UNIT-4

(1) $f = x^2y + 2xz = 4$. $(2,-2,3)$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \bar{i}(2xy + 2z) + \bar{j}(x^2) + \bar{k}(2x)$$

$$(\nabla f)_{(2,-2,3)} = \bar{i}(-8+6) + \bar{j}(4) + \bar{k}(4)$$

$$= -2\bar{i} + 4\bar{j} + 4\bar{k}$$

$$\bar{e} = \text{unit normal vector} = \frac{-2\bar{i}+4\bar{j}+4\bar{k}}{\sqrt{4+16+16}} = \frac{-2\bar{i}+4\bar{j}+4\bar{k}}{6}$$

$$= \frac{-\bar{i}+2\bar{j}+2\bar{k}}{3}$$

(2) $xyz = z^2 \Rightarrow xy = z$ $(4,1,2)$ $(3,3,-3)$

$$f = xy - z$$

$$\nabla f = \bar{i}(y) + \bar{j}(x) + \bar{k}(-1)$$

$$\nabla f = y\bar{i} + x\bar{j} - \bar{k}$$

$$\bar{n}_1 = (\nabla f)_{(4,1,2)} = \bar{i} + 4\bar{j} - \bar{k}$$

$$\bar{n}_2 = (\nabla f)_{(3,3,-3)} = 3\bar{i} + 3\bar{j} - \bar{k}$$

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{(\bar{i}+4\bar{j}-\bar{k})\cdot(3\bar{i}+3\bar{j}-\bar{k})}{\sqrt{1+16+1}\;\sqrt{9+9+1}}$$

$$= \frac{3+12+1}{\sqrt{18}\;\sqrt{19}} = \frac{16}{\sqrt{18}\;\sqrt{19}}$$

③ $f = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal.

$\boxed{\text{Solenoidal} \Rightarrow \operatorname{div}\bar{f} = 0}$

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\frac{\partial f_1}{\partial x} = 1 \qquad \frac{\partial f_2}{\partial y} = 1 \qquad \frac{\partial f_3}{\partial z} = p$$

$\Rightarrow 0 = 1+1+p$

$\Rightarrow p+2 = 0$

$\Rightarrow \underline{\underline{p = -2}}$

④ ⓐ $\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\bar{j} + (2x+cy+3z)\bar{k}$

$\boxed{\bar{f} \text{ is irrotational} \Rightarrow \operatorname{curl}\bar{f} = 0}$

$$\operatorname{curl}\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = 0$$

$= \bar{i}(c-3) - \bar{j}(2-a) + \bar{k}(b-3) = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$\Rightarrow \underline{\underline{c = 3;\ a = 2;\ b = 3}}$

ⓑ $f = xy + yz + zx \qquad \bar{i} + 2\bar{j} + 2\bar{k} \qquad (1,2,0).$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \bar{i}(y+z) + \bar{j}(x+z) + \bar{k}(y+x)$$

$$(\nabla f)_{(1,2,0)} = 2\bar{i} + \bar{j} + 3\bar{k}$$

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1+4+4}} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

$$D.D = \bar{e} \cdot \nabla f = \frac{(2\bar{i} + \bar{j} + 3\bar{k}) \cdot (\bar{i} + 2\bar{j} + 2\bar{k})}{3}$$

$$= \frac{2+2+6}{3} = \underline{\underline{10/3}}$$

⑤ ⓐ $f = x^2 - y^2 + 2z^2 \qquad P = (1,2,3) \qquad Q = (5,0,4)$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = (5\bar{i} + 4\bar{k}) - (\bar{i} + 2\bar{j} + 3\bar{k})$$

$$\overline{PQ} = 4\bar{i} - 2\bar{j} + \bar{k}$$

$$\bar{e} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{16+4+1}} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$$

$$f = x^2 - y^2 + 2z^2 \Rightarrow \nabla f = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$$

$$(\nabla f)_{(1,2,3)} = 2\bar{i} - 4\bar{j} + 12\bar{k}$$

$$\therefore D.D = \bar{e} \cdot \nabla f = \frac{(2\bar{i} - 4\bar{j} + 12\bar{k}) \cdot (4\bar{i} - 2\bar{j} + \bar{k})}{\sqrt{21}} = \frac{8+8+12}{\sqrt{21}} = \underline{\underline{\frac{28}{\sqrt{21}}}}.$$

(6) $\bar{f} = (x^2-y^2-x)\bar{i} - (2xy+y)\bar{j}$ is irrotational

$\Downarrow$

curl $\bar{f} = 0$.

