Unit-5: Complex Integration:

23/09/19

Line Interval:- Let $f(z)$ be a function of complex variable defined in a domain D. Let 'c' be an arc in the domain joining from $z=\alpha$ to $z=\beta$.

$\therefore$ The complex line integration is denoted by $\oint_c f(z)dz$.

① Evaluate $\cancel{\oint_c f(z)dz}$ $\oint_c \bar{z}dz$, where c is the upper half of the unit circle $|z|=1$ taken in anti clockwise direction.

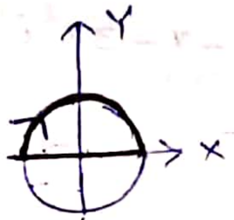
$$\int_c \bar{z}dz = \int_c (x-iy)(dx+idy)$$

$z = x+iy$
$dz = dx+idy$

Given that $|z|=1 \Rightarrow \sqrt{x^2+y^2}=1$

$$x^2+y^2=1$$

$$c=(0,0),\ r=1$$



Take $x = r\cos\theta,\ y = r\sin\theta$

$$x=\cos\theta,\ y=\sin\theta \quad [\because r=1]$$

$$dx = -\sin\theta,\quad dy=\cos\theta$$

$\rightarrow \theta$ varies from π to 0.

$$\int_c \bar{z}dz = \int_{\pi}^{0} [(\cos\theta - i\sin\theta)(-\sin\theta + i\cos\theta)]d\theta$$

$$= \int_{0}^{\pi} [-\sin\theta\cos\theta + i\cos^2\theta + i\sin^2\theta - i^2\sin\theta\cos\theta]d\theta$$

$$= \int_{0}^{\pi} (-\sin\theta\cancel{\cos\theta} + i + \sin\theta\cancel{\cos\theta})d\theta$$

$$= \int_{0}^{\pi} i\,d\theta = i[\theta]_{0}^{\pi}$$

$$= -\pi i.$$

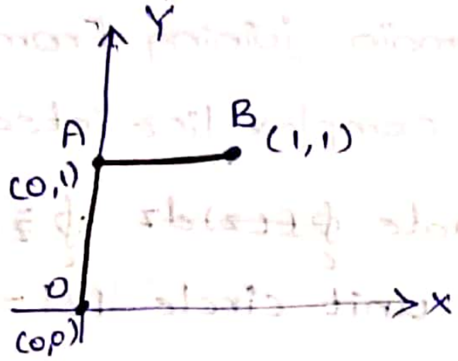
② Evaluate $\int_C (y-x-3x^2 i)\,dz$, where C consists of the line segment $z=0$ to $z=i$ and the other form $z=i$ to $z=1+i$.

$\rightarrow$ $z=0$ to $z=i$ [OA]

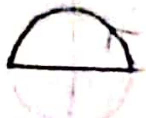
(0,0) (0,1)

$\rightarrow$ $z=i$ to $z=1+i$ [AB]

(0,1) (1,1)



$$\Rightarrow \int_C (y-x-3x^2 i)\,dz = \int_C (y-x-3x^2 i)(dx+i\,dy)$$



$$= \int_{\text{along } OA} (y-x-3x^2 i)(dx+i\,dy) + \int_{\text{along } AB} (y-x-3x^2 i)(dx+i\,dy)$$

On line OA = $x=0$, $y=0$ to 1

along line AB $y=1$, $x=0$ to 1.

$$= \int_0^1 (y-0-0)(0+i\,dy) + \int_0^1 (1-x-3x^2 i)\,dx$$

$$= \int_0^1 iy\,dy + \int_0^1 (1-x-3x^2 i)\,dx$$

$$= \left[\frac{iy^2}{2}\right]_0^1 + \left[x-\frac{x^2}{2}-\frac{\cancel{3}x^3 i}{\cancel{3}}\right]_0^1$$

$$= i\left[\frac{1}{2}-0\right] + \left[1-\frac{1}{2}-i\right]$$

$$= \frac{i}{2}+\frac{1}{2}-i \Rightarrow -\frac{i}{2}+\frac{1}{2}$$

$$= \frac{1}{2}(1-i)$$

③ Evaluate $\int_C (2y+x^2)dx + (3x-y)dy$ along the parabola $x=2t,\ y=t^2+3$ joining the points $(0,3)$ and $(2,4)$.

Given $\int_C (2y+x^2)dx + (3x-y)dy$

Given $x = 2t$, $y = t^2+3$

$dx = 2dt$ $\quad dy = 2t\,dt$

at $x=0,\ t=0$ $\quad$ at $y=3,\ t=0$ at $(0,3)$ then $t=0$

$x=2,\ t=1$ $\quad$ at $y=4,\ t=1$ at $(2,4)$ then $t=1$

$\therefore$ t varies from 0 to 1.

$$\int_C (2y+x^2)\,dx + (3x-y)\,dy = \int_0^1 [2(t^2+3)+(2t)^2]\,2dt + [3(2t)-t^2-3]\,2t\,dt$$

$$= \int_0^1 (2t^2+6+4t^2)2dt + [6t-t^2-3]\,2t\,dt$$

$$= \int_0^1 (4t^2+12+8t^2+12t^2-2t^3-6t)\,dt$$

$$= \left[4\frac{t^3}{3} + 12t + \frac{8t^3}{3} + \frac{12t^3}{3} - \frac{2t^4}{4} - \frac{6t^2}{2}\right]_0^1$$

$$= \left[\frac{4}{3} + 12 + \frac{8}{3} + \frac{12}{3} - \frac{2}{4} - \frac{6}{2}\right] = \frac{33}{2}.$$

④ Evaluate $\int_C f(z)\,dz$ where $f(z) = \frac{z+2}{z}$, where C is

1) the semicircle $z = 2e^{i\theta}$ where θ varies from 0 to π

2) the semicircle $z = 2e^{i\theta}$ where θ varies from 0 to $-\pi$

3) the circle $z = 2e^{i\theta}$, where θ varies from $-\pi$ to π.

Sol:- $f(z) = \frac{z+2}{z}$, Given $z = 2e^{i\theta}$

$$f(z) = \frac{2e^{i\theta}+2}{2e^{i\theta}} = \frac{e^{-i\theta}(2e^{i\theta}+2)}{2}$$

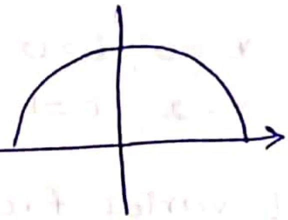
$$= 1 + e^{-i\theta}$$

$$= 1 + (\cos\theta - i\sin\theta)$$

$$f(z) = 1 + \cos\theta - i\sin\theta.$$

i) Given that c is a semi circle where θ varies from 0 to π

$$\int_c f(z)\,dz = \int_0^{\pi} (1+\cos\theta - i\sin\theta)(dx + i\,dy)$$



$$= \int_0^{\pi} (1+\cos\theta - i\sin\theta)(-2\sin\theta + 2i\cos\theta)\,d\theta$$

$z = 2e^{i\theta} = 2(\cos\theta + i\sin\theta)$

$x + iy = 2\cos\theta + i\,2\sin\theta$

$x = 2\cos\theta, \quad y = 2\sin\theta.$

$$= \int_0^{\pi} -2\sin\theta + 2i\cos\theta - \cancel{2\sin\theta\cos\theta} + 2i\cos^2\theta + 2i\sin^2\theta + \cancel{2\sin\theta\cos\theta}$$

$$= \int_0^{\pi} -2\sin\theta + 2i\cos\theta + 2i\cos^2\theta + 2i\sin^2\theta\,d\theta$$

$x^2 + y^2 = 2^2$, $dx = -2\sin\theta\,d\theta$

$r = 2$, $dy = 2\cos\theta\,d\theta.$

$$= 2\int_0^{\pi} (-\sin\theta + i\cos\theta + i)\,d\theta$$

$$= 2\left[\cos\theta + i\sin\theta + i\theta\right]_0^{\pi}$$

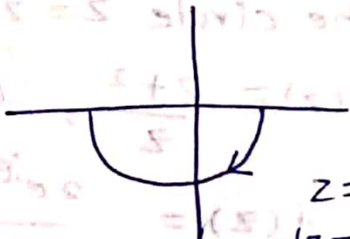
$$= 2\left[[\cos\pi + 0 + i\pi] - (\cos 0 + 0 + 0)\right]$$

$$= 2[-1 + i\pi - 1]$$

$$= 2[-2 + i\pi] = -4 + 2i\pi.$$

ii) Given that c is a semi circle, where θ varies from 0 to $-\pi$.

$$\int_c f(z)\,dz = \int_c \left(\frac{2+2e^{-i\theta}}{\cancel{2}}\right) \cancel{2}i\,e^{i\theta}\,d\theta$$



$z = 2e^{i\theta}$

$dz = 2e^{i\theta}(i\,d\theta)$

$$= \int_0^{-\pi} (2i + 2i\,e^{+i\theta})\,d\theta$$

$$= 2i\int_0^{-\pi}(1+e^{i\theta})\,d\theta = 2i\left[\theta+\frac{e^{i\theta}}{i}\right]_0^{-\pi}$$

$$= 2i\left[-\pi+\frac{e^{-i\pi}}{i}-(0+1)\right]$$

$$= 2i\left[-\pi+\frac{e^{-i\pi}}{i}-1\right]$$

$$= 2i\left[-\pi+\frac{\cos\pi-i\sin\pi}{i}-1\right]$$

$$= -2i\pi+\frac{2(-1)-0}{i}-2$$

$$= -2i\pi-2-4$$

$$= -4-2i\pi.$$

iii) Given that c is a circle $z=2e^{i\theta}$ where θ varies from $-\pi$ to π

$$\therefore \int_C f(z)\,dz = \int_C \left(\frac{2+2e^{-i\theta}}{2}\right)(2ie^{i\theta})\,d\theta$$

$$= \int_{-\pi}^{\pi} 2i(1+e^{i\theta})\,d\theta$$

$$= 2i\left[\theta+\frac{e^{i\theta}}{i}\right]_{-\pi}^{\pi}$$

$$= 2i\left[\pi+\frac{e^{i\pi}}{i}-\left(-\pi+\frac{e^{-i\pi}}{i}\right)\right]$$

$$= 2i\left[\pi+\frac{\cos\pi+i\sin\pi}{i}+\pi-\frac{(\cos\pi-i\sin\pi)}{i}\right]$$

$$= 2i\left[2\pi\cancel{-\frac{1}{i}}\cancel{+\frac{1}{i}}-0\right] = 4i\pi.$$

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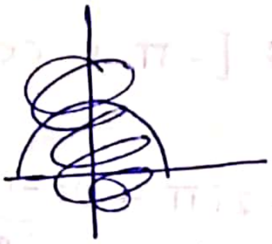
⑤ $\int_C f(z)dz$ where $f(z) = z-1$, and c is the arc from $z=0$ and $z=2$.

i) c is the semicircle $z-1 = e^{i\theta}$, where θ varies from 0 to π.

ii) c is the line segment of the x-axis.

