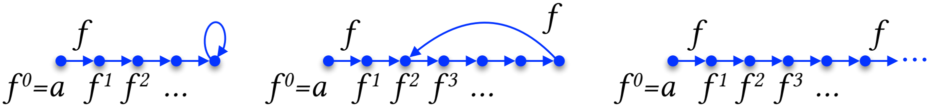
These slides are available at
http://cs.nyu.edu/~pcousot/courses/spring24/CSCI-GA.3140-001/slides/03--2024-02-05-structural-fixpoint-prefix-trace-semantics/slides-10--15--fixpoints-AI.pdf

Chapter 15

Ch. 15, Fixpoints

Iteration

- Iterating an *operator* f on a set S (i.e. $f \in S \to S$) from an element $a \in S$ i.e. $f^0 \triangleq a$ and $f^{n+1} \triangleq f(f^n)$ can



(1) reach a *fixpoint* $f^p = f(f^p)$;

(2) or, reach a *cycle* and iterate for ever along that cycle $f^{p+\ell} = f^p, \ell > 0$;

(3) or, *iterate for ever* along infinitely many different elements $\forall p \neq q \,.\, f^p \neq f^q$.

This last case is impossible if the set S is finite.

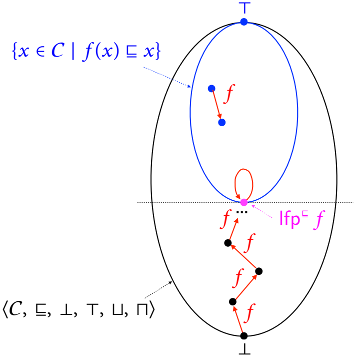
- We study sufficient conditions on S, f, and a ensuring that the iteration reaches a fixpoint.

en.wikipedia.org/wiki/Iterative_method

Fixpoints of increasing function on a complete lattice

There are two reasons why an increasing function $f \in \mathscr{C} \xrightarrow{\nearrow} \mathscr{C}$ on a complete lattice $\langle \mathscr{C}, \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$ has a least fixpoint $\mathsf{lfp}^{\sqsubseteq} f$:

- because it is the glb of the *post-fixpoints* $\{x \in \mathscr{C} \mid f(x) \sqsubseteq x\}$,
- because it is the limit (lub) of the iterates $\bot, f(\bot), \ldots, f^n(\bot), f^{n+1}(\bot), \ldots$



Fixpoint theorems

Tarski's fixpoint theorems (section 15.1)

Fixpoint

- A *fixpoint* of a function $f \in S \to S$ is an element x of S such that $f(x) = x$.
- In general a function may have no, one, or many fixpoints.

`en.wikipedia.org/wiki/Fixed_point_(mathematics)`

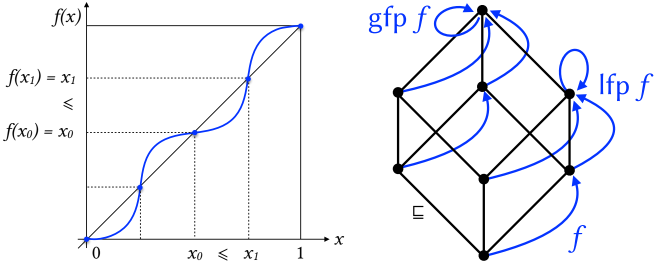
Increasing functions

- A function $f \in S \to S$ on a poset $\langle S, \sqsubseteq\rangle$ is increasing (denoted $f \in S \xrightarrow{\nearrow} S$) if and only if
$$\forall x, y \in S \,.\, (x \sqsubseteq y) \Rightarrow (f(x) \sqsubseteq f(y))$$
- A function f is increasing if and only if, for all $X \in \wp(S)$, when the lubs do exist,
$$\bigsqcup\{f(x) \mid x \in X\} \sqsubseteq f(\bigsqcup\{x \mid x \in X\})$$
- The composition of increasing functions is increasing

