inductor or voltage ...

10.5 Initial Conditions in Elements

Let us study the effect of switching action on basic passive elements such as resistor, inductor and capacitor.

10.5.1 Resistor

For a resistor having value R, the relation between applied voltage and resulting current is given by the equation,

$$V = i \cdot R$$

Above equation is linear and also time independent. This indicates that the current through resistor changes instantaneously if applied voltage changes instantaneously. Thus, in resistor, change in current is instantaneous as there is no storage of energy in it.

10.5.2 Inductor

The relation between current flowing through inductor and voltage across it is given by,

$$v_L = L \frac{d i_L}{dt}$$

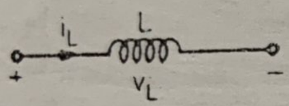

Fig. 10.1

If d.c. current flows through inductor, $\frac{d i_L}{dt}$ becomes zero as d.c. current is constant with respect to time. Hence voltage across inductor, v_L, becomes zero. Thus, as far as d.c. quantities are considered, **in steady state, inductor acts as a short circuit.**

We can express inductor current interms of voltage developed across it as

$$i_L = \frac{1}{L} \int v_L \, dt$$

In above equation, the limits of integration are decided by considering past history. i e. from $-\infty$ to $t\,(0^-)$.

Assuming that switching takes place at $t = 0$, we can split limits into two intervals as $-\infty$ to 0^- and 0^- to t. We have already studied that 0^- is the instant just before switching action takes place, while 0^+ is the instant just after switching action takes place. Hence we can write

$$i_L = \frac{1}{L}\int_{-\infty}^{t} v_L \, dt$$

$$\therefore \quad i_L = \frac{1}{L}\int_{-\infty}^{0^-} v_L \, dt + \frac{1}{L}\int_{0^-}^{t} v_L \, dt$$

First term on RHS of equation represents value of inductor current in history period which is nothing but initial condition of i_L. Let it be denoted by $i_L(0^-)$.

$$\therefore \quad i_L = i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} v_L \, dt$$

At $t = 0^+$, we can write,

$$i_L(0^+) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{0^+} v_L \, dt$$

Initially we have assumed that switch acts in zero time. Thus, integration from 0^- to 0^+ is zero.

$$\therefore \quad i_L(0^+) = i_L(0^-)$$

Thus, **current through inductor cannot change instantaneously.** That means the current through inductor, before and after the switching action is same.

At the time of switching, the voltage across inductor is ideally ∞ as time interval dt is zero. Thus, at the time of switching inductor acts as a open circuit. While in steady state at $t = \infty$ it acts as short circuit as given in Table 10.1.

If inductor carries an initial current I_0 before switching action, then at instant $t = 0^+$ it acts as a constant current source of value I_0, while in steady state at $t = \infty$, it acts as a short circuit across a current source as given in Table 10.1.

10.5.3 Capacitor

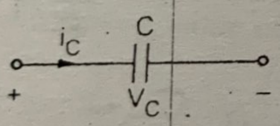

Fig. 10.2

The relationship between current through capacitor and voltage across it is given by,

$$i_C = C\frac{dv_C}{dt}$$

If d.c. voltage is applied to capacitor, $\frac{dv_C}{dt}$ becomes zero as d.c. voltage is constant with respect to time.

Hence current through capacitor, i_C, becomes zero. Thus, as far as d.c. quantities are considered capacitor acts as a open circuit.

We can express voltage across capacitor interms of current flowing through it as

$$v_C = \frac{1}{C}\int i_C \, dt.$$

We can write limits of integration by considering history period:

$$\therefore \quad v_C = \frac{1}{C}\int_{-\infty}^{t} i_C \, dt$$

Splitting limits of integration,

$$\therefore \quad v_C = \frac{1}{C}\int_{-\infty}^{0^-} i_C \, dt + \frac{1}{C}\int_{0^-}^{t} i_C \, dt$$

The first term on RHS of above equation represents initial voltage on capacitor. Let it be denoted by $v(0^-)$.

At $t(0^+)$, equation is given by

$$v_C(0^+) = v_C(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_C \, dt$$

As switch acts in zero time, the integration from 0^- to 0^+ is zero.

$$\therefore \quad v_C(0^+) = v_C(0^-)$$

Element	Behaviour immediately after excitiation is given $t = 0^+$ instant	Behaviour as $t \to \infty$ i.e. steady state
R	R	R
L	O.C.	S.C.
L with I_0	I_0	S.C.
C	S.C.	O.C.
C with V_0 (+ −)	V_0 (+ −)	V_0 (+ −) O.C.

Table 10.1

10.6 D.C. Excitation to Series R-L Circuit

Now we will study the behaviour of series R-L circuit for D.C. excitation. Here we will be having two conditions namely circuit with source which is called driven circuit and other as circuit without source which is called undriven circuit or source free circuit. We will consider both the circuits one by one.

10.6.1 D.C. Response of R-L Series Circuit

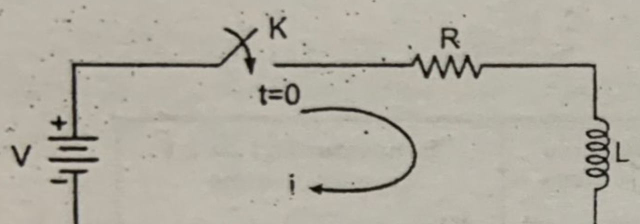

Fig. 10.3

Consider a series R-L circuit as shown in Fig. 10.3.

At $t = 0^-$, switch K is about to close but not fully closed. As voltage is not applied to the circuit, current in the circuit will be zero.

$$\therefore \quad i_L(0^-) = 0$$

In the section 10.5.2 we have studied that current through inductor cannot change instantaneously.

$$\therefore \quad i_L(0^+) = 0$$

Let initial current through inductor be represented as I_0. In above case, I_0 is zero.

Assume that switch K is closed at $t = 0$.

This results in application of d.c. voltage to the series R-L circuit for all $t \geq 0^+$.

Applying KVL,

$$V = i \cdot R + L\frac{di}{dt} \qquad \text{... (1)}$$

$$\therefore \quad \frac{V}{R} = i + \frac{L}{R}\frac{di}{dt}$$

$$\therefore \quad \frac{V}{R} - i = \frac{L}{R}\frac{di}{dt}$$

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Rearranging terms in above equation,

$V = iR + L\cdot\frac{di}{dt}$

$$\frac{R}{L}dt = \frac{di}{\frac{V}{R}-i}$$

$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$

Integrating both sides with respect to corresponding variables,

$$\therefore \quad \frac{R}{L}t = -ln\left[\frac{V}{R}-i\right]+K' \qquad \dots (2)$$

$\frac{dx}{dt} + Px = k$

where K' is constant of integration.

$x = e^{-Pt}\int k e^{Pt} + ce^{-Pt}$

To find K' : At $t = 0$, $i = I_0 = 0$;

Substituting the values in equation (2)

$i = e^{-R/L t}\int \frac{V}{L}\cdot e^{R/L t} + c\cdot e^{-R/L t}$

$$\therefore \quad \frac{R}{L}(0) = -ln\left[\frac{V}{R}-0\right]+K'$$

$$\therefore \quad ln\left[\frac{V}{R}\right] = K' \qquad \dots (3)$$

Substituting value of K' from equation (3) in the equation (2),

$i = e^{-R/L t}\cdot e^{R/L t}\cdot\frac{\cancel{L}}{R}\cdot\frac{V}{\cancel{L}} + c e^{-R/L t}$

$$\therefore \quad \frac{R}{L}t = -ln\left[\frac{V}{R}-i\right]+ln\left[\frac{V}{R}\right]$$

$$\therefore \quad \frac{R}{L}t = ln\left[\frac{\frac{V}{R}}{\frac{V}{R}-i}\right]$$

$= \frac{V}{R} + ce^{-R/L t}$

Taking antilog,

$i = 0, \; t = 0$

$$\therefore \quad e^{\frac{R}{L}t} = \frac{\frac{V}{R}}{\frac{V}{R}-i}$$

$0 = \frac{V}{R} + C$

Rearranging the terms,

$$\frac{V}{R}-i = \frac{V}{R}\cdot e^{-\frac{R}{L}t}$$

$C = -\frac{V}{R}$

$$\therefore \quad i = \frac{V}{R}-\frac{V}{R}e^{-\frac{R}{L}t}$$

$i = \frac{V}{R} - \frac{V}{R}e^{-R/L t}$

$$\therefore \quad i = \frac{V}{R}\left(1-e^{-\frac{R}{L}t}\right)A \qquad \dots (4)$$

$= V/R(1 - e^{-R/L t})$

In above equation, term $\frac{V}{R}$ is the steady state current and the term $\frac{-V}{R}\cdot e^{-\frac{R}{L}t}$ is the transient part of the solution of current.

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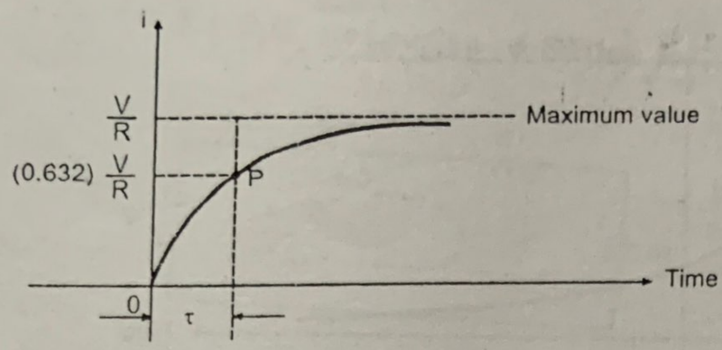

Fig. 10.4

Fig. 10.4 shows variation of current i with respect to time.

From Fig. 10.4 it is clear that current increases exponentially with respect to time. The rising current produces rising flux, which induces e.m.f. in the coil. According to the Lenz's law, the self induced e.m.f. opposes the flow of current. Because of this induced e.m.f. and its opposition, the current in the coil do not reach to its maximum value instantaneously.

The point P shown on the graph in Fig. 10.4 denotes that current in the circuit rises to 0.632 times maximum value of current in steady state. The time required for the current to rise to the 0.632 of its final value is known as time constant of given R-L circuit. The time constant is denoted by τ. Thus for series R-L circuit, time constant is

$$\tau = \frac{L}{R} \text{ sec}$$

The initial rate of rise of current is large upto first time constant. At later stage, the rate of rise of current reduces. Theoretically the current reaches to its maximum value after infinite time.

Voltage across inductor L is given by

$$V_L = L\frac{di}{dt}$$

Substituting value of i from equation (4),

$$\therefore \quad V_L = L\frac{d}{dt}\left[\frac{V}{R}\left(1-e^{-\frac{R}{L}t}\right)\right]$$

$$\therefore \quad V_L = L\left\{\frac{d}{dt}\left(\frac{V}{R}\right)-\frac{d}{dt}\left(\frac{V}{R}\cdot e^{-\frac{R}{L}t}\right)\right\}$$

$$\therefore \quad V_L = L\left\{0-\left(\frac{V}{R}\right)\left(-\frac{R}{L}\right)\left(e^{-\frac{R}{L}t}\right)\right\}$$

$$\therefore \quad V_L = V e^{-\frac{R}{L}t} \text{ volts} \qquad \ldots (5)$$

Consider series R-L circuit shown in Fig. 10.3.

Let us assume that inductor carries initial current I_0 before switching action. Then expression of current flowing through inductor is given by

$$i = \frac{V}{R} - \left(\frac{V}{R} - I_0\right) \cdot e^{-\frac{R}{L}t} \text{ A} \qquad \dots (6)$$

The variation of current i with respect to time is as shown in Fig. 10.4 (a), where current increases from initial current I_0 and not from origin.

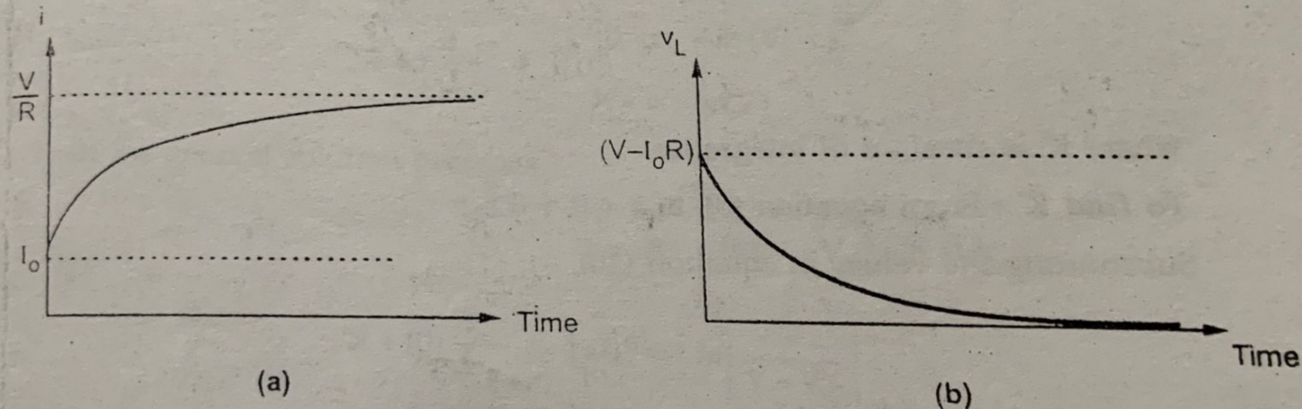

Fig. 10.4

Similarly the expression of voltage generated across inductor is given by,

$$v_L = (V - I_0 \cdot R) \cdot e^{-\frac{R}{L}t} \text{ volts} \qquad \dots (7)$$

The variation of v_L with respect to time is as shown in Fig. 10.4 (b).

The above response is called zero-state response. Because it is a response to a non-zero input to a circuit with zero initial conditions. Also this circuit is called 'driven circuit' because it is driven by voltage source of V volts.

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10.6.2 Current Decay in Series R-L Circuit

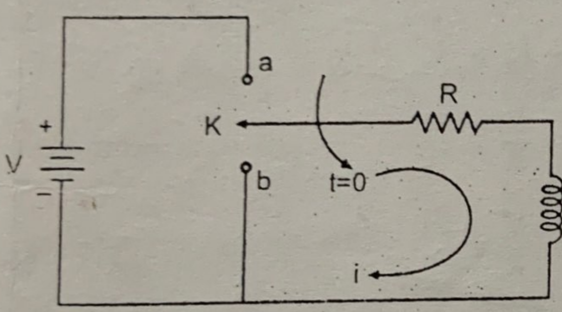

Fig. 10.5

Consider network shown in Fig. 10.5.

