# MECHANICS

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ABSTRACT. These notes embark on an exploration of the geometric and algebraic foundations underpinning special and general relativity. Starting with the fundamental concepts of symmetry groups and their associated algebras, we build from basic spatial rotations to the full symmetries of flat spacetime and their implications for particle properties. The journey then progresses to the curved spacetime of general relativity, aiming to elucidate key physical phenomena such as the anomalous precession of Mercury's orbit, the relativistic intricacies of satellite timekeeping, and the fascinating physics of rotating black holes. The treatment seeks to bridge abstract formalism with physical insight, providing a foundation for readers new to manifolds, Lie algebras, and the geometric description of gravity.

## 1. Introduction

The goal of these notes is to develop a clear and mathematically grounded understanding of the principles underlying classical mechanics, special relativity, and general relativity. We aim to build the formalism necessary to explore key physical phenomena, such as the precession of Mercury's orbit, the intricacies of satellite timekeeping, and the nature of spinning charged black holes.

We begin in Section 2 by introducing continuous symmetries and their algebraic structure, specifically Lie algebras, using the familiar example of spatial rotations and the classical symmetry group of spacetime. Section 3 then transitions to the fabric of flat spacetime and the symmetries of special relativity, exploring the Lorentz group, its algebra, and the fundamental kinematic consequences. The full symmetry group of flat spacetime, the Poincaré group, and its role in defining particle properties like mass and spin through its Casimir invariants, is discussed in Section 4.

Following this, Section 5 discusses the conceptual leap from special to general relativity, motivating the idea of gravity as spacetime curvature. The mathematical language of curved spacetime, differential geometry, is developed in Section 6. This leads to Einstein's field equations in Section 7, which describe how matter and energy dictate this curvature.

With these tools, Section 8 examines the spacetime geometry around a non-rotating, uncharged spherical mass, applying this to understand phenomena like the perihelion precession of Mercury, relativistic effects on clocks, and gravitational lensing. More complex scenarios involving rotating and charged massive objects, including the concept of an ergosphere, are explored in Section 9. Finally, Section 10 briefly touches upon further striking phenomena and future directions in the study of spacetime and gravity.

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Throughout, the emphasis is on building intuition and formalism simultaneously, focusing on precise mathematical definitions and constructive examples. The intended audience is readers with some background in calculus and linear algebra, but no prior exposure to group theory or differential geometry is assumed.

## 2. Continuous Symmetries and Their Algebraic Structure

In physics and mathematics, continuous symmetry groups describe transformations that preserve certain structures. These groups are smooth spaces with operations that vary smoothly. The algebra associated with such a group is the tangent space at the identity element, equipped with a bracket operation that encodes the infinitesimal structure of the group.

### 2.1. Foundational Example: The Group of Spatial Rotations (SO(3)).

A key example is the group of rotations in three dimensions, consisting of all orthogonal $3 \times 3$ matrices with determinant 1. These rotations preserve lengths and orientations in three-dimensional space. We begin with the group of rotations in three dimensions and its algebra, which provides an accessible example to understand generators, commutation relations, and matrix exponentiation. This lays the groundwork for the more complex groups appearing in relativity.

**Definition 2.0.1.** The rotation group is defined as

$$\mathrm{SO}(3) = \left\{ R \in \mathbb{R}^{3\times 3} \middle| R^\top R = I, \quad \det(R) = 1 \right\}.$$

**Theorem 2.1.** *The associated algebra consists of all $3 \times 3$ real skew-symmetric matrices:*

$$\mathfrak{so}(3) = \left\{ X \in \mathbb{R}^{3\times 3} \middle| X^\top = -X \right\}.$$

