UNIT–III

IIR Digital Filters

Introduction:-

Basically a digital filter is a linear time invariant discrete time system. The terms Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) are used to distinguish filter types.

→ The FIR filters are of non recursive type, whereby the present output sample depends on the present input sample and previous input samples.

→ The IIR filters are of recursive type, where by the present output sample depends on the present input, past input samples and output samples.

The impulse Response $h(n)$ for a realizable filter is

$$h(n) = 0 \quad \text{for } n \leq 0$$

and for stability it must satisfy the condition

$$\sum_{n=0}^{\infty} |h(n)| < \infty .$$

IIR Digital filters have the transfer function of the form

$$\boxed{H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}}$$

where a_k, b_k are the filter coefficients.

Frequency Selective Filters :-

A filter is one, which reject unwanted frequencies from the input signal and allow the desired frequencies.

→ The range of frequencies of signal that are passed through the filter is called passband and those frequencies that are blocked is called stopband.

→ The filters are of different types.

1. low pass filter
2. High pass filter
3. Band pass filter
4. Band reject filter.

lowpass filter :-

the magnitude response of an ideal lowpass filter allows low frequencies in the passband $0 < \Omega < \Omega_c$ to pass, whereas the higher frequencies in the stopband $\Omega > \Omega_c$ are blocked.

→ The frequency Ω_c between the two bands is the cut-off frequency where the magnitude $|H(j\Omega)| = 1/\sqrt{2}$.

Highpass filter :-

the highpass filter allows high frequencies above $\Omega > \Omega_c$ and rejects the frequencies between $\Omega = 0$ and $\Omega = \Omega_c$.

Bandpass filter :-

It allows only a band of frequencies Ω_1 to Ω_2 to pass and stops all other frequencies.

Bandreject filter :-

It rejects all the frequencies between Ω_1 and Ω_2 and allows remaining frequencies.

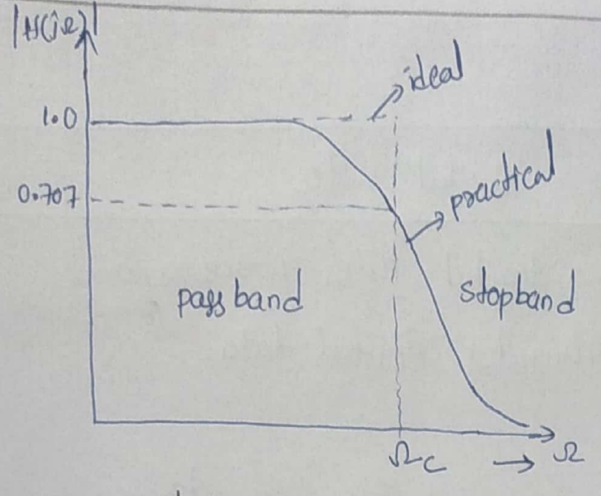


fig: magnitude Response of LPF

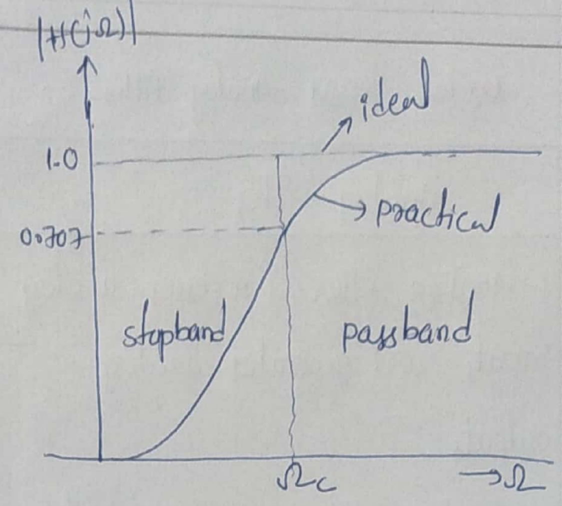


fig: magnitude Response of HPF.

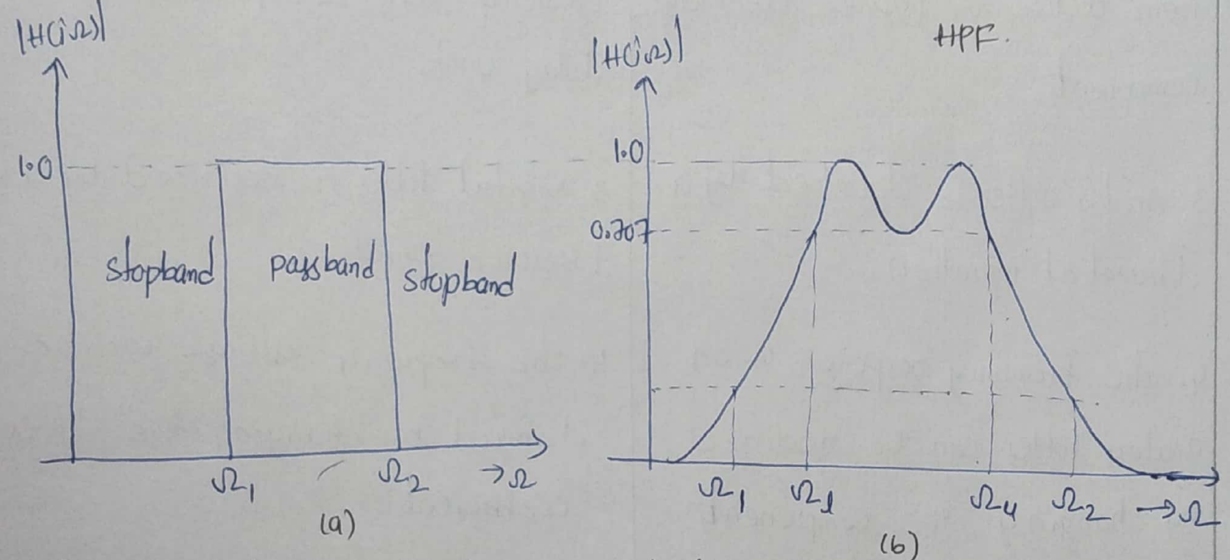


fig: magnitude Response of BPF (a) ideal (b) practical.

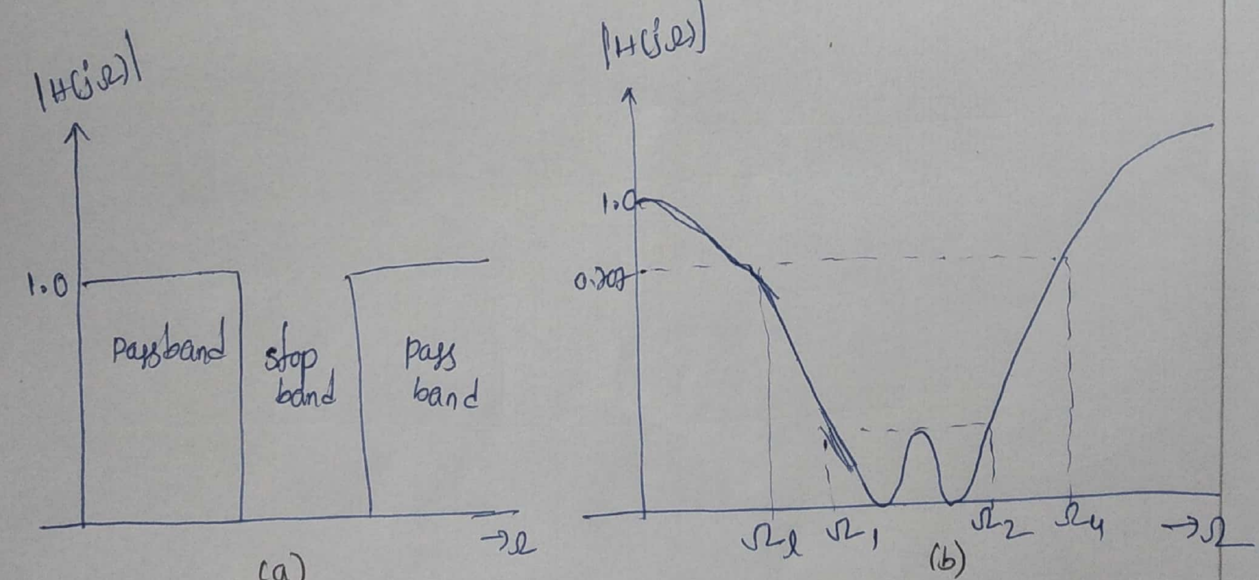


fig: magnitude response of Bandreject filter (a) ideal
(b) practical.

Digital versus Analog Filters:-

Analog Filter	Digital Filter.
1. Analog filter processes analog inputs and generates analog outputs.	1. A digital filter processes and generates digital data.
2. Analog filters are constructed from active or passive electronic components.	2. A digital filter consists of elements like adder, multiplier and delay unit.
3. Analog filter is described by a differential equation.	3. Digital filter is described by a difference equation.
4. The frequency response of an analog filter can be modified by changing the components	4. The frequency response can be changed by changing the filter coefficients.

Advantages of Digital filters :-

1. Unlike analog filter, the digital filter performance is not influenced by component ageing, temperature and power supply variations.

2. A digital filter is highly immune to noise and possesses considerable parameter stability.

3. Digital filter afford a wide variety of shapes for the amplitude and phase responses.

4. There are no problems of input or output impedance matching with digital filters.

5. Digital filters can be operated over a wide range of frequencies.

6. The coefficients of digital filter can be programmed and altered any time to obtain the desired characteristics.

7. multiple filtering is possible only in digital filter.