Given that $z-1 = e^{i\theta}$, ~~where~~

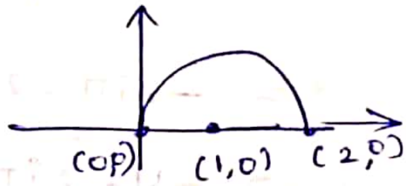
The semicircle $z=0$ to $z=2$.



taking modulous on b.s

$$|z-1| = |e^{i\theta}| \Rightarrow |z-1| = 1$$

$$|x+iy-1| = 1$$



$$(x-1)^2 + y^2 = 1$$

centre $= (1,0)$, $r = 1$.

i) C is the semicircle $z-1 = e^{i\theta}$, θ varies from 0 to π.

$$\therefore dz = e^{i\theta}(i\,d\theta)$$

$$\int_C f(z)dz = \int_C (z-1)dz = \int_0^{\pi} e^{i\theta} \cdot ie^{i\theta} d\theta$$

$$= i\int_0^{\pi} e^{2i\theta} d\theta = \cancel{i}\left[\frac{e^{2i\theta}}{2\cancel{i}}\right]_0^{\pi}$$

$$= \frac{1}{2}\left[e^{2i\pi} - e^{0}\right]$$

$$= \frac{e^{2i\pi} - 1}{2}$$

$$= \frac{\cos(2\pi) + i\sin(2\pi) - 1}{2}$$

$$= \frac{1+0-1}{2} = 0$$

ii) C is the linesegment of the x-axis

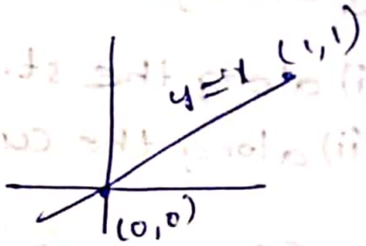
The segment of x-axis $y=0,\ x=x.$
$dy=0,\ dx=dx.$

$$\int_C f(z)\,dz = \int_0^2 (z-1)\,dz = \int_0^2 (x+iy-1)(dx+idy).$$

$$= \int_0^2 (x-1)\,dx = \left[\frac{x^2}{2}-x\right]_0^2 = \frac{4}{2}-2 = 0.$$

⑥ Evaluate $\int_0^{1+i} (x^2-iy)\,dz$ along the paths

i) $y=x$ ii) $y=x^2$.



i) $y=x$

$dy=dx.$, x varies from 0 to 1.

$$\int_0^{1} (x^2-ix)(dx+idy) = \int_0^{1} (x^2-ix)(dx+idx)$$

$$= \int_0^{1} (1+i)(x^2-ix)\,dx = (1+i)\int_0^{1} (x^2-ix)\,dx$$

$$= (1+i)\left[\frac{x^3}{3}-i\frac{x^2}{2}\right]_0^{1} = (1+i)\cancel{\left[\frac{(1+i)^3}{3}-i\frac{(1+i)^2}{2}\right]}.$$

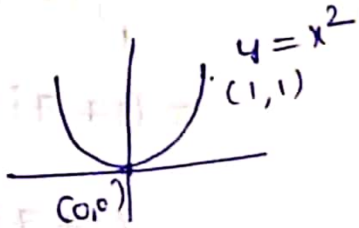
$$= \cancel{1+i}\ (1+i)\left[\frac{1}{3}-\frac{i}{2}\right] = \frac{1}{3}-\frac{i}{2}+\frac{i}{3}+\frac{1}{2}$$

$$= \frac{5}{6}-\frac{i}{6}. = \frac{5-i}{6}.$$

ii) $y=x^2$

$dy = 2x\,dx.$

x varies from 0 to 1.



$$\int_0^1 (x^2-ix^2)(dx+2ix\,dx)$$

$$= \int_0^1 (x^2-ix^2)(1+2ix)\,dx = \int_0^1 (x^2-ix^2+2ix^3+2x^3)\,dx$$

$$\left[\frac{x^3}{3} - i\frac{x^3}{3} + 2i\frac{x^4}{4} + \frac{2x^4}{4}\right]_0^1$$

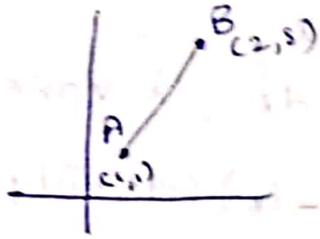
$$\frac{1}{3} - \frac{i}{3} + \frac{2i}{4} + \frac{2}{4} = \frac{1}{3} - \frac{i}{3} + \frac{i}{2} + \frac{1}{2}$$

$$= \frac{5}{6} + \frac{i}{6} = \frac{5+i}{6}$$

(7) Integrate $f(z) = x^2 + ixy$, $A(1,1)$ to $B(2,8)$.

i) along the st. line AB

ii) along the curve $x = t$, $y = t^3$.



Given $f(z) = x^2 + ixy$.

eq^n of AB

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 1 = \frac{8-1}{2-1}(x-1) \Rightarrow y - 1 = \frac{7}{1}(x-1) \Rightarrow y - 1 = 7x - 7$$

$7x - 7 - y + 1$

$7x - y - 6$.

$y = 7x - 6$

$dy = 7\,dx$.

$$\therefore \int_C f(z)\,dz = \int (x^2 + ixy)(dx + i\,dy)$$

$$= \int_1^2 (x^2 + ix(7x-6))(dx + 7i\,dx)$$

$$= \int_1^2 (x^2 + 7ix^2 - 6ix)(1+7i)\,dx.$$

$$= (1+7i)\int_1^2 (x^2 + 7ix^2 - 6ix)\,dx$$

$$= (1+7i)\left[\frac{x^3}{3} + \frac{7ix^3}{3} - \frac{6ix^2}{2}\right]_1^2$$

$$= (1+7i)\left[\left(\frac{8}{3} + \frac{7i \times 8}{3} - \frac{6i \times 4}{2}\right) - \left(\frac{1}{3} + \frac{7i}{3} - \frac{6i}{2}\right)\right]$$

$$= (1+7i)\left[\frac{8}{3} + \frac{56i}{3} - \frac{24}{2}i - \frac{1}{3} - \frac{7i}{3} + \frac{6i}{2}\right]$$

$$= (1+7i)\left[\frac{7}{3} + \frac{22}{3}i\right]$$

$$= \frac{7}{3} + \frac{22i}{3} + \frac{49i}{3} + \frac{22\times 7}{3}i^2$$

$$= \frac{7}{3} - \frac{154}{3} + \frac{71i}{3}$$

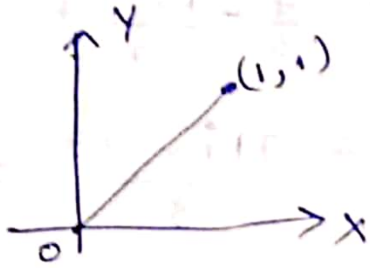
$$= \frac{71i - 147}{3} = \frac{71i}{3} - 49.$$

⑧ ~~Given~~ Evaluate $\int_0^{1+i} (x-y+ix^2)\,dz$

i) Along st. line $z=0$ to $z=1+i$

ii) Along the real axis from $z=0$ to $z=1$ and then, along a line parallel to imaginary axis from $z=1$ to $z=1+i$.

i) $z=0$ to $z=1+i$



eqⁿ of OA is

$$y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-0 = \frac{1-0}{1-0}(x-0)$$

$y=x$; $z=x+iy$

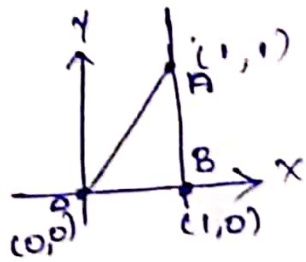
$dy=dx$ $\quad dz=dx+idy.$

$$\int_0^{1+i}(x-y+ix^2)\,dz = \int_0^1(\cancel{x}-\cancel{x}+ix^2)(dx+idy)$$

$$= \int_0^1 ix^2(dx+idx) = (1+i)\int_0^1 ix^2dx. = i(1+i)\left(\frac{x^3}{3}\right)_0^1$$

$$= i(i+1)\left(\frac{1}{3}\right) = (i^2+i)\frac{1}{3} = \frac{i-1}{3}.$$

ii) along real axis $z=0$ to $z=1$



$$\int_0^{1+i}(x-y+ix^2)\,dz = \int_{OB}(x-y+ix^2)\,dz + \int_{BA}(x-y+ix^2)\,dz.$$

On line OB $\Rightarrow$ x varies from 0 to 1, $y=0$, $dy=0$.

on line BA $\Rightarrow$ y varies from 0 to 1, $x=1$, $dx=0$.

$$\int_0^{1+i} (x-y+ix^2)\,dz = \int_0^{1+i} (x-y+ix^2)(dx+idy)$$

$$= \int_0^1 (x-0+ix^2)\,dx + \int_0^1 (1-y+i)(0+idy)$$

$$= \left[\frac{x^2}{2}+i\frac{x^3}{3}\right]_0^1 + i\left[y-\frac{y^2}{2}+iy\right]_0^1$$

$$= \frac{x^2}{2}+i\frac{x^3}{3}+iy-i\frac{y^2}{2}-y.$$

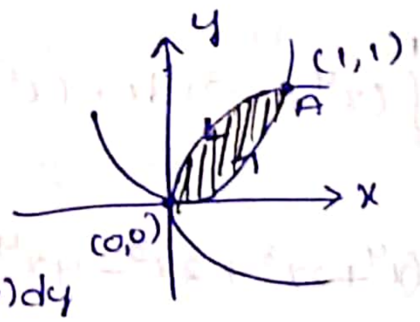
$$= \left(\frac{x^2}{2}-y\right)_0^1 + i\left(\frac{x^3}{3}-\frac{y^2}{2}+y\right)_0^1$$

$$= \left(\frac{1}{2}-1\right)+i\left(\frac{1}{3}-\frac{1}{2}+1\right)$$

$$= \frac{5i}{6}-\frac{1}{2}.$$

⑨ Evaluate $\int_C (y^2+2xy)dx + (x^2-2xy)dy$, where C is the boundary of the region $y=x^2$ and $y^2=x$.

