quadcopter package

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I. Introduction

THE quadcopter package is filled with obtuse and often confusing code derived from various physics equations and mathematical formulas. This paper hopes to bridge the gap between quadcopter theory and the src files included. This paper can also be used as a reference for later expantion of this package.

II. The Model Node

While an actual quadcopter would be helpful, a simulation is the next best thing. The model node simulates the physics of a quadcopter. The physics in this section also provide insight on how motor input and sensor output behave.

A. Rotational Kinetics

The rotation matrices (orthogonal transforms between bases) and their compositions come in handy for later calculations and can be found in the appendix. Throughout this paper, I use notation of the sum of scaled unit vectors (versors), but coding is better described as floats. The dot product of vectors provides these scalaras. The rotation matrices provide a way to move between these arrays of floats. For example, consider the rotation matrix from basis A to B.

$$\begin{pmatrix} \mathbf{v}\cdot\hat{\mathbf{b}}_x \\ \mathbf{v}\cdot\hat{\mathbf{b}}_y \\ \mathbf{v}\cdot\hat{\mathbf{b}}_z \end{pmatrix} = {}^{\mathrm{B}}R^{\mathrm{A}} \begin{pmatrix} \mathbf{v}\cdot\hat{\mathbf{a}}_x \\ \mathbf{v}\cdot\hat{\mathbf{a}}_y \\ \mathbf{v}\cdot\hat{\mathbf{a}}_z \end{pmatrix} \tag{1}$$

The rotation matrices also collect the dot products to push final calculations down the road.

$${}^{\mathrm{B}}R^{\mathrm{A}} = \begin{pmatrix} \hat{\mathbf{a}}_x\cdot\hat{\mathbf{b}}_x & \hat{\mathbf{a}}_y\cdot\hat{\mathbf{b}}_x & \hat{\mathbf{a}}_z\cdot\hat{\mathbf{b}}_x \\ \hat{\mathbf{a}}_x\cdot\hat{\mathbf{b}}_y & \hat{\mathbf{a}}_y\cdot\hat{\mathbf{b}}_y & \hat{\mathbf{a}}_z\cdot\hat{\mathbf{b}}_y \\ \hat{\mathbf{a}}_x\cdot\hat{\mathbf{b}}_z & \hat{\mathbf{a}}_y\cdot\hat{\mathbf{b}}_z & \hat{\mathbf{a}}_z\cdot\hat{\mathbf{b}}_z \end{pmatrix} \tag{2}$$

The rotation of the quadcopter can be described by a composition of three rotations: from a Newtonian frame N to A, by an angle a around their common x axis (the *roll*); from intermediate frames A to B, by angle b around their common y axis (the *pitch*); and from B to the quadcopter body frame C, by angle c around their common z axis (the *yaw*). The angles and bases are chosen to make the angular velocities positive. On the quadcopter frame $\hat{\mathbf{c}}_x$ points to the front motor, $\hat{\mathbf{c}}_y$ points to the left motor, and $\hat{\mathbf{c}}_z$ is the cross product of the two.

$${}_{\mathrm{N}}\boldsymbol{\omega}_{\mathrm{A}} = \dot{a}\hat{\mathbf{n}}_x = \dot{a}\hat{\mathbf{a}}_x \tag{3}$$

$${}_{\mathrm{A}}\boldsymbol{\omega}_{\mathrm{B}} = \dot{b}\hat{\mathbf{a}}_y = \dot{b}\hat{\mathbf{b}}_y \tag{4}$$

$${}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{C}} = \dot{c}\hat{\mathbf{b}}_z = \dot{c}\hat{\mathbf{c}}_z \tag{5}$$

Using the notation of the rotation matrices above, the composition of rotation matrices is the matrix product.

$${}^{\mathrm{C}}R^{\mathrm{N}} = \left({}^{\mathrm{C}}R^{\mathrm{B}}\right)\left({}^{\mathrm{B}}R^{\mathrm{A}}\right)\left({}^{\mathrm{A}}R^{\mathrm{N}}\right) \tag{6}$$

TABLE I
Transforms from the inertial frame N
($s_x = \sin x$, $c_x = \cos x$)

