Unit-4

Functions Of a Complex Variable:-

→ If x and y are real numbers then a number of form $x+iy$ is called complex number, denoted by 'z'.

$x \to$ real part of z, $Re(z)$; $y \to$ imaginary part of z, $img(z)$

* If $y=0$, then $z=x$ is purely real number.
* If $x=0$, then $z=iy$ is purely imaginary number.
* If $z=x+iy$ the $\bar{z}=x-iy$ is called conjugate of z.
* If $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, then

① $z_1=z_2 \Rightarrow x_1=x_2$ & $y_1=y_2$ ⇒ equality of complex no's.

② $z_1 \pm z_2 \Rightarrow (x_1 \pm x_2)+i(y_1 \pm y_2)$ ⇒ sum or diff of complex no.'s.

③ $z_1 \times z_2 \Rightarrow (x_1+iy_1)(x_2+iy_2) = x_1x_2+i(y_1x_2+x_1y_2)-y_1y_2$

④ $$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2} = \frac{x_1x_2+y_1y_2+i(y_1x_2-y_2x_1)}{x_2^2+y_2^2}$$

Polar form Of Complex Variable:-

→ If x is $r\cos\theta$, $y=r\sin\theta$, $z=x+iy$ can be written as $z=r\cos\theta+i\sin\theta = r(\cos\theta+i\sin\theta)$

$z=re^{i\theta}$ → polar form.

→ Here, $r^2=x^2+y^2 \Rightarrow r=|z|=\sqrt{x^2+y^2}$ → mod of complex no.

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ or $\cot^{-1}\left(\frac{x}{y}\right)$ → Arc (z) Amplitude of z.

Complex Function (or) function of Complex Variable:-

'D' is a set of complex numbers, a rule is defined on 'D' which assigns to every z in D.

→ A complex no. w is called a function f, we write $w = f(z)$. i.e., the image of z under the function & we write $w = f(z) = u + iv$

where $u = u(x,y)$, $v = v(x,y)$.

Limit of a complex number:-

→ A function $w = f(z)$ is said to tends to limit L.

→ As z approaches a point z_0 if for every real number, we can find a +ve δ such that

$$|f(z) - L| < \epsilon \text{ for } 0 < |z - z_0| < \delta.$$

It is written as $\underset{z \to z_0}{Lt} f(z) = L$.

(1) Using the def. of limit, find $\underset{z \to 1}{Lt} \frac{z^2 - 1}{z - 1}$

→ The function is not defined at $z = 1$.

$$\underset{z \to 1}{Lt} \frac{z^2 - 1}{z - 1} = \underset{z \to 1}{Lt} \frac{(z+1)(z-1)}{(z-1)} = \underset{z \to 1}{Lt} (z+1) = 1 + 1 = 2.$$

Continuous (or) continuity:-

→ A function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ is defined and $\underset{z \to z_0}{Lt} f(z) = f(z_0)$, then

• we say that 'f' is continuous at z_0.

→ The function must be defined in some neighbourhood of z_0.

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→ show that the function $f(z) = \frac{\bar{z}}{z}$ is not continuous at $z=0$.

Let $z = x+iy$, then $\bar{z} = x-iy$

at $z=0$, $\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} \frac{x+iy}{x-iy}$

$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy}$

$= \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$

$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{\substack{y\to 0\\ x\to 0}} \frac{x+iy}{x-iy} = \lim_{y\to 0} \frac{-iy}{iy}$

$= \lim_{y\to 0} (-1) = -1.$

$\therefore \lim_{\substack{x\to 0\\ y\to 0}} \frac{\bar{z}}{z} \neq \lim_{\substack{y\to 0\\ x\to 0}} \frac{\bar{z}}{z}$

$\therefore$ at $z=0$, $\frac{\bar{z}}{z}$ is not continuous.

② $f(z) = \begin{cases} \frac{z^2+3iz-2}{z+i} & \text{for } z \neq -i \\ i & \text{for } z = -i \end{cases}$ is the function $f(z)$ is continuous

if not, ~~cannot~~ the function be refined to make ~~it continuous~~

continuity at $z = -i$

$\lim_{z\to -\infty} \frac{z^2+3iz-2}{z+i} = \lim_{\substack{x\to 0\\ y\to -1}} \frac{(x+iy)^2+3i(x+iy)-2}{(x+iy)+i}$

$\Rightarrow \lim_{x\to 0} \frac{(x-i)^2+3i(x-i)-2}{(x-\cancel{i}+\cancel{i})}$

$\lim_{x\to 0} \frac{x^2+(-1)-2ix+3ix+3-2}{x} = \lim_{x\to 0} \frac{x^2+5ix}{x}$

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$$\lim_{x \to 0} (x+i) = i$$

→ $f(z) = \frac{g(z)}{h(z)}$ is a continuous when $G(z)$ and $h(z)$ are continuous except at $h(z)=0$. So, $f(z)$ is continuous at every where except at $z=-i$ since $g(z)$ $ $h(z)$ are continuous.

→ $RHL = \lim_{\substack{x \to 0 \\ y \to -1}} f(z) = i$

IIly LHL.

$$\lim_{z \to -i} \frac{(z^2+3iz-2)}{z+i} = \lim_{\substack{y \to -1 \\ x \to 0}} \frac{(x+iy)^2+3i(x+iy)-2}{x+iy+i}$$

$$\Rightarrow \lim_{y \to -1} \frac{i^2y^2+3i(iy)-2}{iy+i}$$

$$\Rightarrow \lim_{y \to -1} \frac{-y^2+3i^2y-2}{i(y+1)} \Rightarrow \lim_{y \to -1} \frac{-y^2-3y-2}{i(y+1)}$$

$$\Rightarrow \underset{y \to -1}{Lt} \frac{-(y^2+3y+2)}{i(y+1)}$$

$$\underset{y \to -1}{Lt} \frac{-(y+1)(y+2)}{i(y+1)}$$

$$\underset{y \to -1}{Lt} \frac{-(y+2)}{i} = -\frac{(-1+2)}{i} = -\frac{1}{i} = i$$

LHL = RHL.

but $f(-i) = 5$

$\therefore$ The function is not continuous.

Derivative of f(z):-

$\rightarrow$ Let $w = f(z)$ defined for all z in the neighbourhood of z_0. If $\underset{\Delta z \to 0}{Lt} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists, then the function $f(z)$ is said to be derivable at z_0 and its is denoted by $f'(z_0)$.