Given curves $y=x^2$ —① and $y^2=x$ —② are parabolas.



$$\int_C (y^2+2xy)dx+(x^2-2xy)dy$$

$$= \int_{OA} (y^2+2xy)dx+(x^2-2xy)dy + \int_{AO} (y^2+2xy)dx+(x^2-2xy)dy$$

From ① & ② $\Rightarrow$ $y=x^2$

$y=(y^2)^2 \Rightarrow y=y^4 \Rightarrow y^3=1$

$y-y^4=0 \Rightarrow y(1-y^3)=0.$

$y=0, \quad y=1$

$x=0, \quad x=1$

$(0,0) \quad (1,1)$

along OA $\Rightarrow x^2 = y$, x varies from 0 to 1.

$2x\,dx = dy$

along AO $\Rightarrow \cancel{4}\cdot y^2 = x$, x varies from 1 to 0.

$y = \sqrt{x}$

$dy = \frac{1}{2\sqrt{x}}\,dx$

$$= \int_0^1 (x^4 + 2x^3)\,dx + (x^2 - 2x^3)(2x\,dx) + \int_1^0 (x + 2x\sqrt{x})\,dx + (x^2 - 2x\sqrt{x})\left(\frac{1}{2\sqrt{x}}\,dx\right).$$

$$= \int_0^1 (x^4 + 2x^3 + 2x^3 - 4x^4)\,dx + \int_1^0 \left(x + 2x\sqrt{x} + \frac{x^2}{2\sqrt{x}} - x\right)dx$$

$$= \left[\frac{x^5}{5} + \frac{2x^4}{4} + \frac{2x^4}{4} - \frac{4x^5}{5}\right]_0^1 + \left[\frac{x^2}{2} + \frac{2x^{5/2}}{5} \times 2 + \frac{2x^{5/2}}{2\times 5} - \frac{x^2}{2}\right]_1^0$$

$$= \frac{1}{5} + \frac{2}{4} + \frac{2}{4} - \frac{4}{5} - \left[\cancel{\frac{1}{2}} + \frac{4}{5} + \frac{2}{10_5} - \cancel{\frac{1}{2}}\right]$$

$$= \cancel{\frac{2}{5} + \frac{4}{4} = \frac{2}{5} + 1 = \frac{7}{5}}$$

$$= \frac{1}{5} + 1 - \frac{4}{5} - \cancel{\frac{1}{2}} - \frac{4}{5} - \frac{1}{5} + \cancel{\frac{1}{2}} = \frac{-3}{5}.$$

(10) Evaluate $\int_c \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along the curve 'c' is given by

a) $z = t^2 + it$.

b) along the line $z=0$ to $z=2i$ and there from $z=2i$ to $z=4+2i$

a) Given curve $z = t^2 + it \Rightarrow dz = (2t+i)dt$

$z=0 \Rightarrow t=0$

$z = 4+2i = (2)^2 + i(2) \Rightarrow t = 2.$

$\bar{z} = t^2 - it$

$$\int \bar{z}\,dz = \int_0^2 (t^2 - it)(2t+i)\,dt$$

$$= \int_0^2 (2t^3 + it^2 - 2it^2 + t)\,dt$$

$$= \left[\cancel{2}\frac{t^4}{\cancel{4}_2} + i\frac{t^3}{3} - 2i\frac{t^3}{3} + \frac{t^2}{2}\right]_0^2$$

$$= 8 + \frac{i\,8}{3} - \frac{2i \times 8}{3} + \frac{\cancel{4}^2}{\cancel{2}} = 10 + \frac{8i}{3} - \frac{16i}{3}$$

$$= 10 - \frac{8i}{3}$$

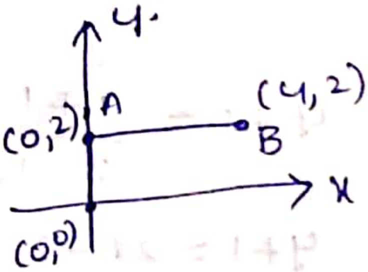
$$= 10 - \frac{8i}{3}.$$

b) along $z=0$ to $z=2i \Rightarrow (0,0)$ to $(0,2)$

and then $z=2i$ to $z=4+2i \Rightarrow (0,2)$ to $(4,2)$.

$\bar{z}/z$ $\quad z = x+iy \Rightarrow dz = dx + i\,dy.$

$\bar{z} = x - iy.$



$$\int_c \bar{z}\,dz = \int_{OA} + \int_{OB}$$

$$= \int_{OA} (x-iy)(dx+idy) + \int_{AB} (x-iy)(dx+idy).$$

along OA $\Rightarrow$ $x=0$, $dx=0$, y varies from 0 to 2.

along AB $\Rightarrow$ $y=2$, $dy=0$, x varies from 0 to 4

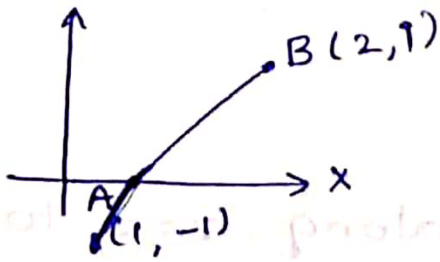
$$= \int_0^2 (0-iy)(0+idy) + \int_0^4 (x-2i)(dx+0)$$

$$= \int_0^2 y\,dy + \int_0^4 (x-2i)dx$$

$$= \left[\frac{y^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2ix\right]_0^4 = \frac{4}{2} + \left(\frac{16}{2} - 8i\right)$$

$$= \frac{20}{2} - 8i = 10 - 8i$$

(11) $\int_{1-i}^{2+i} (2x+iy+1)dz$, along the st. line ~~(+~~ joining $(1,-1)$ and $(2,i)$.



Let $z = x+iy$
$dz = dx+idy$

eqn of AB is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$$y+1 = \frac{1+1}{2-1}(x-1)$$

$$y+1 = 2x-2 \Rightarrow y = 2x-3.$$
$$dy = 2dx.$$

$z = x + iy \Rightarrow z = x + i(2x \mp 3)$.

$dz = dx + i\,dy \Rightarrow dz = dx + i(2dx)$.

x varies from 1 to 2.

$$\int_{1-i}^{2+i} (2x + iy + 1)\,dz = \int_{1}^{2} (2x + i(2x \mp 3) + 1)(dx + 2i\,dx)$$

$$= (1+2i)\int_{1}^{2} (2x + 2ix \mp 3i + 1)\,dx.$$

$$= (1+2i)\left[2\left(\frac{x^2}{2}\right)_1^2 + 2i\left(\frac{x^2}{2}\right)_1^2 \mp 3i(x)_1^2 + (x)_1^2\right]$$

$$= 1+2i\left[2\left(\frac{4}{2} - \frac{1}{2}\right) + 2i\left(\frac{4}{2} - \frac{1}{2}\right) \mp 3i(2-1) + (2-1)\right]$$

$$= (1+2i)\left[\cancel{2}\left(\frac{3}{\cancel{2}}\right) \mp \cancel{2}i\left(\frac{3}{\cancel{2}}\right) + 3i + 1\right]$$

$$= (1+2i)\left[3 + \cancel{3i} \mp \cancel{3i} + 1\right]$$

$$= (1+2i)(4 \cancel{+6i}) \qquad = \cancel{4 + 6i + 8i + 12i^2}$$

$$\qquad = \cancel{4 + 14i - 12}$$

$$= 4 + 8i. \qquad = \cancel{14i - 8}.$$

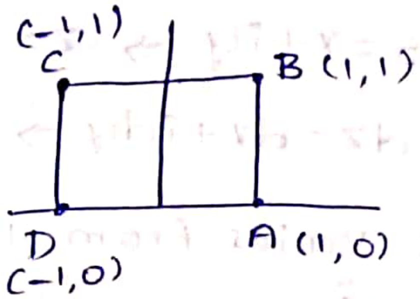
* **Cauchy's integral theorem:** – Let $f(z) = w = u(x,y) + i\,v(x,y)$ be analytic on the circle and within a closed contour "c" [circle] and $f'(z)$ be continuous, then

$$\int_c f(z)\,dz = 0.$$

① Verify Cauchy's theorem for the $\int z^3$ taken over the boundary of the rectangle with vertices $(-1,0)$; $(1,0)$; $(1,1)$; $(-1,1)$.

$$\int_c f(z)\,dz = 0$$

i.e., $\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = 0$



The boundary of rectangle 'c' consisting of 4 curves AB, BC, CD, DA. So that $\int_C f(z)dz = 0$.

i.e., $\int_C z^3 dz = \int_{AB} z^3 dz + \int_{BC} z^3 dz + \int_{CD} z^3 dz + \int_{DA} z^3 dz$

on line AB $\rightarrow x = 1$; $y \rightarrow 0$ to 1, $dx = 0$

on line BC $\rightarrow y = 1$; $x \rightarrow 1$ to -1, $dy = 0$

on line CD $\rightarrow x = -1$; $y \rightarrow 1$ to 0, $dx = 0$

on line DA $\rightarrow y = 0$; $x \rightarrow -1$ to 1; $dy = 0$.

$$\int_C z^3 dz = \int_0^1 (1+iy)^3 (idy) + \int_{-1}^{1} (x+i)^3 dx + \int_0^1 (-1+iy)(idy) + \int_{-1}^{1} (x)^3 dx.$$

$$= \int_0^1 (1 - iy^3 + 3iy - 3y^2) i\,dy + \int_{-1}^{1} (x^3 - i + 3ix^2 - 3x) dx + \int_0^1 (-iy^3 - 1 + 3y^2 + 3iy) i\,dy + \int_{-1}^{1} x^3 dx.$$

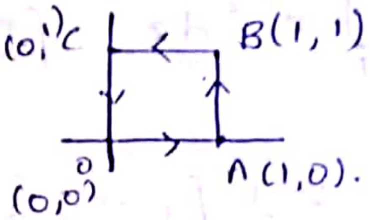
$$= \int_0^1 (i + y^3 - 3y - 3iy^2) dy + \left[\frac{x^4}{4} - ix + ix^3 - \frac{3x^2}{2}\right]_{-1}^{1} + \int_0^1 (y^3 - i + 3iy^2 - 3y) dy + \left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$= \left[iy + \frac{y^4}{4} - \frac{3y^2}{2} - iy^3\right]_0^1 + \left[\cancel{\tfrac{1}{4}} - \cancel{i} + \cancel{i} - \cancel{\tfrac{3}{2}} - \cancel{\tfrac{1}{4}} - \cancel{i} + \cancel{i} + \cancel{\tfrac{3}{2}}\right] + \left[\frac{y^4}{4} - iy + iy^3 - \frac{3y^2}{2}\right]_1^0 + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{4}}\right)$$

$$= \left[\cancel{i} + \frac{1}{4} - \frac{3}{2} - \cancel{i}\right] - \left[\frac{1}{4} - \cancel{i} + \cancel{i} - \frac{3}{2}\right] = \cancel{\tfrac{1}{4}} - \cancel{\tfrac{3}{2}} - \cancel{\tfrac{1}{4}} + \cancel{\tfrac{3}{2}} = 0.$$

∴ Cauchy's integral theorem is verified.

