10.3 STATE-SPACE REPRESENTATION

10.3.1 State-Space Representation Using Physical Variables

Problem1

Consider the *RLC* circuit shown in Figure 10.4 The input is a voltage source $e(t)$. Let us say the input is applied at $t = t_0$. The desired output information is usually the voltages and currents associated with various elements of the network. This information at any time t can be obtained if the initial voltage across the capacitor $e_c(t_0)$ and the initial current through the inductor $i\,(t_0)$ are known in addition to the values of the input $e(t)$ applied for $t > t_0$. The voltage across the capacitor and the current through the inductor thus constitute a set of characterizing variables of the circuit. The initial state of the circuit is given by $e_c(t_0)$ and $i(t_0)$, and the state of the circuit at any time t is given by $e_c(t)$ and $i(t)$. The values of the characterizing variables at time t describe the state of the network at that time. These variables are therefore called state variables of the circuit.

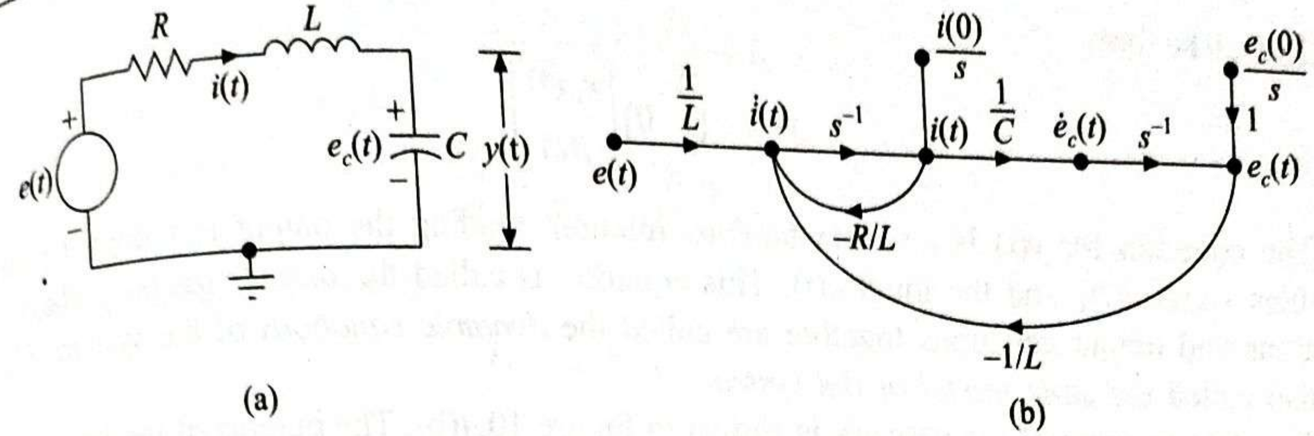


Figure 10.4 (a) *RLC* network and (b) state diagram.

Circuit analysis usually requires setting up of dynamical equations (using Kirchoff's voltage and current laws) in terms of rates of change of capacitor voltages and inductor currents. The solution of these equations describes the state of the network at time t. Desired output information is then obtained from the state using algebraic relation.

For the circuit shown

$$Ri(t) + \frac{Ldi(t)}{dt} + e_c(t) = e(t) \tag{10.6}$$

and

$$\frac{Cde_c(t)}{dt} = i(t) \tag{10.7}$$

Rearrangement of Eqs. (10.6) and (10.7) gives the rates of change of capacitor voltage and inductor current.

$$\frac{de_c(t)}{dt} = \frac{1}{C}i(t) \tag{10.8}$$

$$\frac{di(t)}{dt} = \frac{1}{L}e(t) - \frac{R}{L}i(t) - \frac{1}{L}e_c(t) \tag{10.9}$$

In vector-matrix form, Eqs. (10.8) and (10.9) can be written as

$$\begin{bmatrix} \dfrac{de_c(t)}{dt} \\ \dfrac{di(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} e_c(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} e(t)$$

These equations give the rates of change of state variables (capacitor voltage $e(t)$ and inductor current $i(t)$) in terms of the state variables and the input. These equations are called the *state equations*.