At $t = 0^-$, switch K is at position 'a'.

Assume that switch K is kept at position 'a' for very long time. Thus, the network is in steady state. Initial current through inductor is given as,

$$\therefore \; i_L(0^-) = I_0 = \frac{V}{R} = i_L(0^+) \qquad \text{... (8)}$$

Because current through inductor can not change instantaneously.

Assume that at $t = 0$, switch K is moved to position 'b'.

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Applying KVL,

$$L\frac{di}{dt} + i \cdot R = 0 \qquad \text{... (9)}$$

$$\therefore \quad L\frac{di}{dt} = -i \cdot R$$

Rearranging the terms in above equation,

$$\frac{di}{i} = -\frac{R}{L}\,dt$$

Integrating both sides with respect to corresponding variables,

$$\therefore \quad ln[i] = -\frac{R}{L}t + K' \qquad \text{... (10)}$$

where K' is constant of integration.

To find K' : From equation (8), at $t = 0$, $i = I_0$;

Substituting the values in equation (10),

$$\therefore \quad ln[I_0] = -\frac{R}{L}(0) + K'$$

$$K' = ln[I_0] \qquad \text{... (11)}$$

Substituting value of K' from equation (11) in equation (10),

$$\therefore \quad ln[i] = -\frac{R}{L}t + ln[I_0]$$

$$\therefore \quad ln[i] - ln[I_0] = -\frac{R}{L}t$$

$$\therefore \quad ln\left[\frac{i}{I_0}\right] = -\frac{R}{L}t$$

$$\therefore \quad \frac{i}{I_0} = e^{-\frac{R}{L}t}$$

$$\therefore \quad i = I_0 \cdot e^{-\frac{R}{L}t}$$

But $$I_0 = \frac{V}{R} \qquad \text{..... From equation (8)}$$

$$\therefore \quad \underline{\underline{i = \frac{V}{R} \cdot e^{-\frac{R}{L}t}}} \qquad \text{... (12)}$$

Fig. 10.6 shows variation of current i with respect to time.

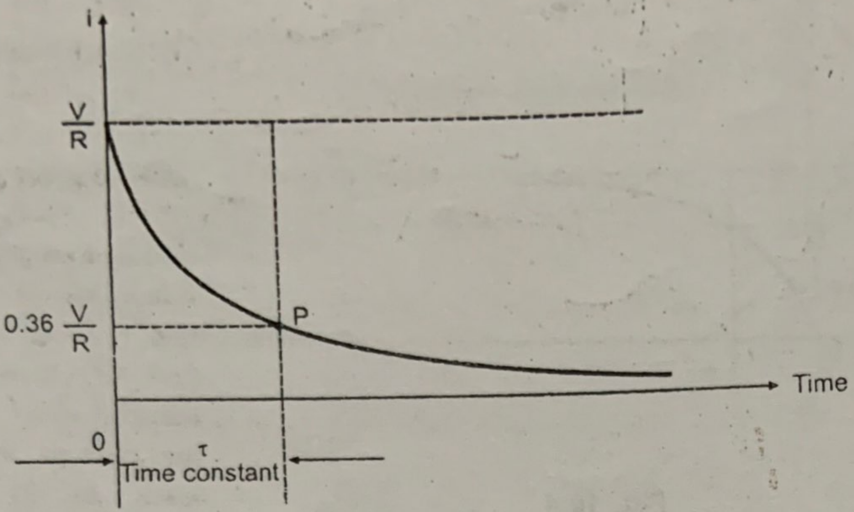

Fig. 10.6

From the graph, it is clear that the current is exponentially decaying. At point P on the graph, the current value is (0.368) times its maximum value. The characteristics of decay is determined by values R and L which are two parameters of network.

The voltage across inductor is given by

$$v_L = L\frac{di}{dt} = L\frac{d}{dt}\left[I_0 \cdot e^{-\frac{R}{L}t}\right] = L \cdot I_0 \cdot \left(\frac{-R}{L}\right) \cdot e^{\frac{-R}{L}t}$$

$$\therefore \quad v_L = -I_0 \cdot R \cdot e^{\frac{-R}{L}t}$$

But $\quad I_0 \cdot R = V$

$$\therefore \quad v_L = -V \cdot e^{-\frac{R}{L}t} \text{ volts} \qquad \dots (13)$$

The variation of v_L with respect to time is as shown in Fig. 10.7.

For physical interpretation of the result, we must consider energy in network. Before the switch is moved to position 'b', the energy stored in inductor $\frac{1}{2}Li^2$ and energy dissipated by resistor is i^2R. After switching, the voltage source is removed and energy stored in the inductor is totally dissipated by resistor as time progresses. Since the energy is dissipated at a maximum rate after switching, the current decreases most rapidly at that time just after switch is closed.

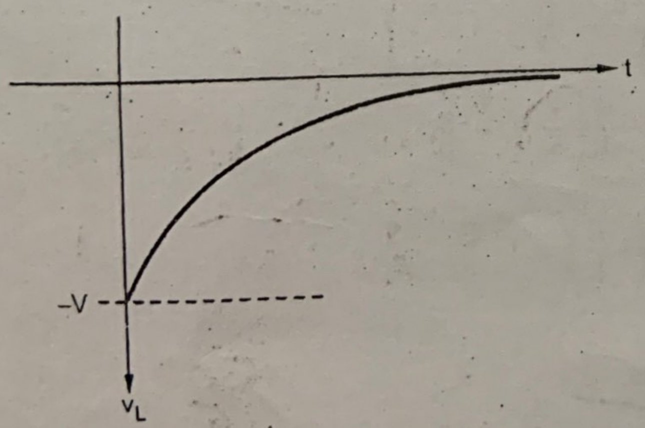

Fig. 10.7

10.7 D.C. Excitation to Series R-C Circuits

In this section, we will analyse series R-C circuit with D.C. excitation.

10.7.1 D.C. Response of Series R-C Circuit

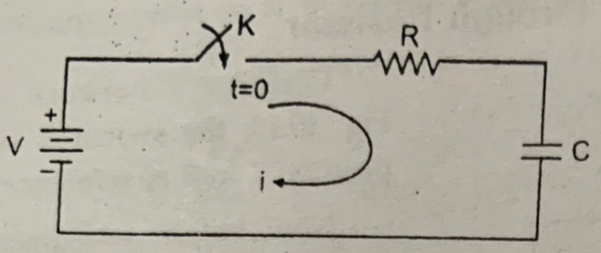

Fig. 10.8

Consider series R-C circuit as shown in Fig. 10.8. The switch K is in open state initially. There is no charge on condenser and no voltage across it. At instant t = 0, switch K is closed.

Immediately after closing a switch, the capacitor acts as a short circuit, so current at the time of switching is high. A voltage across capacitor is zero at $t = 0^+$ as capacitor acts a short circuit, the current is maximum and is given by,

$$i = \frac{V}{R} \text{ Amp}$$

This current is maximum at $t = 0^+$ which is charging current. As the capacitor starts charging, the voltage across capacitor V_C starts increasing and charging current starts decreasing. After some time, when the capacitor charges to V volts, it achieves steady state. In steady state, it acts as a open circuit so current will be zero finally.

After switching instant, applying Kirchhoff's voltage law,

$$V = V_R + V_C$$

where V_R is voltage across resistor and V_C is the voltage across capacitor.

$$\therefore \quad V = i \cdot R + V_C$$

But current i can be written as, $$i = C\frac{dV_C}{dt}$$

Substituting value of i in equation of voltage V,

$$V = RC\frac{dV_C}{dt} + V_C$$

This is the first order linear differential equation. Rearranging the terms in above equation,

$$V - V_C = RC \cdot \frac{dV_C}{dt}$$

Separating the variables,

$$\therefore \quad \frac{dt}{RC} = \frac{dV_C}{V - V_C}$$

Integrating both sides of above equation, we have

$$\frac{t}{RC} = -ln(V - V_C) + K'$$

where K' is constant of integration.

At t = 0, there is no voltage across capacitor i.e. $V_C = 0$.

Substituting in above equation, we have

$$0 = -ln(V) + K'$$

$$\therefore \quad K' = ln(V)$$

Then the general solution becomes,

$$\frac{t}{RC} = -ln(V - V_C) + ln(V)$$

$$\therefore \quad \frac{t}{RC} = ln\left[\frac{V}{V - V_C}\right]$$

$$\therefore \quad \frac{V}{V - V_C} = e^{\frac{+t}{RC}}$$

$$\therefore \quad V - V_C = V \cdot e^{\frac{-t}{RC}}$$

$$\therefore \quad V_C = V - V \cdot e^{\frac{-t}{RC}}$$

$$\therefore \quad V_C = V\left(1 - e^{\frac{-t}{RC}}\right)$$

Above is the solution for voltage across capacitor. First term in the equation gives steady state value of voltage across capacitor. The second term $-Ve^{\frac{-t}{RC}}$ gives transient portion of voltage across capacitor.

When the steady state in achieved, total charge on the capacitor is Q coulombs.

$$\therefore \quad V = \frac{Q}{C}$$

Similarly at any instant, $V_C = \frac{q}{C}$ where q is instantaneous charge.

$$\therefore \quad \frac{q}{C} = \frac{Q}{C}\left[1 - e^{\frac{-t}{RC}}\right]$$

$$\therefore \quad q = Q\left[1 - e^{\frac{-t}{RC}}\right]$$

Thus the charge on the capacitor also behaves similar to the voltage across the capacitor.

Now the current can be expressed as follows.

$$iR = V - V_C$$

Above equation can be written using KVL.

$$\therefore \quad iR = V - \left[V\left(1 - e^{\frac{-t}{RC}}\right)\right]$$

$$\therefore \quad i = \frac{V}{R} \cdot e^{\frac{-t}{RC}}$$

So at $t = 0$, $i = \frac{V}{R}$ is maximum current and in steady state it becomes zero.

The variation of voltage across capacitor and charging current with respect to time is as shown in the Fig. 10.9.

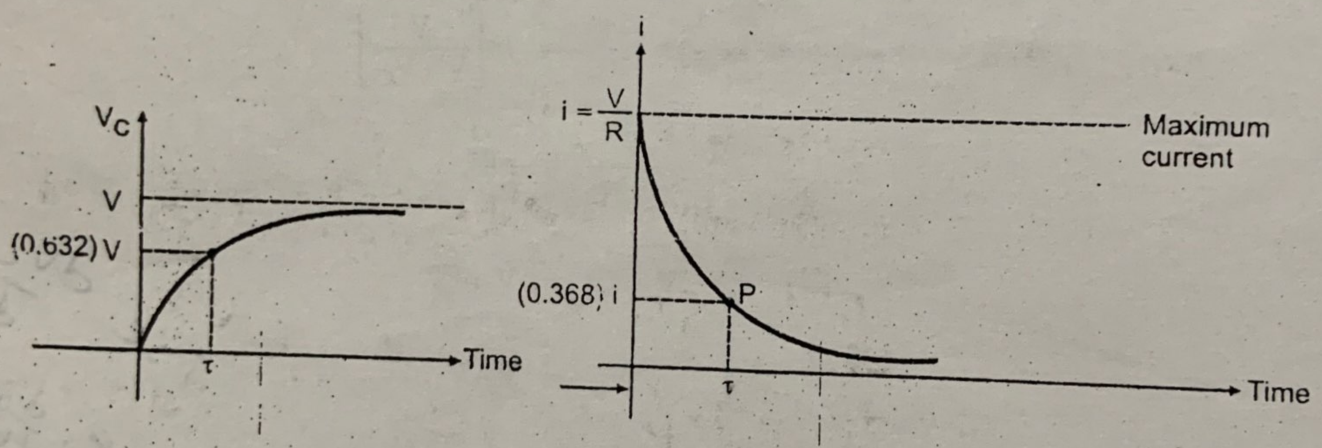

Fig. 10.9

The term RC in equation of V_C or i is called Time constant and denoted by τ, measured in seconds.

when

$$t = R \cdot C = \tau \quad \text{then,}$$

$$V_C = V\left(1 - e^{-1}\right)$$

$$\therefore \quad V_C = (0.632)\, V$$

So time constant of series R-C circuit is defined as time required by the capacitor voltage to rise from zero to 0.632 of its final steady state value during charging.

Incidently after $t = 2\tau, 3\tau, 4\tau$, the capacitor voltage attains the values as 0.865 V, 0.95 V, 0.982 V respectively and practically capacitor requires the time 4 to 5 times the time constant to charge fully.

Thus, time constant ... voltage across capacitor (starting from zero) would reach ... rate of change was maintained constant at its initial value throughout charging period.

Above circuit is called as driven R-C circuit as it is driven by voltage source of V volts.

10.7.2 Discharge of Capacitor through Resistor

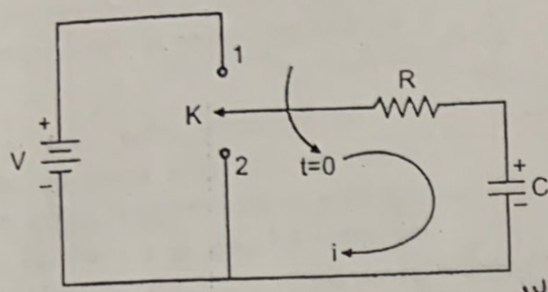

Fig. 10.10

$i \propto \frac{dv}{dt}$

Consider network shown in Fig. 10.10, the switch K is moved from position 1 to 2 at reference time t = 0.

Now before switching takes place, the capacitor C is fully charged to V volts and it discharges through resistance R. As time passes, charge and hence voltage across capacitor i.e. V_C decreases gradually and hence discharge current also decreases gradually from maximum to zero exponentially.

