Unit-2 Numerical Integration 1

Introduction

→ In some cases if the functions have a closed from representation and are unable. 1st standard calculus method then differentiation and Integration can be carried out. In many situations we may not know the exact function we will be knowing only the values of the function at a discrete set of points. In some cases the functions are known but they are so complic that normal integration is difficult. In these situation we take the help of numerical techniques

→ The method of finding the value of an integral of the form $\int_a^b f(x)\,dx$ using Numerical technique is called numerical integration

→ In this chapter we discuss various Numerical Integration method.

* Trapezoidal Rule (2 marks)

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(\text{Sum of first \& last ordinate}) + 2(\text{Sum of remaining ordinate})\right]$$

* Simpson's 1/3 rule (2 marks)

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(\text{Sum of first \& Last ordinate}) + 4(\text{Sum of odd ordinate})\right]$$

* Simpson's 3/8 rule (2 marks)

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(\text{Sum of first \& Last ordinate}) + 3(y_1 + y_2 + \dots y_5 + \dots) + 2(\text{Multiple of 3 ordinates})\right]$$

where $h = \dfrac{x_n - x_0}{n}$

h = step size = Equal difference b/w the Observations.

n = no. of observations.

* Evaluate $\int_0^6 \frac{1}{1+x}\,dx$ by using Trapezodial Rule, simpson's $1/3$ rule and simpson's $3/8$ rule

Sol Given $\int_0^6 \frac{1}{1+x}\,dx$

Here $x_0 = 0 \quad x_n = 6$

or $a = 0 \quad b = 6$

$$h = \frac{x_n - x_0}{n} = \frac{b-a}{n} = \frac{6-0}{6} = \frac{6}{6} = 1$$

0th $\quad$ $n = 6$

x	0	1	2	3	4	5	6
y	1	0.5	0.333	0.25	0.2	0.1666	0.1428
$\frac{1}{1+x}$	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1. Trapezoidal Rule

$$\int_0^6 \frac{1}{1+x}\,dx = h/2 \left[\left(\begin{array}{l} \text{sum of first \& last ordinates} \end{array} \right) + 2 \left(\text{sum of remaining ordinates} \right) \right]$$

$$= \frac{1}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5) \right]$$

$$= \frac{1}{2}\left[(1 + 0.1428) + 2(0.5 + 0.333 + 0.25 + 0.2 + 0.1666) \right]$$

$= 2.021$

2. Simpson's $\frac{1}{3}$ Rule

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{1}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.1428)+4(0.5+0.25+0.1666)+2(0.333+0.2)\right]$$

$$= 1.9584$$

3. Simpson's $\frac{3}{8}$ Rule

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5+\dots)+2(y_3)\right]$$

$$= \frac{3}{8}\left[(1+0.1428)+3(0.5+0.333+0.2+0.1666)+2(0.25)\right]$$

$$= 1.9656$$

* Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ $h=\frac{1}{6}$ using 1^{st} & 2^{nd} & 3^{rd} rules.

Sol Given $\int_0^1 \frac{1}{1+x^2}\,dx$

$h = \frac{1}{6}$

x	0	1/6	2/6	3/6	4/5	5/6	1
y	1	0.9739	0.9	0.8	0.6923	0.5901	0.5

Trapezodiah Rule

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{3}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{3}\left[(1+0.5)+2(0.97293+0.9+0.8+0.6923+0.5901)\right]$$

$$= 0.7842$$

Simpson's 1/3 Rule

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{18}\left[(1+0.5)+(0.9729+0.8+0.5901)+2(0.9+0.6923)\right]$$

$$= 0.7853$$

Simpson's 3/8 Rule

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3h}{8} = \frac{1}{16}\left[(y_0+y_6)+3(y_1+y_2+y_3+y_4+y_5)+2(y_3)\right]$$

$$= \frac{1}{16}\left[(1+0.5)+3(0.49729+0.9+0.6923+0.5901)+2(0.8)\right]$$

$$= 0.7853$$

Note:-

→ In Trapezoidal Rule there is no restructions for taking 'n' value

→ In Simpson's 1/3 rule 'n' value should be taken as even number.

→ In Simpson's 3/8 rule 'n' value should be taken as multiply 3.

→ Its better to consider 'n = 6' always

* Evaluate $\int_0^{\pi/2} e^{\sin x}\,dx$ using Simpson's Rule

Sol Given $\int_0^{\pi/2} e^{\sin x}\,dx$

Here $a=0$ $b=\pi/2$ $n=6$

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \pi/12$$

x	0	$\pi/12$	$2\pi/12$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$6\pi/12$
y	1	1.2953	1.6487	2.0281	2.3774	2.6272	2.7182
$e^{\sin x}$	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $1/3$ Rule

$$\int_0^{\pi} e^{\sin x}\,dx = \frac{\pi}{3\times 12}\Big[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\Big]$$

$$= \frac{\pi}{36}\Big[(1+2.7182) + 4(1.2953+2.0281+2.6272) + 2(1.6487+2.3774)\Big]$$

$$= 3.1043$$

Simpson's $3/8$ Rule

$$\int_0^{\pi} e^{\sin x}\,dx = \frac{3\times\pi}{8\times 12}\Big[(1+2.7182) + 3(1.2953 + 1.6487 + 2.3774 +$$

$2.6272 + 2(2.0281)]$

$= 3.1043$

H/W
problem

1) Evaluate $\int_0^1 \frac{1}{1+x}\,dx$

Sol Given $\int_0^1 \frac{1}{1+x}\,dx$

$a=0 \quad b=1 \quad n=6$

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y	1	0.8571	0.75	0.6666	0.6	0.5454	0.5
$\frac{1}{1+x}$	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Trapezoidal Rule

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{1}{12}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{12}\left[(1+0.5) + 2(0.8571 + 0.75 + 0.6666 + 0.6 + 0.5454)\right]$$

$= 0.6948$

Simpson's 1/3 Rule:-

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{1}{18}\Big[(y_0+y_6)+4(y_1+y_3+y_5)$$
$$+2(y_4+y_6)\Big]$$

$$= \frac{1}{18}\Big[(1+0.5)+4(0.8571+$$
$$0.6666+0.5454)+$$
$$2(0.75+0.6)\Big]$$

$$= 0.6931$$

Simpson's 3/8 Rule

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{1}{16}\Big[(y_0+y_6)+3(y_1+y_2+y_4+y_5)$$
$$+2(y_3)\Big]$$

$$= \frac{1}{16}\Big[(1+0.5)+3(0.8571+0.75+$$
$$0.6+0.5454)+$$
$$2(0.6666)\Big]$$

$$= 0.6931$$

20/1/20

Numerical Solution of Ordinary Differential Equations

Many problems in science & engineering can be formulated into Ordinary Differential Equations.