Disadvantages :-

1. The quantization Error arises due to finite word length in the representation of signals and parameters.

Analog lowpass filter Design :-

The most general form of analog filter transfer function is

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i}$$

where $H(s)$ is the laplace transform of the impulse response $h(t)$.

i.e $H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$.

and $N \geq M$ must be satisfied. For a stable analog filter, the poles of $H(s)$ lie in the left half of the s-plane. there are two types of analog filter design. they are

1. Butterworth filter
2. chebyshev filter.

Analog lowpass Butterworth filter :-

the magnitude function of the Butterworth lowpass filter is given by

$$|H(j\Omega)| = \frac{1}{\left[1 + (\Omega/\Omega_c)^{2N}\right]^{1/2}} \quad N = 1, 2, 3.$$

$$|H(j\Omega)|^2 = \frac{1}{\left[1 + (\Omega/\Omega_c)^{2N}\right]}$$

where N is the order of the filter

Ω_c is the cut-off frequency

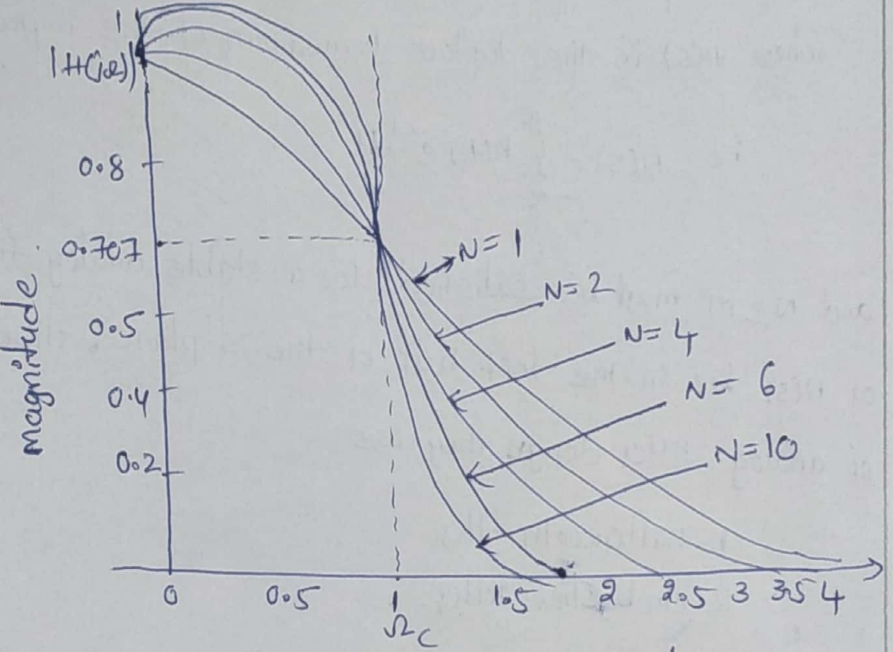
As shown in below fig, the function is monotonically decreasing where the maximum response is unity at $\Omega = 0$. The ideal response is shown by the dash line.

It can be seen that the magnitude response approaches the ideal lowpass characteristics as the order N increases.

For the values $\Omega < \Omega_c$, $|H(j\Omega)| \approx 1$.

For $\Omega > \Omega_c$, the values of $|H(j\Omega)|$ decreases rapidly.

At $\Omega = \Omega_c$ the curves pass through 0.707, which corresponds to -3dB point.



lowpass Butterworth magnitude Response

If $\Omega_c = 1$ rad/sec, then the butterworth filter is called Normalized Butterworth filter.

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2N}} \qquad N = 1, 2, 3 \ldots$$

Now let us derive the transfer function of a stable filter. For this purpose, substituting $\Omega = s/j$

$$\begin{aligned}|H(j\Omega)|^2 &= H(\Omega^2) \\ &= H(s^2/j^2) \\ &= H(-s^2) \\ &= H(s)H(-s)\end{aligned}$$

$$H(s)H(-s) = \frac{1}{1+(s/j)^{2N}} \qquad j=\sqrt{-1}=(-1)^{1/2}$$

$$= \frac{1}{1+(s)^{2N}(-1)^{1/2\times 2N}}$$

$$= \frac{1}{1+(s)^{2N}(-1)^{N}}$$

$$H(s)H(-s) = \frac{1}{1+(-s^2)^N}$$

The above equation tell us that this function has poles in the left Half of s-plane as well as in the Right Half of s-plane, because of the presence of two factors H(s) and H(-s).

If H(s) has roots in the LHP then H(-s) has roots in the RHP.

we can obtain roots by equating the denominator to zero.

i.e. $1+(-s^2)^N = 0$

for N odd, $s^{2N} = 1 = e^{j2\pi k}$

Now the roots are $S_K = e^{j2\pi k/2N} = e^{j\pi k/N}$

$$\therefore \boxed{S_K = e^{j\pi k/N}} \qquad k=1,2,3\ldots 2N$$

For N even, $s^{2N} = -1 = e^{j(2k-1)\pi}$

Now the roots are $S_K = e^{j(2k-1)\pi/2N}$ $\qquad k=1,2,3\ldots 2N$

for $N=3$, $s^6 = 1$

$$S_k = e^{j\pi k/3} \qquad k = 1, 2, 3, 4, 5, 6$$

$$s_1 = e^{j\pi/3} = \cos\frac{\pi}{3} + j\sin \pi/3 = 0.5 + j0.866$$

$$s_2 = e^{j2\pi/3} = \cos(2\pi/3) + j\sin(2\pi/3) = -0.5 + j0.866$$

$$s_3 = e^{j\pi} = \cos\pi + j\sin\pi = -1$$

$$s_4 = e^{j4\pi/3} = \cos\frac{4\pi}{3} + j\sin(4\pi/3) = -0.5 - j0.866$$

$$s_5 = e^{j5\pi/3} = \cos 5\pi/3 + j\sin(5\pi/3) = 0.5 - j0.866$$

$$s_6 = e^{j2\pi} = 1$$

All the above poles are located in the s-plane. It is found that the angular seperation between the poles is given by

$$\frac{360^\circ}{2N} = \frac{2\pi}{2N} = \frac{\pi}{N} = \pi/3 = 180/3 = 60^\circ$$ & all poles lie on unit circle.

To ensure stability & considering only the poles that lie in the LHS of s-plane, we can write the denominator of the Transfer function H(s) as

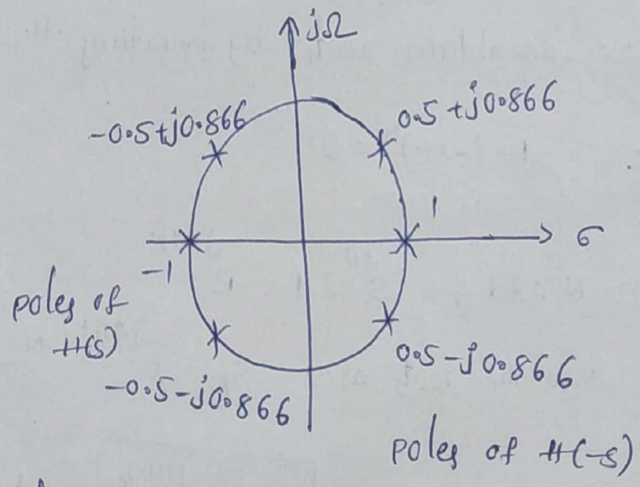


fig: poles locations in the s-plane

$$(s+1)\{(s+0.5-j0.866)(s+0.5+j0.866)\}$$

$$= (s+1)(s^2+s+1)$$

Therefore the transfer function of a 3rd order Butterworth filter for cut-off frequency $\Omega_c = 1$ rad/sec is

$$\boxed{H(s) = \frac{1}{(s+1)(s^2+s+1)}}$$

The roots on LHS of s-plane can be found by the formula

$$S_K = e^{j\phi_K}$$

$$\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N} \qquad K = 1,2,3 \ldots N$$

For $N=4$, The poles can be found $\quad S_K = e^{j(2K-1)\pi/2N}$

$K = 1,2,3,4,5,6,7,8$

$$S_1 = e^{j5\pi/8} = -0.3827 + j0.9239$$

$$S_2 = e^{j7\pi/8} = -0.9239 + j0.3827$$

$$S_3 = e^{j9\pi/8} = -0.9239 - j0.3827$$

$$S_4 = e^{j11\pi/8} = -0.3827 - j0.9239$$

Now the denominator of transfer function $H(s)$ is

$$\{(s+0.3827)^2 + (0.9239)^2\}\{(s+0.9239)^2 + (0.3827)^2\}$$

$$= (s^2 + 1.84776s + 1)(s^2 + 0.76536s + 1)$$

For 4th order Butterworth filter the transfer function for $\Omega_c = 1$ rad/sec is given by

$$H(s) = \frac{1}{(s^2+0.76536s+1)(s^2+1.84776s+1)}$$

list of Butterworth polynomials.

1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.8477s+1)$
5	$(s+1)(s^2+0.618035+1)(s^2+1.61803s+1)$
6	$(s^2+1.931855s+1)(s^2+\sqrt{2}s+1)(s^2+0.517645+1)$
7	$(s+1)(s^2+1.80194s+1)(s^2+1.247s+1)(s^2+0.445s+1)$

The transfer function of Butterworth filter can be obtained by substituting $s \to s/\Omega_c$ in the transfer function of Butterworth filter.

$$\boxed{H_a(s) = H_N(s)\Big|_{s\to s/\Omega_c}}$$

If the maximum passband attenuation in positive dB is α_p (<3dB) at passband frequency Ω_p and Ω_s is the minimum stopband attenuation in positive dB at the stopband frequency Ω_s.

