Distribution Function :-

The probability $P\{X \le x\}$ is the probability of the event $\{X \le x\}$. It is a number that depends on x; that is, it is a function of x. This function is denoted by $F_X(x)$, the cumulative probability distribution function of the random variable X.

Thus

$$F_X(x) = P\{X \le x\} \quad \text{— (1)}$$

$F_X(x)$ is also distribution function of X.

The argument x is any real number ranging from $-\infty$ to ∞.

The distribution function has some specific properties derived from the fact that $F_X(x)$ is a probability. These are:

$$\left.\begin{aligned}
&(1)\ F_X(-\infty) = 0 \\
&(2)\ F_X(\infty) = 1 \\
&(3)\ 0 \le F_X(x) \le 1 \\
&(4)\ F_X(x_1) \le F_X(x_2) \quad \text{if } x_1 < x_2 \\
&(5)\ P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)
\end{aligned}\right\} \text{— (2)}$$

(6) $F_X(x^{-1}) =$

Proofs:-

(1) $F_X(-\infty) = 0$

$F_X(-\infty) =$

the event $\{X \le$

hence $P\{X \le -$

(2) $F_X(\infty) = 1$

$F_X(\infty) =$

$=$

$X \le \infty$ mean

Sample space

(3) $0 \le F_X(x)$

Since $F_X($

$F_X($

$\therefore$ the

(4) $F_X(x_1) \le$

$\{$ It st

(6) $F_X(x^{-1}) = F_X(x)$

Proofs:-

① $F_X(-\infty) = 0$

$$F_X(-\infty) = P\{X \le -\infty\}$$

The event $\{X \le -\infty\}$ is impossible event and hence $P\{X \le -\infty\} = 0$ i.e., $F_X(-\infty) = 0$

② $F_X(\infty) = 1$

$$F_X(\infty) = P\{X \le \infty\}$$

$$= P\{X = -\infty\} + \dots + P\{X = -1\} +$$

$$P\{X = 0\} + P\{X = 1\} + \dots + P\{X = \infty\}$$

$$= P\{\text{sample space}\}$$

$$= 1$$

$X \le \infty$ means all events possible i.e., sample space

③ $0 \le F_X(x) \le 1$

Since $F_X(-\infty) = 0$ (minimum value)

$F_X(\infty) = 1$ (maximum value)

∴ the value of $F_X(x)$ lies between 0 & 1

④ $F_X(x_1) \le F_X(x_2)$ when $x_1 < x_2$.

{ It states that $F_X(x)$ is non-decreasing Function } when $x_1 < x_2$, the events can represent by $\{X \le x_2\} \ge \{X \le x_1\}$

Apply probability on both sides

$$P\{X \le x_2\} \ge P\{X \le x_1\}$$

$$F_X(x_2) \ge F_X(x_1)$$

$$S = \{0, \underset{x_1}{1}, 2, \underset{x_2}{3}\}$$

(5) $P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$

$$\{X \le x_2\} = \{X \le x_1\} \cup \{x_1 < X \le x_2\}$$

$$F_X(x_2) = F_X(x_1) + P\{x_1 < X \le x_2\}$$

$$\therefore P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$$

(6) $F_X(x^+) = F_X(x)$

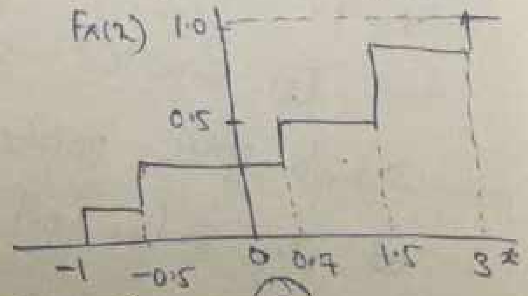
It states that distribution is continuous function.

Properties 1, 2, 4 & 6 may be used as tests to determine whether a function $G_X(x)$ is a valid distribution function.

If X is a discrete random variable then $F_X(x)$ have a stair step form, as shown in Fig. The amplitude of a step will

$F_X(x) = \sum_{i=1} P\{X=x_i\}u(x-x_i)$

equal the probability of occurrence of the value of X where the step occurs. If the values of X are denoted x_i, we may write $F_X(x)$ as



$$F_X(x) = \sum_{i=1}^{N} P\{X=x_i\}u(x-x_i) \quad — (3)$$

where $u(\cdot)$ is the unit step function defined by

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad — (4)$$

and N may be infinite for some random variables. By ~~introducing~~

$$p(x_i) = P\{X=x_i\} \quad — (5)$$

eq^n (3) can be written as

$$F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x-x_i)$$

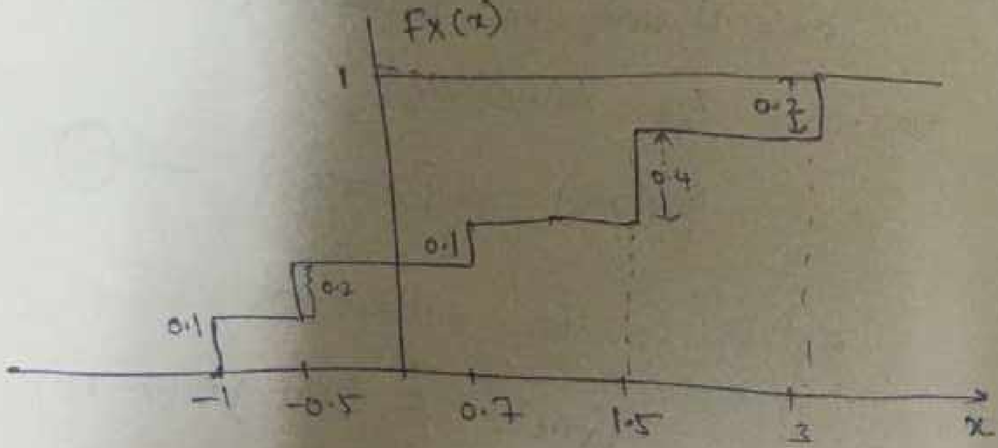
Ex:- Let X have the discrete values in the set $\{-1, -0.5, 0.7, 1.5, 3\}$. The corresponding probabilities are assumed to be $\{0.1, 0.2, 0.1, 0.4, 0.2\}$. Now $P\{X<-1\}=0$ because there are no sample space points in the set $\{X<-1\}$.

only when $X = -1$, we obtain one outcome. Thus there is an immediate jump in probability of 0.1 in the function $F_X(x)$ at the point $x = -1$.

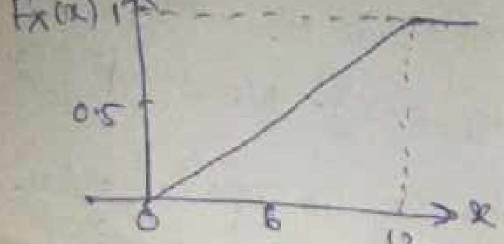
For $-1 < x < -0.5$, there are no additional sample space points so $F_X(x)$ remains constant at the value 0.1.

At $x = -0.5$ there is another jump of 0.2 in $F_X(x)$.

This process continues until all points are included. $F_X(x)$ then equals 1.0 for all x above the last point.



A continuous random variable will have a continuous distribution function. Fig shows $F_X(x)$ is the continuous function



Consider the fair wheel of chance experiment. Let the wheel be numbered from 0 to 12. The probability of the event $\{X \le 0\}$ is 0 because there are no sample space points in this set.

For $0 < x \le 12$ the probability of $\{0 < X \le x\}$ will increase linearly with x for a fair wheel.

The distribution function of a mixed random variable will be a sum of two parts, one is of stairstep form, the other continuous.

Density Function :-

The probability density function, denoted by $f_X(x)$, is defined as the derivative of the distribution function:

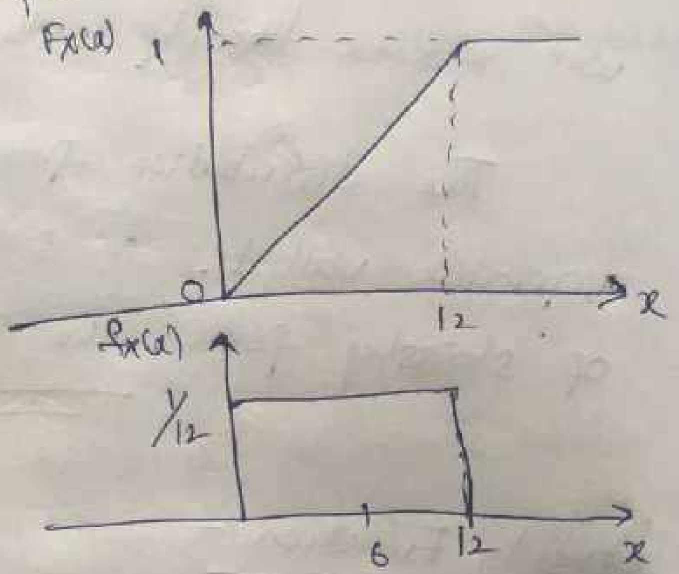
$$f_X(x) = \frac{dF_X(x)}{dx} \quad \text{— (1)}$$

$f_X(x)$ is called density function of the random variable X.

Existence :-

If the derivative of $F_X(x)$ exists, then $f_X(x)$ exists. There may be places where $dF_X(x)/dx$ is not defined.

For example, a continuous random Variable will have a continuous distribution $F_X(x)$, but $F_X(x)$ may have corners i.e., points of abrupt change in slope. For this case, $f_X(x)$ is plotted as a function with step-type discontinuities



For discrete random variables having a stairstep form of distribution function, the unit impulse function $\delta(x)$ is used to describe the derivative of $F_X(x)$ at its stairstep points.

$\delta(x)$ may be defined by its integral property

$$\phi(x_0) = \int_{-\infty}^{\infty} \phi(x)\,\delta(x-x_0)\,dx$$

where $\phi(x)$ is any function continuous at the point $x = x_0$; $\delta(x)$ can be interpreted as a function with infinite amplitude, area of unity, and zero duration.

The unit impulse and the unit-step functions are related by

$$\delta(x) = \frac{du(x)}{dx}$$

or

$$\int_{-\infty}^{x} \delta(\xi)\, d\xi = u(x)$$

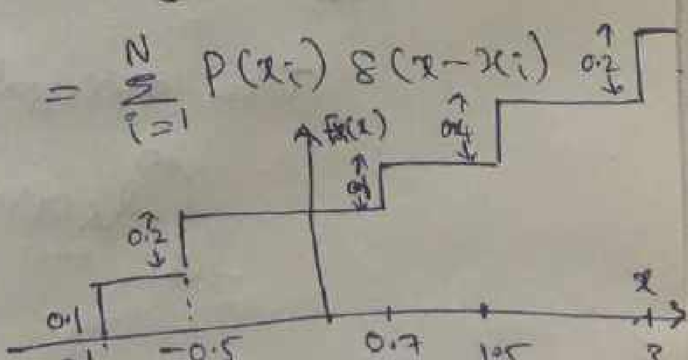
The general impulse function is shown as a vertical arrow occurring at the point $x = x_0$ and having an amplitude equal to the amplitude of the step function for which it is the derivative.

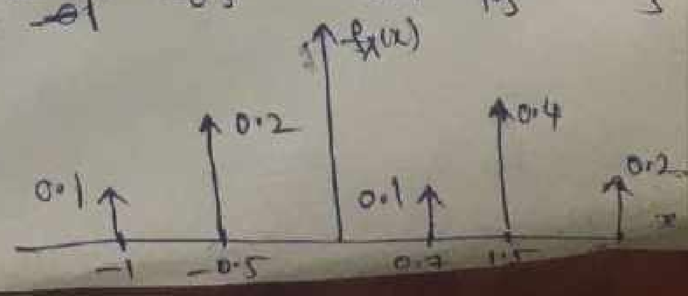
In the case of a discrete random variable, CDF is

$$F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x - x_i)$$

pdf is $f_X(x) = \dfrac{dF_X(x)}{dx}$

$$= \sum_{i=1}^{N} P(x_i)\, \delta(x - x_i)$$





Properties of Density Functions

(1) $0 \leq f_X(x)$ all x {PDF is non negative}

(2) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ {Area under PDF curve is unity}

(3) $F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi$

(4) $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)\,dx$

Proofs ① $f_X(x) \geq 0$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Since $F_X(x)$ is monotonically increasing function, the derivative of $F_X(x)$ is always positive. $F_X(x)$ is also non negative

② $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

~~To~~ We know $f_X(x) = \frac{d}{dx} F_X(x)$

Take integration to above equation

$$\int_{-\infty}^{\infty} f_X(x) = \int_{-\infty}^{\infty} \frac{d}{dx} F_X(x)$$

$$= F_X(\infty) - F_X(-\infty)$$

$$= 1 - 0 = 1$$

(3) $F_X(x) = \int_{-\infty}^{\infty} f_X(v)\,dv$

It is given that CDF is integral of PDF

we know $f_X(x) = \frac{d}{dx} F_X(x)$

integrating on both sides

$$\int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} \frac{d}{dx} F_X(x)$$

$$= F_X(x)\Big|_{-\infty}^{x}$$

$$= F_X(x) - F_X(-\infty)$$

$$= F_X(x) - 0$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

(4) $P\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(x)\,dx$

$$P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$$

$$= \int_{-\infty}^{x_2} f_X(x)\,dx - \int_{-\infty}^{x_1} f_X(x)\,dx$$

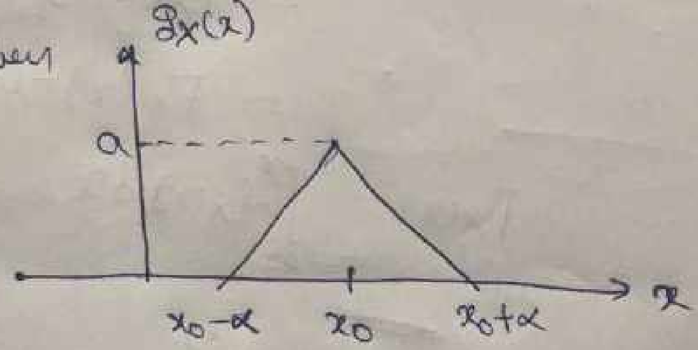
$$= \cancel{\int_{-\infty}^{x_1} f_X(x)\,dx} + \int_{x_1}^{x_2} f_X(x)\,dx$$

$$- \cancel{\int_{-\infty}^{x_1} f_X(x)\,dx}$$

$$P\{x_1 < x \le x_2\} = \int_{x_1}^{x_2} f_X(x)\,dx.$$

Properties 1 & 2 require that the density function be non negative and have an area of unity. These two properties may also be used as tests to see if some function can be a valid probability density function. Both tests must be satisfied for validity.

Ex: Given



Test the function $g_X(x)$ whether it is valid density function. Find distribution function

Ans:- 1. $g_X(x)$ is non negative. it satisfies property 1

2. Its area is $a\alpha$, which must equal unity to satisfy property 2.

$\therefore\ a = \frac{1}{\alpha}$ is necessary if $g_X(x)$ is to be a density.