The Analytical methods of solving Differential Eqn's are applicable only to a selected class of Differential Eqn's

The Problems which cannot be solved by analytical methods we use numerical methods

→ In this chapter we mainly concentrate on the numerical solution of Ordinary Differential Eqn's by using the following methods

1) Taylor's Series method
2) Euler's method
3) Picard's Successive approximation Method
4) Runge - Kutta method (R-K method)

Euler's method.

The formula for Euler's method is,

$$y_{n+1} = y_n + hf(x_n, y_n) \quad n=0,1,2\ldots\ldots$$

Problem

1) If $\frac{dy}{dx} = x+y$ $y(0)=2$ then find $y(0.1)$ & $y(0.2)$ using Euler's Method.

Sol given $\frac{dy}{dx} = f(x,y) = x+y$

$y(0)=2$

$x_0=0 \quad y_0=2$

	x_0	x_1	x_2
x	0	0.1	0.2
y	2	?	?

So $h=0.1$

By Euler's theorem

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $n=0$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 2 + 0.1\,[x_0+y_0]$$

$$= 2 + 0.1\,[0+2]$$

$$y_1 = 2 + 0.2$$

$$y_1 = 2.2$$

for $n=1$

$$y_2 = y_1 + hf(x_1, y_1)$$

$= y_1 + h[x_1 + y_1]$

$= 2.2 + 0.1[0.1 + 2.2]$

$y_2 = 2.43$

	x_0	x_1	x_2
x	0	0.1	0.2
y	2	2.2	2.4

2. solve numerically using Euler's method

$y' = y^2 + x \quad y(0) = 1 \quad$ find $y(0.1)$ & $y(0.2)$

Sol given $\frac{dy}{dx} = f(x,y) = y^2 + x$

$y(0) = 1$

$x_0 = 0 \qquad y_0 = 1$

x	0	0.1	0.2
y	1	?	?

$h = 0.1$

By Euler's theorem

$y_{n+1} = y_n + hf(x_n, y_n)$

for $n = 0$

$y_1 = y_0 + 0.1[\cancel{y^2+x};\ x_0, y_0]$

$y_1 = y_0 + 0.1[y_0^2 + x_0]$

$= 1 + 0.1[1 + 0] = 1.1$

for $n=1$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= y_1 + 0.1(y_1^2 + x_1)$$

$$= 1.1 + 0.1((1.1)^2 + 0.1)$$

$$= 1.23$$

3) $y' = x+y$, $y(0)=1$ find the value of $y(0.4)$ taking $h=0.2$

Sol

$$f(x,y) = x+y$$

$$y(0)=1$$

$$x_0 = 0 \quad y_0 = 1$$

x	0	0.2	0.4
y	1	?	?

$h = 0.2$

By Euler's theorem

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $n=0$

$$y_1 = y_0 + 0.2(x_0 + y_0)$$

$y_1 = 1 + 0.2(0+1)$

$y_1 = 1.2$

for $n = 1$

$y_2 = y_1 + hf(x_1, y_1)$

$y_2 = y_1 + 0.2(x_1 + y_1)$

$y_2 = 1.2 + 0.2(0.2 + 1.2)$

$y_2 = 1.48$

	x_0	x_1	x_2
x	0	0.2	0.4
y	1	1.2	1.48

4) using Euler's method solve for y at $x=2$ from $dy/dx = 3x^2+1$ $y(1)=2$ taking step size i) $h=0.5$ ii) 0.25

Sol $f(x,y) = 3x^2+1$

$y(1) = 2$

$x_0 = 1$ $y_0 = 2$

x	1	1.5	2
y	2	?	

$h = 0.5$

By Euler's theorem

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $n = 0$

$$y_1 = y_0 + 0.5\,(3x_0^2 + 1)$$

$$= 2 + 0.5\,(3(1) + 1)$$

$$y_1 = 4$$

for $n = 1$

$$y_2 = y_1 + hf\,(3x_1^2 + 1)$$

$$= 4 + 0.5\,(3(1.5)^2 + 1)$$

$$= 7.875$$

(ii) $h = 0.25$

$f(x, y) = 3x^2 + 1$

$x_0 = 1 \qquad y_0 = 2$

x	1	1.25	1.5	1.75	2
y	2	? ~~3~~	? ~~4242~~	6 ?	?

By Euler's theorem

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $n=0$

$$y_1 = y_0 + h(3x_0^2+1)$$

$$y_1 = y_0 + 0.25\,(3(1)+1)$$

$$= 2 + 0.25\,(3+1)$$

$$y_1 = 3$$

for $n=1$

$$y_2 = y_1 + 0.25\,(3x_1^2+1)$$

$$= y_1 + 0.25\,(3\,(1.25)^2+1)$$

$$= 3 + 0.25\,(3\,(1.25)^2+1)$$

$$y_2 = 4.42$$

for $n=2$

$$y_3 = y_2 + 0.25\,(3x_2^2+1)$$

$$= y_2 + 0.25\,(3\,(1.75)^2+1)$$

$$= 4.42 + 0.25\,(3\,(1.75)^2+1)$$

$$= 6.35$$

for $n=3$

$$y_4 = y_3 + 0.25\,(3x_3^2+1)$$

$$= 6.35 + 0.25\,(3(1.75)^2+1)$$

$$y_4 = 8.89$$

x	1	1.25	1.5	1.75	2
y	2	3	4.42	6.35	8.89

using Euler's method find $y(0.1)$ & $y(0.2)$ given $y' = (x^3+xy^2)\,e^{-x}$ at $y(0)=1$

Sol

$$f(x,y) = (x^3+xy^2)\,e^{-x}$$

$x_0 = 0 \qquad y_0 = 1$

x	0	0.1	0.2
y	1		

$h = 0.1$

By Euler's theorem

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $n=0$

$$y_1 = y_0 + 0.1\,(x_0^3 + x_0 y_0^2)\,e^{-x_0}$$

$$= 1 + 0.1\,((0)^3 + (0)(1)^2)\,e^{-0}$$

$$y_1 = 1$$

for $n = 1$

$$y_2 = y_1 + 0.1\,(x_1^3 + x_1 y_1^2)\,e^{-x_1}$$

$$= 1 + 0.1\,((0.1)^3 + (0.1)(1)^2)\,e^{-0.1}$$

$$y_2 \Rightarrow 1.009$$

· compute y at $x = 0.25$ by Euler's method given $y' = 2xy$, $y(0) = 1$

Sol $f(x, y) = 2xy$

$x_0 = 0 \quad y_0 = 1$

x	0	0.25
y	1	?