After switching has taken place, applying Kirchhoff's voltage law,

$$0 = V_R + V_C$$

where V_R is voltage across resistor and V_C is voltage across capacitor.

$$\therefore \quad V_C = -V_R = -i \cdot R$$

But

$$i = C\frac{dV_C}{dt}$$

$$\therefore \quad V_C = -R \cdot C \cdot \frac{dV_C}{dt}$$

Above equation is linear, homogeneous first order differential equation. Hence rearranging we have,

$$\frac{dt}{RC} = -\frac{dV_C}{V_C}$$

Integrating both sides of above equations, we have,

$$\frac{t}{RC} = -\ln V_C + K'$$

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Now at $t = 0$, $V_C = V$ which is initial condition. Substituting in equation we have,

$$\therefore \quad 0 = -ln\,V + K'$$

$$\therefore \quad K' = ln\,V$$

Substituting value of K′ in general solution, we have

$$\frac{t}{RC} = -ln\,V_C + ln\,V$$

$$\therefore \quad \frac{t}{RC} = ln\frac{V}{V_C}$$

$$\therefore \quad \frac{V}{V_C} = e^{\frac{t}{RC}}$$

$$\therefore \quad V_C = V \cdot e^{\frac{-t}{RC}}$$

$q = C \frac{d}{dt}\left(V e^{-t/RC}\right)$

$= C V \cdot \frac{-1}{RC} e^{-t/RC}$

$= -\frac{V}{R} e^{-t/RC}$

As $\quad V = \frac{Q}{C}$

where Q is total charge on capacitor.

Similarly, at any instant, $V_C = \frac{q}{C}$ where q is instantaneous charge.

So we have, $\quad \frac{q}{C} = \frac{Q}{C} e^{\frac{-t}{RC}}$

$$q = Q \cdot e^{\frac{-t}{RC}}$$

Thus charge behaves similar to the voltage across capacitor.

Now discharging current i is given by

$$i = \frac{V_R}{R}$$

But $V_R = V_C$ when there is no source is circuit,

$$\therefore \quad i = \frac{V_C}{R}$$

$$\therefore \quad i = \frac{V}{R} e^{\frac{-t}{RC}}$$

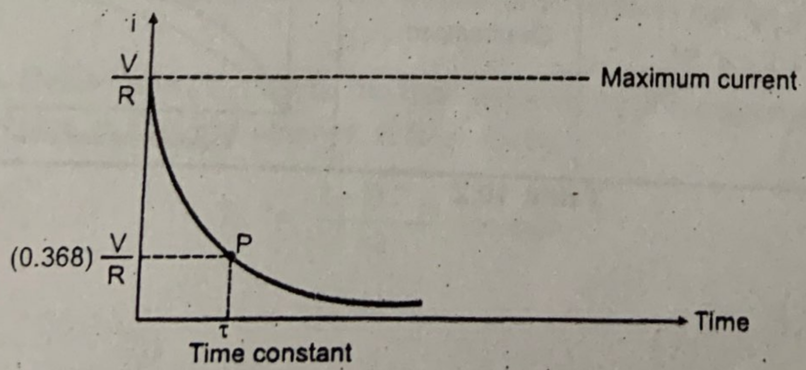

Fig. 10.11

Above expression is nothing but discharge current of capacitor. The variation of this current with respect to time is shown in Fig. 10.11.

This shows that the current is exponentially decaying. At point P on the graph, the current value is (0.368) times its maximum value. The characteristics of decay are the determined by values of R and C, which are the two parameters of the network.

For this network, after the instant t = 0, there is no driving voltage source in circuit, hence it is called undriven R-C circuit.

10.8 Step Response of Series R-L-C Circuit

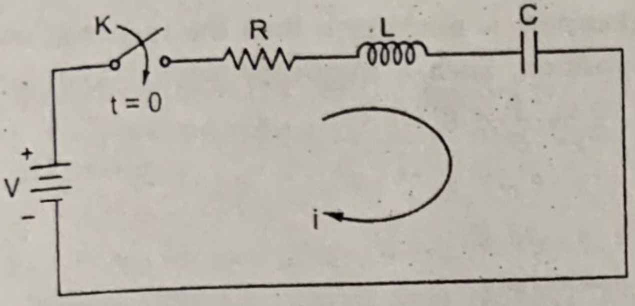

Fig. 10.12

In series R-L-C circuit, there are two energy storing elements which are L and C. Such a circuit gives rise to second order differential equation and hence called **second order circuit.**

Consider a series R-L-C circuit shown in the Fig. 10.12. The switch is closed at t = 0 and a step voltage of V volts gets applied to the circuit.

Applying KVL after switching we get,

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int i\,dt = V$$

As V is step i.e. constant, differentiating both sides of the above equation gives,

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C} = 0$$

$$\therefore \quad \frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i = 0 \quad \Rightarrow \quad i\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)=0$$

$$\therefore \quad s^2 i+\frac{R}{L}si+\frac{1}{LC}i = 0 \qquad \text{where } s=\frac{d}{dt} \qquad s^2+2\xi\omega_n s+\omega_n^2$$

This is called **Characteristic equation** or **auxiliary equation** of the series R-L-C circuit.

The response of the circuit depends on the nature of the roots of the characteristic equation. The two roots are,

$$s_{1,2} = \frac{-\frac{R}{L}\pm\sqrt{\left(\frac{R}{L}\right)^2-\frac{4}{LC}}}{2} = -\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\left(\frac{1}{\sqrt{LC}}\right)^2}$$

Let us define some quantities to find the response according to the nature of the roots.

1. **Critical Resistance** (R_{cr}) : This is the value of the resistance which reduces square root term to zero, giving real, equal and negative roots.

$$\therefore \quad \frac{R_{cr}}{2L} = \frac{1}{\sqrt{LC}}$$

$$\therefore \quad R_{cr} = 2\sqrt{\frac{L}{C}}$$

2. Damping Ratio (ξ) : This ratio is the indication of the opposition from the circuit to cause oscillations in its response. More the value of this ratio, less are the chances of oscillations in the response. It is the ratio of actual resistance in the circuit to the critical resistance. It is denoted by greek letter zeta (ξ).

$$\xi = \frac{R}{R_{cr}} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

3. Natural Frequency (ω_n) : If the damping is made zero then the response oscillates with natural frequency without any opposition. Such a frequency when $\xi = 0$ is called natural frequency of oscillations, denoted as ω_n. It is given by,

$$\omega_n = \frac{1}{\sqrt{LC}}$$

Using these values, the roots of the equation are,

$$s_{1,2} = -\xi\,\omega_n \pm \omega_n\sqrt{\xi^2 - 1} = -\xi\,\omega_n \pm j\,\omega_n\sqrt{1-\xi^2} \qquad s_{1,2} = -\alpha \pm \omega_d j$$

Thus the response is totally dependent on the values of ξ.

Let $\quad \alpha = \xi\,\omega_n \quad$ and $\quad \omega_d = \sqrt{1-\xi^2}\ \omega_n$

where ω_d = actual frequency of oscillations i.e. damped frequency when $\xi = 0$ we get $\omega_d = \omega_n$ i.e. natural frequency.

The general solution of characteristic equation is, $\qquad s_{1,2} = -\alpha \pm j\omega_d$

$$i(t) = K_1\, e^{(-\alpha + j\,\omega_d)\,t} + K_2\, e^{(-\alpha - j\,\omega_d)\,t}$$

It can be seen that for the range $0 < \xi < 1$, imaginary term $j\omega_d$ exists and we get sine and cosine terms in the response as $e^{j\theta} = \cos\theta + j\sin\theta$. Such a network is called **underdamped network** when the roots of the characteristic equation are complex conjugates with negative real part. Due to negative real part, such oscillations are damped and vanish after some time. When $\xi = 1$, the roots are real, equal and negative. In such case the response is exponential and fastest if compared with any other exponential response without oscillations. Such a case is called **critically damped** case. The response for such case takes form,

$$i(t) = K_1\, e^{s_1 t} + K_2\, e^{s_2 t}$$

When $\xi > 1$, then damping becomes high and the response remains exponential but becomes more and more sluggish and slow as ξ increases. Such cases are called **overdamped**.

The response takes the form,

$$i(t) = K_1\, e^{s_1 t} + K_2\, e^{s_2 t}$$

$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

$\dfrac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$

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When $\xi = 0$, the damping is zero and response oscillates with maximum frequency ω_n. Such a case is called **undamped case**. The output is oscillations with constant frequency and amplitude i.e. sustained oscillations.

The cases and the corresponding responses are summarized in the Table 10.2.

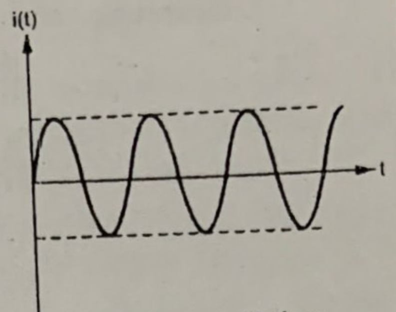
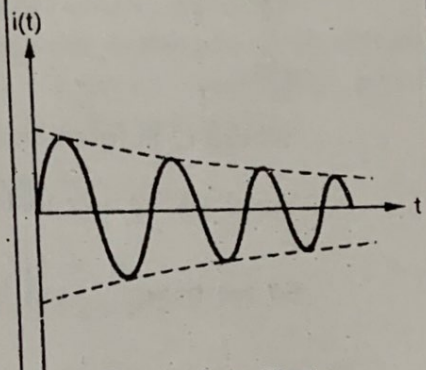
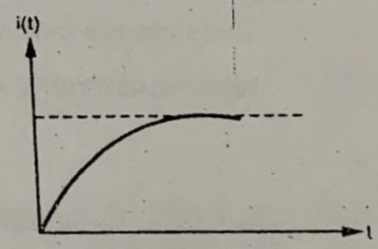
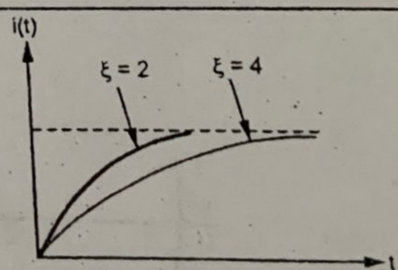
Range of ξ	Nature of roots	Form of response	Circuit classification	Nature of response
$\xi = 0$	Purely imaginary $\pm j\omega_n$	$K_1 \cos \omega t$ $+ K_2 \sin \omega t$	Undamped	 Sustained oscillations
$0 < \xi < 1$	Complex conjugates with negative real part $-\alpha \pm j\omega_d$	$K_1 e^{-\alpha t} \cos \omega t$ $+ K_2 e^{-\alpha t} \sin \omega t$	Underdamped	 Damped oscillations
$\xi = 1$	Real equal negative $-\alpha, -\alpha$	$K_1 e^{-\alpha t}$ $+ K_2 t e^{-\alpha t}$	Critically damped	 Exponential and critical response
$1 < \xi < \infty$	Real unequal negative	$K_1 e^{-s_1 t}$ $+ K_2 e^{-s_2 t}$	Overdamped	 Exponential and slow

Table 10.2

Example 10.1 : *In the circuit shown in Fig. 10.17, 10 V battery is connected to the circuit by closing switch at t = 0. Assume that initial voltage on capacitor is zero. Determine expression for $V_C(t)$ and $i_C(t)$ sketch the waveform.*

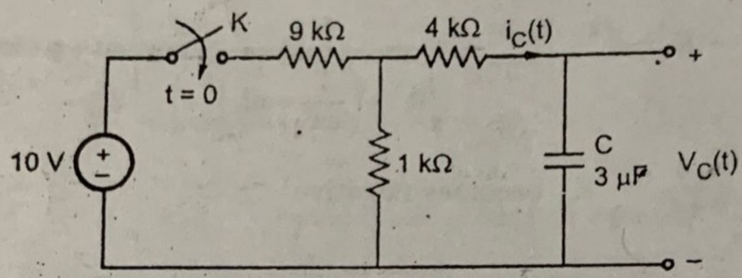


Fig. 10.17

Solution : At $t = 0^-$, switch K is open. Battery of 10 V is not connected to the circuit. Thus initial voltage on the capacitor is given by,

$$V_C(0^-) = V_0 = 0 = V(0^+) \quad \text{... (1)}$$

Because voltage across capacitor cannot change instantaneously.

For all $t \geq 0^+$, switch K is closed. The network can be drawn as shown in Fig. 10.17 (a).

To reduce the network shown in Fig. 10.17 (a) into simple series R-C network, finding Thevenin's equivalent network across terminals (A) - (B).

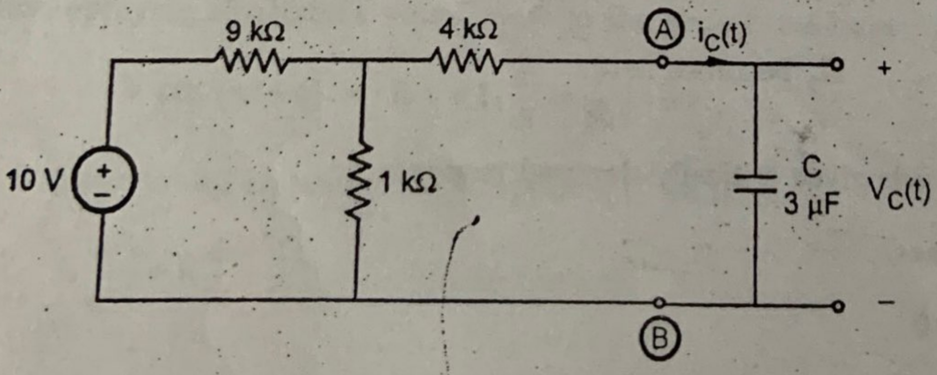


Fig. 10.17 (a)

Step 1 : Open the branch across which voltage is to be calculated as shown in Fig. 10.17 (b).

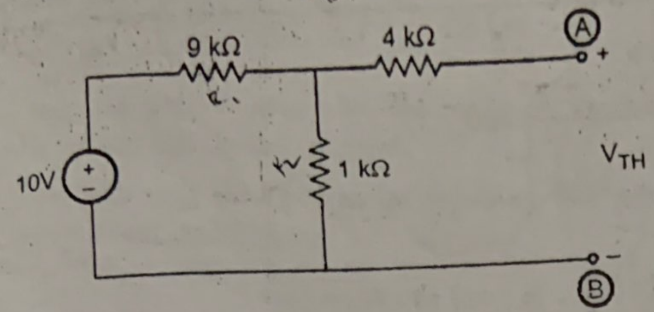


Fig. 10.17 (b)

Step 2 : Using potential divider rule, open circuit voltage V_{OC} is given by

$$V_{OC} = V_{TH} = 10\left[\frac{1\text{ K}}{1\text{ K}+9\text{ K}}\right] = 1\text{ V} \qquad \text{... (2)}$$

Step 3 : To calculate open circuit impedance Z_{TH} looking back into the network from terminals (A)-(B), replacing 10 V independent source by its ideal internal impedance i.e. short circuit as shown in Fig. 10.17 (c).