① Find the derivative of $w = f(z) = z^3 - 2z$ at the point $z = z_0$.

at $z = z_0$

$$f'(z_0) = \underset{\Delta z \to 0}{Lt} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$f(z) = z^3 - 2z$, $f(z_0) = z_0^3 - 2z_0$

$$f(z_0+\Delta z) = (z_0+\Delta z)^3 - 2(z_0+\Delta z)$$

$$= z_0^3 + \Delta z^3 + 3z_0^2 \Delta z + 3z_0 \Delta z^2 - 2z_0 - 2\Delta z$$

$$f'(z_0) = \underset{\Delta z \to 0}{Lt} \frac{\cancel{z_0^3} + \Delta z^3 + 3z_0^2 \Delta z + 3z_0 \Delta z^2 - \cancel{2z_0} - 2\Delta z - \cancel{z_0^3} + \cancel{2z_0}}{\Delta z}$$

$$\underset{\Delta z \to 0}{Lt} \frac{\cancel{\Delta z}(\Delta z^2 + 3z_0^2 + 3z_0 \Delta z - 2)}{\cancel{\Delta z}}$$

$$= (0) + 3z_0^2 + 0 - 2$$

$$f'(z_0) = 3z_0^2 - 2.$$

$\therefore$ at $z = z_0$ the function is derivable.

Note:- * Every differential function is continuous but the converse of the above statement is not true i.e., the function can be continuous at a point but not differentiable at that point.

Analytica Function:-

→ A function $f(z)$ is derivable at every point z in an ϵ neighbourhood of z_0 i.e., $f'(z)$ exists for all z such that $|z - z_0| < \epsilon$, then $f(z)$ is said to be analytic at z_0.

~~→ Cauchy's remanipulations:-~~

→ Cauchy's - Reimann:- iteration Equation:-
(C-R) equations

Suppose $w = u(x,y) + iv(x,y)$ is defined at a point 'z'. The 1st order partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\partial v/\partial x$, $\partial v/\partial y$ exists and continuous and also satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \$ \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$, then the above equations are called cauchy's - Reimann equation.

C-R equations at origin:-

$$\left(\frac{\partial u}{\partial x}\right)_{at\ (0,0)} = \underset{x\to 0}{Lt} \frac{u(x,0)-u(0,0)}{x}$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{Lt} \frac{u(0,y)-u(0,0)}{y}$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \underset{x\to 0}{Lt} \frac{v(x,0)-v(0,0)}{x}$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{Lt} \frac{v(0,y)-v(0,0)}{y}$$

Properties of Analytic function:-

- If $f(z)$ and $g(z)$ are analytic functions, then $f(z)\pm g(z)$, $f(z)\times g(z)$ and $\frac{f(z)}{g(z)}$ are also analytic functions provided $g(z)\neq 0$.

* Analytic function of an analytic function is also analytic.

* Derivative of an analytic function is itself analytic.

Laplacian equation:-

The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called laplacian operator. equation.

→ If $u(x,y)$ is a function of x,y, then

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

→ If $v(x,y)$, then $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$

→ If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain 'D' then u & v satisfies laplacian eqⁿ i.e., $\nabla^2 u = 0$ & $\nabla^2 v = 0$. and have continuous 2nd order partial derivatives in a domain 'D' is called harmonic function.

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Conjugate Harmonic Function:-

→ If two harmonic functions u and v satisfies the C-R equations in a domain 'D' and they are real and imaginary parts of $w = f(z)$, then v is said to be conjugate harmonic of u.

Polar Form of Cauchy's - Reiman Equation:-

→ If $w = f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ and $f(z)$ is differentiable at z_0, then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$; $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$, these equations are called cauchy-Riemann equations in polar form.

① $f(z) = xy + iy$. show that $f(z)$ is continuous but it is not analytic.

Continuity at z_0:- $f(z_0) = x_0 y_0 + iy_0$

$$\underset{z \to z_0}{Lt} f(z_0) = \underset{z \to z_0}{Lt} (x_0 y_0 + iy_0) = \underset{\substack{x \to x_0 \\ y \to y_0}}{Lt} (x_0 y_0 + iy_0)$$

RHL:-

$$\underset{\substack{x \to x_0 \\ y \to y_0}}{Lt} f(z_0) = \underset{x \to x_0}{Lt} (x_0 y_0 + iy_0) = x_0 y_0 + iy_0 = f(z_0)$$

LHL:- $$\underset{(y,x) \to (y_0, x_0)}{Lt} f(z_0) = \underset{\substack{y \to y_0 \\ x \to x_0}}{Lt} (x_0 y_0 + iy_0) = \underset{y \to y_0}{Lt} (x_0 y_0 + iy_0)$$

$= x_0 y_0 + i y_0 = f(z_0)$

$\therefore \lim_{(x,y)\to(x_0,y_0)} f(z_0) = \lim_{(y,x)\to(y_0,x_0)} f(z_0) = f(z_0).$

$\therefore f(z)$ is continuous function.

Analytic condⁿ :- $f(z) = xy + iy$

$u(x,y) + i\,v(x,y) = xy + iy.$

$\Rightarrow u(x,y) = xy \;, \quad v(x,y) = y.$

$\frac{\partial u}{\partial x} = y. \qquad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = x \qquad \frac{\partial v}{\partial y} = 1$

by C-R eqⁿs :- $u_x \neq v_y$ and $u_y \neq -v_x$

$\therefore f(z)$ is not analytic function.

$\therefore f(z)$ is not analytic $\forall$ values of z.

② $f(z) = z^3$. show that $f(z)$ is analytic

Given $w = f(z) = z^3 = (x+iy)^3 = x^3 + i^3 y^3 + 3x^2 iy + 3x i^2 y^2$

$f(z) = x^3 - iy^3 + 3x^2 iy - 3xy^2 = (x^3 - 3xy^2) + i(3x^2 y - y^3)$

$u(x,y) + i\,v(x,y) = (x^3 - 3xy^2) + i(3x^2 y - y^3)$

$u(x,y) = x^3 - 3xy^2 \;; \quad v(x,y) = 3x^2 y - y^3.$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

By C-R eqⁿs :- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow$ C-R eqⁿs are satisfied.

$\therefore f(z)$ is analytic

③ $w = \log z$. Verify the function is analytic or not and also find $\frac{dw}{dz}$.

$$w = f(z) = \log z = \log(x+iy) = \log(re^{i\theta})$$

$$u(x,y) + iv(x,y) = \log r + \log e^{i\theta}$$

$$u(x,y) + iv(x,y) = \log r + i\theta$$

$$u(x,y) + iv(x,y) = \log(x^2+y^2)^{1/2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$u(x,y) = \frac{1}{2}\log(x^2+y^2) \quad ; \quad v(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}} \cdot \frac{1}{x^2+y^2} \cdot \cancel{2}x = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}} \cdot \frac{1}{x^2+y^2} \cdot \cancel{2}y = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

By C-R eqns:-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ $w = \log z$ satisfied C-R eqns.

$\therefore$ $w = \log z$ is analytic at every point of z.

$\frac{dw}{dz}$ $\quad w = f(z) = u(x,y) + iv(x,y)$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$$

$$= \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right).$$

④ $f(z) = z + 2\bar{z}$. show $f(z)$ is analytic or not.

