

QUATERNION ROTATION

ABSTRACT. Quaternions are almost magical with their ability to look like other concepts. They can be described by group/ring theory, or in terms of hyper-complex numbers, or in terms of vector spaces, and they can exist as all three of these at once. There are many resources on the algebra and analysis of quaternions, but often their definitions disagree, making it difficult to compare results. This paper hopes to formally define one type of quaternion operations and use those to derive final results. We will brush over proofs of generic quaternion algebra, and push through to applications ready for implementation in code.

1. THE QUATERNION RING

Quaternions, hopefully obvious from their name, are ordered 4-tuples $q = (x, y, z, w) \in \mathbb{R}^4$. The set of quaternions \mathbb{H} form a ring with two operations \oplus and \otimes

Definition 1.1 (Quaternion Addition). Addition on quaternions is defined easily enough

$$(1.1) \quad \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}$$

Definition 1.2 (Quaternion Multiplication). What makes quaternions special is how multiplication is defined. Multiplication is more complex than addition

$$(1.2) \quad \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1x_2 + x_1w_2 + y_1z_2 - z_1y_2 \\ w_1y_2 - x_1z_2 + y_1w_2 + z_1x_2 \\ w_1z_2 + x_1y_2 - y_1x_2 + z_1w_2 \\ w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2 \end{pmatrix}$$

This formulation of multiplication should coincide with the ROS package `tf2`. A less formal re-creation of some of these definitions can be found in the next section.

Theorem 1.3. *Quaternions form a commutative group with addition, which means they have the following properties*

- *Associativity:* $q_1 \oplus (q_2 \oplus q_3) = (q_1 \oplus q_2) \oplus q_3$
- *Commutativity:* $q_1 \oplus q_2 = q_2 \oplus q_1$
- *Identity:* $q \oplus 0 = q = 0 \oplus q$ where 0 is the 0-quaternion $(0, 0, 0, 0)$
- *Inverse:* $q \oplus (-q) = q \ominus q = 0 = (-q) \oplus q$ where the inverse is defined $-q = (-x, -y, -z, -w)$

Theorem 1.4. *Unlike addition, quaternions form a non-commutative group with multiplication, which gives them the following properties*

- *Associativity:* $q_1 \otimes (q_2 \otimes q_3) = (q_1 \otimes q_2) \otimes q_3$
- *Identity:* $q \otimes 0 = q = 0 \otimes q$ where 0 is the identity quaternion $(0, 0, 0, 1)$
- *Inverse:* $q \otimes q^{-1} = q/q = 1 = q^{-1} \otimes q$ where q^{-1} is the inverse.

Theorem 1.5. *Finally, with the combination of operations, quaternions form a ring, which adds distributive properties*

- *Left distributivity:* $q_1 \otimes (q_2 \oplus q_3) = (q_1 \otimes q_2) \oplus (q_1 \otimes q_3)$
- *Right distributivity:* $(q_1 \oplus q_2) \otimes q_3 = (q_1 \otimes q_3) \oplus (q_2 \otimes q_3)$

Definition 1.6. Before moving too far, defining the scalar multiplication operator will be useful. Multiplication between scalars and quaternions is defined in terms of quaternion multiplication. For a scalar $s \in \mathbb{R}$

$$(1.3) \quad s \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ s \end{pmatrix} \otimes \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} sx \\ sy \\ sz \\ sw \end{pmatrix}$$

Theorem 1.7. *Quaternions form a vector space with the real field, which means they have the following properties*

- (\mathbb{H}, \oplus) forms a commutative group. Details above.
- *Compatibility:* $s_1(s_2q) = (s_1s_2)q$
- *Identity:* $1q = q = q1$ where $1 \in \mathbb{R}$
- *Distributivity over quaternion addition:* $s(q_1 + q_2) = sq_1 + sq_2$
- *Distributivity over scalar multiplication:* $(s_1 + s_2)q = s_1q + s_2q$

Definition 1.8. Quaternions form vector spaces: scaling quaternion q by s is defined in terms of quaternion multiplication $sq = (s, 0, 0, 0) \otimes q$

Definition 1.9. All quaternions have a conjugate $q^* = (q_0, -q_1, -q_2, -q_3)$.

Theorem 1.10. *Conjugation is an involution $(q^*)^* = q$.*

Definition 1.11. The norm is the square root of the sum of the squares $\|q\| = \sqrt{q \otimes q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$.