② show that $\int_C (z+1)dz = 0$ where 'c' is the boundary of the square whose vertices are the points $z=0$; $z=1$; $z=1+i$; $z=i$.



~~z=0~~ $z=0 \rightarrow (x,y)=(0,0)$

$z=1 \rightarrow (x,y)=(1,0)$, $z=i \rightarrow (x,y)=(0,1)$.

$z=1+i \rightarrow (x,y)=(1,1)$.

$$\int_C (z+1)dz = \int_{OA}(z+1)dz + \int_{AB}(z+1)dz + \int_{BC}(z+1)dz + \int_{CO}(z+1)dz$$

on line OA $\rightarrow$ $x \rightarrow 0$ to 1; $y=0 \rightarrow dy=0$

on line AB $\rightarrow$ $x=1$; $y \rightarrow 0$ to 1; $dx=0$

on line BC $\rightarrow$ $x \rightarrow 1$ to 0; $y=1 \rightarrow dy=0$

on line CO $\rightarrow$ $x=0$; $y \rightarrow 1$ to 0; $dx=0$.

$$\int_C (z+1)dz = \int_0^1 (x+1)dx + \int_0^1 (1+iy+1)(i\,dy) + \int_1^0 (x+i+1)\,dx + \int_1^0 (iy+1)\,i\,dy.$$

$$= \left[\frac{x^2}{2}+x\right]_0^1 + \left[iy-\frac{y^2}{2}+iy\right]_0^1 + \left[\frac{x^2}{2}+ix+x\right]_1^0 + \left[-\frac{y^2}{2}+iy\right]_1^0$$

$$=\left[\frac{1}{2}+1\right]+\left[i-\frac{1}{2}+i\right]+\left[-\frac{1}{2}-i-1\right]+\left[\frac{1}{2}-i\right]$$

$$= \cancel{\frac{3}{2}} + \cancel{2i} - \cancel{\frac{1}{2}} - \cancel{\frac{3}{2}} - \cancel{i} + \cancel{\frac{1}{2}} - \cancel{i}$$

$$=0$$

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③ Evaluate (i) $\int_{i}^{i/2} e^{\pi z} dz$ ii) $\int_{0}^{\pi+2i} \cos\frac{z}{2} dz$ iii) $\int_{1}^{3} (z-2)^3 dz$

(i) $\int_{i}^{i/2} e^{\pi z} dz = \left[\frac{e^{\pi z}}{\pi}\right]_{i}^{i/2} = \frac{e^{\pi i/2} - e^{i\pi}}{\pi}$

$= \frac{1}{\pi}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\pi - i\sin\pi\right]$

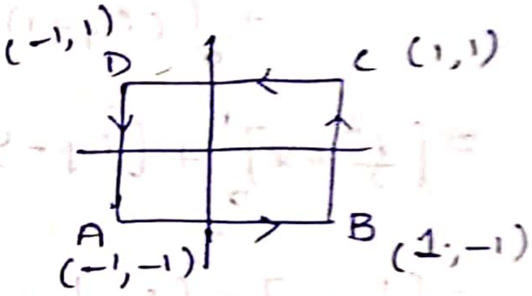
$= \frac{1}{\pi}[i+1].$

(ii) $\int_{0}^{\pi+2i} \cos\left(\frac{z}{2}\right)dz = 2\left[\sin\left(\frac{z}{2}\right)\right]_{0}^{\pi+2i} = 2\sin\left(\frac{\pi+2i}{2}\right)$

$= 2\sin\left(\frac{\pi}{2}+i\right) = 2\cos i$

(iii) $\int_{1}^{3} (z-2)^3 dz = \left[\frac{(z-2)^4}{4}\right]_{1}^{3} = \frac{1-1}{4} = 0.$

④ Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$, if 'c' is a square with vertices at $1 \pm i$ and $-1 \pm i$.



$1 \pm i \Rightarrow (1,1);\ (1,-1)$

$-1 \pm i \Rightarrow (-1,1);\ (-1,-1)$

$\int_c f(z)dz = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

on the line AB $\rightarrow y = -1;\ x \rightarrow -1$ to 1; $dy = 0$

on the line BC $\rightarrow x = 1;\ y \rightarrow -1$ to 1; $dx = 0$

on the line CD $\rightarrow y = 1;\ x \rightarrow 1$ to -1; $dy = 0$

on the line DA $\rightarrow x = -1;\ y \rightarrow 1$ to -1; $dx = 0.$

$f(z) = 3z^2 + iz - 4 = 3(x+iy)^2 + i(x+iy) - 4$

$3x^2 - 3y^2 + 6xiy + ix - y - 4$, $dz = dx + idy$

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$$\int_c f(z)dz = \int_{-1}^{1}(3x^2-3-6xi+ix+1-4)dx + \int_{-1}^{1}(3-3y^2+6iy+i-y-4)i\,dy + \int_{+1}^{-1}(3x^2-3+6xi+ix-1-4)dx + \int_{1}^{-1}(3-3y^2-6iy-i-y-4)i\,dy.$$

$$= \left[x^3-3x-3x^2i+\frac{ix^2}{2}-3x\right]_{-1}^{1} + \int_{-1}^{1}(-i-3iy^2-6y-1-iy)dy + \left[x^3-3x+3x^2i+\frac{ix^2}{2}-5x\right]_{1}^{-1} + \int_{+1}^{-1}(\cancel{+}3i-3iy^2+6y+1-iy-4i)dy$$

$$= \left[(1-3-3i+\frac{i}{2}-3)-(-1+3-3i+\frac{i}{2}+3)\right] + \left[-iy-iy^3-3y^2-y-\frac{iy^2}{2}\right]_{-1}^{1} + \left[(-1+3+3i+\frac{i}{2}+5)-(1-3+3i+\frac{i}{2}-5)\right] + \left[+3iy-iy^3-4iy+3y^2+y-\frac{iy^2}{2}\right]^{+1}$$

$$= \left[-5-\cancel{3i}+\cancel{\frac{i}{2}}-5+\cancel{3i}-\cancel{\frac{i}{2}}\right] + \left[(-i-i-3-1-\frac{i}{2}-(i+i-3+1-\frac{i}{2}))\right]$$

$$\left[7+\cancel{3i}+\cancel{\frac{i}{2}}+7-\cancel{3i}-\cancel{\frac{i}{2}}\right] + \left[(\cancel{+}3i+i+3\cancel{+}1-\frac{i}{2}+4i)-(3i\cancel{-}i+3-4i+1-\frac{i}{2})\right]$$

$$= -10+\left[-2i-4-\cancel{\frac{i}{2}}-2i+2+\cancel{\frac{i}{2}}\right]+14+\left[-8i+8-\cancel{\frac{i}{2}}-8i-4+\cancel{\frac{i}{2}}\right]$$

$$= -\cancel{10}-\cancel{4i}-\cancel{2}+\cancel{14}\,\cancel{-8i+2}+\cancel{4i}-\cancel{2}$$

$$= (\cancel{4-12i})\ 0.$$

Cauchy's theorem for double connected region:- 30/9/2019

→ If $f(z)$ is analytic in the region bounded by simple closed curves C_1 and C_2. Then, $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$

Cauchy's Integral Formula:-

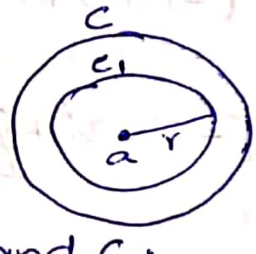
If $f(z)$ is analytic within and on a closed curve C and a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

Proof:-

Consider the function $\phi(z) = \frac{f(z)}{z-a}$ which is analytic at all points within C except at the points $z=a$.

With centre a and radius r, draw a small circle C_1 inside the circle C.



→ Given that $\phi(z)$ is analytic in the region connected by 'C'. $\phi(z)$ is also analytic in the region enclosed by C and C_1.