$h = 0.25$

By Euler's theorem

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $n=0$

$$y_1 = y_0 + 0.25\,(2x_0 y_0)$$

$$= 1 + 0.25\,(2(0)(1))$$

$$= 1 + 0.25\,(0)$$

$$= 1$$

x	0	0.25
y	1	1

23/1/20

Runge- kutta Method (R-k method)

R-k method does not require the determination of higher order derivates give greater accuray & compare to Euler's method. This method have the advantage of requiring only the function values at some selected points on the sub interval & also It has a advantage of whether the operation is Identical since the differential eq'n is linear or Non-Linear.

first order R-K method

$$y_1 = y_0 + hf(x_0, y_0)$$

Second order R-K Method

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

Where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h, y_0 + k_1)$

Third order R-K Method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$

$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$

fourth order R-K method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$

$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2)$

$k_4 = hf(x_0 + h, y_0 + k_3)$

Problems

Using R-K Second order, third, fourth order solve $y' = xy$ $y(1)=2$ at $x=1.2$ with $h=0.2$

Sol: Given $f(x,y)=xy$

$y(1)=2$ $\quad h=0.2$

$x_0=1, \quad y_0=2$

x	1	1.2
y	2	?

Second order R-K method

$$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0+h, y_0+k_1)$

$k_1 = (0.2)(x_0 y_0)$

$= (0.2)(1 \times 2)$

$= 0.2\,(2)$

$k_1 = 0.4$

$k_2 = (0.2)(1+0.2 \times 2+0.4)$

$\Rightarrow (0.2)(1.2 \times 2.4)$

$\Rightarrow (0.2)(2.88) = 0.576$

Scanned by CamScanner

Scanned with CamScanner

$$y_1 = 2 + \frac{1}{2}(0.4 + 0.576)$$

$$= 2.488$$

3^{rd} order R-k method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $k_1 = hf(x_0, y_0)$

$$k_1 = (0.2)(x_0 y_0)$$

$$= 0.2(1 \times 2)$$

$$k_1 = 0.4$$

$$k_2 = hf(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1)$$

$$= (0.2)(1 + \tfrac{1}{2}(0.2) \times 2 + \tfrac{1}{2}(0.4))$$

$$= (0.2)(1.1 \times 2.2)$$

$$= (0.2)(2.42)$$

$$k_2 = 0.484$$

$$k_3 = hf(x_0 + h,\ y_0 + 2k_2 - k_1)$$

$$= (0.2)(1 + (0.2) \times 2 + 2(0.484) - 0.4)$$

$$= (0.2)(1.2 \times 2.568)$$

$$= (0.2)(3.0816)$$

$$= 0.61632$$

fourth order ~~R-K method~~

$y_1 = y_0 + 1/6 (k_1 + 4k_2 + k_3)$

$= 2 + 1/6 (0.4 + 4(0.484) + 0.61632)$

$= 2 + 1/6 (0.4 + 1.936 + 0.61632)$

$= 2 + 1/6 (2.95232)$

$= 2.4920$

fourth order R-K method

$y_1 = y_0 + 1/6 (k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = hf(x_0, y_0)$

$k_1 = (0.2)(x_0 y_0)$

$= (0.2)(1 \times 2)$

$= (0.2)(2)$

$k_1 = 0.4$

$k_2 = hf(x_0 + 1/2 h, y_0 + 1/2 k_1)$

$(x_0 + 1/2 h \times y_0 + 1/2 k_1)$

$= 0.2 (1 + 1/2 (0.2) \times 2 + 1/2 (0.4))$

$k_2 = 0.484$

$$k_3 = hf\,(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2)$$

$$= \cancel{hf}\,(0.2)\,(x_0 + \tfrac{1}{2}h \times y + \tfrac{1}{2}k_2)$$

$$= (0.2)\,(1 + \tfrac{1}{2}(0.2) \times 2 + \tfrac{1}{2}(0.484))$$

$$= 0.2\,(1.1 \times 2.242)$$

$$= 0.2\,(2.4662)$$

$$= 0.49324$$

$$k_4 = hf\,(x_0 + h,\ y_0 + k_3)$$

$$= (0.2)\,(x + h \times y + k_3)$$

$$= (0.2)\,(1 + (0.2) \times 2 + 0.49324)$$

$$= (0.2)\,(1.2 \times 2.49324)$$

$$= 0.2\,(2.991888)$$

$$= 0.5983776$$

$$y_1 = y_0 + \tfrac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2 + \tfrac{1}{6}\,(0.4 + 2\,(0.484) + 2\,(0.49324) + 0.5983776)$$

$$= 2 + \tfrac{1}{6}\,(0.4 + 0.968 + 0.98648 + 0.5983776)$$

$$= 2.492$$

x	1	1.2
y	2	2.492

solve the following using R-K fourth order

$y' = y - x \quad y(0) = 2 \quad h = 0.2$ find $y(0.2)$

sol

$f(x, y) = y - x$

$y(0) = 2$

$x_0 = 0 \quad y_0 = 2 \quad h = 0.2$

x	0	0.2
y	2	?

fourth order R-K method

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = hf(x_0, y_0)$

$= 0.2\ (y - x)$

$= 0.2\ (2 - 0)$

$= 0.2\ (2)$

$k_1 = 0.4$

$k_2 = hf(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1)$

$= (0.2)\left(\cancel{0+\frac{1}{2}}\ 2 + \frac{1}{2}(0.4) - 0 + \frac{1}{2}(0.2)\right)$

$0.2\ (2.2 - 0.1)$

$k_2 = 0.42$

$k_3 = hf\ (x_0 + 1/2\, h,\ y_0 + 1/2\, k_2)$

$= (0.2)\ (2 + 1/2\ (0.42) - 0 + 1/2\ (0.2))$

$= (0.2)\ (2.21 - 0.1)$

$k_3 = 0.422$

$k_4 = hf\ (x_0 + h,\ y_0 + k_3)$

$= (0.2)\ (2 + (0.422) - 0 + 0.2)$

$= (0.2)\ (2.422 - 0.2)$

$k_4 = 0.4444$

$y_1 = y_0 + 1/6\ (k_1 + 2k_2 + 2k_3 + k_4)$

$= 2 + 1/6\ (0.4 + 2(0.42) + 2(0.422) + 0.4444)$

$= 2 + 1/6\ (0.4 + 0.84 + 0.844 + 0.4444)$

$= 2 + 1/6\ (2.5284)$

$= 2.4214$

x	0	1.2
y	2	2.4214

using R-K method of fourth order solve

$dy/dx = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$

given $y' = \frac{y^2 - x^2}{y^2 + x^2}$

$y(0) = 1$

$x_0 = 0 \quad y_0 = 1 \quad h = 0.2$

x	0	0.2
y	1	?