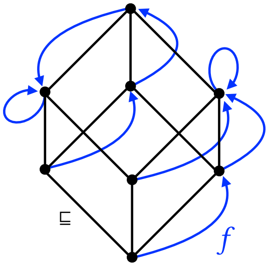
en.wikipedia.org/wiki/Monotonic_function

Fixpoints of increasing functions

- As shown by Alfred Tarski [Tarski, 1955], an increasing function on a complete lattice has at least one fixpoint and has a least one.



increasing function f



non-increasing function f

Least and greatest fixpoint

- $\mathsf{lfp}^{\sqsubseteq} f$ denotes the least fixpoint of $f \in L \to L$ for the partial order $\langle L, \sqsubseteq\rangle$, if any.

- If the partial order $\sqsubseteq$ is understood from the context, we write $\mathsf{lfp}\ f$.

- $\mathsf{lfp}_a^{\sqsubseteq} f$ is the least fixpoint of f for $\sqsubseteq$, which is $\sqsubseteq$-greater than of equal to $a \in L$, if any i.e.
$$a \sqsubseteq \mathsf{lfp}_a^{\sqsubseteq} f = f(\mathsf{lfp}_a^{\sqsubseteq} f) \text{ and if } a \sqsubseteq x = f(x) \text{ then } \mathsf{lfp}_a^{\sqsubseteq} f \sqsubseteq x$$

- Dually for $\mathsf{gfp}_a^{\sqsubseteq} f$, $\mathsf{gfp}^{\sqsubseteq} f$, and $\mathsf{gfp}\ f$

en.wikipedia.org/wiki/Least_fixed_point

Tarski fixpoint theorem

Theorem 15.6 An increasing function $f \in L \xrightarrow{\nearrow} L$ on a complete lattice $\langle L, \sqsubseteq, \bot, \top, \sqcap, \sqcup\rangle$ has a least fixpoint $\mathsf{lfp}^{\sqsubseteq} f = \bigsqcap\{x \in L \mid f(x) \sqsubseteq x\}$.

Tarski fixpoint theorem

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- By duality, f has a greatest fixpoint $\mathsf{gfp}^{\sqsubseteq} f = \bigsqcup\{x \in L \mid x \sqsubseteq f(x)\}$
- The set of fixpoints of f ordered by $\sqsubseteq$ form a complete lattice
- This lattice of fixpoints is not necessary a sublattice of L meaning that the join and meet in this complete lattice of fixpoints may not be $\sqcup$ and $\sqcap$
- Tarski's theorem 15.6 was generalized to cpos [Markowsky, 1976, theorem 9] and [Pataraia, 1997] as reported by [Escardó, 2003, Corollary 2.1.]

en.wikipedia.org/wiki/Alfred_Tarski
en.wikipedia.org/wiki/Knaster-Tarski_theorem

Proof of theorem 15.6 Let $a \triangleq \bigsqcap\{x \in L \mid f(x) \sqsubseteq x\}$ where the greatest lower bound $\bigsqcap$ is well-defined in a complete lattice.

For all $x \in L$,

$$
\begin{array}{ll}
& f(x) \sqsubseteq x \\
\Rightarrow & a \sqsubseteq x \\
& \quad \text{(since } x \in \{x \in L \mid f(x) \sqsubseteq x\} \text{ and the greatest lower bound } a \text{ is smaller than or equal to any element of the set)} \\
\Rightarrow & f(a) \sqsubseteq f(x) \sqsubseteq x \qquad \text{(} f \text{ is increasing and } f(x) \sqsubseteq x \text{)} \\
\Rightarrow & f(a) \sqsubseteq \bigsqcap\{x \in L \mid f(x) \sqsubseteq x\} = a \\
& \quad \text{(since } f(a) \text{ is a lower bound of } \{x \in L \mid f(x) \sqsubseteq x\} \text{ hence smaller than or equal to its greatest lower bound } a \text{)}
\end{array}
$$

