QUATERNION ROTATION

Abstract. I've been distracted by quaternions and quaternion rotation since I first started playing with quadcopters. Hopefully this paper will hide my proofs from clogging up an article about robots. How can you not enjoy them? They're everything all at once.

1. The Quaternion Ring

Definition 1.1 (Quaternion)**.** Formally, a quaternion consists of four values (q_0, q_1, q_2, q_3) along with two operations: Addition $+$ and multiplication $\otimes$. Addition over the quaternions is defined by

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0 + q_0 \\ p_1 + q_1 \\ p_2 + q_2 \\ p_3 + q_3 \end{pmatrix} \tag{1.1}$$

Multiplication is more complex

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \otimes \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 \\ p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2 \\ p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1 \\ p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0 \end{pmatrix} \tag{1.2}$$

Theorem 1.2. *Quaternions form a commutative group with addition.*

Proof. Quaternions exhibit associativity, since their members have associativity

$$\begin{pmatrix} p_0 + (q_0 + r_0) \\ p_1 + (q_1 + r_1) \\ p_2 + (q_2 + r_2) \\ p_3 + (q_3 + r_3) \end{pmatrix} = \begin{pmatrix} (p_0 + q_0) + r_0 \\ (p_1 + q_1) + r_1 \\ (p_2 + q_2) + r_2 \\ (p_3 + q_3) + r_3 \end{pmatrix} \tag{1.3}$$

They exhibit commutativity, since their members are commutative

$$\begin{pmatrix} p_0 + q_0 \\ p_1 + q_1 \\ p_2 + q_2 \\ p_3 + q_3 \end{pmatrix} = \begin{pmatrix} q_0 + q_0 \\ q_1 + q_1 \\ q_2 + q_2 \\ q_3 + p_3 \end{pmatrix} = (p + q) + r \tag{1.4}$$

The additive identity is the *zero quaternion* $0 = (0, 0, 0, 0)$

$$\begin{pmatrix} q_0 + 0 \\ q_1 + 0 \\ q_2 + 0 \\ q_3 + 0 \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{1.5}$$

The additive inverse of quaternion q is $-q = (-q0, -q1, -q2, -q3)$; subtraction is defined by the inverse $p - q = p + (-q)$

$$\begin{pmatrix} q_0 + -q_0 \\ q_1 + -q_1 \\ q_2 + -q_2 \\ q_3 + -q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.6}$$

□

Theorem 1.3. *The quaternion product is associative. The proof is left to the reader.*

Theorem 1.4. *The left and right identity of quaternion multiplication is the* identity quaternion $1 = (1, 0, 0, 0)$

Proof.

$$\begin{pmatrix} 1q_0 - 0q_1 - 0q_2 - 0q_3 \\ 1q_1 + 0q_0 + 0q_3 - 0q_2 \\ 1q_2 - 0q_3 + 0q_0 + 0q_1 \\ 1q_3 + 0q_2 - 0q_1 + 0q_0 \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1q_0 - 0q_1 - 0q_2 - 0q_3 \\ 0q_0 + 1q_1 + 0q_2 - 0q_3 \\ 0q_0 - 0q_1 + 1q_2 + 0q_3 \\ 0q_0 + 0q_1 - 0q_2 + 0q_3 \end{pmatrix} \tag{1.7}$$

□

Definition 1.5. All quaternions have a conjugate $q^* = (q_0, -q_1, -q_2, -q_3)$ that is an involution $(q^*)^* = q$.

Definition 1.6. The norm is the square root of the product of a quaternion and its conjugate $\|q\| = \sqrt{q \otimes q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$

Theorem 1.7. *Every quaternion, besides the zero quaternion, has an inverse. That inverse being* $q^{-1} = q^*/\|q\|$

Theorem 1.8 (Quaternion Commutor)**.** *Quaternion multiplication is not commutative.*

Proof. Let p and q be any two quaternions, then their commutor would be $p \otimes q - q \otimes p$

$$\begin{pmatrix} p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 \\ p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2 \\ p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1 \\ p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0 \end{pmatrix} - \begin{pmatrix} p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 \\ p_1q_0 + p_0q_1 + p_3q_2 - p_2q_3 \\ p_2q_0 - p_3q_1 + p_0q_2 + p_1q_3 \\ p_3q_0 + p_2q_1 - p_1q_2 + p_0q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2p_2q_3 - 2p_3q_2 \\ 2p_3q_1 - 2p_1q_3 \\ 2p_1q_2 - 2p_2q_1 \end{pmatrix} \tag{1.8}$$