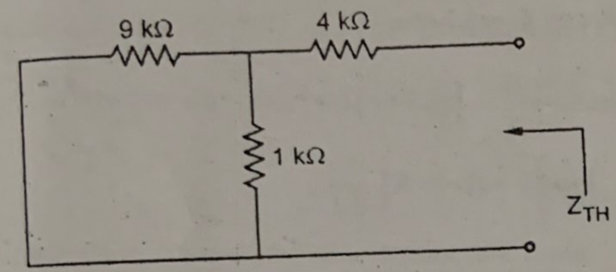


Fig. 10.17 (c)

$$\therefore\ Z_{TH} = (9\text{ K}\,||\,1\text{ K}) + 4\text{ K}$$

$$\therefore\ Z_{TH} = \frac{(9\text{ K})(1\text{ K})}{(9\text{ K}+1\text{ K})} + 4\text{ K}$$

$$\therefore\ Z_{TH} = 4.9\text{ k}\Omega \qquad \text{... (3)}$$

Step 4 : Replacing original network across terminals (A)-(B) by its Thevenin's equivalent network as shown in Fig. 10.17 (d).

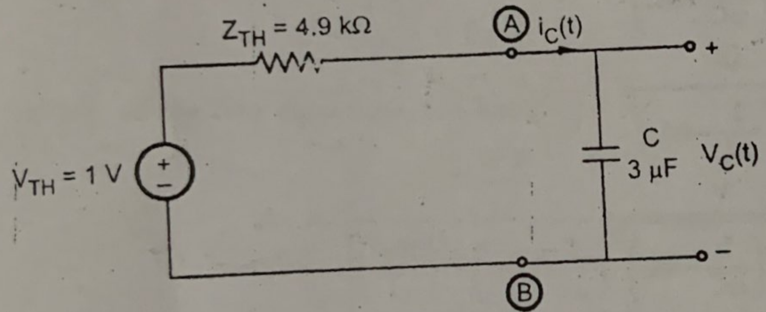


Fig. 10.17 (d)

Applying KVL,

$$(4.9 \times 10^3)\, i_C(t) + V_C = 1 \qquad \text{... (4)}$$

But $i_C = C\dfrac{dV_C}{dt}$

$$\therefore\ (4.9 \times 10^3)(3 \times 10^{-6})\frac{dV_C}{dt} + V_C = 1$$

$$\frac{dV_C}{dt} + 68.027\, V_C = 68.027$$

The above equation is a non-homogeneous equation whose solution is of the form.

$$V_C = e^{-Pt}\int_0^t Q e^{Pt}\,dt + K e^{-Pt}$$

where $\quad P = 68.027$ and $Q = 68.027$

$$\therefore \quad V_C = e^{-68.027t} \int_0^t 68.027\, e^{68.027t} + Ke^{-68.027t}$$

$$\therefore \quad V_C = 68.027 \left[\frac{e^{+68.027t}}{+68.027}\right]_0^t e^{-68.027t} + Ke^{-68.027t}$$

$$\therefore \quad V_C = e^{-68.027t}\left[e^{68.027t} - 1\right] + Ke^{-68.027t}$$

$$\therefore \quad V_C = 1 - e^{-68.027t} + Ke^{-68.027t}$$

To find the value of K, let us use initial conditions i.e. at $t = 0$, $V_C = 0$

$$\therefore \quad 0 = 1 - 1 + K(1)$$

$$\therefore \quad K = 0$$

$$\therefore \quad \mathbf{V_C = 1 - e^{-68.027t} \ V}$$

The expression for current flowing through capacitor is given by,

$$i_C = C\frac{d\,V_C}{dt}$$

Substituting value of V_C from equation (5)

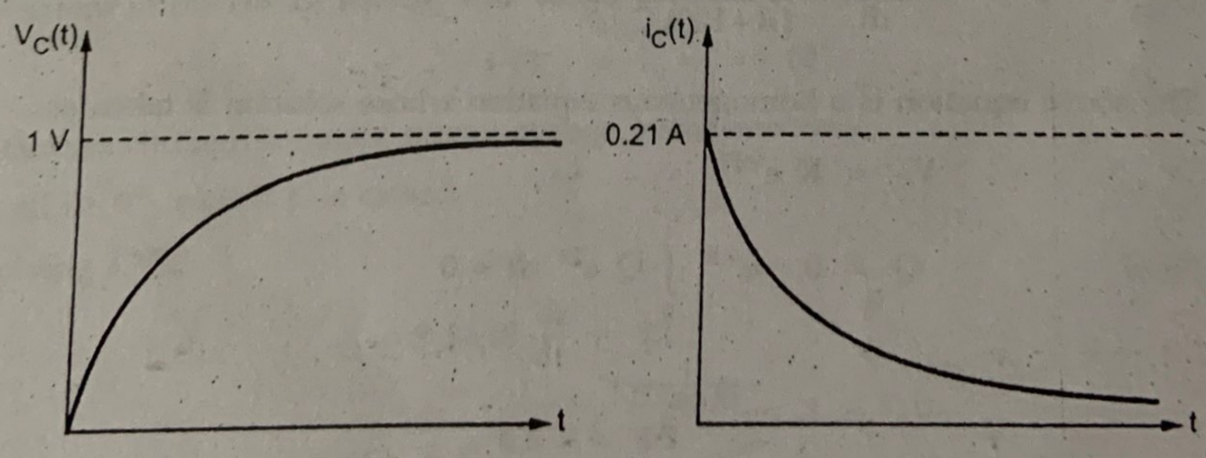
$$\therefore \quad i_C = 3 \times 10^{-6}\left[\frac{d}{dt}\left(1 - e^{-68.027t}\right)\right]$$

$$\therefore \quad i_C = 3 \times 10^{-6}\left[0 - (-68.027)\, e^{-68.027t}\right]$$

$$\therefore \quad i_C = 0.21 \times 10^{-3}\, e^{-68.027t}$$

$$\therefore \quad \mathbf{i_C = 0.21\, e^{-68.027t} \ mA} \qquad \text{... (5)}$$

The waveforms of V_C and i_C against time are as shown below in Fig. 10.17 (e).



Example 10.3 : *In the circuit shown in Fig. 10.19 switch is closed at $t = 0$. Obtain the expression for the current in circuit and find i at $t = 0.25$ sec.*

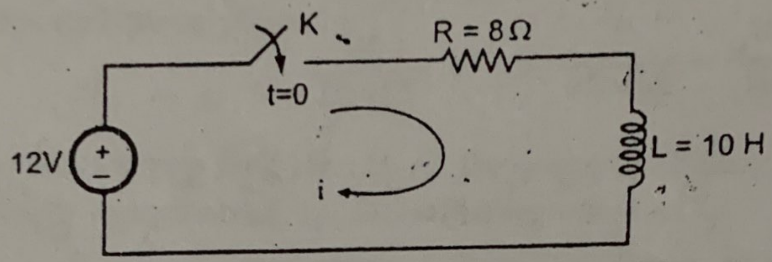


Fig. 10.19

Solution : At $t = 0^-$, switch K is open.

So current in the circuit will be zero. Initial current through inductor I_0 is given by

$$i\ (0^-) = I_0 = 0 = i\ (0^+) \qquad \text{... (1)}$$

As current through inductor cannot change instantaneously

For all $t \geq 0^+$, switch K is closed.

Applying KVL,

$$8\ i + 10\frac{di}{dt} = 12 \qquad \text{... (2)}$$

$$\frac{di}{dt} + 0.8\ i = 1.2$$

The above equation is a non-homogeneous equation whose solution is in the form

$$i = e^{-Pt}\int_0^t Q\ e^{Pt}\ dt + K\ e^{-Pt} \quad \text{C.F}$$

where $\qquad P = 0.8, \quad Q = 1.2$

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$$i = e^{-0.8t} \int_0^t 1.2\, e^{0.8t}\, dt + K\, e^{-0.8t}$$

$$i = 1.2\, e^{-0.8t} \left[\frac{e^{0.8t}}{0.8} \right]_0^t + K\, e^{-0.8t}$$

$$\boxed{i = 1.2\, e^{-0.8t} \left[\frac{e^{0.8t}}{0.8} - \frac{1}{0.8} \right] + K\, e^{-0.8t}}$$

$$i = 1.5 \left[1 - e^{-0.8t}\right] + K\, e^{-0.8t}$$

At $t = 0$, $i = 0$

$$0 = 1.5\,[1-1] + K \cdot 1$$

$\therefore$

$$K = 0$$

$\therefore$

$$i = 1.5 \left[1 - e^{-0.8t}\right] \text{ A}$$

At $t = 0.25$ sec, the value of current i is given by,

$$i = 1.5 \left[1 - e^{-(0.8)\,(0.25)}\right]$$

$\therefore$

$$i = 0.2719 \text{ A}$$

Example 10.4 : *For the circuit shown in Fig. 10.20. Find the current equation for $t > 0$, when the switch K is moved at $t = 0$ from position 'a' to 'b'. Assume that circuit is in steady state before $t = 0$ instant.*

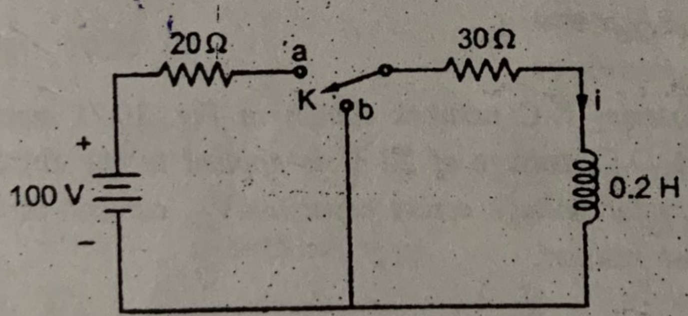

Fig. 10.20

Solution : At $t = 0^-$, switch K is at position 'a'.

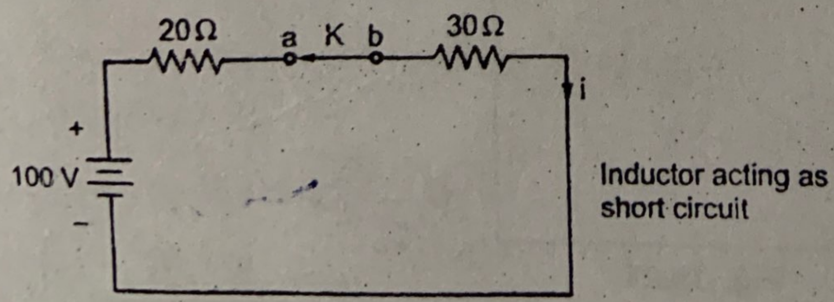

Fig. 10.20 (a)

Initially the circuit is in steady state. So inductor will act as a short circuit for d.c. The circuit can be drawn as shown in Fig. 10.20 (a).

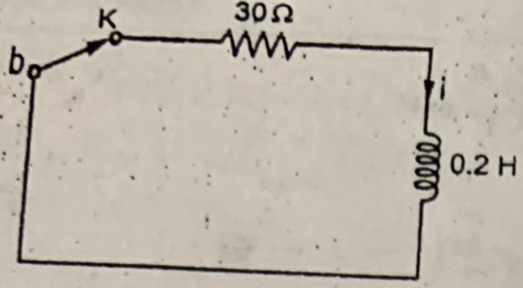

Fig. 10.20 (b)

Initial current in inductor I_0, is given by,

$$i(0^-) = I_0 = \frac{100}{(20+30)} = 2\text{ A} = i(0^+)$$

Because current through inductor cannot be change instantaneously.

For all $t \geq 0^+$, switch K is moved to position 'b'. The circuit can be drawn as shown in Fig. 10.20 (b).

The circuit shown in Fig.10.20(b) is undriven series R-L or source free series R-L circuit.

Applying KVL,

$$30\, i + 0.2 \frac{di}{dt} = 0$$

$$\therefore \quad \frac{di}{dt} + 150\, i = 0$$

This is the homogeneous equation whose solution is given by,

$$i = e^{-Pt} \int_0^t Q\, e^{Pt}\, dt + K\, e^{-Pt}$$

With $P = 150$, $\quad Q = 0$

$$\therefore \quad i = Ke^{-150\,t}$$

To find the value of K let us use the initial conditions i.e. at $t = 0$, $i = 2$A.

$$\therefore \quad 2 = K \cdot 1$$

$$\therefore \quad K = 2$$

$$\therefore \quad i = 2e^{-150\,t}\ \text{A}$$

Example 10.5 : *A series R-C network shown in Fig. 10.21 consists of R = 20 Ω and C = 0.1 F. A constant D.C. voltage of 20 V is applied to the circuit at t = 0 by closing switch. Find expression for voltage across capacitor V_C, current through capacitor i_C. Also determine voltage across resistor.*

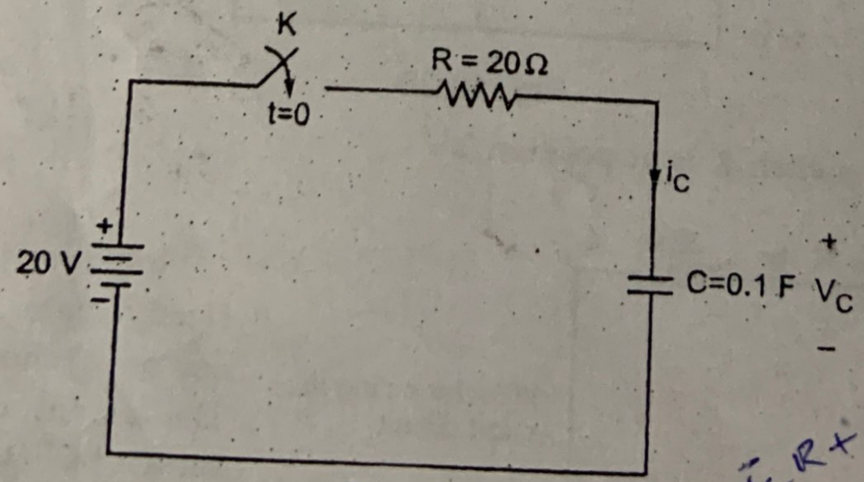

Fig. 10.21

$i_c R + V_c$

Solution : At $t = 0^-$, switch K is open. The initial voltage across capacitor V_0 is given by

$$V_C(0^-) = V_0 = 0 = V_C(0^+) \quad \text{... (1)}$$

As voltage across capacitor cannot change instantaneously.