With $a \triangleq \bigsqcap\{x \in L \mid f(x) \sqsubseteq x\}$, we have proved that

$$
\begin{aligned}
& f(a) \sqsubseteq a \\
\Rightarrow\ & f(f(a)) \sqsubseteq f(a) && \wr\text{since } f \text{ is increasing}\wr \\
\Rightarrow\ & f(a) \in \{x \in L \mid f(x) \sqsubseteq x\} && \wr\text{def. } \in\wr \\
\Rightarrow\ & a = \bigsqcap\{x \in L \mid f(x) \sqsubseteq x\} \sqsubseteq f(a) \\
& \qquad \wr\text{def. greatest lower bound smaller than or equal to any element of } \{x \in L \mid f(x) \sqsubseteq x\}\wr \\
\Rightarrow\ & a = f(a) && \wr\text{by antisymmetry}\wr
\end{aligned}
$$

So a is a fixpoint of f.

Let us prove that it is the least one.

- Let $x = f(x)$ be a fixpoint of f.
- Then $x \in \{x' \in L \mid f(x') \sqsubseteq x'\}$ by reflexivity.
- It follows that $a = \bigsqcap\{x' \in L \mid f(x') \sqsubseteq x'\} \sqsubseteq x$ by def. greatest lower bound.
- Therefore $a = \mathsf{lfp}^{\sqsubseteq} f$ is the least fixpoint of f. $\square$

Optimization problem

- Tarski's theorem 15.6 shows that computing the least fixpoint $\mathsf{lfp}^{\sqsubseteq} f$ can be understood as solving the *optimization problem* [Boyd and Vandenberghe, 2004]

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & f(x) \sqsubseteq x \end{aligned}$$

  where f is the objective function and $f(x) \sqsubseteq x$ is an inequality constraint.

- The optimization problem may be solvable for some complete lattices L [Cousot, 2005; Liberti and Marinelli, 2014] (e.g. finite ones)

- The optimization problem is, in general, uncomputable [Liberti, 2019].

`en.wikipedia.org/wiki/Optimization_problem`

Properties of the fixpoints of increasing functions

Let $f \in L \xrightarrow{\nearrow} L$ be an increasing function on a complete lattice $\langle L, \sqsubseteq, \bot, \top, \sqcap, \sqcup \rangle$

- $\mathsf{lfp}^{\sqsubseteq} f = \bigsqcap\{x \in L \mid f(x) = x\}$

- $\mathsf{lfp}^{\sqsubseteq} f = \mathsf{lfp}^{\sqsubseteq} x \mapsto x \sqcup f(x)$.

- Given a prefixpoint $a \in L$ such that $a \sqsubseteq f(a)$, $\mathsf{lfp}_a^{\sqsubseteq} f = \bigsqcap\{x \in L \mid a \sqsubseteq x \wedge f(x) \sqsubseteq x\}$ is the least fixpoint of f greater than or equal to a.

- For all $a \in L$ such that $a \sqsubseteq f(a)$, $\mathsf{lfp}^{\sqsubseteq} x \mapsto a \sqcup f(x) = \mathsf{lfp}_a^{\sqsubseteq} f$ where $\mathsf{lfp}_a^{\sqsubseteq} f$ is the least fixpoint of f greater than or equal to $a \in L$.

Fixpoint induction

- Let $f \in L \xrightarrow{\nearrow} L$ be an increasing function on the complete lattice $\langle L, \sqsubseteq, \bot, \top, \sqcap, \sqcup\rangle$ and $x \in L$. We have

$$\mathsf{lfp}^{\sqsubseteq} f \sqsubseteq x \text{ if and only if } \exists y \in L \,.\, f(y) \sqsubseteq y \wedge y \sqsubseteq x$$

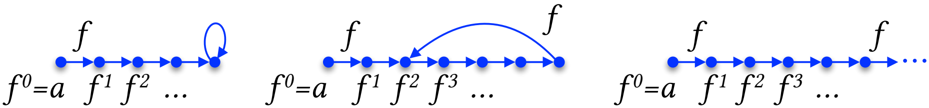
  (This is the basis for the main program verification methods)

- Let $x \in L$. $\mathsf{lfp}^{\sqsubseteq} f \sqsubseteq x$ if and only if $\mathsf{lfp}^{\sqsubseteq} f \sqcap f(x) \sqsubseteq x$ ([Park, 1969, (2.2)]).