$f(z) = (x+iy) + 2(x-iy) = x + iy + 2x - 2iy$.

$u(x,y) + iv(x,y) = 3x - iy$

$u(x,y) = 3x \quad , \quad v(x,y) = -y$.

$\frac{\partial u}{\partial x} = 3 \qquad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} , \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \quad \Rightarrow f(z)$ not satisfy C-R eq^{n}s.

$\therefore f(z)$ is a analytic.

⑤ P.T $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Real } f(x)|^2 = 2|f'(z)|^2$, where $w = f(z)$.

P.T tho w is analytic.

Proof:- $w = f(z) = u(x,y) + iv(x,y)$

Real $f(z) = u$, $|\text{Real } f(z)| = \sqrt{u^2}$

$|\text{Real } f(z)|^2 = u^2$ — ①

$f(z) = u(x,y) + iv(x,y) \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x$

$|f'(z)| = \sqrt{u_x^2 + v_x^2} \Rightarrow |f'(z)|^2 = u_x^2 + v_x^2$ — ②

we have $\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$

$\frac{\partial^2}{\partial x^2}(u^2) = 2\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]$

IIly $\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$\rightarrow \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$w = f(z)$ is analytic

$\therefore$ It satisfies laplace eq^n i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\therefore \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) = 2\left[0 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$\frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) = 2[|f'(z)|]^2 \qquad \{\text{From eq } (2)\}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2|f'(z)|^2$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Real} f(z)|^2 = 2|f'(z)|^2 \qquad \{\text{From eq } (1)\}$$

Hence proved.

(6) $f(z) = e^x(\cos ky + i\sin ky)$. Find 'k' such that $f(z)$ is analytic.

(7) Show that z^2 is analytic for all values of z.

(8) Prove that the function $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & (z \neq 0) \\ 0 & z = 0 \end{cases}$

is continuous and the C-R eq's are satisfied at origin but $f'(0)$ does not exist.

→ Continuity at origin :- (0,0)

RHS $\Rightarrow \lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \lim_{x\to 0} \frac{x^3(1+i)}{x^2}$

$\lim_{x\to 0} x(1+i) = 0.$

LHL $\Rightarrow \lim_{z\to 0} f(z) = \lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \lim_{y\to 0} \frac{-y^3(1-i)}{y^2}$

$= \lim_{y\to 0} y(i-1) = 0.$

$$\therefore \lim_{\substack{(x,y)\to(0,0)}} f(z) = \lim_{(y,x)\to(0,0)} f(z) = 0.$$

at $z=0$, $f(0)=0$.

$\therefore$ At $(0,0)$ $f(z)$ is continuous.

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}$$

$$u(x,y)+iv(x,y) = \frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}$$

$$u(x,y) = \frac{x^3-y^3}{x^2+y^2} \;;\quad v(x,y) = \frac{x^3+y^3}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(3x^2) - (x^3-y^3)(2x)}{(x^2+y^2)^2}$$

$$= \frac{3x^4+3x^2y^2-2x^4+2xy^3}{(x^2+y^2)^2}$$

C-R equations at origin are

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{x^3-0}{x^2+0} - 0}{x}$$

$$= \lim_{x\to 0} \frac{x}{x}$$

$$= 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{\frac{-y^3}{y^2} - 0}{y}$$

$$= \lim_{y\to 0} \frac{-y}{y}$$

$$= -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \underset{x\to 0}{Lt} \frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{Lt} \frac{\frac{x^3+0}{x^2+0} - 0}{x} = \underset{x\to 0}{Lt} \frac{x}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{Lt} \frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{Lt} \frac{\frac{y^3}{y^2} - 0}{y} = \underset{y\to 0}{Lt} \frac{y}{y} = 1.$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \left(\frac{\partial v}{\partial y}\right)_{(0,0)} \quad \text{and} \quad \left(\frac{\partial u}{\partial y}\right)_{(0,0)} = -\left(\frac{\partial v}{\partial x}\right)_{(0,0)}$$

$\therefore$ $f(z)$ satisfies C-R equations at origin.

Derivative:- $f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{Lt} \frac{\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} - 0}{(z)}$

$$= \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)} = \underset{x\to 0}{Lt} \frac{\cancel{x^3}(1+i)}{\cancel{x^3}}$$

$$= 1+i \neq f'(0).$$

$f'(z)$ depends on i and it is not unique.

$\therefore$ $f'(z)$ doesnot exist at $(0,0)$

Note:- Along the path $y=mx$ $\underset{z\to 0}{Lt} f(z) = \underset{(x,y)\to(0,0)}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$

$$\underset{x\to 0}{Lt} \frac{x^3(1+i)-m^3x^3(1-i)}{x^2+m^2x^2} = \underset{x\to 0}{Lt} \frac{x^3[(1+i)-m^3(1-i)]}{x^2(1+m^2)} = 0.$$

(9) If $f(z) = \begin{cases} \frac{x^3y(y-ix)}{x^6+y^2}, & z\neq 0 \\ 0, & z=0 \end{cases}$, prove that the function is continuous or not and also along the curve $y=ax^3$. Verify whether C-R eqn's are satisfied or not at origin.

Continuous:-

RHL :- $\underset{z\to 0}{Lt} f(z) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^3y(y-ix)}{x^6+y^2} = \underset{x\to 0}{Lt} \frac{0}{x^6} = 0.$

LHL :- $\underset{z\to 0}{Lt} f(z) = \underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{x^3y(y-ix)}{x^6+y^2} = \underset{y\to 0}{Lt} (0) = 0.$

$f(0) = \frac{0}{0} = 0.$

$\therefore \underset{(x,y)\to 0}{Lt} f(z) = \underset{(y,x)\to 0}{Lt} f(z) = f(0) = 0.$

$\therefore f(z)$ is continuous at $(0,0)$.

along $y = ax^3$:-

$\underset{z\to 0}{Lt} f(z) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^3y(y-ix)}{x^6+y^2} = \underset{x\to 0}{Lt} \frac{x^3(ax^3)(ax^3-ix)}{x^6+a^2x^6}$

$= \frac{a\cancel{x^6}.\,x(ax^2-i)}{\cancel{x^6}(1+a^2)} = \underset{x\to 0}{Lt} \frac{ax(ax^2-i)}{1+a^2} = \frac{0}{1+a^2} = 0.$

C-R equations:-

$f(z) = \frac{x^3y(y-ix)}{x^6+y^2} \Rightarrow u(x,y)+iv(x,y) = \frac{x^3y^2}{x^6+y^2} - i\frac{x^4y}{x^6+y^2}$

$u(x,y) = \frac{x^3y^2}{x^6+y^2}, \quad v(x,y) = \frac{-x^4y}{x^6+y^2}$

$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \underset{x\to 0}{Lt} \frac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{Lt}$

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(10) Verify the function $f(z)$ satisfies C-R equations or not, Check the function is differentiable or not at $y=mx$.