Theorem 1.12. $\|p\|\|q\| = \|pq\|$

Theorem 1.13. *Every quaternion, besides the zero quaternion, has an inverse $q^{-1} = \frac{q^*}{\|q\|}$*

Theorem 1.14 (Quaternion Commutator). *Quaternion multiplication is not commutative.*

Proof. Let p and q be any two quaternions, then their commutator would be

$$(1.4) \quad \begin{aligned} p \otimes q - q \otimes p &= \begin{pmatrix} p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 \\ p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2 \\ p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1 \\ p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0 \end{pmatrix} - \begin{pmatrix} p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 \\ p_1q_0 + p_0q_1 + p_3q_2 - p_2q_3 \\ p_2q_0 - p_3q_1 + p_0q_2 + p_1q_3 \\ p_3q_0 + p_2q_1 - p_1q_2 + p_0q_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2p_2q_3 - 2p_3q_2 \\ 2p_3q_1 - 2p_1q_3 \\ 2p_1q_2 - 2p_2q_1 \end{pmatrix} \end{aligned}$$

□

1.1. Quaternion Notation. Before this, the notation has been overly explicit to avoid confusion. But the value of quaternions come from their familiar algebraic properties. The motivation for defining quaternions had originally come from these extra special properties

Definition 1.15. Quaternions behave like vectors, with scalars attached. Every vector $\mathbf{v} = (v_1, v_2, v_3)$ can be written as a quaternion $v = (0, v_1, v_2, v_3)$. A quaternion can be described, in turn, as a scalar and a vector $q = (q_0, \mathbf{q})$, where $\mathbf{q} = (q_1, q_2, q_3)$. A quaternion with no vector part is a *scalar quaternion*, able to be treated exactly like a scalar (commutivity, square roots, etc.). A quaternion with no scalar part (but with a vector part) is a *pure quaternion* with additional operations like dot and cross products. Between them, scalars, vectors, and quaternions all have a way to define addition and multiplication. Those quaternion operations can also be re-written in terms of scalar and vector parts:

$$(1.5) \quad \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} + \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} p_0 + q_0 \\ \mathbf{p} + \mathbf{q} \end{pmatrix}$$

and

$$(1.6) \quad \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} \otimes \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} p_0 q_0 - \mathbf{p} \cdot \mathbf{q} \\ p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q} \end{pmatrix}$$

For pure quaternions, that final equation becomes $\mathbf{p} \otimes \mathbf{q} = (-\mathbf{p} \cdot \mathbf{q}, \mathbf{p} \times \mathbf{q})$. The quaternion product \otimes contains both the vector dot \cdot and cross \times products. For the rest of this paper, we will usually omit \otimes , but leave \cdot and \times for clarity.

Definition 1.16. Quaternions can act like complex numbers, by defining four orthogonal unit bases:

$$(1.7) \quad \mathbf{1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{i} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

Each quaternion can now be written out $q = q_1 \mathbf{1} + q_i \mathbf{i} + q_j \mathbf{j} + q_k \mathbf{k}$. The $\mathbf{1}$ is usually omitted, forcing in a definition for adding a scalar and a quaternion. This paper won't use this notation much, but it gives the best way to re-derive the multiplication defined back in equation ???. From Hamilton's bridge, quaternion multiplication can be defined from the identities:

$$(1.8) \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

Please note, some papers do not use this definition and therefor have a different definition of quaternions. The math can all work out the same at the end, but be wary swapping between papers could mean a missed sign or swapped commutativity.

2. THE VERSOR SUBGROUP

Definition 2.1 (Versors). Quaternions with norm $\|q\| = 1$ form a subgroup of the quaternion product.

Theorem 2.2. *The norm of the product of two quaternions is equal to the product of the norm of the two quaterions. So, if the two versors have norm equal to one, the norm of their product must be one. Which makes the resultant quaternion a versor.*

Theorem 2.3. *The identity quaternion $(1, 0, 0, 0)$ belongs to the versor subgroup.*

Theorem 2.4. *The inverse of every versor is a versor. As well, the inverse is equal to the conjugate.*

3. THE ORTHOGONAL GROUP

Definition 3.1. Quaternions can be used to describe rotations in space. A versor rotates between reference frames ${}^A q^B : A \rightarrow B$. Without describing $\mathbf{v} \in \mathbb{R}^3$ in either A or B basis:

$$(3.1) \quad \begin{pmatrix} \mathbf{v} \cdot \hat{\mathbf{b}}_x \\ \mathbf{v} \cdot \hat{\mathbf{b}}_y \\ \mathbf{v} \cdot \hat{\mathbf{b}}_z \end{pmatrix} = {}^B q^A \otimes \begin{pmatrix} \mathbf{v} \cdot \hat{\mathbf{a}}_x \\ \mathbf{v} \cdot \hat{\mathbf{a}}_y \\ \mathbf{v} \cdot \hat{\mathbf{a}}_z \end{pmatrix} \otimes {}^B q^{*A}$$