→ By cauchy's theorem for double connected region, WKT

$$\oint_C \phi(z)dz = \oint_{C_1}\phi(z)dz.$$

$$\therefore \oint_C \frac{f(z)}{z-a}dz = \oint_{C_1}\frac{f(z)}{z-a}dz \quad \text{—①}$$

→ eqn of the circle C_1 is $|z-a| = r \Rightarrow z-a = re^{i\theta}$

$z = a+re^{i\theta}$

$dz = rie^{i\theta}d\theta.$

$$① \Rightarrow \int_{C_1}\frac{f(z)}{z-a}dz = \int_0^{2\pi}\frac{f(a+re^{i\theta})}{\cancel{re^{i\theta}}}\cdot \cancel{r}i\cancel{e^{i\theta}}d\theta$$

$$= i\int_0^{2\pi} f(a+re^{i\theta})d\theta.$$

From the diagram, on the limiting from as c_1 shrinks to the point a, i.e., $r \to 0$

$$\therefore \int_{c_1} \frac{f(z)}{z-a} dz = i\int_0^{2\pi} f(a) d\theta = if(a)(\theta)_0^{2\pi} = 2\pi i f(a).$$

$$① \Rightarrow \oint_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\boxed{f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz}$$

Generalisation of Cauchy's integral formula:-

→ If $f(z)$ is analytic on and within a simple closed curve 'c' and if 'a' is any point within 'c', then

$$f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz.$$

Proof:-

→ WKT, by cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz$$

differentiate w.r.to 'a' on b.s

$$f'(a) = \frac{1}{2\pi i} \int_c \frac{-f(z)(-1)}{(z-a)^2} dz$$

$$f'(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^2} dz.$$

Again differentiate w.r.to 'a' on. bs.

$$f''(a) = \frac{1}{2\pi i} \int_c \frac{f(z)(-2)}{(z-a)^3}(-1) dz$$

$$= \frac{1}{2\pi i} \int_c \frac{2f(z)}{(z-a)^3} dz$$

$$f''(a) = \frac{2!}{2\pi i} \int_c \frac{f(z)}{(z-a)^3}\, dz.$$

$$\therefore\ f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

Problems:-

① Evaluate $\oint_c \frac{z^2+4}{z-3}\, dz$ where 'c' is the circle

(i) $|z| = 5$, (ii) $|z| = 2$ taken in anti-clockwise direction.

~~Given f(z)~~ Let $\frac{f(z)}{z-a} = \frac{z^2+4}{z-3}$

Here, $f(z)$ is analytic at everywhere except at $z=3$

(i) $|z| = 5$.

$|x+iy| = 5 \Rightarrow \sqrt{x^2+y^2} = 5 \Rightarrow x^2+y^2 = 5^2$

$c = (0,0),\ r = 5.$

$\Rightarrow a=3$ is inside the circle.

by cauchy's integral formula, $\int_c \frac{f(z)}{z-a}\, dz = 2\pi i f(a).$

$\int_c \frac{f(z)}{z-3}\, dz = 2\pi i f(3)$ $\qquad f(z) = z^2+4 \Rightarrow f(3) = 13.$

$= 2\pi i (13)$

$= 26\pi i.$

(ii) $|z| = 2 \Rightarrow |x+iy| = 2 \Rightarrow \sqrt{x^2+y^2} = 2 \Rightarrow x^2+y^2 = 2^2$

$c = (0,0),\ r = 2.$

$\Rightarrow a=3$ is outside the circle.

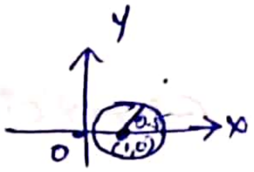
$\therefore$ By cauchy's integral theorem,

$$\int_c \frac{f(z)}{z-a}\, dz = 0.$$

② Evaluate using cauchy's theorem $\oint_C \frac{z^3 e^{-z}}{(z-1)^3} dz$, where 'c' is the circle $|z-1| = \frac{1}{2}$.

Here $f(z) = z^3 e^{-z}$ is an analytic function everywhere expect at $z=1$.

Given circle is $|z-1| = \frac{1}{2} \Rightarrow |x+iy-1| = \frac{1}{2}$



$$\sqrt{(x-1)^2+y^2} = \frac{1}{2} \Rightarrow (x-1)^2+y^2 = \left(\frac{1}{2}\right)^2$$

$$c = (1,0), \ r = \frac{1}{2} = 0.5$$

$\therefore$ by generalised cauchy's integral formula

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz.$$

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

Here $n=2 \Rightarrow \int_C \frac{z^3 e^{-z}}{(z-1)^{2+1}} dz = \frac{2\pi i}{2!} f''(a)$

$$= \frac{\cancel{2}\pi i}{\cancel{2}} (e^{-1})$$

$$= \pi i e^{-1}.$$

$f(z) = z^3 e^{-z}$

$f'(z) = -z^3 e^{-z} + 3z^2 e^{-z}$

$f''(z) = -(-e^{-z} z^3 + 3z^2 e^{-z}) + 3(-e^{-z} z^2 + 2z e^{-z})$

$f''(1) = -(-e^{-1} + 3e^{-1}) + 3(-e^{-1} + 2e^{-1})$

$= e^{-1} - 3\cancel{e^{-1}} - 3\cancel{e^{-1}} + \cancel{6}e^{-1}$

$= e^{-1}$.

③ Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$, where 'c' is the circle $|z| = 3$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$1 = A(z-2) + B(z-1) \Rightarrow$

$z=2 \Rightarrow 1 = B(1) \Rightarrow B = 1$

$z=1 \Rightarrow 1 = A(-1) \Rightarrow A = -1.$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

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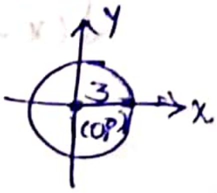
$$\therefore \oint_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \oint_C e^{2z}\left[\frac{-1}{z-1}+\frac{1}{z-2}\right]dz$$

$$= \oint_C \frac{-e^{2z}}{z-1}\,dz + \oint_C \frac{e^{2z}}{z-2}\,dz.$$

Given circle is $|z|=3 \Rightarrow |x+iy|=3 \Rightarrow \sqrt{x^2+y^2}=3$

$x^2+y^2=3^2$, $c=(0,0)$, $r=3$



Singular points are $z=1$ and $z=2$.

1,2 lies ~~b/w~~ inside the circle 'C'.

$\rightarrow$ by cauchy's integral formula.

$\rightarrow f(z) = -e^{2z}$

$f(1) = -e^2$

$= 2\pi i f(1) + 2\pi i f(2)$

$\rightarrow f(z) = e^{2z}$

$f(2) = e^4$.

$= 2\pi i(-e^2) + 2\pi i(e^4)$.

$= 2\pi i\,[e^4 - e^2]$.

④ Evaluate $\oint_C \frac{e^z}{z(1-z)^3}\,dz$, (i) 0 lies inside C. and 1 lies outside C.

ii) 1 lies inside C and 0 lies outside C.

iii) Both lies inside C.

(i) 0 lies inside 'C' and 1 lies outside C.

$$\int_C \frac{e^z}{z(1-z)^3}\,dz = \int_C \frac{\frac{e^z}{(1-z)^3}}{z}\,dz$$

Take $f(z) = \frac{e^z}{(1-z)^3}$

by cauchy's integral formula $\int_C \frac{f(z)}{z-a}\,dz = \int_C \frac{\frac{e^z}{(1-z)^3}}{z}\,dz$

$= 2\pi i f(0)$

$= 2\pi i(1)$

$= 2\pi i.$

ii) 1 lies inside c and 0 lies out side c.

Take $f(z) = \frac{-e^z}{z}$

by cauchy's generalised integral formula.

$$\int_c \frac{\frac{-e^z}{z}}{(1-z)^3} \qquad \int_c \frac{f(z)}{(z-a)^{n+1}} dz = \int_c \frac{\frac{-e^z}{z}}{(z-1)^3} dz$$

$$= \frac{2\pi i}{2!} f^{n}(a)$$

$$= \frac{2\pi i}{2!} f''(1)$$

$$= -\pi i e^1$$

$$= -\pi i e.$$

$$f(a) = \frac{-e^z}{z}$$

$$f'(a) = \frac{-ze^z-(-e^z)}{z^2}$$

$$= \frac{-ze^z+e^z}{z^2}$$

$$f''(a) = \frac{z^2(-ze^z-e^z+e^z) - (-ze^z+e^z)(2z)}{z^4}$$

$$= \frac{-z^3e^z - z^2e^z + z^2e^z + (ze^z - e^z)2z}{z^4}$$

$$= \frac{-z^3e^z + 2z^2e^z - 2ze^z}{z^4}$$

$$f''(1) = \frac{-e^1 + 2e - 2e}{z^4}$$

$$= -e^1$$

iii) both inside 'c'.

$$\frac{1}{z(1-z)^3} \qquad \int_c \frac{f(z)}{(z-a)} dz = \int_c \frac{-e^z}{z(z-1)^3}$$

$$\frac{1}{z(z-1)^3} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3}$$

$$1 = A(z-1)^3 + Bz(z-1)^2 + Cz(z-1) + Dz$$

$$1 = A(z^3-1-3z^2+3z) + Bz(z^2+1-2z) + C(z^2-z) + Dz$$

$$1 = A(z^3-1-3z^2+3z) + B(z^3+z-2z^2) + C(z^2-z) + Dz$$

comparing on b.s

$A+B=0$, $-3A-2B+C=0$,

$3A+B-C+D=0$, $-A=1$

$A=-1$

$A+B=0 \Rightarrow B=-A \Rightarrow B=1$

$-3A-2B+C=0 \Rightarrow -3(-1)-2B(1)+C=0 \Rightarrow 3-2+C=0$

$C=-1.$

$3A+B-C+D=0 \Rightarrow -3+1+1+B=0 \Rightarrow -3+2+D=0$

$D=1$

$$\frac{1}{z(z-1)^3} = \frac{-1}{z} + \frac{1}{z-1} + \frac{-1}{(z-1)^2} + \frac{1}{(z-1)^3}.$$

$-e^{-z}$

$$\int_C \frac{f(z)}{z(z-a)^3} = \int_C \frac{e^z}{z}dz + \int_C \frac{-e^z}{(z-1)}dz + \int_C \frac{e^{+z}}{(z-1)^2}dz + \int_C \frac{-e^{z}}{(z-1)^3}dz$$

From cauchy's generalised integral formula.