$f(z)=\sqrt{xy}$

Given $f(z)=\sqrt{x,y} \Rightarrow u(x,y)+iv(x,y)=\sqrt{xy}$

$u(x,y)=\sqrt{xy}$ $\qquad v(x,y)=0$

$\cancel{\frac{\partial u}{\partial x}=\frac{\sqrt{y}}{2\sqrt{x}}}$ $\qquad \cancel{\frac{\partial v}{\partial x}=0}$ $\qquad \left(\frac{\partial u}{\partial x}\right)_{(0,0)}=\underset{x\to 0}{Lt}\frac{u(x,0)-u(0,0)}{x}$

$\cancel{\frac{\partial u}{\partial y}=\frac{\sqrt{x}}{2\sqrt{y}}}$ $\qquad \cancel{\frac{\partial v}{\partial y}}$ $\qquad =\underset{x\to 0}{Lt}\frac{0-0}{x}=0$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)}=\underset{y\to 0}{Lt}\frac{(0)-0}{y}=0.$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)}=\underset{x\to 0}{Lt}\frac{0-0}{x}=0$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)}=0.$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

$\therefore$ C-R equations are satisfied at origin.

→ along $y = mx$:-

$$f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{Lt} \frac{\sqrt{xy}-0}{x+iy}$$

at $y=mx \Rightarrow \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{\sqrt{x}\sqrt{mx}}{x+imx} \Rightarrow \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{\sqrt{m}\,x}{x(1+im)}$

$$\underset{x\to 0}{Lt} \frac{\sqrt{m}}{(1+im)} = \frac{\sqrt{m}}{1+im}$$

The limit value depends on the 'm' value.

the $f'(z)$ doesnot exists at origin.

⑪ If $w = f(z)$ is analytic function, then P.T the family of curves defined by $u(x,y)=k_1$ \$ $v(x,y)=k_2$ forming an orthogonal system.

$w=f(z)$ is analytic function. That implies it satisfies C-R equations.

~~$u(x,y)=k_1$~~

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \$ \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow ①.$$

given cures are : $u(x,y)=k_1$, $v(x,y)=k_2$

P. Differentiating on b.s w.r.to 'x' $\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0$ | $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{dy}{dx} = 0$

$$m_1 = \frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \rightarrow ②$$

$$\frac{dy}{dx}\,(m_2) = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} \rightarrow ③$$

$$m_1\cdot m_2 = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$ sub CR eqns in it.

$$\frac{-\frac{\partial v}{\partial y}}{+\frac{\partial v}{\partial x}} \times \frac{+\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = -1.$$

$\therefore$ $u(x,y)$ \& $v(x,y)$ form an orthogonal system.

(12) $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$ is analytic. Determine p.

$$u(x,y) + iv(x,y) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$$

$u(x,y) = \frac{1}{2}\log(x^2+y^2)$ $\qquad$ $v = \tan^{-1}\left(\frac{px}{y}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}} \cdot \frac{1}{x^2+y^2} \cdot \cancel{2}x = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{p^2x^2}{y^2}} \cdot \left(\frac{p}{y}\right) = \frac{py}{y^2+p^2x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{p^2x^2}{y^2}} \cdot px\left(\frac{-1}{y^2}\right) = \frac{-px}{y^2+p^2x^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}} \cdot \frac{1}{x^2+y^2} \cdot \cancel{2}y = \frac{y}{x^2+y^2}$$

analytic $\Rightarrow$ satisfy CR eqns.

$$\frac{x}{x^2+y^2} = \frac{-px}{y^2+p^2x^2} \qquad \frac{y}{x^2+y^2} = \frac{-py}{p^2x^2+y^2}$$

comparing on b.s $\qquad$ lb

$-p = 1 \Rightarrow p = -1.$ $\qquad$ $-p = 1 \Rightarrow p = -1.$

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(13) S.T $u = e^{-x}(x\sin y - y\cos y)$ is harmonic or not.

$u = e^{-x}(x\sin y - y\cos y)$

$u = xe^{-x}\sin y - y\cos y\, e^{-x}$

$\frac{\partial u}{\partial x} = \sin y\,(e^{-x} + (-xe^{-x})) - y\cos y(-e^{-x})$

$\frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y.$

$\frac{\partial^2 u}{\partial x^2} = \sin y(-e^{-x}) - \sin y\,(e^{-x} - xe^{-x}) + y\cos y\,(-e^{-x})$

$\frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y - \sin y\, e^{-x} + xe^{-x}\sin y - e^{-x}y\cos y.$

$\frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin y + xe^{-x}\sin y - e^{-x}y\cos y.$

$\rightarrow \frac{\partial u}{\partial y} = xe^{-x}\cos y - e^{-x}(-y\sin y + \cos y)$

$= xe^{-x}\cos y + e^{-x}y\sin y - e^{-x}\cos y.$

$\frac{\partial^2 u}{\partial y^2} = xe^{-x}(-\sin y) + e^{-x}(y\cos y + \sin y) + e^{-x}\sin y$

$= -xe^{-x}\sin y + e^{-x}y\cos y + e^{-x}\sin y + e^{-x}\sin y.$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2e^{-x}\sin y + \cancel{xe^{-x}\sin y} - \cancel{e^{-x}y\cos y} - \cancel{xe^{-x}\sin y}$

$+ \cancel{e^{-x}y\cos y} + e^{-x}\sin y + e^{-x}\sin y.$

$= e^{-x}\sin y - e^{-x}\sin y.$

$= e^{-x}(\sin y - \sin y)$

$= 0. \Rightarrow$ 'u' satisfies laplacian eqn.

$\therefore u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

(14) S.T both real and imaginary parts of an analytic function are harmonic.

Let $f(z) = u + iv$ be an analytic function i.e., it satisfies C-R eqns.

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -(1) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad -(2)$$

P.D w.r.to 'x' on b.s

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad -(3)$$

P.D w.r.to 'y' on b.s

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \quad -(4)$$

eq (3)+(4) $\Rightarrow$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0.$$

$$\Rightarrow \nabla^2 u = 0.$$

$\therefore$ 'u' is harmonic.

P.D eq.(1) w.r.to 'y' on b.s

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \quad -(5)$$

P.D eq. (2) w.r.to 'x' on b.s

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y^2} \quad -(6)$$

eq (5)-(6) $\Rightarrow$ $$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} \Rightarrow \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$$\Rightarrow \nabla^2 v = 0.$$

$\therefore$ 'v' is harmonic

$\therefore$ Real and imaginary parts i.e., 'u' and 'v' are harmonic functions.

(15) $f(x,y) = x^3 + 3kxy^2$ is harmonic function. Then find 'k'.

