Discrete Fourier Transform & Fast Fourier Transform

Discrete Fourier Transform :-

The Discrete Fourier transform is a powerful computation tool which allows to evaluate fourier transform on a digital computer or specially designed digital Hardware.

→ Fourier Transform is continous and periodic, the DFT is obtained by sampling one period of the fourier transform at a finite number of frequency points.

→ DFT is used to perform linear filtering operations in the frequency domain.

Discrete Fourier Series :-

consider a sequence $x_p(n)$ with a period of 'N' samples.

so that $x_p(n) = x_p(n+lN)$.

$x_p(n)$ is a periodic, it can be represented as weighted sum of complex exponentials whose frequencies are integer multiples of the fundamental frequency (i.e $\frac{2\pi}{N}$).

periodic complex exponentials are of the form

$$e^{j\frac{2\pi kn}{N}} = e^{j\frac{2\pi k(n+lN)}{N}} \qquad l \rightarrow \text{integer}.$$

From the periodicity property of the discrete time Fourier transform, we can conclude that there are a finite number of harmonics. The frequencies are $\{ \frac{2\pi}{N}k,\ k = 0, 1, \ldots N-1 \}$.

Therefore a periodic sequence $x_p(n)$ can be expressed as

$$\boxed{x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k)\, e^{j\frac{2\pi nk}{N}}} \quad \text{— (1)} \qquad n = 0, \pm 1 \ldots$$

where $\{X_p(k),\ k = 0, \pm 1, \ldots\}$ are called the Discrete Fourier series coefficients.

To obtain fourier coefficients, multiply with $e^{-j(\frac{2\pi}{N})nm}$ and summing the product from $n=0$ to $n=N-1$, then.

$$\sum_{n=0}^{N-1} x_p(n)\, e^{-j(\frac{2\pi}{N})mn} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} X_p(k)\, e^{j\frac{2\pi}{N}kn}\, e^{-j\frac{2\pi}{N}nm}$$

$$\sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}nm} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} X_p(k)\, e^{j\frac{2\pi}{N}(k-m)n}$$

Interchanging the orders of summation on the right side of above equation.

$$\sum_{n=0}^{N-1} x_p(n)\, e^{-j(\frac{2\pi}{N})nm} = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k) \cdot \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_p(k) \cdot \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

$$\sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})(k-m)n} = N \quad \text{if } k-m = 0, \pm N, \pm 2N$$

$$= 0 \quad \text{otherwise}$$

therefore.

$$\sum_{n=0}^{N-1} x_p(n)\, e^{-j(2\pi/N)nm} = X_p(m)$$

changing the index from m to k, the fourier series coefficients $X_p(k)$ obtained from $x_p(n)$ by the relation.

$$\boxed{X_p(k) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j(2\pi/N)kn}} \quad (2)$$

therefore DFS and Inverse DFS can be expressed as.

$$\text{DFS:} \quad X_p(k) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi kn}{N}} \rightarrow \text{Analysis equation}$$

$$\text{Inverse DFS.} \quad x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k)\, e^{j\frac{2\pi nk}{N}} \rightarrow \text{syntheisis equation.}$$

$$\text{DFS}[x_p(n)] = X_p(k)$$

properties of the DFS :-

1. linearity :-

consider two periodic sequences $x_{1p}(n)$, $x_{2p}(n)$ both with period N, such that

$$DFS[x_{1p}(n)] = X_{1p}(K)$$

$$DFS[x_{2p}(n)] = X_{2p}(K)$$

then

$$DFS[a_1 x_{1p}(n) + a_2 x_{2p}(n)] = a_1 X_{1p}(K) + a_2 X_{2p}(K).$$

2. Time shifting :-

$x_p(n)$ is a periodic with N samples

$$DFS[x_p(n)] = X_p(K)$$

$$DFS[x_p(n-m)] = e^{-j\frac{2\pi}{N}mk} X_p(K)$$

$x_p(n-m)$ is a shifted version of $x_p(n)$.

3. Frequency shifting :-

$$DFS[x_p(n)] = X_p(K)$$

$$DFS[e^{j\frac{2\pi l n}{N}} x_p(n)] = X_p(K-l)$$

4. periodic convolution :-

let $x_{1p}(n)$ and $x_{2p}(n)$ be two periodic sequences with period N

$$DFS[x_{1p}(n)] = X_{1p}(K)$$

$$DFS[x_{2p}(n)] = X_{2p}(K)$$

then periodic sequence $x_{3p}(n)$, $x_{3p}(n) = x_{1p}(n) * x_{2p}(n)$

$$x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m) x_{2p}(n-m)$$

then

$$DFS\left[\sum_{m=0}^{N-1} x_{1p}(m) x_{2p}(n-m)\right] = X_{1p}(K) X_{2p}(K)$$

5. multiplication :-

$$DFS[x_{1p}(n)x_{2p}(n)] = \frac{1}{N}\sum_{l=0}^{N-1} X_{1p}(l)X_{2p}(k-l).$$

6. symmetry property :-

$$DFS[x_p^*(n)] = X_p^*(-k)$$

$$DFS[x_p^*(-n)] = X_p^*(k)$$

$$DFS\{Re[x_p(n)]\} = DFS\left[\frac{x_p(n)+x_p^*(n)}{2}\right]$$

$$= \frac{1}{2}[X_p(k) + X_p^*(-k)]$$

$$= X_{pe}(k)$$

$$DFS\{j\,Im[x_p(n)]\} = DFS\left[\frac{x_p(n)-x_p^*(n)}{2}\right]$$

$$= \frac{1}{2}[X_p(k) - X_p^*(-k)]$$

$$= X_{po}(k)$$

$x_p(n)$ can be written as, $x_p(n) = x_{pe}(n) + x_{po}(n)$

where $x_{pe}(n) = \frac{1}{2}[x_p(n) + x_p^*(-n)]$ $\quad x_{po}(n) = \frac{1}{2}[x_p(n) - x_p^*(-n)]$

$$DFS[x_{pe}(n)] = DFS\{\tfrac{1}{2}[x_p(n) + x_p^*(-n)]\}$$

$$= \frac{1}{2}[X_p(k) + X_p^*(k)]$$

$$= Re\{X_p(k)\}.$$

$$DFS[x_{po}(n)] = DFS\{\tfrac{1}{2}[x_p(n) - x_p^*(-n)]\}$$

$$= \frac{1}{2}[X_p(k) - X_p^*(k)]$$

$$= j\,Im\{X_p(k)\}.$$

Discrete Fourier Transform (DFT) :-

The DFT of a finite duration sequence $x(n)$ is obtained by sampling the Fourier transform $X(e^{jw})$ at N equally spaced points over the interval $0 \le w \le 2\pi$ with a spacing of $2\pi/N$.

The DFT, denoted by $X(K)$ is defined as

$$X(k) = X(e^{jw})\Big|_{w=\frac{2\pi k}{N}} \qquad 0 \le k \le N-1$$

let $x(n)$ is a causal, finite duration sequence containing L samples. Then its fourier transform is given by

$$X(e^{jw}) = \sum_{n=0}^{L-1} x(n)\, e^{-jwn}$$

If we sample $X(e^{jw})$ at 'N' equally spaced points over $0 \le w \le 2\pi$, we get

$$X(k) = X(e^{jw})_{w=\frac{2\pi k}{N}}$$

$$X(k) = \sum_{n=0}^{L-1} x(n)\, e^{-j\frac{2\pi k n}{N}}$$

Since Time domain aliasing occurs if $N < L$, to prevent it, increase the duration of $x(n)$ from L to N samples by appending number of zeros, which is known as zero padding. Then $X(K)$ can be written as

$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}} \qquad 0 \le k \le N-1}$$

Above equation is called N-point DFT.

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Since $x_p(n)$ is periodic extension of $x(n)$ with period N, it can be expressed in fourier series expansion

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k)\, e^{\frac{j2\pi kn}{N}}$$

the DFT $X(k)$ is related to the DFS coefficients $X_p(k)$ by

$$X(k) = X_p(k) \quad 0 \le k \le N-1$$
$$= 0 \quad \text{otherwise}$$

Then,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}$$

which is called as Inverse Discrete Fourier Transform

the formulas for DFT and IDFT are

$$\boxed{\begin{array}{lll} \text{DFT} & X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} & 0 \le k \le N-1 \\ \text{IDFT} & x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} & 0 \le n \le N-1 \end{array}}$$

$X(k) = \text{DFT}[x(n)]$

$x(n) = \text{IDFT}[X(k)]$.

where n, k are ranging from 0 to N−1

$n \rightarrow$ time index (Denotes time instant)

$k \rightarrow$ frequency index. (Denotes Discrete frequency)

let us define a term $\boxed{W_N = e^{-j2\pi/N}}$

which is known as twiddle factor, DFT can be simplified as

$$\boxed{\begin{array}{ll} X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} ; & 0 \le k \le N-1 \\ x(n) = \sum_{k=0}^{N-1} X(k)\, W_N^{-nk} ; & 0 \le n \le N-1 \end{array}}$$

Find the DFT of a sequence $x(n)=\{1,1,0,0\}$ and find the IDFT of $Y(k)=\{1,0,1,0\}$

Assume $N=L=4$.

$$x(n)=\{1,1,0,0\}$$

DFT $$X(k)=\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad k=0,1\ldots N-1$$

$$X(k)=\sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi kn}{4}} \qquad k=0,1,2,3$$

$k=0$,

$$X(0)=\sum_{n=0}^{3} x(n) = x(0)+x(1)+x(2)+x(3)$$
$$=1+1+0+0$$
$$=2$$

$k=1$, $$X(1)=\sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi n}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi n}{2}}$$

$$= x(0)+x(1)e^{-j\pi/2}+x(2)e^{-j\pi}+x(3)e^{-j\frac{3\pi}{2}}$$

$$= 1+1\cdot\left[\cos\tfrac{\pi}{2} - j\sin\tfrac{\pi}{2}\right]+0+0$$

$$X(1)=1+0-j = 1-j$$

$k=2$, $$X(2)=\sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi 2n}{4}} = \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$X(2)=x(0)+x(1)e^{-j\pi}+x(2)e^{-j2\pi}+x(3)e^{-j3\pi}$$

$$=1+1\cdot(\cos\pi - j\sin\pi)+0+0$$

$$=1-1-0$$

$$\underline{\underline{=0}}$$

$k=3$, $$X(3)=\sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi 3n}{4_2}} = \sum_{n=0}^{3} x(n)\, e^{-j\frac{3\pi n}{2}}$$

$$X(3)=x(0)+x(1)e^{-j3\pi/2}+x(2)e^{-j3\pi}+x(3)e^{-j9\pi/2}$$

$$=1+\left[\cos 3\pi/2 - j\sin 3\pi/2\right] = 1+0-j(-1) = \underline{\underline{1+j}}$$

$\therefore$

$X(k) = \{2, 1-j, 0, 1+j\}.$

IDFT :

$Y(k) = \{1, 0, 1, 0\}$, $N = 4$

$Y(0) = 1, \ Y(1) = 0, \ Y(2) = 1 \ \ Y(3) = 0$

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{\frac{j2\pi kn}{N}} \quad n = 0, 1, 2 .. N-1.$$

$n = 0,$

$$y(0) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{\frac{j2\pi k \cdot 0}{4}} = \frac{1}{4} \sum_{k=0}^{3} X(k)$$

$$y(0) = \frac{1}{4}[Y(0) + Y(1) + Y(2) + Y(3)]$$

$$y(0) = \frac{1}{4}[1 + 0 + 1 + 0] = 2/4 = \underline{0.5}$$

$n = 1,$

$$y(1) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{\frac{j2\pi k}{4}} = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j\pi k/2}$$

$$y(1) = \frac{1}{4}[Y(0) + Y(1) e^{j\pi/2} + Y(2) e^{j\pi} + Y(3) e^{j3\pi/2}]$$

$$= \frac{1}{4}[1 + 0 + \cos\pi + j\sin\pi + 0]$$

$$= \frac{1}{4}[1 + 0 - 1 + 0] = 0.$$

$n = 2,$

$$y(2) = \frac{1}{4}\left[\sum_{k=0}^{3} Y(k)\, e^{j\frac{2\pi 2k}{4}}\right] = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, e^{j\pi}$$

$$y(2) = \frac{1}{4}\left[Y(0) + Y(1)\, e^{j\pi} + Y(2)\, e^{j2\pi} + Y(3)\, e^{j3\pi}\right]$$

$$y(2) = \frac{1}{4}\left[1 + 0 + \cos 2\pi + j\sin 2\pi + 0\right]$$

$$y(2) = \frac{1}{4}\left[1 + 0 + 1 + j.0 + 0\right]$$

$$y(2) = 2/4 = \underline{\underline{0.5}}$$

$n = 3$

$$y(3) = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, e^{j\frac{2\pi k(3)}{4\,2}} = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, e^{j3\pi k/2}$$

$$y(3) = \frac{1}{4}\left[Y(0) + Y(1)\, e^{j3\pi/2} + Y(2)\, e^{j3\pi} + Y(3)\, e^{j9\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos 3\pi + j\sin 3\pi + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 + (-1) + j(0) + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 - 1 + 0 + 0\right]$$

$$= \underline{0}$$

$$\therefore\; y(n) = \{0.5, 0, 0.5, 0\}$$

Find the DFT of the following signals.

(i) $x(n) = \delta(n)$

$$x(n) = \delta(n)$$

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$$

$\delta(n) = 1$ for $n=0$
$= 0$ for $n \neq 0$.

$$X(K) = \sum_{n=0}^{N-1} \delta(n) e^{-j2\pi nk/N}$$

$$\underline{\underline{X(K) = 1}}$$

(ii) $x(n) = a^n$

$$X(K) = \sum_{n=0}^{N-1} a^n e^{-j2\pi nk/N}$$

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

$$X(K) = \sum_{n=0}^{N-1} \left[a e^{-j2\pi K/N}\right]^n$$

$$= \frac{1-\left[a e^{-j2\pi K/N}\right]^N}{1-a e^{-j2\pi K/N}}$$

$$X(K) = \frac{1-a^N e^{-j2\pi K}}{1-a e^{-j2\pi K/N}}$$

Determine the 8-point DFT of the sequence

$x(n) = \{1,1,1,1,1,1,0,0\}$

Sol $x(n) = \{1,1,1,1,1,1,0,0\}$, $N=8$

DFT $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}$ $k=0,1,\dots N-1$

$X(k) = \sum_{n=0}^{7} x(n)\, e^{-j\cancel{2}\pi nk/\cancel{8}4}$ $k=0,1,\dots 7$

$$\boxed{X(k) = \sum_{n=0}^{7} x(n)\, e^{-j\pi nk/4} \quad k=0,1,\dots 7}$$

$k=0$, $X(0) = \sum_{n=0}^{7} x(n).\, e^{0} = \sum_{n=0}^{7} x(n)$

$X(0) = x(0)+x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(7)$

$X(0) = 1+1+1+1+1+1+0+0 = \underline{\underline{6}}$

$k=1$, $X(1) = \sum_{n=0}^{7} x(n)\, e^{-j\pi n/4}$ ~~k=0~~

$X(1) = x(0)+x(1)e^{-j\pi/4} + x(2)e^{-j2\pi/4} + x(3)e^{-j3\pi/4} + x(4)e^{-j4\pi/4} +$
$x(5)e^{-j5\pi/4} + x(6)e^{-j6\pi/4} + x(7)e^{-j7\pi/4}$

$X(1) = 1 + e^{-j\pi/4} + e^{-j\pi/2} + e^{-j3\pi/4} + e^{-j\pi} + e^{-j5\pi/4} + 0 + 0$

$X(1) = 1 + \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\pi - j\sin\pi$
$+ \cos\frac{5\pi}{4} - j\sin\frac{\pi}{2}$

$X(1) = 1 + 0.707 - j0.707 + 0 - j + (-0.707 - j0.707) + (-1-0)$
$+ (-0.707 - j(-0.707))$

$X(1) = \cancel{1} + \cancel{0.707} - j0.707 - j - \cancel{0.707} - j\cancel{0.707} - \cancel{1} - 0.707 + j\cancel{0.707}$

$X(1) = -0.707 - j1.707$

$k=2,\quad X(2)=\sum_{n=0}^{7} x(n)\, e^{-j\pi 2n/4_2} = \sum_{n=0}^{7} x(n)\, e^{-j\pi n/2}$

$$X(2)=x(0)+x(1)e^{-j\pi/2}+x(2)e^{-j\pi}+x(3)e^{-j3\pi/2}+x(4)e^{-j2\pi}+x(5)e^{-j5\pi/2}$$
$$+x(6)e^{-j3\pi}+x(7)e^{-j7\pi/2}$$

$$X(2)=1+0-j+(-1+0)+(0+j)+(1+0)+(0-j)+0+0$$

$$X(2)=\cancel{1}-\cancel{j}-\cancel{1}+\cancel{j}+1-j = 1-j$$

$k=3,\quad X(3)=\sum_{n=0}^{7} x(n)\, e^{-j3\pi n/4}$

$$X(3)=x(0)+x(1)e^{-j3\pi/4}+x(2)e^{-j3\pi/2}+x(3)e^{-j9\pi/4}+x(4)e^{-j3\pi}+x(5)e^{-j15\pi/4}$$
$$x(6)e^{-j9\pi/2}+x(7)e^{-j21\pi/4}$$

$$X(3)=\cancel{1}-0.\cancel{7}07-j0.\cancel{7}07+j+0.\cancel{7}07-j0.707-\cancel{1}+0.\cancel{7}07+j0.\cancel{7}07$$

$$X(3)=0.707+j0.293.$$

$k=4,$

$$X(4)=\sum_{n=0}^{7} x(n)\, e^{-j\pi n}$$
$$=x(0)+x(1)e^{-j\pi}+x(2)e^{-j2\pi}+x(3)e^{-j3\pi}+x(4)e^{-j4\pi}+x(5)e^{-j5\pi}$$
$$+x(6)e^{-j6\pi}+x(7)e^{-j7\pi}$$

$$=1-1+1-1+1-1$$

$$X(4)=0$$

$k=5, \quad X(5) = \sum_{n=0}^{7} x(n)\, e^{-j5\pi n/4}$

$X(5) = x(0) + x(1)e^{-j5\pi/4} + x(2)e^{-j5\pi/2} + x(3)e^{-j15\pi/2} + x(4)e^{-j5\pi} +$