Problem:- The probability density function is given as

$$f_X(x) = \begin{cases} 0 & 3 > x \geq 13 \\ (x-3)/25 & 3 \leq x < 8 \\ 0.2-(x-8)/25 & 8 \leq x < 13 \end{cases}$$

Find the probability that X has values greater than 4.5 but not greater than 6.7..

Sol^n $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)\, dx$

from properties

$$P\{4.5 < X \leq 6.7\} = \int_{4.5}^{6.7} \left[(x-3)/25\right] dx$$

$$= \frac{1}{25}\left[\frac{x^2}{2} - 3x\right]\Big|_{4.5}^{6.7}$$

$$= 0.2288$$

Thus the event $\{4.5 < X \leq 6.7\}$ has a probability of 0.2288 or 22.88 %

Problem:- A random variable X is known to have a distribution function

$$F_X(x) = u(x)\left[1 - e^{-x^2/b}\right]$$

where $b > 0$ is a constant. Find its density function.

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$= u(x)\frac{d}{dx}\left[1 - e^{-x^2/b}\right] + \left[1 - e^{-x^2/b}\right]\frac{du(x)}{dx}$$

$$= \left(1 - e^{-x^2/b}\right)\delta(x) + u(x)\frac{2x}{b}e^{-x^2/b}$$

$$= u(x)\frac{2x}{b}e^{-x^2/b}$$

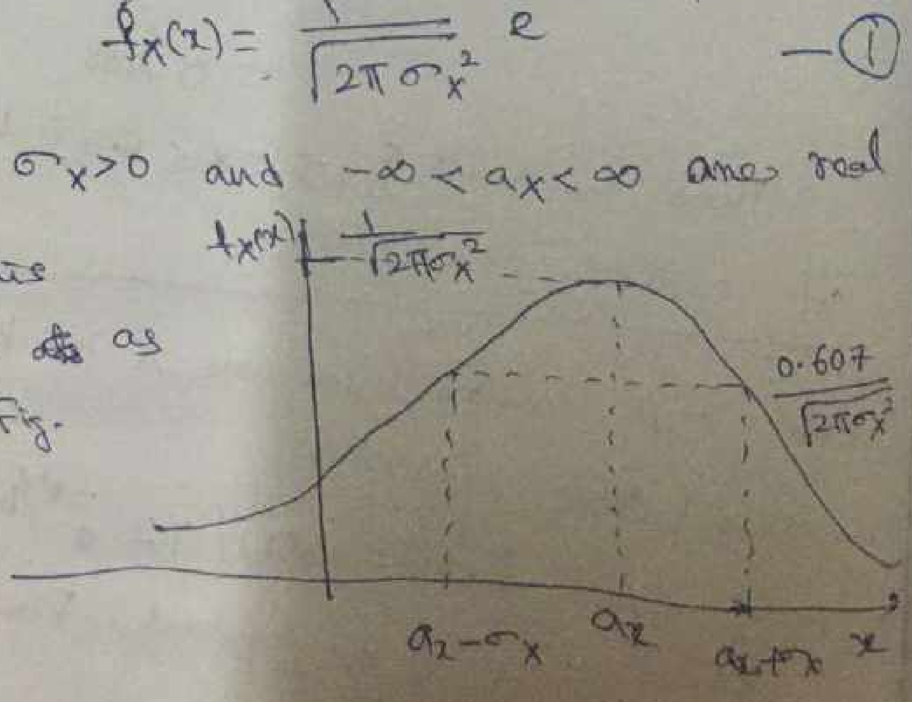
Since $e^{-x^2/b}\delta(x) = 1$

The Gaussian Random Variable :-

A random variable X is called gaussian if its density function has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-a_x)^2/2\sigma_X^2} \quad —(1)$$

where $\sigma_X > 0$ and $-\infty < a_X < \infty$ are real constants. This function is ~~ds~~ as shown in Fig.



Its maximum value $(2\pi\sigma_x^2)^{-1/2}$ occurs at $x = a_x$. Its spread about the point $x = a_x$ is related to σ_x.

The function decreases to 0.607 times its maximum at $x = a_x + \sigma_x$ and $x = a_x - \sigma_x$.

The distribution function is given as

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-(v-a_x)^2/2\sigma_x^2}\, dv \quad -(2)$$

This integral has no known closed form solution & must be evaluated by numerical or approximation methods.

<u>Normalized case :-</u> where $a_x = 0$ & $\sigma_x = 1$, then the distribution function is $F(x)$ is

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-v^2/2}\, dv \quad -(3)$$

which is function of x only.

For negative value of x, we use the relation ship

$$F(-x) = 1 - F(x) \quad -(4)$$

To show that the general distribution function $F_X(x)$ of eq^n (2), can be found interms of $F(x)$ of eq^n (3), the variable is charged as

$$u = (v - a_x)/\sigma_x \quad \text{— (5)}$$

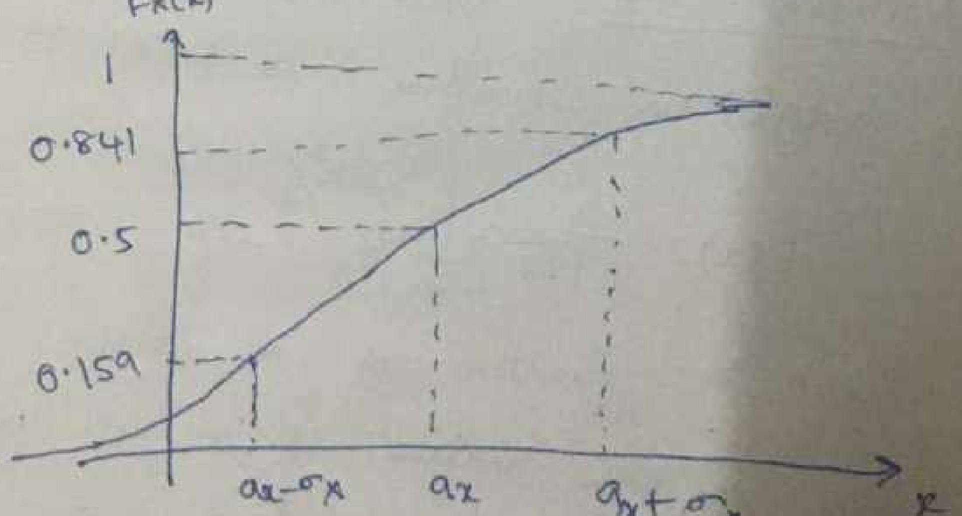
in eq^n (2) to obtain

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-a_x)/\sigma_x} e^{-u^2/2}\, du \quad \text{— (6)}$$

From eq^n (3), this expression is clearly equivalent to

$$F_X(x) = F\left(\frac{x-a_x}{\sigma_x}\right) \quad \text{— (7)}$$

following fig shows the behavior of $F_X(x)$



The function $F(x)$ can also be evaluated by approximation.

First we write $F(x)$ of eqn (3) as

$$F(x) = 1 - Q(x) \quad \text{—— (8)}$$

where $$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\xi^2/2}\, d\xi \quad \text{—— (9)}$$

Is known as the Q-function.

As With $F(x)$, $Q(x)$ has no known closed form solution, but does have an excellent approximation given by

$$Q(x) \simeq \left[\frac{1}{(1-a)x + a\sqrt{x^2+b}}\right] \frac{e^{-x^2/2}}{\sqrt{2\pi}}; \; x \geq 0 \quad \text{—— (10)}$$

where a and b are constants. This approximation has been found to give minimum absolute relative error, for any $x \geq 0$, when $a = 0.339$ & $b = 5.510$ with these values of a & b, the approximation of eqn (10) is said to equal the true value of $Q(x)$ within a maximum absolute error of $Q(x)$ for any $x \geq 0$. of 0.27%

(P) Find the probability of the event $\{X \leq 5.5\}$ for a gaussian random variable having $a_x = 3$ & $\sigma_x = 2$.

Sol^n:-

$$P\{X \leq 5.5\} = F_X(5.5)$$

$$F_X(x) = F\left(\frac{x - a_x}{\sigma_x}\right)$$

$$\frac{x - a_x}{\sigma_x} = \frac{5.5 - 3}{2} = 1.25$$

$$F_X(5.5) = F(1.25)$$

$$F(x) = 1 - Q(x)$$

$$Q(x) = \left[\frac{1}{(1-a)x + a\sqrt{x^2 + b}}\right] \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$x = 1.25$ $\quad a = 0.339$ & $b = 5.510$

$$Q(x) = \left[\frac{1}{(1-0.339)1.25 + 0.339\sqrt{(1.25)^2 + 5.51}}\right] \frac{e^{-(1.25)^2/2}}{\sqrt{2\pi}}$$

$$= \frac{1}{0.82625 + 0.9015} \frac{0.4578}{\sqrt{2\pi}}$$

$$= \frac{1}{1.727} \frac{0.4578}{\sqrt{2\pi}} = \frac{0.1826}{1.727}$$

$$Q(x) = 0.1057$$

$$F(x) = 1 - Q(x)$$

$$= 1 - 0.1057$$

$$= 0.8942$$

$$\therefore P\{X \le 5.5\} = F(1.25) = 0.8942$$

(2) Assume that the height of clouds above the ground at some location is a gaussian random variable X with $a_x = 1830\,m$ & $\sigma_x = 460\,m$. Find the probability that clouds will be higher than $2750\,m$

Solⁿ:- $P\{X > 2750\} = 1 - P\{X \le 2750\}$

$$= 1 - F_X(2750)$$

$$= 1 - F\left\{\frac{2750 - 1830}{460}\right\}$$

$$= 1 - F\{2.0\}$$

$$F\{2.0\} = 1 - Q\{2.0\}$$

$$Q\{2.0\} = \frac{1}{\left((1 - 0.339)2 + 0.339\sqrt{2^2 + 5.51}\right)} \frac{e^{-2^2/2}}{\sqrt{2\pi}}$$

$F\{2.0\} = 0.9772$

$P\{X > 2750\} = 1 - 0.97772$

$= 0.0228$

The probability that clouds are higher than 2750m is therefore about 2.28%.

③ a gaussian random variable for which $a_x = 7$ and $\sigma_x = 0.5$ and find the probability of the event $\{X \leq 7.3\}$.

Sol^n:- $P\{X \leq 7.3\} = F_X(7.3\} = F\left\{\frac{7.3-7}{0.5}\right\}$

$= F(0.6\}$

$F(0.6) = 1 - Q(0.6)$

$\approx 1 - \left(\frac{1}{0.661(0.6) + 0.339\sqrt{0.6^2 + 5.51}}\right) \frac{e^{-(0.6)^2/2}}{\sqrt{2\pi}}$

$P\{X \leq 7.3\} \approx 0.7264$

Application of Gaussian Random Variable:-

The gaussian density is the most important of all densities in the areas of science & engineering. It is used to describe many practical & significant real-world quantities, especially when such quantities are the result of many small independent random effects acting to create the quantity of interest.

Ex:- ① the voltage across a resistor at the output of an amplifier can be random due to a random current that is the result of many ~~small~~ ~~independent~~ contributions from other random currents at various places within the amplifier

② Random thermal agitation of electrons causes the randomness of the various currents.

This type of noise is called gaussian because the random variable representing the noise voltage has the gaussian density

Binomial Density function :-

This is for discrete random variable

Let $0<p<1$, and $N=1,2,\dots$, then the function

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k) \quad -①$$

is called the binomial density function.

The Quantity $\binom{N}{k}$ is the binomial coefficient defined as

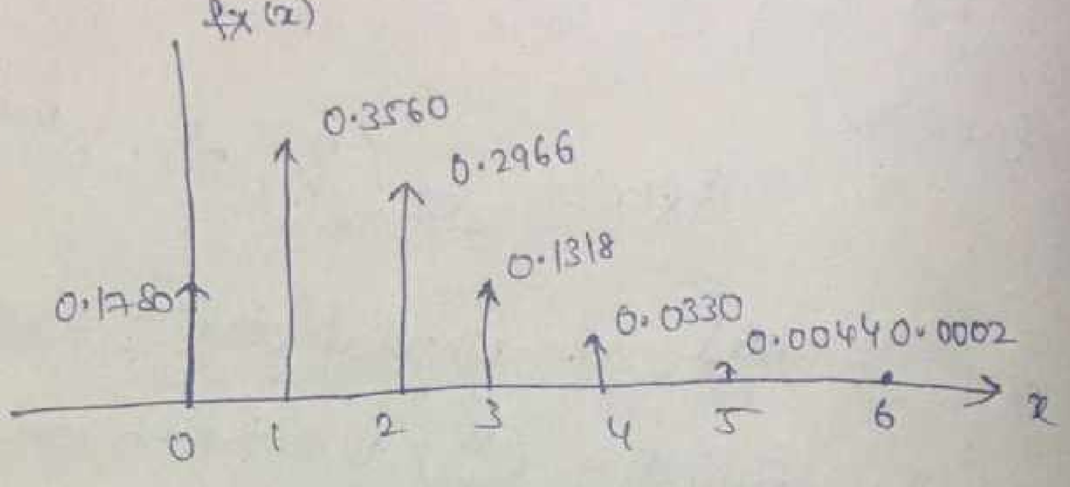
$$\binom{N}{k} = {}^N C_k = \frac{N!}{k!(N-k)!} \quad -②$$

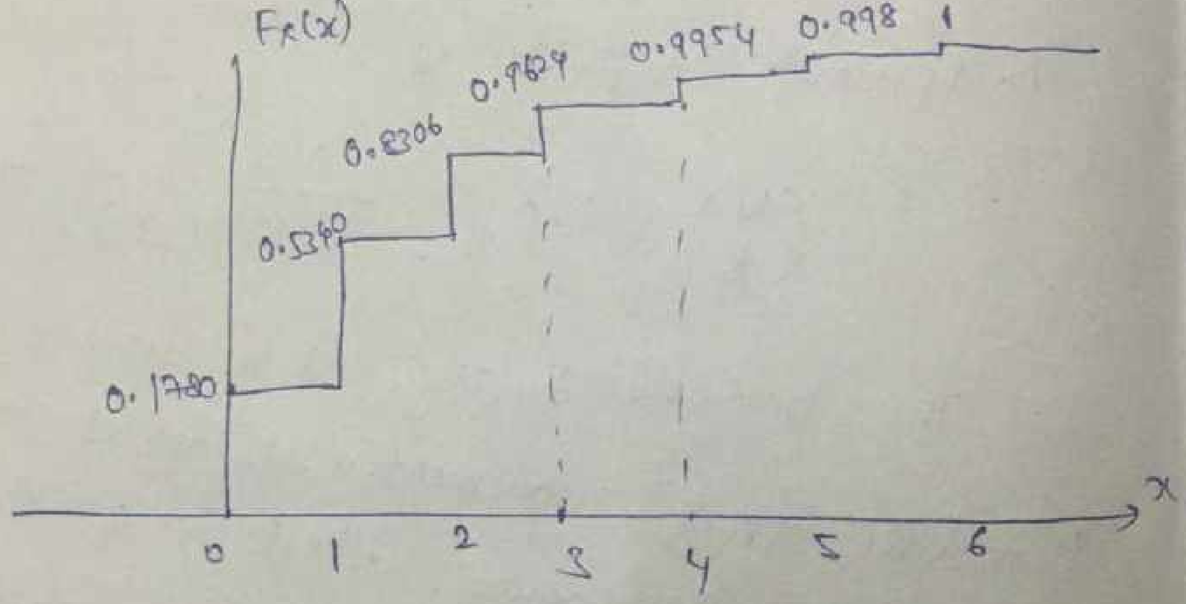
The binomial density can be applied to Bernoulli trial experiment. It applies to many games of chance, detection problems in radar and sonar, and many experiments having only two possible outcomes on any given trail.