$u(x,y) + iv(x,y) = x^3 + 3kxy^2$

~~$u(x,y) =$~~ $f(x,y) = x^3 + 3kxy^2$.

harmonic $\Rightarrow \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

$\dfrac{\partial f}{\partial x} = 3x^2 + 3ky^2$. $\qquad \dfrac{\partial f}{\partial y} = 6kxy$.

$\dfrac{\partial^2 f}{\partial x^2} = 6x \qquad \dfrac{\partial^2 f}{\partial y^2} = 6kx$.

$\cancel{36x^2 + 36k^2x^2 = 0} \Rightarrow \cancel{36x^2(1+k^2) = 0}$. $\quad 6x(1+k) = 0$

$k = -1$.

(16) Construction Of Analytic Function whose real or imaginary Part is Known:-

(1) Milne-Thomson Method:-

Suppose $f(z) = u+iv$ is analytic function whose real part 'u' is known, we can find 'v', (or) whose imaginary part 'v' is known, we can find 'u'.

Let $f(z) = u(x,y) + iv(x,y)$ — (1)

W.K.that $z = x+iy$, $\bar{z} = x-iy$.

$\therefore x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$

$\therefore f(z) = u\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right) + i\, v\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right)$ — (2)

Now considering this result as a formal identity in two dependent variable z and $\bar{z}$.

Put $\bar{z}=z$ in eq. (2)

$$f(z) = u(z,0) + i\,v(z,0). \quad \text{—(3)}$$

Comparing eqns (1) & (3) $\Rightarrow x = z,\ y = 0$

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

Integrating w.r.to 'x' & 'y' on b.s ; we get

$$f(z) = \phi_1(x,y) + \phi_2(x,y)$$

(1) If $u = e^x\left((x^2-y^2)\cos y - 2xy\sin y\right)$ is real part of an analytic function. Find imaginary part 'v.'

$f(z) = u+iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ —(1) [$\because$ By C-R eqns]

Given $u = e^x\left[(x^2-y^2)\cos y - 2xy\sin y\right]$.

$$\frac{\partial u}{\partial x} = e^x[2x\cos y - 2y\sin y] + e^x[(x^2-y^2)\cos y - 2xy\sin y].$$

$$\frac{\partial u}{\partial x} = 2xe^x\cos y - 2e^x y\sin y + x^2e^x\cos y - y^2e^x\cos y - 2xye^x\sin y$$

$\cancel{\frac{\partial u}{\partial x} = 2xe^x\cos y}$

$$\frac{\partial u}{\partial y} = e^x\left[(x^2-y^2)(-\sin y) + \cos y(-2y) - 2xy\cos y - 2x\sin y\right]$$

$$= e^x\left[-x^2\sin y + y^2\sin y - 2y\cos y - 2xy\cos y - 2x\sin y\right]$$

(1) $\Rightarrow f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$= 2xe^x\cos y - 2e^x y\sin y + x^2e^x\cos y - y^2e^x\cos y - 2xye^x\sin y$

$-ie^x[-x^2\sin y + y^2\sin y - 2y\cos y - 2xy\cos y - 2x\sin y].$

$\rightarrow$ by milne - thomson method, $f'(z)$ is expressed in terms of 'z' by replacing

put $x = z, y = 0$.

$= 2ze^z + z^2e^z - 0 - 0 - ie^z[-0 + 0 - 0 - 0 - 0]$

$f'(z) = 2ze^z + z^2e^z.$

Integrate w.r.to 'z' on b.s.

$f(z) = \int(2ze^z + z^2e^z)dz.$

$= 2\int ze^z dz + \int z^2e^z dz$

$= 2[ze^z - \int(1)e^z] + [z^2e^z - \int 2ze^z]$

$= 2[ze^z - e^z] + [z^2e^z - 2(ze^z - \int(1)e^z)]$

$= 2[ze^z - e^z] + [z^2e^z - 2ze^z + 2e^z).$

$= 2[ze^z - e^z] + [z^2e^z - 2ze^z + 2e^z$

$= \cancel{2ze^z} - \cancel{2e^z} + z^2e^z - \cancel{2ze^z} + \cancel{2e^z}$

$= z^2e^z.$

$= (x+iy)^2e^{(x+iy)}$

$= (x+iy)^2e^x.e^{iy}.$ $=$ ~~xe~~ $e\ (x^2 + 2ixy - y^2)e^x.e^{iy}.$

$= (e^x(x^2-y^2) + 2e^x xyi)(\cos y + i\sin y).$

$= e^x(x^2-y^2)\cos y + (2xye^x\cos y)i + e^x(x^2-y^2)\sin y\, i$

$- 2e^x xy\sin y.$

$f(z) = e^x[(x^2-y^2)\cos y - 2xy\sin y] + i[e^x(x^2-y^2)\sin y + 2xye^x\cos y]$

$\therefore v(x,y) = e^x(x^2-y^2)\sin y + 2xye^x\cos y.$

② Find the analytic function whose imaginary part is $v = e^x(x\sin y + y\cos y)$.

$f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Given that $f(z)$ is analytic $\Rightarrow$ satisfy CR eqns.

$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}.$

$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}.$

$v = e^x(x\sin y + y\cos y).$

$\frac{\partial v}{\partial x} = e^x(\sin y + 0) + e^x(x\sin y + y\cos y)$

$\frac{\partial v}{\partial x} = e^x\sin y + xe^x\sin y + e^x y\cos y.$

$\rightarrow \frac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y).$

$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$= e^x(x\cos y + \cos y - y\sin y) + i\,e^x(x\sin y + y\cos y + \sin y)$

$\rightarrow$ By Milne-Thomson method, $f'(z)$ is expressed in terms of 'z' by replacing with $x = z, y = 0.$

$f'(z) = e^z(z + 1 - 0) + i\,e^z(0 + 0 + 0)$

$= \cancel{z}\, e^z(z+1)$

$$f'(z) = ze^z + e^z = (z+1)e^z$$

$$= \cancel{(x+iy)e^x} = \cancel{(x+iy+1)e^{x+iy}}$$

$$f'(z) = ze^z + e^z$$

Integrating on b.s

$$f(z) = \int ze^z + \int e^z = ze^z - \cancel{e^z} + \cancel{e^z}$$

$$= ze^z$$

$$f(z) = (x+iy)e^{x+iy}.$$

$$= (xe^x + ye^x.i)(\cos y + i\sin y)$$

$$= xe^x\cos y + xe^x\sin y.i + ye^x\cos y.i - ye^x\sin y.$$

$$= e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y).$$

$\therefore\ u(x,y) = e^x(x\cos y - y\sin y).$

③ S.T $f(x,y) = x^3y - xy^3 + xy + x + y$ can be the imaginary part of an analytic function of $z = x+iy$. Determine the real part.

$$f(z) = u + iv.$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Given that $f(z)$ is analytic $\Rightarrow$ satisfy CR eq^ns.

