CONTINUOUS MEDIA VIA DISCRETE FIELDS

1. Discrete Calculus

1.1. **Difference Equations.**

Definition 1.1. Let $\{t^n\}$ and $\{x_i\}$ be uniform grids in time and space with spacings Δt and Δx, respectively. For a discrete field u, define $u_i^n := u(t^n, x_i)$. A discrete evolution law is an equation of the type

$$u_i^{n+1} = F(u_i^n, u_i^{n-1}, \dots),$$

which specifies the state of the system at time t^{n+1} as a function of earlier states.

Definition 1.2. The discretization of the first derivative follows directly from the definition of a derivative. The *forward difference* approximation is

$$\frac{u_{i+1} - u_i}{\Delta x} \approx \frac{du}{dx},$$

and the *backward difference* approximation is

$$\frac{u_i - u_{i-1}}{\Delta x} \approx \frac{du}{dx}.$$

Theorem 1.3. *The second derivative of u can be discretized by the finite difference*

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \approx \frac{d^2u}{dx^2}$$

Proof. Recall the forward difference approximation to the first derivative

$$\left(\frac{du}{dx}\right)_i \approx \frac{u_{i+1} - u_i}{\Delta x}.$$

The second derivative is the backward difference of this quantity

$$\frac{d^2u}{dx^2} := \frac{d}{dx}\left(\frac{du}{dx}\right)_i \approx \frac{\left(\frac{du}{dx}\right)_i - \left(\frac{du}{dx}\right)_{i-1}}{\Delta x}.$$

Substituting the forward difference into this expression gives

$$\frac{d^2u}{dx^2} \approx \frac{\frac{u_{i+1}-u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x}}{\Delta x} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}.$$

□

1.2. **Von Neumann Analysis.**

Definition 1.4. A sequence u^n is *stable* if there exists a constant $C > 0$, independent of n, such that for all n,

$$\|u^n\| \leq C\|u^0\|$$

Definition 1.5. A *finite-width stencil* with half-width m and constant coefficients can be written as a scheme of the form

$$u_i^{n+1} = \sum_{j=-m}^{m} a_j u_{i+j}^n$$

Theorem 1.6. *Consider a constant-coefficient, finite-width stencil on a uniform grid with spacing* Δx*:*

$$u_i^{n+1} = \sum_{j=-m}^{m} a_j u_{i+j}^n.$$

For all real wavenumber k*, the discrete Fourier mode*

$$u_i^n = \hat{u}^n e^{iki\Delta x}$$

satisfies

$$\hat{u}^{n+1} = G(k)\hat{u}^n$$

where the growth factor $G(k)$ *is*

$$G(k) = \sum_{j=-m}^{m} a_j e^{ikj\Delta x}.$$

Theorem 1.7. *On an infinite grid, or at interior points of a finite grid away from the boundaries, a constant-coefficient stencil is stable if and only if all Fourier modes are non-growing. Equivalently,*

$$|G(k)| \leq 1 \text{ for all real } k.$$

1.3. **Lagrangian Analysis.**

Definition 1.8. Define the discrete Lagrangian $\mathcal{L}^n$ that depends on consecutive time levels

$$\mathcal{L}^n = \mathcal{L}(u^{n-r}, u^{n-r+1}, \ldots, u^n, \ldots, u^{n+s}).$$

Define the discrete action

$$\mathcal{S}[u] = \sum_n \Delta t \mathcal{L}^n.$$

Definition 1.9. A discrete field u is said to satisfy the *discrete principle of stationary action* if for every variation η vanishing on the temporal and spatial boundaries,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{S}[u + \varepsilon\eta] = 0.$$

Theorem 1.10 (Discrete Euler-Lagrange Equation)**.** *Let* $\mathcal{L}^n = \mathcal{L}(u^{n+1}, u^n, u^{n-1})$*. Then* u *satisfies the discrete stationary condition if and only if, for every* n*,*

$$\frac{\partial \mathcal{L}^{n+1}}{\partial u^n} + \frac{\partial \mathcal{L}^n}{\partial u^n} + \frac{\partial \mathcal{L}^{n-1}}{\partial u^n} = 0.$$