$x(5)e^{-j25\pi/4} + x(6)e^{-j15\pi/2} + x(7)e^{-j35\pi/4}$

$= 1 - 0.707 + j0.707 - j + 0.707 + j0.707 - 1 + 0.707 - j0.707$

$X(5) = 0.707 - j0.293.$

$k=6, \quad X(6) = \sum_{n=0}^{7} x(n)\, e^{-j3\pi n/2}$

$X(6) = x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2} + x(4)e^{-j6\pi} +$

$x(5)e^{-j15\pi/2} + x(6)e^{-j9\pi} + x(7)e^{-j21\pi/2}$

$= 1 + j - 1 - j + 1 + j$

$X(6) = 1 + j$

$k=7,$

$X(7) = \sum_{n=0}^{7} x(n)\, e^{-j7\pi n/4}$

$X(7) = x(0) + x(1)e^{-j7\pi/4} + x(2)e^{-j7\pi/2} + x(3)e^{-j21\pi/4} + x(4)e^{-j7\pi} +$

$x(5)e^{-j35\pi/4} + x(6)e^{-j21\pi/2} + x(7)e^{-j49\pi/2}$

$= 1 + 0.707 + j0.707 + j - 0.707 + j0.707 - 1 - 0.707 - j0.707$

$X(7) = -0.707 + j1.707$

$\therefore X(k) = \{ 6, -0.707 - j1.707, 1 - j, 0.707 + j0.293, 0,$

$0.707 - j0.293, 1 + j, -0.707 + j1.707 \}$

Find the IDFT of the sequence $X(k) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$

sol $X(k) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$

IDFT $\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N} \quad n = 0, 1 \ldots N-1}$

For $N = 8$,

$$x(n) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{+j\frac{2\pi kn}{8}} \quad n = 0, 1, 2 \ldots 7$$

$$x(n) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{+j\frac{\pi kn}{4}} \quad n = 0, 1, 2 \ldots 7$$

for $n = 0$,

$$x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{0} = \frac{1}{8} \sum_{k=0}^{7} X(k)$$

$$x(0) = \frac{1}{8}\left[X(0) + X(1) + X(2) + X(3) + X(4) + X(5) + X(6) + X(7)\right]$$

$$= \frac{1}{8}\left[5 + 0 + 1 - \cancel{j} + 0 + 1 + 0 + 1 + \cancel{j} + 0\right] = 8/8 = \underline{1}$$

for $n = 1$,

$$x(1) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{+j\pi k/4}$$

$$= \frac{1}{8}\left[X(0) + X(1)e^{j\pi/4} + X(2)e^{j2\pi/4} + X(3)e^{j3\pi/4} + X(4)e^{j4\pi/4} + X(5)e^{j5\pi/4} + X(6)e^{j6\pi/4} + X(7)e^{j7\pi/4}\right]$$

$$= \frac{1}{8}\left[5 + 0 + (1-j)(j) + 0 + 1(-1) + 0 + (1+j)(-j) + 0\right]$$

$$= \frac{1}{8}\left[5 + \cancel{j} - j^2 - 1 - \cancel{j} - j^2\right] = 6/8 = 0.75$$

for $n = 2$,

$$x(2) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{j2\pi k/4}$$

$$x(2) = \frac{1}{8}\left[X(0) + X(1)e^{j\pi/2} + X(2)e^{j\pi} + X(3)e^{j3\pi/2} + X(4)e^{j2\pi} + X(5)e^{j5\pi/2} + X(6)e^{j3\pi} + X(7)e^{j7\pi/2}\right]$$

$$= \frac{1}{8}\left[5 + 0 + (1-j)(-1) + 0 + 1(1) + 0 + (1+j)(-1) + 0\right]$$

$$= \frac{1}{8}\left[5 - \cancel{1} + \cancel{j} + \cancel{1} - 1 - \cancel{j}\right] = 4/8 = \underline{\underline{0.5}}$$

for $n=3$ $\quad x(3) = \frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j3\pi k/4}$

$$x(3) = \frac{1}{8}\left[X(0) + X(1)e^{j3\pi/4} + X(2)e^{j3\pi/2} + X(3)e^{j9\pi/2} + X(4)e^{j18\pi} + X(5)e^{j15\pi/4} + X(6)e^{j9\pi/2} + X(7)e^{j21\pi/4}\right]$$

$$x(3) = \frac{1}{8}\left[5 + 0 + (1-j)(-j) + 0 + 1(-1) + 0 + (1+j)j + 0\right]$$

$$= \frac{1}{8}\left[5 - \cancel{j} + j^2 - 1 + \cancel{j} + j^2\right] = 2/8 = 0.25$$

for $n=4$, $\quad x(4) = \frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j4\pi k/4} = \frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j\pi k}$

$$x(4) = \frac{1}{8}\left[X(0) + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi} + X(4)e^{j4\pi} + X(5)e^{j5\pi} + X(6)e^{j6\pi} + X(7)e^{j7\pi}\right]$$

$$x(4) = \frac{1}{8}\left[5 + 0 + (1-j)(1) + 0 + 1(1) + (1+j)(1) + 0\right]$$

$$= \frac{1}{8}\left[5 + 1 - \cancel{j} + 1 + 1 + \cancel{j}\right] = 8/8 = \underline{1}$$

for $n=5$, $\quad x(5) = \frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j5\pi k/4}$

$$x(5) = \frac{1}{8}\left[X(0) + X(1)e^{j5\pi/4} + X(2)e^{j5\pi/2} + X(3)e^{j15\pi/4} + X(4)e^{j5\pi} + X(5)e^{j25\pi/4} + X(6)e^{j15\pi/2} + X(7)e^{j35\pi/4}\right]$$

$$= \frac{1}{8}\left[5 + 0 + (1-j)(j) + 0 + 1(-1) + 0 + (1+j)(-j) + 0\right]$$

$$= \frac{1}{8}\left[5 + \cancel{j} - j^2 - 1 - \cancel{j} - j^2 + 0\right] = 6/8 = \underline{\underline{0.75}}$$

for $n=6$, $\quad x(6) = \frac{1}{8}\sum_{k=0}^{7} X(k)e^{j6\pi k/4} = \frac{1}{8}\sum_{k=0}^{7} X(k)e^{j3\pi k/2}$

$$x(6) = \frac{1}{8}\Big[X(0) + X(1)e^{j3\pi/2} + X(2)e^{j3\pi} + X(3)e^{j9\pi/2} + X(4)e^{j12\pi}$$

$$+ X(5)e^{j15\pi/2} + X(6)e^{j9\pi} + X(7)e^{j21\pi/2}\Big]$$

$$= \frac{1}{8}\left[5 + 0 + (1-j)(-1) + 0 + (1)(1)^{+0} + (1+j)(-1) + 0\right]$$

$$= \frac{1}{8}\left[5 - \cancel{1} + \cancel{j} + \cancel{1} - 1 - \cancel{j}\right] = 4/8 = \underline{\underline{0.5}}$$

for $n=7$, $\quad x(7) = \frac{1}{8}\sum_{k=0}^{7} X(k)e^{j7\pi k/4}$

$$x(7) = \frac{1}{8}\Big[\cancel{8}\, X(0) + X(1)e^{j7\pi/4} + X(2)e^{j7\pi/2} + X(3)e^{j21\pi/4} +$$

$$X(4)e^{j7\pi} + X(5)e^{j35\pi/4} + X(6)e^{j21\pi/2} + X(7)e^{j49\pi/4}\Big]$$

$$x(7) = \frac{1}{8}\left[5 + 0 + (1-j)(-j) + 0 + 1(-1) + 0 + (1+j)(j) + 0\right]$$

$$= \frac{1}{8}\left[5 - \cancel{j} + j^2 - 1 + \cancel{j} + j^2\right] = 2/8 = \underline{\underline{0.25}}$$

$$\therefore\ x(n) = \{1, 0.75, 0.5, 0.25, 1, 0.75, 0.5, 0.25\}$$

DFT $\quad X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \qquad 0 \le k \le N-1$

IDFT $\quad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn} \qquad 0 \le n \le N-1$

Twiddle factor $W_N = e^{-j\frac{2\pi}{N}}$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \quad ; \quad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \quad ; \quad 0 \le n \le N-1$$

magnitude of twiddle factor,

$$|W_N| = \left|e^{-j\frac{2\pi}{N}}\right| = \left|\cos\frac{2\pi}{N} - j\sin\frac{2\pi}{N}\right|$$

phase angle $\quad \angle e^{-j2\pi/N} = \frac{-2\pi}{N}.$

consider the term W_N^{kn}, where $kn = r$,

term W_N^{r}

Symmetry property :-

$$W_N^{r} = -W_N^{r \pm N/2}$$

proof:

$$W_N^{r} = e^{-j\frac{2\pi r}{N}}$$

$$W_N^{r\pm N/2} = e^{-j\frac{2\pi}{N}\left[r \pm \frac{N}{2}\right]}$$

$$= e^{-j\frac{2\pi}{N}r \pm j\frac{2\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{2}}$$

$$= e^{-j\frac{2\pi}{N}r}\left(e^{\pm j\pi}\right)$$

$$= -e^{-j\frac{2\pi}{N}r}$$

$$\boxed{W_N^{r} = -W_N^{r\pm N/2}}$$

Note:

$W_8^0 = -W_8^4$

$W_8^1 = -W_8^5$

$W_8^2 = -W_8^6$

$W_8^3 = -W_8^7$

$W_4^0 = -W_4^2$

$W_4^1 = -W_4^3$

periodicity property:-

$$W_N^n = W_N^{n \pm lN}.$$

<u>proof:</u>

$$W_N^n = e^{-j\frac{2\pi n}{N}}$$

$$W_N^{n \pm lN} = e^{-j\frac{2\pi}{N}(n \pm lN)}$$

$$= e^{-j\frac{2\pi}{N}n} \, e^{\pm j\frac{2\pi}{N}lN}$$

$$= e^{-j\frac{2\pi}{N}n} \, e^{\pm j2\pi l} \qquad (\because e^{\pm j2\pi l} = 1)$$

$$= e^{-j\frac{2\pi}{N}n}$$

$$\therefore \boxed{W_N^n = W_N^{n \pm lN}}$$

$$W_8^0 = W_8^8 = W_8^{16} = W_8^{24} = W_8^{32}$$

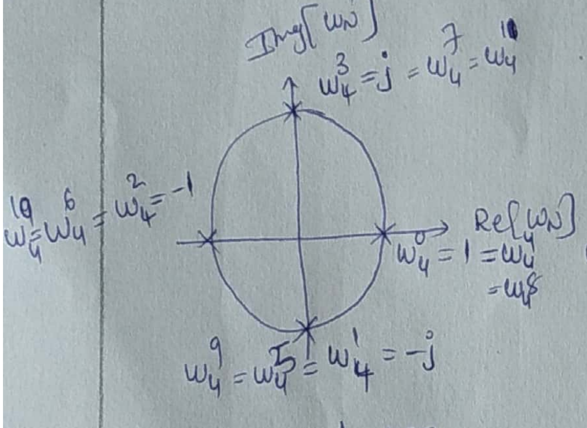


fig: 4 point DFT

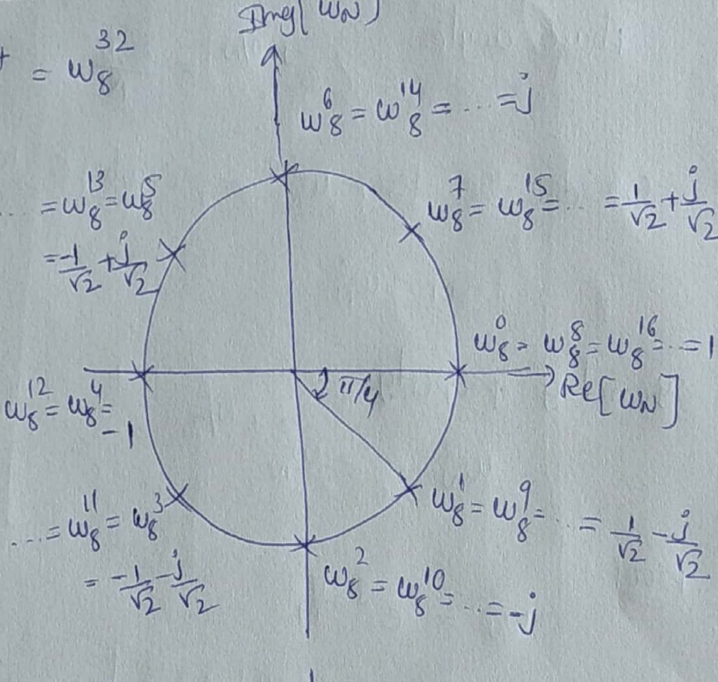


fig: 8-point DFT

<u>Note:</u>

$$W_4^9 = W_4^{1+2(4)} = W_4^1$$

$$W_4^{15} = W_4^{3+3(4)} = W_4^3$$

$$W_8^{17} = W_8^{1+2(8)} = W_8^1$$

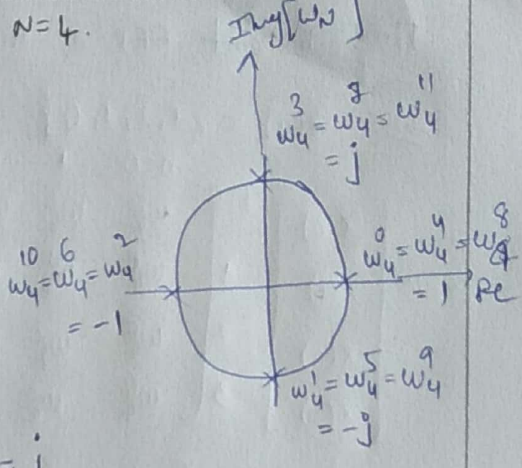
$$W_8^{49} = W_8^{1+6(8)} = W_8^1$$

$$W_8^{27} = W_8^{3+3(8)} = W_8^3$$

$kn = r$	$W_8^r = (e^{-j\pi/4})^r$	magnitude	phase angle
0	$W_8^0 = 1$	1	0
1	$W_8^1 = e^{-j\pi/4} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	1	$-\pi/4$
2	$W_8^2 = e^{-j\pi/2} = -j$	1	$-\pi/2$
3	$W_8^3 = e^{-j3\pi/4} = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	1	$-3\pi/4$
4	$W_8^4 = e^{-j\pi} = -1$	1	$-\pi$
5	$W_8^5 = e^{-j5\pi/4} = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$	1	$-5\pi/4$
6	$W_8^6 = e^{-j6\pi/4} = j$	1	$-3\pi/2$
7	$W_8^7 = e^{-j7\pi/4} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$	1	$-7\pi/4$.

Matrix method:

① Find the DFT of $x(n) = \{1, 2, 3, 4\}$. for $N = 4$.



$N = 4 \quad L = 4,$

$N \geq L,$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \qquad 0 \leq k \leq 3$$

$$X(k) = \sum_{n=0}^{3} x(n)\, W_4^{nk} \qquad 0 \leq k \leq 3.$$

$$W_4^0 = 1, \quad W_4^1 = -j \quad W_4^2 = -1, \quad W_4^3 = j$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{array}{c c} & \begin{array}{cccc} n=0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} k=0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}_{4\times 4} \end{array} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+j-3-4j \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$\therefore \; X(k) = \{10, -2+2j, -2, -2-2j\}$$

Find 4 point DFT of the sequence $x(n) = \{1, 2, 1, 0\}$.

$$W_N^{r} = \left[e^{-j\frac{2\pi}{N}}\right]^{r}$$

Sol $x(n) = \{1, 2, 1, 0\}$. $N = 4$.

$$W_4^0 = e^{-j\left(\frac{2\pi}{4}\right)0} = 1$$

$$W_4^1 = e^{-j\frac{2\pi}{4}1} = -j$$

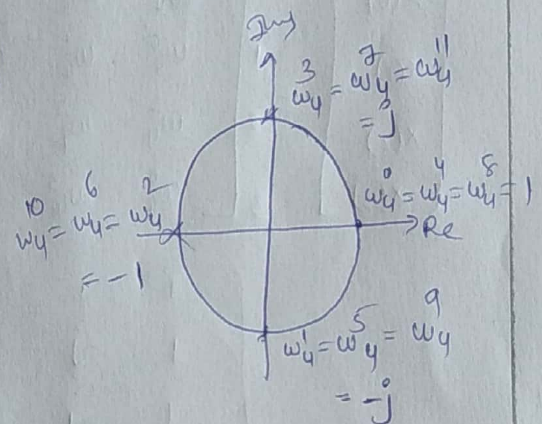
$$W_4^2 = e^{-j\frac{2\pi}{4}2} = -1$$

$$W_4^3 = e^{-j\frac{2\pi}{4}3} = j$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{array}{c|cccc} & n=0 & 1 & 2 & 3 \\ \hline K=0 & W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ 1 & W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ 2 & W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ 3 & W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{array} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1+2+1+0 \\ 1-2j-1+0 \\ 1-2+1+0 \\ 1+2j-1+0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$



$\therefore \; X(k) = \{4, -j2, 0, j2\}$

Find 8-point DFT of the sequence $\{1, 0, -1, 1\}$.

$x(n) = \{1, 0, -1, 1\}$. $\quad N = 8$.

