QUATERNION ROTATION

ABSTRACT. Quaternions are almost magical with their ability to look like other concepts. They can be described by group/ring theory, or in terms of hypercomplex numbers, or in terms of vector spaces, and they can exist as all three of these at once. There are many resources on the algebra and analysis of quaternions, but often their definitions disagree, making it difficult to compare results. This paper hopes to formally define one type of quaternion operations and use those to derive final results. We will brush over proofs of generic quaternion algebra, and push through to applications ready for implementation in code. The language in this paper is more formal than other treatments, but often formal thinking is useful for defining your quaternion class in the programming language of choice.

1. The Quaternion Ring

Definition 1.1 (Quaternion)**.** Quaternions, hopefully obvious from their name, are ordered 4-tuples

$$\mathrm{q} = (x, y, z, w) \in \mathbb{H} \subset \mathbb{R}^4. \tag{1.1}$$

The set of quaternions $\mathbb{H}$ form a division ring with two operations $\oplus$ and $\otimes$

Definition 1.2 (Quaternion Addition)**.** Addition on quaternions is defined easily enough

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix} \tag{1.2}$$

Definition 1.3 (Quaternion Multiplication)**.** However, multiplication is more complex

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1x_2 + x_1w_2 + y_1z_2 - z_1y_2 \\ w_1y_2 - x_1z_2 + y_1w_2 + z_1x_2 \\ w_1z_2 + x_1y_2 - y_1x_2 + z_1w_2 \\ w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2 \end{pmatrix} \tag{1.3}$$

But multiplication is what makes quaternions, quaternions. Other sources will often have a different definition of multiplication for quaternions that is incompatible with this one. All quaternion multiplication definitions should have similar properties, but when applying this to other topics, the definition of This formulation of multiplication was chosen to coincide with the ROS package `tf2`. A less formal re-creation of some of these definitions can be found in the next section.

Theorem 1.4. *Quaternions form a commutative group with addition, which means they have the following properties*

- *Associativity:* $\mathrm{q}_1 \oplus (\mathrm{q}_2 \oplus \mathrm{q}_3) = (\mathrm{q}_1 \oplus \mathrm{q}_2) \oplus \mathrm{q}_3$
- *Commutativity:* $\mathrm{q}_1 \oplus \mathrm{q}_2 = \mathrm{q}_2 \oplus \mathrm{q}_1$
- *Identity:* $\mathrm{q} \oplus \mathrm{e}_0 = \mathrm{q} = \mathrm{e}_0 \oplus \mathrm{q}$ *where* e_0 *is the 0-quaternion* $(0, 0, 0, 0)$

- *Inverse:* $\mathrm{q} \oplus (-q) = \mathrm{q} \ominus \mathrm{q} = \mathrm{e}_0 = (-\mathrm{q}) \oplus \mathrm{q}$ *where the inverse is defined* $-\mathrm{q} = (-x, -y, -z, -w)$

Theorem 1.5. *Unlike addition, quaternions for a* non-commutative *group with multiplication, which gives them the following properties*

- *Associativity:* $\mathrm{q}_1 \otimes (\mathrm{q}_2 \otimes \mathrm{q}_3) = (\mathrm{q}_1 \otimes \mathrm{q}_2) \otimes \mathrm{q}_3$
- *Identity:* $\mathrm{q} \otimes \mathbf{1} = \mathrm{q} = \mathbf{1} \otimes \mathrm{q}$ *where* $\mathbf{1}$ *is the identity quaternion* $(0, 0, 0, 1)$
- *Inverse:* $\mathrm{q} \otimes \mathrm{q}^{-1} = \mathrm{q}/\mathrm{q} = \mathbf{1} = \mathrm{q}^{-1} \otimes \mathrm{q}$ *unless* q *is the zero quaternion.*

Theorem 1.6. *With the combination of operations, quaternions form a division ring, which adds distributive properties*

- *Left distributivity:* $\mathrm{q}_1 \otimes (\mathrm{q}_2 \oplus \mathrm{q}_3) = (\mathrm{q}_1 \otimes \mathrm{q}_2) \oplus (\mathrm{q}_1 \otimes \mathrm{q}_3)$
- *Right distributivity:* $(\mathrm{q}_1 \oplus \mathrm{q}_2) \otimes \mathrm{q}_3 = (\mathrm{q}_1 \otimes \mathrm{q}_3) \oplus (\mathrm{q}_2 \otimes \mathrm{q}_3)$