The solution of these equations for given input $e(t)$ applied at $t = 0$ and given initial state $[e_c(0), i(0)]$ yields the state $[e_c(t), i(t)]$ for $t > 0$. If $y(t)$ shown in Figure 10.4(a) is the desired output information, we have the following algebraic relation to obtain $y(t)$.

$$y(t) = e_c(t)$$

In matrix form

$$y(t) = [1 \quad 0]\begin{bmatrix} e_c(t) \\ i(t) \end{bmatrix}$$

The equation for $y(t)$ is an instantaneous relation, reading the output $y(t)$ from the state variables $\{e_c(t),\ i(t)\}$ and the input $e(t)$. This equation is called the *output equation*. The state equations and output equations together are called the *dynamic equations* of the system. They are also called the *state model of the system*.

Example 10.4 Obtain the state-space representation of the *RLC* network shown in Figure 10.8.

Problem 2

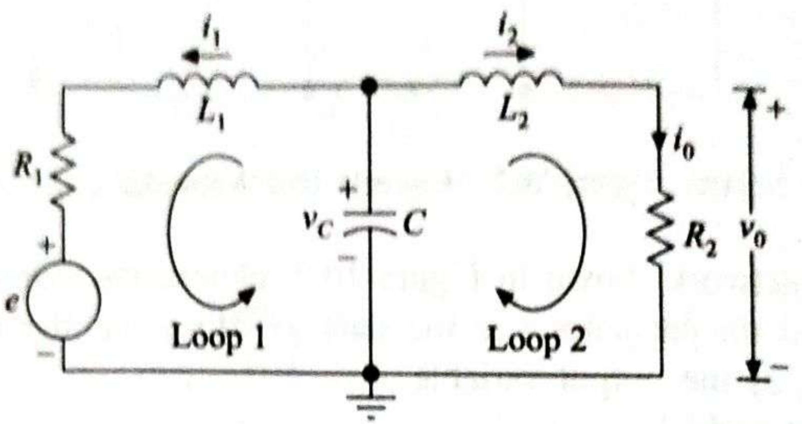


Figure 10.8 Example 10.4: Network.

Solution: The network shown in Figure 10.8 has three energy storage elements, a capacitor C and two inductors L_1 and L_2. History of the network is completely specified by the voltage across the capacitor and the currents through the inductors at $t = 0$. If we have a knowledge of initial conditions $i_1(0)$, $i_2(0)$, $v_c(0)$ and the input signal $e(t)$ for $t \geq 0$, then the behaviour of the network is completely specified for $t \geq 0$. However, if one (or more) of the initial conditions is not known, we are unable to determine the complete response of the network to a given input. Therefore, the initial conditions $i_1(0)$, $i_2(0)$, $v_c(0)$ together with the input signal $e(t)$ for $t \geq 0$ constitute the minimum information needed. Hence select the currents through the inductors $i_1(t)$ and $i_2(t)$ and the voltage across the capacitor $v_c(t)$ as the state variables. The output variables are the current through R_2, i.e. $i_0(t)$ and the voltage across R_2, i.e. $v_0(t)$ and the input variable is $e(t)$. Hence let

$$x_1(t) = i_1(t) \qquad y_1(t) = i_0(t)$$
$$x_2(t) = i_2(t) \qquad y_2(t) = v_0(t)$$
$$x_3(t) = v_c(t) \qquad u(t) = e(t)$$

The differential equations governing the behaviour of the *RLC* network are obtained by writing the KCL equation at the node and the KVL equations around the two loops.