$x(n) = \{1, 0, -1, 1, 0, 0, 0, 0\}$.

$$
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix}
=
\begin{array}{c}
 \\ k=0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} n=0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\
W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\
W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\
W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\
W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\
W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49}
\end{bmatrix}
\end{array}
\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
=
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{bmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\
W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\
W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1
\end{bmatrix}
\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}} & -1 & \frac{-1}{\sqrt{2}}+\frac{j}{\sqrt{2}} & j & \frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & \frac{-1}{\sqrt{2}}-\frac{j}{\sqrt{2}} & j & \frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}} & -j & \frac{-1}{\sqrt{2}}+\frac{j}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1}{\sqrt{2}}+\frac{j}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}} & j & \frac{-1}{\sqrt{2}}-\frac{j}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}} & j & \frac{-1}{\sqrt{2}}+\frac{j}{\sqrt{2}} & -1 & \frac{-1}{2}-\frac{j}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-\cancel{1}+\cancel{1}+0+0+0+0 \\ 1+0+j-\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}+0+0+0+0 \\ 1+0+1+j+0+0+0+0 \\ 1+0-j+\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}+0+0+0+0 \\ \cancel{1}+0-\cancel{1}-1+0+0+0+0 \\ 1+0+j+\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}+0+0+0+0 \\ 1+0+1-j+0+0+0+0 \\ 1+0-j-\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}+0+0+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.29+j0.29 \\ 2+j \\ 1.707-1.707j \\ -1 \\ 1.707+1.707j \\ 2-j \\ 0.29-j0.29 \end{bmatrix}$$

$$\therefore x(K) = \{1,\ 0.29+j0.29,\ 2+j,\ 1.707-j1.707,\ -1,\ 1.707+j1.707,\ 2-j,\ 0.29-j0.29\}$$

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Find IDFT of $x(k) = \{1, -j, 5, j\}$.

$$X(K) = \{1, -j, 5, j\}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn} \qquad 0 \le n \le N-1$$

$$x(n) = \frac{1}{4}\sum_{k=0}^{N-1} X(k)\, W_4^{-kn} \qquad 0 \le n \le 3$$

$$W_4^0 = e^{-j\frac{2\pi}{4}(0)} = 1, \quad W_4^{-1} = e^{-j\frac{2\pi}{4_2}(-1)} = e^{j\pi/2} = j$$

$$W_4^{-2} = e^{-j\left(\frac{2\pi}{4}\right)(-2)} = e^{j\pi} = -1 \qquad W_4^{-3} = e^{-j\left(\frac{2\pi}{4}\right)(-3)} = e^{j3\pi/2} = -j$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ W_4^0 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ W_4^0 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix}\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 1 \\ -j \\ 5 \\ j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 - \cancel{j} + 5 + \cancel{j} \\ 1 - j^2 - 5 - j^2 \\ 1 + \cancel{j} + 5 - \cancel{j} \\ 1 + j^2 - 5 + j^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6 \\ -2 \\ 6 \\ -6 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \\ 3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \\ 1.5 \\ -1.5 \end{bmatrix}$$

$$x(n) = \{1.5, -0.5, 1.5, -1.5\}$$

Find IDFT of $X(k) = \{1,0,0,1\}$.

$$X(k) = \{1,0,0,1\} \quad , \; N = 4$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N} \qquad 0 \le n \le N-1$$

$$x(n) = \frac{1}{4}\sum_{k=0}^{N-1} X(k)\, W_4^{-kn} \qquad 0 \le n \le 3$$

$$W_4^0 = 1, \quad W_4^{-1} = \left[e^{-j\frac{2\pi}{4}}\right]^{-1} = e^{j\pi/2} = j$$

$$W_4^{-2} = \left[e^{-j\frac{2\pi}{4}}\right]^{-2} = e^{j\pi} = -1, \quad W_4^{-3} = \left[e^{-j\frac{2\pi}{4}}\right]^{-3} = e^{j3\pi/2} = -j$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ W_4^0 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ W_4^0 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix} \end{array} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1+0+0-1 \\ 1+0+0+j \\ 1+0+0+1 \\ 1+0+0-j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 1+j \\ 2 \\ 1-j \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1+j}{4} \\ 1/2 \\ \frac{1-j}{4} \end{bmatrix}$$

$$x(k) = \left\{ 0, \frac{1+j}{4}, \frac{1}{2}, \frac{1-j}{4} \right.$$

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Relationship of the DFT to the Fourier Transform:-

The Fourier Transform of a finite duration sequence $x(n)$ having length 'N' is given by

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \quad \text{—}(1)$$

The discrete fourier transform of $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad k=0,1,2,\dots N-1 \quad \text{—}(2)$$

By comparing equation (1) and (2), we can conclude that the DFT of $x(n)$ is sampled version of the fourier transform of the sequence and is given by

$$\boxed{X(k) = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} \qquad k=0,1,2\dots N-1}$$

Relationship of the DFT to the Z-Transform:-

consider a sequence $x(n)$ of finite duration 'N' with the z-transform,

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \quad \text{—}(1)$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

substitute $x(n)$ in equation (1),

$$X(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}\right] z^{-n}$$

$$\sum_{n=0}^{N} a^n = \frac{1-a^N}{1-a}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1}\left[e^{j2\pi k/N} z^{-1}\right]^n$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \frac{1-\left[e^{j2\pi k/N}\cdot z^{-1}\right]^N}{1-e^{j2\pi k/N} z^{-1}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \frac{1 - e^{j2\pi k} \cdot z^{-N}}{1 - e^{j2\pi k/N} z^{-1}} \cdot X(k)$$

$$x(z) = \frac{(1 - z^{-N})}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j2\pi k/N} z^{-1}}$$

Calculate the DFT of the sequence $x(n) = \left(\frac{1}{4}\right)^n$ for $N=16$,

Sol $\quad x(n) = \left(\frac{1}{4}\right)^n \quad , N=16$

DFT $\quad \boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad k=0,1,2 \ldots N-1}$

$$X(k) = \sum_{n=0}^{15} x(n)\, e^{-j2\pi kn/16} \qquad k=0,1,2\ldots15$$

$$= \sum_{n=0}^{15} \left(\frac{1}{4}\right)^n e^{-j\pi kn/8}$$

$$= \sum_{n=0}^{15} \left[\frac{1}{4} e^{-j\pi k/8}\right]^n$$

$$= \frac{1-\left[\frac{1}{4} e^{-j\pi k/8}\right]^{16}}{1-\frac{1}{4} e^{-j\pi k/8}}$$

$$= \frac{1-(1/4)^{16}\, e^{-j\pi k.16/8}}{1-\frac{1}{4} e^{-j\pi k/8}}$$

$$X(k) = \frac{1-(1/4)^{16}\, e^{-j2\pi k}}{1-1/4\, e^{-j\pi k/8}} \qquad k=0,1,2\ldots15$$

properties of the DFT :-

property	Time Domain	Frequency Domain
linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(k)+a_2X_2(k)$.
periodicity	$x(n)=x(n+N)$	$X(K)=X(k+N)$
Time reversal	$x(N-n)$	$X(N-k)$
circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
circular convolution	$x_1(n) \; (N) \; x_2(n)$	$X_1(k)X_2(k)$
circular correlation	$x_1(n) \; (N) \; y^*(-n)$	$X(k)Y^*(k)$
multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}[X(k) \; (N) \; X_2(k)]$
complex conjugate	$x^*(n)$	$X^*(N-k)$
parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{K=0}^{N-1} X(k)Y^*(k)$

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properties of the Discrete Fourier Transform:-

1. periodicity :-

If $X(k)$ is N-point DFT of the sequence $x(n)$, then

$$x(n+N) = x(n) \quad \forall n$$

$$X(k+N) = X(k) \quad \forall k$$

proof:

$$X(k+N) = X(k)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi (k+N)n/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} e^{-j2\pi \cancel{N} n/\cancel{N}}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{kn}{N}} e^{-j2\pi n}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{kn}{N}} \quad \left(e^{-j2\pi n} = 1, \forall n \right)$$

$$\therefore X(k+N) = X(k)$$

linearity property :-

If $x_1(n)$ and $x_2(n)$ are two finite duration sequences, and

if $DFT[x_1(n)] = X_1(k)$

$DFT[x_2(n)] = X_2(k)$

then for any real valued or complex valued constants a & b.

$$DFT[a x_1(n) + b x_2(n)] = a X_1(k) + b X_2(k).$$

proof:

$$DFT[a x_1(n)+b x_2(n)] = \sum_{n=0}^{N-1} [a x_1(n)+b x_2(n)] e^{-j2\pi nk/N}$$

$$= a \sum_{n=0}^{N-1} a x_1(n) e^{-j2\pi nk/N} + \sum_{n=0}^{N-1} b x_2(n) e^{-j2\pi nk/N}$$

$$= a \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N} + b \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi nk/N}$$

$$= a X_1(k) + b X_2(k)$$

$\therefore$ $DFT[a x_1(n)+b x_2(n)] = a X_1(k) + b X_2(k)$.

Time reversal of the sequence:-

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x((-n))_N$.

$$x((-n))_N = x(N-n) \quad 0 \le n \le N-1$$

If $DFT[x(n)] = X(k)$

then $DFT[x((-n))_N] = X(N-k)$.

proof:

$$DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n) e^{-j2\pi \frac{nk}{N}}$$

changing index n to m, $m = N-n$, $n = N-m$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi \frac{kN}{N}} e^{+j2\pi km/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{j2\pi km/N} \quad (\because e^{-j2\pi k} = 1)$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi m(N-k)/N}$$

$$= X(N-k)$$

$\therefore$ $DFT[x((-n))_N] = X((-k))_N = X(N-k)$

Circular frequency shift :-

If $\quad DFT[x(n)] = X(k)$

then $\quad DFT[x(n)e^{j2\pi ln/N}] = X((k-l))_N$

Proof!

$$DFT[x(n)e^{j2\pi ln/N}] = \sum_{n=0}^{N-1} x(n) e^{j2\pi ln/N} e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(k-l)/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N+k-l)/N}$$

$$= X(N+k-l)$$

$$= X((k-l))_N$$

$$\therefore DFT[x(n)e^{j2\pi ln/N}] = X((k-l))_N$$

complex conjugate property :-

If $\quad x(n) \xleftrightarrow{DFT} X(k)$

$$x^*(n) \xleftrightarrow{DFT} X^*(N-k) = X^*((-k))_N$$

$$\cancel{x^*(n)}\ x^*(N-n) \xleftrightarrow{DFT} X^*(k)$$

Proof:

$$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi nN/N} e^{-j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)N}\right]^*$$

$$= X^*(N-k)$$

$$= X^*((-k))_N$$

$$x^*(N-n) \xleftrightarrow{DFT} X^*(k)$$

$$IDFT[X^*(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{j2\pi kn/N}$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k)e^{-j2\pi kn/N}\right]^*$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k)e^{j2\pi kN/N}e^{-j2\pi kn/N}\right]^*$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k)e^{j2\pi k(N-n)/N}\right]^*$$

$$= x^*(N-n)$$

$$\therefore D[x^*(N-n)] = X^*(k)$$

circular correlation:-

For complex valued sequences $x(n)$ & $y(n)$,

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$y(n) \xleftrightarrow{DFT} Y(k)$$

then

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)y^*((n-l))_N$$

$$DFT\left\{\sum_{n=0}^{N-1} x(n)y^*((n-l))_N\right\} = X(k)Y^*(k)$$

$$\therefore DFT[\overset{*}{r}_{xy}(l)] = X(k)Y^*(k)$$

circular convolution :-

The convolution property of DFT says that, the multiplication of DFT's of two sequences is equivalent to the DFT of the circular convolution of the two sequences.

If $\quad DFT[x_1(n)] = X_1(k)$ &

$\quad\quad DFT[x_2(n)] = X_2(k)$

then $\quad DFT[x_1(n) \circledast x_2(n)] = X_1(k)X_2(k)$

proof: let $x_1(n)$ & $x_2(n)$ are two finite duration sequences of length N.

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N} \quad k=0,1,\ldots N-1$$

$$X_2(k) = \sum_{l=0}^{N-1} x_2(l) e^{-j2\pi kl/N} \quad k=0,1,\ldots N-1$$

on multiplying above two sequences,

$$X_3(k) = X_1(k)X_2(k)$$

$$x_3(m) = IDFT\{X_3(k)\}.$$

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k) e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2(k) e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N}\right)\left(\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi kl/N}\right).e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n).\sum_{l=0}^{N-1} x_2(l)\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} 1\right]$$

$$= \frac{1}{\cancel{N}}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l).\cancel{N}$$

$\left(\because e^{j2\pi k(m-n-l)/N} = 1\right)$

$m-n-l = PN$

$\left.P \rightarrow \text{is integer}\right)$

$e^{j2\pi kPN/N}$

$e^{j2\pi k.P} = (e^{j2\pi P})^k = 1^k$

$= 1$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) . \sum_{l=0}^{N-1} x_2(l).$$

$$= \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N$$

$$x_3(m) = x_1(n) \, Ⓝ \, x_2(n)$$

$$\therefore \quad X_1(k) X_2(k) = DFT[\, x_1(n) \, Ⓝ \, x_2(n) \,].$$

multiplication of Two sequences:

If $DFT[\, x_1(n) \,] = X_1(k)$

$DFT[\, x_2(n) \,] = X_2(k)$

then $DFT[\, x_1(n) x_2(n) \,] = \frac{1}{N} [\, X_1(k) \, Ⓝ \, X_2(k) \,]$.

Circular time shift of a sequence :-

If $x(n) \xleftrightarrow{DFT} X(k)$

then $x((n-l))_N \xleftrightarrow{DFT} X(k)\, e^{-j2\pi k l/N}$,

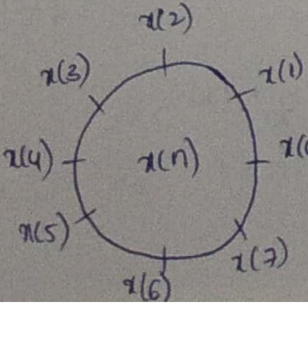
Proof:

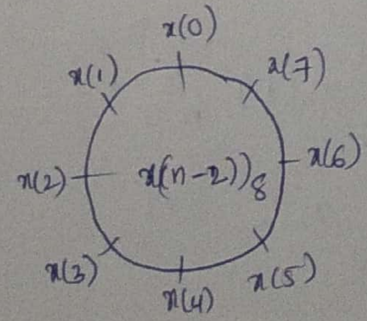
circular shift.

$x(n) = \{ x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7) \}$

$x((n-1))_8 = \{ x(7), x(0), x(1), x(2), x(3), x(4), x(5), x(6) \}$

$x((n-2))_8 = \{ x(6), x(7), x(0), x(1), x(2), x(3), x(4), x(5) \}$





$$DFT[x((n-l))_N] = \sum_{n=0}^{N-1} x((n-l))_N \, e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{l-1} x((n-l))_N \, e^{-j2\pi kn/N} + \sum_{n=l}^{N-1} x(n-l) \, e^{-j2\pi kn/N} \quad (1)$$

summation (1)

$$\sum_{n=0}^{l-1} x((n-l))_N \, e^{-j2\pi kn/N} = \sum_{n=0}^{l-1} x(N+n-l) \, e^{-j2\pi kn/N}$$

$(\because x((n-l))_N = x(N+n-l))$

let $N+n-l = m$,

$$= \sum_{m=N-l}^{N-1} x(m) \, e^{-j2\pi k(l+m-N)/N}$$

$$= \sum_{m=N-l}^{N-1} x(m) \, e^{-j2\pi k(l+m)/N} \quad (2)$$

$(\because e^{j2\pi k} = 1$ for $k = 1, 2, 3)$

summation (2), let $n-l = m$

$$\sum_{n=l}^{N-1} x(n-l) \, e^{-j2\pi kn/N} = \sum_{n=l}^{N-1} x(n-l) \, e^{-j2\pi kn/N}$$

$$= \sum_{m=0}^{N-1-l} x(m) \, e^{-j2\pi k(l+m)/N}$$

$$= \sum_{m=0}^{N-1-l} x(m) \, e^{-j2\pi k(l+m)/N} \quad (3)$$

substitute eq (2) & (3) in (1),

$$DFT[x((n-l))_N] = \sum_{m=N-l}^{N-1} x(m) e^{-j2\pi k(l+m)/N} + \sum_{m=0}^{N-1-l} x(m) e^{-j2\pi k(l+m)/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi km/N} \, e^{-j2\pi kl/N}$$

$$= e^{-j2\pi kl/N} \left[\sum_{m=0}^{N-1} x(m) e^{-j2\pi km/N} \right]$$

$$DFT[x((n-l))_N] = e^{-j2\pi kl/N} X(k)$$

Real valued sequence :-

If $x(n)$ is real then

$$X(n-k) = X^*(k) = X(-k)$$

proof

$$X(k) = DFT\{x(n)\}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad .. \quad 0 \le k \le N-1$$

$$X(N-k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{(N-k)n}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi \cancel{N} n/\cancel{N}} \cdot e^{j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n} e^{-j2\pi(-k)n/N}$$

$$X(N-k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi(-k)n/N} \quad (\because e^{-j2\pi n} = 1)$$

$$X(N-k) = X(-k) \quad \text{--- (1)}$$

$$X^*(k) = \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}\right]^*$$

$$= \sum_{n=0}^{N-1} x^*(n) e^{j2\pi kn/N} \qquad x(n) = x^*(n) = \text{real valued}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi(-k)n/N}$$

$$X^*(k) = X(-k) \quad \text{--- (2)}$$

from eq (1) & (2)

$$\boxed{X(N-k) = X^*(k) = X(-k)}$$

The DFT of real valued sequence satisfy conjugate symmetry about the midpoint if n is odd, conjugate symmetry is about N/2. The index N/2 is called folding index. Similarly, if $x(k)$ is real & IDFT satisfies symmetric condition about midpoint.

Parseval's theorem:-

$$x(n) \xleftrightarrow{DFT} X(k)$$
$$y(n) \xleftrightarrow{DFT} Y(k)$$

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, Y^*(k).$$

If $x(n) = y(n)$,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

proof:-

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \sum_{n=0}^{N-1} x(n) \cdot \left[\frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{j2\pi kn/N}\right]^*$$

$$= \sum_{n=0}^{N-1} x(n) \cdot \frac{1}{N}\sum_{k=0}^{N-1} Y^*(k)\, e^{-j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} Y^*(k) \cdot \left[\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}\right]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} Y^*(k) \cdot X(k)$$

$$\therefore \sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, Y^*(k) \quad \text{—} (1)$$

from eq (1), $x(n) = y(n)$

$X(k) = Y(k)$

$$\sum_{n=0}^{N-1} x(n)\, x^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, X^*(k)$$

$x(n)^* x(n) = |x(n)|^2$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2.$$

$X(k)^* X(k) = |X(k)|^2$

$$\therefore \boxed{\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2}$$

Find the circular convolution of two finite duration sequences

$x_1(n) = \{1, -1, -2, 3, -1\}$ $x_2(n) = \{1, 2, 3\}$.

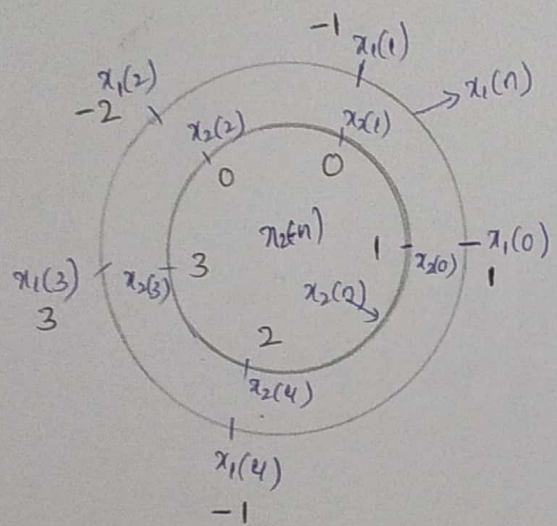
Sol

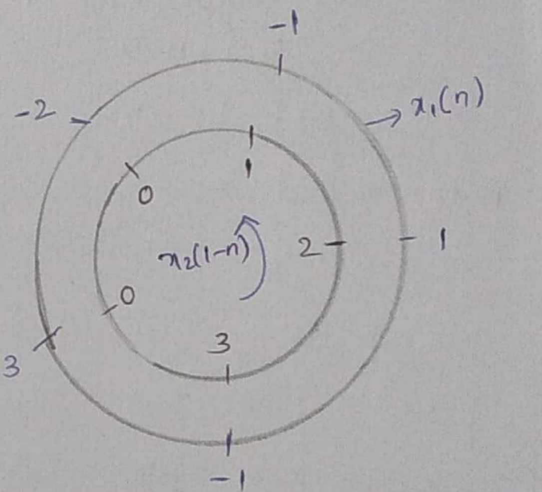
$x_1(n) = \{1, -1, -2, 3, -1\}$ $x_2(n) = \{1, 2, 3\}$

To find circular convolution, both sequences must be of same length.

therefore we append two zeros to the sequence $x_2(n)$.