fourth order R-K method

$y_1 = y_0 + 1/6 (k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = hf(x_0, y_0)$

$= 0.2 \left(\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2} \right)$

$= 0.2 \left(\frac{1 - 0}{1 + 0} \right)$

$k_1 = 0.2 \left(1/1 \right) = 0.2$

$k_2 = hf\,(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1)$

$= (0.2)\left(0 + \frac{1}{2}(0.2),\ 1 + \frac{1}{2}(0.2)\right)$

$= 0.2\ (0.1,\ 1.1)$

$= 0.2\left(\dfrac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right)$

$= \cancel{0.2\,(0.833)}$

$k_2 = 0.19672$

$k_3 = hf\,(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2)$

$= (0.2)\left(0 + \frac{1}{2}(0.2),\ 1 + \frac{1}{2}(0.19672)\right)$

$= (0.2)\ (0.1,\ 1.09836)$

$= (0.2)\left(\dfrac{(1.09836)^2 - (0.1)^2}{(1.09836)^2 + (0.1)^2}\right)$

$= 0.19671$

$k_4 = hf\,(x_0 + h,\ y_0 + k_3)$

$= (0.2)\ (0 + 0.2,\ 1 + 0.19671)$

$= (0.2)\ (0.2,\ 1.19671)$

$= (0.2)\left(\dfrac{(1.19671)^2 - (0.2)^2}{(1.19671)^2 + (0.2)^2}\right)$

$= 0.189131$

27/1/20

Runge-Kutta method

Use R-K method 4th order to evaluate $y(0.1)$ & $y(0.2)$ given $y' = x+y$, $y(0) =$

Sol Given $y' = x+y$

$y(0)=1$ $\quad x_0 = 0 \quad y_0 = 1$

$h = 0.1$

x	0	0.1	0.2
y	1	?	?

fourth order

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where $k_1 = hf(x_0, y_0)$

$$k_1 = (0.1)(x_0 + y_0)$$

$$= (0.1)(0+1)$$

$$= (0.1)(1)$$

$$= 0.1$$

$$k_2 = hf\left[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right]$$

$$= 0.1\left[0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(0.1)\right]$$

$$= 0.1[0.05 + 1.05] = 0.11$$

$k_3 = hf\,(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2)$

$= (0.1)\,[0 + \tfrac{1}{2}(0.1),\ 1 + \tfrac{1}{2}\,(0.11)]$

$= 0.1\ [0.05 + 1.055]$

$k_3 = 0.1105$

$k_4 = hf\ (x_0 + h,\ y_0 + k_3)$

$= 0.1\ [0 + 0.1 + 1 + 0.1105]$

$= 0.12105$

$y_1 = y_0 + \tfrac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \tfrac{1}{6}\,[0.1 + 2\,(0.11) + 2\,(0.1105) + 0.12105]$

$y_1 = 1.11034$

TO find $y(0.2)$

$x_1 = 0.1 \qquad y_1 = 1.11034$

$k_1 = hf\ (x_1, y_1)$

$= 0.1\ (0.1 + 1.11034)$

$k_1 = 0.121034$

$k_2 = hf\ (x_1 + \tfrac{1}{2}h,\ y_1 + \tfrac{1}{2}k_1)$

$= (0.1)\,[0.1 + \tfrac{1}{2}(0.1) + 1.11034 + \tfrac{1}{2}\,(0.121034)]$

$= 0.1320857$

$k_3 = hf[x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2]$

$= (0.1)[0.1 + \frac{1}{2}(0.1) + 1.11034 + \frac{1}{2}(0.1320857)]$

$= 0.132638$

$k_4 = hf(x_1 + h, y_1 + k_3]$

$= 0.1[0.1 + 0.1 + 1.11034 + 0.132638]$

$k_4 = 0.1442978$

$y_2 = 1.11034 + \frac{1}{6}(0.121034 + 2(0.1320857)$

$+ 2(0.132638) + 0.1442978]$

$y_2 = 1.2428032$

x	0	0.1	0.2
y	1	1.11034	1.2428

Using R-K Method find $y(0.1), y(0.2), y(0.3)$

given that $dy/dx = 1 + xy$, $y(0) = 2$

Given $dy/dx = 1 + xy$

$x_0 = 0 \quad y_0 = 2 \quad h = 0.1$

x	0	0.1	0.2	0.3
y	2	?	?	?

$$y_1 = y_0 + 1/6\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 2.110357$$

4th order for $x = 0.1$ $\quad y_1 = 2.110357$

$$k_1 = hf\,(x_1, y_1)$$

$$= 0.1\,(1 + (0.1)(2.1103)]$$

$$k_1 = 0.121103$$

$$k_2 = hf\,[x_1 + 1/2 h,\ y_1 + 1/2 k_1]$$

$$= 0.1\,[0.1 + 1/2\,(0.1), 2.110357 + 1/2\,(0.121103)]$$

$$= 0.1\,[0.15, 2.1709085]$$

$$= 0.1\,[1 + (0.15)(2.1709085)]$$

$$k_2 = 0.132563$$

$$k_3 = hf\,(x_1 + 1/2 h,\ y_1 + 1/2 k_2)$$

$$= 0.1\,[0.1 + 1/2(0.1), 2.110357 + 1/2\,(0.132563)]$$

$$= 0.1\,[0.15, 2.17663]$$

$$= 0.1\,[1 + (0.15)(2.17663)]$$

$$= 0.132649$$

$$k_4 = hf\,(x_1 + h,\ y_1 + k_3)$$

$$= 0.1\,[0.1 + 0.1, 2.110357 + 0.132649]$$

$$= 0.1\,[0.2, 2.243006]$$

Scanned by CamScanner

Scanned with CamScanner

$= 0.1 [1 + (0.2)(2.243006)]$

$= 0.14486$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 2.110357 + \frac{1}{6}[0.121103 + 2(0.132563)$

$+ 2(0.132649) + 0.14486)]$

$y_2 = 2.24308.$

4th order for $x_2 = 0.2$ $y_2 = 2.24308$

$k_1 = hf(x_2, y_2)$

$= (0.2)(1 + (0.2)(2.24308)]$

$= 0.14486$

$k_2 = hf(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1)$

$= 0.1 [0.2 + \frac{1}{2}(0.1), 2.24308 +$

$\frac{1}{2}(0.14486)]$

$= 0.1 [1 + (0.25)(2.31551)]$

$= 0.1578$

$k_3 = hf(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2)$

$= 0.1 [0.2 + \frac{1}{2}(0.1), 2.24308 + \frac{1}{2}$

$(0.1578)]$

$= 0.1 [1 + (0.25)(2.32198)]$

$= 0.15804$

$k_4 = hf\ (x_2+h, y_2+k_3)$

$= hf\ (0.2+0.1, 2.24308 + 0.15804)$

$= 0.1\ [1+(0.3)(2.40112)]$

$= 0.17203$

$y_3 = y_2 + 1/6\ [k_1 + 2k_2 + 2k_3 + k_4]$

$= 2.401175$

x	0	0.1	0.2	0.3
y	2	2.1103	2.2430	2.4011

Taylor's Series Method

The formula is

$$y_{n+1} = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots$$

→ Using Taylor's Series method solve the Eqn $\frac{dy}{dx} = x^2+y^2$ for $x = 0.4$ given that $y=0$ when $x=0$

Sol:- Given $y' = x^2+y^2$ & $x_0 = 0$ $y_0 = 0$.