Now the magnitude function can be written as

$$|H(j\Omega)| = \frac{1}{\left[1+\varepsilon^2(\Omega/\Omega_p)^{2N}\right]^{1/2}}$$

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}}$$

Taking logarithm on both sides,

$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\varepsilon^2(\Omega/\Omega_p)^{2N}\right]$$

at $\Omega=\Omega_p$, the attenuation equal to α_p.

$$20\log|H(j\Omega_p)| = 0 - 10\log(1+\varepsilon^2(1)^{2N})$$

$$\cancel{-}\alpha_p = \cancel{-}10\log(1+\varepsilon^2)$$

$$0.1\alpha_p = \log(1+\varepsilon^2)$$

taking antilog on both sides

$$10^{0.1\alpha_p} = 1+\varepsilon^2$$

$$\varepsilon^2 = (10^{0.1\alpha_p}-1)$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p}-1}$$

at $\Omega=\Omega_s$, the maximum stopband attenuation is equal to

$$20\log|j\Omega_s| = -\alpha_s = -10\log\left[1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}\right]$$

$$\alpha_s = 10\log\left[1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}\right]$$

$$0.1\alpha_s = \log\left[1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}\right]$$

taking antilog on bothsides,

$$10^{0.1\alpha_s} = 1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{\varepsilon^2}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{N} = \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}$$

Apply logarithm on both sides

$$N\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right) = \log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}$$

$$N = \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log_{10}(\Omega_s/\Omega_p)}$$

$$\boxed{N = \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log(\Omega_s/\Omega_p)}}$$

Since this expression normally doesn't result in an integer value, therefore roundoff 'N' to the next highest integer.

$$\boxed{N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left[\Omega_s/\Omega_p\right]}}$$

where

$$\varepsilon = \sqrt{10^{0.1\alpha_p}-1}$$

$$\lambda = \sqrt{10^{0.1\alpha_s}-1}$$

$$\boxed{N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)}}$$

For simplicity of notation, we define the parameters A & k as follows

$$A = \frac{\lambda}{\varepsilon} = \left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]^{1/2} \qquad k = \frac{\Omega_p}{\Omega_s}$$

$$\therefore \boxed{N \geq \frac{\log A}{\log(1/k)}}$$

Cut-off frequency (Ω_c) :-

The magnitude square function of Butterworth analog lowpass filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \quad \text{—} ①$$

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}} \quad \text{—} ②$$

By comparing equation ① & ②

$$\cancel{1} + \left(\frac{\Omega}{\Omega_c}\right)^{2N} = \cancel{1} + \varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\left(\frac{\Omega}{\Omega_c}\right)^{2N} = \varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \varepsilon^2$$

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{0.1\alpha_p} - 1$$

$$\left(\frac{\Omega_p}{\Omega_c}\right) = \left[10^{0.1\alpha_p} - 1\right]^{1/2N}$$

$$\boxed{\Omega_c = \frac{\Omega_p}{\left[10^{0.1\alpha_p} - 1\right]^{1/2N}}} \qquad \Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}}$$

we know that,

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

$$\Omega_s = \Omega_p\left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right]^{1/2N}$$

$$\Omega_s = \Omega_c\left[\cancel{10^{0.1\alpha_p} - 1}\right]^{1/2N}\left[\frac{10^{0.1\alpha_s} - 1}{\cancel{10^{0.1\alpha_p} - 1}}\right]^{1/2N}$$

$$\Omega_s = \Omega_c\left[10^{0.1\alpha_s} - 1\right]^{1/2N}$$

$$\boxed{\Omega_c = \frac{\Omega_s}{\left[10^{0.1\alpha_s} - 1\right]^{1/2N}}}$$

Therefore,

$$\Omega_c = \frac{\Omega_p}{\left[10^{0.1\alpha_p} - 1\right]^{1/2N}} = \frac{\Omega_s}{\left[10^{0.1\alpha_p} - 1\right]^{1/2N}}$$

Steps to design an Analog Butterworth lowpass filter :-

1. From the given specifications find the order of the filter N.

2. Round off it to the next highest integer.

3. Find the transfer function H(s) for $\Omega_c = 1$ rad/sec for the value of N.

4. calculate the value of cut-off frequency Ω_c.

5. Find the transfer function $H_a(s)$ for the above value of Ω_c by substituting s/Ω_c in H(s).

problem

Design an analog butterworth filter that has a -2dB passband attenuation at a frequency of 20rad/sec and atleast -10dB stopband attenuation at 30 rad/sec.

given
$\alpha_p = 2dB \quad \Omega_p = 20 rad/sec$
$\alpha_s = 10dB \quad \Omega_s = 30 rad/sec$

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\frac{\Omega_s}{\Omega_p}}$$

$$N \geq \frac{\log\sqrt{\frac{10^{0.1(10)}-1}{10^{0.1(2)}-1}}}{\log\left(\frac{30}{20}\right)}$$

$$N \geq \frac{\log\sqrt{\frac{10-1}{10^{0.2}-1}}}{\log\left(\frac{3}{2}\right)}$$

$N \geq 3.37$

$\boxed{N=4}$

the Normalized lowpass butterworth filter for $N=4$,

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

cut-off frequency (Ω_c)

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}}$$

$$= \frac{20}{\left(10^{0.1\times 2}-1\right)^{1/8}}$$

$$\Omega_c = 21.3868$$

$$H_a(s) = H_N(s)\Big|_{s\to s/\Omega_c}$$

$$= \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2+0.76537\left(\frac{s}{21.3868}\right)+1\right]\left[\left(\frac{s}{21.3868}\right)^2+1.8477\left(\frac{s}{21.3865}\right)+1\right]}$$

$$H_a(s) = \frac{21.3863\times 10^4}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

For the given specifications design an analog butterworth filter.

$0.9 \le |H(j\Omega)| \le 1$ for $0 \le \Omega \le 0.2\pi$, $|H(j\Omega)| \le 0.2$ for $0.4\pi \le \Omega \le \pi$.

Sol

$\Omega_p = 0.2\pi$, $\Omega_s = 0.4\pi$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9 \qquad \frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\sqrt{1+\varepsilon^2} = \frac{1}{0.9} \qquad \sqrt{1+\lambda^2} = \frac{1}{0.2}$$

$$1+\varepsilon^2 = \left(\frac{1}{0.9}\right)^2 \qquad 1+\lambda^2 = \left(\frac{1}{0.2}\right)^2$$

$$\underline{\underline{\varepsilon = 0.484}} \qquad \underline{\underline{\lambda = 4.898}}$$

$$N \ge \frac{\log(\lambda/\varepsilon)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \ge \frac{\log\left(\frac{4.898}{0.484}\right)}{\log\left(\frac{0.4\pi}{0.2\pi}\right)}$$

$N \ge 3.34$ i.e $\underline{\underline{N=4}}$

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.84776s+1)}$$

~~H(s)~~

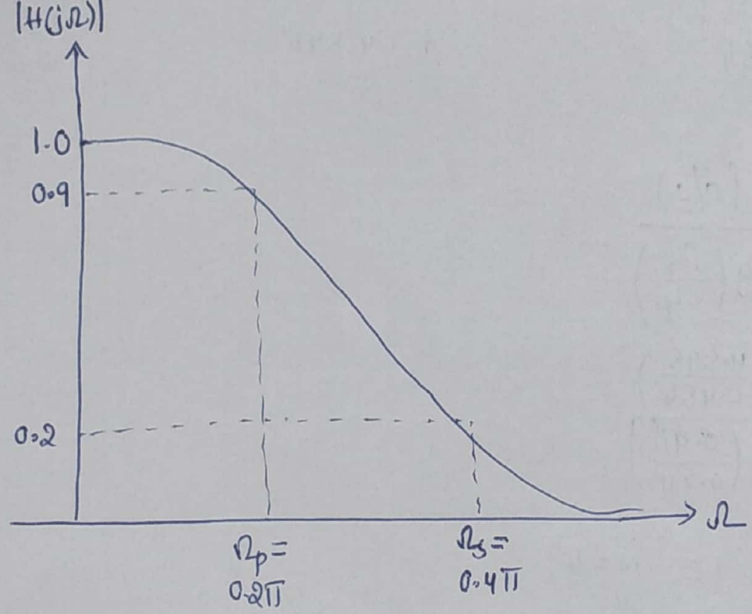
$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}$$

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/2N}} = \frac{0.2\pi}{(0.484)^{1/4}} = \underline{\underline{0.24\pi}}$$

$$H_a(s) = H(s)\Big|_{s \to s/\Omega_c} = H(s)\Big|_{s \to s/0.24\pi}$$

$$H_a(s) = \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2 + 0.76537\left(\frac{s}{0.24\pi}\right) + 1\right\}\left\{\left(\frac{s}{0.24\pi}\right)^2 + 1.8477\left(\frac{s}{0.24\pi}\right) + 1\right\}}$$

$$H_a(s) = \frac{0.323}{(s^2 + 0.577s + 0.0576\pi^2)(s^2 + 1.393s + 0.0576\pi^2)}$$