*Proof.* Consider a smooth curve $R(t) : \mathbb{R} \to \mathrm{SO}(3)$ such that $R(0) = I$. Since $R(t)$ is orthogonal,

$$R(t)^\top R(t) = I.$$

Differentiate both sides with respect to $t$:

$$\frac{d}{dt}\left(R(t)^\top R(t)\right) = \frac{d}{dt} I = 0.$$

Applying the product rule,

$$\dot{R}(t)^\top R(t) + R(t)^\top \dot{R}(t) = 0.$$

Evaluating at $t = 0$, where $R(0) = I$,

$$\dot{R}(0)^\top I + I^\top \dot{R}(0) = \dot{R}(0)^\top + \dot{R}(0) = 0.$$

Define $X := \dot{R}(0)$. The above equation implies

$$X^\top = -X,$$

so $X \in \mathfrak{so}(3)$ is skew-symmetric. □

**Definition 2.1.1.** The basis elements of algebra $\mathfrak{so}(3)$ are

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Theorem 2.2.** *These satisfy the commutation relations:*

$$[J_i, J_j] = \epsilon_{ijk} J_k,$$

*where $\epsilon_{ijk}$ is the fully antisymmetric symbol with*

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{if any two indices are equal.} \end{cases}$$

*Proof.* This can be proven explicitly by multiplying each pair of generators and computing their commutators. For example,

$$[J_1, J_2] := J_1 J_2 - J_2 J_1 = J_3,$$

and similarly for the other pairs, confirming the relation

$$[J_i, J_j] = \epsilon_{ijk} J_k.$$

□

**Theorem 2.3.** *The image of the exponential map of $\mathfrak{so}(3)$ is contained in* $\mathrm{SO}(3)$*:*

$$\forall X \in \mathfrak{so}(3) : \ \exp(X) \in \mathrm{SO}(3).$$

*Proof.* Let $R = \exp(X)$ for some $X \in \mathfrak{so}(3)$. Since $\mathfrak{so}(3)$ consists of skew-symmetric matrices, we have $X^\top = -X$. Using the identity $(\exp(X))^\top = \exp(X^\top)$, it follows:

$$R^\top R = \exp(X^\top) \exp(X) = \exp(-X) \exp(X).$$

Since $X$ commutes with $-X$ (as $[X, -X] = 0$ for all matrices), we may combine the exponents:

$$R^\top R = \exp(-X + X) = \exp(0) = I,$$

so $R$ is orthogonal. Next, we use the identity $\det(\exp(X)) = \exp(\mathrm{tr}(X))$, valid for all square matrices $X$. Since every $X \in \mathfrak{so}(3)$ is skew-symmetric, we have $\mathrm{tr}(X) = 0$, so:

$$\det(\exp(X)) = \exp(\mathrm{tr}(X)) = \exp(0) = 1.$$

Therefore, $R = \exp(X)$ is a special orthogonal matrix: $\exp(X) \in \mathrm{SO}(3)$. □

### 2.2. Broader Example: The Classical Symmetry Group of Spacetime.

**Definition 2.3.1.** The *classical symmetry group*, denoted CSG(3), is the group associated with classical physics (pre-relativity). It acts on $\mathbb{R}^4$ (space and time) of the form:

$$(t, \vec{x}) \mapsto (t + \tau, R\vec{x} + \vec{v}t + \vec{a}),$$

where $R \in \mathrm{SO}(3)$ is a rotation described in the previous subsection, $\vec{v} \in \mathbb{R}^3$ is a velocity (a "classical boost" or change of inertial frame), $\vec{a} \in \mathbb{R}^3$ is a spatial translation, and $\tau \in \mathbb{R}$ is a time translation. These transformations are also known as Galilean transformations.