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$F_X(x) = \sum_{k=0}^{N} {}^N C_k \, p^k (1-p)^{N-k} u(x-k) \quad -③$$

Fig shows the binomial density & distribution functions for $N=6$ & $p=0.25$





Poisson:- This is for discrete random variable

The Poisson random variable X has a density & distribution given by

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(n-k) \quad \text{—} (4)$$

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(n-k) \quad \text{—} (5)$$

where $b>0$ is a real constant.

When plotted, these functions appear quite similar to those for the binomial random variable. In fact, if $N \to \infty$ & $p \to 0$ for the binomial case in such a way that $Np = b$, a constant, the Poisson case results.

Applications

1. The Poisson random variable applies to a wide variety of counting-type applications.

2. It describes the number of defective units in a sample taken from a production line, the number of telephone calls made during a period of time, the number of electrons emitted from a small section of a cathode in a given time interval, etc.

If the time interval of interest has duration T, and the events being counted are known to occur at an average rate λ and have a poisson distribution, then b in eq^n (4) is given by

$$b = \lambda T$$

Uniform:-

The uniform probability density & distribution functions are defined by

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

$F_X(x) = \int_a^x f_X(x)\,dx$

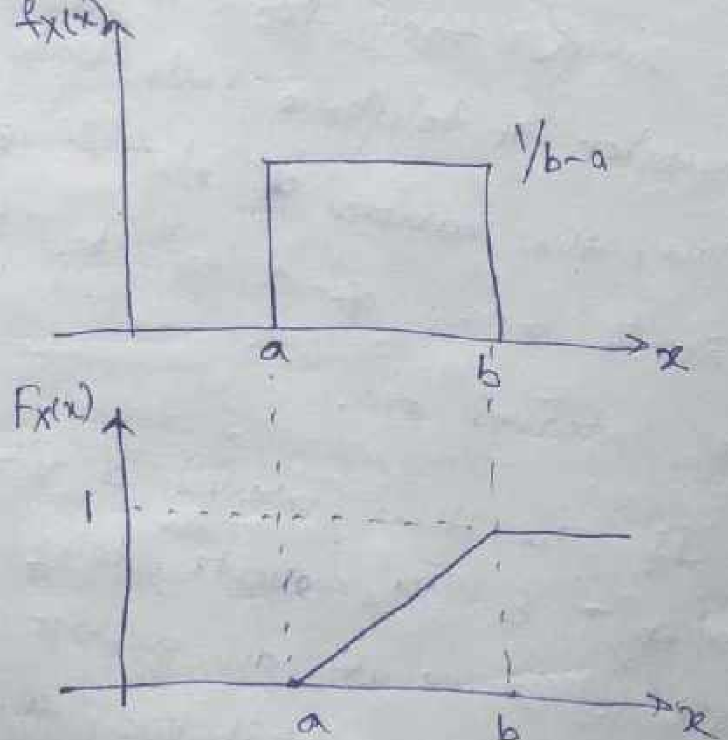
$= \int_a^x \frac{1}{b-a}\,dx$

$= \frac{x-a}{b-a}$

$F_X(a) = 0,\ F_X(b) = 1$

$$F_X(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$

for real constants $-\infty < a < \infty$ and $b > a$



Applications:-

① The uniform density finds in the quantization of signal samples prior to encoding in digital communication systems.

· Quantization amounts to "rounding off" the actual sample to the nearest of a large number of discrete "quantum levels". The errors introduced in the round-off process are uniformly distributed.

Exponential :-

The exponential density & distribution functions are:

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & x > a \\ 0 & x < a \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

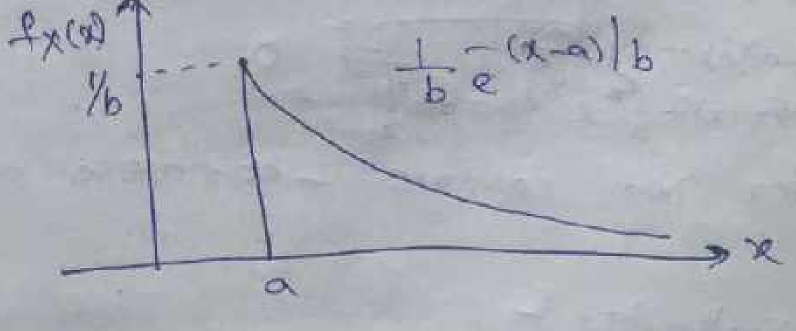
$$F_X(x) = \int_{a}^{x} \frac{1}{b} e^{-(x-a)/b}\,dx$$

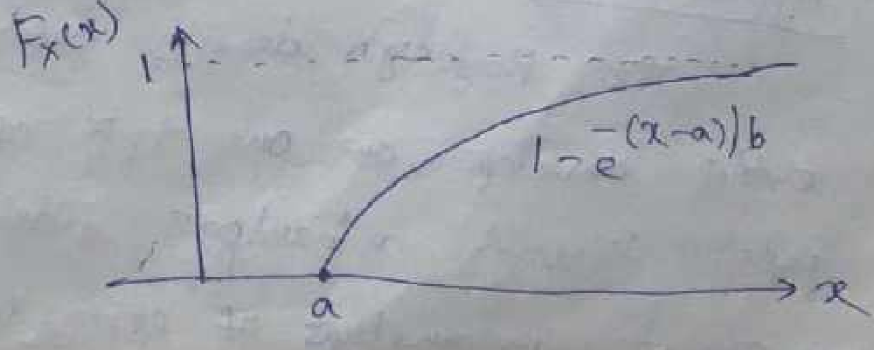
$$= \frac{1}{b}\, b\, e^{-(x-a)/b} \Big|_{a}^{x}$$

$$F_X(x) = 1 - e^{-(x-a)/b}$$

$$F_X(x) = \begin{cases} 1 - e^{-(x-a)/b} & x > a \\ 0 & x < a \end{cases}$$

for real numbers $-\infty < a < \infty$ and $b > 0$.





Applications:-

① The exponential density is useful in describing raindrop sizes when a large number of rain storm measurements are made.

② It is also approximately describe the fluctuations in signal strength received by radar from certain types of aircraft.

~~Rata~~ Rayleigh:-

The Rayleigh density & distribution functions are:

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)\, e^{-(x-a)^2/b} & x \geq a \\ 0 & x < a \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_{a}^{x} \frac{2}{b}(x-a)\, e^{-(x-a)^2/b}\, dx$$

$$\frac{(x-a)^2}{b} = y$$

$$\frac{2}{b}(x-a)\,dx = dy$$

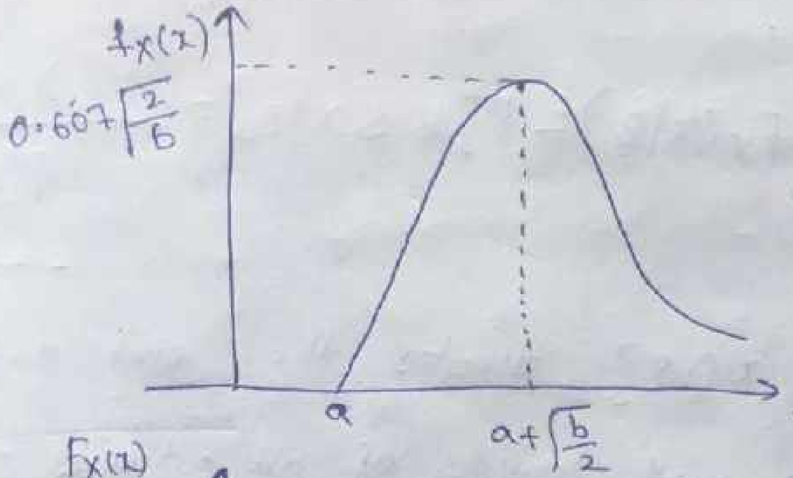
$$F_X(x) = \int_{a}^{(x-a)^2/b} e^{-y}\,dy = -e^{-y}\Big|_{a}^{(x-a)^2/b}$$

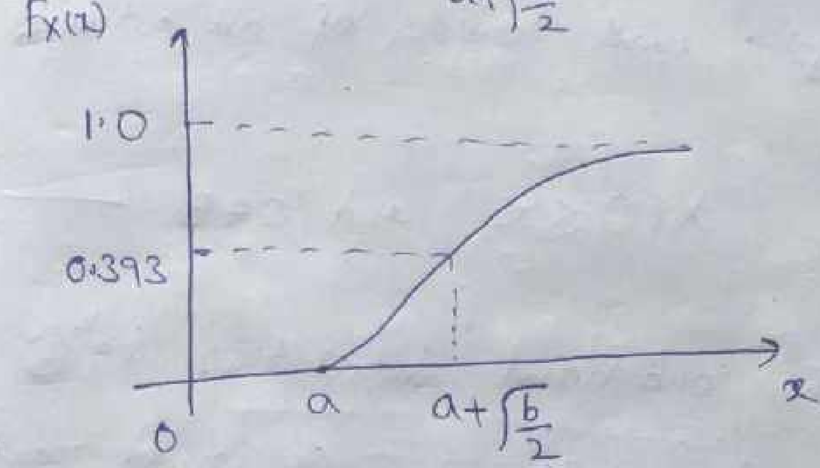
$$F_X(x) = \begin{cases} 1 - e^{-(x-a)^2/b} & x \geq a \\ 0. & x < a \end{cases}$$

for real constants $-\infty < a < \infty$ and $b > 0$.

Application:-

The Rayleigh density describes the ~~radar~~ envelope of one type of noise when passed through a bandpass filter. It is also important in analysis of errors in various measurement systems.





Conditional Distribution & Density Function :-

For two events A and B where $P(B) \neq 0$, The conditional probability of A given B is given as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{— (1)}$$

Conditional Distribution :-

Let A in eqn (1) be identified as the event $\{X \leq x\}$ for the random variable X. The resulting probability $P\{X \leq x | B\}$ is defined as conditional distribution ~~of~~ function of X, which is denoted as $F_X(x|B)$.

$$F_X(x|B) = P\{X \le x|B\} = \frac{P\{X \le x \cap B\}}{P\{B\}} \quad \text{--- (2)}$$

here $\{X \le x \cap B\}$ denotes the joint event $\{X \le x\} \cap B$

This joint event consists of all outcomes s such that

$$X(s) \le x \quad \text{and} \quad s \in B \quad \text{--- (3)}$$

The conditional distribution eq^n (2) applies to discrete, continuous, or mixed random variables.

<u>Properties of conditional Distribution:-</u>

All the properties of ordinary distributions apply to $F_X(x|B)$. It has the following characteristics;

1. $F_X(-\infty|B) = 0$
2. $F_X(\infty|B) = 1$
3. $0 \le F_X(x|B) \le 1$
4. $F_X(x_1|B) \le F_X(x_2|B)$ if $x_1 < x_2$

(5) $P\{x_1 < X \leq x_2 | B\} = F_X(x_2|B) - F_X(x_1|B)$

(6) $F_X\{x^+|B) = F_X(x|B)$

Conditional Density:-

conditional density function of the random variable X as the derivative of the conditional distribution function. $f_X(x|B)$ is denoted as

$$f_X(x|B) = \frac{dF_X(x|B)}{dx}$$

If $F_X(x|B)$ contains step discontinuities, as when X is a discrete or mixed random variable, and impulse functions are present in $f_X(x|B)$ to describe the derivatives at the discontinuities.

Properties of conditional Density:-

It satisfies the same properties as the ordinary density function. They are;

(1) $f_X(x|B) \geq 0$

(2) $\int_{-\infty}^{\infty} f_X(x|B)\, dx = 1$

(3) $F_X(x|B) = \int_{-\infty}^{x} f_X(\xi|B)\, d\xi$

(4) $P\{x_1 < X \leq x_2 | B\} = \int_{x_1}^{x_2} f_X(x|B)\, dx$

Example:-

In an experiment, there are two boxes. Each box containing balls as shown in Table. The event is to select a box randomly and then select a ball from the selected box". If the probability of selecting the first box is 0.3, then find

(i) the conditional probability distribution & density function

(ii) the probability distribution & density function

(iii) plot the functions.

Ball colour	Boxes 1	2	Total
Red	10	50	60
Blue	20	40	60
white	50	30	80
Total	80	120	200

Soln: Let the first box be B_1, and the second box be B_2.

Given, $P(B_1) = 0.3$

Since both boxes cannot be selected at a time

$$P(B_2) = 1 - P(B_1) = 0.7$$

Now consider that the discrete random variable X is the event of selecting a coloured ball. The values of X are $x_1=1$, $x_2=2$ and $x_3=3$, when a red, a blue, and a white ball is selected respectively.