Writing the KCL equation at the node

$$i_1 + i_2 + C\frac{dv_c}{dt} = 0$$

Writing KVL equation around loop 1

$$L_1\frac{di_1}{dt} + R_1 i_1 + e - v_c = 0$$

Writing KVL equation around loop 2

$$L_2\frac{di_2}{dt} + R_2 i_2 - v_c = 0$$

Expressing the first derivatives of the state variables $\frac{di_1}{dt}, \frac{di_2}{dt}$ and $\frac{dv_c}{dt}$ as linear combinations of the state variables i_1, i_2, v_c, and the input variable e, the state equations are

$$\frac{di_1}{dt} = \frac{-R_1}{L_1}i_1 + \frac{1}{L_1}v_c - \frac{1}{L_1}e$$

$$\frac{di_2}{dt} = \frac{-R_2}{L_2}i_2 + \frac{1}{L_2}v_c$$

$$\frac{dv_c}{dt} = \frac{-1}{C}i_1 - \frac{1}{C}i_2$$

and the output equations are

$$i_0(t) = i_2(t)$$

and

$$v_0(t) = i_2(t)\,R_2$$

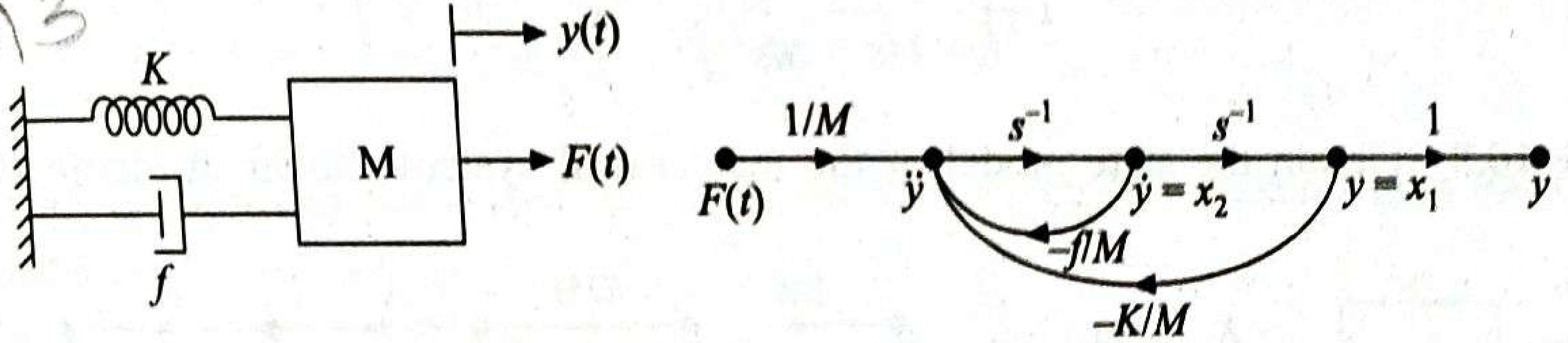
In terms of the state variables and the outputs defined earlier, the state variable formulation in matrix form is

$$\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_c \end{bmatrix} + \begin{bmatrix} -\frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} e, \text{ i.e. } \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & \frac{-R_2}{L_2} & \frac{1}{L_2} \\ -\frac{1}{C} & \frac{-1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$\begin{bmatrix} i_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_c \end{bmatrix} \text{ i.e. } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 10.7 Obtain a state model for the mechanical system shown in Figure 10.12(a).

Problem 3



(a) (b)

Figure 10.12 Example 10.7: (a) mechanical system and (b) state diagram.