**Definition 2.3.2.** If we use an augmented spacetime 5-vector in homogeneous coordinates,

$$\begin{pmatrix} t \\ x \\ y \\ z \\ 1 \end{pmatrix},$$

we can define the classical symmetry algebra $\mathfrak{csg}(3)$ in terms of its generators. The rotation generators $J_1$, $J_2$, and $J_3$ are:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the boost generators $K_1$, $K_2$, $K_3$ are:

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the translation generators $P_1$, $P_2$, $P_3$ are:

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the time translation generator $H$ is:

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Theorem 2.4** (Commutator relations of the classical symmetry algebra $\mathfrak{csg}(3)$)**.** *The generators $J_i$ (rotations), $K_i$ (boosts), $P_i$ (spatial translations), and $H$ (time translation) satisfy the following commutation relations:*

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, K_j] = \epsilon_{ijk} K_k, \quad [J_i, P_j] = \epsilon_{ijk} P_k,$$
$$[K_i, K_j] = 0, \quad [P_i, P_j] = 0, \quad [K_i, P_j] = 0,$$
$$[H, J_i] = 0, \quad [H, P_i] = 0, \quad [H, K_i] = P_i,$$

*where $\epsilon_{ijk}$ is the fully antisymmetric symbol.*

## 3. The Fabric of Flat Spacetime and its Symmetries (Special Relativity)

### 3.1. Redefining Space and Time: The Postulates.

The theory of Special Relativity, formulated by Albert Einstein in 1905, revolutionized our understanding of space and time. It is built upon two fundamental postulates:

1. **The Principle of Relativity:** The laws of physics take the same form in all inertial frames of reference. This means there is no "absolute" rest frame; any inertial frame is equally valid for describing physical phenomena.
2. **The Principle of the Constancy of the Speed of Light:** The speed of light in a vacuum, denoted $c$, has the same value for all inertial observers, regardless of the motion of the light source or the motion of the observer.

These postulates, particularly the second one, have profound consequences, leading to a departure from classical (Galilean) notions of space and time and requiring a new set of transformation laws between inertial frames—the Lorentz transformations. The invariance of the speed of light implies the invariance of the spacetime interval, which is the cornerstone for defining the geometry of flat spacetime and its symmetry group.

### 3.2. Symmetries of Flat Spacetime: The Proper Orthochronous Spacetime Symmetry Group ($\mathrm{SO}^+(1,3)$).

**Definition 3.0.1** (Proper orthochronous spacetime symmetry group)**.** Let $\eta$ be the flat spacetime metric tensor:

$$\eta = \mathrm{diag}(-1, +1, +1, +1).$$

The group of real $4 \times 4$ matrices $\Lambda$ preserving this bilinear form satisfies

$$\Lambda^\top \eta \Lambda = \eta.$$

The subgroup of these matrices with determinant $+1$ and with $\Lambda^0{}_0 \geq 1$ (preserving time orientation) is called the *proper orthochronous spacetime symmetry group*, denoted

$$\mathrm{SO}^+(1,3) = \left\{\Lambda \in \mathrm{GL}(4,\mathbb{R}) \middle| \Lambda^\top \eta \Lambda = \eta,\ \det \Lambda = 1,\ \Lambda^0{}_0 \geq 1\right\}.$$

**Theorem 3.1.** *Rotations in* $\mathrm{SO}(3)$ *are a subgroup of* $\mathrm{SO}^+(1,3)$. *For* $R_3 \in \mathrm{SO}(3)$, *the* $4 \times 4$ *matrix*

$$\Lambda_R = \begin{pmatrix} 1 & 0 \\ 0 & R_3 \end{pmatrix} \text{ is in } \mathrm{SO}^+(1,3).$$

*(Here,* $0$ *denotes row or column vectors of zeros of appropriate size).*

*Proof.* We check the conditions for $\Lambda_R \in \mathrm{SO}^+(1,3)$:

(1) $\Lambda_R^\top \eta \Lambda_R = \eta$:

$$\begin{aligned} \Lambda_R^\top \eta \Lambda_R &= \begin{pmatrix} 1 & 0 \\ 0 & R_3 \end{pmatrix}^\top \begin{pmatrix} -1 & 0 \\ 0 & I_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & R_3^\top \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & I_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & R_3^\top \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & R_3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & R_3^\top R_3 \end{pmatrix}. \end{aligned}$$

Since $R_3 \in \mathrm{SO}(3)$, we have $R_3^\top R_3 = I_3$. Thus,

$$\Lambda_R^\top \eta \Lambda_R = \begin{pmatrix} -1 & 0 \\ 0 & I_3 \end{pmatrix} = \eta.$$

(2) $\det(\Lambda_R) = 1$: The determinant of a block diagonal matrix is the product of the determinants of its diagonal blocks.

$$\det(\Lambda_R) = \det(1) \cdot \det(R_3) = 1 \cdot 1 = 1,$$

since $\det(R_3) = 1$ for $R_3 \in \mathrm{SO}(3)$.