From Table, the conditional probabilities are:

Probability of "getting a red ball when box B_1 is selected" is

$$P(x_1|B_1) = \frac{10}{80} = 0.125$$

Probability of "getting a blue ball, when box B_1 is selected" is

$$P(x_2|B_1) = \frac{20}{80} = 0.25$$

Probability of "getting a white ball when box B_1 is selected" is

$$P(x_3|B_1) = \frac{50}{80} = 0.625$$

Similarly, when box B_2 is selected; the probabilities are

$$P(x_1|B_2) = \frac{50}{120} = 0.4167$$

$$P(x_2|B_2) = \frac{40}{120} = 0.3333$$

$$P(x_3|B_2) = \frac{30}{120} = 0.25$$

(i) The conditional probability density & distribution functions are given by

$$f_X(x|B_1) = \sum_{i=1}^{3} P(x_i|B_1)\,\delta(x-x_i)$$

$$f_X(x|B_1) = P(x_1|B_1)\,\delta(x-x_1) + P(x_2|B_1)\,\delta(x-x_2) + P(x_3|B_1)\,\delta(x-x_3)$$

$$= \frac{1}{8}\delta(x-1) + \frac{2}{8}\delta(x-2) + \frac{5}{8}\delta(x-3)$$

$$F_X(x|B_1) = \frac{1}{8}u(x-1) + \frac{2}{8}u(x-2) + \frac{5}{8}u(x-3)$$

Similarly,

$$f_X(x|B_2) = \frac{5}{12}\delta(x-1) + \frac{4}{12}\delta(x-2) + \frac{3}{12}\delta(x-3)$$

$$F_X(x|B_2) = \frac{5}{12}u(x-1) + \frac{4}{12}u(x-2) + \frac{3}{12}u(x-3)$$

(ii) Using the total probability theorem, the probabilities of X are

$$P(x_1) = P(x_1|B_1)P(B_1) + P(x_1|B_2)P(B_2)$$

$$= \frac{10}{80}\times 0.3 + \frac{50}{120}\times 0.7 = 0.33$$

$$P(x_2) = P(x_2|B_1)P(B_1) + P(x_2|B_2)P(B_2)$$

$$= \frac{20}{80}\times 0.3 + \frac{40}{120}\times 0.7 = 0.308$$

$$P(x_3) = P(x_3|B_1)P(B_1) + P(x_3|B_2)P(B_2)$$

$$= \frac{50}{80} \times 0.3 + \frac{30}{120} \times 0.7 = 0.3625$$

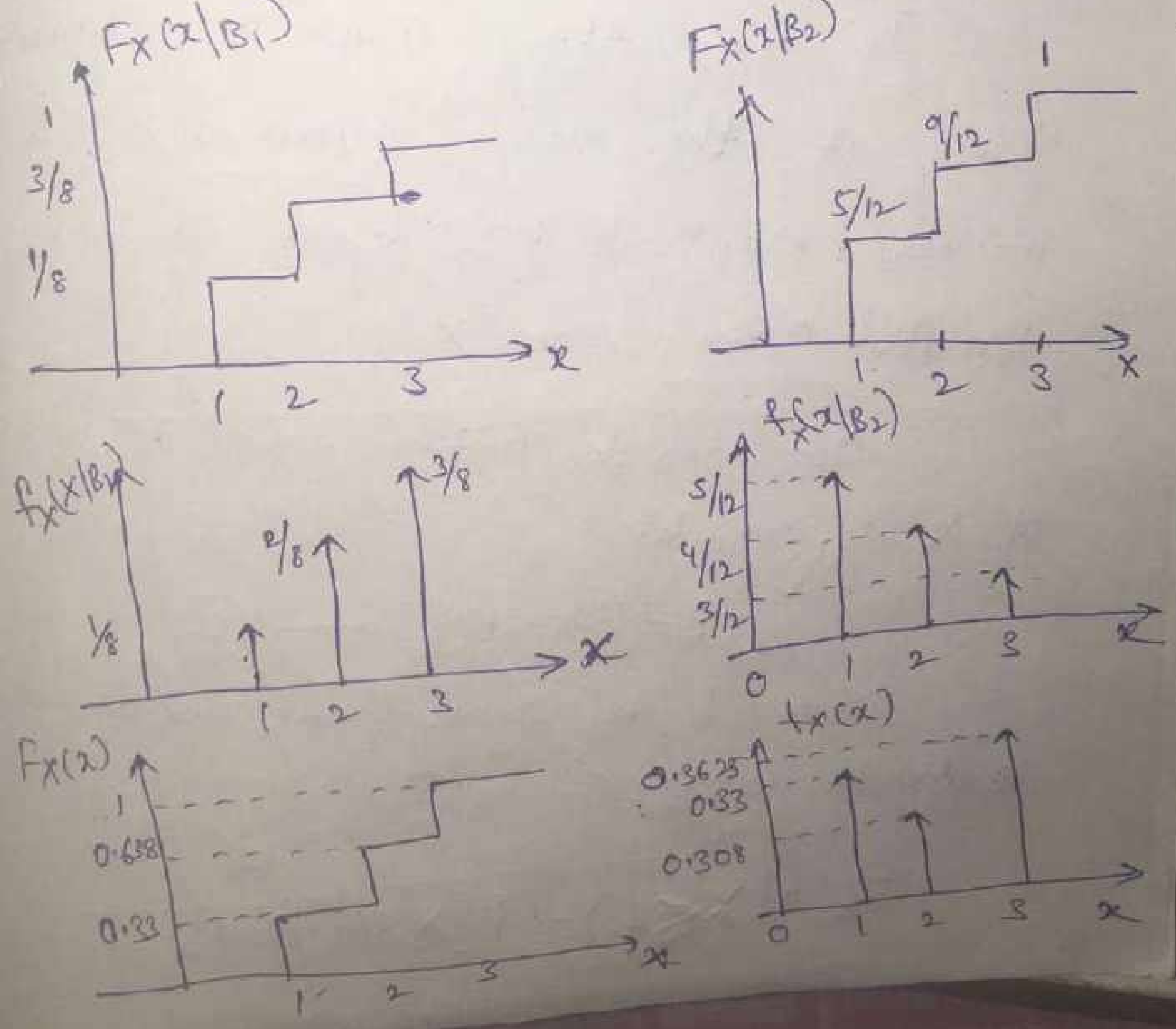
The probability density & distribution functions are

$$f_X(x) = \sum_{i=1}^{3} P(x_i)\,\delta(x - x_i)$$

$$f_X(x) = 0.33\,\delta(x-1) + 0.308\,\delta(x-2) + 0.3625\,\delta(x-3)$$

$$F_X(x) = 0.33\,u(x-1) + 0.308\,u(x-2) + 0.3625\,u(x-3)$$

(iii)



Methods of Defining Conditioning Event :-

Conditional Distribution function

$$F_X(x|B) = P\{X \le x | B\}$$

$$= \frac{P\{X \le x \cap B\}}{P\{B\}} \quad \text{--- (1)}$$

where we use the notation $\{X \le x \cap B\}$ to imply the joint event $\{X \le x\} \cap B$

B is called conditioning event. There are two ways to ~~defining~~ define conditioning event.

In one method, event B is defined in terms of the random variable X. In another method, event B may depend on some random variables other than X.

One way to define event B interms of X is to let

$$B = \{X \le b\} \quad \text{--- (2)}$$

where b is some real number $-\infty < b < \infty$

Substitute eqn (2) in eqn (1)

$$F_X(x | X \le b) = P\{X \le x | X \le b\}$$

$$= \frac{P\{X \le x \cap X \le b\}}{P\{X \le b\}} \quad \text{--- (3)}$$

for all events $\{X \le b\}$ for which $P\{X \le b\} \ne 0$

case 1:- $b \le x$

If $b \le x$, the event $\{X \le b\}$ is a subset of the event $\{X \le x\}$, so $\{X \le x\} \cap \{X \le b\} = \{X \le b\}$.

eqn (3) becomes

$$F_X(x \mid X \le b) = \frac{P\{X \le x \cap X \le b\}}{P\{X \le b\}}$$

$$= \frac{P\{X \le b\}}{P\{X \le b\}} = 1 \qquad b \le x \quad \text{--- (4)}$$

case 2:- when $x < b$, the event $\{X \le x\}$ is a subset of the event $\{X \le b\}$, so

$\{X \le x\} \cap \{X \le b\} = \{X \le x\}$ and eqn (3)

becomes

$$F_X(x \mid X \le b) = \frac{P\{X \le x \cap X \le b\}}{P\{X \le b\}}$$

$$= \frac{P\{X \le x\}}{P\{X \le b\}} = \frac{F_X(x)}{F_X(b)} \quad \text{--- (5)}$$

$x < b$

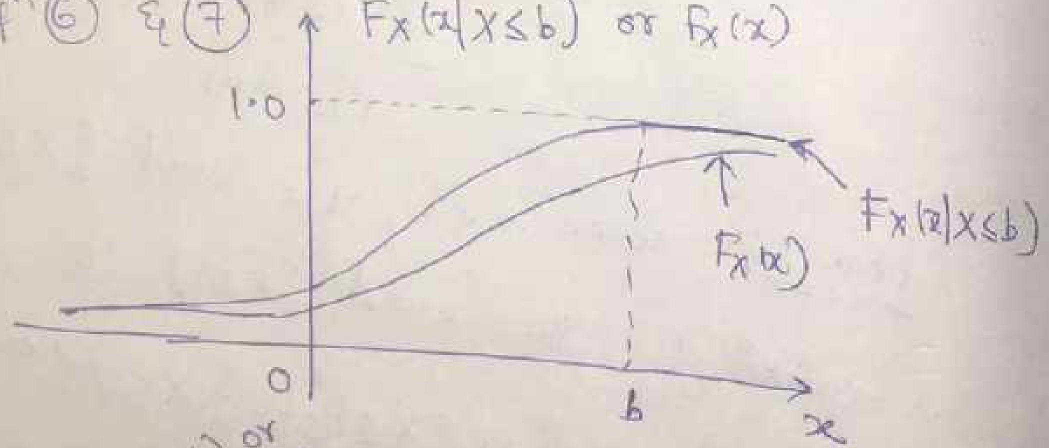
From eqⁿs (4) & (5)

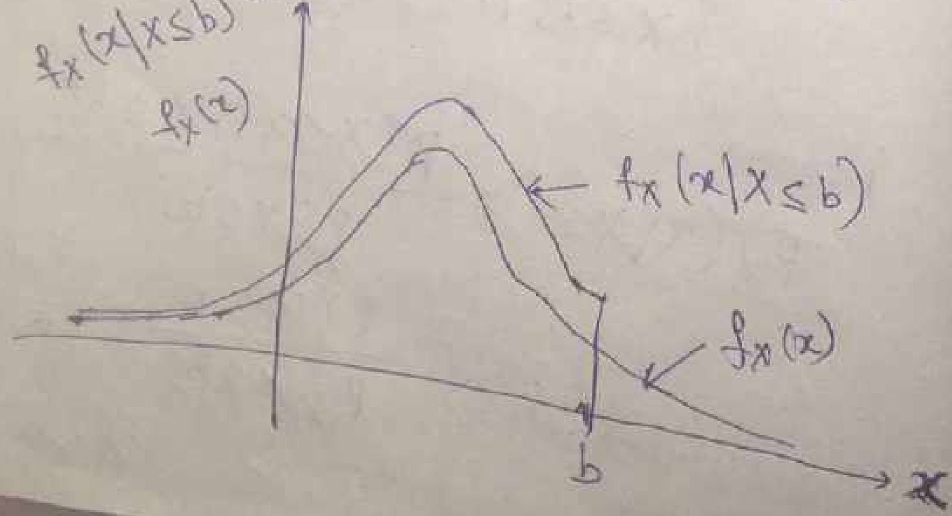
$$F_X(x|X \le b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x < b \\ 1 & b \le x \end{cases} \quad -(6)$$

The conditional density function derives from the derivative of eq^n (6)

$$f_X(x|X \le b) = \begin{cases} \dfrac{f_X(x)}{F_X(b)} = \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\,dx} & x<b \\ 0 & x \ge b \end{cases} \quad -(7)$$

Figures shows possible functions representing eq^n (6) & (7)





From our assumptions that the conditioning event has non zero probability, we have $0 < F_X(b) \leq 1$, so from eqn (3), the conditional distribution function is never smaller than the ordinary distribution function:

$$F_X(x|X \leq b) \geq F_X(x) \quad \text{— (8)}$$

Similarly from eqn (7), conditional density function

$$f_X(x|X \leq b) \geq f_X(x) \quad x < b \quad \text{— (9)}$$

Prob:- If the probability density of a random variable is given by

$$f_X(x) = k(1-x^2) \qquad 0 < x < 1$$

find the value k and $F_X(x)$

Soln:- Given the probability density function

$$f_X(x) = k(1-x^2) \qquad 0 < x < 1$$

(a) value of k,

If the pdf is valid, then

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_0^1 k(1-x^2)\,dx = 1$$

$$k\left(x-\frac{x^3}{3}\right)\Big|_0^1 = 1$$

$$k\left(1-\tfrac{1}{3}\right) = 1$$

$$\frac{2k}{3} = 1$$

$$k = \frac{3}{2} = 1.5$$

$$f_X(x) = \tfrac{3}{2}(1-x^2) \quad 0 < x < 1$$

(b) The Probability distribution function is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_0^x \frac{3}{2}(1-x^2)\,dx$$

$$F_X(x) = \frac{3}{2}\left(x-\frac{x^3}{3}\right)$$

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ \frac{3}{2}\left(x-\frac{x^3}{3}\right) & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

Exp:-

Find a constant $b>0$, so that the function

$$f_X(x) = \begin{cases} \frac{1}{10} e^{3x} & 0 \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

is a valid pdf.

Solⁿ:-

$$f_X(x) = \begin{cases} \frac{1}{10} e^{3x} & 0 \le x < b \\ 0 & \text{otherwise} \end{cases}$$

If it is a valid density function, then

$f_X(x) \ge 0$ is true, and

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\int_{0}^{b} \frac{1}{10} e^{3x}\, dx = 1$$

$$\frac{1}{10} \left[\frac{e^{3x}}{3} \right]_0^b = 1$$

$$\frac{1}{30} \left[e^{3b} - 1 \right] = 1$$

$$e^{3b} = 31$$

$$3b = \ln(31)$$

$$b = \tfrac{1}{3} \ln(31) = 1.1446$$

Example: A Gaussian random variable X with $\mu_x = 4$ & $\sigma_x = 3$ is generated. Find the probability of $X \le 7.75$. Write down the function & draw the graph.

Sol^n:

Given a Gaussian random variable with $\mu_x = 4$, $\sigma_x = 3$. Given the event $\{x \le 7.75\}$, then

$$P\{X \le 7.75\} = F_X(7.75)$$

$$F_X(x) = F\left(\frac{x - \mu_x}{\sigma_x}\right)$$

$$F_X(7.75) = F\left(\frac{7.75 - 4}{3}\right)$$

$$= F\left(\frac{3.75}{3}\right) = F(1.25)$$

Using the Q-function approximation

$$F(1.25) = 1 - Q(1.25)$$

$$= 1 - \frac{1}{(0.66)(1.25) + 0.34\sqrt{(1.25)^2 + 5.51}} \cdot \frac{e^{-1.25^2/2}}{\sqrt{2\pi}}$$

$$F(1.25) = 0.8944$$

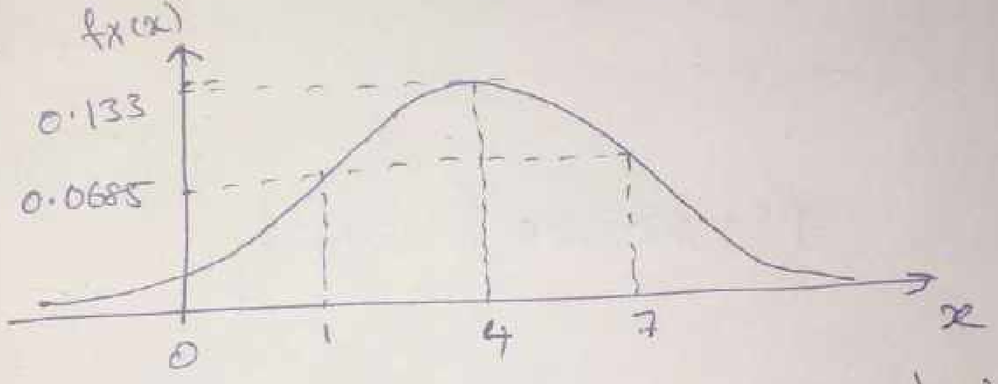
$P\{X \le 7.75\} = 0.8944$

The Gaussian density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{\cancel{x}-\mu_x}{2\sigma_x^2}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi(9)}} e^{-\frac{(x-4)^2}{18}}$$

$$= 0.133\, e^{-(x-4)^2/18}$$



<u>Example</u>:- Assume that the height of clouds σ_x, above the ground at some location is a gaussian random variable X with mean value 2km and $\sigma_x = 0.25$ km. Find the probability of clouds higher than 2.5 km