$$\therefore\ f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$v = x^3y - xy^3 + xy + x + y$$

$$\frac{\partial v}{\partial x} = 3x^2y - y^3 + y + 1 \quad , \quad \frac{\partial v}{\partial y} = x^3 - 3y^2x + x + 1$$

$$\therefore\ f'(z) = i(3x^2y - y^3 + y + 1) + (x^3 - 3xy^2 + x + 1)$$

$\rightarrow$ put $x = z, y = 0$

$f'(z) = i(0 - 0 + 0 + 1) + (z^3 - 0 + z + 1)$

$f'(z) = i + (z^3 - z + 1) \Rightarrow$ ~~f'(z) = 1 + i((x+iy)^3 - x - iy + 1)~~

~~Integrating on b.s~~

~~$f(z) = \int dz + i[\int z^3 dz +$~~

$f'(z) = z^3 - z + 1 + i$

I.O.B.S

$f(z) = \frac{z^4}{4} - \frac{z^2}{2} + z + iz$

$= \frac{(x+iy)^4}{4} - \frac{(x+iy)^2}{2} + (1+i)(x+iy).$

$= \frac{(x^2 - y^2 + 2ixy)(x^2 - y^2 + 2ixy)}{4} - \frac{(x^2 + 2ixy - y^2)}{2} + (x + xi + iy - y)$

$= \frac{x^4 - x^2y^2 + 2ix^3y - x^2y^2 + y^4 - 2ixy^3 + 2ix^3y - 2ixy^3 - 4x^2y^2}{4}$

$+ \frac{2x + 2ix + 2iy - y - x^2 - 2ixy + y^2}{2}$ +

$= \frac{x^4 + y^4 - 6x^2y^2 + 4ix^3y - 4ixy^3 + 4x + 4ix + 4iy - 2y - 2x - 4ixy + 2y^2}{4}$

$= \frac{x^4 + y^4 - 6x^2y^2 + 4x - 2y - 2x + 2y^2}{4} + i\frac{(4x^3y - 4xy^3 + 4x + 4y - 4xy)}{4}$

$=$ ~~$x^4 + y^4 - 6x^2y^2 + 2x - 2y + i(x^3y - xy^3 + x + y - xy)$~~

$$= \frac{x^4+y^4-6x^2y^2+2x^2-2y^2+4x-4y}{4} + i\,(4x^3y-4xy^3+4x+4y-4xy).$$

$$\therefore u(x,v) = \frac{x^4+y^4-6x^2y^2+2x^2-2y^2+4x-4y}{4}.$$

(24) $f(z) = (x^2-2xy+ay^2)+i(bx^2-y^2+2xy)$ is analytic. Find $f(z)$ in terms of z. Find a, b.

$u(x,y)+iv(x,y) = (x^2-2xy+ay^2)+i(bx^2-y^2+2xy)$

$u(x,y) = x^2-2xy+ay^2$, $\quad v(x,y) = bx^2-y^2+2xy.$

$\frac{\partial u}{\partial x} = 2x-2y$ $\qquad \frac{\partial v}{\partial x} = 2bx+2y$

$\frac{\partial u}{\partial y} = -2\cancel{y}x+2ay$ $\qquad \frac{\partial v}{\partial y} = -2y+2x.$

analytic $\Rightarrow$ satisfy CR eqns.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\cancel{-2y+2ay = -2y+2x}$ $\qquad \cancel{-2y+2ay = -2bx-2y}$

$\cancel{2ay = 2x}$ $\qquad \cancel{ay = -bx.}$

$\cancel{x = ay}$ $\qquad \cancel{-2y+2ay}$

$2x-2y = -2y+2x$ $\qquad -2x+2ay = -2bx-2y.$

comparing on b.s

$\boxed{b=1,\ a=-1}$

sub a, b in eqn f(z).

$f(z) = (x^2-2xy-y^2)+i(x^2-y^2+2xy).$

~~$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$~~

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= (2x - 2y) + i(2x + 2y).$$

replace $x = z, y = 0$

$$f'(z) = 2z + i2z.$$

Integrate on b.s

$$f(z) = \frac{\cancel{2}z^2}{\cancel{2}} + i\frac{\cancel{2}z^2}{\cancel{2}}$$

$$\boxed{f(z) = z^2(1+i)}$$

3 $u = e^{x^2-y^2}\cos 2xy$. Hence find $f(z)$ in terms of z.

$$u = e^{x^2-y^2}\cos 2xy.$$

$$\frac{\partial u}{\partial x} = e^{x^2-y^2}(2x)\cdot\cos 2xy + e^{x^2-y^2}(-\sin 2xy)\cdot(2y)$$

$$= 2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy.$$

$$\frac{\partial u}{\partial y} = e^{x^2-y^2}(-2y)\cos 2xy + e^{x^2-y^2}(-\sin 2xy)\cdot(+2x)$$

$$= -2ye^{x^2-y^2}\cos 2xy - 2xe^{x^2-y^2}\sin 2xy.$$

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = (2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy) +$$
$$i(-2ye^{x^2-y^2}\cos 2xy - 2xe^{x^2-y^2}\sin 2xy).$$

put $x = z, y = 0$

$f'(z) = (2ze^{z^2} - 0) + i(-0+0).$

$f'(z) = 2ze^{z^2}.$

I.O.B.S.

~~$f(z) = 2[z$~~

Let $z^2 = t$

$2z\,dz = dt$

$f(z) = \int e^t dt$

$= e^t$

$\boxed{f(z) = e^{z^2}}$

(6) Find the conjugate harmonic function of the harmonic function $u = x^2 - y^2$

$u = x^2 - y^2$ is harmonic function.

$\Rightarrow \cancel{\nabla^2 u = 0}$ $\rightarrow$ conjugate harmonic function of 'u' is v.

$u = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = -2y$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$f'(z) = 2x + 2iy.$

put $x = z, y = 0$

$f'(z) = 2z$

I.O.B.S $\Rightarrow f(z) = z^2 = (x+iy)^2$

$u(x,y) + iv(x,y) = x^2 - y^2 + i(2xy)$

$\boxed{\therefore v = 2xy}$

(7) Find the analytic function of $f(z)=u+iv$, if $u=a(1+\cos\theta)$.

$u(r,\theta) = a(1+\cos\theta)$

$$\frac{\partial u}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = -a\sin\theta$$

C-R eq^ns in polar form are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$f(z) = u(r,\theta) + i v(r,\theta)$.

$$f'(z) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$f'(z) = \frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$f'(z) = 0 - \frac{1}{r}(-a\sin\theta)$$

$$f'(z) = \frac{a\sin\theta}{r}$$

by milne-thomson method, put $r=z, \theta=0$.

$f'(z) = 0$.

Integrating on b.s

$f(z) = 0$.

(8) ~~Find the~~

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(8) $V(x,y) = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$. P.T $V(x,y)$ satisfies laplace equations & find its conjugate harmonic.