(3) $(\Lambda_R)^0{}_0 \geq 1$: The $(0,0)$ component of $\Lambda_R$ is $(\Lambda_R)^0{}_0 = 1$, which satisfies $1 \geq 1$.

All conditions are met, so $\Lambda_R \in \mathrm{SO}^+(1,3)$. $\square$

**Lemma 3.1.1.** *For a boost in the* $x$*-direction that preserves the interval defined by the flat metric, the transformation matrix* $\Lambda \in \mathrm{SO}(1,1)$ *(a subgroup of* $\mathrm{SO}^+(1,3)$ *acting on* $t, x$*) can be written as*

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix},$$

*where* $\gamma = \frac{1}{\sqrt{1-v^2}}$ *and the* speed $v \in (-1,1)$ *is expressed in natural units* $(c = 1)$. *Alternatively, in terms of the* rapidity *parameter* $\phi := \operatorname{arctanh} v$, *the matrix becomes*

$$\Lambda = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}.$$

*Proof.* Consider a transformation acting only on the $t$ (or $x^0$) and $x$ (or $x^1$) coordinates:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}.$$

Let $\Lambda_{1+1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We require this transformation to preserve the 2D flat spacetime interval, defined by $\eta_{1+1} := \mathrm{diag}(-1, 1)$. The condition is $\Lambda_{1+1}^\top \eta_{1+1} \Lambda_{1+1} = \eta_{1+1}$. Expanding this:

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -A^2 + C^2 & -AB + CD \\ -AB + CD & -B^2 + D^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This yields the system of equations:

$$\begin{aligned} -A^2 + C^2 &= -1 \\ -AB + CD &= 0 \\ -B^2 + D^2 &= 1 \end{aligned}$$

And the determinant condition for a special transformation (unit determinant):

$$AD - BC = 1$$

From $-A^2 + C^2 = -1$, we rearrange to $A^2 = 1 + C^2$. Since $C^2 \geq 0$, this implies $A^2 \geq 1$, so $|A| \geq 1$. To ensure that the transformation does not reverse the direction of time (i.e., it is orthochronous), we impose the physical condition $A > 0$. Combined with $|A| \geq 1$, this leads to the requirement $A \geq 1$. This is a defining characteristic of the orthochronous Lorentz transformations we are considering.

Similarly, from $-B^2 + D^2 = 1$, we rearrange to $D^2 = 1 + B^2$. Since $B^2 \geq 0$, this implies $D^2 \geq 1$, so $|D| \geq 1$. The choice between $D \geq 1$ and $D \leq -1$ determines the relative orientation of the spatial $x'$-axis. For a "standard" boost along the positive $x$-direction, which does not involve an additional spatial reflection (e.g., $x \to -x$), we choose the convention $D \geq 1$. If $D \leq -1$, the transformation would correspond to a boost combined with such a reflection.

Given $A \geq 1$, the equation $A^2 - C^2 = 1$ allows us to parameterize $A = \cosh \phi_A$ and $C = s_C \sinh \phi_A$ for some real parameter $\phi_A$ and $s_C = \pm 1$. Similarly, given $D \geq 1$, the equation $D^2 - B^2 = 1$ allows us to parameterize $D = \cosh \phi_D$ and $B = s_B \sinh \phi_D$ for some real parameter $\phi_D$ and $s_B = \pm 1$.