<u>Sol^n</u>:- Given a Gaussian random variable X,

Mean value $\mu_x = 2$ km

Spread $\sigma_x = 0.25$ km

$$P\{X > 2.5\text{km}\} = 1 - P\{X \le 2.5\text{ km}\}$$

$$= 1 - F_X(2.5)$$

$$= 1 - F\left(\frac{2.5-2}{0.25}\right) = 1 - F(2)$$

$$= 1 - (1 - Q(2))$$

$$= Q(2)$$

Using Q-function

$$P\{X > 2.5\,km\} = Q(2) = \frac{1}{0.66 \times 2 + 0.34\sqrt{2^2 + 5.5}} \times \frac{e^{-2^2/2}}{\sqrt{2\pi}}$$

$$P\{X > 2.5\,km\} = 0.0228$$

Prob :- A continuous random variable X has a PDF $f(x) = 3x^2, \quad 0 < x < 1$. Find a and b such that

(i) $P\{X = a\} = P\{X > a\}$ and (ii) $P\{X > b\} = 0.05$

Sol:- Given the probability density function

$$f_X(x) = 3x^2, \quad 0 < x < 1$$

(i) $P\{X > a\} = 1 - P\{X \leq a\}$

$$= 1 - F_X(a)$$

$$= 1 - \int_0^a 3x^2\, dx$$

$$= 1 - x^3 = 1 - a^3$$

$$P\{X = a\} = f_X(a) = 3a^2$$

Given that $P\{X=a\} = P\{X>a\}$

$$3a^2 = 1-a^3$$

$$a^2(a+3) = 1$$

(ii) $P\{X>b\} = 1-P\{X\leq b\} = 0.05$

$$1-F_X(b) = 0.05$$

$$F_X(b) = \int_0^b 3x^2\,dx = b^3$$

$$1-b^3 = 0.05$$

$$b^3 = 0.05$$

$$b = \sqrt[3]{0.05} = 0.3684$$

Operations on a single Random Variable

The random variable was introduced as a means of providing a systematic definition of events defined on a sample space. Specifically, it formed a mathematical model for describing characteristics of some real, physical world random phenomenon. In this chapter, we discuss some important operations that can perform on a random variable.

① Expectation :-

Expectation is the name given to the process of averaging when a random variable is involved. For a random variable X, we use the notation E[X], which may be read "the mathematical expectation of X", "the expected value of X", "the mean value of X", and "the statistical average of X". Occasionally we also use the notation $\bar{X}$.

$$\bar{X} = E[X]$$

Expected value of a Random variable :-

If X is a continuous random variable with a valid probability density function $f_X(x)$, then the expected value of X or the mean value of

X is defined as

$$E[X] = \overline{X} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \quad -(1)$$

If X is a discrete random variable with a set of elements $\{x_1, x_2, x_3, \ldots x_N\}$ and a set of corresponding probabilities $\{P(x_1), P(x_2), P(x_3), \ldots P(x_N)\}$, then the expected value of X is

$$E[X] = \sum_{i=1}^{N} x_i P(x_i) \quad -(2)$$

where N is an integer and may be infinite.

For equiprobable elements, where $P(x_1) = P(x_2) = \ldots \ldots = P(x_N) = \frac{1}{N}$, the expected value is

$$E[X] = \sum_{i=1}^{N} \frac{x_i}{N} = \frac{1}{N}(x_1 + x_2 + x_3 + \ldots + x_N) \quad -(3)$$

which is the arithmetic mean value of X.

If a random variable's density is symmetrical about a line $x = a$, then $E[X] = a$, that is

$$E[X] = a \quad \text{if } f_X(x+a) = f_X(-x+a)$$

Expected value of a function of a Random variable:

Consider a random variable X with probability density function $f_X(x)$. The Expected value of a real function $g(\cdot)$ of a continuous random variable X is defined as

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

If X is a discrete random variable, then

$$E[g(X)] = \sum_{i=1}^{N} g(x_i) P(x_i)$$

where N may be infinite for some random variables.

Conditional Expected value of a Random variable:-

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \quad \text{—} \; ①$$

In eqⁿ ①, $f_X(x)$ is replaced by the conditional density $f_X(x|B)$, where B is any event defined on the sample space, The conditional expected value of X, denoted $E[X|B]$

$$E[X|B] = \int_{-\infty}^{\infty} x f_X(x|B)\, dx \quad \text{—} \; ②$$

Let event B depend on the random variable X, then

$$B = \{X \le b\} \qquad -\infty < b < \infty \quad — (3)$$

Then

$$f_X(x | X \le b) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\,dx} & x < b \\ 0 & x \ge b \end{cases} \quad — (4)$$

Thus by substituting eqn (4) into eqn (2)

$$E[X | X \le b] = \frac{\int_{-\infty}^{b} x\, f_X(x)\,dx}{\int_{-\infty}^{b} f_X(x)\,dx} \quad — (5)$$

this is the expected value of X when the event $\{B \le b\}$ is already known.

Problem:- Find the expected value of a uniformly distributed random variable

Soln:- For a uniform distribution,

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

where a, b are real constants

$$E[X] = \int x\, f_X(x)\,dx$$

$$E[x] = \int_a^b \frac{x}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

$$= \frac{b+a}{2}$$

$\therefore$ The Expected value of uniformly distributed random variable is

$$E[x] = \frac{a+b}{2}$$

Properties of Expectation :-

(1) If a random variable X is a constant, i.e., X = a, then E[a] = a, where a is a constant.

Proof :-

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} a f_X(x)\,dx$$

$$= a\int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= a \times 1 = a \qquad \boxed{\because \int_{-\infty}^{\infty} f_X(x)\,dx = 1}$$

$$E[a] = a$$

(2) If a is any constant, then

$$E[aX] = aE[X]$$

Proof:- $E[aX] = \int_{-\infty}^{\infty} ax f_X(x)\, dx$

$$= a \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$E[aX] = aE[X]$$

(3) If a and b are any two constants, then

$$E[aX+b] = aE[X]+b$$

Proof:- $E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$

$$= \int_{-\infty}^{\infty} a x f_X(x)\, dx + b \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= aE[X] + b$$

(4) If $X \geq 0$, then $E[X] \geq 0$

Proof:- If X is a continuous random variable such that $x \geq 0$, then

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{0} x f_X(x)\, dx + \int_{0}^{\infty} x f_X(x)\, dx$$

$$= 0 + \int_0^{\infty} x f_X(x)\, dx$$

Since $x > 0$ and $f_X(x) \geq 0$, then

$$E[X] = \int_0^{\infty} x f_X(x)\, dx \geq 0$$

Hence if $x \geq 0$, then $E[X] \geq 0$

(5) If X is any random variable, then the inequality $|E[X]| \leq E|X|$ exists

Proof: we know that

$$X \leq |X|$$

and also

$$-X \leq |X|$$

Then $E[X] \leq E|X|$, and $E[-X] \leq E|X|$

or $-E[X] \leq E|X|$

$$\therefore |E[X]| \leq E|X|$$

(6) If $g_1(x)$ and $g_2(x)$ are two functions of a random variable X, then

$$E[g_1(x) + g_2(x)] = E[g_1(x)] + E[g_2(x)]$$

Proof:- $E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$

$$E[g_1(x) + g_2(x)] = \int_{-\infty}^{\infty} [g_1(x) + g_2(x)]\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} g_1(x)\, f_X(x)\, dx + \int_{-\infty}^{\infty} g_2(x)\, f_X(x)\, dx$$

$$= E[g_1(x)] + E[g_2(x)]$$

Similarly for n functions,

$$E[g_1(x) + g_2(x) + \dots + g_n(x)] = E[g_1(x)] + E[g_2(x)] + \dots + E[g_n(x)]$$

Prob:- If X is a discrete random variable with probability mass function given in table, find (i) $E[X]$, (ii) $E[2X+3]$, (iii) $E[X^2]$, and (iv) $E[(2X+1)^2]$

X	-2	-1	0	1	2
P(X)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$

Sol^n:- (i) X is a discrete random variable,

$$E[X] = \sum_{i=1}^{N} x_i P(x_i) = \sum_{i=1}^{5} x_i P(x_i)$$

Since $N = 5 =$ No of variables

from Table

$$E[X] = -2\left(\frac{1}{5}\right) + (-1)\left(\frac{2}{5}\right) + 0\left(\frac{1}{10}\right) + 1\left(\frac{1}{10}\right) + 2\left(\frac{1}{5}\right)$$

$$= -\frac{2}{5} - \frac{2}{5} + \frac{1}{10} + \frac{2}{5}$$

$$E[X] = \frac{-3}{10} = -0.3$$

(ii) $E[2X+3]$

from the properties, we know that

$$E[2X+3] = 2E[X] + 3$$

$$= 2(-0.3) + 3$$

$$= -0.6 + 3 = 2.4$$

(iii) To find out $E[X^2]$, let $g(x) = x^2$

$$E[g(x)] = \sum g(x_i)\, p(x_i)$$

$$= \sum_{i=1}^{5} x_i^2\, p(x_i)$$

$$E[X^2] = (-2)^2\left(\frac{1}{5}\right) + (-1)^2\left(\frac{2}{5}\right) + 0\left(\frac{1}{10}\right) + 1^2\left(\frac{1}{10}\right) + 2^2\left(\frac{1}{5}\right)$$

$$= \frac{4}{5} + \frac{2}{5} + \frac{1}{10} + \frac{4}{5} = \frac{21}{10}$$

$$E[X^2] = 2.1$$

(iv) $E[(2X+1)^2]$, let $g(x) = (2x+1)^2$

$$= 4x^2 + 4x + 1$$

$$E[4x^2+4x+1] = 4E[x^2] + 4E[x] + 1$$

$$= 4\left(\frac{21}{10}\right) + 4\left(\frac{-3}{10}\right) + 1$$

$$= \frac{84}{10} - \frac{12}{10} + 1$$

$$= \underline{8.2}$$

Moments

There are two types of moments for a function of a random variable X.

(1) Moments about the origin
(2) Moments about the mean or central moments.

Moments about the origin:-

Let $g(x)$ be a real function of the random variable X, such that $g(x) = x^n$ for $n = 0, 1, 2, 3, \cdots$. Then the expected value of the function $g(x)$ is called the moments about the origin of a random variable X. It is denoted as m_n, where n indicates the order of the moment.

Mathematically, the nth moment is defined as

$$m_n = E[x^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

For a discrete random variable

$$E[x^n] = \sum x_i^n p(x_i)$$

If $n=0$,

$$m_0 = \int_{-\infty}^{\infty} x^0 f_X(x)\,dx = 1$$

$\therefore$ The zeroth moment of X is $m_0 = 1$. It gives the area of the pdf

If $n=1$,

$$m_1 = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = E[X]$$

The first moment is equal to its mean value, $m_1 = E[X] = \overline{X}$

If $n=2$,

$$m_2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = E[X^2]$$

It is called the second moment of X or the mean square value of X

The physical significance is that if X is any random signal, then the m_2 of X gives the average power of the signal, and the square root of m_2 gives the rms value of the signal.

Moments about the mean :-

Let $g(x)$ be a real function of a random variable X, such that

$$g(x) = (x - \bar{x})^n \text{ for } n = 0, 1, 2, 3, \ldots$$

where $\bar{X}$ is the mean of X. Then the expected value of the function $g(x)$ is called moments about the mean of the random variable X. It is denoted as μ_n. where n indicates the order of the moment. It is also called central moments of the random variable X.

The n^{th} central moment of X is given as

$$\mu_n = E[(x - \bar{x})^n]$$

$$= \int_{-\infty}^{\infty} (x - \bar{x})^n f_X(x)\, dx$$

For a discrete random variable,

$$\mu_n = \sum (x_i - \bar{x})^n P(x_i)$$

If $n = 0$

$$\mu_0 = E[(x - \bar{x})^0] = \int_{-\infty}^{\infty} (x - \bar{x})^0 f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

That is, the zeroth central moment of X is

$$\mu_0 = m_0 = 1 = \text{area of the pdf.}$$

If $n = 1$

$$\mu_1 = E[(X - \bar{X})] = \int_{-\infty}^{\infty} (x - \bar{X}) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx - \int_{-\infty}^{\infty} \bar{X} f_X(x)\,dx$$

$$= \bar{X} - \bar{X} \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$\mu_1 = \bar{X} - \bar{X} = 0$$

That is, the first central moment of X is always equal to zero.

Variance:-

The variance of the density function $f_X(x)$ for a random variable X is defined as the second central moment μ_2 of X. It is also denoted as σ_X^2 or var(X) and is given by

$$\mu_2 = \sigma_X^2 = E[(X - \bar{X})^2] = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x)\,dx$$

For a discrete random variable,

$$\sigma_X^2 = \Sigma (x_i - \bar{X})^2 P(x_i)$$

Standard deviation:

The standard deviation of a random variable is defined as the square root of the variance, i.e., σ_X. It is expressed as

$$\sigma_X = \sqrt{E[(X-\bar{X})^2]}$$

Prob:- Find the variance of X for a uniform probability density function.

Soln:-

The uniform pdf is given by

$$f_X(x) = \frac{1}{b-a} \qquad a \le x \le b$$

Variance

$$\sigma_X^2 = E[(X-\bar{X})^2]$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (X-\bar{X})^2 f_X(x)\, dx$$

$$\bar{X} = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{b-a}\, dx = \frac{x^2}{2(b-a)} \Big|_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

and $$\sigma_X^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \times \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\int_a^b \left(x^2 + \left(\frac{a+b}{2}\right)^2 - x(a+b)\right] dx\right]$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3} + \frac{(a+b)^2}{4}x - \frac{(a+b)x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^3-a^3}{3} + \frac{(a+b)^2(b-a)}{4} - \frac{(a+b)(b^2-a^2)}{2}\right]$$

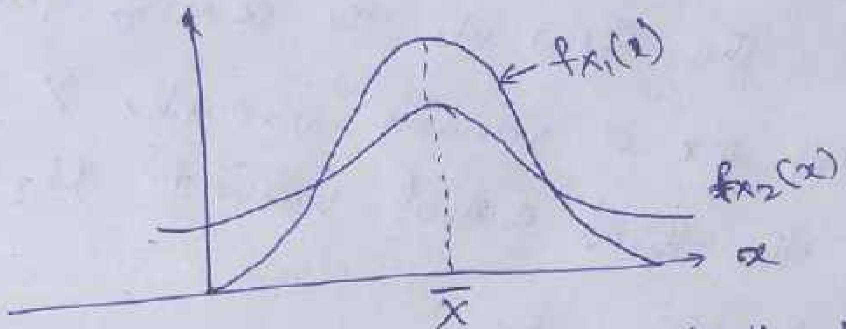
$$= \frac{b-a}{b-a}\left[\frac{a^2+b^2+ab}{3} + \frac{(a+b)^2}{4} - \frac{(a+b)^2}{2}\right]$$

$$= \frac{a^2+b^2+ab}{3} - \frac{a^2+b^2+2ab}{4}$$

$$\sigma_X^2 = \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

Physical Significance of variance & standard deviation:-

Consider two random variables X_1 and X_2. The Probability density functions of X_1 and X_2 are shown in Fig. as $f_{X_1}(x)$ and $f_{X_2}(x)$ with the same mean value $\bar{X}$.