$x_2(n) = \{1, 2, 3, 0, 0\}$.



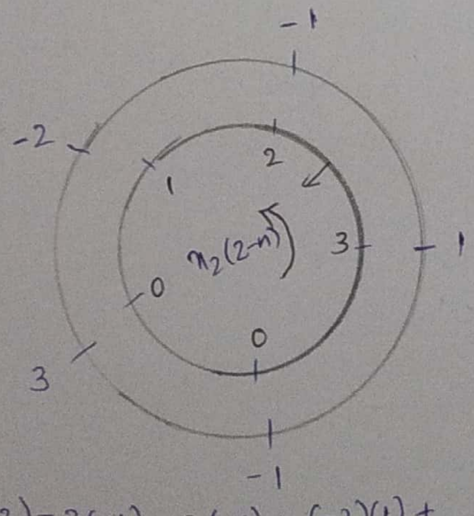


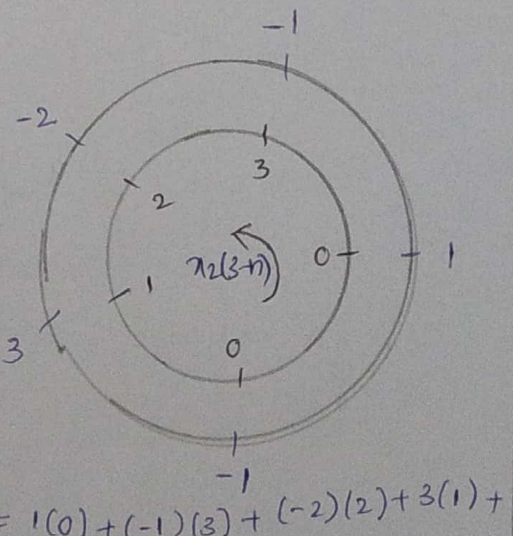
$y(0) = 1(1) + 0(-1) + 0(-2) + 3(3) + 2(-1)$

$y(0) = 1 + 9 - 2 = \underline{8}$

$y(1) = 1(2) + 1(-1) + 0(-2) + 3(0) + (-1)(3)$

$y(1) = 2 - 1 - 3 = \underline{\underline{-2}}$





$y(2) = 3(+1) + 2(-1) + (-2)(1) + 3(0) + 1(0)$

$y(2) = +3 - 2 - 2 + 0 + 0 = \cancel{4}$

$= -1$

$y(3) = 1(0) + (-1)(3) + (-2)(2) + 3(1) + (-1)(0)$

$y(3) = -\cancel{3} - 4 + \cancel{3} = \underline{\underline{-4}}$

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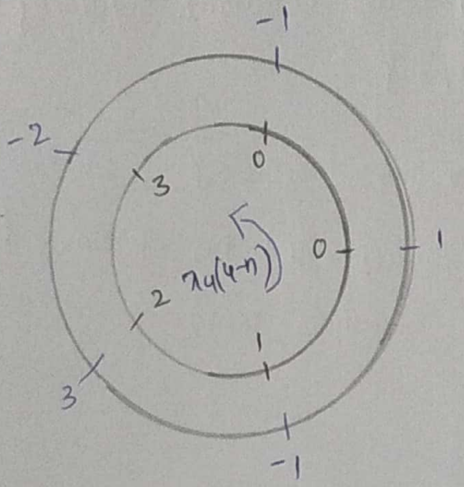


$y(0) = 8, \quad y(1) = -2, \quad y(2) = -1,$

$y(3) = -4, \quad y(4) = -1.$

$y(n) = \{8, -2, -1, -4, -1\}.$

$y(4) = 1(0) + (-1)(0) + (-2)(3) + 3(2) + 1(-1)$

$y(4) = -6 + 6 - 1 = \underline{\underline{-1}}$

matrix method :-

$x_1(n) = \{1, -1, -2, 3, -1\} \quad x_2(n) = \{1, 2, 3, 0, 0\}$

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

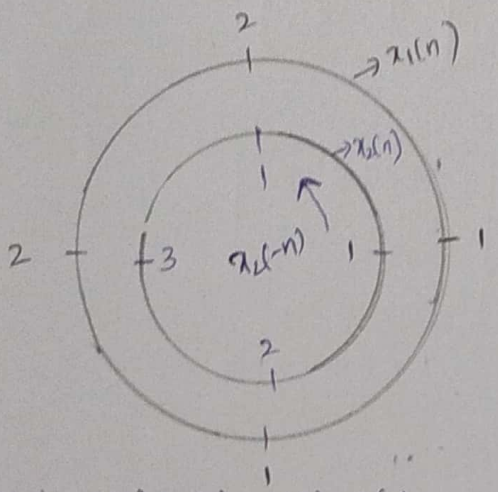
$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+0+0+9-2 \\ 2-1+0+0-3 \\ 3+2-2+0+0 \\ 0-3-4+3+0 \\ 0+0-6+6-1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 3 \\ -4 \\ -1 \end{bmatrix}$$

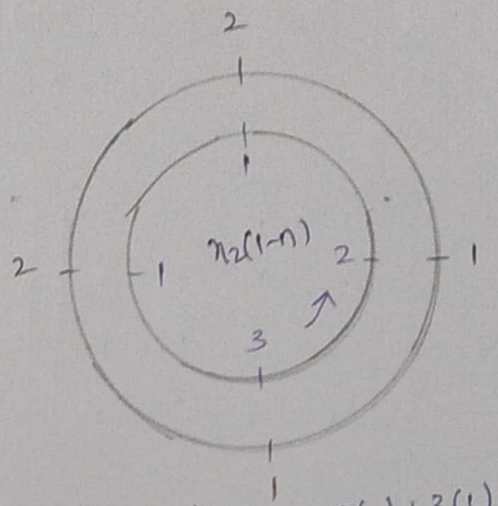
$\therefore \; y(n) = \{8, -2, -1, -4, -1\}$

Find the circular convolution of the two sequences $x_1(n) = \{1, 2, 2, 1\}$, $x_2(n) = \{1, 2, 3, 1\}$. using (a) concentric circle method (b) matrix method.

concentric circle method :-

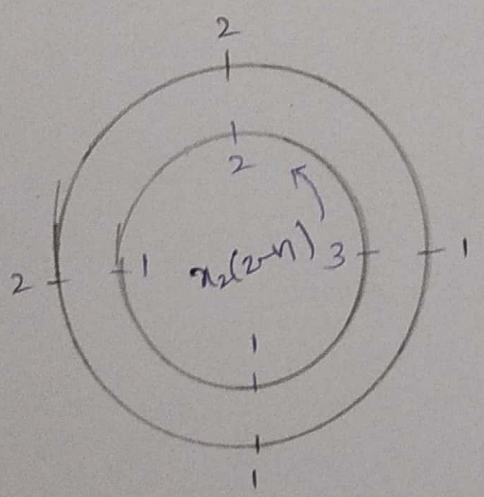
$x_1(n) = \{1, 2, 2, 1\}$ $\quad$ $x_2(n) = \{1, 2, 3, 1\}$.

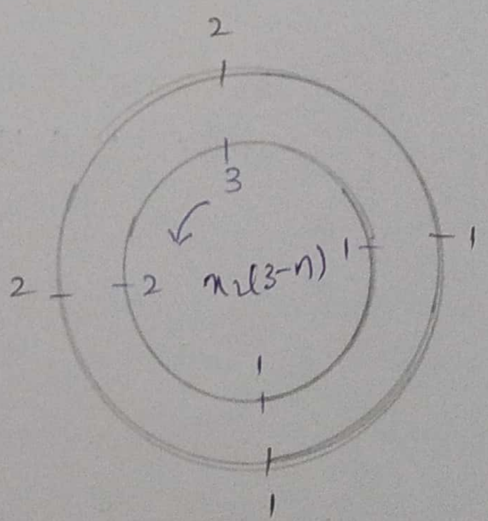




$y(0) = 1(1) + 2(1) + 2(3) + 2(1)$
$= 1 + 2 + 6 + 2 = \underline{11}$

$y(1) = 2(1) + 1(2) + 2(1) + 3(1)$
$y(1) = \underline{9}$





$y(2) = 1(3) + 2(2) + 2(1) + 1(1)$
$y(2) = \underline{10}$

$y(3) = 1 + 6 + 2(2) + 1$
$y(3) = \underline{12}$

matrix method :-

$x_1(n) = \{1, 2, 2, 1\}$ $x_2(n) = \{1, 2, 3, 1\}$.

$$\begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2+6+2 \\ 2+2+2+3 \\ 3+4+2+1 \\ 1+6+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ 9 \\ 10 \\ 12 \end{bmatrix}$$

$\therefore$ $y(n) = \{11, 9, 10, 12\}$

perform the circular convolution of the following sequences $x(n) = \{1,1,2,1\}$ & $h(n) = \{1,2,3,4\}$ using DFT & IDFT method.

<u>sol</u>

$x_1(n) = x(n) = \{1,1,2,1\}$

$x_2(n) = h(n) = \{1,2,3,4\}$

$x_3(n) = x_1(n) * x_2(n)$

$DFT[x_3(n)] = X_3(k) = X_1(k) X_2(k).$

$$\boxed{X_1(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \quad k=0,1,2..N-1}$$

$N=4,$ $$X_1(k) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi kn/4} \quad k=0,1,2,3$$

$$X_1(k) = \sum_{n=0}^{3} x(n)\, e^{-j\pi kn/2} \quad k=0,1,2,3$$

$k=0 \Rightarrow$ $$X_1(0) = \sum_{n=0}^{3} x(n)\, e^{0} = \sum_{n=0}^{3} x(n)$$

$$X_1(0) = x(0)+x(1)+x(2)+x(3)$$
$$= 1+1+2+1$$
$$= \underline{\underline{5}}$$

$k=1 \Rightarrow$ $$X_1(1) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n/2}$$

$$X_1(1) = x(0) + x(1)\, e^{-j\pi/2} + x(2)\, e^{-j\pi} + x(3)\, e^{-j3\pi/2}$$
$$= 1 + 1(-j) + 2(-1) + 1(+j)$$
$$= 1 - \cancel{j} - 2 + \cancel{j} = \underline{\underline{-1}}$$

$k=2 \Rightarrow$ $$X_1(2) = \sum_{n=0}^{3} x(n)\, e^{-j\cancel{2}\pi n/\cancel{2}} = \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$X_1(2) = x(0) + x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$
$$= 1 + 1(-1) + 2(1) + 1(-1)$$
$$= \cancel{1} - \cancel{1} + 2 - 1 = \underline{\underline{1}}$$

$k=3, \quad X_1(3) = \sum_{n=0}^{3} x(n) e^{-j3\pi n/2}$

$X_1(3) = x(0) + x(1) e^{-j3\pi/2} + x(2) e^{-j3\pi} + x(3) e^{-j9\pi/2}$

$X_1(3) = 1 + 1(j) + 2(-1) + 1(-j)$

$= 1 + \cancel{j} - 2 - \cancel{j} = -1$

$$\boxed{X_1(k) = \{5, -1, 1, -1\}}$$

$\underline{\underline{X_2(k)}}$

$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N} \quad k = 0, 1, \ldots N-1$

$N=4, \quad x_2(n) = \{1, 2, 3, 4\}$

$X_2(k) = \sum_{n=0}^{3} x_2(n) e^{-j2\pi kn/4} \quad k = 0, 1, 2, 3$

$X_2(k) = \sum_{n=0}^{3} x_2(n) e^{-j\pi kn/2} \quad k = 0, 1, 2, 3$

$k=0, \quad X_2(0) = \sum_{n=0}^{3} x_2(n)$

$X_2(0) = x_2(0) + x_2(1) + x_2(2) + x_2(3)$

$= 1 + 2 + 3 + 4 = \underline{10}$

$k=1, \quad X_2(1) = \sum_{n=0}^{3} x_2(n) e^{-j\pi n/2}$

$X_2(1) = x_2(0) + x_2(1) e^{-j\pi/2} + x_2(2) e^{-j\pi} + x_2(3) e^{-j3\pi/2}$

$= 1 + 2(-j) + 3(-1) + 4(j)$

$= 1 - 2j - 3 + 4j$

$X_2(1) = -2 + 2j$

$K=2,\quad X_2(2) = \sum_{n=0}^{3} x_2(n)\, e^{-j\pi n}$

$X_2(2) = x_2(0) + x_2(1)e^{-j\pi} + x_2(2)e^{-j2\pi} + x_2(3)e^{-j3\pi}$

$= 1 + 2(-1) + 3(1) + 4(-1)$

$= 1 - 2 + 3 - 4$

$= \underline{\underline{-2}}$

$K=3,\quad X_2(3) = \sum_{n=0}^{3} x_3(n)\, e^{-j3\pi n/2}$

$X_2(3) = x_3(0) + x_3(1)e^{-j3\pi/2} + x_3(2)e^{-j3\pi} + x_3(3)e^{-j9\pi/2}$

$X_2(3) = 1 + 2(j) + 3(-1) + 4(-j)$

$= 1 + 2j - 3 - 4j = \underline{\underline{-2 - j2}}.$

$\therefore \boxed{X_2(k) = \{10, -2+j2, -2, -2-j2\}.}$

$X_3(K) = X_1(K).X_2(K)$

$= \{5, -1, 1, -1\}.\{10, -2+j2, -2, -2-j2\}$

$\boxed{X_3(K) = \{50, 2-j2, -2, 2+j2\}}$

IDFT

$IDFT[X_3(K)] = x_3(K)$

$x_3(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K)\, e^{j2\pi Kn/N} \quad n = 0, 1, 2 \ldots N-1$

$N = 4,$

$x_3(n) = \frac{1}{4} \sum_{K=0}^{3} X_3(K)\, e^{j2\pi Kn/4} \quad n = 0, 1, 2, 3$

$\boxed{x_3(n) = \frac{1}{4} \sum_{K=0}^{3} X_3(K)\, e^{j\pi Kn/2} \quad n = 0, 1, 2, 3.}$

$n=0, \quad x_3(0) = \frac{1}{4} \sum_{k=0}^{3} X_3(k)$

$x_3(0) = \frac{1}{4}[X_3(0) + X_3(1) + X_3(2) + X_3(3)]$

$= \frac{1}{4}[50 + \cancel{2} - \cancel{j2} + \cancel{2} + 2 + \cancel{j2}] = 52/4 = \underline{\underline{13}}$

$n=1, \quad x_3(1) = \frac{1}{4} \sum_{k=0}^{3} X_3(k) e^{j\pi k/2}$

$x_3(1) = \frac{1}{4}[X_3(0) + X_3(1) e^{j\pi/2} + X_3(2) e^{j\pi} + X_3(3) e^{j3\pi/2}]$

$= \frac{1}{4}[50 + (2-j2)j + (-2)(-1) + (2+j2)(-j)]$

$x_3(1) = \frac{1}{4}[50 + \cancel{2j} - 2j^2 + 2 - \cancel{2j} - 2j^2] = \frac{56}{4} = \underline{14}$

$n=2, \quad x_3(2) = \frac{1}{4} \sum_{k=0}^{3} X_3(k) e^{j\pi k}$

$x_3(2) = \frac{1}{4}[X_3(0) + X_3(1) e^{j\pi} + X_3(2) e^{j2\pi} + X_3(3) e^{j3\pi}]$

$= \frac{1}{4}[50 + (2-j2)(-1) + (-2)(1) + \cancel{(-2)(1)} + (2+j2)(-1)]$

$x_3(2) = \frac{1}{4}[50 - 2 + \cancel{j2} - 2 - 2 - \cancel{j2}] = \frac{44}{4} = 11,$

$n=3, \quad x_3(3) = \frac{1}{4} \sum_{k=0}^{3} X_3(k) e^{j3\pi k/2}$

$x_3(3) = \frac{1}{4}[X_3(0) + X_3(1) e^{j3\pi/2} + X_3(2) e^{j3\pi} + X_3(3) e^{j9\pi/2}]$

$= \frac{1}{4}[50 + (2-j2)(-j) + (-2)(-1) + (2+j2)(j)]$

$= \frac{1}{4}[50 - \cancel{2j} + 2j^2 + 2 + \cancel{2j} + 2j^2]$

$x_3(3) = 48/4 = \underline{\underline{12}}$

$\therefore \; x_3(n) = \{13, 14, 11, 12\}$

① Find the output $y(n)$ of a filter whose impulse response is $h(n) = \{1,1,1\}$ & $x(n) = \{3,-1,0,1,3,2,0,1,2,1\}$ using overlap Add and overlap save method.

Sol given $x(n) = \{3,-1,0,1,3,2,0,1,2,1\}$.

$h(n) = \{1,1,1\}$ length of $h(n) = M = 3$.

→ $(M-1)$ zero's appended to block of each sequence.

let the length of data block be 3. i.e $L=3$.

then $L+M-1 = 5$ (To bring the length to 5).

$x_1(n) = \{3,-1,0,\underbrace{0,0}_{M-1=2 \text{ zeros}}\}$ $x_2(n) = \{1,3,2,0,0\}$

$x_4(n) = \{1,0,0,0,0\}$.

$x_3(n) = \{0,1,2,0,0\}$

$L-1 = 3-1 = 2$ zero's appended to impulse sequence i.e

$h(n) = \{1,1,1,0,0\}$.

then $y_1(n) = x_1(n) \, (N) \, h(n) = \{3,-1,0,0,0\} \, (N) \, \{1,1,1,0,0\}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3-1 \\ 3-1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

$y_1(n) = \{3,2,2,-1,0\}$.