Iterative fixpoint theorems (section 15.2)

Iteration

- In general, the iteration $f^0 = a$ and $f^{n+1} = f(f^n)$ for $n \in \mathbb{N}$ of a function $f \in S \to S$ on a set S from an element $a \in S$ either reaches a fixpoint, a cycle, or loops for ever.

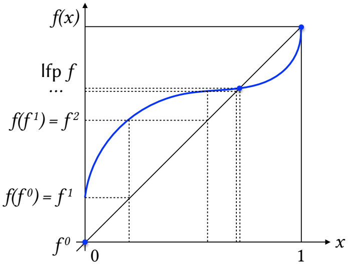


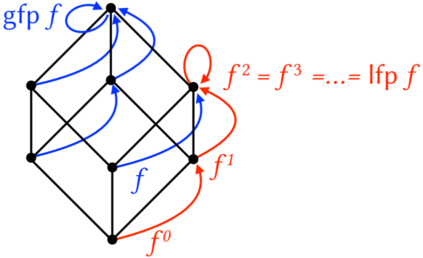
- In the case of an increasing function on a poset $\langle P, \sqsubseteq \rangle$, the iterates from $a \in S$ such that $a \sqsubseteq f(a)$ are increasing so reaching a cycle is impossible by antisymmetry.

- If limits exist and have a continuity property then a fixpoint is always reached.

`hen.wikipedia.org/wiki/Fixed-point_theorem`

Examples of iteration of increasing functions





Iterates

Definition 15.20 Let $f \in P \to P$ be a function on a poset $\langle P, \sqsubseteq, \perp, \sqcup\rangle$ with infimum $\perp$ and lub $\sqcup$.

The (partially defined) *iterates* of f from $a \in P$ are $\langle f^n,\ n \in \mathbb{N} \cup \{\omega\}\rangle$ such that

- $f^0 \triangleq a$,
- $f^{n+1} \triangleq f(f^n)$ for $n \in \mathbb{N}$
- $f^\omega \triangleq \bigsqcup\{f^n \mid n \in \mathbb{N}\}$

(where, to denote the limit of infinite iterations, the naturals are extended by the infinity ω such that $\forall n \in \mathbb{N} \,.\, n < \omega \leqslant \omega$).

By default, $a = \perp$.

We write $\langle f^n(a),\ n \in \mathbb{N} \cup \{\omega\}\rangle$ to make clear that the iterates start from $a \neq \perp$.

The iterates *converge* in finitely many steps or are *ultimately stationary* at $\ell \in \mathbb{N}$ if and only if $\forall n \geqslant \ell \,.\, f^n = f^\ell$.

Tarski-Kantorovich iterative fixpoint

Theorem 15.21 Let $f \in P \xrightarrow{\nearrow} P$ be an increasing function on a poset $\langle P, \sqsubseteq, \bot, \sqcup\rangle$ with infimum $\bot$ and lub $\sqcup$.

Let $\langle f^n, n \in \mathbb{N} \cup \{\omega\}\rangle$ be the iterates of f.

Assume that the least upper bound $f^\omega = \bigsqcup\{f^n \mid n \in \mathbb{N}\}$ exists and $f(\bigsqcup\{f^n \mid n \in \mathbb{N}\}) = \bigsqcup\{f(f^n) \mid n \in \mathbb{N}\}$ then

- $\langle f^n, n \in \mathbb{N} \cup \{\omega\}\rangle$ is an increasing chain
- f has a least fixpoint $\mathsf{lfp}^{\sqsubseteq} f = \bigsqcup\{f^n \mid n \in \mathbb{N}\}$.

en.wikipedia.org/wiki/Kleene_fixed-point_theorem

The iterative fixpoint theorem is often proved with stronger hypotheses, see below.

