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* Prove that the set of rational numbers $\mathbb{Q}$, equipped with two binary options of addition and multiplication, forms a field.

Answer: $\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \}$ with ordinary addition and multiplication is a field.

Proof: We verify the field axioms.

1. **Well-definess of operations:**

A rational number is an equivalence class of pairs (p, q) with $q \neq 0$ under $\frac{p}{q} = \frac{p'}{q'} \Leftrightarrow pq' = p'q$. The usual formulas,

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}, \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

respect this equivalance, so addition and multi-plication are well-defined on $\mathbb{Q}$.

2. **Closure:** If $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ (with $q, s \neq 0$), then

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} \in \mathbb{Q}, \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs} \in \mathbb{Q}.$$

since integers are closed under + and $\cdot$ $qs \neq 0$.

3. <u>Associativity of + and $\cdot$</u>: Associativity follows from associativity in $\mathbb{Z}$ and the formulas for sum/product of fractions; e.g for addition computer both $\left(\frac{p}{q}+\frac{r}{s}\right)+\frac{t}{u}$. and $\frac{p}{q}+\left(\frac{r}{s}+\frac{t}{u}\right)$ and simplify to the same fraction $\frac{psu+rqu+tqs}{qsu}$. similar for multiplication.

4. <u>commutarity of + and $\cdot$</u>: for commutarity in $\mathbb{Z}$:

$$\frac{p}{q}+\frac{r}{s}=\frac{ps+rq}{qs}=\frac{rq+ps}{sq}=\frac{r}{s}+\frac{p}{q},$$

and likewise $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}=\frac{rp}{sq}=\frac{r}{s}\cdot\frac{p}{q}$

5. <u>Identities:</u>

a) Additive identity: $0=\frac{0}{1}$. for any $\frac{p}{q}$,

$$\frac{p}{q}+\frac{0}{1}=\frac{p.1+0.q}{q.1}=\frac{p}{q}$$

b) Multiplicative Identity: $1 = \frac{1}{1}$. for any $\frac{p}{q}$

$$\frac{p}{q} \cdot \frac{1}{1} = \frac{p \cdot 1}{q \cdot 1} = \frac{p}{q}$$

6. Additive Inverse: For $\frac{p}{q} \in \mathbb{Q}$, the additive inverse is $-\frac{p}{q} = \frac{-p}{q}$ since

$$\frac{p}{q} + \frac{-p}{q} = \frac{pq + (-p)q}{q^2} = \frac{0}{q^2} = 0$$

7. Multiplicative inverse: If $\frac{p}{q} \in \mathbb{Q}$ and $\frac{p}{q} \neq 0$, then $p \neq 0$. The inverse is

$$\left(\frac{p}{q}\right)^{-1} = \frac{q}{p} \text{ and indeed } \frac{p}{q} \cdot \frac{q}{p} = \frac{pq}{qp} = 1$$

8. Distributive law: for any $\frac{p}{q}, \frac{r}{s}, \frac{t}{u} \in \mathbb{Q}$

$$\frac{p}{q}\left(\frac{r}{s} + \frac{t}{u}\right) = \frac{p}{q} \cdot \frac{ru + ts}{su} = \frac{p(ru + ts)}{qsu}$$

$$= \frac{pru + pts}{qsu}$$

$$= \frac{pr}{qs} + \frac{pt}{qu}$$

$$= \frac{p}{q} \cdot \frac{r}{s} + \frac{p}{q} \cdot \frac{t}{u}$$

So multiplication distributes over addition.
All field axioms hold, so $(\Theta, +, \cdot)$ is a field.