For all $t \geq 0^+$, switch K is closed.

Applying KVL,

$$V_R + V_C = 20$$

i.e.

$$i_C R + V_C = 20 \quad \text{... (2)}$$

But

$$i_C = C \frac{dV_C}{dt}$$

$$\therefore \quad \left(C \frac{dV_C}{dt}\right) R + V_C = 20$$

$$RC \frac{dV_C}{dt} + V_C = 20$$

$$\therefore \quad \frac{dV_C}{dt} + \frac{1}{RC} V_C = \frac{20}{RC}$$

$$\therefore \quad \frac{dV_C}{dt} + \frac{1}{20 \times 0.1} V_C = \frac{20}{20 \times 0.1}$$

$$\therefore \quad \frac{dV_C}{dt} + 0.5\, V_C = 10$$

$\frac{dx}{dt} + Px = Q$

$x = e^{-Pt} \int Q e^{Pt} + K e^{-Pt}$

The above equation is a non-homogeneous differential equation whose solution is given in the form

$$V_C = e^{-Pt} \int_0^t Q\, e^{Pt}\, dt + K\, e^{-Pt}$$

where $\quad P = 0.5, \quad Q = 10$

$$\therefore \quad V_C = e^{-0.5t} \int_0^t 10\, e^{0.5t}\, dt + K\, e^{-0.5t}$$

$$\therefore \quad V_C = 10\, e^{-0.5t} \left[\frac{e^{0.5t}}{0.5}\right]_0^t + K\, e^{-0.5t}$$

$$\therefore \quad V_C = 20\, e^{-0.5t} \left[e^{0.5t} - 1\right] + K\, e^{-0.5t}$$

$$\therefore \quad V_C = 20\left[1 - e^{-0.5t}\right] + K\, e^{-0.5t}$$

To find the value of K, let us use the initial condition i.e. at $t = 0$, $V_C = 0$

$$\therefore \quad 0 = 20\,[1-1] + K \cdot 1$$

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$$\therefore \quad K = 0$$

$$\therefore \quad V_C = 20\,[1 - e^{-0.5t}]$$

The current flowing through the capacitor is given by,

$$i_C = C\frac{dV_C}{dt}$$

Substituting value of V_C

$$\therefore \quad i_C = 0.1\frac{d}{dt}[20\,(1 - e^{-0.5t})]$$

$$\therefore \quad i_C = (0.1)\,(20)\left[\frac{d}{dt}(1) - \frac{d}{dt}(e^{-0.5t})\right]$$

$$\therefore \quad i_C = 2\,[0 - (-0.5)\,e^{-0.5t}]$$

$$\therefore \quad i_C = e^{-0.5t}\ \text{A}$$

Voltage across resistor is given by

$$V_R = i_C\,R$$

As above circuit is series R-C circuit, so the current flowing through resistor is same as that flowing through capacitor i.e. i_C. Substituting value of i_C

$$V_R = [e^{-0.5t}]\,[20]$$

$$\therefore \quad V_R = 20\,e^{-0.5t}\ \text{V}$$

Example 10.6 : *In the circuit shown in Fig. 10.22 the switch is closed at $t = 0$. Determine and sketch $i_L(t)$ and $v_L(t)$ for $t > 0$. Assume at $t = 0$, the current in the inductance is zero.*

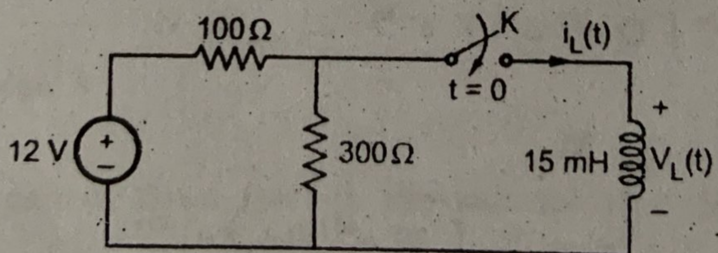


Fig. 10.22

Solution : At $t = 0^-$, switch K is open. Inductor is not connected in the circuit.

Hence $$i_L(0^-) = 0 = i_L(0^+) \qquad \text{... (1)}$$

Because current through inductor cannot be changed instantaneously.

For all $t \geq 0^+$, switch K is closed. The network can be drawn as shown in Fig. 10.22 (a).

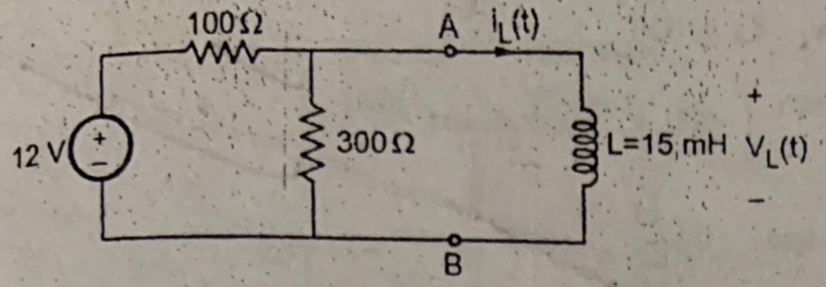


Fig. 10.22 (a)

Finding Thevenin's equivalent network to reduce the given network into simple series R-L network.

Step 1 : Open the branch through which current is to be calculated i.e. branch consisting L.

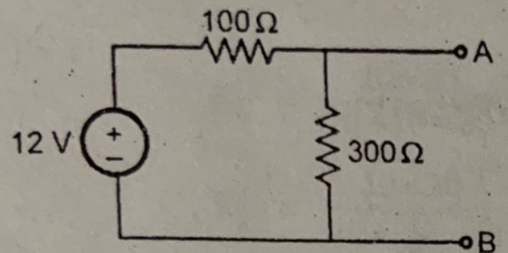


Fig. 10.22 (b)

Step 2 : Using potential divider rule, open circuit voltage V_{OC} is given by

$$V_{OC} = V_{TH} = 12\left[\frac{300}{300+100}\right] = 9\ \text{V} \qquad \text{... (2)}$$

Step 3 : To calculate open circuit impedance Z_{TH} looking back into the network from terminals A-B, replacing the independent voltage source 9 V by its ideal internal impedance i.e. short circuit as shown in Fig. 10.22 (c).

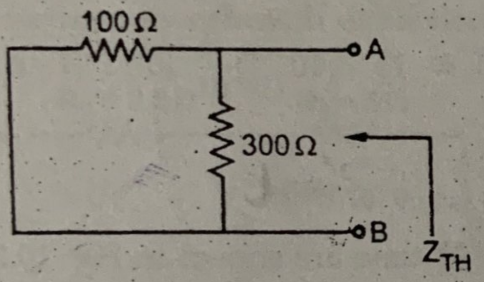


Fig. 10.22 (c)

$$Z_{TH} = (100)||(300)$$

$$= \frac{(100)(300)}{(100+300)} = \frac{30000}{400}$$

$$\therefore\ Z_{TH} = 75\ \Omega \qquad \text{... (3)}$$

Step 4 : Replacing original network across terminals A - B by its Thevenin's equivalent network as shown in Fig. 10.22 (d).

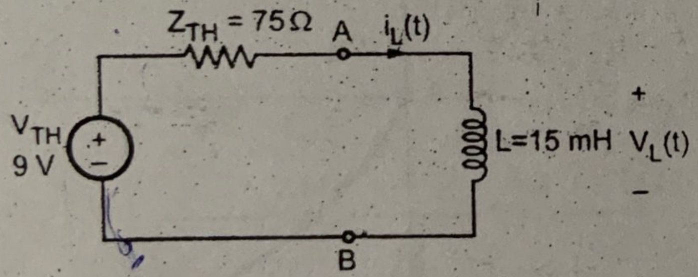


Fig. 10.22 (d)

Consider network shown in Fig. 10.22 (d).

Applying KVL,

$$75\ i_L(t) + 15\times 10^{-3}\frac{di_L(t)}{dt} = 9$$

$$\therefore \quad \frac{di_L(t)}{dt} + 5000\ i_L(t) = 600$$

This is a non-homogeneous equation whose solution is of the form.

$$i_L(t) = e^{-Pt}\int_0^t Q\, e^{Pt}\, dt + K\, e^{-Pt}$$

where $P = 5000, \quad Q = 600$

$$\therefore \quad i_L(t) = e^{-5000t}\int_0^t 600\, e^{5000t}\, dt + Ke^{-5000t}$$

$$= 600\, e^{-5000t}\left[\frac{e^{5000t}}{5000}\right]_0^t + Ke^{-5000t} = 0.12\, e^{-5000t}\,[e^{5000t} - 1] + Ke^{-5000t}$$

At $t = 0, \; i_L = 0$

$$0 = 0.12\,[1-1] + K$$

$$K = 0$$

$$\therefore \quad i_L(t) = 0.12\, e^{-5000t}\,[e^{5000t} - 1]$$

$$\therefore \quad i_L(t) = 0.12\,[1 - e^{-5000t}]\ \text{A}$$

Voltage across inductor $V_L(t)$ is given by

$$V_L(t) = L\frac{di_L(t)}{dt}$$

Substituting value of $i_L(t)$

$$V_L(t) = 15 \times 10^{-3}\frac{d}{dt}\,[0.12\,(1 - e^{-5000t})]$$

Solving above derivative we get,

$$V_L(t) = 9\, e^{-5000t}\ \text{V}$$

The waveforms of $i_L(t)$ and $V_L(t)$ against time are shown in Fig. 10.22 (e).

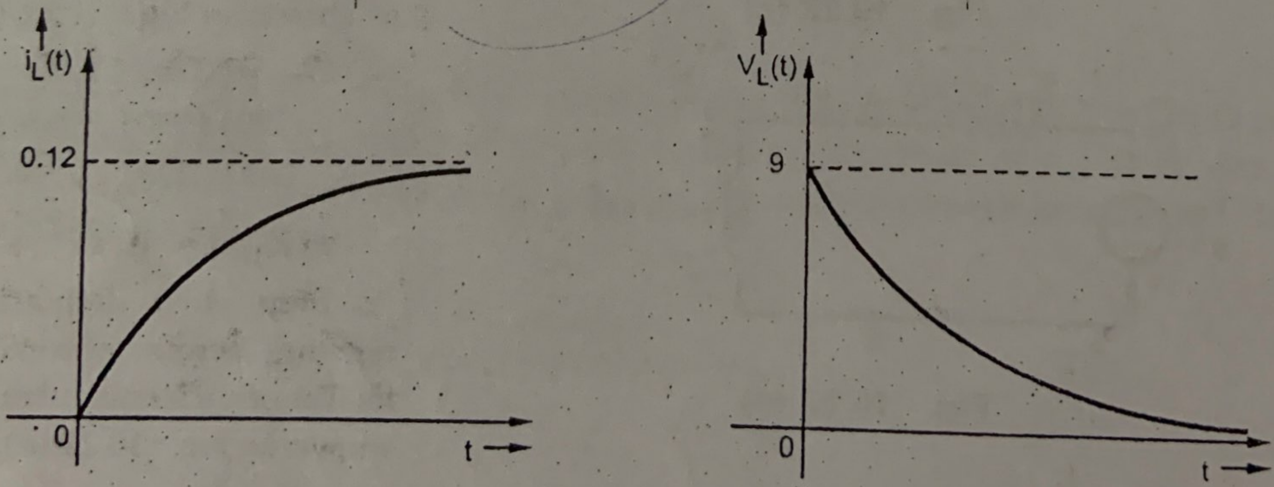


Fig. 10.22 (e)

Example 10.8 : *In the network shown in Fig. 10.24 switch K is initially kept open and network reaches steady state. At t = 0, switch K is closed. Find an expression for current through inductor for t > 0. Sketch current waveform.*

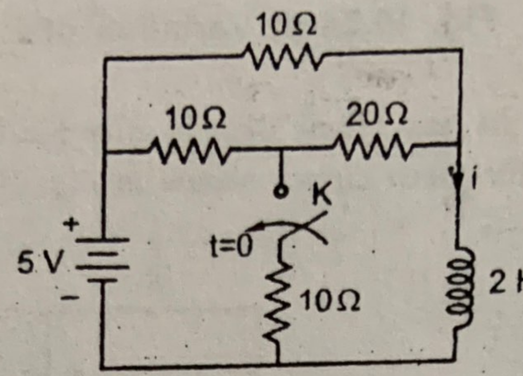


Fig. 10.24

Solution : At $t = 0^-$, switch K is open. It is kept open till network reaches steady state. In steady state network inductor acts as short circuit for d.c. The network can be drawn as shown in Fig. 10.24 (a).

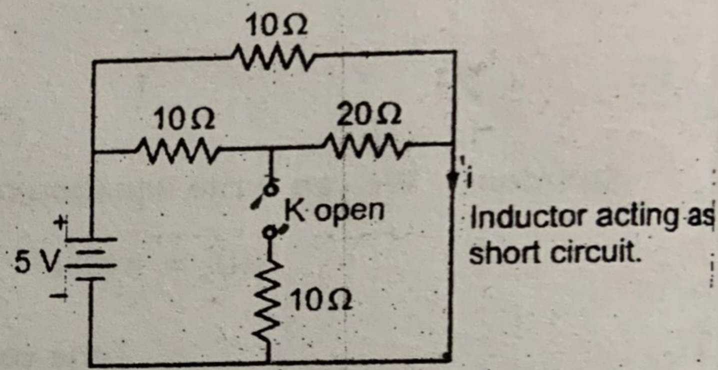


Fig. 10.24 (a)

$$R_{eq} = (10) \| (10+20) = \frac{(10)(30)}{(10+30)} = 7.5\ \Omega$$

The total current supplied by 5 V source is given by

$$I = \frac{5}{R_{eq}} = \frac{5}{7.5} = 0.6666\ \text{A}$$

Above calculated current flows through the inductor also. So the initial current of inductor is given by

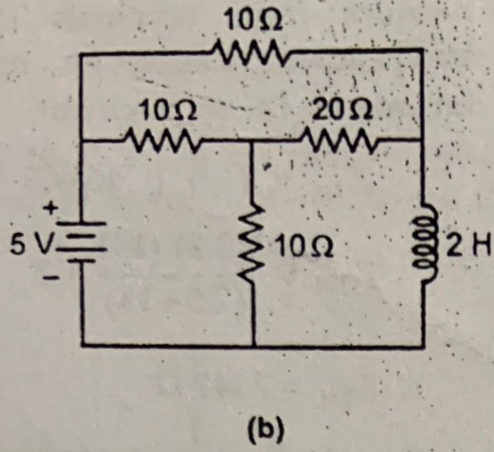
$$i(0^-) = I_0 = 0.6666\ \text{A} = i(0^+) \qquad \text{... (1)}$$

Because current through inductor cannot change instantaneously.