As shown in the figure, both functions are symmetrical about mean $\bar{X}$. The values of $f_{X_1}(x)$ deviate less from $\bar{X}$, whereas the values of $f_{X_2}(x)$ deviate more from $\bar{X}$. The amount of deviation about the mean of X_1 is different from X_2.

This amount of deviation is called spread of the function $f_X(x)$. The standard deviation of X is a measure of the spread in the function $f_X(x)$ about the mean.

The ~~dexia~~ deviation of X about mean, i.e., $X-\bar{X}$, may be positive $(X>\bar{X})$ or negative $(X<\bar{X})$. Therefore, the expected value of the square of the deviation of X about $\bar{X}$ is called variance and its square root is called standard deviation.

Hence any pdf can be characterized by its mean & variance.

Skew and coefficient of skewness :-

Skew :-

The skew of the density function $f_X(x)$ for a random variable X is defined as the third central moment μ_3 of X.

It is given by.

$$\mu_3 = E[(X-\bar{X})^3] = \int_{-\infty}^{\infty} (x-\bar{X})^3 f_X(x)\,dx$$

For a discrete random variable,

$$\mu_3 = \sum_i (x_i-\bar{X})^3 P(x_i)$$

The physical significance of the third central moment or skew is that it is a measure of the asymmetry of $f_X(x)$ about its mean. It is the amount of deviation of symmetry from the mean value.

For example, if the density function is symmetrical about its mean, then the skew is zero.

Coefficient of skewness :-

The normalised third central moment or the ratio of the third central ~~moment~~ moment to the cube of standard deviation is called skewness of the density function or the coefficient of skewness. It is given by

$$\text{Skewness} = \frac{\mu_3}{\sigma_x^3} = \frac{E[(X-\bar{X})^3]}{E[(X-\bar{X})^2]^{3/2}}$$

Properties of variance:-

(1) The variance of a constant is zero, i.e, if k is a constant, then $Var(k)=0$

Proof:- $Var(X) = E[(X-\bar{X})^2]$

$$Var(k) = E[(k-\bar{k})^2] = E[(k-k)^2]$$
$$= E[0] = 0$$

(2) If k is a constant, then for a random variable X

$$Var(kX) = k^2 Var(X)$$

Proof:-

The variance of kX is given by

$$Var(kX) = E[(kX - k\bar{X})^2]$$
$$= E[k^2(X-\bar{X})^2]$$
$$= k^2 E[(X-\bar{X})^2]$$

$$Var(kX) = k^2 Var(X)$$

③ For a given random variable X, the relationship between the variance & the moments is given by

$$\sigma_X^2 = m_2 - m_1^2$$

<u>Proof:-</u> Let X be a random variable, The variance is

$$\sigma_X^2 = E[(X-\bar{X})^2]$$

$$= E[X^2 + \bar{X}^2 - 2X\bar{X}]$$

$$= E[X^2] + E[\bar{X}^2] - 2E[X\bar{X}]$$

$$= E[X^2] + \bar{X}^2 - 2\bar{X}E[X]$$

$$= E[X^2] + \bar{X}^2 - 2\bar{X}\bar{X}$$

$$\therefore \sigma_X^2 = E[X^2] - \bar{X}^2$$

$$\text{or } \sigma_X^2 = m_2^2 - m_1^2$$

④ If X is a random variable and a, b are real constants, then

$$var(ax+b) = a^2 var(x)$$

<u>Proof:-</u> The variance of (ax+b) is

$$var(ax+b) = E[((ax+b) - (\overline{ax+b}))^2]$$

$$(\overline{ax+b}) = aE[x] + b$$

$$var(ax+b) = E[(ax+b-aE[x]-b)^2]$$

$$= E[(ax-aE[x])^2]$$

$$= E[a^2(x-\bar{x})^2]$$

$$= a^2 E[(x-\bar{x})^2]$$

$$= a^2 var(X)$$

(5) If two random variables X_1 and X_2 are independent, then

$$var(X_1+X_2) = var(X_1) + var(X_2)$$

and $$var(X_1-X_2) = Var(X_1) + Var(X_2)$$

Proof :- $var(X_1+X_2) = E[((X_1+X_2) - (\overline{X_1+X_2}))^2]$

we know that

$$\overline{X_1+X_2} = \bar{X}_1 + \bar{X}_2$$

so $$Var(X_1+X_2) = E[(X_1+X_2-\bar{X}_1-\bar{X}_2)^2]$$

$$= E[((X_1-\bar{X}_1)+(X_2-\bar{X}_2))^2]$$

$$var(X_1+X_2) = E[(X_1-\bar{X}_1)^2] + E[(X_2-\bar{X}_2)^2]$$
$$+2E[(X_1-\bar{X}_1)(X_2-\bar{X}_2)]$$

since X_1 and X_2 are independent

$$E[(X_1-\bar{X}_1)(X_2-\bar{X}_2)] = E[X_1-\bar{X}_1]E[X_2-\bar{X}_2]$$

$$= (\overline{x_1} - \overline{x_1})(\overline{x_2} - \overline{x_2}) = 0$$

Therefore, $var(x_1 + x_2) = var(x_1) + var(x_2)$

similarly $var(x_1 - x_2) = var(x_1) + var(-x_2)$

$$= var(x_1) + (-1)^2 var(x_2)$$

$\because var(kx) = k^2 var(x)$

$$var(x_1 - x_2) = var(x_1) + var(x_2)$$

Relationship between central Moments and moments about the origin:-

Let X be a random variable. Then the nth central moment is given by

$$\mu_n = E[(x - \overline{x})^n]$$

we know from binomial theorem that

$$(x - \overline{x})^n = \sum_{k=0}^{n} (-1)^k \, nC_k \, x^{n-k} \, \overline{x}^k$$

The expected value is

$$E[(x - \overline{x})^n] = E\left[\sum_{k=0}^{n} (-1)^k \, nC_k \, x^{n-k} \, \overline{x}^k\right]$$

$$= \sum_{k=0}^{n} (-1)^k \, nC_k \, \overline{x}^k \, E[x^{n-k}]$$

We know that

$$\bar{X} = m_1 = \text{mean value}$$

$$E[X^{n-k}] = m_{n-k} = (n-k)^{th} \text{ moment}$$

$$E[(X-\bar{X})^n] = \mu_n = \sum_{k=0}^{n} (-1)^k \, nC_k \, m_1^k \, m_{n-k}$$

This expression is used to find the central moments of a random variable when the moments about the origin are known.

For example, if $n=2$, the variance is

$$\sigma_x^2 = \mu_2 = \sum_{k=0}^{2} (-1)^k \, 2C_k \, m_1^2 \, m_{2-k}$$

$$= (-1)^0 \, 2C_0 \, m_1^0 \, m_2 +$$

$$(-1)^1 \, 2C_1 \, m_1 m_1 +$$

$$(-1)^2 \, 2C_2 \, m_1^2 m_0$$

$$\sigma_x^2 = m_2 - 2m_1^2 + m_1^2 m_0$$

But $m_0 = 1$

$$\sigma_x^2 = m_2 - m_1^2$$

for $n=3$

$$\mu_3 = \sum_{k=0}^{3} (-1)^k \, 3C_k \, m_1^k \, m_{3-k}$$

$= (-1)^0 \, 3C_0 \, m_1^0 m_3 + (-1)^1 \, 3C_1 \, m_1^1 m_2 +$

$(-1)^2 \, 3C_2 \, m_1^2 m_1 + (-1)^3 \, 3C_3 \, m_1^3 \, m_0$

we know that $m_0 = 1$

$$\mu_3 = m_3 - 3m_1m_2 + 3m_1^3 - m_1^3$$

$$= m_3 - 3m_1(m_2 - m_1^2) - m_1^2$$

In terms of σ_x^2, we know that

$$\sigma_x^2 = m_2 - m_1^2$$

$$\mu_3 = m_3 - 3m_1\sigma_x^2 - m_1^3$$

$$\mu_3 = \overline{X^3} - 3\overline{X}\,\sigma_x^2 - \overline{X}^3$$

(or) μ_3 can also be obtained directly from

$$\mu_3 = E[(X - \overline{X})^3]$$

$$= E[X^3 - \overline{X}^3 - 3\overline{X}X^2 + 3\overline{X}^2 X]$$

$$= E[X^3] - E[\overline{X}^3] - 3E[\overline{X}X^2]$$

$$+ 3E[\overline{X}^2 X]$$

$$= E[X^3] - \overline{X}^3 - 3\overline{X}E[X^2] +$$

$$3\overline{X}^2 E[X]$$

$$= E[X^3] - \bar{X}^3 - 3\bar{X}\,E[X^2 - \bar{X}^2]$$

$$\mu_3 = \overline{X^3} - 3\bar{X}\,\sigma_X^2 - \bar{X}^3$$

This expression gives skew of the density function when moments are known.

Prob:- Find out the skew and skewness of a uniform probability density function for a random variable X.

Sol^n:- we know that for a uniform pdf, the mean value is

$$\bar{X} = \frac{a+b}{2}$$

the variance is $\sigma_X^2 = \frac{(a-b)^2}{12}$

and the skew is

$$\mu_3 = \overline{X^3} - 3\bar{X}\,\sigma_X^2 - \bar{X}^3$$

Now the third moment is

$$\overline{X^3} = \int_{-\infty}^{\infty} x^3 f_X(x)\,dx$$

$$= \frac{1}{(b-a)}\int_a^b x^3\,dx = \frac{1}{4(b-a)}[b^4 - a^4]$$

$$\overline{X^3} = \frac{b^4 - a^4}{4(b-a)}$$

The skew is

$$\mu_3 = \overline{X^3} - 3\overline{X}\,\sigma_X^2 - \overline{X}^3$$

$$= \frac{b^4 - a^4}{4(b-a)} - 3\left(\frac{b+a}{2}\right)\left(\frac{(a-b)^2}{12}\right) - \frac{(a+b)^3}{8}$$

$$= \frac{b^4 - a^4}{4(b-a)} - \frac{(b+a)(a-b)^2}{8} - \frac{(a+b)^3}{8}$$

$$= \frac{(b^2-a^2)(b^2+a^2)}{4(b-a)} - \frac{(b+a)(a-b)^2}{8} - \frac{(a+b)^3}{8}$$

$$= \frac{(b+a)(b^2+a^2)}{4} - \frac{(b+a)(a-b)^2}{8} - \frac{(a+b)^3}{8}$$

$$= \frac{a+b}{8}\left[2b^2 + 2a^2 - (a-b)^2 - (a+b)^2\right]$$

$$= \frac{a+b}{8}\left[2b^2 + 2a^2 - 2a^2 - 2b^2\right]$$

$$\mu_3 = 0$$

$\therefore$ the skew of the uniform random variables is zero.

Functions for Moments:-

To calculate the n^{th} moments of a random variable X, two functions are generally used (1) characteristic function and (2) moment generation function.

characteristic function:-

consider a random variable X with a probability density function $f_X(x)$. Then the expected value of the function $e^{j\omega x}$ is called the characteristic function. It is expressed as

$$\phi_X(\omega) = E\left[e^{j\omega X}\right]$$

It is a function of the real variable, $-\infty < \omega < \infty$, where j is an imaginary operator.

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

and for a discrete random variable

$$\phi_X(\omega) = \sum_i e^{j\omega x_i} P(x_i)$$

The characteristic function transforms the random variable X into another real variable ω. It can be expressed as a Fourier transform of $f_X(x)$ with the sign of ω reversed.

∴ the inverse Fourier transform of $\phi_X(\omega)$ gives the probability density function with the sign of x is reversed. Hence

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega) e^{-j\omega x} d\omega$$

So the functions $\phi_X(\omega)$ and $f_X(x)$ are Fourier transform pairs with the sign of the variable reversed.

Theorem :-

If $\phi_X(\omega)$ is a characteristic function of a random variable X, then the nth moment of X is given by

$$m_n = (-j)^n \left. \frac{d^n \phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$

Proof:-

Consider a random variable X with the characteristic function

$$\phi_X(\omega) = E\left[e^{j\omega X}\right]$$

we know that the series expansion of $e^{j\omega x}$ is

$$e^{j\omega x} = 1 + j\omega x + \frac{(j\omega x)^2}{2!} + \frac{(j\omega x)^3}{3!} + \cdots$$

$$+ \frac{(j\omega x)^n}{n!} + \cdots$$

$$e^{j\omega x} = 1 + j\omega x - \frac{\omega^2 x^2}{2!} - j\frac{\omega^3 x^3}{3!} + \cdots$$

Then $\phi_x(\omega) = E\left[e^{j\omega x}\right]$

$$= E\left[1 + j\omega x - \frac{\omega^2 x^2}{2!} - j\frac{\omega^3 x^3}{3!} + \cdots\right]$$

$$\phi_x(\omega) = 1 + j\omega E[x] - \frac{\omega^2}{2} E[x^2] - j\frac{\omega^3}{3!} E[x^3]$$

$$+ \cdots$$

$$\phi_x(\omega) = m_0 + j\omega m_1 - \frac{\omega^2}{2} m_2 - j\frac{\omega^3}{3!} m_3 + \cdots$$

To find the moments, substitute $\omega = 0$

So, $\phi_x(0) = m_0 = 1$

Differentiation of $\phi_x(\omega)$ with respect to ω gives

$$\frac{d\phi_x(\omega)}{d\omega} = jm_1 - \omega m_2 - j\frac{3\omega^2}{3!} m_3 + \cdots$$

At $\omega = 0$ $\left.\frac{d\phi_x(\omega)}{d\omega}\right|_{\omega=0} = jm_1$

$\therefore$ first order moment is $m_1 = (-j) \left.\dfrac{d\phi_x(\omega)}{d\omega}\right|_{\omega=0}$

Second time differentiation of $\phi_x(\omega)$ w.r.to ω is

$$\frac{d^2\phi_x(\omega)}{d\omega^2} = -m_2 - j\omega m_3$$

At $\omega=0$

$$\left.\frac{d^2\phi_x(\omega)}{d\omega^2}\right|_{\omega=0} = -m_2 = j^2 m_2$$

$\therefore$ second order moment is $m_2 = \left.\dfrac{-d^2\phi_x(\omega)}{d\omega^2}\right|_{\omega=0}$

Similarly, the n^{th} time differentiation of $\phi_x(\omega)$ w.r.to ω is

$$\left.\frac{d^n\phi_x(\omega)}{d\omega^n}\right|_{\omega=0} = j^n m_n$$

$$m_n = (-j)^n \left.\frac{d^n\phi_x(\omega)}{d\omega^n}\right|_{\omega=0}$$

Thus the n^{th} moments of a random variable X can be obtained from the n^{th} differentiation of the characteristic function at $\omega=0$

Properties of the characteristic Function :-

(1) The characteristic function is unity at $\omega=0$, and given by

$$\phi_x(\omega)\Big|_{\omega=0} = \phi_x(0) = 1$$

Proof :- The characteristic function is given by

$$\phi_x(\omega) = [e^{j\omega x}]$$

At $\omega=0$, $\phi_x(0) = E[e^0] = E[1] = 1$

(2) The maximum amplitude of the characteristic function is unity at $\omega=0$

i.e., $|\phi_x(\omega)| \le \phi_x(0)$

or $|\phi_x(\omega)| \le 1$

Proof :- Let the characteristic function

$$\phi_x(\omega) = E[e^{j\omega x}]$$

The amplitude of $\phi_x(\omega)$ is

$$|\phi_x(\omega)| = |E[e^{j\omega x}]|$$
$$= \left|\int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx\right|$$

Since $|xy| \le |x||y|$

$$|\phi_x(\omega)| \leq \int_{-\infty}^{\infty} |e^{j\omega x}| \, |f_X(x)| \, dx$$

$$|\phi_x(\omega)| \leq \int_{-\infty}^{\infty} |f_X(x)| \, dx$$

Since $|e^{j\omega x}| = 1$

$$\therefore |\phi_X(\omega)| \leq 1$$

③ $\phi_x(\omega)$ is a continuous function of ω in the range $-\infty < \omega < \infty$

Proof:- we know that

$$\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x) \, dx$$

Since ω is continuous, $e^{j\omega x}$ is also continuous.

