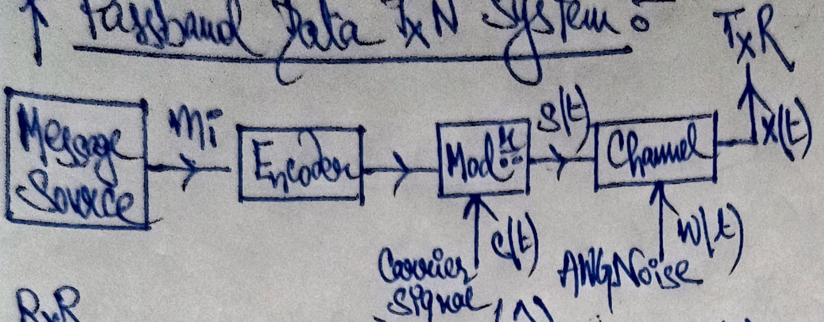
Binary Phase Shift Keying (BPSK):-

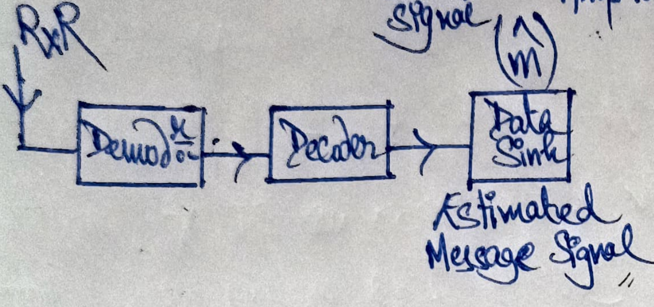
* It is one of the passband modulation techniques for passband communication (with carrier signal).

Binary Passband Model

A binary message signal of M nos, i.e. $m_1, m_2, \dots, m_M$ for any $i = 1, 2, 3, \dots M$ for message signal m_i.

* Passband Data TxN System:-





$X(t) = S(t) + W(t)$, where

$S(t)$ = Txd signal (Modulated signal)

$W(t)$ = AWGN signal with PSD $\frac{N_0}{2}$ Watt/Hz. WSS zero-mean

Received signal $X(t)$ is a random signal/process.

* BPSK Mathematical Model:- Coherent ≡ Phase of $X(t)$ known to RxR.

This is a coherent BPSK. We have a pair of signals $S_1(t)$ and $S_2(t)$, i.e. $i = 1, 2$. to represent '1' and '0' respectively.

$$S_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \phi), \text{ here } \phi = 0^\circ \text{ for } S_1(t).$$

$\phi = 180^\circ$ for $S_2(t)$

or, $\boxed{S_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)}$ (to represent '1').

$0 \leq t \leq T_b$

and, $S_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \phi)$, here $\phi = 180^\circ$ for $S_2(t)$

or, $\boxed{S_2(t) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)}$ (to represent '0')

$0 \leq t \leq T_b$

Here, $S_2(t) = -S_1(t)$ since one is negative of the other, they are called antipodal signals.

Amplitude of modulated signal $= \sqrt{\frac{2E_b}{T_b}}$: Why?

- The amplitude of carrier signal is chosen as follows:—

* Let the carrier be $A_c \cos(2\pi f_c t)$.

$\therefore$ Normalized power of carrier, $= \frac{A_c^2}{2}$.

Normalized power $\equiv$ Corresponding amplitude sq. value

The message bit/symbol duration (Bit ~~rat~~ duration) is called T_b.

$\therefore$ Normalized power within T_b duration $= E_b$.

or, $\left(\frac{A_c^2}{2}\right) \times T_b = E_b$ or, $A_c = \sqrt{\frac{2E_b}{T_b}}$.

* Here, $S_1(t) = \sqrt{E_b} \cdot \left(\sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)\right) = \sqrt{E_b} \cdot \phi_1(t)$

where $\boxed{\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)}$ is the orthonormal basis f^n—

By Gram-Schmidt Orthogonalization Procedure, we have

1st Basis f^n, $\phi_1(t) = \frac{S_1(t)}{\sqrt{E_b}} = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$.

Also, $S_2(t) = \sqrt{E_b}\, \phi_1(t)$

* Here, no. of Basis function $= 1 \Rightarrow$ BPSK is 1-D signal space system

$\therefore \boxed{S_1(t) = \sqrt{E_b}\, \phi_1(t)}$ and $\boxed{S_2(t) = -\sqrt{E_b}\, \phi_1(t)}$, by standard

Convent^n, $S_i(t) = \sum_{j=1}^{N} S_{ij} \phi_j(t) = S_{i1} \phi_1(t) \Rightarrow \boxed{\begin{aligned} S_1(t) &= S_{11} \phi_1(t) \\ S_2(t) &= S_{21} \phi_1(t) \end{aligned}}$

Hence, comparing we get $\boxed{S_{11} = \sqrt{E_b},\ S_{21} = -\sqrt{E_b}}$

Message Points.