For all $t \geq 0^+$, switch K is closed. The network can be drawn as shown in Fig. 10.24 (b).

Converting delta formed by 10 Ω, 10 Ω, 20 Ω into its equivalent star as shown in Fig. 10.24 (c).

Scanned with CamScanner



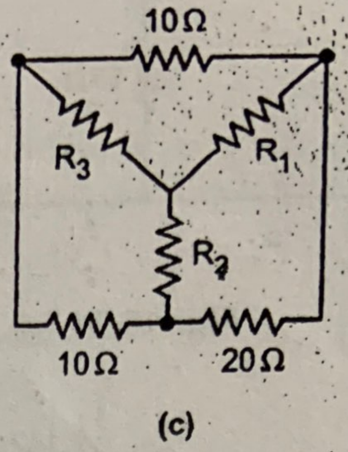


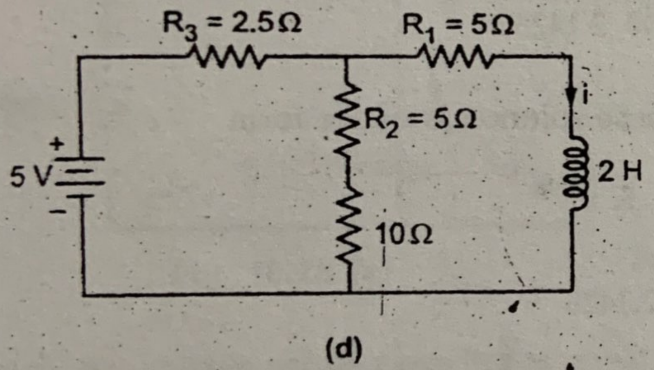
Fig. 10.24

$$R_1 = \frac{10 \times 20}{10+10+20} = \frac{200}{40} = 5\ \Omega$$

$$R_2 = \frac{10 \times 20}{10+10+20} = \frac{200}{40} = 5\ \Omega$$

$$R_3 = \frac{10 \times 10}{10+10+20} = \frac{100}{40} = 2.5\ \Omega$$

Redrawing network with equivalent star as shown in Fig. 10.24 (d).



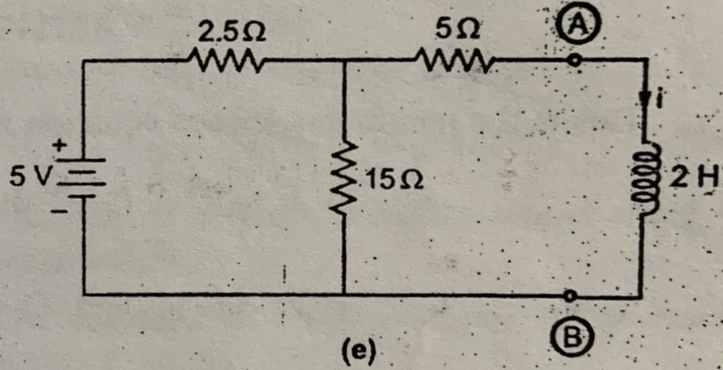


Fig. 10.24

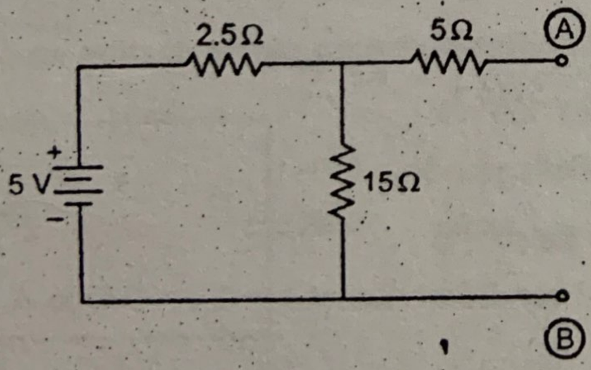


Fig. 10.24 (f)

To reduce the network shown in Fig. 10.24 (e) into simple series R-L network, finding Thevenin's equivalent network across terminals (A)-(B).

Step 1 : Open circuit the branch through which current is to be calculated as shown in Fig. 10.24 (f).

Step 2 : Using potential divider rule, open circuit voltage is given by,

$$V_{OC} = V_{TH} = 5\left[\frac{15}{15+2.5}\right] = 4.2857\ V \quad \dots (2)$$

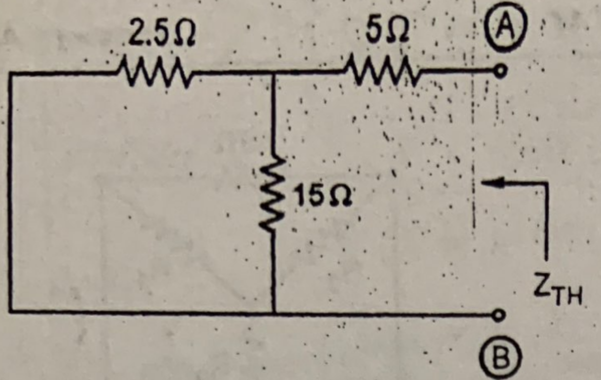


Fig. 10.24 (g)

Step 3 : To calculate open circuit impedance Z_{TH} looking back into the network from terminals (A)-(B), replacing independent source by its ideal internal impedance i.e. short circuit.

$$Z_{TH} = (2.5 \parallel 15)+5$$

$$\therefore\ Z_{TH} = \frac{(2.5)(15)}{(2.5+15)}+5$$

$$\therefore\ Z_{TH} = 7.142\ \Omega \qquad \text{... (3)}$$

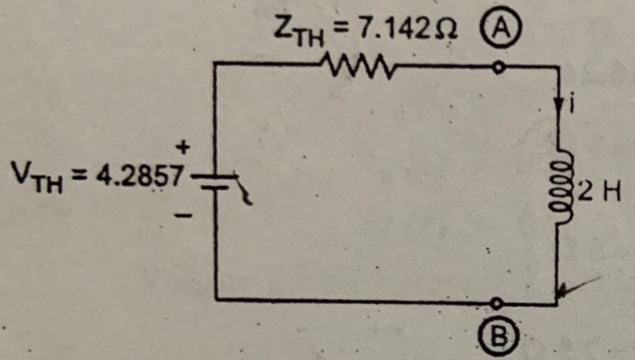


Fig. 10.24 (h)

Step 4 : Replacing original network across terminals (A)-(B) by its Thevenin's equivalent network as shown in Fig. 10.24 (h).

Applying KVL,

$$i(7.142)+2\frac{di}{dt} = 4.2857 \qquad \text{... (4)}$$

Dividing both the sides by (7.142),

$$\therefore\ i+0.28\frac{di}{dt} = 0.6$$

$$\therefore\ \frac{di}{dt}+3.5714\,i = 2.1428$$

This is the non-homogeneous equation whose solution is of the form

$$i = e^{-Pt}\int_0^t Q\,e^{Pt}\,dt+K\,e^{-Pt}$$

where $P = 3.5714$ and $Q = 2.1428$

$$\therefore\ i = e^{-3.5714t}\int_0^t 2.1428\,e^{+3.5714t}\,dt+Ke^{-3.5714t}$$

$$= 2.1438\,e^{-3.5714t}\left[\frac{e^{3.5714t}}{3.5714}\right]_0^t+Ke^{-3.5714t}$$

$$= 0.6\,e^{-3.5714t}\left[e^{3.5714t}-1\right]+Ke^{-3.5714t}$$

$$\therefore\ i = 0.6\left[1-e^{-3.5714t}\right]+Ke^{-3.5714t}$$

To find the value of K let us use the initial conditions i.e. at $t = 0$, $i = 0.6666$ A.

$$\therefore\ 0.6666 = 0.6[1-1]+K\,1$$

$$\therefore\ K = 0.6666$$

$$\therefore\ i = 0.6\left[1-e^{-3.5714t}\right]+0.6666\,e^{-3.5714t}$$

$\therefore \qquad i = 0.6 + 0.0666 \; e^{-3.5714\,t} \; A$

To plot variation of i consider different instants of time as $t = 0$ and $t = \infty$.

At $t = 0$, $i = 0.6 + 0.0666 \;\; e^{-0} = 0.6 + 0.0666 = 0.6656$ A

At $t = \infty$, $i = 0.6 + 0.0666 \;\; e^{-\infty} = 0.6 + 0 = 0.6$ A

The variation of i with time is as shown in Fig. 10.24 (i).

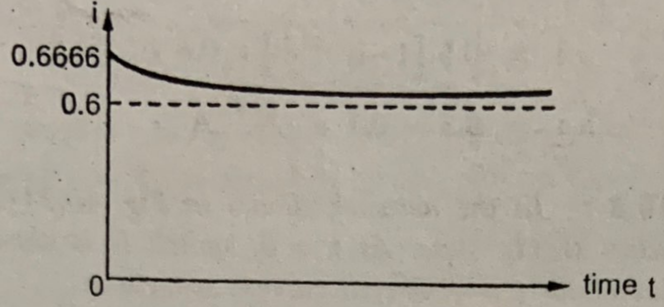


Fig. 10.24 (i) Variation of i with time t

Example 10.9 : *In how many seconds after t = 0 has the current i(t) become one half of its initial value in the given circuit shown in Fig. 10.25.*

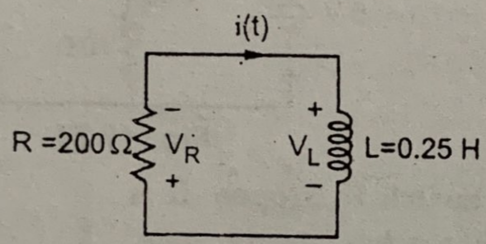


Fig. 10.25

Solution : We can write equation of i(t) as follows,

$$i(t) = i(0)\, e^{-t/T}$$

$$T = \text{Time constant} = \frac{L}{R}$$

$$T = \frac{0.25}{200} = 1.25 \times 10^{-3}$$

At $t = t_1$, $i(t) = \dfrac{i(0)}{2}$ current becomes half of its initial value,

$$\therefore \qquad \frac{i(0)}{2} = i(0)\, e^{-t_1/T}$$

$$\therefore \qquad 0.5 = e^{-t_1/1.25 \times 10^3}$$

Taking natural logarithm on both the sides,

$$\therefore \qquad ln(0.5) = \frac{-t_1}{1.25 \times 10^{-3}}$$

$$\therefore \quad t_1 = -(1.25 \times 10^{-3}) \cdot ln(0.5)$$

$$\therefore \quad t_1 = 8.66 \times 10^{-4} \text{ sec.}$$

$$\therefore \quad t_1 \doteq \mathbf{866\ \mu sec.}$$

So at t = 866 μsec, current in above circuit becomes half of its initial value.

Example 10.20 : *Obtain current i(t) for t ≥ 0, using time domain approach.*

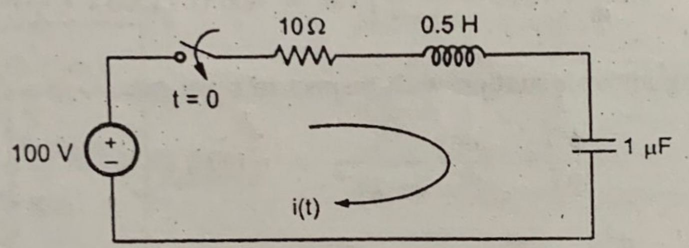


Fig. 10.37

(JNTU : June-2004, Set-2, Dec.-2004, Set-4)

Solution : At $t = 0^-$, switch is open. Hence,

$$i_L(0^-) = 0 = i_L(0^+) \qquad \ldots(1)$$

$$v_C(0^-) = 0 = v_C(0^+) \qquad \ldots(2)$$

For all $t \geq 0^+$, switch is closed. Hence applyig KVL we get,

$$10\, i(t) + 0.5 \frac{di(t)}{dt} + \frac{1}{1 \times 10^{-6}} \int_{-\infty}^{t} i(t)dt = 100$$

$$\therefore 10\, i(t) + 0.5 \frac{di(t)}{dt} + \frac{1}{1 \times 10^{-6}} \int_{-\infty}^{0} i(t)dt + \frac{1}{1 \times 10^{-6}} \int_{0^-}^{t} i(t)dt = 100$$

Initial voltage across C from equation (2) is zero,

$$10\, i(t) + 0.5 \frac{di(t)}{dt} + \frac{1}{1 \times 10^{-6}} \int_{0}^{t} i(t)dt = 100 \qquad \ldots(3)$$

Differentiating equation (3) with respect to t, we get,

$$10 \frac{di(t)}{dt} + 0.5 \frac{d^2 i(t)}{dt^2} + \frac{i(t)}{1 \times 10^{-6}} = 0$$

Assume $\frac{d}{dt} = s$

Dividing both sides by 0.5 and rearranging terms, we get,

$$\frac{d^2 i(t)}{dt^2} + 20 \frac{di(t)}{dt} + 2 \times 10^6 = 0 \qquad \ldots(4)$$

Above equation is second order homogeneous linear differential equation. Hence the auxilliary equation is given by,

$$S^2 + 20\, S + 2 \times 10^6 = 0$$

Finding roots of auxillary equation, we get,

$$S_{1,2} = \frac{-20 \pm \sqrt{(20)^2 - 4(1)(2 \times 10^6)}}{2(1)} = \frac{-20 \pm j\, 2828.36}{2}$$

$$\therefore \quad S_{1,2} = -10 \pm j\, 1414.18$$

i.e.