Now substitute these parameterizations into $-AB + CD = 0$ ($AB = CD$):

$$(\cosh \phi_A)(s_B \sinh \phi_D) = (s_C \sinh \phi_A)(\cosh \phi_D).$$

Assuming a non-trivial boost (so that $\sinh \phi_A \neq 0$ and $\sinh \phi_D \neq 0$), we can divide by the product $\cosh \phi_A \cosh \phi_D$. Since $\cosh \phi_A \geq 1$ and $\cosh \phi_D \geq 1$, this yields:

$$s_B \tanh \phi_D = s_C \tanh \phi_A.$$

For a single, simple transformation (represented by a single rapidity $\phi$), we expect $\phi_A$ and $\phi_D$ to be the same or directly related. If we set $\phi_A = \phi_D =: \phi$, then this equation implies $s_B = s_C =: s$. So our parameters become: $A = \cosh \phi$, $C = s \sinh \phi$, $D = \cosh \phi$, $B = s \sinh \phi$.

Next, use the determinant condition $AD - BC = 1$:

$$(\cosh \phi)(\cosh \phi) - (s \sinh \phi)(s \sinh \phi) = \cosh^2 \phi - s^2 \sinh^2 \phi = 1.$$

Since $s^2 = (\pm 1)^2 = 1$, this simplifies to $\cosh^2\phi - \sinh^2\phi = 1$. This is the fundamental hyperbolic identity, which is always true for any real $\phi$. Thus, the parameters $A = \cosh\phi, D = \cosh\phi, B = s\sinh\phi, C = s\sinh\phi$ satisfy all conditions derived from metric preservation and unit determinant. The matrix is $\Lambda_{1+1} = \begin{pmatrix} \cosh\phi & s\sinh\phi \\ s\sinh\phi & \cosh\phi \end{pmatrix}$.

The specific form stated in the lemma, $\Lambda = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$, corresponds to choosing the sign factor $s = -1$. This means $A = \cosh\phi, B = -\sinh\phi, C = -\sinh\phi, D = \cosh\phi$. This is the standard parameterization for a boost.

To relate $\phi$ to the speed $v$: consider the origin of the $S'$ frame ($x' = 0$). It moves with velocity $v$ in the $S$ frame, so its worldline is $x = vt$. Substituting into $x' = Ct + Dx$:

$$0 = Ct + D(vt) = (C + Dv)t.$$

For this to hold for all $t \neq 0$, we need $C+Dv = 0$, so $v = -C/D$. Using $C = -\sinh\phi$ and $D = \cosh\phi$ (which corresponds to our choice $s = -1$):

$$v = -(-\sinh\phi)/(\cosh\phi) = \tanh\phi.$$

Thus, the rapidity $\phi = \operatorname{arctanh} v$. Given $v = \tanh\phi$:

$$1 - v^2 = 1 - \tanh^2\phi = \operatorname{sech}^2\phi = \frac{1}{\cosh^2\phi}.$$

So, $\cosh\phi = \frac{1}{\sqrt{1-v^2}}$, which is defined as the Lorentz factor $\gamma$. And $\sinh\phi = \tanh\phi\cosh\phi = v\gamma$. This implies that the speed $|v| < 1$ (since $\phi$ is real, $\tanh\phi \in (-1, 1)$). Substituting $\cosh\phi = \gamma$ and $\sinh\phi = v\gamma$ (with $s = -1$ implicitly included in the signs of $B$ and $C$) back into the matrix form $A = \cosh\phi, B = -\sinh\phi, C = -\sinh\phi, D = \cosh\phi$ gives

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix}.$$

□

**Theorem 3.2.** *A general boost in an arbitrary direction $\hat{n}$ (a unit 3-vector) with speed $v$ is given by the $4 \times 4$ matrix $\Lambda_B \in \mathrm{SO}^+(1, 3)$:*

$$\Lambda_B = \begin{pmatrix} \gamma & -\gamma v\hat{n}^\top \\ -\gamma v\hat{n} & I_3 + (\gamma - 1)\,\hat{n}\hat{n}^\top \end{pmatrix}$$