Proof of theorem 15.21 Fixpoint:

- Observe that

$$
\begin{aligned}
& \bigsqcup\{f^n \mid n \in \mathbb{N}\} \\
= {} & \bigsqcup(\{f^{n+1} \mid n \in \mathbb{N}\} \cup \{f^0\}) && \text{(extract case } n = 0\text{)} \\
= {} & (\bigsqcup\{f^{n+1} \mid n \in \mathbb{N}\}) \sqcup \bot && \text{(since } f^0 = \bot \text{ is the infimum)} \\
= {} & \bigsqcup\{f(f^n) \mid n \in \mathbb{N}\} && \text{(since } f^{n+1} = f(f^n) \text{ for } n \in \mathbb{N}\text{)}
\end{aligned}
$$

- Then

$$
\begin{aligned}
& f(\bigsqcup\{f^n \mid n \in \mathbb{N}\}) \\
= {} & \bigsqcup\{f(f^n) \mid n \in \mathbb{N}\} && \text{(by hypothesis)} \\
= {} & \bigsqcup\{f^n \mid n \in \mathbb{N}\}
\end{aligned}
$$

- It follows that $\bigsqcup\{f^n \mid n \in \mathbb{N}\}$ is a fixpoint of f. $\square$

Least fixpoint:

- Let $x = f(x)$ be a fixpoint of f. $f^0 = \bot \sqsubseteq x$ since $\bot$ is the infimum (smallest element) of P.
- Assume that $f^n \sqsubseteq x$ by induction hypothesis.
- Then $f^{n+1} = f(f^n) \sqsubseteq f(x) = x$ by def. of the iterates, f increasing, and fixpoint property.
- By recurrence $\forall n \in \mathbb{N} \,.\, f^n \sqsubseteq x$.
- So x is an upper bound of the $\{f^n \mid n \in \mathbb{N}\}$
- Hence x is larger than or equal to the least upper bound $\bigsqcup\{f^n \mid n \in \mathbb{N}\}$,
- It follows that $\bigsqcup\{f^n \mid n \in \mathbb{N}\} = \mathsf{lfp}^{\sqsubseteq} f$ is the least fixpoint of f. $\square$

Increasing chain:

- We have $f^0 = \bot \sqsubseteq f^1$ by def. infimum.
- Assume $f^n \sqsubseteq f^{n+1}$.
- By definition 15.20 of the iterates and f increasing, we have $f^{n+1} = f(f^n) \sqsubseteq f(f^{n+1}) = f^{n+2}$, proving $\langle f^n, n \in \mathbb{N}\rangle$ to be increasing.
- The lub $f^\omega = \bigsqcup\{f^n \mid n \in \mathbb{N}\}$ is assumed to exist so $\langle f^n, n \in \mathbb{N} \cup \{\omega\}\rangle$ is an increasing chain.

$\square$

Chain (recall)

- A subset $\{x_i \mid i \in \mathbb{N}\} \subseteq P$ of a poset $\langle P, \sqsubseteq\rangle$ is an (enumerable) increasing chain if and only if $\forall i, j \in \mathbb{N} . (i \leqslant j) \Rightarrow (x_i \sqsubseteq x_j)$.
- The family $x_i, i \in \mathbb{N}$ considered as a function of i is therefore increasing.