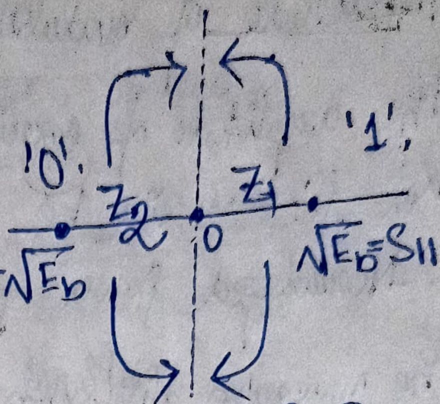
* General method to find message pts.:— $S_{ij} = \int S_i(t) \cdot \phi_j(t)\, dt$

$\therefore S_{11} = \int^{T_b} S_1(t) \cdot \phi_1(t)\, dt = \int^{T_b} \sqrt{E_b} \cdot \phi_1(t) \cdot \phi_1(t)\, dt$ or, $S_{i1} = \int S_i(t) \cdot \phi_1(t)\, dt$

or, $S_{11} = \sqrt{E_b} \left[\int_0^{T_b} \phi_1^2(t)\, dt\right]$ or, $S_{11} = \sqrt{E_b}$.

* Signal-Space Diagram :-

1-D signal-space. When point lies on the ⊥ar bisector line, then RxR will take random guess, i.e., error may occur. $S_{21} = -\sqrt{E_b}$



* Error - Probability :-

When '1' is recognized as '0' or vice-versa, then RxR commits an error. $x_1 = S_{11} + W_1$

observation element

For symbol '1' or signal $S_1(t)$, $Z_1 : 0 < x_1 < \infty$, $X_1 = S_{11} + W_1$

Here, X_1 is an observation vector and it's a r.v.,

The conditional PDF of the r.v. X_1 when symbol say '0' was transmitted is $f_{X_1}(x_1|0) = \frac{1}{\sqrt{2\pi\sigma_{X_1}^2}} e^{-\frac{(x_1 - \mu_{X_1})^2}{2\sigma_{X_1}^2}}$

Here, $\sigma_{X_1}^2 = \frac{N_0}{2}$, mean of x_1, $E[x_1] = E[S_{21} + W_1] = S_{21}$

$$\therefore f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_1 - S_{21})^2}{N_0}\right).$$

$$\text{or, } \boxed{f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_1 + \sqrt{E_b})^2}{N_0}\right)}$$

$\therefore$ Probability of error, $P_{10} = \int_{Z_1} f_{X_1}(x_1|0)\,dx_1 = \int_0^\infty f_{X_1}(x_1|0)\,dx_1$

(i.e., '1' received when '0' was transmitted)

$$\text{or, } P_{10} = \frac{1}{\sqrt{\pi N_0}} \int_0^\infty \exp\left(\frac{-(x_1 + \sqrt{E_b})^2}{N_0}\right) dx_1$$

Substitute $z = \left(\frac{x_1 + \sqrt{E_b}}{\sqrt{N_0}}\right) \Rightarrow dz = \frac{1}{\sqrt{N_0}} dx_1 \Rightarrow dx_1 = \sqrt{N_0}\,dz.$

$$\therefore P_{10} = \frac{1}{\sqrt{\pi N_0}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp(-z^2) \sqrt{N_0}\,dz$$

or, $P_{10} = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} \exp(-x^2)\,dx = \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} \exp(-x^2)\,dx$

or, $\boxed{P_{10} = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)}$

Also, for $P_{01} = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$

* **Average Probability of symbol error :-**

$P_e = p_0 P_{10} + p_1 P_{01}$ — Equiprobable $\frac{1}{M}$; here $M = 2 \Rightarrow p_0 = p_1 = \frac{1}{2}$

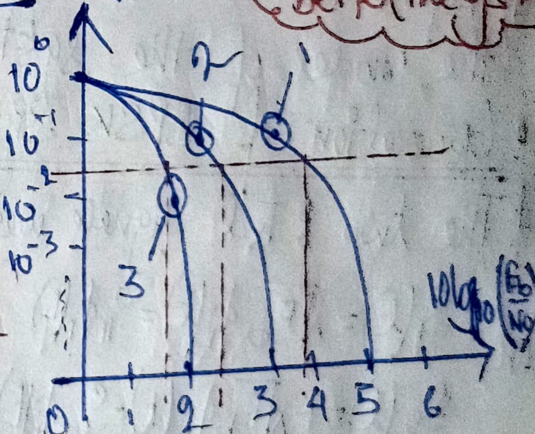
$= \frac{1}{2} \cdot \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) + \frac{1}{2} \cdot \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$

or, $\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)}$ — Imp. formula.