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \oint_C \frac{2\pi i}{n!} f^n(a).$$

For $\int_C \frac{e^z}{z}dz.$, we use cauchy's integral formula.

$$\Rightarrow \int_C \frac{e^z}{z}dz = 2\pi i f(0)$$

$$= 2\pi i(1) = 2\pi i$$

$f(z)=e^z \Rightarrow f(0)=e^0=1$

$$\Rightarrow \int_C \frac{-e^z}{(z-1)}dz = 2\pi i f(1)$$

$$= -2\pi i e^1.$$

$f(z) = -e^z \Rightarrow -e^1$

$$\Rightarrow \int_C \frac{e^{+z}}{(z-1)^2} = \frac{2\pi i}{n!} f^n(a)$$

$$= \frac{2\pi i}{1!} f'(a)$$

$$= 2\pi i e^1.$$

$f(a) = e^{+z}$

$f'(z) = +e^{+z}$

$f'(1) = +e^{+1}$

$$\Rightarrow \int_C \frac{-e^z}{(z-1)^3}dz = \frac{\cancel{2}\pi i}{\cancel{2!}} f''(a)$$

$f(z) = -e^z$

$f'(z) = -e^z$

$f''(z) = -e^z$

$f''(1) = -e^1$

$$= \pi i f''(1)$$

$$= \pi i(-e^1)$$

$$= -\pi i e^1.$$

$$\therefore \int_C \frac{f(z)}{z-a} = \int_C \frac{e^z}{z(z-a)^3} = 2\pi i - \cancel{2\pi i e^1} + \cancel{2\pi i e^1} - \pi i e^1$$

$$= \pi i(2 - e^1).$$

01/10/2019
Tuesday

(5) Evaluate $\int_C \frac{\log z}{(z-1)^3}dz$, where 'C' is the circle $|z-1| = \frac{1}{2}$

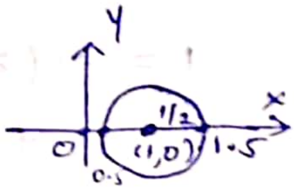
$f(z) = \log z$

Given circle is $|z-1| = \frac{1}{2} \Rightarrow |x+iy-1| = \frac{1}{2}$

$\sqrt{(x-1)^2+y^2} = \frac{1}{2} \Rightarrow (x-1)^2 + y^2 = \left(\frac{1}{2}\right)^2$

$C = (1,0)$, $r = \frac{1}{2}$

$r = 0.5$



Here, $f(z)$ is analytic everywhere except at $z=1$.

$z=1$ lies inside circle

$\therefore$ By cauchy's generalised integral formula.

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^n(a).$$

$f(z) = \log z$

$f'(z) = \frac{1}{z}$

$f''(z) = \frac{-1}{z^2}$

$f''(1) = \frac{-1}{1}$

$= -1$

$$\int_C \frac{\log z}{(z-1)^3} = \frac{\cancel{2}\pi i}{\cancel{2!}} f''(1)$$

$$= \pi i(-1)$$

$$\therefore \int_C \frac{\log z}{(z-1)^3}dz = -\pi i$$

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⑥ Evaluate $\int_C \frac{\cos\pi z^2}{(z-1)(z-2)^3}$, where C is the circle $|z|=3$.

$$\int_C \frac{f(z)}{(z-a)} = \int_C \frac{\cos\pi z^2}{(z-1)(z-2)^3}$$

$f(z) = \cos\pi z^2$.

$$\frac{1}{(z-1)(z-2)^3} = \frac{A}{z-1} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2} + \frac{D}{(z-2)^3}$$

$1 = A(z-2)^3 + B(z-1)(z-2)^2 + C(z-1)(z-2) + D(z-1)$

$z=2 \Rightarrow 1 = 0+0+0+D(2-1) \Rightarrow D=1$

$z=1 \Rightarrow 1 = A(1-2)^3+0+0+0 \Rightarrow 1 = A(-1) \Rightarrow A=-1.$

$1 = -1(z^3-8-6z^2-12z) + B(z-1)(z^2+4-4z) + C(z^2-2z-z+2) + z-1$

$1 = -z^3+8+6z^2+12z + B(z^3+4z-4z^2-z^2-4+4z) + C(z^2-3z+2)$

$1 = (-1+B)z^3 + (6-5B+C)z^2 + (12+4B+4B-3C)z + (8-4B+2C)$

$B-1=0 \Rightarrow B=1.$

$6-4B+C=0 \Rightarrow 6-5+C=0 \Rightarrow 1+C=0 \Rightarrow C=-1.$

$$\frac{1}{(z-1)(z-2)^3} = \frac{-1}{z-1} + \frac{1}{(z-2)} + \frac{-1}{(z-2)^2} + \frac{1}{(z-2)^3}$$

$$\int_C \frac{\cos\pi z^2}{(z-1)(z-2)^3}dz = \int_C \frac{-\cos\pi z^2}{(z-1)}dz + \int_C \frac{\cos\pi z^2}{z-2}dz - \int_C \frac{\cos\pi z^2}{(z-2)^2} + \int_C \frac{\cos\pi z^2}{(z-2)^3}$$

By cauchy's integral formual and generalised cauchy's integral formula.

$= -2\pi i f(1) + 2\pi i f(2) - \frac{2\pi i}{1!} f'(2) + \frac{2\pi i}{2!} f''(2).$

$= -2\pi i(-1) + 2\pi i(1) - 2\pi i(0) + \pi i(-16\pi^2)$

$= \cancel{2\pi i + 2\pi i - 16\pi i} = 2\pi i + 2\pi i - 16\pi^3 i$

$= \cancel{20\pi i} \; \cancel{-12\pi i} = 4\pi i - 16\pi^3 i$

⑦ Evaluate $\int_c \frac{z+4}{z^2+2z+5} dz$, where 'c' is the circle i) $|z| = 1$, ii) $|z+1-i| = 2$

iii) $|z+1+i| = 2$

$\int_c \frac{z+4}{(z^2+2z+5)} dz = \int_c \frac{z+4}{(z+1)^2+2^2} dz$

$= \int_c \frac{z+4}{[(z+1)+2i][(z+1)-2i]}$

$= \int_c \frac{z+4}{[z-(-1-2i)][z-(-1+2i)]}$

$\frac{\cancel{z+4} \; 1}{[z-(-1-2i)][z-(-1+2i)]} = \frac{A}{[z-(-1-2i)]} + \frac{B}{[z-(-1+2i)]}$

$1 \; \cancel{z+4} = A[z+1-2i] + B(z+1+2i)$

$z = -1+2i \Rightarrow +1 + \cancel{2i+4} = 0 + B(-1+2i+1+2i)$

$1 \; \cancel{3+2i} = B(4i).$

$\Rightarrow B = \frac{1}{4i}$

$z = -1-2i \Rightarrow 1 = A(-1-2i+1-2i) + 0;$

$A = -1/4i$

$f(z) = \cos\pi z^2$

$f'(z) = -2\pi z \sin\pi z^2$

$f''(z) = -4\pi^2 z^2 \cos\pi z^2 - 2\sin\pi z^2$

$f''(2) = -16\cos 4\pi - 2\sin\pi$

$= -16\pi^2 0 = -16.$

$f'(2) = -4\pi\sin 4\pi$

$= 0$

$f(2) = \cos 4\pi$

$= 1$

$f(1) = \cos\pi$

$= -1$

$\rightarrow f(z) = \cos\pi z^2$

$f(1) = \cos\pi = -1$

$f(2) = \cos 4\pi = 1$

$\rightarrow f'(z) = -2z\pi\sin\pi z^2$

$f'(2) = -2\pi z\sin 4\pi$

$= 0$

$\rightarrow f''(z) = -2\pi[\sin\pi z^2 + 2\pi z^2\cos\pi z^2]$

$f''(2) = 0 - 4\pi^2 z^2\cos\pi z^2$

$= 0 - 4\pi^2(4)\cos 4\pi$

$= -16\pi^2\cos 4\pi$

$= -16\pi^2$

$$\int_c \frac{z+4}{[z-(-1-2i)][z-(-1+2i)]}\,dz = \int_c \frac{-(z+4)\,dz}{4i[z-(-1-2i)]} + \int_c \frac{z+4}{4i[z-(-1+2i)]}$$

iii

i) $|z|=1 \Rightarrow |x+iy|=1 \Rightarrow x^2+y^2=1^2$

$(0,0),\ r=1$

The singular points are $(-1,-2),\ (-1,2)$

These two points lie outside the circle.

$\therefore$ By cauchy's integral theorem.

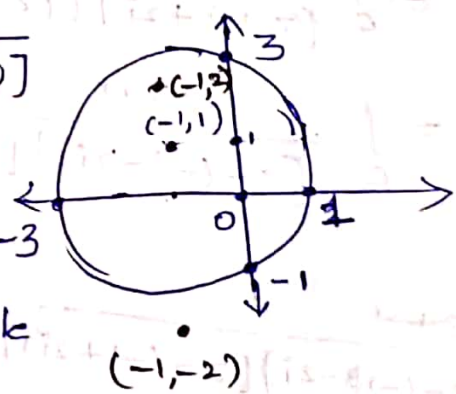
$$\int_c \frac{z+4}{z^2+2z+5} = 0.$$

ii) Given circle is $|z+1-i|=2 \Rightarrow |x+iy+1-i|=2$

$$|x+1+i(y-1)| = 2 \Rightarrow (x+1)^2+(y-1)^2=2^2.$$

$c=(-1,1),\ r=2.$

$$\int_c \frac{-(z+4)}{4i[z-(-1,-2i)]}\,dz + \int_c \frac{z+4}{4i[z-(-1+2i)]}$$



The singular points are

$(-1,-2),\ (-1,2)$

$\rightarrow$ The point $(-1,-2)$ lies outside circle and $(-1,2)$ lies inside circle.

$\therefore$ By cauchy's integral theorem

$$\int_c \frac{z+4}{z^2+2z+5}\,dz = 0 + \frac{1}{4i}\int_c \frac{z+4}{z-(-1+2i)}\,dz$$

$$= \frac{1}{\cancel{4i}_{\,2}}\,\cancel{2}\pi\cancel{i}\, f(-1,2).$$

$$= \frac{\pi}{2}(3+2i)$$

$f(z)=z+4$
$=x+iy+4$
$=-1+2i+4$
$=3+2i$

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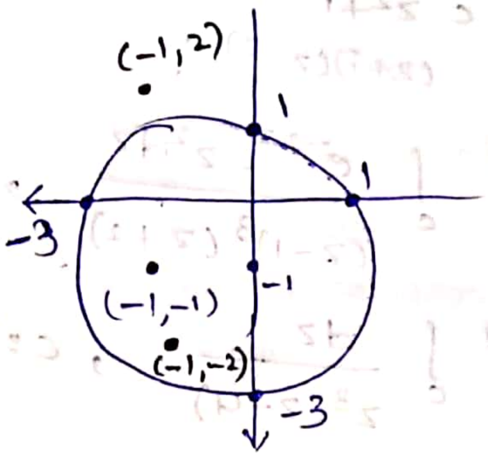
iii) $|z+1+i|=2 \Rightarrow |x+iy+1+i|=2 \Rightarrow |(x+1)+i(y+1)|=2$

$$(x+1)^2+(y+1)^2=2^2$$

$$c=(-1,-1),\ r=2.$$



Singular points are $(-1,-2),(-1,2)$

→ $(-1,-2)$ is inside circle and

$(-1,2)$ is outside the circle.

$$\int_c \frac{z+4}{z^2+2z+5}\,dz = 2\pi i f(-1,-2)+0$$

$$= \cancel{2}\pi i\left(\frac{-1}{\cancel{4i}_{2}}\right)(3-2i)$$

$$= \frac{-\pi}{2}(3-2i)$$

$f(z)=z+4$

$=x+iy+4.$

$=-1-2i+4$

$=3-2i$

⑧ $\int_c \frac{e^{2z}dz}{(z+1)^4}$, where c is the circle $|z-1|=3$.