Proof. We introduce the varied field $v^m := u^m + \varepsilon\eta^m$ where η^m vanishes on the temporal boundaries. Consider the varied action

$$\mathcal{S}[v] = \sum_n \Delta t \mathcal{L}(v^{n+1}, v^n, v^{n-1}).$$

Differentiate both sides by ε, by the multivariable chain rule

$$\frac{d\mathcal{S}[v]}{d\varepsilon} = \sum_n \Delta t \left(\frac{\partial \mathcal{L}^n}{\partial v^{n+1}} \frac{dv^{n+1}}{d\varepsilon} + \frac{\partial \mathcal{L}^n}{\partial v^n} \frac{dv^n}{d\varepsilon} + \frac{\partial \mathcal{L}^n}{\partial v^{n-1}} \frac{dv^{n-1}}{d\varepsilon} \right).$$

After taking the derivatives of v, using the principle of stationary action, and taking where $\varepsilon = 0$ implies $u = v$

$$\left.\frac{d\mathcal{S}[u+\varepsilon\eta]}{d\varepsilon}\right|_{\varepsilon=0} = 0 = \sum_n \Delta t \left(\frac{\partial \mathcal{L}^n}{\partial u^{n+1}} \eta^{n+1} + \frac{\partial \mathcal{L}^n}{\partial u^n} \eta^n + \frac{\partial \mathcal{L}^n}{\partial u^{n-1}} \eta^{n-1} \right).$$

Summing over all n, there are three terms that contain a factor of η^n:

$$\begin{aligned} n \mapsto n+1 &\Rightarrow \frac{\partial \mathcal{L}^{n+1}}{\partial u^{n+1-1}} \eta^{n+1-1}, \\ n \mapsto n &\Rightarrow \frac{\partial \mathcal{L}^n}{\partial u^n} \eta^n, \text{ and} \\ n \mapsto n-1 &\Rightarrow \frac{\partial \mathcal{L}^{n-1}}{\partial u^{n-1+1}} \eta^{n-1+1}. \end{aligned}$$

Combining the three terms, we get the Discrete Euler-Lagrange equations

$$\frac{\partial \mathcal{L}^{n+1}}{\partial u^n} + \frac{\partial \mathcal{L}^n}{\partial u^n} + \frac{\partial \mathcal{L}^{n-1}}{\partial u^n} = 0.$$

□

2. The Perfect Steak - The One-Dimensional Heat Equation

We consider the one-dimensional heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2},$$

where $T(t, y)$ is the temperature and α is the thermal diffusivity.

2.1. Discrete Scheme.

On a uniform grid with spacing Δy and time step Δt, the explicit finite-difference scheme is

$$T_i^{n+1} = T_i^n + A\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right), \quad A := \alpha \frac{\Delta t}{\Delta y^2}.$$

Boundary values T_0^n and $T_{N_y}^n$ are defined by the cooking "recipe". The initial temperature T_0 specifies T_i^0. These boundary values are trivially stable: they are prescribed and therefore cannot grow. The stability of the interior points is determined by Fourier Analysis.

2.2. Fourier Analysis of the Scheme.

Assume a discrete Fourier mode

$$T_j^n = G^n e^{iky_j}, \quad y_j = j\Delta y.$$

Then

$$T_{j\pm1}^n = G^n e^{ik(y_j \pm \Delta y)} = G^n e^{iky_j} e^{\pm ik\Delta y}.$$

Substituting into the update rule

$$G^{n+1} e^{iky_j} = G^n e^{iky_j} + AG^n e^{iky_j}\left(e^{ik\Delta y} - 2 + e^{-ik\Delta y}\right).$$