Hence $\phi_x(\omega)$ is a continuous function.

④ $\phi_x(-\omega)$ and $\phi_x(\omega)$ are conjugate functions.

That is, $\phi_x(-\omega) = \phi_x^*(\omega)$

and $\phi_x^*(-\omega) = \phi_x(\omega)$

Proof:- we know that

$$\phi_x(\omega) = E[e^{j\omega x}]$$

$$\phi_x(-\omega) = E[e^{j(-\omega)x}]$$

$$\phi_x(-\omega) = E\left[e^{-j\omega x}\right] = E\left[e^{j\omega x}\right]^*$$

$$\phi_x(-\omega) = \phi_x^*(\omega)$$

similarly,

$$\phi_x(-\omega) = E\left[e^{j(-\omega)x}\right]$$

$$\phi_x^*(-\omega) = E\left[e^{j(-\omega)x}\right]^*$$

$$\phi_x^*(-\omega) = E\left[e^{(-j)(-\omega)x}\right]$$

$$\phi_x^*(-\omega) = E\left[e^{j\omega x}\right]$$

$$\underline{\underline{\phi_x^*(-\omega) = \phi_x(\omega)}}$$

(5) If $\phi_x(\omega)$ is a characteristic function of a random variable X, then the characteristic function of $Y = aX+b$ is given by

$$\phi_y(\omega) = e^{j\omega b}\,\phi_x(a\omega)$$, where a & b are real constants.

Proof:- Given $Y = aX+b$

Then $\phi_y(\omega) = E\left[e^{j\omega(aX+b)}\right]$

$$= E\left[e^{j\omega b}\, e^{j\omega aX}\right]$$

$$= e^{j\omega b} E\left[e^{j\omega a X}\right]$$

$$\phi_Y(\omega) = e^{j\omega b}\phi_X(\omega a)$$

(6) If $\phi_X(\omega)$ is a characteristic function of a random variable X, then $\phi_X(c\omega) = \phi_X(\omega)$ where C is a real constant.

<u>Proof:-</u>

we know that

$$\phi_X(\omega) = E\left[e^{j\omega X}\right]$$

$$\phi_X(c\omega) = E\left[e^{jc\omega X}\right]$$

$$= E\left[e^{j\omega c X}\right]$$

$$\phi_X(c\omega) = \phi_{cX}(\omega)$$

(7) If X_1 and X_2 are two independent random variables, then

$$\phi_{X_1+X_2}(\omega) = \phi_{X_1}(\omega)\,\phi_{X_2}(\omega)$$

<u>Proof:-</u> Given $\phi_X(\omega) = E\left[e^{j\omega X}\right]$

Then $\phi_{X_1+X_2}(\omega) = E\left[e^{j\omega(X_1+X_2)}\right]$

$$= E\left[e^{j\omega X_1} \cdot e^{j\omega X_2}\right]$$

Since X_1 and X_2 are independent,

$$= E\left[e^{j\omega X_1}\right] E\left[e^{j\omega X_2}\right]$$

$$\phi_{X_1+X_2}(\omega) = \phi_{X_1}(\omega)\, \phi_{X_2}(\omega)$$

Example :- Find the characteristic function of a uniformly distributed random variable X in the range [0,1] and hence find m_1.

sol^n :- For a uniformly distributed random variable, the density function is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

In the range [0,1], $a=0$, $b=1$

$$\therefore f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The characteristic function is

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_0^1 e^{j\omega x}\, dx = \frac{1}{j\omega}\left[e^{j\omega x}\right]_0^1$$

$$\phi_X(\omega) = \frac{1}{j\omega}\left[e^{j\omega} - 1\right]$$

To find m_1, we know that

$$m_1 = (-j) \left. \frac{d\phi_X(\omega)}{d\omega} \right|_{\omega=0}$$

So, $$\frac{d\phi_X(\omega)}{d\omega} = \frac{d}{d\omega}\left[\frac{e^{j\omega}-1}{j\omega}\right]$$

$$\frac{d\phi_X(\omega)}{d\omega} = -j\left[(e^{j\omega}-1)\left(\frac{-1}{\omega^2}\right)+j\left(\frac{e^{j\omega}}{\omega}\right)\right]$$

$$= -j\left[\frac{1}{\omega^2} - \frac{e^{j\omega}}{\omega^2} + j\frac{e^{j\omega}}{\omega}\right]$$

At $\omega=0$,

$$m_1 = (-j)\left(\frac{-j}{2}\right) = \frac{1}{2}$$

Moment Generating Function :-

The moment generating function (MGF) of a random variable is also used to generate the n^{th} moments about the origin.

Consider a random variable X with a probability density function $f_X(x)$. Then the moment generating function of X is ~~depen~~ defined as the expected value of the function e^{vx}. It ~~can~~ can be expressed as

$$M_X(v) = E[e^{vX}]$$

where v is a real variable $-\infty < v < \infty$.

Therefore $M_X(v) = \int_{-\infty}^{\infty} e^{vx} f_X(x)\,dx$

and for a discrete random ~~varia~~ variable X.

$$M_X(v) = \sum_i e^{vx_i} p(x_i)$$

The n^{th} moments of X can be derived from the moment generating function. Here the j operator does not exist. The main disadvantage of the moment generating function is that it may not exist for all random variables and all values of v.

But the characteristic function exists for all values of X and ω.

<u>Theorem :-</u>

If $M_X(v)$ is a moment generating function of a random variable X, then the n^{th} moment of X is given by

$$m_n = \left.\frac{d^n M_X(v)}{dv^n}\right|_{v=0}$$

<u>Proof:-</u>

Consider a random variable X with moment generating function

$$M_x(v) = E[e^{vx}]$$

we know that the series expansion of e^{vx} is

$$e^{vx} = 1 + vx + \frac{(vx)^2}{2!} + \frac{(vx)^3}{3!} + \cdots + \frac{(vx)^n}{n!} + \cdots$$

$$M_x(v) = E[e^{vx}]$$

$$= E\left[1 + vx + \frac{v^2x^2}{2} + \frac{v^3x^3}{3!} + \cdots + \frac{v^nx^n}{n!} + \cdots\right]$$

$$M_x(v) = E[1] + vE[x] + \frac{v^2}{2}E[x^2] + \frac{v^3}{3!}E[x^3] + \cdots + \frac{v^n}{n!}E[x^n] + \cdots$$

$$M_x(v) = m_0 + vm_1 + \frac{v^2}{2}m_2 + \frac{v^3}{3!}m_3 + \cdots + \frac{v^n}{n!}m_n + \cdots$$

To find the moments, substitute $v=0$

so $M_x(0) = m_0 = 1$

Differentiating $M_x(v)$ w.r. to v

$$\frac{dM_x(v)}{dv} = m_1 + vm_2 + \frac{v^2}{2}m_3 + \cdots$$

At $v=0$,

$$m_1 = \left.\frac{d\, m_x(v)}{dv^2}\right|_{v=0}$$

second differentiation of $m_x(v)$ w.r.to v is

$$\frac{d^2 m_x(v)}{dv^2} = m_2 + v m_3 + \cdots$$

At $v=0$,

$$m_2 = \left.\frac{d^2 m_x(v)}{dv^2}\right|_{v=0}$$

Similarly, at the nth differentiation of $m_x(v)$ w.r. to v, we get

$$\left.\frac{d^n m_x(v)}{dv^n}\right|_{v=0} = m_n$$

The nth moment of X is given by

$$m_n = \left.\frac{d^n m_x(v)}{dv^n}\right|_{v=0}$$

Thus the nth moments of X can be obtained from the nth differentiation of MGF at $v=0$

Properties of the moment Generating Function :-

(1) The moment generating function at $v=0$ is unity. It is given as

$$M_X(v)\Big|_{v=0} = 0 = M_X(0) = 1$$

<u>Proof :-</u>

The moment generating function is given by

$$M_X(v) = E\left[e^{vX}\right]$$

At $v=0$, $M_X(0) = E\left[e^{0}\right] = E[1] = 1$

$$M_X(0) = 1$$

(2) Let X be a random variable with moment generating function $M_X(v)$. Then the moment generating function of $Y = aX+b$ is given by

$$M_Y(v) = \cancel{e^{bv} M_X(av)} \; e^{vb} M_X(av)$$

<u>Proof :-</u> we know that

$$M_X(v) = E\left[e^{vX}\right]$$

$$M_Y(v) = E\left[e^{vY}\right]$$

$$= E\left[e^{v(aX+b)}\right]$$

$$= E\left[e^{avX}\, e^{vb}\right]$$

$$= e^{vb}\left[e^{avX}\right]$$

$$M_Y(v) = e^{vb} M_X(av)$$

(3) If $M_X(v)$ is a moment generating function of a random variable X, then $M_X(cv) = M_{cX}(v)$ where c is a real constant.

Proof:-
$$M_X(v) = E[e^{vX}]$$

Then $M_X(cv) = E[e^{cvX}] = E[e^{vcX}]$

$$\therefore M_X(cv) = M_{cX}(v)$$

(4) If X_1 and X_2 are two independent random variables with moment generating function $M_{X_1}(v)$ and $M_{X_2}(v)$, then
$$M_{X_1+X_2}(v) = M_{X_1}(v) M_{X_2}(v)$$

Proof:-
$$M_X(v) = E[e^{vX}]$$

for $M_{X_1+X_2}(v) = E[e^{v(X_1+X_2)}]$

$$M_{X_1+X_2}(v) = E[e^{vX_1} e^{vX_2}]$$

Since X_1 and X_2 are independent random variables
$$E[X_1 X_2] = E[X_1] E[X_2]$$

$$M_{X_1+X_2}(v) = E[e^{vX_1}] E[e^{vX_2}]$$

$$M_{X_1+X_2}(v) = M_{X_1}(v) M_{X_2}(v)$$

Similarly, if there are n independent random variables $X_1, X_2, X_3, \ldots X_n$ with moment generating functions $M_{X_1}(v), M_{X_2}(v) \ldots M_{X_n}(v)$ respectively, then

$$M_{X_1+X_2+\cdots+X_n}(v) = M_{X_1}(v)\, M_{X_2}(v) \cdots M_{X_n}(v)$$

Example :-

If pdf of a random variable is given by

$$f_X(x) = e^{-x} \quad \text{for } x \geq 0$$

find $M_X(v)$, m_1 & m_2

Soln:-
$$M_X(v) = E\left[e^{vX}\right] = \int_{-\infty}^{\infty} e^{vx} f_X(x)\,dx$$

$$= \int_{0}^{\infty} e^{vx} e^{-x}\,dx$$

$$= \int_{0}^{\infty} e^{-(1-v)x}\,dx$$

$$= \left.\frac{-e^{-(1-v)x}}{1-v}\right|_{0}^{\infty}$$

$$\therefore M_X(v) = \frac{1}{1-v}$$

Now $m_1 = \left.\frac{d}{dv} M_X(v)\right|_{v=0} = \left.\frac{1}{(1-v)^2}\right|_{v=0} = 1$

$$m_2 = \left.\frac{d^2 m_x(v)}{dv^2}\right|_{v=0} = \left.\frac{2}{(1-v^3)}\right|_{v=0} = 2$$

Inequalities:-

There are three important inequalities which are very useful in solving some types of probability problems.

(1) Chebyshev's inequality

(2) Markov's inequality

(3) Chernoff's inequality & bound.

Chebychev's inequality:-

Statement:-

For a given random variable X with mean value $\overline{X}$ and variance σ_x^2, it states that

$$P\{|X-\overline{X}| \geq \epsilon\} = \frac{\sigma_x^2}{\epsilon^2}$$

where ϵ is a very small +ve number

Proof:-

We know that the probability density function of a random variable X is given by

$$P\{X \leq x\} = F_X(x) = \int_{-\infty}^{\infty} f_X(x)\,dx$$

Now expand

$$P\{|X-\overline{X}| \geq \epsilon\} = P\{(x-\overline{x}) \leq -\epsilon\} + P\{(x-\overline{x}) \geq \epsilon\}$$

$$= P\{X \leq \overline{X} - \epsilon\} + P\{X \geq \overline{X} + \epsilon\}$$

$$= \int_{-\infty}^{\overline{X}-\epsilon} f_X(x)\,dx + \int_{\overline{X}+\epsilon}^{\infty} f_X(x)\,dx$$

$$\therefore P\{|X - \overline{X}| \geq \epsilon\} = \int_{|x-\overline{x}| \geq \epsilon}^{\infty} f_X(x)\,dx$$

we know that

$$\sigma_X^2 = \int_{-\infty}^{\infty} (X - \overline{X})^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{|x-\overline{x}| \geq \epsilon} (x - \overline{x})^2 f_X(x)\,dx$$

$$+ \int_{|x-\overline{x}| \geq \epsilon}^{\infty} (x - \overline{x})^2 f_X(x)\,dx$$

$$\sigma_X^2 \geq \int_{|x-\overline{x}| \geq \epsilon}^{\infty} (x - \overline{x})^2 f_X(x)\,dx$$

If $\quad X - \overline{X} = \epsilon$

then $\quad \sigma_X^2 \geq \int_{|x-\overline{x}| \geq \epsilon}^{\infty} \epsilon^2 f_X(x)\,dx$

$$\sigma_X^2 \geq \epsilon^2 \int_{|x-\bar{x}|\geq \epsilon}^{\infty} f_X(x)\, dx$$

$$\sigma_X^2 \geq \epsilon^2 P\{|X-\bar{X}| \geq \epsilon\}$$

$$\therefore P\{|X-\bar{X}| \geq \epsilon\} \leq \frac{\sigma_X^2}{\epsilon^2}$$

Similarly, chebychev's inequality theorem also states that

$$P\{|X-\bar{X}| < \epsilon\} \geq 1 - \frac{\sigma_X^2}{\epsilon^2} \quad \text{for any } \epsilon > 0$$

Take the total probability as

$$P\{|X-\bar{X}| < \epsilon\} + P\{|X-\bar{X}| \geq \epsilon\} = 1$$

or $P\{|X-\bar{X}| < \epsilon\} = 1 - P\{(X-\bar{X}) \geq \epsilon\}$

$$P\{|X-\bar{X}| < \epsilon\} \geq 1 - \frac{\sigma_X^2}{\epsilon^2}$$

Case 1

If $\sigma_X^2 \to 0$, then $P\{|X-\bar{X}| < \epsilon\} \to 1$ for any ϵ,

If $\epsilon = 0$, then $P\{X = \bar{X}\} \to 1$

i.e., If the variance of a random variable tends to zero, then the probability at mean value tends to one.