(Upper) continuity

Definition 15.22 A function $f \in P \to P$ on a poset $\langle P, \sqsubseteq \rangle$ is upper continuous (denoted $f \in P \xrightarrow{uc} P$) if and and only if

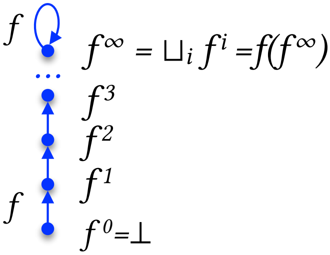
- for any enumerable increasing chain $\{x_i \mid i \in \mathbb{N}\} \subseteq P$ of P which has a least upper bound $\bigsqcup\{x_i \mid i \in \mathbb{N}\} \in P$ then $\bigsqcup\{f(x_i) \mid i \in \mathbb{N}\}$ exists in P
- and $f(\bigsqcup\{x_i \mid i \in \mathbb{N}\}) = \bigsqcup\{f(x_i) \mid i \in \mathbb{N}\}$.

The dual is lower-continuity (denoted $f \in P \xrightarrow{lc} P$).

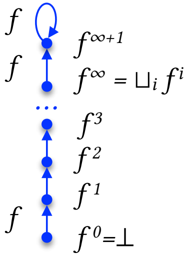
Observe that join preservation implies upper continuity.

`en.wikipedia.org/wiki/Scott_continuity`

Examples of upper-continuous functions



continuous



not continuous

Examples of upper-continuous functions

- An upper continuous function on a poset is increasing (but not all increasing functions are upper continuous).

- The composition of continuous functions is continuous.

CPO (recall)

- A complete partial order (cpo) is a poset $\langle P, \sqsubseteq, \perp\rangle$ with infimum $\perp$ such that any increasing chain $\{x_i \mid i \in \mathbb{N}\}$ has a least upper bound $\bigsqcup\{x_i \mid i \in \mathbb{N}\} \in P$.

en.wikipedia.org/wiki/Chain-complete_partial_order
en.wikipedia.org/wiki/Complete_partial_order

Kleene/Scott iterative fixpoint theorem

Theorem 15.26 If $f \in L \xrightarrow{uc} L$ be an upper continuous function on a cpo $\langle L, \sqsubseteq, \bot, \sqcup \rangle$ then f has a least fixpoint $\mathsf{lfp}^{\sqsubseteq} f = \bigsqcup_{n\in\mathbb{N}} f^n(\bot)$.
en.wikipedia.org/wiki/Kleene_fixed-point_theorem

The proof is similar to that of theorem 15.21.

Continuity of the least fixpoint of continuous functions

Theorem 15.32 If $F \in L \xrightarrow{uc} L \xrightarrow{uc} L$ be an upper continuous function on a cpo $\langle L, \sqsubseteq, \bot\rangle$ then $R \in L \mapsto \mathsf{lfp}^{\sqsubseteq} F(R) \in L \xrightarrow{uc} L$.

Proof of theorem 15.32

- $F(R) \in L \xrightarrow{uc} L$ so $\mathsf{lfp}^{\sqsubseteq} F(R) = \bigsqcup_{n\in\mathbb{N}} F(R)^n(\bot)$ exists by theorem 15.26.
- Let us prove, by recurrence, that $R \in L \mapsto F(R)^n(\bot) \in L \xrightarrow{uc} L$.
  - For the basis, $R \in L \mapsto F(R)^0(\bot) = R \in L \mapsto \bot$ is constant so continuous.
  - For the induction, $R \in L \mapsto F(R)^n(\bot)$ is continuous by induction hypothesis

    so $R \in L \mapsto F(R)^{n+1}(\bot) = F(R) \circ R \in L \mapsto F(R)^n(\bot)$ is continuous since $F(R) \in L \xrightarrow{uc} L$ and,

    the composition of continuous functions is continuous. $\square$

- For $F \in L \xrightarrow{uc} L \xrightarrow{uc} L$, $R \in L \mapsto \mathsf{lfp}^{\sqsubseteq} F(R)$ is continuous. Let $\langle R_i,\ i \in \mathbb{N}\rangle$ be a $\sqsubseteq$-increasing chain.