• **Bit error Rate v/s E_b/N_0 curve :-** P_e

Sharpen the curve better the system

* This is a semi-log plot where x-axis is log-scale, $\frac{E_b}{N_0}\Big|_{dB} = 10\log_{10}\left(\frac{E_b}{N_0}\right)$.



For same Bit error Rate, we observe that when $\frac{E_b}{N_0} \downarrow \Rightarrow N_0 \uparrow$ but still we are getting Bit error Rate (BER) quite enough $\Rightarrow$ Better performance.

Steps to study digital modulation schemes:-

(1). Form modulated signal equations, $S_i(t)$, $i=1,2,\dots,M$

(2). Find basis functions ϕ_j and dimension $j=1,2,\dots,N$ of signal space.

(3) Find message points, S_{ij}.

(4) Sketch signal space diagram.

(5) Find conditional PDF for all the i's when k's were sent where $i \neq k$.

(6) Find prob. of bit/symbol error and then average prob. of error P_e.

(7) Propose or sketch Generation & Detection diagrams as per the model of the system.

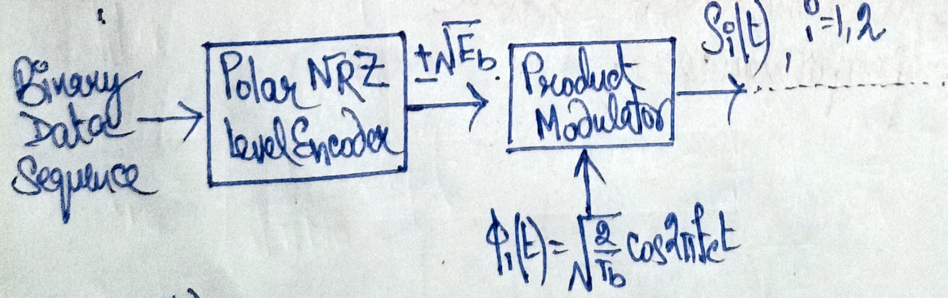
- **Generation of BPSK Signal :- (TxR)**

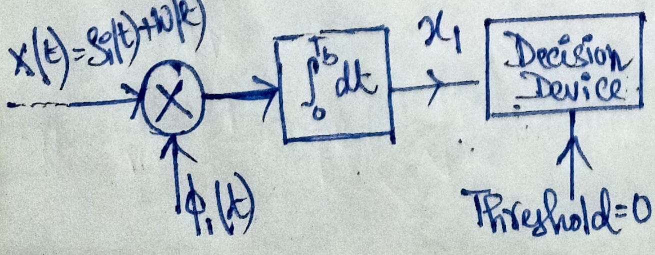
* The two Txd signals were

$$\left.\begin{array}{l} S_1(t)=\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t) \quad \text{'1'} \\ S_2(t)=-\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t) \quad \text{'0'} \end{array}\right\} \text{—(1)}$$

or,

$$\left.\begin{array}{l} S_1(t)=\sqrt{E_b}\,\phi_1(t)=S_{11}\phi_1(t) \\ S_2(t)=-\sqrt{E_b}\,\phi_2(t)=S_{21}\phi_1(t) \end{array}\right\} \text{—(2)}$$





Here, when $x_1 > 0$, symbol is '1' and when $x_2 < 0$, symbol is '0'. But, when $x_1 = 0$, RxR makes a random guess either in favour of '0' or in favour of '1', which is decided by flipping a coin.

* How to decide carrier-signal frequency :- $f_c = \frac{n_c}{T_b}$, $n_c = \pm 1, \pm 2, ...$

- Binary Amplitude Shift Keying :- (BASK)

* It is again one of the digital modulation schemes where amplitude of the carrier is varied wrt binary message.

- Amplitude, A_c will be appropriately mapped wrt '0' or '1' as for the binary message sequence and no freq. shifting in Fourier domain. Hence, shift keying.

* Here, the modulated signals are,

$\boxed{S_1(t) = A_c \cos 2\pi f_c t = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)}$ to represent '1' in $0 \leq t \leq T_b$

$f_c = \frac{n_c}{T_b}$

$\boxed{S_2(t) = 0}$ to represent '0' in $0 \leq t \leq T_b$.