*where $\gamma = \frac{1}{\sqrt{1-v^2}}$. In terms of rapidity $\phi = \operatorname{arctanh} v$:*

$$\Lambda_B = \begin{pmatrix} \cosh\phi & -\sinh\phi\hat{n}^\top \\ -\sinh\phi\hat{n} & I_3 + (\cosh\phi - 1)\hat{n}\hat{n}^\top \end{pmatrix}.$$

*(Proof omitted here but involves verifying $\Lambda_B^\top\eta\Lambda_B = \eta$, $\det(\Lambda_B) = 1$, and $(\Lambda_B)^0{}_0 \geq 1$).*

3.3. **The Algebra of Spacetime Symmetries ($\mathfrak{so}(1, 3)$).** The Lie algebra associated with the Lorentz group $\mathrm{SO}^+(1, 3)$ is denoted $\mathfrak{so}(1, 3)$. It consists of all $4 \times 4$ real matrices $X$ such that a curve $\Lambda(s) = \exp(sX)$ is in $\mathrm{SO}^+(1, 3)$ for small $s$. Differentiating $\Lambda(s)^\top\eta\Lambda(s) = \eta$ at $s = 0$ (where $\Lambda(0) = I$) yields the condition for elements $X \in \mathfrak{so}(1, 3)$:

$$X^\top\eta + \eta X = 0.$$

This condition means $X$ is ”$\eta$-antisymmetric” or ”$\eta$-skew-adjoint”.

A convenient basis for this 6-dimensional algebra is given by:

- Three generators of spatial rotations, $J_1, J_2, J_3$:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

  These are analogous to the $\mathfrak{so}(3)$ generators, embedded in $4 \times 4$ matrices.
- Three generators of boosts, $K_1, K_2, K_3$:

$$K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Note: These generators $X$ indeed satisfy $X^\top \eta + \eta X = 0$ with $\eta = \mathrm{diag}(-1, 1, 1, 1)$. For example, for $K_1$:

$$\begin{aligned} K_1^\top \eta + \eta K_1 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

**Theorem 3.3** (Commutator relations of the algebra $\mathfrak{so}(1,3)$)**.** *The generators $J_i$ (rotations) and $K_i$ (boosts) satisfy the following commutation relations:*

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ijk} J_k, \\ [J_i, K_j] &= \epsilon_{ijk} K_k, \\ [K_i, K_j] &= -\epsilon_{ijk} J_k, \end{aligned}$$

*where $\epsilon_{ijk}$ is the Levi-Civita symbol.*

(Proof involves direct computation of the matrix commutators.)

Finite Lorentz transformations can be constructed by exponentiating linear combinations of these generators:

$$\Lambda = \exp\left( \sum_i \theta_i J_i + \sum_i \phi_i K_i \right),$$

where $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ are rotation parameters (angle and axis) and $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ are boost parameters (rapidity and direction).

3.4. **Kinematic Consequences.** The Lorentz transformations have profound implications for how different inertial observers measure time intervals and spatial lengths. These effects are not mere illusions but are fundamental to the nature of spacetime.

3.5. **Structure of Flat Spacetime.** Special relativity operates in a four-dimensional **flat spacetime**. Its geometric structure is defined by a metric tensor, $\eta_{\mu\nu}$, which is used to calculate the spacetime interval between events.