${}^{\mathrm{U}}R^{\mathrm{N}}$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$	$\hat{\mathbf{n}}_z$
$\hat{\mathbf{a}}_x$	1	0	0
$\hat{\mathbf{a}}_y$	0	c_a	s_a
$\hat{\mathbf{a}}_z$	0	$-s_a$	c_a
$\hat{\mathbf{b}}_x$	c_b	$s_a s_b$	$-s_a c_a$
$\hat{\mathbf{b}}_y$	0	c_a	s_a
$\hat{\mathbf{b}}_z$	s_b	$-s_a c_b$	$c_a c_b$
$\hat{\mathbf{c}}_x$	$c_b c_c$	$-c_a s_c + s_a s_b c_c$	$-s_a s_c - c_a s_b c_c$
$\hat{\mathbf{c}}_y$	$c_b s_c$	$c_a c_c + s_a s_b s_c$	$s_a c_c - c_a s_b s_c$
$\hat{\mathbf{c}}_z$	s_b	$-s_a c_b$	$c_a c_b$

TABLE II
Transforms from the robot frame C
($s_x = \sin x$, $c_x = \cos x$)

${}^{\mathrm{U}}R^{\mathrm{C}}$	$\hat{\mathbf{c}}_x$	$\hat{\mathbf{c}}_y$	$\hat{\mathbf{c}}_z$
$\hat{\mathbf{b}}_x$	c_c	s_c	0
$\hat{\mathbf{b}}_y$	$-s_c$	c_c	0
$\hat{\mathbf{b}}_z$	0	0	1
$\hat{\mathbf{a}}_x$	$c_b c_c$	$c_b s_c$	s_b
$\hat{\mathbf{a}}_y$	$-s_c$	c_c	0
$\hat{\mathbf{a}}_z$	$s_b c_c$	$-s_b s_c$	c_b
$\hat{\mathbf{n}}_x$	$c_b c_c$	$c_b s_c$	s_b
$\hat{\mathbf{n}}_y$	$-c_a s_c + s_a s_b s_c$	$c_a c_c + s_a s_b s_c$	$-s_a c_b$
$\hat{\mathbf{n}}_z$	$-s_a s_c - c_a s_b c_c$	$s_a c_c - c_a s_b s_c$	$c_a c_b$

The important combinations of dot products can be found in tables I and II.

B. Linear Equations of Motion

If you choose the right reference frames the linear equations of motion are easy to express. First, the motion of the quadcopter from its origin point is defined simply:

$$\mathbf{r}_{c/o} = x\hat{\mathbf{n}}_x + y\hat{\mathbf{n}}_y + z\hat{\mathbf{n}}_z \tag{7}$$

$${}_{\mathrm{N}}\mathbf{v}_c = \dot{x}\hat{\mathbf{n}}_x + \dot{y}\hat{\mathbf{n}}_y + \dot{z}\hat{\mathbf{n}}_z \tag{8}$$

$${}_{\mathrm{N}}\mathbf{a}_c = \ddot{x}\hat{\mathbf{n}}_x + \ddot{y}\hat{\mathbf{n}}_y + \ddot{z}\hat{\mathbf{n}}_z \tag{9}$$

For our model we assume each motor puts out a force proportional to its square. If we let m be the total mass of the robot and k be the proportionality constant:

$$m_{\mathrm{N}}\mathbf{a}_c = k\left(\omega_f^2 + \omega_l^2 + \omega_b^2 + \omega_r^2\right)\hat{\mathbf{c}}_y - mg\hat{\mathbf{n}}_y \tag{10}$$

Where f, l, b, and r stand for the front, right, back and left motors respectively. Expressed in the inertial frame N the equations of motion are:

$$\ddot{x} = \frac{k}{m}\Omega\cos b\sin c \tag{11}$$

$$\ddot{y} = \frac{k}{m}\Omega\cos a\cos c + \frac{k}{m}\Omega\sin a\sin b\sin c - g \tag{12}$$

$$\ddot{z} = \frac{k}{m}\Omega\sin a\cos c - \frac{k}{m}\Omega\cos a\sin b\sin c \tag{13}$$

where Ω is the sum of the squares of the motor inputs.