$h = 0.4$

x	0	0.4
y	0	?

$y' = x^2+y^2$ —(1) $\Rightarrow x_0^2 + y_0^2 = 0+0 = 0$.

$y' = 0$.

diff (1) w.r.to. 'x.'

$$y'' = 2x + 2y\frac{dy}{dx} = 2x+2yy' \quad \text{—(2)}$$

$$= 2x_0 + 2y_0y_0' \Rightarrow 2(0)+2(0)(0)$$

$$y'' = 0$$

diff (2) w.r.to. 'x.

$$y''' = 2+2[yy''+y'y']$$

$$= 2+2yy''+2(y')^2 \quad \text{—— (3)}$$

$$= 2+2y_0y_0''+2(y_0')^2$$

$$= 2+2(0)(0)+2(0)^2 = 2.$$

$$y''' = 2.$$

diff (3) w.r.to. 'x'

$$y^{iv} = 0+2[yy'''+y''y']+4y'.y''$$

$$= 2y_0y_0'''+2y_0''y_0'+4y_0'y_0'' \Rightarrow 2y_0y_0'''+6y_0'y_0''$$

$$\Rightarrow 2(0)(2)+6(0)(0) = 0.$$

Now Sub. y', y'', y''', y^{iv} in ~~Equ (1)~~ Taylor's series formula

$$y_{n+1} = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots \quad n = 0, 1, 2, \ldots$$

Put $n = 0$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots \quad h = 0.4$$

$$y_1 = 0 + \frac{0.4}{1}(0) + \frac{(0.4)^2}{2!}(0) + \frac{(0.4)^3}{3!}(2) + \frac{(0.4)^4}{4!}(0) + \cdots$$

$$y_1 = 0.0213.$$

→ Solve $y' = x - y^2$, $y(0) = 1$ using taylor's series method and compute $y(0.1)$ & $y(0.2)$

Sol:- Given $y' = x - y^2$, and $x_0 = 0$ $y_0 = 1$.

x	0	0.1	0.2
y	1	?	?

$\therefore h = 0.1$.

$$y' = x - y^2 \quad \text{—— (1)}$$
$$y' = x_0 - y_0^2 = 0 - 1 = -1.$$

diff (1) w.r. to. x.

$$y'' = 1 - 2yy' \quad \text{—— (2)}$$
$$= 1 - 2y_0 y_0' \Rightarrow 1 - 2(1)(-1) = 3.$$

diff (2) w.r.to. x.

$$y''' = 0 - 2[yy'' + y' \cdot y']$$
$$= -2yy'' - 2(y')^2 \quad \text{—— (3)}$$
$$= -2y_0 y_0'' - 2(y_0')^2 \Rightarrow -2(1)(3) - 2(-1)^2$$
$$= -6 - 2 = -8.$$

$$y''' = -8$$

diff (3) w.r.to. x.

$y^{iv} = -2(y y''' + y'' y') - 4 y' . y''$

$= -2 y_0 y_0''' - 2 y_0'' y_0' - 4 y_0' y_0''$ (or) $-2 y_0 y_0''' - 6 y_0' y_0''$

$= -2(1)(-8) - 2(3)(-1) - 4(-1)(3)$

$= 34$

By Taylor's Series

$y_{n+1} = y_1 = y(0.1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$ (*)

$y(0.1) = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(-8) + \frac{(0.1)^4}{4!}(34) + \cdots$

$= 0.91381.$

To find next value i-e $x = 0.2$

just take $2 \times h \Rightarrow 2(0.1) = 0.2.$

$h = 0.2.$

Now sub $h = 0.2$ in (*) no need of applying formule again

$y(0.2) = 1 + \frac{0.2}{1!}(-1) + \frac{(0.2)^2}{2!}(3) + \frac{(0.2)^3}{3!}(-8) + \frac{(0.2)^4}{4!}(34) + \cdots$

$= 0.8512.$

$$y_2 = 1 + \frac{0.2}{1!}(-1) + \frac{(0.2)^2}{2!}(3) + \frac{(0.2)^3}{3!}(-8) + \frac{(0.2)^4}{4!}(34) + \ldots$$

$$y_2 = 0.8516$$

x	0	0.1	0.2
y	1	0.9138	0.8516

3) calculate $y(0.1)$ $y(0.2)$ $y(0.3)$ using Taylor's method $y' = y^2 + x$ & $y(0) = 1$

sol Given $y' = y^2 + x$ $y(0) = 1$

$x_0 = 0$ $y_0 = 1$ $h = 0.1$

x	0	0.1	0.2	0.3
y	1	?	?	?

$y' = y^2 + x$

$= (1)^2 + 0$

$y' = 1$

$y'' = 2y\,y' + (1)$

$= 2(1)(1) + 1$

$y'' = 2 + 1 = 3$

$(uv)' = uv' + vu'$

$y''' = 2[yy'' + y'y'] + 0$

$= 2[yy'' + (y')^2]$ $\rightarrow 2(1)(1) + 2(1)^2$

$= 2[1(3) + (1)^2]$ $\Rightarrow 2 + 2$

$\Rightarrow 4$

$= 2[3+1]$

$\Rightarrow 2[4]$

$y''' = 8$

$y^{iv} = 2[yy''' + y''y'] + 2[2y'y'']$

$2[1(8) + 3(1)] + 4(1)(3) \Rightarrow$

$= 2yy''' + 2y''y' + 4y'y''$ $\quad 2[8+3]$

$2[11]+12$

$= 2(1)(8) + 2(3)(1) + 4(1)(3)$ $\quad 22+12$

$= 34$

$\Rightarrow 34$

$y^{iv} = 34$

By Taylor's serious

$$y_{n+1} = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0 + \frac{h^4}{4!} y_0^{iv} +$$