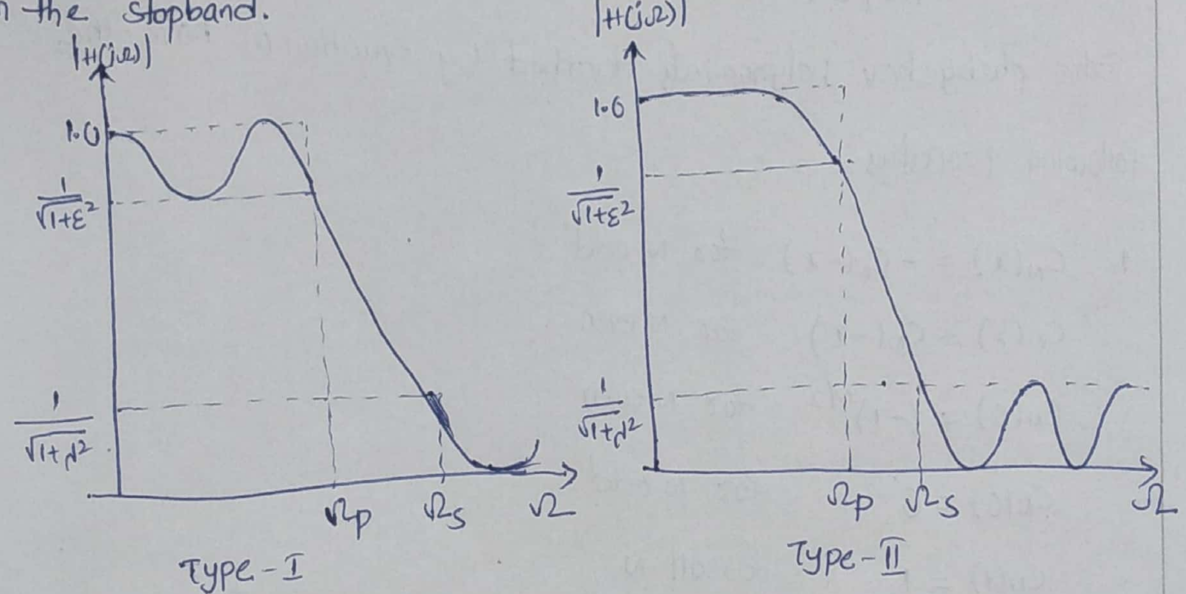
magnitude Response of filter.

Analog lowpass chebyshev filters :-

there are two types of chebyshev filters

→ Type I chebyshev filters are all pole filters that exhibit equiripple behaviour in the passband and a monotonic characteristics in the stopband.

→ Type II chebyshev filter contains both poles and zeros and exhibit a monotonic behaviour in the passband and an equiripple behaviour in the stopband.



characteristics of chebyshev filters

The magnitude square response of N^{th} order type I filter can be Expressed as

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)} \quad \text{—(1)} \quad N = 1, 2 \ldots$$

ε → parameter of the filter related to the ripple in the passband

$C_N(x)$ → is the N^{th} order chebyshev polynomial defined as

$C_N(x) = \cos(N\cos^{-1}x)$, $|x| \leq 1$ passband

$C_N(x) = \cosh(N\cosh^{-1}x)$, $|x| > 1$ stopband

the chebyshev polynomial is defined by the recussive formula

$$C_N(x) = 2x\,C_{N-1}(x) - C_{N-2}(x) \quad (2) \quad N > 1$$

where $C_0(x) = 1$,

$C_1(x) = x$

the chebyshev polynomials described by equation (2) have the following properties

1. $C_N(x) = -C_N(-x)$ for N odd

$C_N(x) = C_N(-x)$ for N even

$C_N(0) = (-1)^{N/2}$ for N even

$C_N(0) = 0$ for N odd

$C_N(1) = 1$ for all N

$C_N(-1) = 1$ for N even

$C_N(-1) = -1$ for N odd.

2. $C_N(x)$ oscillates with equal ripple between ± 1 for $|x| \leq 1$.

3. For all N $\quad 0 \leq |C_N(x)| \leq 1$ for $0 \leq |x| \leq 1$

$|C_N(x)| > 1$ for $|x| \geq 1$.

4. $C_N(x)$ is monotonically increasing for $|x| > 1$ for all N.

Below figure shows the equiripple characteristic of chebyshev filter. For odd values of N, the oscillatory curve starts from unity and for even values of N, the oscillatory curve starts from $\frac{1}{\sqrt{1+\varepsilon^2}}$

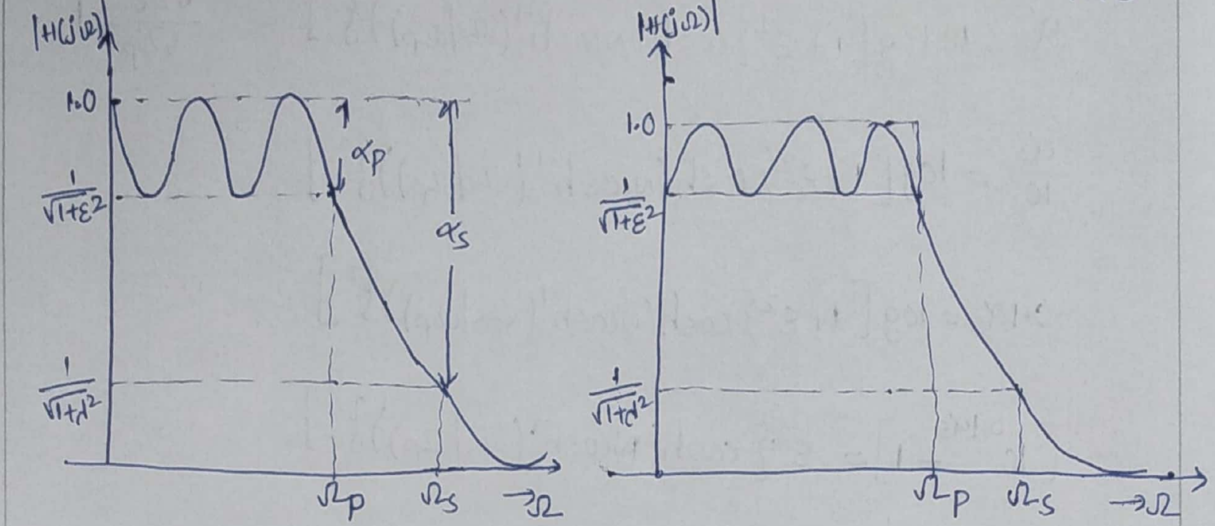


fig: lowpass chebyshev filter magnitude Response.

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)} \qquad N = 1, 2 \ldots$$

Taking logarithm for above equation

$$20\log|H(j\Omega)| = 10\log(1) - 10\log\left[1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)\right]$$

let α_p is the attenuation in positive dB at the passband freq (Ω_p)

α_s is the attenuation in positive dB at the stopband freq (Ω_s)

At $\Omega = \Omega_p$,

$$\alpha_p = 10\log(1+\varepsilon^2) \qquad C_N(1) = 1$$

$$0.1\alpha_p = \log(1+\varepsilon^2)$$

$$\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{1/2} \Rightarrow \underline{\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{1/2}}$$

At $\Omega = \Omega_s$,

$$\alpha_s = 10 \log\left[1 + \varepsilon^2 C_N^2 \left(\frac{\Omega_s}{\Omega_p}\right)\right]$$

$$\alpha_s = 10 \log\left[1 + \varepsilon^2 \{\cosh(N\cosh^{-1}(\Omega_s/\Omega_p))\}^2\right] \quad ; \quad \frac{\Omega_s}{\Omega_p} > 1$$

$$\frac{\alpha_s}{10} = \log\left[1 + \varepsilon^2 \{\cosh(N\cosh^{-1}(\Omega_s/\Omega_p))\}^2\right]$$

$$0.1\alpha_s = \log\left[1 + \varepsilon^2 \{\cosh(N\cosh^{-1}(\Omega_s/\Omega_p))\}^2\right]$$

$$\left[10^{0.1\alpha_s} - 1\right] = \varepsilon^2 \{\cosh(N\cosh^{-1}(\Omega_s/\Omega_p))\}^2$$

$$\{\cosh(N\cosh^{-1}(\Omega_s/\Omega_p))\}^2 = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \qquad \varepsilon^2 = 10^{0.1\alpha_p} - 1$$

$$\cosh(N\cosh^{-1}(\Omega_s/\Omega_p)) = \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}$$

$$N\cosh^{-1}(\Omega_s/\Omega_p) = \cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}$$

$$N = \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

Solving for N and rounding it to the next highest integer,

$$\boxed{N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}} \qquad \cosh^{-1}x = \ln\left[x + \sqrt{x^2 - 1}\right]$$

$$A = \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}$$

$$N \geq \frac{\cosh^{-1} A}{\cosh^{-1}(1/k)} \qquad k = \frac{\Omega_p}{\Omega_s}$$

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Steps to design an analog chebyshev lowpass filter :-

1. From the given specifications find the order of the filter N.

2. Round off it to the next highest integer.

3. Using the following formula's find the values of a and b, which are minor and major axis of the ellipse respectively.

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} \qquad b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

where

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2}+1}$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$$

Ω_p = passband frequency

α_p = maximum allowable attenuation in the passband.

(for normalized chebyshev filter $\Omega_p = 1$ rad/sec)

4. Calculate the poles of chebyshev filter which lie on an ellipse by using the formula.

$$S_k = a\cos\phi_k + jb\sin\phi_k \qquad k = 1, 2 \ldots N$$

$$\text{where } \phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi \qquad k = 1, 2 \ldots N$$

5. Find the denominator polynomial of the transfer function using the above poles

6. the numerator of the transfer function depends on the value of N

(a) For N odd substitute s=0 in the denominator polynomial and find the value. This value is equal to the numerator of the transfer function. (For N odd the magnitude response $|H(j\Omega)|$ starts at 1

(b) For N even substitute $s=0$ in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$. This value is equal to the numerator.