III. Rotational Momentum

Supposing that most of the mass of the quadcopter is in the motors, the inertia dyadics for the four motors is:

$$\mathbf{I}^{f/c} = \frac{mL^2}{4}\hat{\mathbf{c}}_y\hat{\mathbf{c}}_y + \frac{mL^2}{4}\hat{\mathbf{c}}_z\hat{\mathbf{c}}_z \quad (14)$$

$$\mathbf{I}^{l/c} = \frac{mL^2}{4}\hat{\mathbf{c}}_x\hat{\mathbf{c}}_x + \frac{mL^2}{4}\hat{\mathbf{c}}_z\hat{\mathbf{c}}_z \quad (15)$$

$$\mathbf{I}^{b/c} = \frac{mL^2}{4}\hat{\mathbf{c}}_y\hat{\mathbf{c}}_y + \frac{mL^2}{4}\hat{\mathbf{c}}_z\hat{\mathbf{c}}_z \quad (16)$$

$$\mathbf{I}^{r/c} = \frac{mL^2}{4}\hat{\mathbf{c}}_x\hat{\mathbf{c}}_x + \frac{mL^2}{4}\hat{\mathbf{c}}_z\hat{\mathbf{c}}_z \quad (17)$$

The total inertial dyadic of the robot is then

$$\mathbf{I}^{R/c} = \frac{mL^2}{2}\left(\hat{\mathbf{c}}_x\hat{\mathbf{c}}_x + \hat{\mathbf{c}}_y\hat{\mathbf{c}}_y + 2\hat{\mathbf{c}}_z\hat{\mathbf{c}}_z\right) \quad (18)$$

The angular velocity is the compositions of simple velocities given in equations 3-5.

$${}^{\mathrm{N}}\boldsymbol{\omega}_C = \dot{a}\hat{\mathbf{a}}_x + \dot{b}\hat{\mathbf{b}}_y + \dot{c}\hat{\mathbf{c}}_z \quad (19)$$

And the angular momentum is the dot product of the two

$$\begin{aligned} {}^{\mathrm{N}}\mathbf{H}^{R/c} &= \mathbf{I}^{R/c}\cdot{}^{\mathrm{N}}\boldsymbol{\omega}_{\mathrm{C}} \\ &= \frac{mL^2}{4}\left(\dot{a}\cos b\cos c - \dot{b}\sin c\right)\hat{\mathbf{c}}_x \\ &\quad + \frac{mL^2}{4}\left(\dot{a}\cos b\sin c + \dot{b}\cos c\right)\hat{\mathbf{c}}_y \\ &\quad + \frac{mL^2}{4}\left(\dot{a}\sin b + 2\dot{c}\right)\hat{\mathbf{c}}_z \end{aligned} \quad (20)$$

The angular acceleration (in the inertial frame) can be found with some manipulation of derivatives in reference frames.

$$\begin{aligned} {}^{\mathrm{N}}\boldsymbol{\alpha}_{\mathrm{C}} &= \frac{{}^{\mathrm{N}}\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{N}}\boldsymbol{\omega}_{\mathrm{C}} \\ &= \frac{{}^{\mathrm{N}}\mathrm{d}}{\mathrm{d}t}\dot{a}\hat{\mathbf{n}}_x + \frac{{}^{\mathrm{A}}\mathrm{d}}{\mathrm{d}t}\dot{b}\mathbf{a}_y + {}^{\mathrm{N}}\boldsymbol{\omega}_{\mathrm{A}}\times\dot{b}\hat{\mathbf{a}}_y \\ &\quad + \frac{{}^{\mathrm{B}}\mathrm{d}}{\mathrm{d}t}\dot{c}\hat{\mathbf{b}}_z + {}^{\mathrm{N}}\boldsymbol{\omega}_{\mathrm{B}}\times\dot{c}\hat{\mathbf{b}}_z \\ &= \ddot{a}\hat{\mathbf{n}}_x + \ddot{b}\hat{\mathbf{a}}_y + \ddot{c}\hat{\mathbf{c}}_z \\ &\quad + \dot{a}\dot{b}\hat{\mathbf{a}}_z + \dot{b}\dot{c}\hat{\mathbf{b}}_x + \dot{a}\dot{c}\left(\hat{\mathbf{n}}_x\times\hat{\mathbf{c}}_z\right) \end{aligned} \quad (21)$$