Definition 1.7.

$$\mathrm{q}^* = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}^* = \begin{pmatrix} -x \\ -y \\ -z \\ w \end{pmatrix} \tag{1.4}$$

Theorem 1.8. *As a convolution algebra, quaternions have the following properties*

- *Conjugate Involution:* $(\mathrm{q}^*)^* = \mathrm{q}$
- $\|\mathrm{q}_1 \otimes \mathrm{q}_2\| = \|\mathrm{q}_1\| \|\mathrm{q}_2\|$

2. Scalar Quaternions

This section is overly formal and could probably be left out, still...with our strict rules of quaternions, we will force the subject

Definition 2.1 (Quaternion Scalars)**.** Multiplication between scalars and quaternions is defined in terms of quaternion multiplication. For a scalar $s \in \mathbb{R}$

$$s \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ s \end{pmatrix} \otimes \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} sx \\ sy \\ sz \\ sw \end{pmatrix} \tag{2.1}$$

For good measure, lets define addition of a scalar $s \in \mathbb{R}$ and a quaternion $q \in \mathbb{H}$ similarily

$$s + \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ s \end{pmatrix} \oplus \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ s + w \end{pmatrix} \tag{2.2}$$

With those two definitions, scalars can be readily transformed back and forth between *scalar quaternions*: quaternions of the form $(0, 0, 0, s)$. The algebraic properties of the real field $(\mathbb{R}, +, \times)$ and the quaternion ring $\mathbb{H}, \oplus, \otimes$ behave similiarly enough, that from here on, we will use the notation for the real field: $+$ and $\times$.

3. The Quaternion Space

Theorem 3.1. *Quaternions form a vector space with the real field, which means they have the following properties*

- $(\mathbb{H}, \oplus)$ *forms a commutative group. Details above.*
- *Compatibility:* $s_1(s_2q) = (s_1s_2)q$
- *Scalar Identity:* $1q = q$ *where* $1 \in \mathbb{R}$
- *Distributivity over quaternion addition:* $s(q_1 + q_2) = sq_1 + sq_2$
- *Distributivity over scalar multiplication:* $(s_1 + s_2)q = s_1q + s_2q$

Definition 3.2. In the quaternion vector space we can define four orthogonal unit bases:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.1}$$

Each quaternion can now be written out $q = x\mathbf{i} + y\mathbf{j} + z\mathbf{z} + w$. The $\mathbf{1}$ is the unit quaternion and can be dropped after our previous formalizations of scalar/quaternion addition. Since we previously defined the addition of a scalar with a quaternion the $\mathbf{1}$ can also be dropped, i.e. $w + \mathrm{q} = w\mathbf{1} + \mathrm{q}$ for all scalars $w \in \mathbb{R}$

Theorem 3.3. *The definition of quaternion multiplication in equation 1.3 is unintuitive and hard to remember. Historically quaternions were used to extend complex numbers and as such, using the bases definition above, the complex* $i^2 = -1$ *can be extended for quaternion hypercomplex numbers*

$$\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = \mathbf{ijk} = -\mathbf{1} \tag{3.2}$$

Please note, some papers do not use this definition and therefor have a different definition of quaternions. The math can all work out the same at the end, but be wary swapping between papers could mean a missed sign or swapped commutativity.

4. Vector Quaternions

In the previous section we said quaternions form a vector space, so technically every quaternion is a 4-vector. But in practice, quaternions are lovely because of how they work alongside 3-vectors.