Solution: The differential equation governing the behaviour of the given mechanical system is

$$F(t) = M\frac{d^2y}{dt^2} + f\frac{dy}{dt} + Ky$$

$$\therefore \qquad \frac{d^2y}{dt^2} = -\frac{f}{M}\frac{dy}{dt} - \frac{K}{M}y + \frac{1}{M}F(t)$$

The state diagram of the system is constructed as shown in Figure 10.12(b). By defining the outputs of the integrators on the state diagram as state variables x_1 and x_2, the state equations are as follows:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\frac{K}{M}x_1 - \frac{f}{M}x_2 + \frac{1}{M}F(t)$$

and the output equation is

$$y(t) = x_1(t)$$

In vector-matrix form, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{f}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} F(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From the state diagram, the transfer function of the system is

$$\frac{Y(s)}{F(s)} = \frac{\dfrac{1}{Ms^2}}{1-\left(-\dfrac{f}{Ms} - \dfrac{K}{Ms^2}\right)} = \frac{1}{Ms^2 + fs + K}$$

Example 10.8 Obtain the state model of the mechanical system shown in Figure 10.13(a).

Problem 4

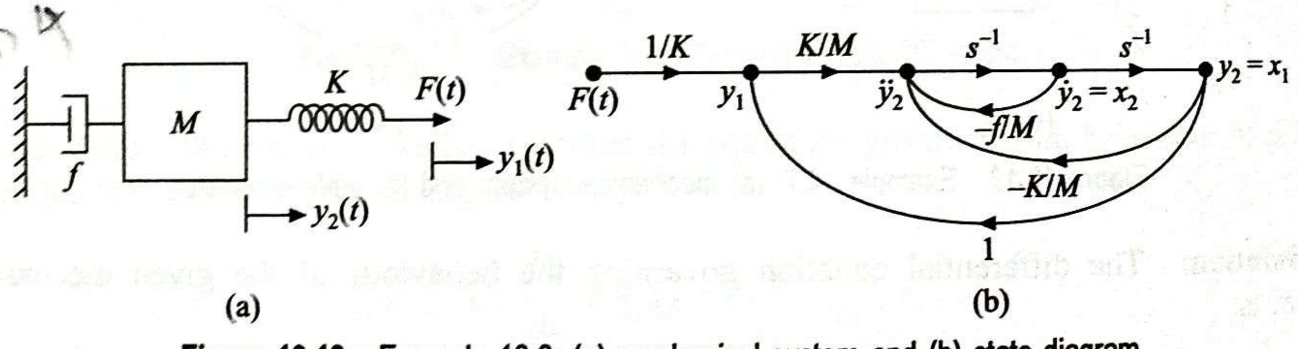


Figure 10.13 Example 10.8: (a) mechanical system and (b) state diagram.

Solution: The differential equations governing the behaviour of the given mechanical system are

$$F(t) = K[y_1(t) - y_2(t)]$$

$$M\frac{d^2y_2(t)}{dt^2} + K(y_2(t) - y_1(t)) + f\frac{dy_2(t)}{dt} = 0$$

Let the output be $y_1(t)$. The equations are arranged as

$$\frac{d^2y_2(t)}{dt^2} = \frac{K}{M}y_1(t) - \frac{K}{M}y_2(t) - \frac{f}{M}\frac{dy_2(t)}{dt}$$

and
$$y_1(t) = \frac{1}{K}F(t) + y_2(t)$$

Using the last two equations, the state diagram of the system is drawn as shown in Figure 10.13(b). The outputs of the integrators are taken as the state variables. So defining the state variables as $x_1(t) = y_2(t)$ and $x_2(t) = \dot{y}_2(t)$, the state equations and the output equation written directly from the state diagram are

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = -\frac{f}{M}x_2(t) + \frac{1}{M}F(t)$$

and
$$y_1(t) = y_2(t) + \frac{1}{K}F(t)$$

So in vector-matrix form, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{f}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{K}F(t)$$

Example 10.10 Obtain a state model for the system described by

Problem 5

$$T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 6s^2 + 10s + 5}$$

Solution: The differential equation corresponding to the given transfer function is obtained by cross-multiplying and taking the inverse Laplace transform. So, we have

$$\dddot{y} + 6\ddot{y} + 10\dot{y} + 5y = u$$

Since the derivatives of the input are not present in the differential equation, phase variables can be selected as the state variables. Therefore,

$$x_1 = y \qquad \text{i.e.} \qquad y = x_1$$

$$x_2 = \dot{y} = \dot{x}_1 \qquad \dot{x}_1 = x_2$$

$$x_3 = \ddot{y} = \dot{x}_2 \qquad \dot{x}_2 = x_3$$

$$\dddot{y} = -6\ddot{y} - 10\dot{y} - 5y + u \qquad \dot{x}_3 = -5x_1 - 10x_2 - 6x_3 + u$$