The moment from the reference frame $\mathbf{M}^* = \mathbf{I}\cdot\boldsymbol{\alpha} + \boldsymbol{\omega}\times\mathbf{H}$ was calculated with a script

$$\begin{aligned} \mathbf{M}^*\cdot\hat{\mathbf{c}}_x &= \ddot{a}\cos b\cos c + \ddot{b}\sin c - \dot{a}\dot{b}\sin b\cos c \\ &\quad -\dot{a}\dot{c}\sin c\cos b + \dot{b}\dot{c}\cos c \end{aligned} \quad (22)$$

$$\begin{aligned} \mathbf{M}^*\cdot\hat{\mathbf{c}}_y &= -\ddot{a}\sin c\cos b + \ddot{b}\cos c + \dot{a}\dot{b}\sin b\sin c \\ &\quad -\dot{a}\dot{c}\cos b\cos c - \dot{b}\dot{c}\sin c \end{aligned} \quad (23)$$

$$\mathbf{M}^*\cdot\hat{\mathbf{c}}_z = \ddot{a}\sin b + \ddot{c} + \dot{a}\dot{b}\cos b \quad (24)$$

Quaternion Definition

Quadcopters rotate. And flip. Sometimes they even hover. In order to do any of these things, the rotation of the quadcopter *must* be tracked. A great way for tracking rotations is unit quaternions. This section covers the basic algebra the quaternion ring $(\mathbb{H}, +, \otimes)$. A quaternion is a member of a 4-dimensional vector space

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0 + q_0 \\ p_1 + q_1 \\ p_2 + q_2 \\ p_3 + q_3 \end{pmatrix} \quad (25)$$

Multiplication is more complex

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \otimes \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 \\ p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2 \\ p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1 \\ p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0 \end{pmatrix} \quad (26)$$

Please note, quaternion multiplication is not commutative. Some other algebraic properties: Every scalar can be expressed as a quaternion $(s, 0, 0, 0)$ and every (3-)vector can be a quaternion $(0, v1, v2, v3)$. The 0-quaternion $(0, 0, 0, 0)$ is the additive identity and the 1-quaternion the multiplicative identity $(1, 0, 0, 0)$. The conjugate of a quaternion has inverted signs on the vector components. $q^* = (q0, q1, q2, q3)^* = (q0, -q1, -q2, -q3)$. The norm of a quaternion is the square root of the components, or the square of the quaternion multiplied by its conjugate (if we treat a purely scalar quaternion as a scalar).

$$\|q\| = \sqrt{q\otimes q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \quad (27)$$

Every quaternion has an inverse q^{-1} such that $q\otimes q^{-1} = 1 = q^{-1}\otimes q$. The inverse is $q^{-1} = q^*/\|q\|$. A quaternion with norm equal to one is called a unit quaternion, or versor. The quaternions covered outside this appendix can be considered unit quaternions. The $\otimes$ will be omitted too. We've seen that quaternion arithmetic can be applied to scalars and vectors. Pure vector multiplication notation will remain $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$ for clarification, despite these also being quaternion operations.

A. Quaternion Kinematics

The ring of unti quaternions can be used to describe rotations in space. This quaternion is expressed by the following

$$q = \exp\left(\frac{\theta}{2}\hat{\mathbf{u}}\right) = \cos\left(\frac{\theta}{2}\right) + \hat{\mathbf{u}}\sin\left(\frac{\theta}{2}\right) \quad (28)$$

where $\hat{\mathbf{u}}$ is a unit vector axis of rotation and θ is the angle it rotated.

Quaternion rotation from angular velocity

$$\dot{q} = \frac{1}{2}q\boldsymbol{\omega} \quad (29)$$

Linear interpolation

$$q_{k+1} = \frac{1}{2}q_k\boldsymbol{\omega}\Delta t + q_k \quad (30)$$

Spherical interpolation?

$$q_{k+1} = \exp\left(\frac{1}{2}\boldsymbol{\omega}\Delta t\right)q_k \quad (31)$$