Given the specifications $\alpha_p = 3dB$, $\alpha_s = 16dB$, $f_p = 1KHz$ and $f_s = 2KHz$. Determine the order of the filter using chebyshev approximation. Find $H(s)$.

sol

$$\Omega_p = 2\pi \times f_p = 2\pi \times 1000 = 2000\pi \text{ rad/sec}$$

$$\Omega_s = 2\pi \times f_s = 2\pi \times 2000 = 4000\pi \text{ rad/sec}$$

$$\alpha_p = 3dB \qquad \alpha_s = 16dB.$$

step1:

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{1.6}-1}{10^{0.3}-1}}}{\cosh^{-1}\left(\frac{\overset{2}{\cancel{4000\pi}}}{\cancel{2000\pi}}\right)}$$

$$N \geq 1.91$$

step2: Rounding 'N' to next higher value, $\underline{\underline{N=2}}$

for N even, the oscillatory curve starts from $\frac{1}{\sqrt{1+\varepsilon^2}}$

step3: The values of minor axis and major axis can be found as below

$$\varepsilon = \left(10^{0.1\alpha_p}-1\right)^{0.5} = \left(10^{0.3}-1\right)^{0.5} = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 1+\sqrt{1+1} = \underline{\underline{2.414}}$$

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} = 2000\pi \frac{[(2.414)^{1/2} - (2.414)^{-1/2}]}{2}$$

$$= 910\pi$$

$$b = \Omega_p \frac{[\mu^{1/N} + \mu^{-1/N}]}{2} = 2000\pi \frac{[(2.414)^{1/2} + (2.414)^{-1/2}]}{2}$$

$$= 2197\pi$$

<u>step4</u>: the poles are given by

$$S_k = a\cos\phi_k + jb\sin\phi_k \qquad k = 1,2..$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1,2$$

$$\phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{2(2)} = \frac{\pi}{2} + \frac{\pi}{4} = 135^0$$

$$\phi_2 = \frac{\pi}{2} + \frac{(4-1)\pi}{2(2)} = \frac{\pi}{2} + \frac{3\pi}{4} = 225^0$$

$$S_1 = a\cos\phi_1 + jb\sin\phi_1 = -643.46\pi + j\,1554\pi$$

$$S_2 = a\cos\phi_2 + jb\sin\phi_2 = -643.46\pi + j\,1554\pi$$

<u>step5</u>: the denominator of $H(s)$

$$= (s + 643.46\pi)^2 + (1554)^2$$

<u>step6</u>: the numerator of $H(s) = \frac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\varepsilon^2}}$

$$= (1414.38)^2\pi^2$$

$$\therefore \text{Transfer function } H(s) = \frac{(1414.38)^2\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}$$

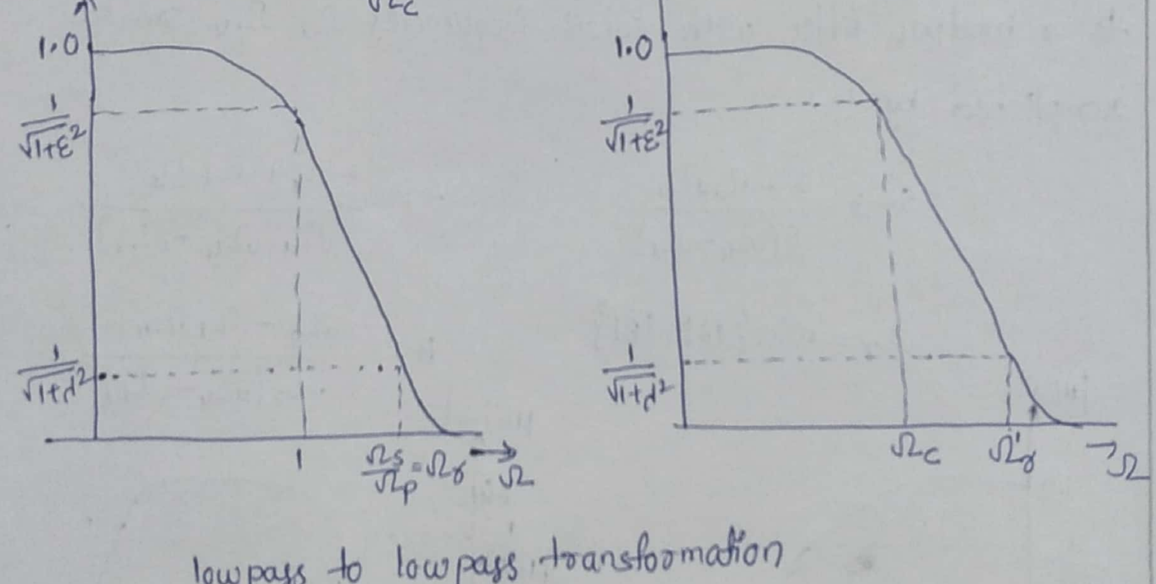
Frequency Transformation in Analog Domain :-

The frequency transformations can be used to design low pass filters with different passband frequencies, high pass filters, band pass filters and bandstop filters from a normalized lowpass analog filter ($\Omega_c = 1$ rad/sec).

lowpass to lowpass Filter :-

Given a normalized lowpass filter, it is desirable to have a low pass filter with a different cut-off frequency Ω_c or passband frequency Ω_p. This can be accomplished by the transformation given in below

$$s \longrightarrow \frac{s}{\Omega_c}$$



lowpass to lowpass transformation

lowpass to Highpass :-

Given a normalized low pass filter, it is desirable to have a highpass filter with cut-off frequency Ω_c. then the transformation is

$$s \longrightarrow \frac{\Omega_c}{s}$$

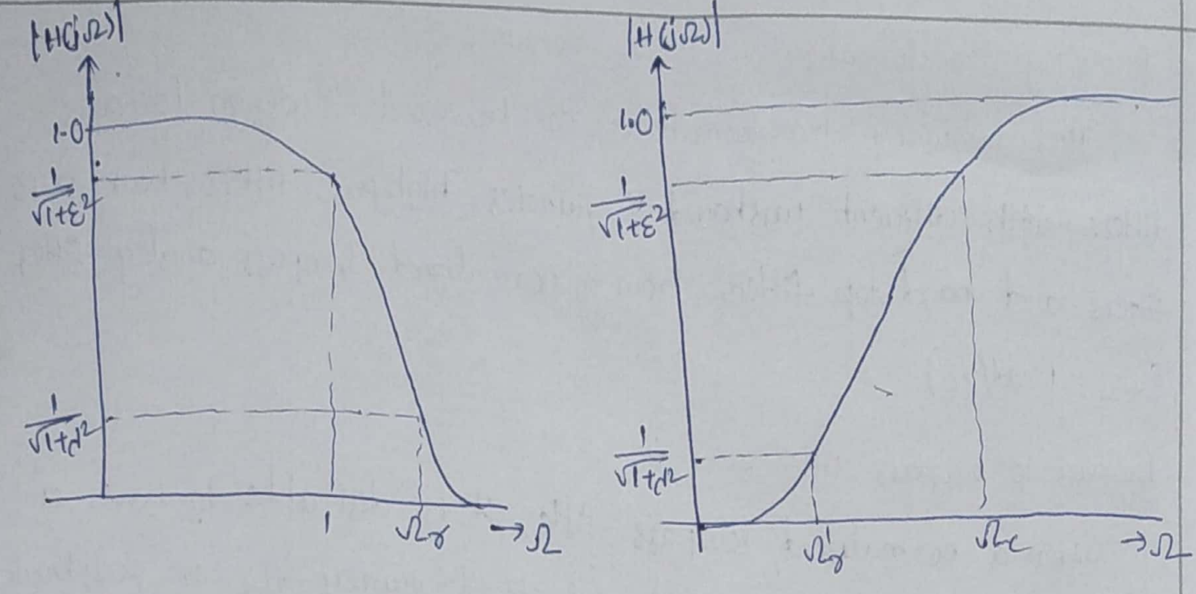


fig: lowpass to Highpass transformation.

lowpass to Bandpass :-

The transformation for converting a normalized lowpass filter to a bandpass filter with cutoff frequencies Ω_l, Ω_u can be accomplished by

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \qquad A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$$

$$\Omega_r = \min\{|A|, |B|\} \qquad B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$