$$S_1 = -\alpha + j\,\omega_d = -10 + j\, 1414.18$$

$$S_2 = -\alpha - j\,\omega_d = -10 - j\, 1414.18$$

So the roots are complex conjugate with negative real parts. Hence the solution of equation (4) is given by,

$$i(t) = K_1 e^{-\alpha t} \cos\omega_d t + K_2 e^{-\alpha t} \sin\omega_d t$$

i.e.

$$i(t) = K_1 e^{-10t} \cos 1414.18\, t + K_2 \cdot e^{-10t} \sin 1414.18\, t \qquad \ldots(5)$$

At t = 0, i(t) = 0, putting values in equation (5), we get,

$$0 = K_1 e^{-0} \cos(0) + K_2 e^{-0} \sin(0)$$

$$\therefore \quad K_1 = 0 \qquad \ldots(6)\ldots \ (\because e^{-0} = 1, \cos 0 = 1, \sin 0 = 0)$$

Thus equation (5) becomes,

$$i(t) = K_2 e^{-10t} \sin 1414.18\, t \qquad \ldots(7)$$

Differentiating equation (7) with respect to t, we get,

$$\frac{di(t)}{dt} = K_2 \left[e^{-10t} (1414.18) \cos 1414.18\, t + \sin 1414.18\, t\, (-10)\, e^{-10t} \right]$$

$$\therefore \frac{di(t)}{dt} = K_2 \cdot e^{-10t} \left[(1414.18) \cos 1414.18\, t - 10 \sin 1414.18\, t \right]$$

At t = 0,

$$\frac{di}{dt}(0) = K_2 \cdot e^{-0} \left[(1414.18) \cos 0 - 10 \sin 0 \right] = K_2 (1414.18) \qquad \ldots(8)$$

At t = 0, equation (3) becomes,

$$10\, i(0) + 0.5 \frac{di}{dt}(0^+) + \frac{1}{1 \times 10^{-6}} \int_0^0 i(0)\, dt = 100$$

$$\therefore \quad 0 + 0.5 \frac{di}{dt}(0^+) + 0 = 100$$

$$\therefore \quad \frac{di}{dt}(0^+) = 200 \qquad \ldots(9)$$

Equating equations (8) and (9), we get,

$$K_2 (1414.18) = 200$$

$$\therefore \quad K_2 = 0.1414 \qquad \ldots(10)$$

Hence substituting value of K_2 in equation (7), the expression for i(t) is given by,

$$i(t) = \mathbf{0.1414\, e^{-10t} \sin 1414.18\, t\ A}$$

Example 10.21 : *Using classical method, find i(t) for t = 0*

$K_1 e^{-\alpha t j \omega_d} + K_2 e^{-\alpha - j \omega_d}$

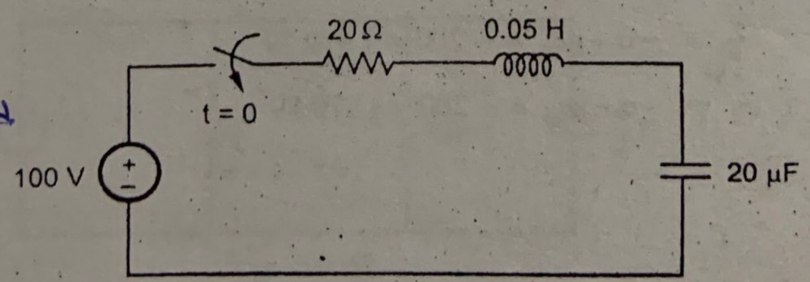


Fig. 10.38

(JNTU : Dec.-2004, Set-2)

$K_1 e^{-\alpha t} \cdot (\cos \alpha + j \sin \omega_d t) + K_2 e^{\alpha t} \cos \alpha t + j \sin \omega_d t$

Solution : At $t = 0^-$, switch is kept open.

Hence $\quad i(0^-) = 0 = i(0^+)$ ...(1)

$v(0^-) = 0 = v_C(0^+)$ ...(2)

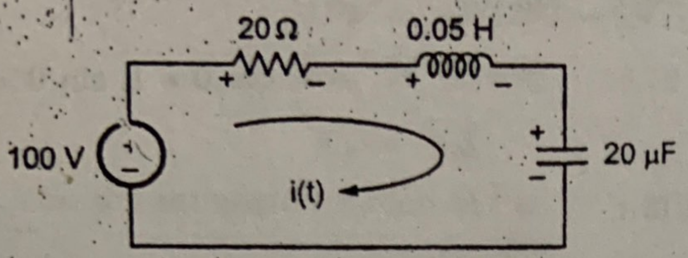


Fig. 10.38 (a)

For all $t \geq 0^+$, switch is closed as shown in the Fig. 10.38 (a).

Applying KVL we get,

$$-20\ i(t) - 0.05\frac{di(t)}{dt} - \frac{1}{20\times10^{-6}}\int_{-\infty}^{t} i(t)dt - 100 = 0$$

Separating limits of integration,

$$-20\ i(t) - 0.05\frac{di(t)}{dt} - \frac{1}{20\times10^{-6}}\int_{-\infty}^{t} i(t)dt - \frac{1}{20\times10^{-6}}\int_{0}^{t} i(t)dt = -100$$

Now initial voltage across C is zero from equation (2). So changing signs of all terms and putting value of third term on L.H.S. as zero, we get,

$$20\ i(t) + 0.05\frac{di(t)}{dt} + \frac{1}{20\times10^{-6}}\int_{0}^{t} i(t)dt = 100 \quad ...(3)$$

Differentiating equation (3) with respect to t, we get,

$$20\frac{di(t)}{dt} + 0.05\frac{d^2i(t)}{dt^2} + \frac{i(t)}{20\times10^{-6}} = 0$$

Dividing both sides by factor 0.05 and rearranging the terms, we get,

$$\frac{d^2i(t)}{dt^2} + 400\frac{di(t)}{dt^2} + 1\times10^6\, i(t) = 0 \quad ...(4)$$

Above equation is second order, homogeneous linear differential equation. The auxillary equation is given by,

$$S^2 + 400\ S + 1\times10^6 = 0 \quad ...(5)$$

The roots are given by,

$$S_{1,2} = \frac{-400 \pm \sqrt{(400)^2 - 4(1)(1\times10^6)}}{2(1)} = \frac{-400 \pm j1959.59}{2}$$

$$\therefore \quad S_{1,2} = (-200 \pm j\ 979.79)$$

i.e.

$$S_1 = -\alpha + j\omega_d = -200 + j\ 979.79$$

$$S_2 = -\alpha - j\omega_d = -200 - j\ 979.79$$

So roots are complex conjugates with negative real parts. Hence the solution for the equation (4) is given by,

$$i(t) = K_1 e^{-\alpha t} \cos \omega_d t + K_2 e^{-\alpha t} \sin \omega_d t$$

i.e $$i(t) = e^{-200t} [K_1 \cos 979.79\, t + K_2 \sin 979.79\, t] \quad \ldots(6)$$

Now At t = 0, i(t) = 0, so substituting these values in equation (6), we get,

$$0 = e^{-0} [K_1 \cos 0 + K_2 \sin 0]$$

$$\therefore \quad K_1 = 0 \quad \ldots(7) \ldots [\because \cos 0 = 1, \sin 0 = 0, e^{-0} = 1]$$

Hence equation (6) gets modified to new equation given by,

$$i(t) = e^{-200t} [K_2 \sin 979.79\, t] \quad \ldots(8)$$

Differentiating equation (8) with respect to t, we get,

$$\frac{di(t)}{dt} = K_2 [e^{-200t} (\cos 979.99\, t)(979.99) + (-200)\, e^{-200t} (\sin 979.99\, t)]$$

At t = 0,

$$\therefore \quad \frac{di}{dt}(0) = K_2 [e^{-0} (979.99) \cos 0 - (200) \cdot e^{-0} (\sin 0)]$$

$$\therefore \quad \frac{di}{dt}(0) = 979.99\, K_2 \quad \ldots(9) \ldots (\because e^{-0} = 1, \cos 0 = 1, \sin 0 = 0)$$

Now at t = 0, equation (3) becomes,

$$20\, i(0) + 0.05 \frac{di}{dt}(0) + 0 = 100$$

$$\therefore \quad \frac{di}{dt}(0) = 2000 \text{ A/sec} \quad \ldots(10)$$

Equating equations (9) and (10) we get,

$$K_2 = 2.0408 \quad \ldots(11)$$

Hence the solution of the i(t) is given by,

$$i(t) = [2.048 \sin 979.99\, t]\, e^{-200\, t} \text{ A}$$

Example 10.22 : *Find the expression for the current in a series RLC circuit fed by a d.c. voltage of 20 V with R = 4 Ω, L = 1 H and C =* $\frac{1}{4}$ *F. Assume initial conditions to be zero.*

(JNTU : Dec.-2005/ NR)

Solution : Assuming zero initial conditions, i.e.,

$$i_L(0^-) = 0 = i_L(0^+) \quad \ldots(1)$$

$$v_C(0^-) = 0 = v_C(0^+) \quad ...(2)$$

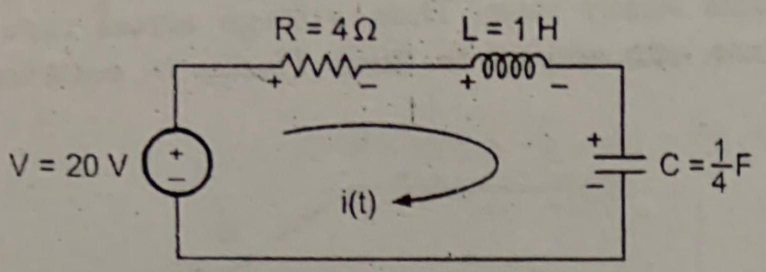


Fig. 10.39

Because current through L and voltage across C can not change instantaneously.

To find expression for i(t) through series RLC circuit is as shown in the Fig. 10.39.

Applying KVL to loop we get,

$$-4i(t) - 1\frac{di(t)}{dt} - \frac{1}{(1/4)}\int_{-\infty}^{t} i(t)\,dt + 20 = 0$$

$$\therefore -4i(t) - \frac{di(t)}{dt} - \left[4\int_{-\infty}^{0^-} i(t)\,dt + 4\int_{0^-}^{t} i(t)\,dt\right] = -20$$

Now changing signs of all the terms and putting first integral term value zero from equation (2), we get,

$$4i(t) + \frac{di(t)}{dt} + 4\int_{0}^{t} i(t)\,dt = 20 \quad ...(3)$$

Differentiating both the sides of above equation with respect to t, we get,

$$4\frac{di(t)}{dt} + \frac{d^2i(t)}{dt^2} + 4\,i(t) = 0$$

Rearranging terms, we get,

$$\frac{d^2i(t)}{dt^2} + 4\frac{di(t)}{dt} + 4\,i(t) = 0 \quad ...(4)$$

Above is second order, homogeneous differential equation which has complementary function as solution.

The auxillary equation is given by,

$$s^2 + 4s + 4 = 0 \quad ...(5)$$

$\frac{b \pm \sqrt{b^2-4ac}}{2a}$

Finding roots of equation,

$$s_{1,2} = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(4)}}{2(1)} = \frac{-4 \pm 0}{2} = -2$$

$$\therefore \quad s_1 = s_2 = -2$$

So the roots are real, equal and negative. Hence solution is given by,

$$i(t) = K_1 e^{-st} + K_2 t\, e^{-st}$$

i.e.
$$i(t) = K_1 e^{-2t} + K_2 t\, e^{-2t} \quad \text{...6)}$$

At $t = 0$, $i(t) = 0$, putting values in equation (6), we get,

$$\therefore \quad 0 = K_1 e^{-0} + K_2 (0)\, e^{-0}$$

$$\therefore \quad K_1 = 0 \quad \text{...(7)}$$

Hence
$$i(t) = K_2\, t\, e^{-2t} \quad \text{...(8)}$$

Now put $t = 0$ in equation (3), we get,

$$4i(0) + \frac{di}{dt}(0) + 4\,(0) = 20$$

$$\therefore \quad \frac{di}{dt}(0) = 20 \text{ A/sec} \quad \text{...(9)}$$

Differentiating equation (8) with respect to t, we get, — (1)

$$\frac{di(t)}{dt} = K_2\,[t\,(-2)\, e^{-2t} + e^{-2t}\,(1)]$$

$$\therefore \quad \frac{di(t)}{dt} = K_2\,(-2\,t\, e^{-2t}) + K_2\, e^{-2t} \quad \text{...(10)}$$

At $t = 0$, equation (10) becomes,

$$\boxed{\frac{di}{dt}(0)} = 20 = K_2\,[-2(0)\, e^{-0}] + K_2\, e^{-0}$$

$$\therefore \quad K_2 = 20 \quad \text{...(11)}$$

Hence the expression for the current in series RLC circuit is given by,

$$i(t) = \mathbf{20\, t\, e^{-2t}\ A}$$

f(t)	F(s)
1	$\frac{1}{s}$.
Constant K	$\frac{K}{s}$
K f(t), K is constant	K F(s)
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
e^{at}	$\frac{1}{s-a}$
$e^{-at}\ t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$

$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$t.e^{-at}$	$\frac{1}{(s+a)^2}$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$

Example 11.14 : *For series RLC circuit, the capacitor is initially charged to 1 V, find the current $i(t)$ when the switch K is closed at $t = 0$. Use Laplace transform.*

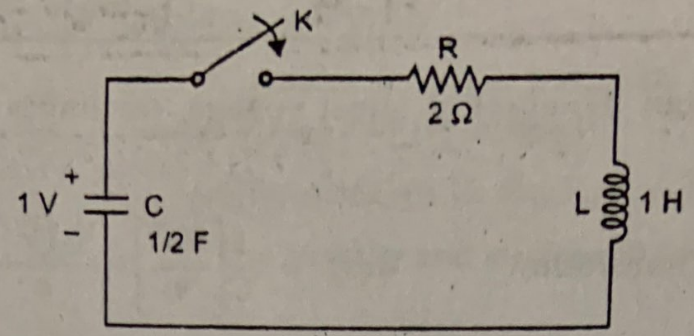


Fig. 11.30

Solution : Assume the current i(t) as shown in the Fig. 11.30 (a).