Definition 4.1 (Vector Quaternion)**.** Quaternions behave like vectors with scalars attached. More formally, every quaternion can be described by a *vector* or *pure quaternion* $\mathbf{v} = (x, y, z, 0)$ plus a scalar quaternion $s = (0, 0, 0, s)$: either $s + \mathbf{v}$ or, like below, $(\mathbf{v}, s)$. Addition is then just:

$$\begin{pmatrix} \mathbf{v}_1 \\ s_1 \end{pmatrix} + \begin{pmatrix} \mathbf{v}_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} p_0 + q_0 \\ \mathbf{p} + \mathbf{q} \end{pmatrix} \tag{4.1}$$

and multiplication can be defined from the vector dot and cross product

$$\begin{pmatrix} \mathbf{v}_1 \\ s_1 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{v}_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \\ s_1s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \end{pmatrix} \tag{4.2}$$

For pure quaternions, that final equation becomes

$$\mathbf{v}_1 \otimes \mathbf{v}_2 = \begin{pmatrix} -\mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_1 \times \mathbf{v}_2 \end{pmatrix} \tag{4.3}$$

The scalar part is minus the dot product of the two vectors and the vector part is the cross product–another way to re-derive equation 1.3 from a more readable/memorable place.

Theorem 4.2 (Quaternion Commutor)**.** *Quaternion multiplication is not commutative. Its commutor is given by*

$$\mathrm{q}_1 \otimes \mathrm{q}_2 - \mathrm{q}_2 \otimes \mathrm{q}_1 = 2\mathrm{q}_1 \times \mathrm{q}_2 \tag{4.4}$$

Proof. Let q_1 and q_2 be any two quaternions. In terms of vectors, their commutor would be

$$q_1 \otimes q_2 - q_2 \otimes q_1 = \begin{pmatrix} v_1 \times v_2 - v_2 \times v_1 \\ -v_1 \cdot v_2 + v_2 \cdot v_1 \end{pmatrix} \tag{4.5}$$

The vector dot product is commutative $v_1 \cdot v_2 = v_2 \cdot v_1$, but the vector cross product is *anti*-commutative $v_1 \times v_2 = -v_2 \times v_1$. Thus

$$q_1 \otimes q_2 - q_2 \otimes q_1 = \begin{pmatrix} 2v_1 \times v_2 \end{pmatrix} \tag{4.6}$$

If we allow $q_1 \cdot q_2$ and $q_1 \times q_2$ be the dot and cross product on their *vector* component, respectively, then the commuter is simply: $2q_1 \times q_2$ □

5. Rotation Group

Theorem 5.1. *Quaternions with norm* $\|\mathrm{q}\| = 1$ *form a subgroup with the quaternion product.*

- *Closure: If* $\|\mathrm{q}_1\| = 1$ *and* $\|\mathrm{q}_2\| = 1$ *then* $\|\mathrm{q}_1\mathrm{q}_2\| = 1$
- *Identity: The identity quaternion is a versor* $\|\mathbf{1}\| = 1$
- *Inverse: The inverse of a versor is its conjugate* $\mathrm{q}\mathrm{q}^* = 1 = \mathrm{q}^*\mathrm{q}$

Definition 5.2. Quaternions can be used to describe rotations in space. A versor rotates between reference frames $R : A \to B$. Without describing $\mathbf{v} \in \mathbb{R}^3$ in either A or B basis:

$$\begin{pmatrix} \mathbf{v} \cdot \hat{\mathbf{b}}_x \\ \mathbf{v} \cdot \hat{\mathbf{b}}_y \\ \mathbf{v} \cdot \hat{\mathbf{b}}_z \end{pmatrix} = \mathrm{q}_{A2B} \otimes \begin{pmatrix} \mathbf{v} \cdot \hat{\mathbf{a}}_x \\ \mathbf{v} \cdot \hat{\mathbf{a}}_y \\ \mathbf{v} \cdot \hat{\mathbf{a}}_z \end{pmatrix} \otimes \mathrm{q}^*_{A2B} \tag{5.1}$$

Theorem 5.3. *These transformations also form a group: the orthogonal group. If* $R(\mathrm{q}, \mathbf{v}) = \mathrm{q}\mathbf{v}\mathrm{q}^*$

- *Orthogonality: For all rotations* $\|\mathbf{v}\| = \|R(\mathrm{q}, \mathbf{v})\|$
- *Cover: For all vectors where* $\|\mathbf{u}\| = \|\mathbf{v}\|$ *there is a rotation* $\mathbf{u} = R(\mathrm{q}, \mathbf{v})$
- *Double Cover: Rotations are not unique* $R(\mathrm{q}, \mathbf{v}) = R(-\mathrm{q}, \mathbf{v})$
- *Closure:* $R(\mathrm{q}_1, R(\mathrm{q}_2, \mathbf{v})) = R(\mathrm{q}_1\mathrm{q}_2, \mathbf{v})$
- *Identity: The identity quaternion forms the identity rotation* $\mathbf{v} = R(\mathbf{1}, \mathbf{v})$
- *Inverse:* $R(\mathrm{q}, R(\mathrm{q}^*, \mathbf{v})) = R(\mathbf{1}, \mathbf{v})$