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$V(x,y) = \sin x \cosh y + 2\cos x \sinh y + x^2 \mp y^2 + 4xy.$$

$$\frac{\partial v}{\partial x} = +\cos x \cosh y - 2\sin x \sinh y + 2x + 4y.$$

$$\frac{\partial v}{\partial y} = +\sin x \sinh y \pm 2\cos x \cosh y - 2y + 4x.$$

$$\boxed{\begin{array}{l} \frac{d}{dx}(\sinh x) = +\cosh x \\ \frac{d}{dx}(\cosh x) = +\sinh x \end{array}}$$

$$\rightarrow \frac{\partial^2 v}{\partial x^2} = -\sin x \cos hy - 2\cos x \sinh y + 2.$$

$$\frac{\partial^2 v}{\partial y^2} = +\sin x \cosh y + 2\cos x \sinh y - 2$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\sin x \cosh y - 2\cos x \sinh y + 2 + \sin x \cosh y + 2\cos x \sinh y - 2$$

$$= 0$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}.$$

$$= \sin x \sinh y + 2\cos x \cosh y - 2y + 4x + i(\cos x \cosh y - 2\sin x \sinh y + 2x + 4y)$$

Replace $x = z$, $y = 0$

$$f'(z) = 2\cos z + 4z + i(\cos z + 2z).$$

Integrate on b.s

$$f(z) = 2\int \cos z\,dz + 4\int z\,dz + i\int \cos z\,dz + 2i\int z\,dz$$

$$= 2\sin z + \cancel{4}^{2}\frac{z^2}{\cancel{2}} + i\left[\sin z + \cancel{2}\frac{z^2}{\cancel{2}}\right]$$

$$f(z) = 2(\sin z + 2z^2) + i(\sin z + z^2)$$

$$= 2\sin(x+iy) + 2(x+iy)^2 + i\sin(x+iy) + i(x+iy)^2$$

$$= \sin(x+iy)[2+i] + 2(x^2-y^2+2ixy) + i(x^2-y^2+2ixy)$$

$$= \sin(x+iy)(2+i) + (2+i)(x^2-y^2+2ixy).$$

$$= [\sin x\cos(iy) + \cos x\sin(iy)](2+i) + (2+i)(x^2-y^2+2ixy)$$

$$= [\sin x\cosh y + \cos x(i\sinh y)](2+i) + \quad ''$$

$$= [\sin x\cosh y + i(\cos x\sinh y)](2+i) + (2+i)(x^2-y^2+2ixy).$$

$$= 2\sin x\cosh y + i\sin x\cosh y + 2i\cos x\sinh y - \cos x\sinh y$$
$$+ 2x^2 - 2y^2 + 4ixy + ix^2 - iy^2 - 2xy.$$

$$= (2\sin x\cosh y - \cos x\sinh y + 2x^2 - 2y^2 - 2xy) +$$
$$i(\sin x\cosh y + 2\cos x\sinh y - y^2 + 4xy + x^2).$$

$\therefore$ Conjugate harmonic of v i.e., real part is

$$u = 2\sin x\cosh y - \cos x\sinh y + 2x^2 - 2y^2 - 2xy.$$

Elementary Function:-

→ Let $z = x+iy$, then complex exponential function is written as $e^z = e^{x+iy} = e^x e^{iy}$

$$= e^x(\cos y + i\sin y)$$

(or)

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

Complex Trigonometric Functions:-

$$e^{iy} = \cos y + i\sin y$$

$$e^{-iy} = \cos y - i\sin y$$

$$\cos y = \frac{e^{iy}+e^{-iy}}{2}$$

$$\sin y = \frac{e^{iy}-e^{-iy}}{2i}$$

Illy $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\tan z = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$$

$$\cot z = \frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$$

$$\sec z = \frac{2}{e^{iz}+e^{-iz}}$$

$$\operatorname{cosec} z = \frac{2i}{e^{iz}-e^{-iz}}$$

→ $\sin^2 z + \cos^2 z = 1$

$1+\tan^2 z = \sec^2 z$

$1+\cot^2 z = \operatorname{cosec}^2 z$

$\sin 2z = 2\sin z\cos z$

→ $\sin z$ is odd function, $\sin(-z) = -\sin z$.

$\cos z$ is even function, $\cos(-z) = \cos z$

Hyperbolic Functions:-

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\coth z = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$\operatorname{sech} z = \frac{2}{e^z + e^{-z}}$$

$$\operatorname{cosech} z = \frac{2}{e^z - e^{-z}}$$

Relationship b/w complex, trigonometric and hyperbolic functions:-

$\cosh(iz) = \cos z$

$\sinh(iz) = i\sin z$

$\cos(iz) = \cosh z$

$\sin(iz) = i\sinh z$

$\tan(iz) = i\tanh z$

$\sinh z = \sinh(x+iy)$

$= \sinh x \cosh iy + \cosh x \sinh iy$

$= \sinh x \cos y + i\cosh x \sin y.$

$\cosh z = \cosh(x+iy)$

$= \cosh x \cosh iy + \sinh x \sinh iy$

$= \cosh x \cos y + i \sinh x \sin y.$

Q1) Find the real and imaginary part of i) $\sin z$ ii) $\cos z$ iii) $\tan z$.

i) $\sin(z) = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy.$

$\sin z = \sin x \cosh y + i\cos x \sinh y$

~~ii) cos z~~

$Re(\sin z) = \sin x \cosh y$, $Im(\sin z) = \cos x \sinh y.$

ii) $\cos z = \cos(x+iy) = \cos x \cos iy \mp \sin x \sin iy$

$= \cos x \cosh y \mp i \sin x \sinh y$

$Re(\cos z) = \cos x \cosh y$, $Im(\cos z) = -\sin x \sinh y.$

iii) $\tan z = \dfrac{\sin z}{\cos z} = \dfrac{\sin(x+iy)}{\cos(x+iy)} \times \dfrac{2\cos(x-iy)}{2\cos(x-iy)}$

$= \dfrac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)} = \dfrac{\sin(2x) + \sin(+2iy)}{\cos(2x)+\cos(2iy)}$

$= \dfrac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}.$

$\therefore Re(\tan z) = \dfrac{\sin 2x}{\cos 2x + \cosh 2y}$, $Im(\tan z) = \dfrac{\sinh 2y}{\cos 2x + \cosh 2y}.$

$\rightarrow \cosh^2 z - \sinh^2 z = 1$

$\rightarrow \sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$

$\rightarrow \cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

EO

② seperate the real and imaginary parts i) cot z

ii) cosec z.