Therefore, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(6) The Transfer function of a control system is given by

$$\frac{Y(s)}{U(s)} = \frac{s^2+3s+4}{s^3+2s^2+3s+2}$$

Obtain a state model

Soln:
$$\frac{Y(s)}{U(s)} = \frac{s^2+3s+4}{s^3+2s^2+3s+2}$$

let
$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{B(s)} \times \frac{B(s)}{U(s)}$$

where
$$\frac{Y(s)}{B(s)} = s^2+3s+4 \quad \text{---} (1)$$

&
$$\frac{B(s)}{U(s)} = \frac{1}{s^3+2s^2+3s+2} \quad \text{---} (2)$$

From eqn (1)

$$Y(s) = B(s)[s^2+3s+4]$$

$$Y(s) = s^2B(s)+3sB(s)+4B(s)$$

By taking I.L.T

$$y(t) = \ddot{b}(t)+3\dot{b}(t)+4b(t)$$

(or)
$$y = \ddot{b}+3\dot{b}+4b \quad \text{---} (3)$$

From eqn (2)

$$U(s) = B(s)[s^3+2s^2+3s+2]$$

$$U(s) = s^3B(s)+2s^2B(s)+3sB(s)+2B(s)$$

By taking I.L.T

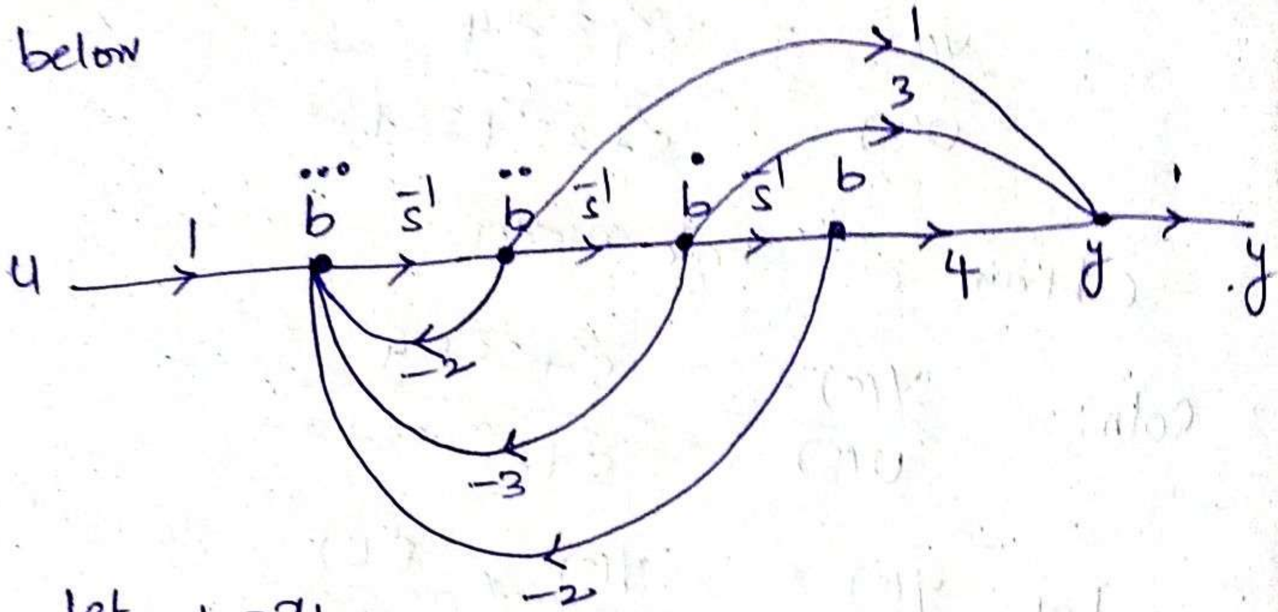
$$u(t) = \dddot{b}(t)+2\ddot{b}(t)+3\dot{b}(t)+2b(t)$$