□

Please note, quaternion multiplication is not commutative (For most quaternions: $p \otimes q \neq q \otimes p$). Some other algebraic properties: Every scalar can be expressed as a quaternion $(s, 0, 0, 0)$ and every (3-)vector can be a quaternion $(0, v1, v2, v3)$. The 0-quaternion $(0, 0, 0, 0)$ is the additive identity and the 1-quaternion the multiplicative identity $(1, 0, 0, 0)$. The conjugate of a quaternion has inverted signs on the vector components: $q^* = (q0, q1, q2, q3)^* = (q0, -q1, -q2, -q3)$. The norm of a quaternion is the square root of the components, or the square of the quaternion multiplied by its conjugate (if we treat a purely scalar quaternion as a scalar).

$$\|q\| = \sqrt{q \otimes q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \tag{1.9}$$

Every quaternion has an inverse q^{-1} such that $q \otimes q^{-1} = 1 = q^{-1} \otimes q$. The inverse is $q^{-1} = q^*/\|q\|$. A quaternion with norm equal to one is called a unit quaternion, or versor. The quaternions covered outside this appendix can be considered unit quaternions. The $\otimes$ will be usually omitted too. We've seen that quaternion algebra can be applied to scalars and vectors. Pure vector multiplication notation will remain $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ for clarification, despite these also being quaternion operations.

1.1. **Quaternion Notation.**

2. The Versor Subgroup

Definition 2.1 (Versors)**.** Quaternions with norm $\|q\| = 1$ form a subgroup of the quaternion product.

Theorem 2.2. *The norm of the product of two quaternions is equal to the product of the norm of the two quaterions. So, if the two versors have norm equal to one, the norm of their product must be one. Which makes the resultant quaternion a versor.*

Theorem 2.3. *The identity quaternion* $(1, 0, 0, 0)$ *belongs to the versor subgroup.*

Theorem 2.4. *The inverse of every versor is a versor. As well, the inverse is equal to the conjugate.*

3. The Orthogonal Group

A unit quaternion can be used to describe a rotation between bases. A vector described in unit basis A would be notated ${}^{A}\mathbf{v}$. The quaternion ${}^{A}q^{B}$ transforms ${}^{A}\mathbf{v}$ to ${}^{B}\mathbf{v}$ by the following formula:

$$ {}^{B}\mathbf{v} = \left({}^{A}q^{B}\right)\left({}^{A}\mathbf{v}\right)\left({}^{A}q^{B*}\right) \tag{3.1} $$

By pre- and post-multiplying quaternion conjugates to invert the rotation $\left({}^{A}q^{B*}\right)\left({}^{B}\mathbf{v}\right)\left({}^{A}q^{B}\right) = {}^{A}\mathbf{v}$, we see that the inverse of the rotation is just its conjugate. Since we are only talking about two reference frames in this paper: the inertial and the robot; rotating between the two can be described with a quaternion and its conjugate. Rotating from the inertial to the robot frame is chosen to be ${}^{N}q^{R} = q$ and the robot to the inertial frame ${}^{R}q^{N} = q^*$.

By Euler's rotation theorem, every rotation in 3-space can be described by a rotation axis and angle. For the transformation above, that quaternion is

$$ q = \exp\left(\frac{\theta}{2}\hat{\mathbf{u}}\right) = \cos\left(\frac{\theta}{2}\right) + \mathbf{u}\sin\left(\frac{\theta}{2}\right) \tag{3.2} $$

where $\hat{\mathbf{u}}$ is the axis of rotation (scaled so that $\|q\| = 1$) and θ is the rotated angle. Every rotation can be composed of small rotations when differentiated w.r.t. time gives:

$$ \dot{q} = \frac{1}{2}\dot{\theta}\ \exp\left(\frac{\theta}{2}\hat{\mathbf{u}}\right) \tag{3.3} $$

Quaternion Kinematics

Integrating the quaternion into

The quaternion is related to angular velocity by the following formula

$$\dot{q} = \frac{1}{2} q \otimes \boldsymbol{\omega} \tag{3.4}$$

Definition 3.1. A quaternion rotates a 3-dimensional vector $\mathbf{u} \mapsto \mathbf{v}$

$$\mathbf{v} = \mathrm{q}\mathbf{u}\mathrm{q}^* \tag{3.5}$$

Quaternion rotation from angular velocity

$$\dot{q} = \frac{1}{2} q \boldsymbol{\omega} \tag{3.6}$$

Linear interpolation

$$q_{k+1} = \frac{1}{2} q_k \boldsymbol{\omega} \Delta t + q_k \tag{3.7}$$

Spherical interpolation?

$$q_{k+1} = \exp\left(\frac{1}{2} \boldsymbol{\omega} \Delta t\right) q_k \tag{3.8}$$