i) $\cot z = \dfrac{\cos z}{\sin z} = \dfrac{\cos(x+iy)}{\sin(x+iy)} \times \dfrac{2\sin(x-iy)}{2\sin(x-iy)}$

$$= \frac{\sin(2x) + \sin(2iy)}{\cos(2iy) - \cos(2x)}$$

$$= \frac{\sin 2x \mp i\sinh 2y}{\cosh 2y - \cos 2x}$$

$Re(\cot z) = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$, $Im(\cot z) = \dfrac{-\sinh 2y}{\cosh 2y - \cos 2x}$

iv) $\operatorname{cosec} z = \dfrac{1}{\sin(x+iy)} = \dfrac{1}{\sin(x+iy)} \times \dfrac{2\sin(x-iy)}{2\sin(x-iy)}$

$$= \frac{2[\sin x \cos(iy) - \cos x \sin(iy)]}{\cos(2iy) - \cos(2x)}$$

$$= \frac{2[\sin x \cosh y - i\cos x \sinh y]}{\cosh 2y - \cos 2x.}$$

$Re(\operatorname{cosec} z) = \dfrac{2\sin x \cosh y}{\cosh 2y - \cos 2x}$, $Im(\operatorname{cosec} z) = \dfrac{-2\cos x \sinh y}{\cosh 2y - \cos 2x}$

③ If $\sin(A+iB) = x+iy$. P.T i) $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$

ii) $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$.

Sol:- Given $\sin(A+iB) = x+iy$.

$\sin A \cos iB + \cos A \sin iB = x+iy$.

$\sin A \cosh B + i\cos A \sinh B = x+iy$.

C.O.B.S $\Rightarrow$ $\therefore$ $x = \sin A \cosh B$, $y = \cos A \sinh B$.

$\Rightarrow \sin A = \frac{x}{\cosh B}$ —① $\cos A = \frac{y}{\sinh B}$ —③

$\cosh B = \frac{x}{\sin A}$ —② $\sinh B = \frac{y}{\cos A}$ —④

i) eq. $①^2 + ③^2 \Rightarrow \sin^2 A + \cos^2 A = \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B}$

$$1 = \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B}$$

ii) $②^2 - ④^2 \Rightarrow \cosh^2 B - \sinh^2 B = \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A}$

$$1 = \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A}.$$

④ If $(x+iy)^{1/3} = a+ib$, then P.T $4(a^2-b^2) = \frac{x}{a} + \frac{y}{b}$

$(x+iy)^{1/3} = a+ib \Rightarrow (x+iy) = (a+ib)^3$.

$x+iy = (a+ib)^3$ —① IIly $x-iy = (a-ib)^3$ —②

~~x+iy~~ = ① + ② $\Rightarrow$ $2x = (a+ib)^3 + (a-ib)^3$

$= a^3 - \cancel{ib^3} + \cancel{3a^2ib} - 3ab^2 + a^3$

$+ \cancel{ib^3} - \cancel{3a^2ib} - 3ab^2$

$$\cancel{2}x = \cancel{2}(a^3 - 3ab^2)$$

$$x = a^3 - 3ab^2 \Rightarrow \frac{x}{a} = a^2 - 3b^2$$

$$① - ② \Rightarrow x + iy - x + iy \Rightarrow 2iy = (a+ib)^3 - (a-ib)^3$$

$$2iy = \cancel{a^3} - ib^3 + 3a^2ib - \cancel{3ab^2}$$
$$- \cancel{a^3} - ib^3 + 3a^2ib + \cancel{3ab^2}$$

$$\cancel{2}\cancel{i}y = -\cancel{i}(-\cancel{2}b^3 + \cancel{6}^{3}a^2b)$$

$$y = 3a^2b - b^3$$

$$\frac{y}{b} = 3a^2 - b^2$$

$$\frac{x}{a} + \frac{y}{b} = 3a^2 - b^2 + a^2 - 3b^2 = 4a^2 - 4b^2$$

$$\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$$

23/09/2019

Logarithmic Functions:-

→ Let z be a complex variable, such that $z = re^{i\theta}$, where θ is measured in radians. We define $\log z$ by the equation $\log z = \log(re^{i\theta})$
$$= \log r + i\theta.$$

The above definition is single valued.

If $z = x+iy$, $\omega = u+iv$, $e^{\omega} = z$, then ω is said to be logarithm of z and also $e^{\omega + i2n\pi} = e^{\omega} \cdot e^{i2n\pi}$
$$= z \quad [\because e^{i2n\pi} = 1]$$

$\boxed{\therefore \log z = \omega + 2in\pi}$ is called logarithm of a complex number having infinite no. of values. The value $\omega + 2n\pi$ is called general ~~term of~~ value of z.

① ~~Fo~~ Find all solutions of $e^z = 3+4i$

$$|e^z| = |3+4i|$$

$$e^x = \sqrt{9+16}$$

$$e^x = 5$$

$$x = \log_e 5 = 1.609$$

$$e^z = e^{x+iy} = e^x . e^{iy}$$

$$e^z = e^x(\cos y + i \sin y)$$

$$e^z = e^x \cos y + i e^x \sin y.$$

$$|e^z| = \sqrt{(e^x \cos y)^2 + (e^x \sin y)^2}$$

$$= \sqrt{e^{x^2}(\cos^2 y + \sin^2 y)}$$

$$= \sqrt{e^{x^2}} \Rightarrow |e^z| = e^x.$$

$$\rightarrow e^z = 3+4i$$

$$e^z = e^x \cos y + i e^x \sin y$$

1) $e^x \cos y = 3 \Rightarrow \cos y = 3e^{-x}$

$$= 3e^{-1.609}$$

$$\cos y = 0.6002$$

$$y = \cos^{-1}(0.6002) = 0.927$$

2) $e^x \sin y = 4 \Rightarrow \sin y = 4e^{-x} = 4e^{-1.609}$

$$\sin y = 0.8003$$

$$y = \sin^{-1}(0.8003) = 0.927.$$

$\therefore$ The general solⁿ of $z = x + iy \pm 2in\pi$, $n = 0, 1, 2, 3 \ldots$

$$z = 1.609 + i0.927 \pm 2in\pi.$$

② Find all solutions of $\cos z = 0$.

$$\cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

$$|\cos z|^2 = \sqrt{(\cos x)^2 \cosh^2 y + (\sin x)^2 \sinh^2 y}$$

$$= \sqrt{\cos^2 x(1+\sinh^2 y) + \sin^2 x \sinh^2 y}$$

$= \sqrt{\cos^2 x + \cos^2 x \sinh^2 y + \sin^2 x \sinh^2 y}$

$= \sqrt{\cos^2 x + \sinh^2 y (\sin^2 x + \cos^2 x)}$

$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}.$

$\rightarrow$ ~~Taki~~ squaring on b.s

$|\cos z|^2 = \cos^2 x + \sinh^2 y.$

$\therefore \cos z = 0 \Rightarrow \cos x = 0, \sinh y = 0$

$x = (2n+1)\frac{\pi}{2}, \quad y = 0$

~~n = 1, 2,~~ 3 - - - -

$n \in \mathbb{Z}$