or
$$u = \dddot{b}+2\ddot{b}+3\dot{b}+2b \quad \text{---} (4)$$

$$\dddot{b} = u-2\ddot{b}-3\dot{b}-2b \quad \text{---} (4)$$

from eqn (3) & eqn (4)

State diagram can be drawn as shown below



let $b = x_1$

$\dot{b} = x_2$

$\ddot{b} = x_3$

$\dot{x}_1 = \dot{b} = x_2$

$\dot{x}_2 = \ddot{b} = x_3$

$\dot{x}_3 = \dddot{b}$

from eqn (4)

$$\dot{x}_3 = -2x_1 - 3x_2 - 2x_3 + u$$

and $y = 4x_1 + 3x_2 + x_3$

stat eqn is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

and o/p eqn is

$$y = \begin{bmatrix} 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For a system described by the state equation

(1) $\dot{x}(t) = A\,x(t)$

The response is $x(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ when $x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

and $x(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ when $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Determine the system matrix A and the state transmission matrix

Soln: let the STM be $\phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix}$ $\boxed{\phi(t) = e^{At}}$ STM

We know that $x(t) = \phi(t)\,x(0)$

$$\therefore \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) - 2\phi_{12}(t) \\ \phi_{21}(t) - 2\phi_{22}(t) \end{bmatrix}$$

$$\therefore \phi_{11}(t) - 2\phi_{12}(t) = e^{-2t} \quad \text{--- (1)}$$

and $$\phi_{21}(t) - 2\phi_{22}(t) = -2e^{-2t} \quad \text{--- (2)}$$

Ily $$\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) - \phi_{12}(t) \\ \phi_{21}(t) - \phi_{22}(t) \end{bmatrix}$$

$$\therefore \phi_{11}(t) - \phi_{12}(t) = e^{-t} \quad \text{--- (3)}$$

and $$\phi_{21}(t) - \phi_{22}(t) = -e^{-t} \quad \text{--- (4)}$$

Subtracting eqn (5) from eqn (1)

$$\cancel{\phi_{11}(t)} - 2\phi_{12}(t) = e^{-2t}$$
$$\cancel{\phi_{11}(t)} - \phi_{12}(t) = e^{-t}$$
$$- \quad + \quad -$$

$$-\phi_{12}(t) = e^{-2t} - e^{-t}$$
$$\therefore \phi_{12}(t) = e^{-t} - e^{-2t} \text{ ——— } (5)$$

eqn (1) − eqn (3) ×2

$$\phi_{11}(t) - \cancel{2\phi_{12}(t)} = e^{-2t}$$
$$2\phi_{11}(t) - \cancel{2\phi_{12}(t)} = 2e^{-t}$$
$$- \quad + \quad -$$

$$-\phi_{11}(t) = e^{-2t} - 2e^{-t}$$
$$\phi_{11}(t) = 2e^{-t} - e^{-2t} \text{ ——— } (6)$$

Similary by solving eqn (2) & (4)

$$\phi_{21}(t) = 2e^{-t} + 2e^{-2t} \text{ ——— } (7)$$
$$\phi_{22}(t) = 2e^{-2t} - e^{-t} \text{ ——— } (8)$$