$y_2(n) = x_2(n) \, (N) \, h(n) = \{1,3,2,0,0\} \, (N) \, \{1,1,1,0,0\}$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 1+3 \\ 1+3+2 \\ 3+2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 5 \\ 2 \end{bmatrix}$$

$y_2(n) = \{1,4,6,5,2\}$.

$y_3(n) = x_3(n) \; (N) \; h_2(n) = \{0,1,2,0,0\} \; (N) \; \{1,1,1,0,0\}.$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1+2 \\ 1+2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

$y_3(n) = \{0,1,3,3,2\}$

$y_4(n) = x_4(n) \; (N) \; h(n) = \{1,0,0,0,0\} \; (N) \; \{1,1,1,0,0\}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$y_4(n) = \{1,1,1,0,0\}.$

($m-1=2$ overlapping points.

$\boxed{3 \quad 2 \quad 2 \quad -1 \quad 0}$

add

$\boxed{1 \quad 4 \quad 6 \quad 5 \quad 2}$

add

$\boxed{0 \quad 1 \quad 3 \quad 3 \quad 2}$

add

$\boxed{1 \quad 1 \quad 1 \quad 0 \quad 0}$

$y(n) = \{3,2,2,0,4,6,5,3,3,4,3,1,\}.$

overlap save method:

Sol $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$.

$h(n) = \{1, 1, 1\}$ $\quad m = 3$.

input sequence can be divided into blocks of data as follows.

$x_1(n) = \{\underbrace{0, 0}_{\substack{m-1=2 \\ \text{zero's}}}, \underbrace{3, -1, 0}_{\substack{L=3 \text{ data} \\ \text{points}}}\}$. $\quad x_2(n) = \{\underbrace{-1, 0}_{\substack{\text{Two data} \\ \text{points from} \\ \text{previous block}}}, \underbrace{1, 3, 2}_{\substack{3 \text{ new} \\ \text{data points}}}\}$

$x_3(n) = \{3, 2, 0, 1, 2\}$

$x_4(n) = \{1, 2, 1, 0, 0\}$.

$L-1 = 3-1 = 2$ zero's appended to $h(n)$.

$h(n) = \{1, 1, 1, 0, 0\}$.

$y_1(n) = x_1(n) \text{ (N) } h(n) = \{0, 0, 3, -1, 0\} \text{ (N) } \{1, 1, 1, 0, 0\}$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$y_1(n) = \{-1, 0, 3, 2, 2\}$.

$y_2(n) = x_2(n) \text{ (N) } h(n) = \{-1, 0, 1, 3, 2\} \text{ (N) } \{1, 1, 1, 0, 0\}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+3+2 \\ -1+2 \\ -1+1 \\ 1+3 \\ 1+3+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

$y_2(n) = \{4, 1, 0, 4, 6\}$.

$y_3(n) = x_3(n) \text{ Ⓝ } h(n) = \{3, 2, 0, 1, 2\} \text{ Ⓝ } \{1, 1, 1, 0, 0\}$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+1+2 \\ 3+2+2 \\ 3+2 \\ 2+1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \\ 3 \\ 3 \end{bmatrix}$$

$y_3(n) = \{6, 7, 5, 3, 3\}$.

$y_4(n) = x_4(n) \text{ Ⓝ } h(n) = \{1, 2, 1, 0, 0\} \text{ Ⓝ } \{1, 1, 1, 0, 0\}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+2 \\ 1+2+1 \\ 2+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

$y_4(n) = \{1, 3, 4, 3, 1\}$.

$\boxed{\underbrace{-1, 0}_{\text{discard}}, 3, 2, 2}$

$\boxed{\underbrace{4, 1}_{\text{discard}}, 0, 4, 6}$

$\boxed{\underbrace{6, 7}_{\text{discard}}, 5, 3, 3}$

$\boxed{\underbrace{1, 3}_{\text{discard}}, 4, 3, 1}$

$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$.

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Ex) Using overlap save and overlap add method, find the output $y(n)$ for the sequences $x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$ and $h(n) = \{1, 2\}$.

(i) overlap save method:-

$$x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$$

$h(n) = \{1, 2\}$, length $M = 2$, $L = 3$ data points

$x_1(n) = \{\underbrace{0}_{\substack{M-1 \\ \text{zero}}}, \underbrace{1, 2, -1}_{\text{'L' data points}}\}$ $\qquad$ $x_2(n) = \{\underbrace{-1}_{\substack{M-1=1 \\ \text{data from} \\ \text{previous block}}}, \underbrace{2, 3, -2}\}$

$x_3(n) = \{-2, -3, -1, 1\}$ $\qquad$ $x_4(n) = \{1, 1, 2, -1\}$ $\qquad$ $x_5(n) = \{-1, 0, 0, 0\}$.

$h(n) = \{1, 2, 0, 0\}$ Append two zeros to the sequence,

$y_1(n) = x_1(n) \circledN h(n) = \{0, 1, 2, -1\} \circledN \{1, 2, 0, 0\}$

$y_2(n) = x_2(n) \circledN h(n) = \{-1, 2, 3, -2\} \circledN \{1, 2, 0, 0\}$

$y_3(n) = x_3(n) \circledN h(n) = \{-2, -3, -1, 1\} \circledN \{1, 2, 0, 0\}$

$y_4(n) = x_4(n) \circledN h(n) = \{1, 1, 2, -1\} \circledN \{1, 2, 0, 0\}$

$y_5(n) = x_5(n) \circledN h(n) = \{-1, 0, 0, 0\} \circledN \{1, 2, 0, 0\}$.

$y_1(n)$:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \\ 3 \end{bmatrix}$$

$y_2(n)$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 7 \\ 4 \end{bmatrix}$$

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$y_3(n)$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ -7 \\ -1 \end{bmatrix}$$

$y_4(n)$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 4 \\ 3 \end{bmatrix}$$

$y_5(n)$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{-2 \quad 1 \quad 4 \quad 3}$
discard

$\boxed{-5 \quad 0 \quad 7 \quad 4}$
discard

$\boxed{0 \quad -7 \quad -7 \quad -1}$
discard

$\boxed{-1 \quad 3 \quad 4 \quad 3}$
discard

$\boxed{-1 \quad -2 \quad 0 \quad 0}$
discard.

$$y(n) = \{1, 4, 3, 0, 7, 4, -7, -7, -1, 3, 4, 3, -2\}.$$

overlap Add method :-

$x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$. $h(n) = \{1, 2\}$.

$M = 2,\ L = 3$

$x_1(n) = \{1, 2, -1, 0\}$

$\downarrow$

$M - 1 = 1$

zero appended.

$x_2(n) = \{2, 3, -2, 0\}$ $x_3(n) = \{-3, -1, 1, 0\}$, $x_4(n) = \{1, 2, -1, 0\}$.

$h(n) = \{1, 2, 0, 0\}$

$y_1(n) = x_1(n) \text{ Ⓝ } h(n) = \{1, 2, -1, 0\} \text{ Ⓝ } \{1, 2, 0, 0\}$

$y_2(n) = x_2(n) \text{ Ⓝ } h(n) = \{2, 3, -2, 0\} \text{ Ⓝ } \{1, 2, 0, 0\}$

$y_3(n) = x_3(n) \text{ Ⓝ } h(n) = \{-3, -1, 1, 0\} \text{ Ⓝ } \{1, 2, 0, 0\}$

$y_4(n) = x_4(n) \text{ Ⓝ } h(n) = \{1, 2, -1, 0\} \text{ Ⓝ } \{1, 2, 0, 0\}$.

$$y_1(n) \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ -2 \end{bmatrix}$$

$$y_2(n) \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 4 \\ -4 \end{bmatrix}$$

$$y_3(n) \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \\ -1 \\ 2 \end{bmatrix}$$

$y_4(n) = x_4(n) \, \textcircled{N} \, h(n).$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ -2 \end{bmatrix}$$

$y_1(n) = \{1, 4, 3, -2\}$

$+$

$y_2(n) = \{2, 7, 4, -4\}$

$+$

$y_3(n) = \{-3, -7, -1, 2\}$

$+$

$y_4(n) = \{1, 4, 3, -2\}$

$y(n) = \{1, 4, 3, 0, 7, 4, -7, -7, -1, 3, 4, 3, -2\}.$

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Fast Fourier Transform.

Introduction

The N-point DFT of a sequence $x(n)$ converts the time domain N-point sequence $x(n)$ to a frequency domain N-point sequence $X(k)$.

→ The direct computation of an N-point DFT requires $N \times N$ complex multiplications and $N(N-1)$ complex additions.

→ many methods were developed for reducing the number of calculations involved. The most popular of these is the Fast Fourier Transform.

→ FFT proposed by cooley and Tukey in 1965. The FFT is a highly efficient ~~produ~~ procedure for computing the DFT of a finite series and requires less ~~time~~ number of computations that that of direct evaluation of DFT.

→ The FFT is based on decomposition and breaking the sequence transform into smaller transforms and combining them to get the total transform.

→ FFT reduces the computation time required to compute a discrete Fourier transform and improves the performance by a factor 100 or more over direct evaluation of the DFT..

Direct Evaluation of the DFT :-

The DFT of a sequence $x(n)$ can be evaluated using the formula

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \qquad 0 \le k \le N-1 \quad \text{(1)}$$

substitute $W_N = e^{-j2\pi/N}$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \qquad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + j\, x_I(n)][W_{RN}^{nk} + j\, W_{IN}^{nk}] \quad \text{--- ②}$$

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) W_{RN}^{nk} - x_I(n) W_{IN}^{nk}] + j \sum_{n=0}^{N-1} \{x_I(n) W_{RN}^{nk} + x_R(n) W_{IN}^{nk}\} \quad \text{--- ③}$$

To evaluate one value of $X(k)$,

No of complex multiplications $= N$

No of complex Addions $= (N-1)$

To evaluate all N value of $X(k)$,

No of complex multiplications $= N \times N = N^2$

No of complex Additions $= N \times (N-1) = N(N-1)$

From equation ③,

To evaluate all N value of $X(k)$

Total numbers of real multiplications $= 4N \times N = 4N^2$

Total numbers of real Additions $= (4N-2) \times N = N(4N-2)$
$= 2N(2N-1)$

Symmetry property : $W_N^{k+\frac{N}{2}} = -W_N^{k}$

periodicity property : $W_N^{k+N} = W_N^{k}$.

Fast Fourier Transform :-

FFT algorithms are based on the fundamental principle of decomposing the computation of discrete Fourier transform of a sequence of length 'N' into successively smaller discrete Fourier transforms.

There are basically two FFT algorithms. (Radix - 2 Algorithms).

1. Decimation in time (DIT) FFT algorithm.
2. Decimation in Frequency (DIF) FFT algorithm.

Decimation - in - time (DIT) FFT Algorithm :-

In Decimation in time (DIT) algorithm, the time domain sequence $x(n)$ is decimated and smaller point DFT's are computed and they are combined to get the result of N-point DFT.

→ In general, the N-point DFT can be realized from two numbers of $\frac{N}{2}$ point DFT's, the $\frac{N}{2}$ point DFT can be realized from two numbers of $N/4$ point DFT's.

let $x(n)$ be an N-sample sequence, where 'N' is a power of 2. Decimate or break this sequence into two sequences $f_1(n)$ and $f_2(n)$ of length $N/2$, one composed of the even indexed values of $x(n)$ and the other of odd indexed values of $x(n)$.

$$x(n) = \{x(0), x(1), x(2), x(3) \ldots x(N-1)\}.$$

Even indexed sequence $f_1(n) = x(2n) = \{x(0), x(2), x(4) \ldots\}$, $n = 0, 1, 2 \ldots \left(\frac{N}{2}-1\right)$

odd indexed sequence $f_2(n) = x(2n+1) = \{x(1), x(3), x(5), x(7) \ldots\}$, $n = 0, 1, \ldots \left(\frac{N}{2}-1\right)$

The N-point DFT of $x(n)$ can be written as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \qquad k = 0, 1, 2 \ldots N-1 \quad (1)$$

$$X(k) = \sum_{\substack{n=0 \\ \text{even}}}^{N-1} x(n) W_N^{nk} + \sum_{\substack{n=0 \\ \text{odd}}}^{N-1} x(n) W_N^{nk}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{(2n+1)k}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_N^{2nk}$$

Rearranging each part of $X(k)$ into $\frac{N}{2}$ point transform using

$$W_N^{2nk} = (W_N^2)^{nk} = \left[e^{-j\frac{2\pi}{N}} \right]^{2nk} = \left[e^{-j\frac{2\pi}{N/2}} \right]^{nk} = W_{\frac{N}{2}}^{nk}$$

$$\therefore \quad W_N^{2nk} = W_{\frac{N}{2}}^{nk} \quad \& \quad W_N^{(2n+1)k} = W_N^{k} \cdot W_{N/2}^{nk}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n) W_{N/2}^{nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} f_2(n) W_{\frac{N}{2}}^{nk}$$

By definition of DFT, the $\frac{N}{2}$-point DFT of $f_1(n)$ and $f_2(n)$ is given by

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n) W_{N/2}^{nk} \qquad F_2(k) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n) W_{N/2}^{kn}$$

$$\boxed{X(k) = F_1(k) + W_N^{k} F_2(k) \quad k = 0,1,2,3 \dots N-1} \quad (2)$$

The DFT of a sequence is periodic, with period of $N/2$.

$$F_1\left(k+\frac{N}{2}\right) = F_1(k), \quad F_2\left(k+\frac{N}{2}\right) = F_2(k),$$

$$\boxed{W_N^{\left(k+\frac{N}{2}\right)} = -W_N^{k}}$$

$$X(k+\tfrac{N}{2}) = F_1(k+\tfrac{N}{2}) + W_N^{k+\frac{N}{2}} F_2(k+\tfrac{N}{2})$$

$$\boxed{X(k+\tfrac{N}{2}) = F_1(k) - W_N^{k} F_2(k)} \quad \text{---} \ (3)$$

let $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$ be the 8-point sequence i.e $N=8$.

$f_1(n) = x(2n) = \{x(0), x(2), x(4), x(6)\}$ $\quad \frac{N}{2} = \frac{8}{2} = 4$ samples.

$f_2(n) = x(2n+1) = \{x(1), x(3), x(5), x(7)\}$. $\quad \frac{N}{2} = \frac{8}{2} = 4$ samples.

1st stage of Decimation:

From equation (2), $N=8$

$X(k) = F_1(k) + W_N^{k} F_2(k)$

from equation (3)

$X(k+\frac{N}{2}) = F_1(k) - W_N^{k} F_2(k)$

$k=0$:	$X(0) = F_1(0) + W_8^0 F_2(0)$	$k=0$;	$X(4) = F_1(0) - W_8^0 F_2(0)$
$k=1$	$X(1) = F_1(1) + W_8^1 F_2(1)$	$k=1$	$X(5) = F_1(1) - W_8^1 F_2(1)$
$k=2$	$X(2) = F_1(2) + W_8^2 F_2(2)$	$k=2$	$X(6) = F_1(2) - W_8^2 F_2(2)$
$k=3$	$X(3) = F_1(3) + W_8^3 F_2(3)$	$k=3$	$X(7) = F_1(3) - W_8^3 F_2(3)$.

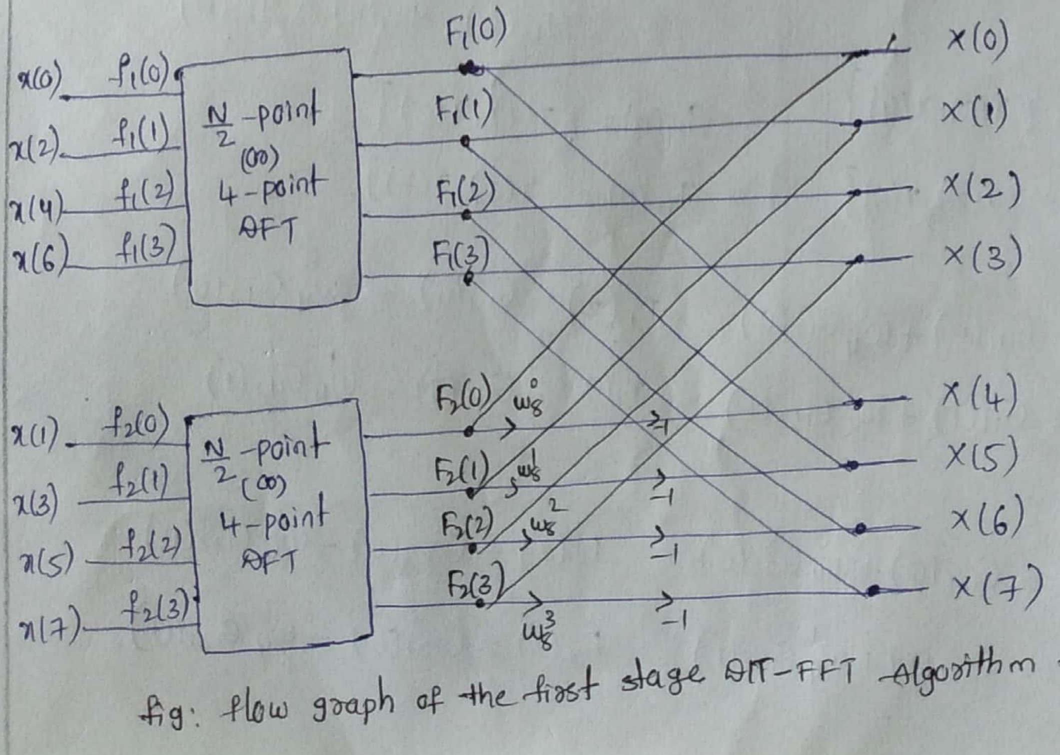


fig: flow graph of the first stage DIT-FFT Algorithm for N=8.

