

## CONTINUOUS MEDIA VIA DISCRETE FIELDS

### 1. DISCRETE CALCULUS

#### 1.1. Difference Equations.

**Definition 1.1.** Let  $\{t^n\}$  and  $\{x_i\}$  be uniform grids in time and space with spacings  $\Delta t$  and  $\Delta x$ , respectively. For a discrete field  $u$ , define  $u_i^n := u(t^n, x_i)$ . A discrete evolution law is an equation of the type

$$u_i^{n+1} = F(u_i^n, u_i^{n-1}, \dots),$$

which specifies the state of the system at time  $t^{n+1}$  as a function of earlier states.

**Definition 1.2.** The discretization of the first derivative follows directly from the definition of a derivative. The *forward difference* approximation is

$$\frac{u_{i+1} - u_i}{\Delta t} \approx \frac{du}{dt},$$

and the *backward difference* approximation is

$$\frac{u_i - u_{i-1}}{\Delta t} \approx \frac{du}{dt}.$$

**Theorem 1.3.** The second derivative of  $u$  can be discretized as approximated by the finite difference

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2} \approx \frac{d^2u}{dt^2}$$

*Proof.* Recall the forward difference approximation to the first derivative

$$\left( \frac{du}{dt} \right)_i \approx \frac{u_{i+1} - u_i}{\Delta t}.$$

The second derivative is the backwards difference of this quantity

$$\frac{d^2u}{dt^2} := \frac{d}{dt} \left( \frac{du}{dt} \right)_i \approx \frac{\left( \frac{du}{dt} \right)_i - \left( \frac{du}{dt} \right)_{i-1}}{\Delta t}.$$

Substituting the forward difference into this expression gives

$$\frac{d^2u}{dt^2} \approx \frac{\frac{u_{i+1} - u_i}{\Delta t} - \frac{u_i - u_{i-1}}{\Delta t}}{\Delta t} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2}.$$

□

#### 1.2. Von Neumann Analysis.

**Definition 1.4.** A sequence  $u^n$  is *stable* if there exists a constant  $C > 0$ , independent of  $n$ , such that for all  $n$ ,

$$\|u^n\| \leq C\|u^0\|$$

**Definition 1.5.** A *finite-width stencil* with half-width  $m$  and constant coefficients can be written as a scheme of the form

$$u_i^{n+1} = \sum_{j=-m}^m a_j u_{i+j}^n$$

**Theorem 1.6.** Consider a constant-coefficient, finite-width stencil on a uniform grid with spacing  $\Delta x$ :

$$u_i^{n+1} = \sum_{j=-m}^m a_j u_{i+j}^n.$$

For all real wavenumber  $k$ , the discrete Fourier mode

$$u_i^n = \hat{u}^n e^{ik i \Delta x}$$

satisfies

$$\hat{u}^{n+1} = G(k) \hat{u}^n$$

where the growth factor  $G(k)$  is

$$G(k) = \sum_{j=-m}^m a_j e^{ik j \Delta x}.$$

**Theorem 1.7.** On an infinite grid, or at interior points of a finite grid away from the boundaries, the scheme is stable if the Fourier mode is non-growing a constant-coefficient stencil is stable if and only if all Fourier modes are non-growing. Equivalently,

$$|G(k)| \leq 1 \text{ for all real } k.$$

### 1.3. Lagrangian Analysis.

**Definition 1.8.** Define the discrete Lagrangian  $\mathcal{L}^n$  that depends on consecutive time levels

$$\mathcal{L}^n = \mathcal{L}(u^{n-r}, u^{n-r+1}, \dots, u^n, \dots, u^{n+s}).$$

Define the discrete action

$$\mathcal{S}[u] = \sum_n \Delta t \mathcal{L}^n.$$

**Definition 1.9.** A discrete field  $u$  is said to satisfy the *discrete principle of stationary action* if for every variation  $\eta$  vanishing on the temporal and spatial boundaries,

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{S}[u + \varepsilon\eta] = 0.$$

**Theorem 1.10** (Discrete Euler-Lagrange Equation). Let  $\mathcal{L}^n = \mathcal{L}(u^{n+1}, u^n, u^{n-1})$ . Then  $u$  satisfies the discrete stationary condition if and only if, for every  $n$ ,

$$\frac{\partial \mathcal{L}^{n+1}}{\partial u^n} + \frac{\partial \mathcal{L}^n}{\partial u^n} + \frac{\partial \mathcal{L}^{n-1}}{\partial u^n} = 0.$$