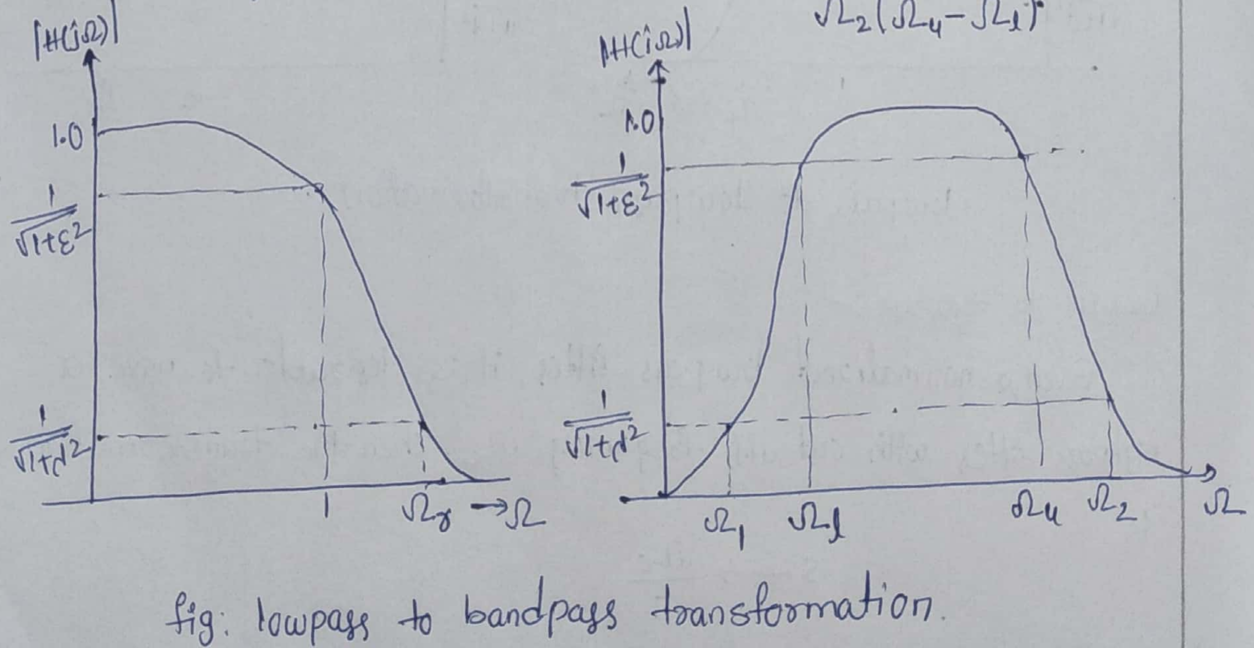


fig: lowpass to bandpass transformation.

lowpass to Bandstop :-

The transformation to convert a normalized lowpass filter to a bandstop filter is

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u} \qquad A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l \Omega_u}$$

$$\Omega_r = \min\{|A|, |B|\} \qquad B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l \Omega_u}$$

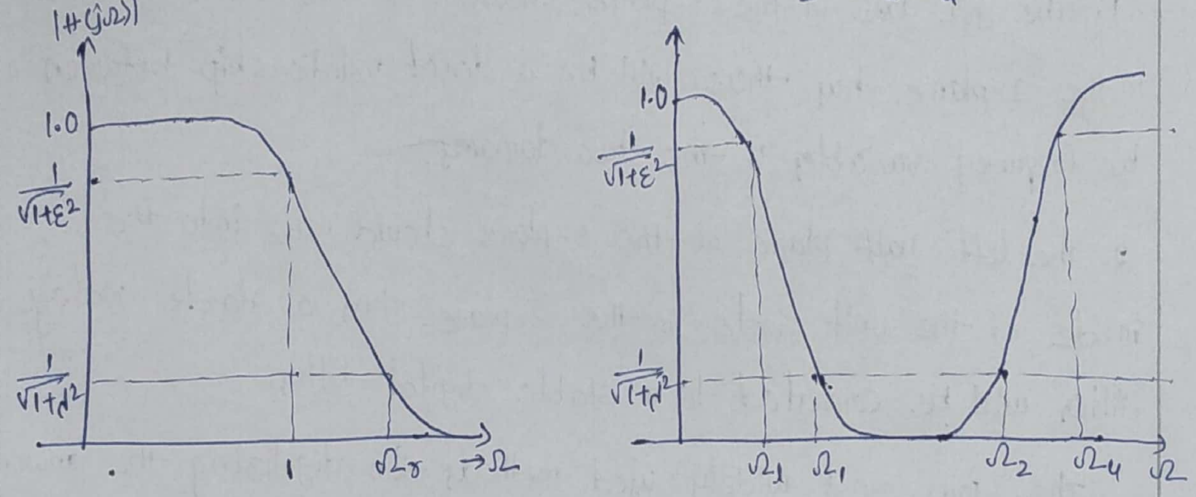


fig: low pass to Bandstop transformation.

Design of IIR filters from Analog filters :-

There are several methods that can be used to ~~find~~ design digital filters having an infinite duration unit sample response. The techniques described are all based on converting an analog filter into a digital filter. If the conversion technique is to be effective it should possess the following desirable properties.

1. The $j\Omega$ axis in the s-plane should map into the unit circle in the z-plane. thus there will be a direct relationship between two frequency variables in the two domains.

2. the left half plane of the s-plane should map into the inside of the unit circle in the z-plane. Thus a stable analog filter will be converted to a stable digital filter.

The four most widely used methods for digitizing the analog filter into a digitizing the analog filter into a digital filter include

1. Approximation of derivatives
2. The impulse invariant transformation
3. The bilinear transformation
4. the matched z-transformation technique.

Design of IIR Filter Using Impulse Invariance Technique :-

In impulse invariance method the IIR filter is designed such that the unit impulse response h(n) of digital filter is the sampled version of the impulse response of analog filter.

The z-transform of an infinite impulse response is given by

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \quad \text{— (1)}$$

$$H(z)\Big|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n)\, e^{-sTn}$$

let us consider the mapping of points from the s-plane to the z-plane implied by the relation

$$z = e^{sT}$$

If we substitute $s = \sigma + j\Omega$ and express the complex variable 'z' in polar form as $z = re^{j\omega}$ we get

$$re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$re^{j\omega} = e^{\sigma T} \cdot e^{j\Omega T}$$

which gives, $\boxed{r = e^{\sigma T}}$, $\boxed{\omega = \Omega T}$

→ $e^{\sigma T}$ has a magnitude of $e^{\sigma T}$ and an angle of 0 - a real number.

→ $e^{j\Omega T}$ has unity magnitude and an angle of ΩT.

∴ Analog pole is mapped to a place in the z-plane of magnitude $e^{\sigma T}$ and angle ΩT.

The real part of the analog pole determines the radius of the z-plane pole and the imaginary part of the analog pole dictates the angle of the digital pole.

Consider any pole on the $j\Omega$-axis, where $\sigma = 0$. These poles map to the z-plane at radius $r = e^{0 \cdot T} = 1$. Therefore, the impulse invariant mapping map poles from the s-plane's $j\Omega$ axis to the z-plane's unit circle.

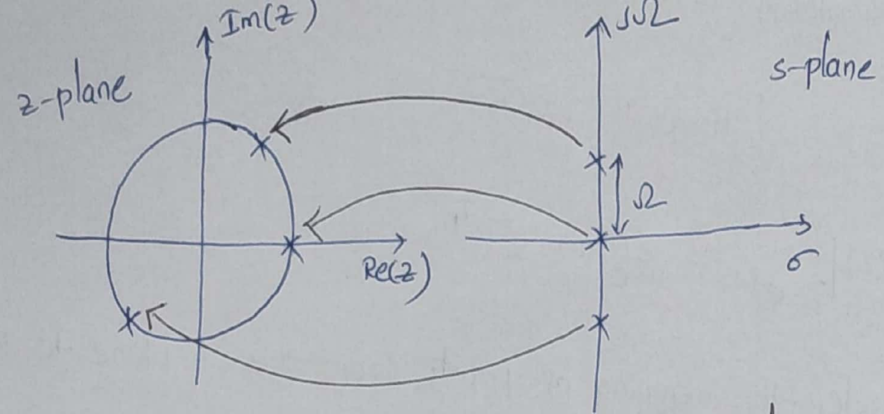


fig: $j\Omega$ axis mapping to the unit circle.

Now consider the poles in the left half of s-plane where $\sigma < 0$. These poles map inside the unit circle, because $r = e^{\sigma T} < 1$ for $\sigma < 0$.

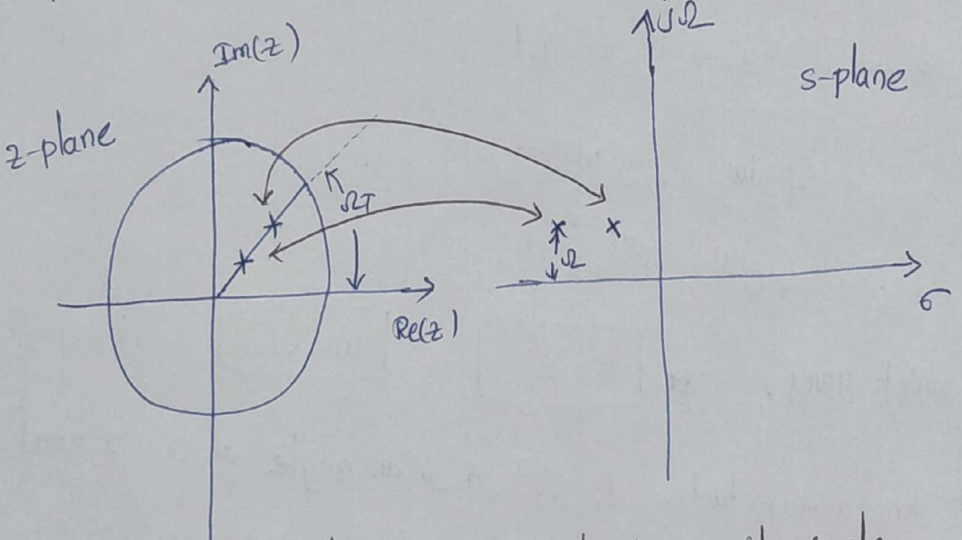


fig: stable poles mapping inside the unit circle.

Therefore, all s-plane poles with negative real parts map to z-plane poles inside the unit circle - stable analog poles are mapped to stable digital poles. The impulse invariant mapping preserves the stability of the filter.