* Orthonormal basis f^ns :- By Gram Schmidt Orthogonalization Procedure, Here, $m = 2$, and

$$\phi_1(t) = \frac{S_1(t)}{\sqrt{E_b}} = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \quad \text{or,} \quad \boxed{\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)}$$

and $\phi_2(t) = 0 \Rightarrow$ Only 1 Basis f^n $\Rightarrow$ One-dimensional signal space, $N = 1$

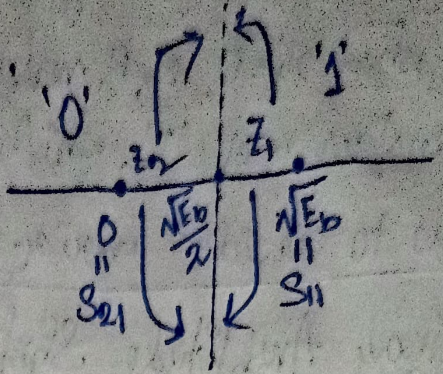
$$\therefore S_1(t) = \sqrt{E_b} \cdot \left(\sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)\right) = \sqrt{E_b} \cdot \phi_1(t)$$

and $S_2(t) = 0 = 0 \cdot \phi_1(t)$.

* Message Points :- $S_{11} = \sqrt{E_b}$, $S_{21} = 0$.

Signal-Space Diagram:

Here, we have $N=1$, 1-D signal space.
When point lies on $\frac{\sqrt{E_b}}{2}$, RxR will take random guess.



$$\underset{('1')}{Z_1} : \left(\frac{\sqrt{E_b}}{2}, \infty\right), \quad \underset{('0')}{Z_2} : \left(-\infty, \frac{\sqrt{E_b}}{2}\right).$$

* Conditional PDF:

When '0' is transmitted and it is received as '1', we get an error ~~or vice versa~~.

$\therefore$ Conditional PDF, $$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_1 - \mu_{x_1})^2}{N_0}\right).$$

Mean of x_1, $E[x_1] = E[S_{21} + W_1]$ ($\because$ '0' transmitted $\Rightarrow S_{21}$).
$$= S_{21} = 0.$$

$$\Rightarrow f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_1^2}{N_0}\right)$$

To find the probability of error, we integrate $f_{X_1}(x_1|0)$ in Z_1.

$$\therefore P_{10} = \int_{Z_1} f_{X_1}(x_1|0)\,dx_1 = \int_{\frac{\sqrt{E_b}}{2}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_1^2}{N_0}\right) dx_1.$$

Substitute, $z = \frac{x_1}{\sqrt{N_0}}$ or, $dx_1 = \sqrt{N_0}\,dz$.

$x_1 \to \infty$, $z \to \infty$
$x_1 = \frac{1}{2}\sqrt{E_b}$, $z = \frac{1}{2}\sqrt{\frac{E_b}{N_0}}$.

$$\text{or, } P_{10} = \int_{\frac{1}{2}\sqrt{\frac{E_b}{N_0}}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp(-z^2) \sqrt{N_0}\,dz = \frac{1}{2} \int_{\frac{1}{2}\sqrt{\frac{E_b}{N_0}}}^{\infty} \frac{2}{\sqrt{\pi}} e^{-z^2} dz.$$

$$\text{or, } P_{10} = \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$ and ~~$P_{01} \neq \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\right)$~~

When '1' is transmitted and it is received as '0', we get an error.

$\therefore$ Conditional PDF, $f_{X_2}(x_2|1) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_2-\mu_{x_2})^2}{N_0}\right)$

$\therefore$ Mean of x_2, $E[x_2] = E[S_{11}+W_1] = S_{11} = \sqrt{E_b}$.

$$\Rightarrow f_{X_2}(x_2|1) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_2-\sqrt{E_b})^2}{N_0}\right).$$

To find the probability of error, we integrate $f_{X_2}(x_2|1)$ in Z_2

$$\therefore P_{01} = \int_{Z_2} f_{X_2}(x_2|1)\,dx_2 = \int_{-\infty}^{\sqrt{E_b}/2} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_2-\sqrt{E_b})^2}{N_0}\right) dx_2$$

Substitute, $v = \frac{x_2-\sqrt{E_b}}{\sqrt{N_0}}$, $x_2 \to -\infty, v \to -\infty$. $x_2 \to \sqrt{E_b}/2, v \to -\frac{1}{2}\sqrt{\frac{E_b}{N_0}}$.

$$\Rightarrow P_{01} = \int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{E_b}{N_0}}} \frac{1}{\sqrt{\pi}} \exp(-v^2)\,dv = \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{E_b}{N_0}}}^{\infty} \frac{2}{\sqrt{\pi}} \exp(-v^2)\,dv$$

or, $P_{01} = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$.

* <u>Average Probability of Symbol Error</u> :-

$P_e = p_0 P_{10} + p_1 P_{01}$

or, $\boxed{P_e = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)}$