Hence the State Transmission matrix is

$$\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

By Taking L.T on both sides we get

$$\phi(s) = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{2}{s+2} - \frac{2}{s+1} & \frac{2}{s+2} - \frac{1}{s+1} \end{bmatrix}$$

but

$$\phi(s) = \frac{1}{sI-A} = [sI-A]^{-1} \qquad \phi(t) = e^{At}$$

$$[sI-A]^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{\begin{array}{cc} d & -c \\ \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} & \\ -b & a \end{array}}{|\Delta|} = \frac{\text{adj } sI-A}{|sI-A|}$$

where $\Delta = |sI-A| = (s+1)(s+2)$

$$\boxed{\begin{array}{l} \text{if } \quad X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad X^{-1} = \dfrac{\text{adj } X}{|X|} \\ \text{where } \quad \text{adj } X = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{array}}$$

$$sI-A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

(8) Obtain the time response of the sys described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

with the intial conditions $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and $y = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Soln: State eqn is

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

Apply LT

$$\mathcal{L}[\dot{x}(t)] = A\,\mathcal{L}[x(t)] + B\,\mathcal{L}[u(t)].$$

$$sX(s) - x(0) = A\,X(s) + B\,U(s)$$

$$[sI - A]\,X(s) = x(0) + B\,U(s)$$

$$X(s) = [sI-A]^{-1} x(0) + [sI-A]^{-1} B\,U(s)$$

where $\phi(s) = [sI-A]^{-1}$

$$\therefore \quad X(s) = \phi(s)\,x(0) + \phi(s)\,B\,U(s)$$

$$X(s) = \phi(s)\left[x(0) + B\,U(s)\right]$$

$$= \phi(s)\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\left(\frac{1}{s}\right)\right]$$

$\because$ $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $U(s) = \frac{1}{s}$

$$= \phi(s)\begin{bmatrix} 0 + \frac{1}{s} \\ 1 - 1\left(\frac{1}{s}\right) \end{bmatrix}$$

$$\therefore \quad X(s) = \phi(s)\begin{bmatrix} \frac{1}{s} \\ 1-\frac{1}{s} \end{bmatrix} \quad \text{—— ①}$$

$$\phi(s) = [sI-A]^{-1}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad (sI-A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}$$

$$\text{adj } sI-A = \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}$$

$$|sI-A| = s(s+2)-(-1)(1) = s^2+2s+1 = (s+1)^2$$

$$\phi(s) = [sI-A]^{-1} = \frac{\text{adj } sI-A}{|sI-A|}$$

$$\phi(s) = \frac{\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}}{(s+1)^2} \quad \text{—— ②}$$

$$\therefore X(s) = \frac{\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}}{(s+1)^2}\begin{bmatrix} \frac{1}{s} \\ 1-\frac{1}{s} \end{bmatrix}$$

By solving $\quad X(s) = \begin{bmatrix} \frac{2s+1}{s(s+1)^2} \\ \frac{s^2-s-1}{s(s+1)^2} \end{bmatrix}$

$$X(s) = \begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2} \\ -\frac{1}{s} + \frac{2}{s+1} - \frac{1}{(s+1)^2} \end{bmatrix}$$

By taking inverse L.T

$$x(t) = \begin{bmatrix} 1 - e^{-t} + te^{-t} \\ -1 + 2e^{-t} - te^{-t} \end{bmatrix}$$

o/p response is

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In S-domain

$$Y(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

$$Y(s) = X_2(s) = -\frac{1}{s} + \frac{2}{s+1} - \frac{1}{(s+1)^2}$$

By taking I.L.T

the o/p eqn is

$$y(t) = -1 + 2e^{-t} - te^{-t}$$

(9) A linear Time Invariant sys is characterized by the homogeneous state eqn

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(a) Compute the soln of the homogeneous equation assuming intial state vector

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) Compute the soln of the non homogeneous equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where u is unit step function

assume the initial conditions mentioned in part (a)

Soln: (a) For the homogeneous eqn.

soln is $x(t) = \phi(t)\, x(0)$

where $\phi(t) = \mathcal{L}^{-1}[\phi(s)]$

& $\phi(s) = [sI - A]^{-1}$

(b) For Non-homogeneous soln

Similar to the problem (8)