→ By using generalised cauchy's integral formula,

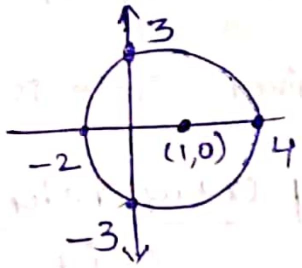
$f(z)=e^{2z}$, circle is $|z-1|=3 \Rightarrow |x+iy-1|=3$

$(x-1)^2+(y)^2=3^2$

$c=(1,0),\ r=3$

→ The $f(z)$ is analytic everywhere except at the point $z=-1$.

$z=-1$ lies inside the circle.



$$\int_c \frac{e^{2z}}{(z+1)^4}dz = \int_c \frac{e^{2z}}{[z-(-1)]^4}dz.$$

∴ By cauchy's generalised integral formula.

$$\int_c \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!}f'''(-1)$$

$$= \frac{2\pi i}{\cancel{6}_{3}}\times \cancel{8}^{4}e^{-2} = \frac{8\pi i}{3}e^{-2}.$$

$f(z)=e^{2z}$

$f'(z)=2e^{2z}$

$f''(z)=4e^{2z}$

$f'''(z)=8e^{2z}$

$f'''(-1)=8e^{-2}.$

(9) $\int_C \frac{z}{z^2+1}dz$, $C: |z+\frac{1}{2}|=2$.

$(z+i)(z-i)$

(10) $\int_C \frac{e^{-2z}\cdot z^2 dz}{(z-1)^3(z+2)}$, $C: |z+2|=1$

(11) $\int_C \frac{dz}{z^3(z+4)}$, $C: |z|=2$

(12) $\int_C ze^z dz$

$(x(x^2+y^2))^2$

(9) $\Rightarrow \int_C \frac{z}{z^2+1}dz = \int_C \frac{z}{(z+i)(z-i)}dz$

$\frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i} \Rightarrow 1 = A(z-i)+B(z+i)$

$z=i \Rightarrow 1 = 0 + B(2i) \Rightarrow B = \frac{1}{2i}$, $z=-i \Rightarrow 1 = A(-2i)+0$

$A = -\frac{1}{2i}$

$\int_C \frac{z}{(z+i)(z-i)}dz = \frac{-1}{2i}\int_C \frac{z}{z+i}dz + \frac{1}{2i}\int_C \frac{z}{z-i}dz$

Given circle is $|z+\frac{1}{z}|=2 \Rightarrow |x+iy+\frac{1}{x+iy}|=2$.

$\left|\frac{(x+iy)(x+iy)+1}{x+iy}\right| = 2 \Rightarrow \left|\frac{x^2+ixy+ixy-y^2+1}{x+iy}\right| = 2$.

$\left|\frac{x^2-y^2+2ixy}{x+iy} \times \frac{x-iy}{x-iy}\right| = 2 \Rightarrow \left|\frac{x^3-ix^2y+iy^3-xy^2+2ix^2y+2xy^2}{x^2-y^2}\right| = 2$

$= \left|\frac{x^3-xy^2+2xy^2+i(y^3+2x^2y-x^2y)}{x^2-y^2}\right| = 2$

$= \left|\frac{x^3+xy^2}{x^2-y^2} + \frac{i(y^3+x^2y)}{x^2-y^2}\right| = 2$

$$\left(\frac{x^3+xy^2}{x^2-y^2}\right)^2+\left(\frac{y^3+x^2y}{x^2-y^2}\right)^2=2^2.$$

$$\Rightarrow \frac{x^2(x^2+y^2)^2}{(x^2-y^2)^2}+\frac{y^2(x^2+y^2)^2}{(x^2-y^2)^2}=2.$$

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Complex Power Series

① **Sequence:-** A function defined on the set of natural numbers into the set of complex no.'s is called a sequence of complex numbers.

→ Denoted by $\{z_n\}$ or (z_n)

Here, z_n is called the n^{th} term of the sequence.

② **Series:-** Sum of the complex sequences is called series of complex numbers.

→ Let $z_1 + z_2 + z_3 + \cdots z_n + \cdots = \Sigma z_n$ is an infinite series of complex numbers. Here, $z_n = a_n + ib_n$.

where, a_n and b_n are real numbers

$\therefore \quad \Sigma z_n = (a_1 + ib_1) + (a_2 + ib_2) + \cdots$

If the series of real numbers $\Sigma a_n = a_1 + a_2 + \cdots$ and $\Sigma b_n = b_1 + b_2 + b_3 + \cdots$ both converges to the limit l_1 and l_2. Then, we say that Σz_n is converges to $l_1 + il_2$.

③ **Absolute convergent:-** If Σz_n is a alternating series, ~~of~~ then Σz_n and $\Sigma |z_n|$ are convergent. Then Σz_n is absolute convergent.

④ **Complex power Series:-** A power series is defined as

$$\sum_{n=0}^{\infty} a_n (z-b)^n = a_0 + a_1(z-b) + a_2(z-b)^2 + \cdots + a_n(z-b)^n + \cdots$$

where, b is any fixed point in the complex plane [b is also called centre of power series] and $a_0, a_1, \ldots a_n$ are called coefficients of power series.

→ power series is convergent for $|z-b|<R$ and is divergent for $|z-b|>R$.

→ If $|z-b|=R$, then it denotes a circle. This circle is called a circle of convergence of the power series and R is called radius of convergence of power series

⇒ $\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$ is called a power series whose centre is at the origin.

(5) <u>Taylor's Series</u>:- If $f(z)$ is analytic inside a circle 'c' with centre a, then $\forall z$ inside C, ~~f(z)~~

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots \frac{(z-a)^n f^n(a)}{n!} + \cdots$$

is called taylor's series.

→ put $a=0$ in taylor's series, then it is called <u>Maclaurin's series</u> i.e.,

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots + \frac{z^n}{n!} f^n(a) + \cdots$$

<u>Note</u>:-

1) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\forall z, |z|<\infty$

2) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ for $|z|<\infty$.

3) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2n!}$

4) $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

5) $\cosh z = 1 + z^2/2! + z^4/4! + \cdots = \sum_{n=0}^{\infty} z^{2n}/2n!$

6) $\dfrac{1}{1+z} = (1+z)^{-1} = 1 - z + \dfrac{z^2}{\cancel{2!}} - \dfrac{z^3}{\cancel{3!}} + \cdots = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^n}{\cancel{n!}} \quad \forall\ |z|<1$

7) $\dfrac{1}{1-z} = (1-z)^{-1} = 1 + z + \dfrac{z^2}{\cancel{2!}} + \dfrac{z^3}{\cancel{3!}} + \cdots = \sum_{n=0}^{\infty} \dfrac{z^n}{\cancel{n!}} \quad \forall\ |z|<1$

problems:-

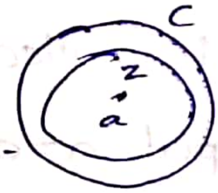
① Obtain a taylor's series expansion of e^z above the point $z=1$.

Let $f(z)=e^z$.

At $z=1$, $f(z)$ is analytic.

The point $z=0$ is the only singular point and is at a distance of 1 unit from $z=1$. Hence, the taylor's series expansion around $z=1$ is valid in ~~$|z-1|<1$~~ $|z-1|<1$.

∴ By taylor's series,



$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

$f(z) = e^z$

about $z=1$

$z=a$

$\Rightarrow a=1$

$f'(z) = f''(z) = e^z$

$f(1) = f'(1) = f''(1)$

$\cdots\cdots = e.$

$$e^z = e + (z-1)e^{a} + \frac{(z-1)^2}{2!} e^{a} + \cdots$$

$$e^z = e\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right]$$

$$e^z = e + e\left[(z-1) + \frac{(z-1)^2}{2!} + \cdots\right]$$

$$e^z = e + e \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}.$$

(or)

Given $f(z) = e^z$, about $z=1$

Take $w = z-1 \Rightarrow z = w+1$

$$e^z = e^{w+1} = e\,e^{w} = e[e^{w}] = e\left[1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right]$$

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$$= e\left[1+(z-1)+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\cdots\right]$$

$$= e + e\sum_{n=1}^{\infty}\frac{(z-1)^n}{n!}$$

② Using taylor's, expand $f(z) = \frac{1}{z^2}$ i) in power of $z+1$ ii) in power of $z-2$. State the region of validity of the expansion in each case.