Consistent with the definition of the Lorentz group $\mathrm{SO}^+(1,3)$ used earlier, we adopt the metric signature $(-1,+1,+1,+1)$. In standard Cartesian coordinates $(x^0,x^1,x^2,x^3)=(ct,x,y,z)$ (with $c=1$ in natural units), the metric tensor is:

$$\eta_{\mu\nu}=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

The **invariant spacetime interval**, $s^2$ (or often $ds^2$ for infinitesimal separations), between two events separated by $dx^\mu=(dx^0,dx^1,dx^2,dx^3)$ is given by:

$$ds^2=\eta_{\mu\nu}dx^\mu dx^\nu=-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2.$$

This interval is invariant under Lorentz transformations (rotations and boosts). The nature of the interval between two events determines their causal relationship:

- **Timelike interval**: $ds^2<0$. It is possible for a massive particle to travel between these events; one event can causally affect the other. The proper time $d\tau$ is defined by $d\tau^2=-ds^2/c^2$ (or $d\tau^2=-ds^2$ if $c=1$).
- **Spacelike interval**: $ds^2>0$. It is not possible for a massive particle to travel between these events; they are causally disconnected in the sense that neither can lie in the past or future light cone of the other. The proper distance $d\sigma$ is $d\sigma^2=ds^2$.
- **Null (or lightlike) interval**: $ds^2=0$. Only massless particles (like photons) can travel along such an interval; these define the light cones.

At every point in this spacetime, a **tangent space** can be defined. This is a vector space whose elements are 4-vectors, such as 4-displacement $dx^\mu$, 4-velocity $v^\mu$, 4-momentum $p^\mu$, etc. These vectors represent physical quantities or directions in spacetime. The metric tensor $\eta_{\mu\nu}$ defines an inner product on this tangent space: for two 4-vectors $A^\mu$ and $B^\nu$, their inner product is $\langle A,B\rangle=\eta_{\mu\nu}A^\mu B^\nu=A_\mu B^\mu=A^\mu B_\mu$.

A **4-velocity** $v^\mu=\frac{dx^\mu}{d\tau}$ describes the rate of change of spacetime coordinates $x^\mu$ with respect to the proper time $\tau$ along a particle's worldline. Proper time is the time measured by a clock moving with the particle. For a massive particle, its 4-velocity is normalized such that:

$$\eta_{\mu\nu}v^\mu v^\nu=v^\mu v_\mu=-1\quad(\text{with } c=1).$$

This normalization reflects that in the particle's own rest frame, its 4-velocity is purely in the time direction ($v^\mu_{rest}=(1,0,0,0)$ if $c=1$), and $d\tau^2=-(dx^0_{rest})^2$. The metric and the concept of the invariant interval are thus fundamental to understanding motion and causality in special relativity.

## 4. Symmetries in Special Relativity: The Poincaré Group and Particle States

This section explores the full symmetry group of flat spacetime, the Poincaré group, and its profound implications for defining fundamental particle properties.

4.1. **Introduction to the Poincaré Group.**

4.2. **The Lie Algebra of the Poincaré Group.**

4.3. **Noether's Theorem and Conservation Laws from Poincaré Invariance.**

4.4. **Casimir Invariants: Defining Intrinsic Particle Properties.** This subsection discusses operators that commute with all generators of the Poincaré algebra, leading to fundamental invariant properties of particles.

4.4.1. *What are Casimir Invariants?*

4.4.2. *The $P^2$ Invariant: Mass.* The first key Casimir invariant is $P^2 = P_\mu P^\mu$.
Proof that $P^2$ is a Casimir Invariant: *[Proof to be filled in by user based on our previous discussion.]*
Physical Significance of $P^2$: The eigenvalue of $P^2$ is $m^2$ (or $-m^2c^2$ depending on metric and $c$ conventions), where $m$ is the invariant rest mass of the particle. This demonstrates that mass is a fundamental Poincaré-invariant characteristic.

4.4.3. *The $W^2$ Invariant: Spin.* The second key Casimir invariant is $W^2 = W_\mu W^\mu$, where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma$ is the Pauli-Lubanski pseudovector.

4.5. **The Role of Schur's Lemma: Why Casimir Eigenvalues Label Particles.** This subsection explains the mathematical theorem that underpins why Casimir invariants provide constant labels for irreducible representations.