Here $h = 0.1$

$$y_1 = 1 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(34)$$

$= 1.1164$

$h \times 2 = 0.1 \times 2$

$= 0.2$

$$y_2 = 1 + \frac{(0.2)}{1!}(1) + \frac{(0.2)^2}{2!}(3) + \frac{(0.2)^3}{3!}(8) + \frac{(0.2)^4}{4!}(34)$$

$= 1.2729$

$y_3 = h \times 3 = 0.1 \times 3$

$= 0.3$

$$y_3 = 1 + \frac{(0.3)}{1!}(1) + \frac{(0.3)^2}{2!}(3) + \frac{(0.3)^3}{3!}(8) + \frac{(0.3)^4}{4!}(34)$$

$y_3 \Rightarrow 1.4824$

x	0	0.1	0.2	0.3
y	1	1.1164	1.2729	1.4824

Using Taylor's series method find an approximate value of y at $x = 0.1$ & $x = 0.2$ for the differential eq^n $y' = y^2 + x$ & $y(0) = 1$

$y' = 2y + 3e^x$ at $y(0) = 0$

Sol Given $y' = 2y + 3e^x$

$y(0) = 0$

$x_0 = 0 \quad y_0 = 0 \quad h = 0.1$

x	0	0.1	0.2
y	0	?	?

$y' = 2y + 3e^x$

$= 2(0) + 3e^x$

$= 2(0) + 3e^0$

$y' = 3$

$y'' = 2y' + 3e^x$

$= 2(3) + 3e^0$

$= 6 + 3e^0$

$y'' = 9$

$y''' = 2y'' + 3e^x$

$= 2(9) + 3e^0$

$= 18 + 3e^0$

$y''' = 21$

$$y^{iv} = 2y''' + 3e^{x}$$

$$= 2y''' + 3e^{0}$$

$$= 2(21) + 3e^{0}$$

$$y^{iv} = 45$$

By Taylors method

$$y_{n+1} = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots$$

$$y_1 = 0 + \frac{(0.1)}{1!}(3) + \frac{(0.1)^2}{2!}(9) + \frac{(0.1)^3}{3!}(21) + \frac{(0.1)^4}{4!}(45)$$

$$y_1 = 0.3486$$

$$h \times 2 = 0.1 \times 2$$
$$= 0.2$$

$$y_2 = 0 + \frac{(0.2)}{1!}(3) + \frac{(0.2)^2}{2!}(9) + \frac{(0.2)^3}{3!}(21) + \frac{(0.2)^4}{4!}(45)$$

$$y_2 = 0.811$$

x	0	0.1	0.2
y	0	0.3486	0.811

find $y(0.1)$ given that $\frac{dy}{dx} = e^x - y^2$ at

$y(0)=1$

Sol Given $y' = e^x - y^2$

$y(0) = 1$

$x_0 = 0 \quad y_0 = 1$

x	0	0.1
y	1	?

$h = 0.1$

$y' = e^x - y^2$

$= e^0 - (1)^2$

$= e^0 - 1$

$y' = 0$

$y'' = e^x - 2y\,y'$ (u, v)

$= e^0 - 2(1)(0)$

$= e^0 - 2(1)(0)$

$= e^0 - 2(1)(0)$

$y'' = 1$

$d(uv)' = uv' + vu'$

$y''' = e^x - 2[yy'' + y'(y')]$

$= e^x - 2[yy'' + (y')^2]$

$= e^0 - 2[(1)(1) + 0]$

$= e^0 - 2[1] \quad = e^0 - 2 = -1$

Scanned by CamScanner

Scanned with CamScanner

$y^{iv} = e^x - 2[yy''' + y''(1) + y'']$

$= e^x - 2[(1)(-1) + (1) + (1)]$

$= e^x - 2[-1 + 1 + 1]$

$e^x - 2$

$= e^0 - 2$

$y''' = e^x - 2[yy'' + y'y']$

$= e^x - 2[yy'' + (y')^2]$

$= e^x - 2yy'' - 2(y')^2$

$\cancel{= e^x - 2yy'' - 2(2y'y'')}$

$\cancel{= e^x - 2yy'' - 4y'y''}$

$= e^0 - 2(1)(1) - 2(0)^2$

$= -1$

$y^{iv} = e^x - 2[yy''' + y''y'] - 2(2y'y'')$

$= e^0 - 2[(1)(-1) + (1)(0)] - 4(0)(1)$

$= e^0 - 2[-1]$

$= e^0 + 2$

$= 3$

By Taylor's Methods

$$y_{n+1} = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \dots$$

$$y_1 = 1 + \frac{(0.1)}{1!}(0) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(-1) + \frac{(0.1)^4}{4!}(3)$$

$$= 1.00484$$

27/1/20

Use Taylor series method to find the value of y at $x = 0.1$ given $y(0) = 1$ & $y' = 3x + y^2$

Sol given $y' = 3x + y^2$

$y(0) = 1$

$x_0 = 0$ $\quad y_0 = 1$ $\quad h = 0.1$

x	0	0.1
y	1	?

$y' = 3x + y^2$

$= 3(0) + (1)^2$

$y' = 1$

$y'' = 3(1) + 2yy'$

Scanned by CamScanner

$= 3 + 2(1)(1)$

$= 3 + 2$

$= 5$

$y''' = 0 + 2[y\,y'' + y'y']$ d(uv) = uv' + vu'

$= 2[(1)(5) + (1)(1)]$

$= 2[5 + 1]$

$= 2[6]$

$y''' = 12$

$y^{iv} = 0 + 2[y\,y'' + (y')^2]$

$= 2(y\,y'') + 2(y')^2$

$= 2[y\,y''' + y''y'] + 2(2y'y'')$

$= 2[(1)(12) + 5(1)] + 2(2(1)(5))$

$= 2[12 + 5] + 2(10)$

$= 2[17] + 20$

$= 34 + 20$

$y^{iv} = 54$

By Taylor's method

→

Picard's Method of Successive approximations :-

formula :- $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \qquad n=1,2,3,\ldots$

1) Given that $\frac{dy}{dx} = 1+xy$ and $y(0)=1$ compute $y(0.1)$ and $y(0.2)$ using picard's method.

Sol:- Given $\frac{dy}{dx} = 1+xy$ and $y(0)=1$.

i.e. $x_0=0 \quad y_0=1$.