Sol:- Given $f(z) = \frac{1}{z^2}$

$z=0$ has its singular point.

i) To expand in powers of $z+1$. Put $z+1 = \omega \Rightarrow z = \omega - 1$

$$\therefore f(z) = \frac{1}{z^2} = \frac{1}{(\omega-1)^2} = \frac{1}{(1-\omega)^2} = (1-\omega)^{-2}.$$

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots\ldots \quad |x|<1.$$

$$(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots\ldots \quad |x|<1.$$

$$f(z) = (1-\omega)^{-2} = 1+2\omega+3\omega^2+4\omega^3\cdots\ldots \quad |\omega|<1$$

$$f(z) = \frac{1}{z^2} = 1+2(z+1)+3(z+1)^2+4(z+1)^3+\cdots\ldots \quad |z+1|<1$$

ii) put $z-2 = \omega \Rightarrow z = \omega+2$

$$\therefore f(z) = \frac{1}{z^2} = \frac{1}{(\omega+2)^2} = \frac{1}{4(1+\frac{\omega}{2})^2} = \frac{1}{4}(1+\frac{\omega}{2})^{-2}.$$

$$f(z) = \frac{1}{4}(1+\frac{\omega}{2})^{-2} = 1-2(\frac{\omega}{2})+3(\frac{\omega}{2})^2-4(\frac{\omega}{2})^3\cdots\ldots \quad |\frac{\omega}{2}|<1$$

$$f(z) = \frac{1}{z^2} = 1-(z-2)+3(\frac{z-2}{2})^2-4(\frac{z-2}{2})^3\cdots\ldots \quad |\frac{z-2}{2}|<1$$

③ Using taylor's series, expand $f(z) = \sinh z$ about $z = \pi i$

Given $f(z) = \sinh z$ about $z = \pi i$

Let $\omega = z - \pi i \Rightarrow z = \omega + \pi i$

$\therefore f(z) = \sinh z = \sinh(\omega + \pi i)$

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$= \sinh\omega \cosh\pi i + \cosh\omega \sinh\pi i$

$= -\sinh\omega + 0$

$\therefore f(z) = -\sinh\omega.$

$f(z) = -\left[\omega + \frac{\omega^3}{3!} + \frac{\omega^5}{5!} + \cdots\right].$

$f(z) = -\left[z - \pi i + \frac{(z-\pi i)^3}{3!} + \frac{(z-\pi i)^5}{5!} + \cdots\right].$

④ Find the taylor's series expansion of $\cosh z$ about $z = \pi i$

Given $f(z) = \cosh z$ about $z = \pi i$

Let $\omega = z - \pi i \Rightarrow z = \omega + \pi i$

$\therefore f(z) = \cosh z = \cosh(\omega + \pi i) = \cosh\omega \cosh\pi i - \sinh\omega \sinh\pi i$

$= -\cosh\omega - 0$

$\therefore f(z) = -\cosh\omega$

$f(z) = -\left[1 + \frac{\omega^2}{2!} + \frac{\omega^4}{4!} + \cdots\right]$

$f(z) = -\left[1 + \frac{(z-\pi i)^2}{2!} + \frac{(z-\pi i)^4}{4!} + \cdots\right].$

⑤ Expand $f(z) = \sin z$ in taylor's series about $z = \frac{\pi}{4}$.

Given $f(z) = \sin z$ about $z = \frac{\pi}{4}$

Let $\omega = z - \frac{\pi}{4} \Rightarrow z = \omega + \frac{\pi}{4}$

$\therefore f(z) = \sin(\omega + \frac{\pi}{4}) = \sin\omega \cos\frac{\pi}{4} + \cos\omega \sin\frac{\pi}{4}$

$\therefore f(z) = \frac{\sin\omega}{\sqrt{2}} + \frac{\cos\omega}{\sqrt{2}}$

$f(z) = \frac{1}{\sqrt{2}}\left[\omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \cdots\right] + \frac{1}{\sqrt{2}}\left[1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \frac{\omega^6}{6!} + \cdots\right]$

$= \frac{1}{\sqrt{2}}\left[1 + \omega - \frac{\omega^2}{2!} - \frac{\omega^3}{3!} + \frac{\omega^4}{4!} + \frac{\omega^5}{5!} - \frac{\omega^6}{6!} - \cdots\right]$

$= \frac{1}{\sqrt{2}}\left[1 + (z - \frac{\pi}{4}) - \frac{(z-\frac{\pi}{4})^2}{2!} - \frac{(z-\frac{\pi}{4})^3}{3!} + \frac{(z-\frac{\pi}{4})^4}{4!} + \cdots\right]$

⑥ $f(z) = \frac{1}{(z-1)(z-2)}$, obtain the taylor's series expansion of $f(z)$ about $z=0$.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \Rightarrow 1 = A(z-2) + B(z-1)$$

put $z=2 \Rightarrow 1 = B$, put $z=1 \Rightarrow 1 = -A \Rightarrow A = -1$.

$$\therefore f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{1}{1-z} + \frac{-1}{2(1-\frac{z}{2})}$$

$$= (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= (1+z+z^2+z^3+\cdots) - \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right],$$

where $|z|<1$ \$ $|z/2|<1$.

⑦ Obtain the taylor's series expansion of $f(z) = \frac{1}{z^2-z-6}$
i) about $z=-1$, ii) about $z=1$

i) about $z=-1 \Rightarrow \omega = z+1 \Rightarrow z = \omega - 1$.

$$f(z) = \frac{1}{z^2-z-6} = \frac{1}{z^2-3z+2z-6} = \frac{1}{(z+2)(z-3)}$$

$$\frac{1}{(z+2)(z-3)} = \frac{A}{z+2} + \frac{B}{z-3} \Rightarrow 1 = A(z-3) + B(z+2)$$

$$A = \frac{1}{5}, \; B = \frac{-1}{5}.$$

$$\therefore f(z) = \frac{1}{(z-3)(z+2)} = \frac{1}{5(z-3)} - \frac{1}{5(z+2)}$$

$$= \frac{1}{5(\omega-1-3)} - \frac{1}{5(\omega-1+2)} = \frac{1}{5(\omega-4)} - \frac{1}{5(\omega+1)}$$

$$= \frac{-1}{5(4)(1-\frac{\omega}{4})} - \frac{1}{5(1+\omega)} = \frac{-1}{20}\left(1-\frac{\omega}{4}\right)^{-1} - \frac{1}{5}(1+\omega)^{-1}$$

$$= \frac{-1}{20}\left[1+\left(\frac{\omega}{4}\right)^2+\left(\frac{\omega}{4}\right)^3+\cdots\right] - \frac{1}{5}\left[1-\omega+\omega^2-\omega^3+\cdots\right]$$

$$-\frac{1}{20}\left[1+\left(\frac{z+1}{4}\right)^2+\left(\frac{z+1}{4}\right)^3+\cdots\right]-\frac{1}{5}\left[1-(z+1)+(z+1)^2-(z+1)^3+\cdots\right].$$

ii) about $z=1 \Rightarrow$ Let $\omega = z-1$

$z = \omega + 1$

$$\therefore f(z)=\frac{1}{z^2-z-6}=\frac{1}{5(z-3)}-\frac{1}{5(z+2)}$$

$$=\frac{1}{5(\omega+1-3)}-\frac{1}{5(\omega+1+2)}=\frac{1}{5(\omega-2)}-\frac{1}{5(\omega+3)}$$

$$=\frac{-1}{5(2)(1-\frac{\omega}{2})}-\frac{1}{5(3)(1+\frac{\omega}{3})}$$

$$=-\frac{1}{10}\left(1-\frac{\omega}{2}\right)^{-1}-\frac{1}{15}\left(1+\frac{\omega}{3}\right)^{-1}$$

$$=-\frac{1}{10}\left[1+\frac{\omega}{2}+\left(\frac{\omega}{2}\right)^2+\left(\frac{\omega}{2}\right)^3+\cdots\right]-\frac{1}{15}\left[1-\frac{\omega}{3}+\left(\frac{\omega}{3}\right)^2-\left(\frac{\omega}{3}\right)^3+\cdots\right]$$

$$=-\frac{1}{10}\left[1+\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2+\left(\frac{z-1}{2}\right)^3+\cdots\right]-\frac{1}{15}\left[1-\frac{(z-1)}{3}+\left(\frac{z-1}{3}\right)^2-\left(\frac{z-1}{3}\right)^3+\cdots\right].$$

(8) Expand $\log z$ by taylor's series about $z=1$.

$f(z)=\log z$, $f'(z)=\frac{1}{z}$, $f''(z)=\frac{-1}{z^2}$, $f'''(z)=\frac{2}{z^3}$ ---

T.S for $f(z) = f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\cdots$

about $z=1 \Rightarrow a=1$

$f(1)=0$, $f'(1)=1$, $f''(1)=-1$, $f'''(1)=2$

$$\therefore f(z)=f(1)+(z-1)f'(1)+\frac{(z-1)^2}{2!}f''(1)+\frac{(z-1)^3}{3!}f'''(1)+\cdots$$

$$f(z)=(z-1)-\frac{(z-1)^2}{2}+\frac{2(z-1)^3}{3}+\cdots\cdots$$

⑨ $f(z) = \frac{z-1}{z^2}$ about $z=1$ using T.S

Let $z-1=\omega \Rightarrow z=1+\omega$

$\therefore f(z) = \frac{z-1}{z^2} = \frac{\omega}{(1+\omega)^2} \Rightarrow \omega(1+\omega)^{-2}$

$f(z) = \omega(1-2\omega+3\omega^2-4\omega^3+ - - - -)$

$= z-1[1-2(z-1)+3(z-1)^2 \mp 4(z-1)^3 + - - -]$

$f(z) = (z-1)-2(z-1)^2+3(z-1)^3-4(z-1)^4+ - - - |z|<1$

$f(z) = \sum_{n=1}^{\infty} (-1)^{n+1} n(z-1)^n, \quad |z-1|<1.$

25|10|19

⑩ obtain the taylor series expansion of the $\frac{z^2-1}{(z+2)(z+3)}$, $|z|<2$.

$f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{z^2+5z+6}$

$z^2+5z+6) z^2-1 (1$
$\quad -z^2+5z+6$
$\quad -5z-7$

$= 1 - \frac{5z+7}{z^2+5z+6}$ $\qquad = 1 + \frac{(-5z-7)}{z^2+5z+6}.$

$= 1 - \frac{5z+7}{(z+2)(z+3)}$

$\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$5z+7 = A(z+3)+B(z+2)$

$z=-3 \Rightarrow -15+7 = -B$
$-8=-B$
$B=8$

$z=-2 \Rightarrow -10+7 = A(1)$
$A=-3.$

$= 1 - \left[\frac{-3}{z+2} + \frac{8}{z+3}\right]$

$= 1 + \frac{3}{z+2} - \frac{8}{z+3}.$

Given $|z|<2 \Rightarrow \frac{|z|}{2}<1$

$f(z) = 1 + \frac{3}{2(1+\frac{z}{2})} - \frac{8}{3(1+\frac{z}{3})} = 1 + \frac{3}{2}(1+\frac{z}{2})^{-1} - \frac{8}{3}(1+\frac{z}{3})^{-1}$

$= 1 + \frac{3}{2}[1-\frac{z}{2}+(\frac{z}{2})^{2}-(\frac{z}{2})^3+ - - -] - \frac{8}{3}[1-\frac{z}{3}+(\frac{z}{3})^2-(\frac{z}{3})^3+ - -]$

$|\frac{z}{2}|<1$ and $|\frac{z}{3}|<1.$