Case 2:-

If $\epsilon = k\sigma_x$, where k is any real number,

then $P\{|X-\bar{X}| \geq k\sigma_x\} \leq \frac{1}{k^2}$

$P\{|X-\bar{X}| \geq \epsilon\} \leq \frac{\sigma_x^2}{\epsilon^2}$

$P\{|X-\bar{X}| \geq k\sigma_x\} \leq \frac{\sigma_x^2}{k^2\sigma_x^2}$

or $P\{|X-\bar{X}| < k\sigma_x\} \geq 1 - \frac{1}{k^2}$

Markov's Inequality:-

Consider a continuous random variable X with pdf $f_X(x)$. If $f_X(x)=0$ for $x<0$, then the markov's inequality states that,

$$P\{X \geq a\} \leq \frac{\bar{X}}{a} \quad \text{for } a>0$$

Proof:- we know that

$$P(X \geq a) = \int_a^{\infty} f_X(x)\, dx$$

also since $f_X(x)=0$, for $x<0$

$$\bar{X} = E[X] = \int_0^{\infty} x f_X(x)\, dx$$

$$= \int_0^{a} x f_X(x)\, dx + \int_a^{\infty} x f_X(x)\, dx$$

$$\bar{X} \geq \int_a^{\infty} x f_X(x)\, dx$$

Let $x \geq a$

$$\bar{X} \geq \int_a^{\infty} a f_X(x)\, dx$$

$$\therefore a \int_a^\infty f_X(x)\,dx \le \overline{X}$$

$$\int_a^\infty f_X(x)\,dx \le \frac{\overline{X}}{a}$$

$$P(X \ge a) \le \frac{\overline{X}}{a}$$

Note

If $a = \overline{X}$, then $P(X \ge \overline{X}) \le 1$

Problem:- Find the largest probability that any random variable's values are smaller than its mean by 4 standard deviations or larger than its mean by the same amount.

Soln:- Let X be any random variable

The probability of X being smaller than $\overline{X} - 4\sigma_X$ or the probability of X being larger than $\overline{X} + 4\sigma_X$ is, given by

$$P\{X \ge \overline{X} + 4\sigma_X\} + P\{X \le \overline{X} - 4\sigma_X\}$$

$$= P\{|X - \overline{X}| \ge 4\sigma_X\}$$

where $\overline{X}$ = mean and σ_X is standard deviation of X. Now using Chebyshev's inequality

$$P\{|X - \overline{X}| \ge \epsilon\} \le \frac{\sigma_X^2}{\epsilon^2}$$

Here $\epsilon = 4\sigma_x$

$$P\{|X-\bar{X}| \geq 4\sigma_x\} \leq \frac{\sigma_x^2}{4^2\sigma_x^2} = \frac{1}{16}$$

$\therefore$ The maximum probability is

$$P\{|X-\bar{X}| \geq 4\sigma_x\} = \frac{1}{16} = 0.0625 = 6.25\%$$

<u>Transformations of a Random variable :-</u>

Transformation is used to convert a given random variable X into another random variable Y. It is denoted as

$$Y = T(X)$$

where T represents transformation. It may be linear, non-linear, stair case or segmented

The transformation of X to Y is shown in Fig

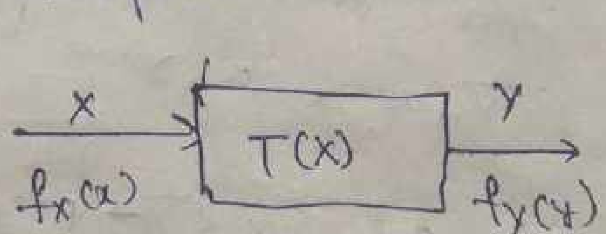


Fig: Transformation of X to Y

Here we need to consider three cases:

(1) Both X and T are continuous and T is either monotonically increasing or decreasing with X.

(2) Both X and T are continuous and T is non-monotonic.

(3) X is discrete & T is continuous.

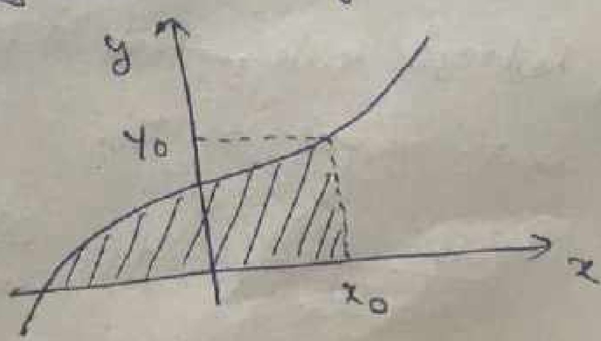
Monotonic Transformation of a continuous Random variable:-

Consider a random variable X. If the transformation is $T(x_1) < T(x_2)$ for any $x_1 < x_2$, then it is called a monotonically increasing transformation.

For the transformation to be monotonically decreasing, the condition is $T(x_1) > T(x_2)$ for any $x_1 < x_2$.

Monotonically Increasing function:-

Assume that the transformation T is continuous and differentiable for all values of x with $f_X(x) \neq 0$. Let another random variable Y have a value y_0 corresponding to x_0 of X as shown in fig



the transformation is given as

$$y = T(x)$$

$$y_0 = T(x_0) \text{ or}$$

$$x_0 = T^{-1}(y_0)$$

where T^{-1} represents the inverse of the transformation T.

Since transformation provides a one-to-one correspondence between X and Y, the probability of the event $\{Y \le y_0\}$ must be equal to the probability of the event $\{X \le x_0\}$

Thus,
$$P\{Y \le y_0\} = P\{X \le x_0\}$$

$$F_Y(y_0) = F_X(x_0)$$

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = \int_{-\infty}^{x_0} f_X(x)\,dx$$

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = \int_{-\infty}^{T^{-1}(y_0)} f_X(x)\,dx$$

Using Leibniz's rule, and differentiating both sides with respect to y_0,

$$\frac{d}{dy_0}\int_{-\infty}^{y_0} f_Y(y)\,dy = \frac{d}{dy_0}\int_{-\infty}^{T^{-1}(y_0)} f_X(x)\,dx$$

$$f_Y(y_0) = f_X(T^{-1}(y_0)) \frac{dT^{-1}(y_0)}{dy_0}$$

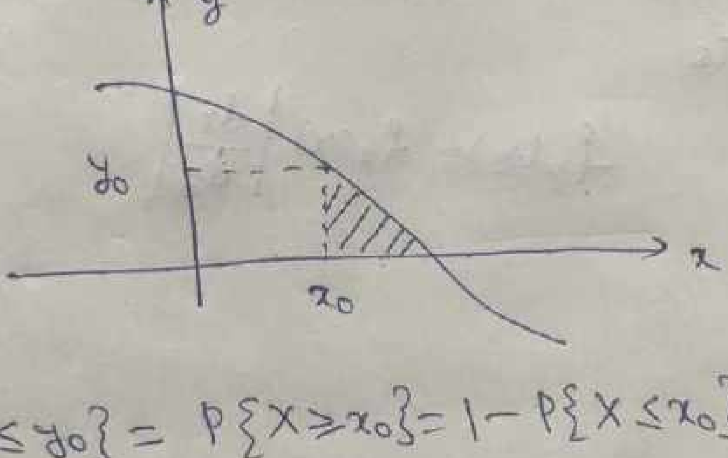
for any y_0,

$$f_Y(y) = f_X(T^{-1}(y)) \frac{dT^{-1}(y)}{dy}$$

$$f_Y(y) = f_X(x) \frac{dx}{dy}$$

Monotonically Decreasing function:-

Similarly, for a monotonically decreasing function as shown in Fig.



$$P\{Y \le y_0\} = P\{X \ge x_0\} = 1 - P\{X \le x_0\}$$

$$F_Y(y_0) = 1 - F_X(x_0)$$

or $$\int_{-\infty}^{y_0} f_Y(y)\,dy = 1 - \int_{-\infty}^{x_0} f_X(x)\,dx$$

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = 1 - \int_{-\infty}^{x_0 = T^{-1}(y_0)} f_X(x)\,dx$$

Using Leibniz's rule & differentiating w.r.t y_0

$$f_y(y_0) = -f_x\left(T^{-1}(y_0)\right) \cdot \frac{dT^{-1}(y_0)}{dy_0}$$

or for any y_0,

$$f_y(y) = -f_x\left(T^{-1}(y)\right) \frac{dT^{-1}(y)}{dy}$$

$\therefore$

$$f_y(y) = f_x(x)\left(-\frac{dx}{dy}\right)$$

$\therefore$ For a monotonic transformation, either increasing or decreasing, the density function of Y is

$$f_y(y) = f_x(x)\left|\frac{dx}{dy}\right|$$

Leibnize integral rule:-

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)\,dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}\,dx + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z}$$

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x,b(x))\cdot\frac{d\,b(x)}{dx} - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt$$

Prob:- show that the linear transformation of a Gaussian Random variable produces another Gaussian random variable.

Sol^n:- consider a Gaussian random variable X. Let another random variable Y be a linear transformation of X, with function

$$Y = T[X] = aX + b$$

where a and b are real constants.

Therefore $$X = \frac{Y-b}{a} = \frac{Y}{a} - \frac{b}{a}$$

$$\frac{dX}{dY} = \frac{1}{a}$$

From transformation,

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\left|\frac{1}{a}\right|$$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

If X is a Gaussian random variable, then

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$$

where a_x is mean & σ_x^2 is variance

$$f_Y(y) = \frac{1}{a}\frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{y-b}{a} - a_x\right)^2 / 2\sigma_x^2}$$

$$f_Y(y) = \frac{1}{a\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{y-(aa_x+b)}{a}\right)^2 / 2\sigma_x^2}$$

$$= \frac{1}{\sqrt{2\pi(a\sigma_x)^2}} e^{-(y-(aa_x+b))^2 / 2(a\sigma_x)^2}$$

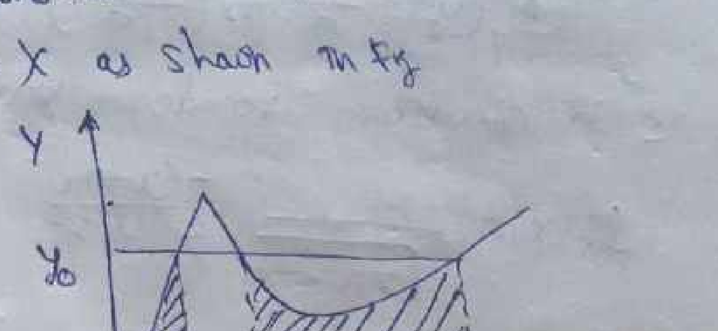
$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-(y-a_y)^2 / 2\sigma_y^2}$$

This is another Gaussian density function having variance $\sigma_y^2 = (a\sigma_x)^2$ and mean $a_y = aa_x + b$.

Hence Y is a Gaussian random variable

Non-monotonic Transformation of a continuous Random variable:-

Consider that a random variable Y is a non-monotonic transformation of a random variable X as shown in fig



For a given event $\{y \le y_0\}$, there is more than one value of X. From Fig, it is found that the event $\{y \le y_0\}$ corresponds to the events $\{X \le x_1$ and $x_2 \le X \le x_3\}$

Thus the probability of the event $\{Y \le y_0\}$ is equal to the probability of the event $\{X | Y \le y_0\}$

$$P\{Y \le y_0\} = P\{X | Y \le y_0\}$$

$$F_Y(y_0) = \int_{(X|Y \le y_0)} f_X(x)\, dx$$

By differentiating, the density function is given by

$$\frac{dF_Y(y_0)}{dy_0} = f_Y(y_0) = \frac{d}{dy_0} \int_{X|Y \le y_0} f_X(x)\, dx$$

or

$$f_Y(y) = \frac{d}{dy} \int_{(X|Y \le y)} f_X(x)\, dx$$

This can be simplified by using Leibniz's rule as

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{\left| \frac{dT(x)}{dx} \right|_{x = x_n}}$$

$$f_Y(y) = f_X(x_1)\left| \frac{dx_1}{dy} \right| + f_X(x_2)\left| \frac{dx_2}{dy} \right| + f_X(x_3)\left| \frac{dx_3}{dy} \right| + \cdots$$

where x_n, $n = 1, 2, \dots$ are real roots of the equation $y = T(x)$

Transformation of a Discrete Random Variable:-

If X is a discrete random variable and the transformation is monotonic, then the set $\{y_n\}$ of a random variable Y has a one-to-one correspondence with the set $\{x_n\}$, through the equation

$$y_n = T(x_n)$$

and

$$P(y_n) = P(x_n)$$

Therefore

$$f_Y(y) = \sum_n P(y_n)\,\delta(y - y_n)$$

$$F_Y(y) = \sum_n P(y_n)\,u(y - y_n)$$

If the transformation is non-monotonic, there may exist more than one value x_n which corresponds to y_n. $P(y_n)$ equals the sum of all probability of values x_n for which $y_n = T(x_n)$

Prob:- A discrete random variable X with pdf is given in Table. Find the density function of Y for the transformation

$$Y = 3X^3 - 3X^2 + 2$$

X	0	1	2	3	4
P(X)	0.2	0.15	0.3	0.15	0.2

Sol^n Given from the Table, the pdf is

$$f_X(x) = 0.2\,\delta(x) + 0.15\,\delta(x-1) + 0.3\,\delta(x-2)$$
$$+ 0.15\,\delta(x-3) + 0.2\,\delta(x-4)$$

and the transformation is

$$Y = 3X^3 - 3X^2 + 2$$

The corresponding values of Y for each X is

$$Y = \{2, 2, 14, 56, 730\}$$

Here the two values of $x=0$ & $x=1$ map onto only one value of $Y=2$

The probability of

$$P\{Y=2\} = P\{X=0\} + P\{X=1\}$$
$$= 0.2 + 0.15 = 0.35$$

∴ Y values are given in Table

Y	2	14	56	730
P(Y)	0.35	0.3	0.15	0.2

$$f_Y(y) = 0.35\,\delta(y-2) + 0.3\,\delta(y-14) + 0.15\,\delta(y-56)$$
$$+ 0.2\,\delta(y-730).$$