By Picard's method we have

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \qquad n=1,2,3,\ldots$$

$$y_n = 1 + \int_0^x (1+xy_{n-1})\,dx. \qquad \text{———} (1)$$

Sub $n=1$ in (1)

$$y_1 = 1 + \int_0^x (1+xy_0)\,dx.$$

$$= 1 + \int_0^x (1+x)\,dx = 1 + \int_0^x dx + \int_0^x x\,dx.$$

$$= 1 + (x)_0^x + \left(\frac{x^2}{2}\right)_0^x = 1+x+\frac{x^2}{2}$$

Sub $n=2$ in (1).

$$y_2 = 1 + \int_0^x (1+xy_1)\,dx.$$

$$= 1 + \int_0^x \left[1 + x\left(1+x+\frac{x^2}{2}\right)\right]dx.$$

$$= 1 + \int_0^x \left(1+x+x^2+\frac{x^3}{2}\right)dx.$$

$$= 1 + (x)_0^x + \left(\frac{x^2}{2}\right)_0^x + \left(\frac{x^3}{3}\right)_0^x + \left(\frac{x^4}{8}\right)_0^x = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}$$

Sub $n=3$ in Eu (1)

$$y_3 = 1 + \int_0^x (y_2 - x^2)\, dx.$$

$$= 1 + \int_0^x \left(1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12} - x^2\right) dx.$$

$$= 1 + \int_0^x \left(1 + x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12}\right) dx \qquad \left[\because \frac{x^2}{2} - x^2 = -\frac{x^2}{2}\right]$$

$$= 1 + (x)_0^x + \left(\frac{x^2}{2}\right)_0^x - \left(\frac{x^3}{6}\right)_0^x - \left(\frac{x^4}{12}\right)_0^x - \left(\frac{x^5}{60}\right)_0^x.$$

$$y_2 = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{60}.$$

Here 2nd & 3rd approximation '3' terms are equal

so, stop the procedure.

At $x = 0.1$ & $x = 0.2$.

$$y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{6} - \frac{(0.1)^4}{12} - \frac{(0.1)^5}{60}$$

$$= 1.1048$$

$$y(0.2) = 1 + (0.2) + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{6} - \frac{(0.2)^4}{12} - \frac{(0.2)^5}{60}$$

$$= 1.21853.$$

→ find the value of 'y' at $x = 0.1$ by Picard's method

Given that $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0) = 1$.

Sol:- Here $f(x,y) = \frac{y-x}{y+x}$ & $x_0 = 0$ $y_0 = 1$.

By Picard's method $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx.$

Sub $n=3$ in Equ (1)

$$y_3 = 1+\int_0^x (1+xy_2)\,dx.$$

$$= 1+\int_0^x 1+x\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}\right)\Big]\,dx.$$

$$= 1+\int_0^x \left(1+x+x^2+\frac{x^3}{2}+\frac{x^4}{3}+\frac{x^5}{8}\right)dx.$$

$$= 1+(x)_0^x + \left(\frac{x^2}{2}\right)_0^x + \left(\frac{x^3}{3}\right)_0^x + \left(\frac{x^4}{8}\right) + \left(\frac{x^5}{15}\right)_0^x + \left(\frac{x^6}{48}\right)_0^x.$$

$$= \underbrace{1+x+\frac{x^2}{2}+\frac{x^3}{3}}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}$$

In 2^{nd} & 3^{rd} Approximations the first '4' term's are Equal so, stop the process.

Now sub $x=0.1$ & $x=0.2$ in 3^{rd} approximation.

i.e
$$y_3 = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}$$

i.e $$y(0.1) = 1+0.1+\frac{(0.1)^2}{2}+\frac{(0.1)^3}{3}+\frac{(0.1)^4}{8}+\frac{(0.1)^5}{15}+\frac{(0.1)^6}{48}$$

$$= 1.10534$$

$$y(0.2) = 1+0.2+\frac{(0.2)^2}{2}+\frac{(0.2)^3}{3}+\frac{(0.2)^4}{8}+\frac{(0.2)^5}{15}+\frac{(0.2)^6}{48}$$

$$= 1.222868$$

→ Solve $y' = y - x^2$ $y(0) = 1$ Hence find $y(0.1)$ & $y(0.2)$

Sol:- Given $y' = y - x^2$ & $y(0) = 1$.

i.e $x_0 = 0$ $y_0 = 1$.

By Picard's method.

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx.$$

$$y_n = 1 + \int_{0}^{x} f(x, y_{n-1})\, dx.$$

$$y_n = 1 + \int_{0}^{x} (y_{n-1} - x^2)\, dx. \quad \text{———— (1)}$$

Sub $n=1$ in Eu (1)

$$y_1 = 1 + \int_0^x (y_0 - x^2)\, dx.$$

$$= 1 + \left[\int_0^x (1 - x^2)\right] dx \Rightarrow 1 + \int_0^x dx - \int_0^x x^2 dx.$$

$$= 1 + (x)_0^x - \left(\frac{x^3}{3}\right)_0^x \Rightarrow 1 + x - \frac{x^3}{3}$$

Sub $n=2$ in Eu (1)

$$y_2 = 1 + \int_0^x (y_1 - x^2)\, dx.$$

$$= 1 + \int_0^x \left(1 + x - \frac{x^3}{3} - x^2\right) dx$$

$$= 1 + \int_0^x dx + \int_0^x x\, dx - \int_0^x \frac{x^3}{3} dx - \int_0^x x^2 dx.$$

$$= 1 + (x)_0^x + \left(\frac{x^2}{2}\right)_0^x - \left(\frac{x^4}{12}\right)_0^x - \left(\frac{x^3}{3}\right)_0^x.$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3}.$$

$$y_2 = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12}$$

$$y_n = y_0 + \int_0^x \frac{y_0 - x}{y_0 + x}\, dx. \quad \text{———} (1)$$

for $n=1$. in Eqn (1)

$$y_1 = 1 + \int_0^x \frac{1-x}{1+x}\, dx.$$

$$y_1 = 1 + \int_0^x \left(-1 + \frac{2}{1+x}\right) dx$$

$\left[\because \frac{1-x}{1+x} \text{ can be writ} \right.$
$\left. -1 + \frac{2}{1+x} \right]$

$$= 1 + \left[-x + 2\log(1+x)\right]_0^x.$$

$$= 1 + [-x + 2\log(1+x)] - [0 + 2\log(1+0)]$$

$$= 1 - x + 2\log(1+x). \qquad [\because \log(1) = 0]$$

for $n=2$ in Eqn (1)

$$y_2 = 1 + \int_0^x \frac{1 - x + 2\log(1+x) - x.}{1 - x + 2\log(1+x) + x}\, dx.$$

$$= 1 + \int_0^x \frac{1 - 2x + 2\log(1+x)}{1 + 2\log(1+x)}\, dx.$$

which is very difficult to integrate.

Here we use the first approximation itself as the va

$$\therefore y(x) = y_1 = 1 - x + 2\log(1+x)$$

Putting $x = 0.1$, we obtain.

$$y(0.1) = 1 - 0.1 + 2\log(1+0.1)$$

$$= 1 - 0.1 + 0.1906203$$

$$\therefore y(0.1) = 1.0906$$

2) Given that $dy/dx = 1+xy$ and $y(0)=1$

Compute $y(0.1)$ & $y(0.2)$ using piccard's method.