2^{nd} stage of Decimation :-

$f_1(n)$ and $f_2(n)$ results in two $\frac{N}{4}$ point sequences.

let the decimated $(\frac{N}{4})$ point sequences of $f_1(n)$ be $g_{11}(n)$ & $g_{12}(n)$

$g_{11}(n) = f_1(2n)$; $n = 0, 1, 2 \ldots (\frac{N}{4} - 1)$

$g_{12}(n) = f_1(2n+1)$; $n = 0, 1, 2 \ldots (\frac{N}{4} - 1)$

let the decimated $(\frac{N}{4})$ point sequences of $f_2(n)$ be $g_{21}(n)$ & $g_{22}(n)$

$g_{21}(n) = f_2(2n)$; for $n = 0, 1 \ldots (\frac{N}{4} - 1)$

$g_{22}(n) = f_2(2n+1)$; for $n = 0, 1 \ldots (\frac{N}{4} - 1)$.

$$\left[\because W_N^{k+\frac{N}{4}} = -W_{\frac{N}{2}}^{k} \right]$$

$$F_1(k) = G_{11}(k) + W_{\frac{N}{2}}^{k} G_{12}(k) \quad ; 0 \le k \le \frac{N}{2} - 1 \quad (3)$$

$$F_1(k+\frac{N}{4}) = G_{11}(k) - W_{\frac{N}{2}}^{k} G_{12}(k) \quad ; 0 \le k \le \frac{N}{2} - 1 \quad (4)$$

Similarly

$$F_2(k) = G_{21}(k) + W_{\frac{N}{2}}^{k} G_{22}(k) \quad ; 0 \le k \le \frac{N}{2} - 1 \quad (5)$$

$$F_2(k+\frac{N}{4}) = G_{21}(k) - W_{\frac{N}{2}}^{k} G_{22}(k) \quad ; 0 \le k \le \frac{N}{2} - 1 \quad (6)$$

$G_{11}(k), G_{12}(k), G_{21}(k)$ & $G_{22}(k)$ are $\frac{N}{4}$-point DFT of $g_{11}(n)$, $g_{12}(n)$, $g_{21}(n)$ & $g_{22}(n)$.

$g_{11}(n) = \{x(0), x(4)\}$ $\qquad$ $g_{21}(n) = \{x(1), x(5)\}$

$g_{12}(n) = \{x(2), x(6)\}$ $\qquad$ $g_{22}(n) = \{x(3), x(7)\}$.

$F_1(0) = G_{11}(0) + W_4^0 G_{12}(0)$ $\qquad$ $F_1(2) = G_{11}(0) - W_4^0 G_{12}(0)$

$F_1(1) = G_{11}(1) + W_4^1 G_{12}(1)$ $\qquad$ $F_1(3) = G_{11}(0) - W_4^1 G_{12}(1)$

Similarly

$F_2(0) = G_{21}(0) + W_4^0 G_{22}(0)$ $\qquad$ $F_2(2) = G_{21}(0) - W_4^0 G_{22}(0)$

$F_2(1) = G_{21}(1) + W_4^1 G_{22}(1)$ $\qquad$ $F_2(3) = G_{21}(0) - W_4^1 G_{22}(0)$.

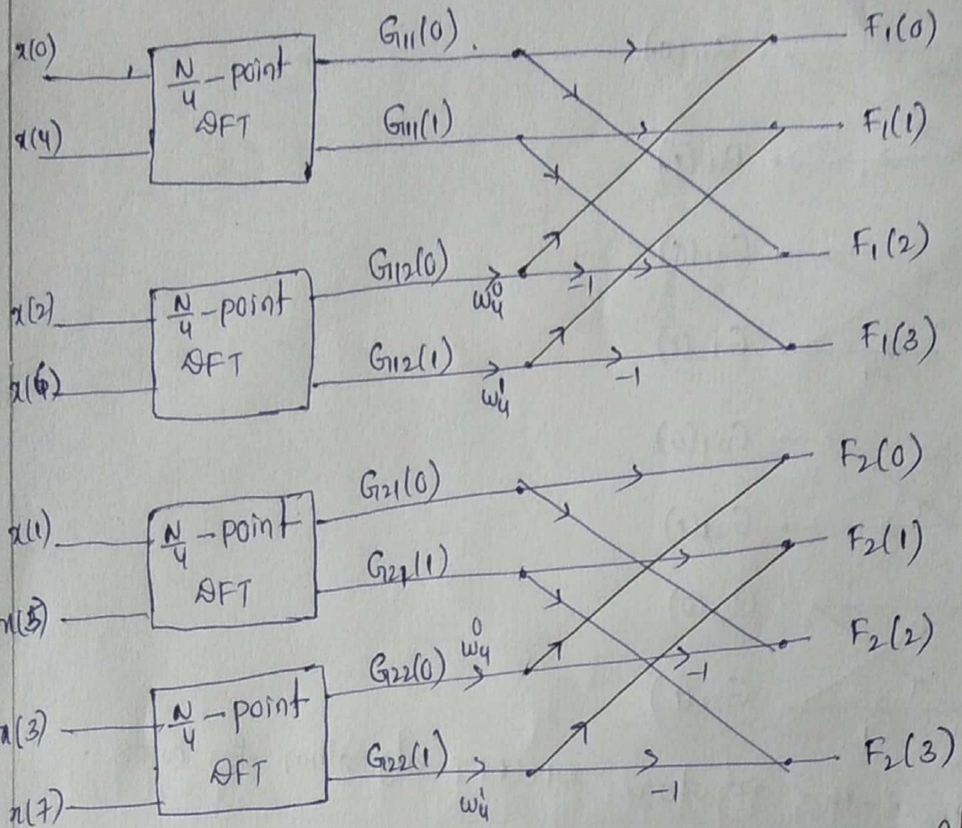


fig: Flowgraph of the second stage DIT-FFT Algorithm for $N=8$.

3rd stage of Decimation :-

let $G_{11}(k)$ is the DFT of $g_{11}(n)$.

2-point DFT of $g_{11}(n)$ is given by

$$\boxed{G_{11}(k) = \sum_{n=0}^{1} g_{11}(n) W_{\frac{N}{4}}^{nk} \quad \text{for } k=0,1}$$

$k=0 \Rightarrow G_{11}(0) = g_{11}(0) W_2^0 + g_{11}(1) W_2^0$

$$\boxed{G_{11}(0) = g_{11}(0) + g_{11}(1)}$$

$k=1 \Rightarrow G_{11}(1) = \sum_{n=0}^{1} g_{11}(n) W_2^n \quad [\because W_2^1 = -1]$

$G_{11}(1) = g_{11}(0) W_2^0 + g_{11}(1) W_2^1$

$$\boxed{G_{11}(1) = g_{11}(0) - g_{11}(1)}$$

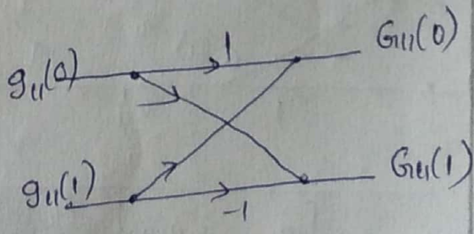


fig: Butterfly diagram.

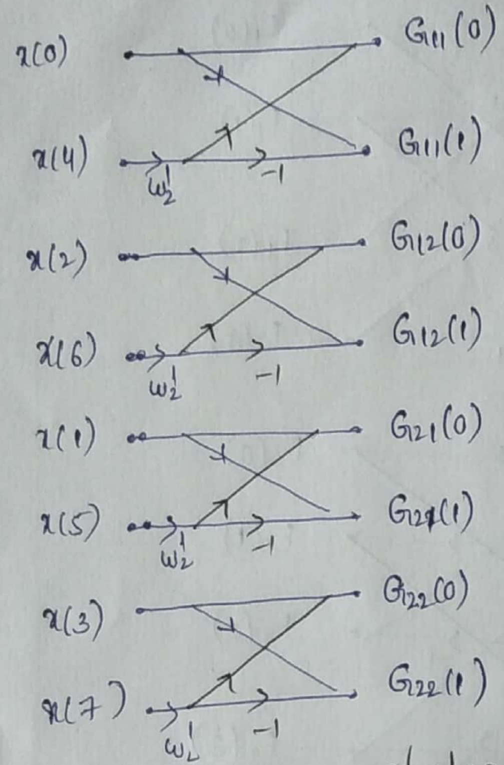


fig: flow graph of the 3rd stage DIT FFT algorithm for N=8.

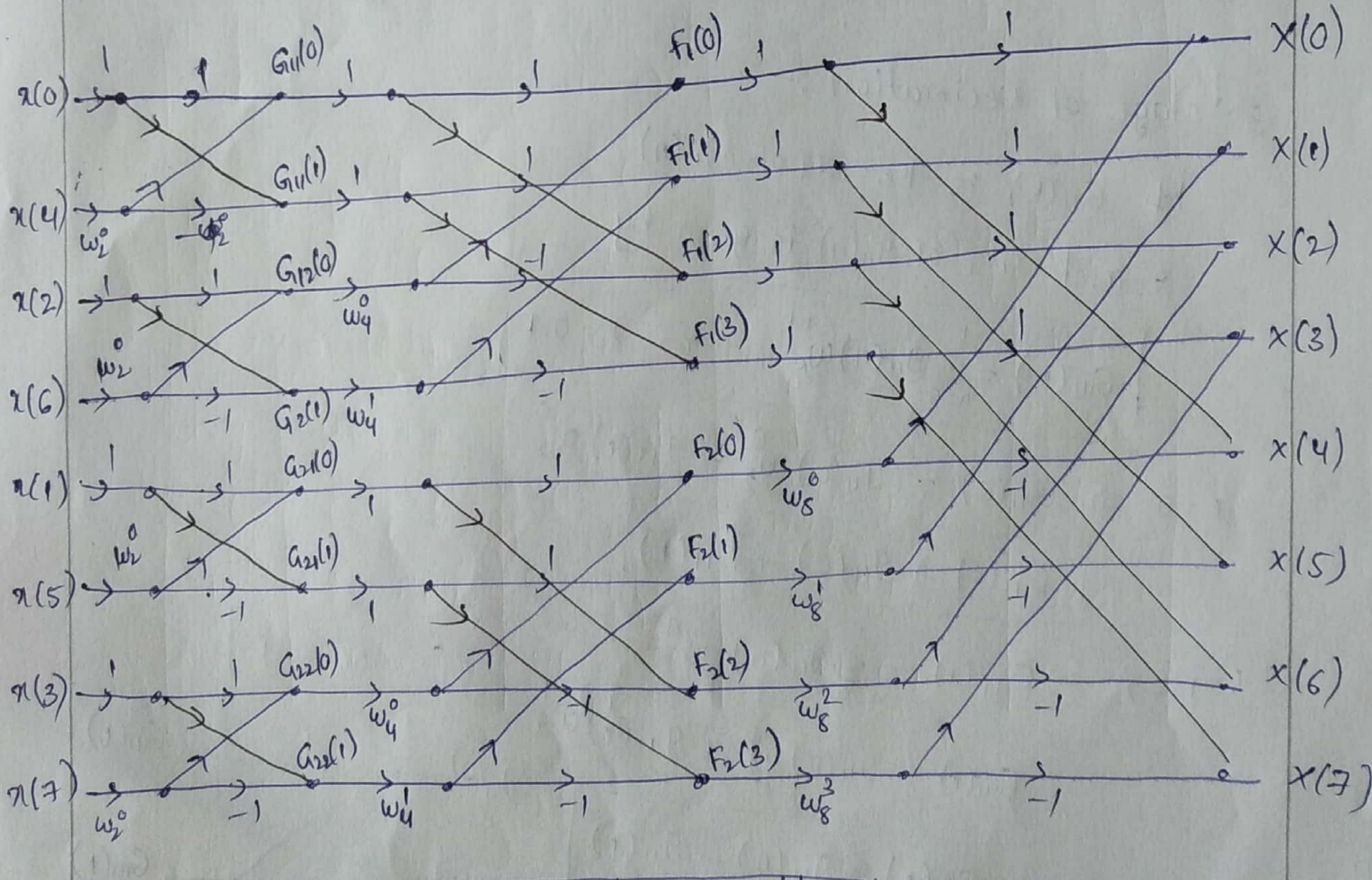


fig: signal flow graph or Butterfly diagram for 8-point radix-2 DIT-FFT.

1. Bit reversal.

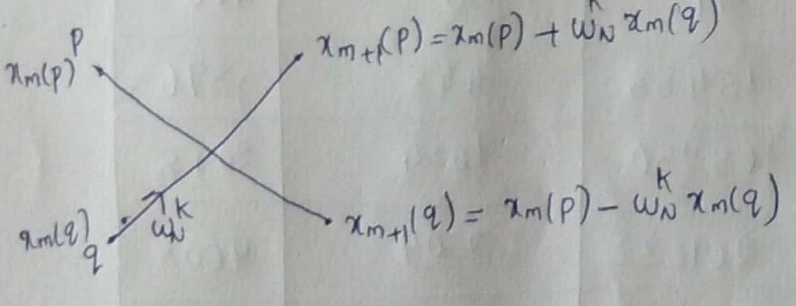
In DIT, the output sequence to be in natural order, the input sequence has to be stored in shuffled order.

→ the input sequence must be stored in bit reversal order for the output to be computed in a natural order.

I/P samples Index	Binary Representation	Bit reversed binary	Bit reversed Sample index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Basic operation:-

the basic computation block in the diagram is the butterfly in which two inputs are combined to give two outputs.



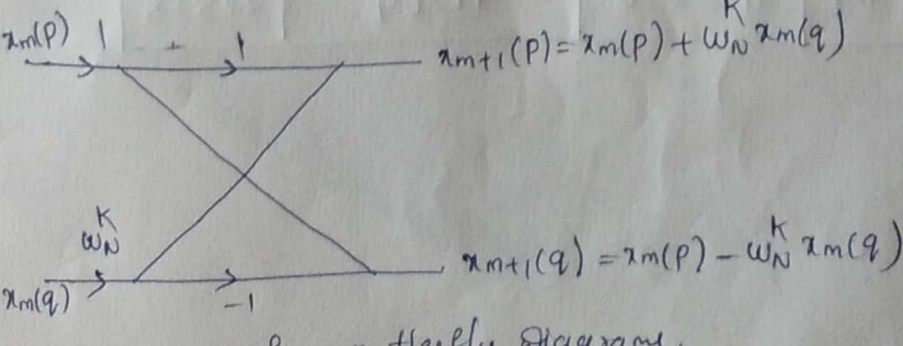


fig: Butterfly Diagrams.

No of stages :-

Number of stages in the flowgraph is given by

$$N = 2^m$$

$$\log_2 N = m$$

$$\therefore \boxed{m = \log_2 N}$$

→ The total number of complex multiplications required for calculating DIT-FFT : $\boxed{\frac{N}{2}\log_2 N}$

→ The total number of complex additions for evaluating a DFT using DIT-FFT is $\boxed{N\log_2 N}$

Comparision of computational complexity of Direct DFT and DIT-FFT

N	Direct DFT: complex multiplication N^2	Direct DFT: complex Additions $N(N-1)$	DIT-FFT: complex multiplications $\frac{N}{2}\log_2 N$	DIT-FFT: complex Additions $N\log_2 N$
8	64	56	12	24
16	256	240	32	64
32	1024	992	80	160
128	16384	15488	448	896
256	65536	65280	1024	2048
512	262144	261632	2304	4608
2048	4194304	4192256	11264	22528

Decimation In frequency (DIF) - FFT Algorithm :-

In DIF, the frequency domain sequence $X(K)$ is decimated. In this Algorithm, the N-point time domain sequence is converted to two numbers of $\frac{N}{2}$ point sequences. Then each $\frac{N}{2}$-point sequence is converted to two numbers of $N/4$ point sequences. This process is continued until we get $N/2$ numbers of 2-point sequences. Finally the 2-point DFT of each 2-point sequence is computed.

consider N-point sequence

$$x(n) = \{x(0), x(1), x(2), x(3) \ldots x(N-1)\}.$$

DFT,
$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2}) W_N^{K(n+\frac{N}{2})}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2}) W_N^{nk} \cdot W_N^{K\cdot(N/2)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{(\frac{N}{2})K} \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2}) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + (-1)^K \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2}) W_N^{nk}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^K x(n+\tfrac{N}{2}) \right] W_N^{nk}$$

split $X(k)$ into even and odd numbered samples.

For even values of k, $X(K)$ can be written as

$$X(2K) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k} x(n+\tfrac{N}{2}) \right] W_N^{2nk}$$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x(n+\tfrac{N}{2}) \right] W_{\frac{N}{2}}^{kn} \quad ; \quad k = 0, 1, 2 \ldots (\tfrac{N}{2}-1)$$

For odd values of K, the x(K) can be written as

$$x(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k+1} x(n+\tfrac{N}{2}) \right] W_N^{(2k+1)n}$$

$$x(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x(n+\tfrac{N}{2}) \right] W_N^{n} W_{N/2}^{kn} \quad ; \quad K = 0,1,2 \dots (\tfrac{N}{2}-1)$$

If we define new time domain sequences $u_1(n)$ and $u_2(n)$ consisting of $\frac{N}{2}$ samples, such that,

$$u_1(n) = x(n) + x(n+\tfrac{N}{2}) \quad ; \quad n = 0,1,2, \dots \tfrac{N}{2}-1 \qquad \text{—(1)}$$

$$u_2(n) = \left[x(n) - x(n+\tfrac{N}{2}) \right] W_N^n \quad ; \quad n = 0,1,2 \dots \tfrac{N}{2}-1 \qquad \text{—(2)}$$

consider 8-point sample sequence,

$$x(n) = \{ x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7) \}$$

Ist stage

$u_1(n) = x(n) + x(n+\frac{N}{2})$ —(1)

$u_1(n) = x(n) + x(n+4)$: for $n = 0,1,2,3$

$u_1(0) = x(0) + x(4)$

$u_1(1) = x(1) + x(5)$

$u_1(2) = x(2) + x(6)$

$u_1(3) = x(3) + x(7)$

$u_2(n) = [x(n) - x(n+\frac{N}{2})] W_N^n$ —(2)

$u_2(n) = [x(n) - x(n+4)] W_8^n$ for $n = 0,1,2,3$

$u_2(0) = [x(0) - x(4)] W_8^0$

$u_2(1) = [x(1) - x(5)] W_8^1$

$u_2(2) = [x(2) - x(6)] W_8^2$

$u_2(3) = [x(3) - x(7)] W_8^3$

2nd stage :

In this, four numbers of 2-point sequences $V_{11}(n)$ ~~and~~ $V_{12}(n)$ $V_{21}(n)$ & $V_{22}(n)$ are obtained from the two 4-point sequences $u_1(n)$ and $u_2(n)$ obtained in stage one as follows.

$$\boxed{v_{11}(n) = u_1(n) + u_1\left(n+\frac{N}{4}\right)}$$

$v_{11}(n) = u_1(n) + u_1(n+2)$, for $n = 0, 1$

$v_{11}(0) = u_1(0) + u_1(2)$

$v_{11}(1) = u_1(1) + u_1(3)$

$$\boxed{v_{12}(n) = \left[u_1(n) - u_1\left(n+\frac{N}{4}\right)\right] W_{\frac{N}{2}}^{n}}$$