Applying KVL to the loop,

$$-2i(t)-1\frac{di(t)}{dt}-\frac{1}{\left(\frac{1}{2}\right)}\int_{-\infty}^{t} i(t)\,dt = 0$$

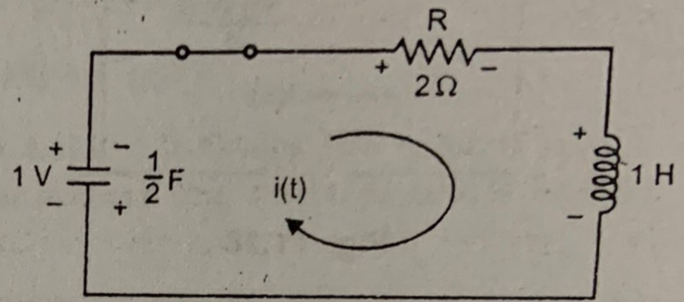


Fig. 11.30 (a)

The inductor initial current is zero

$$\therefore \quad i(0^-) = 0$$

and

$$V_C(0^-) = 1\text{ V} \quad \text{... Given}$$

$$\therefore \quad 2\,i(t)+\frac{di(t)}{dt}+2\int_{-\infty}^{t} i(t)\,dt = 0$$

The limits $-\infty$ to t can be split as $-\infty$ to 0^- and 0 to t so we get,

$$2i(t)+\frac{di(t)}{dt}+2\left[\int_{-\infty}^{0^-} i(t)\,dt+\int_{0}^{t} i(t)\,dt\right] = 0$$

Now

$$2\int_{-\infty}^{0^-} i(t)\,dt = V_C(0^-) = 1\text{ V.}$$

But as the **polarities** of $V_C(0^-)$ are **opposite** to that produced by i(t), the term must be written as negative.

$$\therefore \; 2i(t)+\frac{di(t)}{dt}-1+2\int_{0}^{t} i(t)\,dt = 0$$

Taking Laplace transform of the equation,

$$2\,I(s)+[sI(s)-i(0^-)]-\frac{1}{s}+\frac{2\,I(s)}{s} = 0$$

$$\therefore \; I(s)\left[2+s+\frac{2}{s}\right] = \frac{1}{s} \quad \text{... } i(0^-) = 0$$

$$\therefore \; I(s) = \frac{s}{s(s^2+2s+2)} = \frac{1}{s^2+2s+2} = \frac{1}{(s^2+2s+1)+1}$$

... Completing square

$$\therefore \; I(s) = \frac{1}{(s+1)^2+(1)^2}$$

Taking Laplace inverse we get,

$$\therefore \; i(t) = e^{-t} \sin 1t = e^{-t} \sin (t) \text{ A}$$

Example 11.15 : *In the RL circuit shown in the Fig. 11.31 the switch is in position 1 long enough to establish the steady state conditions. At t = 0, the switch is thrown to position 2. Find the expression for the resulting current.*

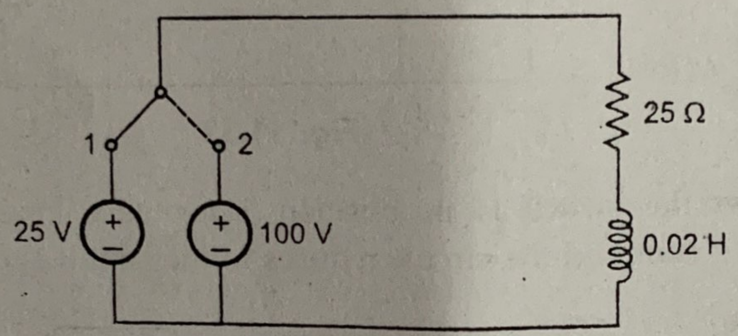


Fig. 11.31

Solution : When the switch is in the position 1, the circuit reduces to the circuit shown in the Fig. 11.31 (a).

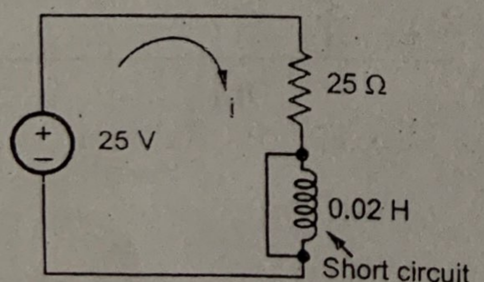


Fig. 11.31 (a)

Under steady state conditions the inductance acts a **short circuit** hence,

$$i = \frac{25}{25} = 1 \text{ A}$$

This becomes initial current through the circuit, just before the switch is to be thrown to position 2.

$$\therefore \; i(0^-) = 1 \text{ A}$$

When the switch is in the position 2, the circuit reduces to as shown in the Fig. 11.31 (b).

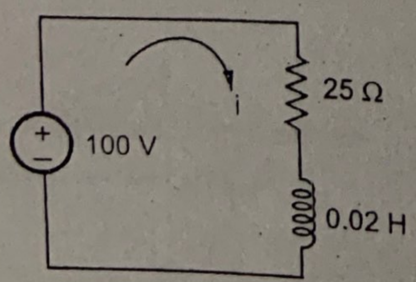


Fig. 11.31 (b)

Applying KVL,

$$-25\, i(t) - 0.02 \frac{di(t)}{dt} + 100 = 0$$

$$\therefore \quad 0.02 \frac{di(t)}{dt} + 25\ i(t) = 100$$

Taking Laplace transform of the equation we get,

$$0.02\ s\ I(s) - 0.02\ i(0^-) + 25\ I(s) = \frac{100}{s}$$

$$\therefore \quad 0.02\ s\ I(s) - 0.02\ (1) + 25\ I(s) = \frac{100}{s} \qquad \ldots\ i(0^-) = 1\ A$$

$$\therefore \quad I(s)\ [0.02\ s + 25] = \frac{100}{s} + 0.02$$

$$\therefore \quad I(s) = \frac{(100 + 0.02\ s)}{s(25 + 0.02\ s)}$$

$$\therefore \quad I(s) = \frac{0.02(s+5000)}{0.02\ s(s+1250)} = \frac{s+5000}{s(s+1250)}$$

$$\therefore \quad \frac{s+5000}{s(s+1250)} = \frac{A}{s} + \frac{B}{s+1250}$$

Now $\quad A = \left.\frac{s+5000}{s+1250}\right|_{s=0} = 4$

and $\quad B = \left.\frac{s+5000}{s}\right|_{s=-1250} = -3$

$$\therefore \quad I(s) = \frac{4}{s} - \frac{3}{s+1250}$$

Taking Laplace inverse of I(s),

$$\therefore \quad i(t) = L^{-1}\ [I(s)]$$

$$= 4 - 3 \cdot e^{-1250\ t}\ A$$

Example 11.16 : *In the network shown in the Fig. 11.32, the switch K is thrown from the position a to b at t = 0. Just before the switch is thrown to b, the initial conditions are $i(0^-) = 2$ A and $V_C(0^-) = 2$ V. Find the current i(t). Assume L = 1 H, R = 3 Ω, C = 0.5 μF and $V_1 = 5$ V.*

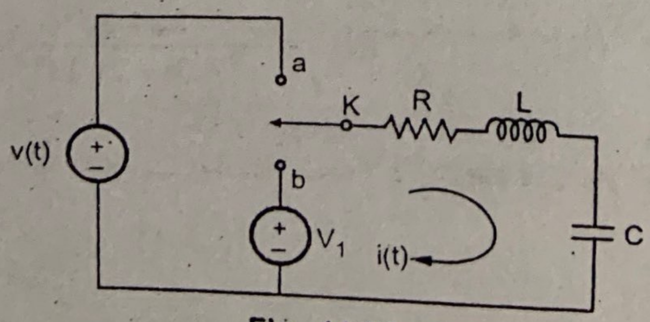


Fig. 11.32

Solution : When switch is thrown to position b, the circuit becomes as shown in th Fig. 11.32 (a).

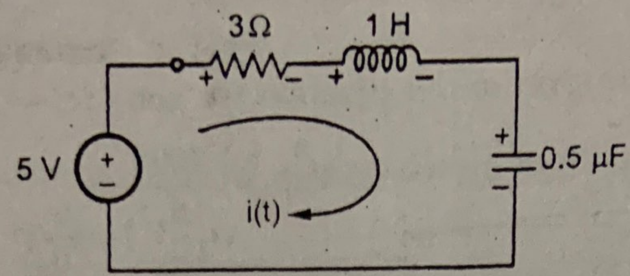


Fig. 11.32 (a)

Applying KVL,

$$-3i(t) - \frac{1\,di(t)}{dt} - \frac{1}{0.5\times 10^{-6}} \int_{-\infty}^{t} i(t)\,dt + 5 = 0$$

Splitting the limits of integration,

$$3\,i(t) + \frac{di(t)}{dt} + \frac{1}{0.5\times 10^{-6}} \int_{-\infty}^{0-} i(t)\,dt + \frac{1}{0.5\times 10^{-6}} \int_{0}^{t} i(t)\,dt = 5$$

Now $$\frac{1}{0.5\times 10^{-6}} \int_{-\infty}^{0-} i(t)\,dt = V_C(0^-) = 2$$

Assume the polarities of $V_C(0^-)$ same as the polarities of the voltage drop across C due to i(t).

$$\therefore \quad 3\,i(t) + \frac{di(t)}{dt} + 2 + \frac{1}{0.5\times 10^{-6}} \int_{0}^{t} i(t)\,dt = 5$$

Taking Laplace transform,

$$3I(s) + s\,I(s) - i(0^-) + \frac{2}{s} + \frac{1}{0.5\times 10^{-6}} \frac{I(s)}{s} = \frac{5}{s}$$

$$\therefore \quad I(s)\left[3 + s + \frac{2\times 10^{6}}{s}\right] = \frac{5}{s} - \frac{2}{s} + 2$$

$$\therefore \quad I(s)\left[\frac{3s + s^2 + 2\times 10^{6}}{s}\right] = \frac{(3+2s)}{s}$$

$$I(s) = \frac{2s+3}{s^2 + 3s + 2\times 10^{6}} = \frac{2s+3}{s^2 + 3s + \frac{9}{4} + 2\times 10^{6} - \frac{9}{4}}$$

$$= 2\left\{\frac{\left(s+\frac{3}{2}\right)}{\left(s+\frac{3}{2}\right)^2 + (1.4142\times 10^3)^2}\right\}$$

$$\therefore \quad i(t) = 2\,e^{-\frac{3}{2}t}\cos(1.4142\times 10^3\,t)\ \text{A}$$

Example 11.18 : *In the network shown in the Fig. 11.34, the switch K is moved from position 'a' to 'b' at t = 0 (a steady state existing in position 'a' before t = 0). Solve for the current i(t), using Laplace Transformation.*

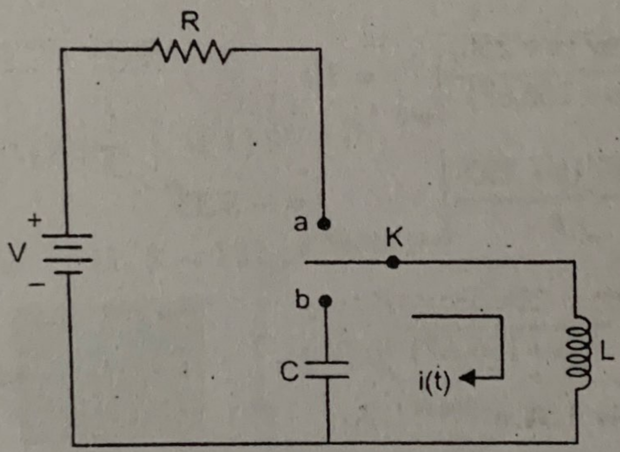


Fig. 11.34

Solution : When switch is in position 'a', steady state is achieved and inductor L acts as short circuit in the steady state. So circuit reduces as shown in the Fig. 11.34 (a).

$$\therefore \qquad i'(t) = \frac{V}{R}$$

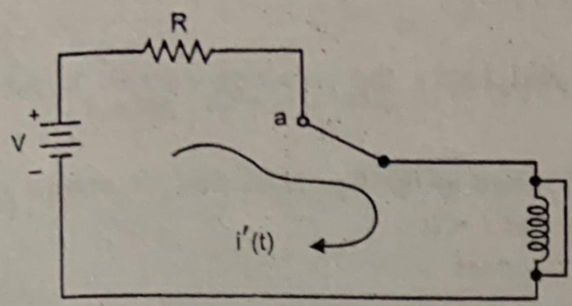


Fig. 11.34 (a)

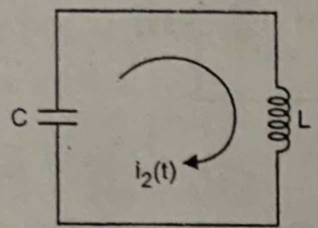


Fig. 11.34 (b)

Same current flows through the inductor L just before switch is closed to 'b' at t = 0.

$$\therefore i_L(0^-) = \frac{V}{R}\ A$$

... Initial current through L

As capacitor is out of the circuit when switch is in position 'a',

$$\therefore V_C(0^-) = 0\ V$$

When switch is in position 'b' circuit reduces as shown in the Fig. 11.34 (b).

Applying KVL,

$$-L\frac{di(t)}{dt} - \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt = 0$$

But as $V_C(0^-) = 0$, limits can be changed to 0 to t.

$$\therefore \quad L\frac{di(t)}{dt} + \frac{1}{C}\int_{0}^{t} i(t)\,dt = 0$$

$$\therefore \quad L\,[s\,I(s) - i_L(0^-)] + \frac{1}{C}\cdot\frac{I(s)}{s} = 0 \qquad \text{... Taking Laplace transform}$$

$$\therefore \quad I(s)\left[sL + \frac{1}{sC}\right] - L\times\frac{V}{R} = 0 \qquad \text{... } i_L(0^-) = \frac{V}{R}$$

$$\therefore \quad I(s)\left[\frac{s^2LC+1}{sC}\right] = \frac{VL}{R}$$

$$I(s) = \frac{\left(\frac{VL}{R}\right)C\,s}{s^2\,LC+1} = \frac{\frac{VLC}{R}}{LC}\times\frac{s}{s^2+\frac{1}{LC}}$$

$$\therefore \quad I(s) = \left(\frac{V}{R}\right)\frac{s}{s^2+\left(\frac{1}{\sqrt{LC}}\right)^2}$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R} \cos \frac{t}{\sqrt{LC}} \text{ A}$$

Note that LC circuit forms a tank circuit and hence the resulting current is purely oscillatory.