Sol Given $dy/dx = 1+xy$

$y(0)=1$

$x_0=0 \quad y_0=1$

By picard's method

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \qquad n=1,2,3 \ldots \text{(1)}$$

Substitute $n=1$ in eqn (1)

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

$$y_1 = 1 + \int_0^x f(x,1)\,dx$$

$$= 1 + \int_0^x (1+x)\,dx$$

$$= 1 + \int_0^x (1)\,dx + \int_0^x x\,dx$$

$$= 1 + x + \frac{x^2}{2}$$

Substitute $n=2$ In Eq n ①

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1 + \int_0^x f\left(x, 1+x+\frac{x^2}{2}\right) dx$$

$$= 1 + \int_0^x \left(x + 1 + x + x^2/2\right) dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2/2\right) dx$$

$$= 1 + \int_0^x (1)\,dx + \int_0^x 2x\,dx + \frac{1}{2}\int_0^x x^2\,dx$$

$$= 1 + (x)_0^x + 2\left(x^2/2\right)_0^x + \frac{1}{2}$$

$n=2$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1 + \int_0^x f\left(x, 1+x+x^2/2\right) dx$$

$$= 1 + \int_0^x \left(1 + x\left(1+x+x^2/2\right)\right) dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + x^3/2\right) dx$$

$$= 1 + \int_0^x 1\,dx + \int_0^x x\,dx + \int_0^x x^2\,dx + \frac{1}{2}\int_0^x x^3\,dx$$

$$= 1 + (x)_0^x + \left(x^2/2\right)_0^x + \left(\frac{x^3}{3}\right)_0^x + \left(\frac{x^4}{4}\right)_0^x \cdot \frac{1}{2}$$

$1+(x-0)+\frac{1}{2}(x^2-0)+\frac{1}{3}(x^3-0)+\frac{1}{8}(x^4-0)$

$y_2 = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8} \longrightarrow (2)$

$n=3$

$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$

$= 1 + \int_0^x (x, 1+x+x^2/2 + x^3/3 + \frac{x^4}{8})\,dx$

$= 1 + \int_0^x (1 + x(1+x+x^2/2 + x^3/3 + x^4/8$

Atleast 3 terms are Equals in y_1 & y_2

Hence stop the procedure

Now find y values at $x=0.1$, $x=0.2$

by substituting x values in the last

approximation i.e in Eq^n (2)

$y_2 = 1+x+x^2/2 + x^3/3 + x^4/8$

$y(0.1) = 1+(0.1)+\frac{(0.1)^2}{2}+\frac{(0.1)^3}{3}+\frac{(0.1)^4}{8}$

$y_2(0.1) = 1.1053$

$$y(0.2) = 1 + (0.2) + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{8}$$

$$y(0.2) = 1.2228$$

3) Solve

3) Solve $y' = y - x^2$ $y(0) = 1$ Hence find $y(0.1)$ & $y(0.2)$

Sol Given $y' = y - x^2$

$$y(0) = 1$$

$$x_0 = 0 \qquad y_0 = 1$$

By picard's method

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$$

$n = 1$

$$y_1 = y_0 + \int_0^x f(x, y)\,dx$$

$$= y_0 + \int_0^x (x, 1)\,dx$$

$$= y_0 + \int_0^x (1 - x^2)\,dx$$

$$= 1 + \int_0^x (1 - x^2)\,dx$$

$$= 1 + \int_0^x 1\,dx - \int_0^x x^2\,dx$$

$$1+(x)_0^x-\left(\frac{x^3}{3}\right)_0^x$$

$$1+(x-0)-\left(\frac{x^3}{3}-\frac{0}{3}\right)$$

$$y_1 = 1+x-\frac{x^3}{3}$$

$n=2$

$$y_2 = y_0+\int_{x_0}^{x} f(x,y_1)\,dx$$

$$= 1+\int_0^x f(x, 1+x-x^3/3)\,dx$$

$$= 1+\int_0^x (1+x-x^3/3 - x^2)\,dx$$

$$= 1+\int_0^x 1\,dx+\int_0^x x\,dx + \int_0^x \frac{x^3}{3}\,dx-\int_0^x x^2\,dx$$

$$= 1+(x)_0^x+(x^2/2)_0^x-\frac{1}{3}(x^4/4)_0^x - (x^3/3)_0^x$$

$$= 1+(x-0)+\frac{1}{2}(x^2-0)-\frac{1}{12}(x^4-0) - \frac{1}{3}(x^3-0)$$

$$= 1+x+x^2/2-\frac{x^4}{12}+\frac{x^3}{3}.$$

Scanned by CamScanner

Scanned with CamScanner

$n=3$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= 1 + \int_0^x f(x, 1 + x + x^2/2 - x^4/12 - x^3/3)\,dx$$

$$= 1 + \int_0^x (1 + x + x^2/2 - x^4/12 - x^3/3 - x^2)\,dx$$

$$= 1 + \int_0^x 1\,dx + \int_0^x x\,dx + \int_0^x \frac{x^2}{2}\,dx - \int_0^x \frac{x^4}{12}\,dx$$

$$- \int_0^x \frac{x^3}{3}\,dx - \int_0^x x^2\,dx$$

$$= 1 + (x)_0^x + \left(\frac{x^2}{2}\right)_0^x + \frac{1}{2}\left(\frac{x^3}{3}\right)_0^x - \frac{1}{12}\left(\frac{x^5}{5}\right)_0^x$$

$$- \frac{1}{3}\left(\frac{x^4}{4}\right)_0^x - \left(\frac{x^3}{3}\right)_0^x$$

$$= 1 + (x) + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{60} - \frac{x^4}{12} - \frac{x^3}{3}$$

Atleast 3 terms are Equal in y_1 & y_2

Hence stop the procedure

Now find y values at $x=0.1$, $x=0.2$

by substituting x values in the last

$$y(0\cdot1) = 1 + (0\cdot1) + \frac{(0\cdot1)^2}{2} + \frac{(0\cdot1)^3}{6} + \frac{(0\cdot1)^5}{60} - \frac{(0\cdot1)^4}{12} - \frac{(0\cdot1)^3}{3}$$

$$y(0\cdot1) = 1\cdot1048$$

$$y(0\cdot2) = 1 + (0\cdot2) + \frac{(0\cdot2)^2}{2} + \frac{(0\cdot2)^3}{6} + \frac{(0\cdot2)^5}{60} - \frac{(0\cdot2)^4}{12} - \frac{(0\cdot2)^3}{3}$$

$$= 1\cdot21853$$