$v_{12}(n) = [u_1(n) - u_1(n+2)]\, W_4^{n}$; $n = 0, 1$

$v_{12}(0) = [u_1(0) - u_1(2)]\, W_4^{0}$

$v_{12}(1) = [u_1(1) - u_1(3)]\, W_4^{1}$

$$\boxed{v_{21}(n) = u_2(n) + u_2\left(n+\frac{N}{4}\right)}$$

$v_{21}(n) = u_2(n) + u_2(n+2)$; $n = 0, 1$

$v_{21}(0) = u_2(0) + u_2(2)$

$v_{21}(1) = u_2(1) + u_2(3)$

$$\boxed{v_{22}(n) = \left[u_2(n) - u_2\left(n+\frac{N}{4}\right)\right] W_{\frac{N}{2}}^{n}}$$

$v_{22}(n) = [u_2(n) - u_2(n+2)]\, W_4^{n}$; $n = 0, 1$

$v_{22}(0) = [u_2(0) - u_2(2)]\, W_4^{0}$

$v_{22}(1) = [u_2(1) - u_2(3)]\, W_4^{1}$

3rd stage:

In this stage, the 2-point DFT's of the 2-point sequences obtained in the second stage are computed as follows.

$$V_{11}(k) = DFT[v_{11}(n)] = \sum_{n=0}^{1} v_{11}(n)\, W_2^{kn} \quad ; k = 0, 1$$

$V_{11}(0) = \sum_{n=0}^{1} v_{11}(n)\, W_2^{0}$

$$\boxed{V_{11}(0) = v_{11}(0) + v_{11}(1)}$$

$V_{11}(1) = \sum_{n=0}^{1} v_{11}(n)\, W_2^{n}$

$V_{11}(1) = v_{11}(0)\, W_2^{0} + v_{11}(1)\, W_2^{1}$

$$\boxed{V_{11}(1) = [v_{11}(0) - v_{11}(1)]\, W_2^{0}}$$

$V_{12}(k) = DFT[v_{12}(n)] = \sum_{n=0}^{1} v_{12}(n)\, W_2^{nk}$; $k = 0, 1$

$V_{12}(0) = \sum_{n=0}^{1} v_{12}(n)\, W_2^{0}$

$$\boxed{V_{12}(0) = v_{12}(0) + v_{12}(1)}$$

$V_{12}(1) = \sum_{n=0}^{1} v_{12}(n)\, W_2^{n}$

$V_{12}(1) = v_{12}(0)\, W_2^{0} + v_{12}(1)\, W_2^{1}$

$$\boxed{V_{12}(1) = [v_{12}(0) - v_{12}(1)]\, W_2^{0}}$$

Similarly,

$$V_{21}(0) = V_{21}(0) + V_{21}(1)$$

$$V_{21}(1) = [V_{21}(0) - V_{21}(1)]\, W_2^0$$

$$V_{22}(0) = V_{22}(0) + V_{22}(1)$$

$$V_{22}(1) = [V_{22}(0) - V_{22}(1)]\, W_2^0$$

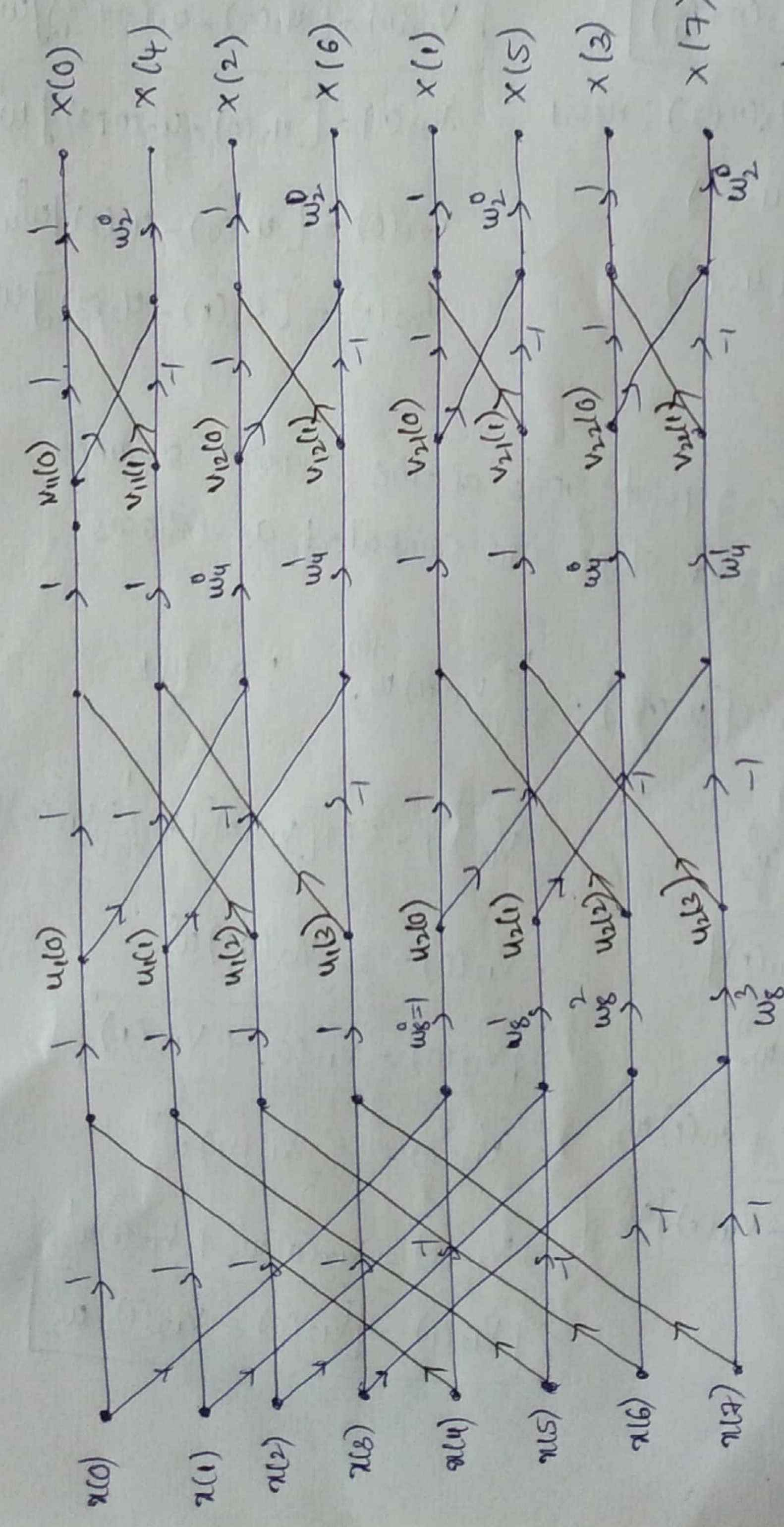


fig: Butterfly diagram or signal-flow graph for the 8-point radix-2 DIF-FFT Algorithm.

① Find the 4-point DFT of the sequence $x(n) = \{2,1,4,3\}$ by

(a) DIT-FFT Algorithm. (b) DIF-FFT Algorithm.

n			$x_r(n)$
0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3

Sol $x(n) = \{2,1,4,3\}$.

$x(n) = \{x(0), x(1), x(2), x(3)\}$.

bit reversed order $x_r(n) = \{x(0), x(2), x(1), x(3)\}$

$x_r(n) = \{2,4,1,3\}$.

→ $W_4^0 = 1,\ W_4^1 = -j,\ W_2^0 = 1$

i.e $W_4^0 = e^{-j\frac{2\pi}{4}(0)} = 1, \quad W_4^1 = e^{-j\left(\frac{2\pi}{4}\right)\cdot(1)} = -j$

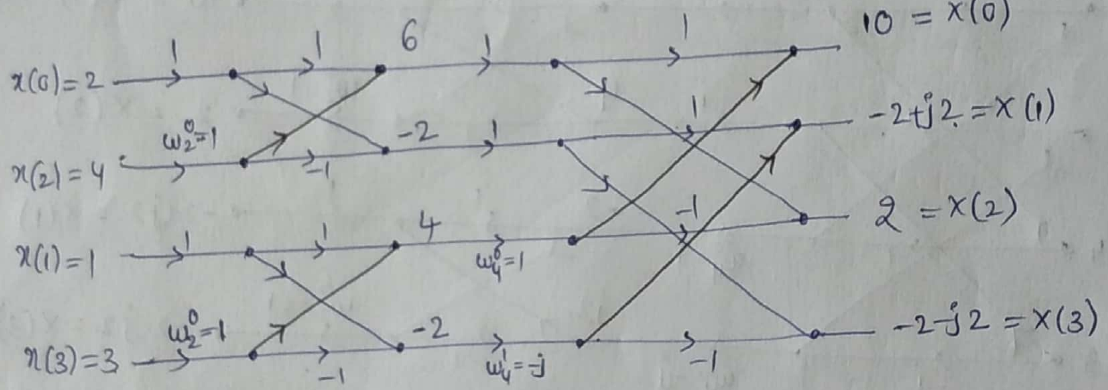


fig: 4-point DFT by DIT-FFT Algorithm.

∴ 4-point DFT of $x(n)$ is

$X(k) = \{10, -2+j2, 2, -2-j2\}$.

No of stages $M = \log_2 N = \log_2 4 = \log_2 2^2 = 2$

complex multiplications $= \frac{N}{2}\log_2 N = \frac{4}{2}\log_2 4 = 2\times 2 = 4$

complex Additions $= N\log_2 N = 4\log_2 4 = 8$

DIF-FFT Algorithm:-

To compute the DFT by DIF-FFT, the input sequence is to be in normal order and the output sequence will be in bit reversed order.

$$x(n) = \{2, 1, 4, 3\}$$

$$x(n) = \{x(0), x(1), x(2), x(3)\}.$$

$$W_2^0 = 1, \quad W_4^0 = 1, \quad W_4^1 = -j$$

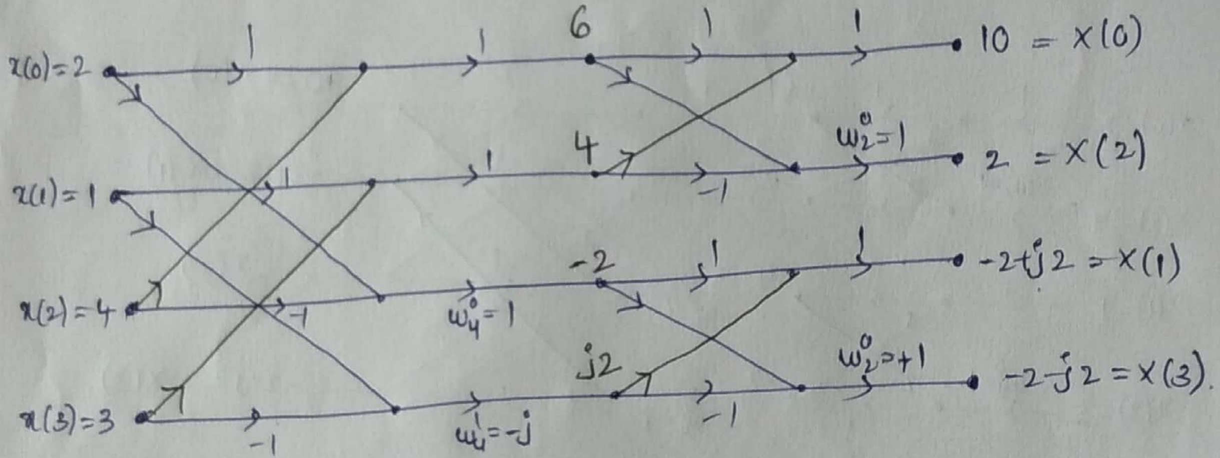


fig: 4-point DFT by DIF-FFT

$\therefore$ 4-point DFT of $x(n)$ is $X(k)$.

$$X(k) = \{10, -2+j2, 2, -2-j2\}$$

① Find the 8-point DFT by radix-2 DIT FFT Algorithm.

$x(n) = \{2,1,2,1,2,1,2,1\}$.

<u>sol</u> $x(n) = \{2,1,2,1,2,1,2,1\}$.

$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$.

The input sequence must be in bit reversed order.

$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$.

$x_r(n) = \{2,2,2,2,1,1,1,1\}$.

$\rightarrow$ $W_2^0 = 1,\ W_4^0 = 1,\ W_4^1 = -j,\ W_8^0 = 1,\ W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}},\ W_8^2 = -j,\ W_8^3 = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

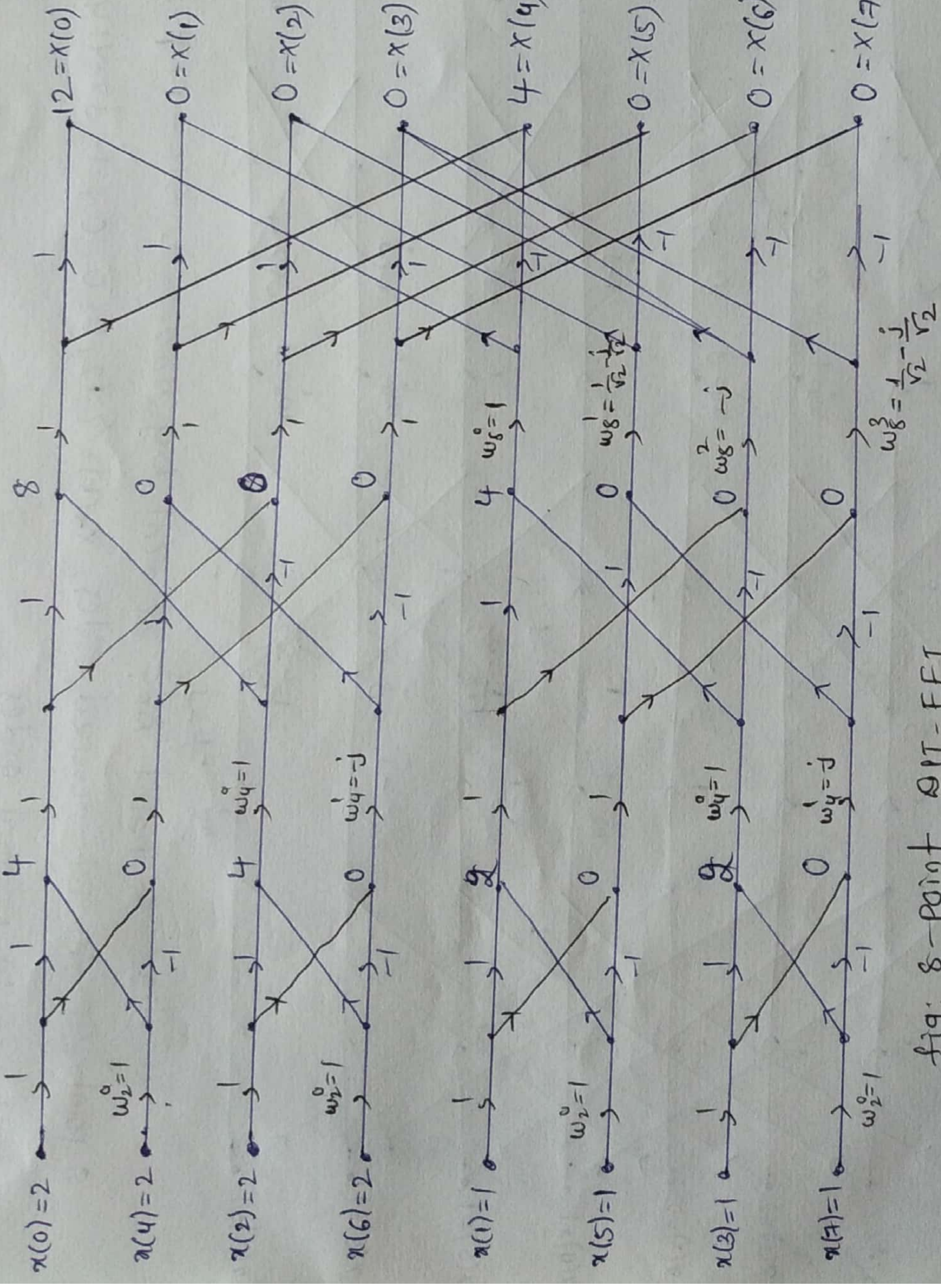


fig: 8-point DIT-FFT.

$\therefore$ 8-point DFT of $x(n)$ by radix-2 DIT FFT Algorithm is

$X(k) = \{12,0,0,0,4,0,0,0\}$

(b) Find the 8-point DFT by Radix-2 DIF-FFT Algorithm.

$x(n) = \{2,2,2,2,1,1,1,1\}$.

Sol For DIF-FFT Algorithm, the input is normal order & output is in bit reversed order.

$x(n) = \{2,2,2,2,1,1,1,1\}$.

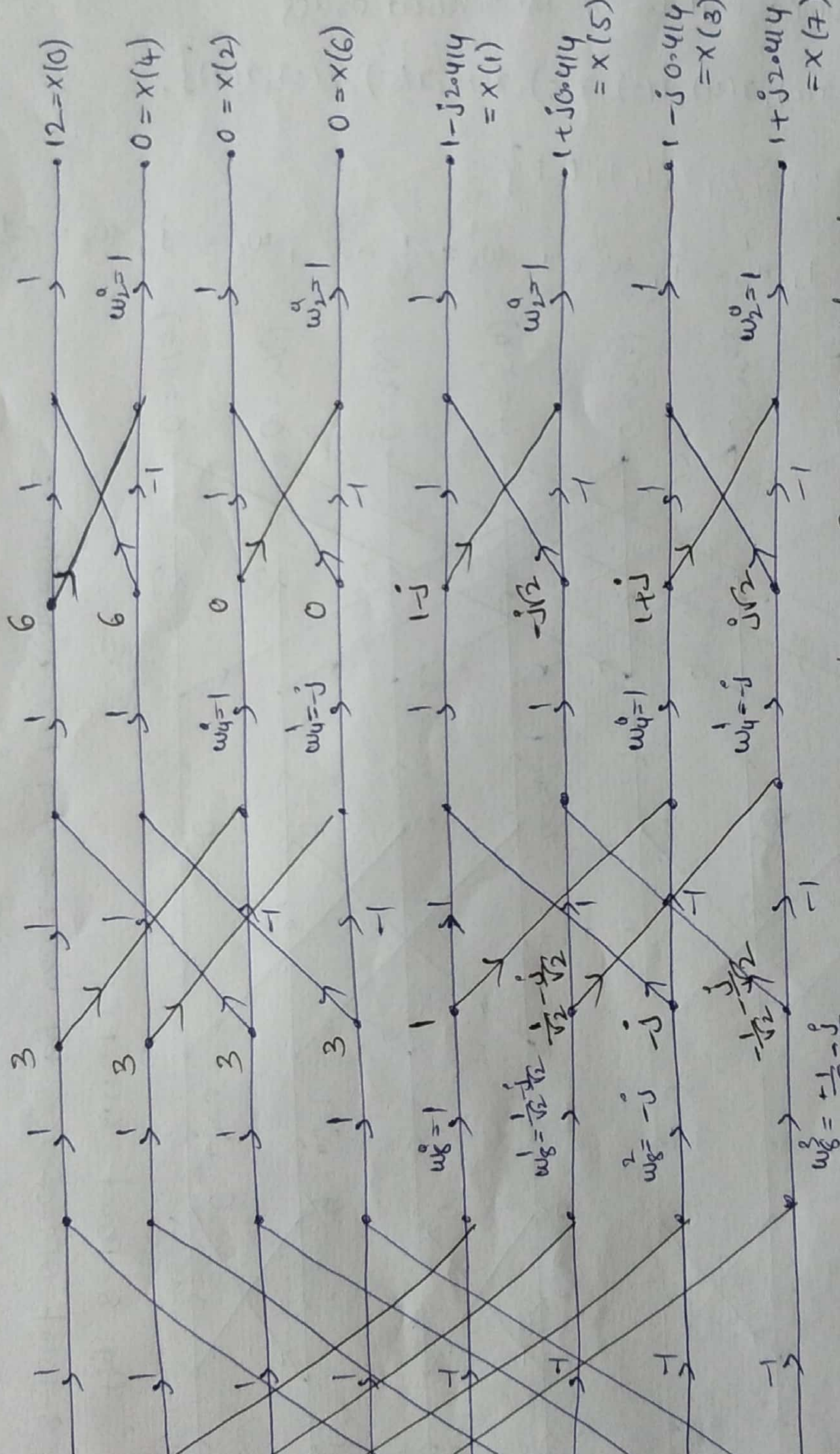


fig: 8-point DFT of $x(n)$ by radix-2 DIF-FFT Algorithm

8-point DFT in bit reversed order, $X_r(k) = \{12, 0, 0, 0, 1-j2.414, 1+j0.414, 1-j0.414, 1+j2.414\}$.