4.5.1. *Statement of Schur's Lemma.*

4.5.2. *Proof of Schur's Lemma. [Proof to be filled in by user based on our previous discussion. This involves showing that the eigenspace of the commuting operator is an invariant subspace, then using irreducibility.]*

4.5.3. *Implication for Casimir Invariants.* Since Casimir invariants commute with all generators (and thus all elements of a representation), Schur's Lemma implies they act as a constant scalar (their eigenvalue) on any irreducible representation.

4.6. **Consequence: Classification of Particles in Special Relativity.**

## 5. Transition to Curved Spacetime (General Relativity)

The theory of special relativity, while revolutionary, is fundamentally a theory of flat spacetime. It does not incorporate gravity. Newtonian gravity, on the other hand, describes gravity as a force acting instantaneously at a distance, which is incompatible with the finite speed of light posited in special relativity. General relativity emerges from the need to reconcile these ideas.

5.1. **The Principle of Equivalence.**

5.2. **Gravity as Spacetime Curvature.**

## 6. The Mathematical Language of Curved Spacetime (Differential Geometry)

To describe gravity as the curvature of spacetime, we need the tools of differential geometry. Spacetime is no longer assumed to be flat but is instead a more general mathematical object called a manifold, equipped with a metric tensor that can vary from point to point.

### 6.1. Manifolds and Charts.

### 6.2. Tangent Spaces, Vectors, and Tensors in GR.

### 6.3. The Metric Tensor $g_{\mu\nu}$.

### 6.4. Covariant Derivative and Parallel Transport.

### 6.5. Geodesics: Paths of Freely Falling Particles.

### 6.6. Curvature: Riemann Tensor, Ricci Tensor, Ricci Scalar.

## 7. The Field Equations of Gravity: How Matter and Energy Dictate Spacetime Curvature

### 7.1. The Stress-Energy Tensor $T_{\mu\nu}$.

### 7.2. The Einstein Tensor $G_{\mu\nu}$.

### 7.3. The Equations: $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$.

## 8. Gravity of a Non-Rotating, Uncharged Spherical Mass (and its Observational Tests)

One of the first and most important solutions to Einstein's field equations describes the spacetime outside a static, spherically symmetric, uncharged massive object. This is known as the Schwarzschild solution.

### 8.1. The Metric for a Spherical Mass (Schwarzschild Metric).

### 8.2. Properties: Event Horizon, Gravitational Singularity.

**8.3. Application 1: Perihelion Precession of Orbits (e.g., Mercury).** The perihelion precession of Mercury's orbit is a seminal observational phenomenon that challenged Newtonian gravity and provided empirical support for the relativistic description of gravitation. Classical Newtonian mechanics predicts orbital motion based on an inverse-square central force, resulting in closed elliptical orbits. However, observations show an additional precession unaccounted for by classical perturbations from other planets. General relativity explains this anomaly through the curvature of spacetime caused by the Sun's mass. By modeling Mercury's trajectory as a geodesic in the Schwarzschild metric, one derives corrections to the Newtonian orbital elements, resulting in a predicted advance of the perihelion that matches observations to high precision. This section presents the derivation of the relativistic corrections to Mercury's orbit, starting from the Schwarzschild solution to Einstein's field equations and applying perturbation techniques to the geodesic equations of motion.

### 8.4. Application 2: Relativistic Effects on Clocks and Timekeeping (e.g., for Satellites/GPS).

### 8.5. Application 3: The Bending of Light and Gravitational Lensing.

#### 8.5.1. *The "Einstein Cross" and other lensing phenomena.*

## 9. Beyond Simple Cases: Rotating and Charged Massive Objects

### 9.1. Need for More General Solutions.

### 9.2. Rotating Massive Objects (e.g., Kerr solution).

#### 9.2.1. *The Ergosphere.*

### 9.3. Charged, Non-Rotating Massive Objects (e.g., Reissner-Nordström solution).

### 9.4. Charged AND Rotating Massive Objects (e.g., Kerr-Newman solution).

## 10. Further Striking Phenomena and Future Directions