*Proof.* We introduce the varied field  $v^m := u^m + \varepsilon\eta^m$  where  $\eta^m$  vanishes on the temporal boundaries. Consider the varied action

$$\mathcal{S}[v] = \sum_n \Delta t \mathcal{L}(v^{n+1}, v^n, v^{n-1}).$$

Differentiate both sides by  $\varepsilon$ , by the multivariable chain rule

$$\frac{d\mathcal{S}[v]}{d\varepsilon} = \sum_n \Delta t \left( \frac{\partial \mathcal{L}^n}{\partial v^{n+1}} \frac{dv^{n+1}}{d\varepsilon} + \frac{\partial \mathcal{L}^n}{\partial v^n} \frac{dv^n}{d\varepsilon} + \frac{\partial \mathcal{L}^n}{\partial v^{n-1}} \frac{dv^{n-1}}{d\varepsilon} \right).$$

After taking the derivatives of  $v$ , using the principle of stationary action, and taking where  $\varepsilon = 0$  implies  $u = v$

$$\frac{d\mathcal{S}[u + \varepsilon\eta]}{d\varepsilon} \Big|_{\varepsilon=0} = 0 = \sum_n \Delta t \left( \frac{\partial \mathcal{L}^n}{\partial u^{n+1}} \eta^{n+1} + \frac{\partial \mathcal{L}^n}{\partial u^n} \eta^n + \frac{\partial \mathcal{L}^n}{\partial u^{n-1}} \eta^{n-1} \right).$$

Summing over all  $n$ , there are three terms that contain a factor of  $\eta^n$ :

$$\begin{aligned} n \mapsto n+1 &\Rightarrow \frac{\partial \mathcal{L}^{n+1}}{\partial u^{n+1}} \eta^{n+1}, \\ n \mapsto n &\Rightarrow \frac{\partial \mathcal{L}^n}{\partial u^n} \eta^n, \text{ and} \\ n \mapsto n-1 &\Rightarrow \frac{\partial \mathcal{L}^{n-1}}{\partial u^{n-1}} \eta^{n-1}. \end{aligned}$$

Combining the three terms, we get the Discrete Euler-Lagrange equations

$$\frac{\partial \mathcal{L}^{n+1}}{\partial u^n} + \frac{\partial \mathcal{L}^n}{\partial u^n} + \frac{\partial \mathcal{L}^{n-1}}{\partial u^n} = 0.$$

□

## 2. THE PERFECT STEAK - THE ONE-DIMENSIONAL HEAT EQUATION

We consider the one-dimensional heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2},$$

where  $T(t, y)$  is the temperature and  $\alpha$  is the thermal diffusivity.

**2.1. Discrete Scheme.** On a uniform grid with spacing  $\Delta y$  and time step  $\Delta t$ , the explicit finite-difference scheme is

$$T_i^{n+1} = T_i^n + A (T_{i+1}^n - 2T_i^n + T_{i-1}^n), \quad A := \alpha \frac{\Delta t}{\Delta y^2}.$$

Boundary values  $T_0^n$  and  $T_{N_y}^n$  are defined by the cooking “recipe”. The initial temperature  $T_0$  specifies  $T_i^0$ . These boundary values are trivially stable: they are prescribed and therefore cannot grow. The stability of the interior points is determined by Fourier Analysis.

**2.2. Fourier Analysis of the Scheme.** Assume a discrete Fourier mode

$$T_j^n = G^n e^{iky_j}, \quad y_j = j\Delta y.$$

Then

$$T_{i\pm 1}^n = G^n e^{ik(y_j \pm \Delta y)} = G^n e^{iky_j} e^{\pm ik\Delta y}.$$

Substituting into the update rule

$$G^{n+1} e^{iky_j} = G^n e^{iky_j} + AG^n e^{iky_j} (e^{ik\Delta y} - 2 + e^{-ik\Delta y}).$$