$$
\begin{aligned}
& (R \in L \mapsto \mathsf{lfp}^{\sqsubseteq} F(R))(\bigsqcup_{i\in\mathbb{N}} R_i) \\
= {} & (R \in L \mapsto \bigsqcup_{n\in\mathbb{N}} F(R)^n(\bot))(\bigsqcup_{i\in\mathbb{N}} R_i) && \text{(theorem 15.26)} \\
= {} & \bigsqcup_{n\in\mathbb{N}} F(\bigsqcup_{i\in\mathbb{N}} R_i)^n(\bot) && \text{(function application)} \\
= {} & \bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i\in\mathbb{N}} F(R_i)^n(\bot) && \text{(}R \in L \mapsto F(R)^n(\bot) \in L \xrightarrow{uc} L \text{ is continuous)} \\
= {} & \bigsqcup_{i\in\mathbb{N}} \bigsqcup_{n\in\mathbb{N}} F(R_i)^n(\bot) && \text{(def. } \sqcup\text{)} \\
= {} & \bigsqcup_{i\in\mathbb{N}} (R \in L \mapsto \bigsqcup_{n\in\mathbb{N}} F(R)^n(\bot))(R_i) && \text{(def. function application)} \\
= {} & \bigsqcup_{i\in\mathbb{N}} (R \in L \mapsto \mathsf{lfp}^{\sqsubseteq} F(R))(R_i) && \text{(theorem 15.26)}
\end{aligned}
$$

Conjugate fixpoint theorem

David Park conjugate fixpoint theorem [Park, 1969], [Park, 1979, (4.1.2)]

Theorem 15.33 Let S be a set and $f \in \wp(S) \stackrel{\nearrow}{\longrightarrow} \wp(S)$ be $\subseteq$-increasing on the complete lattice $\langle \wp(S), \subseteq, \varnothing, S, \cap, \cup \rangle$. Let $\neg X \triangleq S \setminus X$ be the set complement. Define $\tilde{f} \triangleq X \mapsto \neg f(\neg X)$. Then $\mathsf{gfp}^{\subseteq} f = \neg \mathsf{lfp}^{\subseteq} \tilde{f}$.

- $\mathsf{lfp}^{\subseteq} f \cap \mathsf{lfp}^{\subseteq} \tilde{f} = \varnothing$ [Park, 1969, (2.2.3)]
- $\mathsf{lfp}^{\subseteq} f \cup \mathsf{lfp}^{\subseteq} \tilde{f} = S$ if and only if $\mathsf{lfp}^{\subseteq} f$ is the unique fixpoint of f [Park, 1969, (2.2.4)]
- This generalizes to complete boolean lattices [Monk and Bonnet, 1983].

en.wikipedia.org/wiki/Complete_Boolean_algebra

Proof of theorem 15.33

$$
\begin{array}{ll}
& \neg\mathsf{lfp}^{\subseteq}\, \tilde{f} \\
= & \neg\mathsf{lfp}^{\subseteq}\, X \mapsto \neg f(\neg X) \hfill \text{(def. } \tilde{f}\text{)} \\
= & \neg \bigcap\{X \mid \neg f(\neg X) \subseteq X\} \\
& \hfill \text{(}f \text{ is increasing so } X \mapsto \neg f(\neg X) \text{ is increasing and theorem 15.6)} \\
= & \bigcup\{\neg X \mid f(\neg X) \supseteq \neg X\} \hfill \text{(De Morgan laws)} \\
= & \bigcup\{Y \mid Y \subseteq f(Y)\} \hfill \text{(letting } Y \triangleq \neg X\text{)} \\
= & \mathsf{gfp}^{\subseteq}\, f \hfill \text{(dual of theorem 15.6)} \quad \square
\end{array}
$$

Conclusion

Conclusion

- Fixpoint theorems are the mathematical foundations of program semantics, verification, and static analysis;
- Fixpoint theorems will be extensively used in the following chapters of the book;
- They deserve to be studied carefully, e.g. by trying to redo the proofs by yourself.

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The End, Thank you