8-point DFT in normal order

$X(k) = \{12, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$.

Computation of IDFT through FFT:-

The IDFT of an N-point sequence $\{X(K)\}$: $K=0,1 \ldots N-1$

is defined as

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi}{N}nk}$$

$$\boxed{x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-nk}}$$

Taking the conjugate of the above equation for $x(n)$,

$$x^*(n) = \left[\frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-nk}\right]^*$$

$$x^*(n) = \frac{1}{N}\sum_{K=0}^{N-1} X^*(K)\, W_N^{nk}$$

Taking the conjugate of the above equation for $x^*(n)$,

$$\boxed{x(n) = \frac{1}{N}\left[\sum_{K=0}^{N-1} X^*(K)\, W_N^{nk}\right]^*}$$

procedure to compute the IDFT of $X(K)$:-

1. Take conjugate of $X(K)$ i.e determine $X^*(K)$.
2. Compute the N-point DFT of $X^*(K)$ using radix-2 FFT.
3. Take conjugate of the output sequence of FFT.
4. Divide the sequence obtained in step-3 by N.

find the IDFT of the following by DIT-FFT.

(a) $x(k) = \{1, 1-j2, -1, 1+j2\}$.

0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3

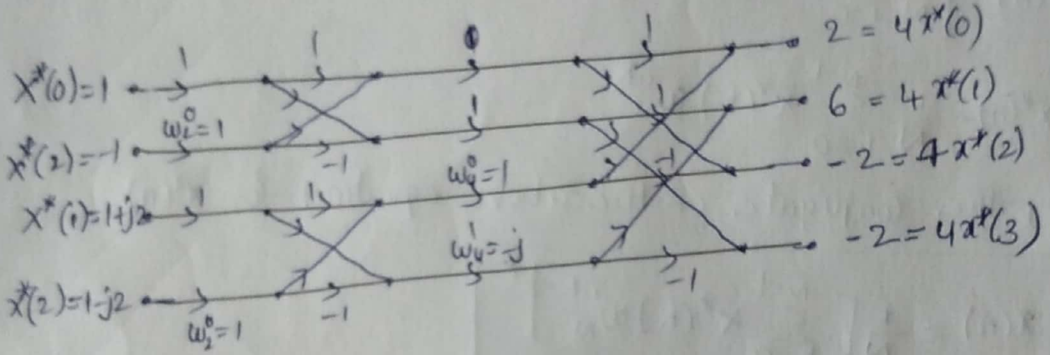
Sol

$X(k) = \{1, 1-j2, -1, 1+j2\}$

$X^*(k) = \{1, 1+j2, -1, 1-j2\}$

bit reverse order $X_0^*(k) = \{1, -1, 1+j2, 1-j2\}$

$W_2^0 = 1, \; W_4^0 = 1, \; W_4^1 = -j$



$4x^*(n) = \{2, 6, -2, -2\}$

$x^*(n) = \frac{1}{4}\{2, 6, -2, -2\}$

$x^*(n) = \{0.5, 1.5, -0.5, -0.5\}$

$x(n) = \{0.5, 1.5, -0.5, -0.5\}$

Find the response of LTI system with impulse response $h(n)=\{2,2,1,1\}$ when the input sequence $x(n)=\{1,2,0,1\}$.

the Response $y(n)$ of LTI system is given by

$$y(n) = x(n) \circledast h(n)$$

$$x(n) = \{1,2,0,1\} \qquad h(n)=\{2,2,1,1\}$$

$$DFT\{x(n)\} = X(k), \quad DFT\{h(n)\} = H(k), \quad DFT\{y(n)\} = Y(k)$$

By convolution theorem of DFT,

$$DFT\{x(n) \circledast h(n)\} = X(k)H(k)$$

$$y(n) = IDFT\{X(k)H(k)\} = IDFT\{Y(k)\}.$$

step1: Determine $X(k)$ using radix-2 DIT-FFT Algorithm

step2: Determine $H(k)$ using radix-2 DIT-FFT Algorithm

step3: Determine $X(k)H(k)$

step4: Take the IDFT of the product $X(k)H(k)$ using radix-2 DIT-FFT Algorithm.

$\rightarrow X(k)$

$$x(n) = \{1,2,0,1\}$$

$$x_r(n) = \{1,0,2,1\}$$

$x(n)$			$x_r(n)$
0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3

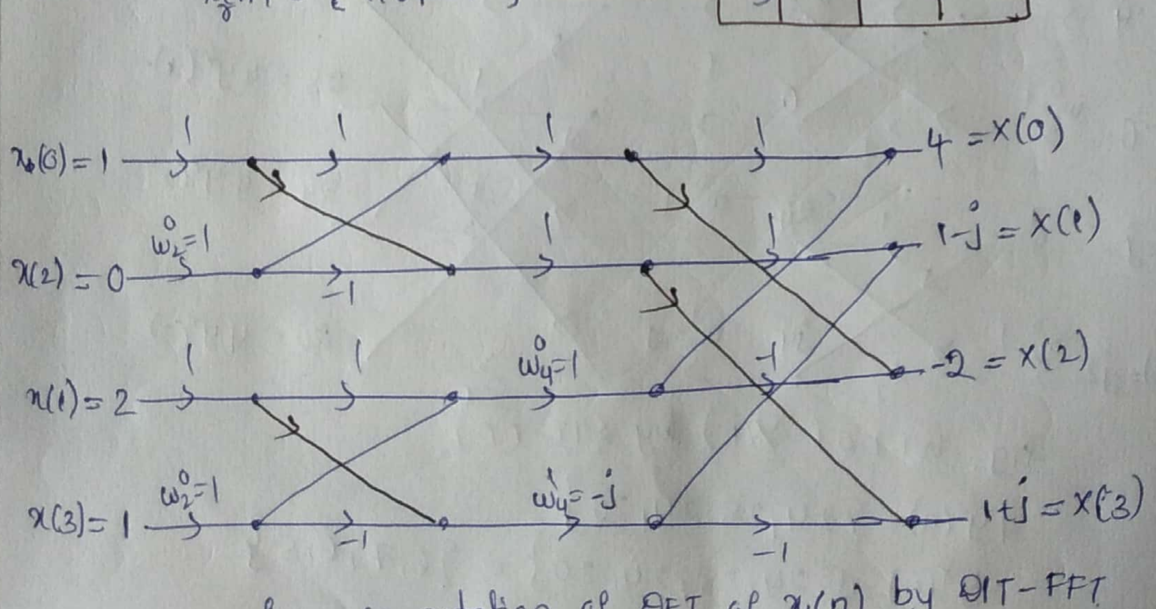


fig: computation of DFT of $x_1(n)$ by DIT-FFT

$$X(k) = \{4, 1-j, -2, 1+j\}$$

H(k) $h(n) = \{2, 2, 1, 1\}$,

$h_r(n) = \{2, 1, 2, 1\}$

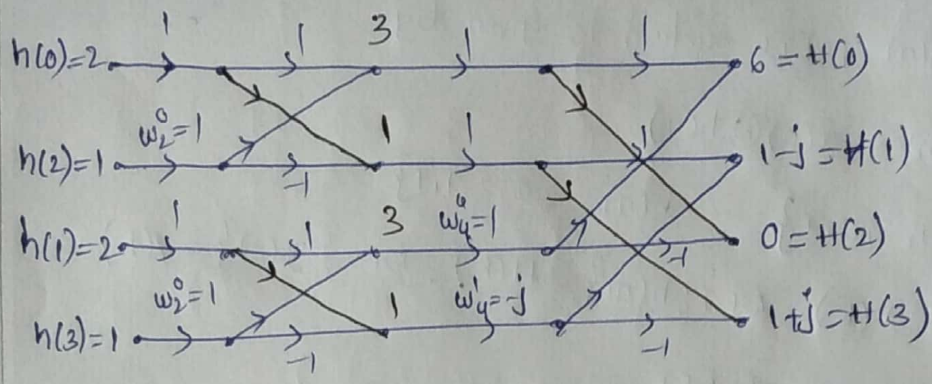


fig: DFT of h(n) by DIT-FFT.

$H(K) = \{6, 1-j, 0, 1+j\}$

→ Y(k) $Y(K) = X(K)H(K)$

$= \{4, 1-j, -2, 1+j\}\{6, 1-j, 0, 1+j\}$

$Y(k) = \{24, -j2, 0, j2\}$.

→ IDFT of Y(K).

$Y^*(K) = \{24, j2, 0, -j2\}$.

$Y_r^*(k) = \{24, 0, j2, -j2\}$.

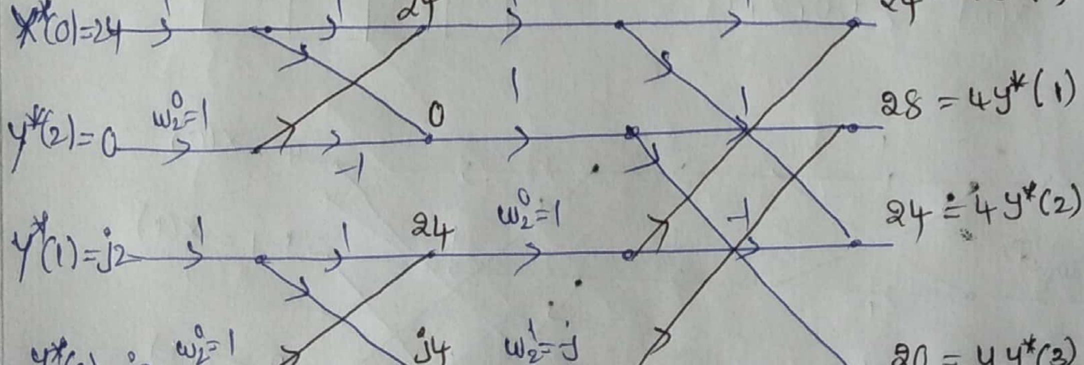


fig: IDFT of Y(K) by DIT-FFT.

$4y^*(n) = \{24, 28, 24, 20\}$

$y^*(n) = \frac{1}{4}\{24, 28, 24, 20\} = \{6, 7, 6, 5\}$

conjugate of $[y^*(n)]^* = \{6, 7, 6, 5\}^* \Rightarrow y(n) = \{6, 7, 6, 5\}$

Develop DIT-FFT Algorithms for decomposing the DFT for $N=6$ draw the flow diagrams for given sequence $x(n)=\{1,1,1,2,2,2\}$.

$x(n)=\{1,1,1,2,2,2\}$

$N=6=3\times 2=P_1N_1 \quad , P_1=3, N_1=2.$

$x(n)$ is splited into 3 sequences each of two samples.

$$X(k)=\sum_{n=0}^{N-1} x(n)\,W_N^{nk}$$

$$X(k)=\sum_{n=0}^{5} x(n)\,W_6^{nk}$$

$$X(k)=\sum_{n=0}^{1} x(3n)\,W_6^{3nk}+\sum_{n=0}^{1} x(3n+1)\,W_6^{(3n+1)k}+\sum_{n=0}^{1} x(3n+2)\,W_6^{(3n+2)k}$$

$$X(k)=\sum_{n=0}^{1} x(3n)\,W_6^{3nk}+W_6^{k}\sum_{n=0}^{1} x(3n+1)\,W_6^{3nk}+W_6^{2k}\sum_{n=0}^{1} x(3n+2)\,W_6^{3nk}$$

$$X(k)=X_1(k)+W_6^{k}X_2(k)+W_6^{2k}X_3(k)$$

$$X_1(k)=\sum_{n=0}^{1} x(3n)\,W_6^{3nk}=x(0)+x(3)W_6^{3k}$$

$$X_2(k)=\sum_{n=0}^{1} x(3n+1)\,W_6^{3nk}=x(1)+x(4)W_6^{3k}$$

$$X_3(k)=\sum_{n=0}^{1} x(3n+2)\,W_6^{3nk}=x(2)+x(5)W_6^{3k}$$

$X_1(0)=x(0)+x(3)W_6^{0}=1+2(1)=3$

$X_1(1)=x(0)+x(3)W_6^{3}=1+2(-1)=-1$

$X_2(0)=x(1)+x(4)W_6^{0}=1+2(1)=3$

$X_2(1)=x(1)+x(4)W_6^{3}=1+2(-1)=-1$

$X_3(0)=x(2)+x(5)W_6^{0}=1+2(1)=3$

$X_3(1)=x(2)+x(5)W_6^{3}=1+2(-1)=-1$

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$$X(k) = X_1(k) + W_6^k X_2(k) + W_6^{2k} X_3(k) \qquad (\because X_1(k+2) = X(k))$$

$$k=0 \Rightarrow X(0) = X_1(0) + W_6^0 X_2(0) + W_6^0 X_3(0) = 1 + (1)(3) + 1(3) = 9$$

$$X(1) = X_1(1) + W_6^1 X_2(1) + W_6^2 X_3(1) = -1 + (0.5 - j0.866)(-1) + (-0.5 - j0.866)(-1)$$

$$= -1 + j1.732$$

$$X(2) = X_1(2) + W_6^2 X_2(2) + W_6^4 X_3(2) = X_1(0) + W_6^2 X_2(0) + W_6^4 X_3(0)$$

$$= 3 + (-0.5 - j0.866)(3) + (-0.5 + j0.866)(3)$$

$$= 0$$

$$X(3) = X_1(3) + W_6^3 X_2(3) + W_6^6 X_3(3) = -1 + (-1)(-1) + 1(-1) = -1$$

$$= X_1(1) + W_6^3 X_2(1) + W_6^6 X_3(1)$$

$$X(4) = X_1(4) + W_6^4 X_2(4) + W_6^8 X_3(4) = X_1(0) + W_6^4 X_2(0) + W_6^2 X_3(0)$$

$$= 3 + (-0.5 + j0.866)3 + (-0.5 - j0.866)(3) = 0$$

$$X(5) = X_1(1) + W_6^5 X_2(1) + W_6^4 X_3(1)$$

$$X(5) = -1 + (0.5 + j0.866)(-1) + (-0.5 + j0.866)(-1) = -1 - j1.732$$

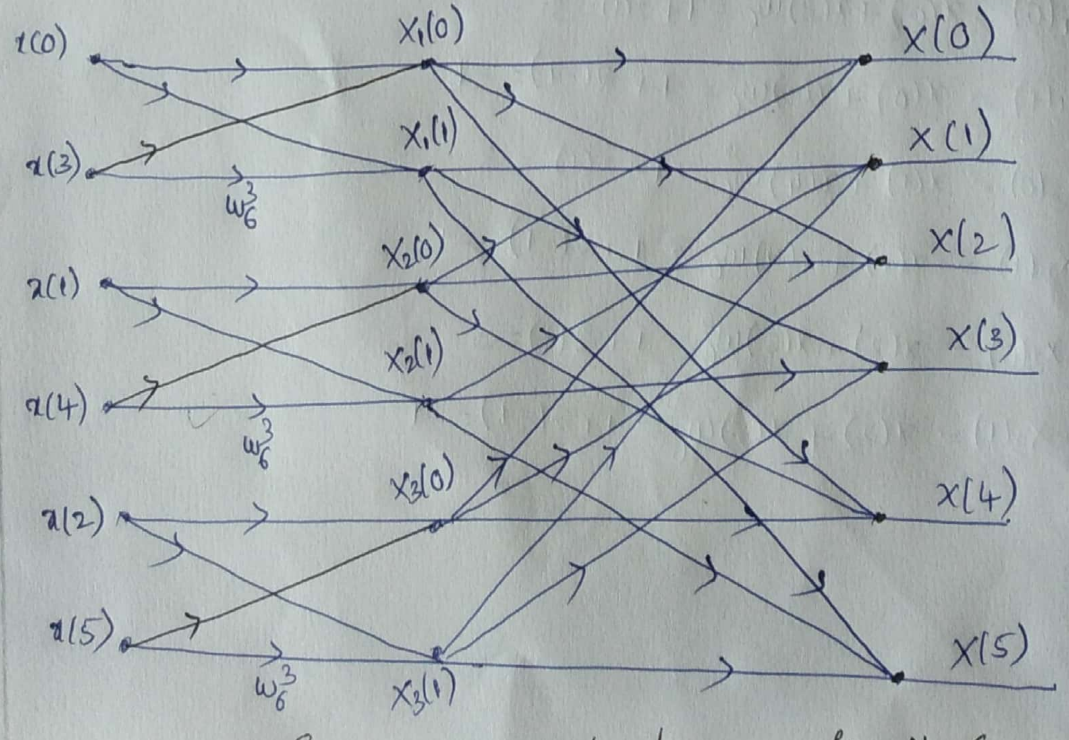

fig: DIT-FFT flow diagram for N=6.