Cancelling the common factor  $G^n e^{iky_j}$  gives the growth factor

$$G(k) = 1 + A (e^{ik\Delta y} - 2 + e^{-ik\Delta y}).$$

Using the trigonometric identities

$$e^{ik\Delta y} + e^{-ik\Delta y} = 2 \cos(k\Delta y), \quad \frac{1 - \cos(k\Delta y)}{2} = \sin^2(k\Delta y),$$

we obtain

$$G = 1 + A(2 \cos(k\Delta x) - 2) = 1 - 4A \sin^2\left(\frac{k\Delta x}{2}\right).$$

Since  $\sin^2 \theta \in [0, 1]$ ,  $G(k)$  ranges over the interval

$$G \in [1 - 4A, 1].$$

**2.3. Stability Condition.** Von Neumann stability requires  $|G| \leq 1$  for all real wavenumbers  $k$ . The upper bound is automatic since  $G(k) \leq 1$ . The nontrivial constraint is

$$|1 - 4A| \leq 1.$$

This inequality is equivalent to

$$-1 \leq 1 - 4A \leq 1 \Leftrightarrow 0 \geq A \geq \frac{1}{2}$$

Therefore the scheme is stable exactly when

$$\alpha \frac{\Delta t}{\Delta y^2} \leq \frac{1}{2}$$

#### 2.4. Pseudocode Implementation.

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for  $i = 0$  to  $N_y$  do                                 $\triangleright$  Initial temperature.
     $T[0, i] = T_0$ 
end for                                          $\triangleright$  Boundary values from the recipe.

for  $n = 0$  to  $N_t$  do
     $T[n, 0] \leftarrow T(n\Delta t, 0)$ 
     $T[n, N_y] \leftarrow T(n\Delta t, L)$ 
end for                                          $\triangleright$  Explicit diffusion step.

for  $n = 0$  to  $N_t - 1$  do
    for  $i = 1$  to  $N_y - 1$  do
         $T[n + 1, i] \leftarrow T[n, i] + A(T[n, i + 1] - 2T[n, i] + T[n, i - 1])$ 
    end for
end for
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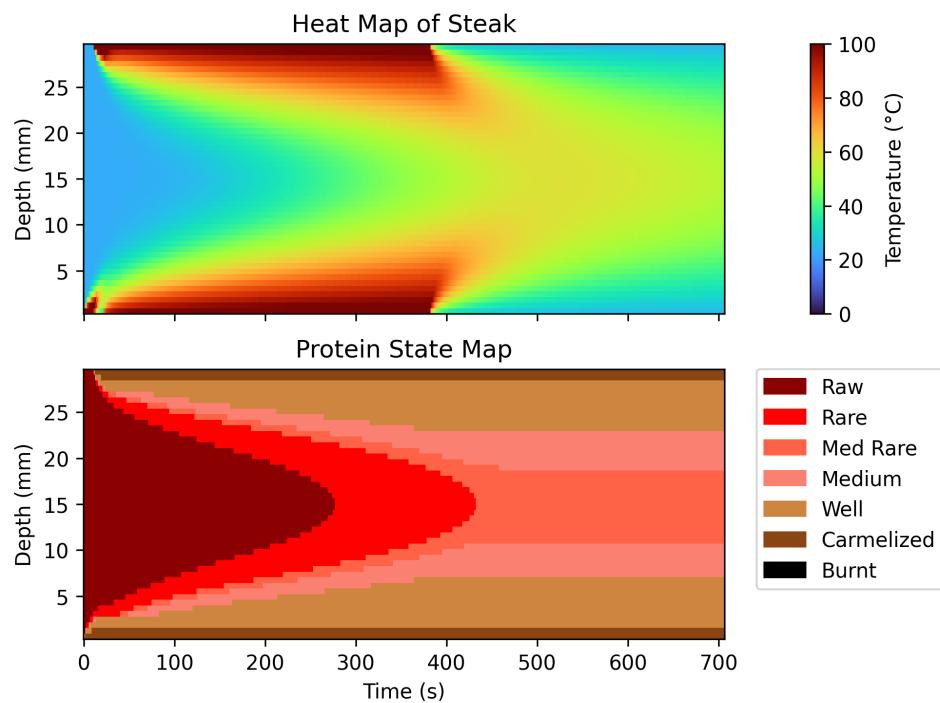


FIGURE 1. Reverse Seared Steak

## 3. A GUITAR STRING - THE ONE-DIMENSIONAL WAVE EQUATION

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$