All poles in the right half of the s-plane map to digital poles outside the unit circle.

$$\boxed{r = e^{\sigma T} > 1 \quad \text{for } \sigma > 0} \quad (2)$$

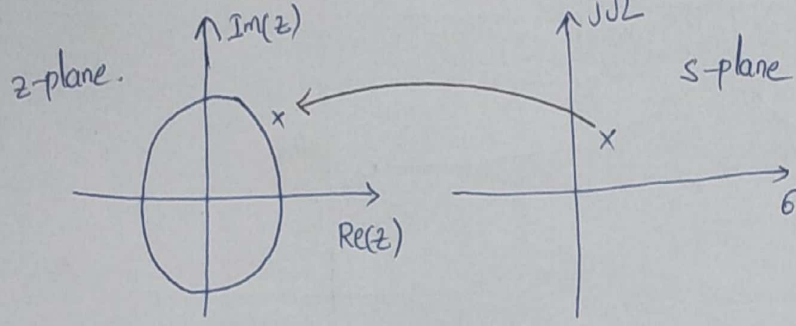


fig: unstable poles mapping, outside the unit circle.

Although the jΩ axis is mapped into the unit circle, it is not one to one mapping rather it is many to one mapping, where many points in the s-plane are mapped to a single point in the z-plane. the easiest way to explain this is to consider two poles in the s-plane with identical real parts, but with imaginary components differing by $\frac{2\pi}{T}$.

let the poles be

$$S_1 = \sigma + j\Omega$$

$$S_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right).$$

these poles map to z-plane poles z_1 and z_2, via impulse invariant mapping,

$$z_1 = e^{S_1 T} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T} \quad (3)$$

$$z_2 = e^{S_2 T} = e^{[\sigma + j(\Omega + \frac{2\pi}{T})]T} = e^{\sigma T} \cdot e^{j\Omega T + j2\pi}$$

$$= e^{\sigma T} e^{j\Omega T} e^{j2\pi} \quad (\because e^{j2\pi} = 1)$$

$$z_2 = e^{\sigma T} \cdot e^{j\Omega T} \quad (4)$$

where P_k are the poles of the analog filter

C_k are the coefficients in the partial fraction expansion.

Apply I.L.T on both sides

$$h_a(t) = \sum_{k=1}^{N} C_k e^{P_k t} \quad t \geq 0$$

If we sample $h_a(t)$ periodically at $t = nT$, we have

$$h(n) = h_a(nT)$$

$$= \sum_{k=1}^{N} C_k e^{P_k nT} \quad \cancel{t \geq 0}$$

$$= \sum_{k=1}^{N} C_k e^{P_k nT} \quad \text{--- (6)}$$

we know, $$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \quad \text{--- (7)}$$

Substitute equation (6) in (7).

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} C_k e^{P_k nT} z^{-n}$$

$$= \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} \left(e^{P_k T} z^{-1}\right)^n$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

i.e if $H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}$

then $H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$

for high sampling rates (for small T), the digital filter gain is high.

Therefore, $$H(z) = \sum_{k=1}^{N} \frac{T c_k}{1 - e^{p_k T} z^{-1}}$$

Due to the presence of aliasing, the impulse invariant method is appropriate for the design of lowpass and bandpass filters only. The impulse invariance method is unsuccessful for implementing digital filters such as a highpass filter.

Steps to design a digital filter using impulse invariance method :-

1. For the given specifications find $H_a(s)$, the transfer function of an analog filter.

2. Select the sampling rate of the digital filter, 'T' seconds per sample

3. Express the analog filter transfer function as the sum of single pole filters.

$$H_a(s) = \sum_{K=1}^{N} \frac{C_K}{s-P_K}$$

4. Compute the z-transform of the digital filter by using the formula

$$H(z) = \sum_{K=1}^{N} \frac{C_K}{1-e^{P_K T} z^{-1}}$$

For high sampling rates use

$$H(z) = \sum_{K=1}^{N} \frac{T C_K}{1-e^{P_K T} z^{-1}}$$

→ For the analog transfer function $H(s) = \dfrac{2}{(s+1)(s+2)}$ determine $H(z)$ using impulse invariance method. Assume $T = 1$ sec.

$$H(s) = \frac{2}{(s+1)(s+2)}$$

By using partial fractions

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$A = \cancel{(s+1)} \cdot \frac{2}{\cancel{(s+1)}(s+2)} \Bigg|_{s=-1} = \frac{2}{-1+2} = 2$$

$$B = \cancel{(s+2)} \frac{2}{(s+1)\cancel{(s+2)}} \Bigg|_{s=-2} = \frac{2}{-2+1} = -2$$

$$H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)}$$

$$H(s) = \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

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Using impulse invariance technique we have, if

$$H(s) = \sum_{k=1}^{N} \frac{c_k}{s-P_k}$$

Then $H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{P_k T} z^{-1}}$

i.e $(s-P_k)$ is transformed to $1-e^{P_k T} z^{-1}$.

there are two poles $P_1=-1$ & $P_2=-2$, so

$$H(z) = \frac{2}{1-e^{-T} z^{-1}} - \frac{2}{1-e^{-2T} z^{-1}}$$

for $T = 1\,sec$

$$H(z) = \frac{2}{1-e^{-1} z^{-1}} - \frac{2}{1-e^{-2} z^{-1}}$$

$$H(z) = \frac{2}{1-0.3678 z^{-1}} - \frac{2}{1-0.1353 z^{-1}}$$

$$H(z) = \frac{0.465 z^{-1}}{1-0.503 z^{-1} + 0.04976 z^{-2}}$$

Using impulse invariance with $T = 1sec$ determine $H(z)$ if $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$.

Given $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$

$$h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s^2+\sqrt{2}s+1}\right]$$

$$= L^{-1}\left[\frac{1}{(s+\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}\right]$$

$$= L^{-1}\left[\sqrt{2}.\ \frac{1/\sqrt{2}}{(s+\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}\right]$$

$$= \sqrt{2}\, L^{-1}\left[\frac{1/\sqrt{2}}{(s+\frac{1}{\sqrt{2}})^2+(1/\sqrt{2})^2}\right]$$

$$h(t) = \sqrt{2}\, e^{-t/\sqrt{2}} \sin(t/\sqrt{2})$$

let $t = nT$

$$h(nT) = \sqrt{2}\, e^{-nT/\sqrt{2}} \sin\left(\frac{nT}{\sqrt{2}}\right)$$

if $T = 1sec$

$$h(n) = \sqrt{2}\, e^{-n/\sqrt{2}} \sin\left(\frac{n}{\sqrt{2}}\right)$$

$$H(z) = z[h(n)] = z\left[\sqrt{2}.\, e^{-n/\sqrt{2}} \sin\left(\frac{n}{\sqrt{2}}\right)\right]$$

$$= \sqrt{2}\left[\frac{e^{-1/\sqrt{2}} z^{-1} \sin(1/\sqrt{2})}{1-2e^{-1/\sqrt{2}} \cos(1/\sqrt{2}) + e^{-\sqrt{2}} z^{-2}}\right]$$

$$H(z) = \frac{0.453z^{-1}}{1-0.7497z^{-1}+0.2432z^{-2}}$$

Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period T=1 sec.

Normalized transfer function of a 3rd order Butterworth filter is given by

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$H(s) = \frac{A}{s+1} + \frac{B}{(s+0.5+j0.866)} + \frac{C}{(s+0.5-j0.866)}$$

$$A = \cancel{(s+1)} \frac{1}{\cancel{(s+1)}(s^2+s+1)}\Bigg|_{s=-1} = \frac{1}{1-1+1} = \underline{1}$$

$$B = \cancel{(s+0.5+j0.866)} \frac{1}{(s+1)\cancel{(s+0.5+j0.866)}(s+0.5-j0.866)}\Bigg|_{s=-0.5-j0.866}$$

$$B = \frac{1}{(-0.5-j0.866+1)(-\cancel{0.5}-j0.866+\cancel{0}.5-j0.866)}$$

$$B = \frac{1}{(0.5-j0.866)(-j1.732)}$$

$$B = \frac{1}{-1.5-j0.866}$$

$$B = \frac{-1.5+j0.866}{(-1.5-j0.866)(-1.5+j0.866)}$$

$$B = -0.5+j0.288$$

$$C = B^* = -0.5-j0.288$$

$$H(s) = \frac{1}{(s+1)} + \frac{-0.5+0.288j}{s+0.5+j0.866} + \frac{-0.5-0.288j}{s+0.5-j0.866}$$

$$H(s) = \frac{1}{(s-(-1))} + \frac{-0.5+0.288j}{s-(-0.5-j0.866)} + \frac{-0.5-0.288j}{(s-(-0.5+j0.866)}$$

In impulse invariant Technique.

if $H(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k}$

Then $H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_k T} z^{-1}}$

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} + \frac{-0.5+j0.288}{1-e^{-0.5}e^{-j0.866}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5}e^{j0.866}z^{-1}}$$

$$H(z) = \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$$

Design of IIR filter using Bilinear Transformation :-

The bilinear transformation is a conformal mapping that transforms the $j\Omega$ axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Furthermore, all points in the LHP of 's' are mapped inside the unit circle in the z-plane and all points in the RHP of 's' are mapped into corresponding points outside the unit circle in the z-plane.

Let us consider an analog linear filter with system function

$$H(s) = \frac{b}{s+a} \quad \text{--- (1)}$$

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$sY(s) + aY(s) = bX(s)$$

this can be characterized by the differential equation

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t) \quad \text{--- (2)}$$

y(t) can be approximated by the trapezoidal formula

$$y(t) = \int_{t_0}^{t} y'(\tau)\,d\tau + y(t_0) \quad \text{--- (3)}$$

where $y'(t)$ denotes the derivative of $y(t)$.