Given,

$\bar{f}$ is irrotational vector,

then there exists ϕ such that $\bar{f} = \nabla\phi$.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (x^2-y^2-x)\bar{i} - (2xy+y)\bar{j}.$$

$$\frac{\partial\phi}{\partial x} = x^2-y^2-x$$

I.O.B.S w.r.t x.

$$\int\frac{\partial\phi}{\partial x}dx = \int(x^2-y^2-x)dx$$

$$\Rightarrow \phi = \frac{x^3}{3} - y^2x - \frac{x^2}{2} + \text{a const. independent of } x \quad \text{--- (1)}$$

$$\frac{\partial\phi}{\partial y} = -(2xy+y)$$

I.O.B.S w.r.t y.

$$\int\frac{\partial\phi}{\partial y}dy = -\int(2xy+y)dy$$

$$\Rightarrow \phi = -\frac{\cancel{2}xy^2}{\cancel{2}} - \frac{y^2}{2} + \text{a const. independent of } y \quad \text{--- (2)}$$

(1) & (2) $\Rightarrow$ $$\phi = \frac{x^3}{3} - xy^2 - \frac{x^2}{2} - \frac{y^2}{2} + c.$$

⑦ $f = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$.

$\boxed{\text{irrotational} \Rightarrow \text{curl } \bar{f} = 0}$.

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \bar{i}(-1+1) - \bar{j}(3z^2 - 3z^2) + \bar{k}(6x - 6x)$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}$$

$\therefore$ curl $\bar{f} = \bar{0} \Rightarrow$ it is irrotational

Then there exists atleast one ϕ such that $f = \nabla\phi$.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}.$$

$\frac{\partial\phi}{\partial x} = 6xy + z^3 \rightarrow$ I.O.B.S $\Rightarrow \phi = \frac{\cancel{6}^{3} y x^2}{\cancel{2}} + z^3 x +$ a const. —①

$\frac{\partial\phi}{\partial y} = 3x^2 - z \rightarrow$ I.O.B.S $\Rightarrow \phi = 3x^2 y - zy +$ a const. —②

$\frac{\partial\phi}{\partial z} = 3xz^2 - y \rightarrow$ I.O.B.S $\Rightarrow \phi = \frac{\cancel{3}xz^3}{\cancel{3}} - yz +$ a const. —③

From ①, ② & ③,

$$\underline{\underline{\phi = 3x^2 y + xz^3 - yz + c.}}$$

⑧ $f = (y^2\cos x + z^3)\bar{i} + (2y\sin x - 4)\bar{j} + 3xz^2\bar{k}$

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$= \bar{i}(0-0) - \bar{j}(3z^2-3z^2) + \bar{k}(2y\cos x - 2y\cos x)$

$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}.$

curl $\bar{f} = \bar{0} \Rightarrow$ Irrotational vector

Then there exists atleast one ϕ, $\ni$ $\bar{f} = \nabla\phi$.

$\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (y^2\cos x + z^3)\bar{i} + (2y\sin x - 4)\bar{j} + 3xz^2\bar{k}$

$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \rightarrow$ I.O.B.S $\Rightarrow \phi = y^2\sin x + z^3x +$ a const. —①

$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \rightarrow$ I.O.B.S $\Rightarrow \phi = \frac{\cancel{2}y^2\sin x}{\cancel{2}} - 4y +$ a const. —②

$\frac{\partial\phi}{\partial z} = 3xz^2 \rightarrow$ I.O.B.S $\Rightarrow \phi = \frac{\cancel{3}xz^3}{\cancel{3}} +$ a const. —③

①, ②, ③ $\Rightarrow$

$\therefore \phi = y^2\sin x + xz^3 - 4y + c.$

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