Theorem 5.4 (Quaternion to Matrix)**.** *Every rotation* $\mathrm{A} \to \mathrm{B}$ *can be described as both a quaternion and a rotation matrix*

$$(q_{A2B}) \otimes {}^{\mathrm{A}}\mathbf{v} \otimes (q^*_{A2B}) = (R_{A2B})\, {}^{A}\mathbf{v} \tag{5.2}$$

that rotation matrix also collects the dot products of the rotation.

R_{A2B}	$\hat{\mathbf{a}}_x$	$\hat{\mathbf{a}}_y$	$\hat{\mathbf{a}}_z$
$\hat{\mathbf{b}}_x$	$x^2 - y^2 - z^2 + w^2$	$2xy - 2zw$	$2xz + 2yw$
$\hat{\mathbf{b}}_y$	$2xy + 2zw$	$-x^2 + y^2 - z^2 + w^2$	$-2xw + 2yz$
$\hat{\mathbf{b}}_z$	$2xz - 2wy$	$2xw + 2yz$	$-x^2 - y^2 + z^2 + w^2$

Proof. R can be formed from algebra.

$$R \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \otimes \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix} \otimes \begin{pmatrix} -x \\ -y \\ -z \\ w \end{pmatrix} \tag{5.3}$$

□

Theorem 5.5. *For a constant angular velocity, the time derivative*

$$\dot{q} = \frac{1}{2} q \otimes \boldsymbol{\omega} \tag{5.4}$$

Proof. Let $\boldsymbol{\omega}$ be the angular velocity from frame A to B. Let ${}^A\mathbf{v}$ and ${}^B\mathbf{v}$ be any vector described in basis A and B respectively, then

$$\frac{d}{dt}\mathbf{v}_B = \frac{d}{dt}\mathbf{v}_A + \boldsymbol{\omega}_{A2B} \times \mathbf{v}_A \tag{5.5}$$

Affix the vector in B perpendicular to the angular velocity such that $\mathbf{v} \cdot \boldsymbol{\omega} = 0$. Since the dot product is zero, the cross product is equal to the quaternion product. Along with $\mathbf{v}$ being affixed in B gives

$$\frac{d}{dt}\mathbf{v}_B = \boldsymbol{\omega}_{A2B} \times \mathbf{v}_A \tag{5.6}$$

And

$$\frac{d}{dt}\mathbf{v}_B = \dot{\mathrm{q}}_{\mathrm{A2B}} \otimes \mathbf{v}_A \otimes \mathrm{q}^*_{A2B} + \mathrm{q}_{A2B} \otimes \mathbf{v} \otimes \dot{\mathrm{q}}^*_{A2B} \tag{5.7}$$

After removing references to $\mathbf{v}_B$ we can clean up those two equations giving

$$\dot{\mathrm{q}}\mathbf{v}\mathrm{q}^* + \mathrm{q}\mathbf{v}\dot{\mathrm{q}}^* = \boldsymbol{\omega}\mathbf{v} \tag{5.8}$$

Since the conjugate is the inverse of the quaternion

$$\mathrm{q}^*\mathrm{q} = 1 \tag{5.9}$$

Taking the time derivative gives

$$\dot{\mathrm{q}}^*\mathrm{q} - \mathrm{q}^*\dot{\mathrm{q}} = 0 \tag{5.10}$$

With some manipulation the equation relating to the angular velocity becomes

$$h \tag{5.11}$$

□

Linear interpolation

$$q_{k+1} = \frac{1}{2} q_k \boldsymbol{\omega} \Delta t + q_k \tag{5.12}$$

Spherical interpolation?

$$\mathrm{q}_{k+1} = \exp\left(\frac{1}{2}\boldsymbol{\omega}\Delta t\right) q_k \tag{5.13}$$