Theorem 3.2. *The unit quaternion $\mathbf{1}$ is the rotation identity*

$$(3.2) \quad \mathbf{1} \mathbf{v} \mathbf{1}^* = \mathbf{v}$$

Theorem 3.3. *By pre- and post-multiplying quaternion conjugates to invert the rotation*

$$(3.3) \quad ({}^A q^{B*}) ({}^B \mathbf{v}) ({}^A q^B) = {}^A \mathbf{v}$$

we see that the inverse of the rotation is just its conjugate. When there is just one rotation, the rotation will be denoted as q and its inverse q^ .*

Theorem 3.4 (Quaternion to Matrix). *Every rotation $A \rightarrow B$ can be described as both a quaternion and a rotation matrix*

$$(3.4) \quad ({}^B q^A) \otimes {}^A \mathbf{v} \otimes ({}^A q^B) = ({}^B R^A) {}^A \mathbf{v}$$

that rotation matrix also collects the dot products of the rotation.

${}^B R^A$	$\hat{\mathbf{a}}_x$	$\hat{\mathbf{a}}_y$	$\hat{\mathbf{a}}_z$
$\hat{\mathbf{b}}_x$	$q_0^2 + q_1^2 - q_2^2 - q_3^2$	$-2q_0q_3 + 2q_1q_2$	$2q_0q_2 + 2q_1q_3$
$\hat{\mathbf{b}}_y$	$2q_0q_3 + 2q_1q_2$	$q_0^2 - q_1^2 + q_2^2 - q_3^2$	$-2q_0q_1 + 2q_2q_3$
$\hat{\mathbf{b}}_z$	$-2q_0q_2 + 2q_1q_3$	$2q_0q_1 + 2q_2q_3$	$q_0^2 - q_1^2 - q_2^2 + q_3^2$

Proof. R can be formed from algebra. The notation used implies that a pure quaternion is equal to its vector part.

$$(3.5) \quad R \mathbf{v} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \otimes \begin{pmatrix} q_0 \\ -q_1 \\ -q_2 \\ -q_3 \end{pmatrix}$$

$$R = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & -2q_0q_3 + 2q_1q_2 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & -2q_0q_1 + 2q_2q_3 \\ -2q_0q_2 + 2q_1q_3 & 2q_0q_1 + 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

□

4. SMOOTH TRANSFORMS

Theorem 4.1. *The quaternion is related to angular velocity by the following formula*

$$(4.1) \quad \dot{q} = \frac{1}{2} q \otimes \boldsymbol{\omega}$$

Proof. Start with the equation from 2-point theory, which says, for a vector \mathbf{v} , described in A (as ${}^A\mathbf{v}$) and in B (as ${}^B\mathbf{v}$):

$$(4.2) \quad \frac{d}{dt} {}^B\mathbf{v} = \frac{d}{dt} {}^A\mathbf{v} + {}^B\boldsymbol{\omega}^A \times {}^A\mathbf{v}$$

Strategically substituting our rotation quaternion

$$(4.3) \quad \frac{d}{dt} [({}^Bq^A) ({}^A\mathbf{v}) ({}^Aq^B)] = \frac{d}{dt} {}^A\mathbf{v} + {}^B\boldsymbol{\omega}^A \times {}^A\mathbf{v}$$

After this, we'll drop the A/B bases for clarity. Choose a vector \mathbf{v} in A where \mathbf{v} is fixed, then:

$$(4.4) \quad \dot{q}\mathbf{v}q^* + q\mathbf{v}\dot{q}^* = \boldsymbol{\omega} \times \mathbf{v}$$

Manipulating the left hand side with identities

$$(4.5) \quad \begin{aligned} q^*\dot{q}\mathbf{v}q^*\dot{q} + q^*q\mathbf{v}\dot{q}^*\dot{q} &= q^*(\boldsymbol{\omega} \times \mathbf{v})\dot{q} \\ q^*\dot{q}\mathbf{v}q^*\dot{q} + \mathbf{v} &= q^*(\boldsymbol{\omega} \times \mathbf{v})\dot{q} \end{aligned}$$

□

Linear interpolation

$$(4.6) \quad q_{k+1} = \frac{1}{2}q_k\boldsymbol{\omega}\Delta t + q_k$$

Spherical interpolation?

$$(4.7) \quad q_{k+1} = \exp\left(\frac{1}{2}\boldsymbol{\omega}\Delta t\right) q_k$$