Cancelling the common factor $G^n e^{iky_j}$ gives the growth factor

$$G(k) = 1 + A\left(e^{ik\Delta y} - 2 + e^{-ik\Delta y}\right).$$

Using the trigonometric identities

$$e^{ik\Delta y} + e^{-ik\Delta y} = 2\cos(k\Delta y), \quad \frac{1-\cos(k\Delta y)}{2} = \sin^2(\frac{k\Delta y}{2}),$$

we obtain

$$G = 1 + A(2\cos(k\Delta y) - 2) = 1 - 4A\sin^2\left(\frac{k\Delta y}{2}\right).$$

Since $\sin^2\theta \in [0,1]$, $G(k)$ ranges over the interval

$$G \in [1-4A, 1].$$

2.3. **Stability Condition.** Von Neumann stability requires $|G| \leq 1$ for all real wavenumbers k. The upper bound is automatic since $G(k) \leq 1$. The nontrivial constraint is

$$|1-4A| \leq 1.$$

This inequality is equivalent to

$$-1 \leq 1 - 4A \leq 1 \Leftrightarrow 0 \leq A \leq \frac{1}{2}$$

Therefore the scheme is stable exactly when

$$\alpha\frac{\Delta t}{\Delta y^2} \leq \frac{1}{2}$$

2.4. **Pseudocode Implementation.**

▷ Initial temperature.

for $i = 0$ to N_y **do**
  $T[0, i] = T_0$
end for

▷ Boundary values from the recipe.

for $n = 0$ to N_t **do**
  $T[n, 0] \leftarrow T(n\Delta t, 0)$
  $T[n, N_y] \leftarrow T(n\Delta t, N_y\Delta y)$
end for

▷ Explicit diffusion step.

for $n = 0$ to $N_t - 1$ **do**
  for $i = 1$ to $N_y - 1$ **do**
    $T[n+1, i] \leftarrow T[n, i] + A(T[n, i+1] - 2T[n, i] + T[n, i-1])$
  end for
end for

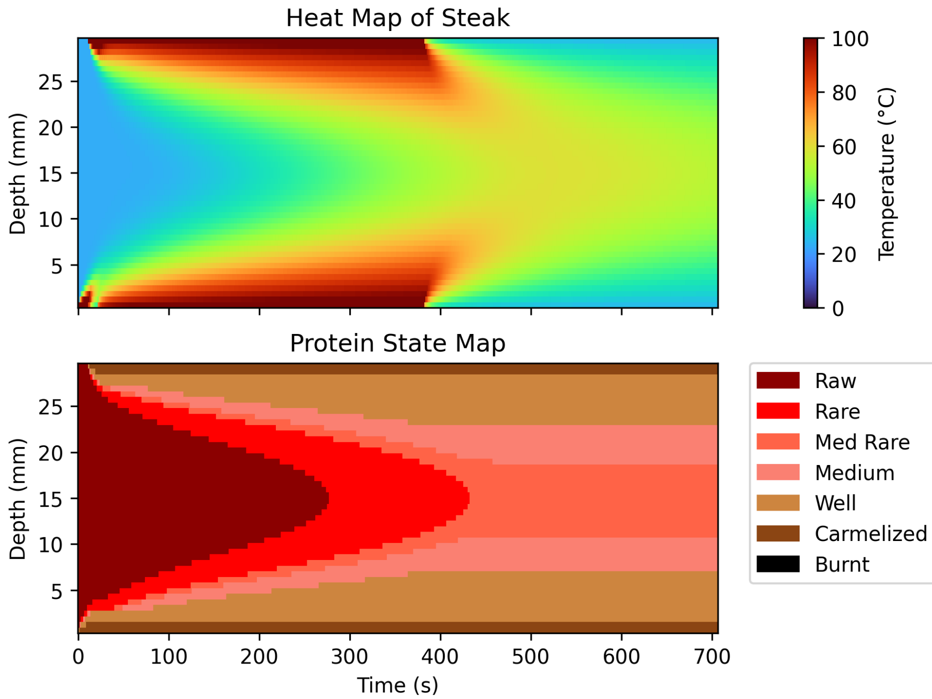

FIGURE 1. Reverse Seared Steak

3. A Guitar String - The One-Dimensional Wave Equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$