The approximation of the integral in eq (3) by the trapezoidal formula at $t = nT$ and $t_0 = nT - T$ yields

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T) \quad \text{--- (4)}$$

from equation ②,

$$y'(nT) = -a\,y(nT) + b\,x(nT) \quad -⑤$$

substitute eq ⑤ in equation ④

$$y(nT) = \frac{T}{2}\left[-a\,y(nT) + b\,x(nT) - a\,y(nT-T) + b\,x(nT-T)\right] + y(nT-T)$$

$$y(nT) = -\frac{aT}{2}y(nT) + \frac{bT}{2}x(nT) - \frac{aT}{2}y(nT-T) + \frac{bT}{2}x(nT-T) + y(nT-T)$$

$$y(nT) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T) - y(nT-T) = \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

$$y(nT) + \frac{aT}{2}y(nT) - \left(1 - \frac{aT}{2}\right)y(nT-T) = \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

$$\left(1 + \frac{aT}{2}\right)y(nT) - \left(1 - \frac{aT}{2}\right)y(nT-T) = \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

with $y(n) = y(nT)$, $x(n) = x(nT)$

$$\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right].$$

Apply Z-Transform to above difference equation.

$$\left(1 + \frac{aT}{2}\right)Y(z) - \left(1 - \frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}\left[X(z) + z^{-1}X(z)\right]$$

$$\left[\left(1 + \frac{aT}{2}\right) - \left(1 - \frac{aT}{2}\right)z^{-1}\right]Y(z) = \frac{bT}{2}\left[1 + z^{-1}\right]X(z)$$

the system function of the digital filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}(1 + z^{-1})}{1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right)z^{-1}} \quad -⑥$$

$$H(z) = \frac{\frac{bT}{2}(1+z^{-1})}{1+\frac{aT}{2} - z^{-1} + \frac{aT}{2}z^{-1}}$$

$$H(z) = \frac{\frac{bT}{2}(1+z^{-1})}{(1-z^{-1}) + \frac{aT}{2}(1+z^{-1})}$$

dividing numerator and denominator by $\frac{T}{2}(1+z^{-1})$ we get

$$H(z) = \frac{\frac{\cancel{bT/2}\cancel{(1+z^{-1})}}{\cancel{\frac{T}{2}}\cancel{(1+z^{-1})}}}{\frac{(1-z^{-1})}{\frac{T}{2}(1+z^{-1})} + \frac{\cancel{aT/2}\cancel{(1+z^{-1})}}{\cancel{\frac{T}{2}}\cancel{(1+z^{-1})}}}$$

$$H(z) = \frac{b}{\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + a} \quad \text{—(7)}$$

comparing equation ① & ⑦, mapping from s-plane to the z-plane can be obtained as

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

this relationship between s & z is known as bilinear transformation.

let $z = re^{j\omega}$

$$s = \sigma + j\Omega$$

$$s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

$$s = \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right]$$

$$S = \frac{2}{T}\left[\frac{r(\cos w + j\sin w) - 1}{r(\cos w + j\sin w) + 1}\right]$$

$$S = \frac{2}{T}\left[\frac{r\cos w - 1 + jr\sin w}{r\cos w + 1 + jr\sin w}\right]$$

$$S = \frac{2}{T}\left[\left(\frac{r\cos w - 1 + jr\sin w}{r\cos w + 1 + jr\sin w}\right)\left(\frac{r\cos w + 1 - jr\sin w}{r\cos w + 1 - jr\sin w}\right)\right]$$

$$S = \frac{2}{T}\left[\frac{r^2\cos^2 w + \cancel{r\cos w} - \cancel{jr^2\sin w\cos w} - \cancel{r\cos w} - 1 + jr\sin w + \cancel{jr^2\sin w\cos w} + jr\sin w + r^2\sin^2 w}{(r\cos w + 1)^2 + r^2\sin^2 w}\right]$$

$$S = \frac{2}{T}\left[\frac{r^2\cos^2 w - 1 + r^2\sin^2 w + j2r\sin w}{r^2\cos^2 w + 1 + 2r\cos w + r^2\sin^2 w}\right]$$

$$S = \frac{2}{T}\left[\frac{r^2(\cos^2 w + \sin^2 w) - 1 + j2r\sin w}{r^2(\cos^2 w + \sin^2 w) + 1 + 2r\cos w}\right]$$

$$S = \frac{2}{T}\left[\frac{r^2 - 1 + j2r\sin w}{1 + r^2 + 2r\cos w}\right]$$

$$S = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos w} + j\,\frac{2r\sin w}{1 + r^2 + 2r\cos w}\right] \quad -(8)$$

compare above equation with $s = \sigma + j\Omega$

$$\boxed{\sigma = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos w}\right]} \qquad \boxed{\Omega = \frac{2}{T}\left[\frac{2r\sin w}{1 + r^2 + 2r\cos w}\right]} \quad -(9)$$

From equation (9), we find that

if $r \leq 1$ then $\sigma < 0$

if $r > 1$ then $\sigma > 0$

consequently the LHP in 's' maps into the inside of the unit circle in the z-plane and the RHP in the 's' maps into the outside ~~the~~ of the unit circle.

when $r = 1$, then $\sigma = 0$ and

$$\Omega = \frac{2}{T}\left[\frac{\sin\omega}{1+\cos\omega}\right]$$

$$\Omega = \frac{2}{T}\left[\frac{\cancel{2}\sin\frac{\omega}{2}\cancel{\cos\frac{\omega}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\omega}{2}}\right]$$

$$\boxed{\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)}$$

$$\frac{\Omega T}{2} = \tan\left(\frac{\omega}{2}\right)$$

$$\tan^{-1}\left(\frac{\Omega T}{2}\right) = \frac{\omega}{2}$$

$$\boxed{\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)}$$

The warping effect :-

let Ω & ω represent the frequency variables in the analog filter and the derived digital filter respectively.

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

for small value of ω,

$$\Omega = \frac{2}{T}\cdot\frac{\omega}{2} = \frac{\omega}{T}$$

$$\Rightarrow \boxed{\omega = \Omega T}$$

For low frequencies the relationship between Ω & ω are linear, as a result, the digital filter have the same amplitude response as the analog filter. For high frequencies the relationship between ω and Ω becomes non linear and distortion is introduced in the frequency scale of the digital filter to that of the analog filter. This is known as the warping effect.

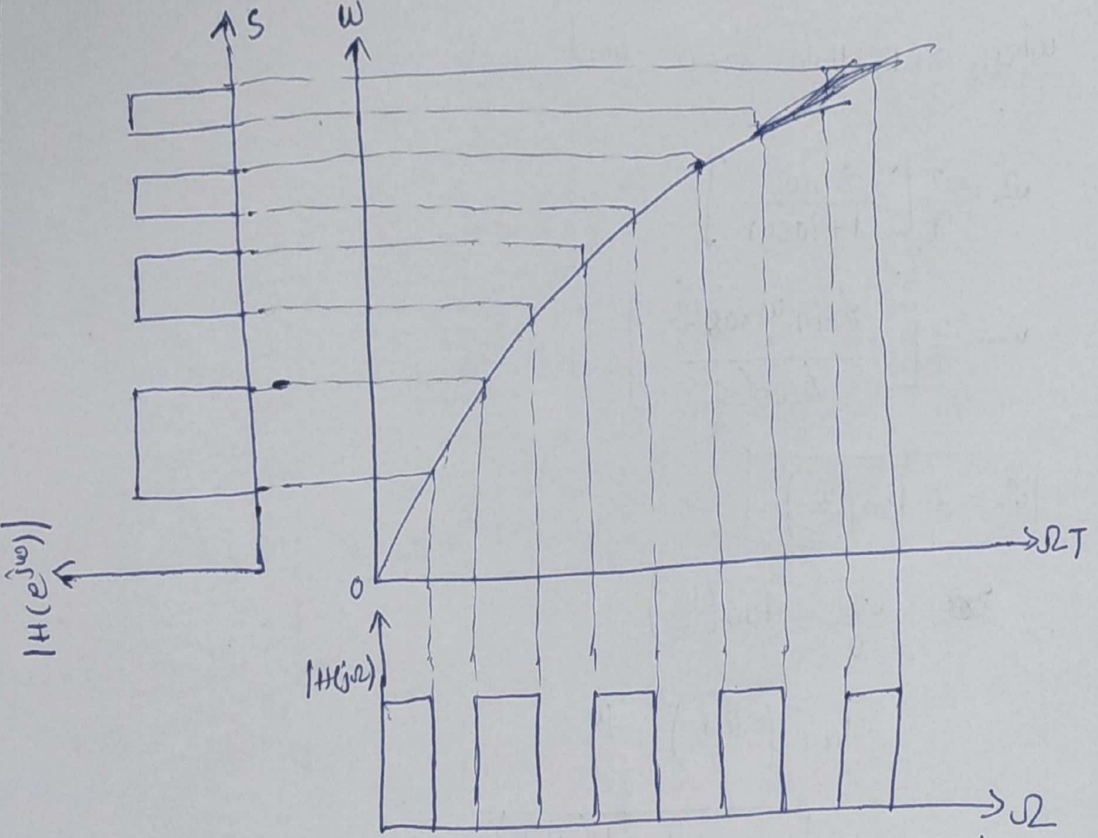

fig: the effect on magnitude response due to warping effect.

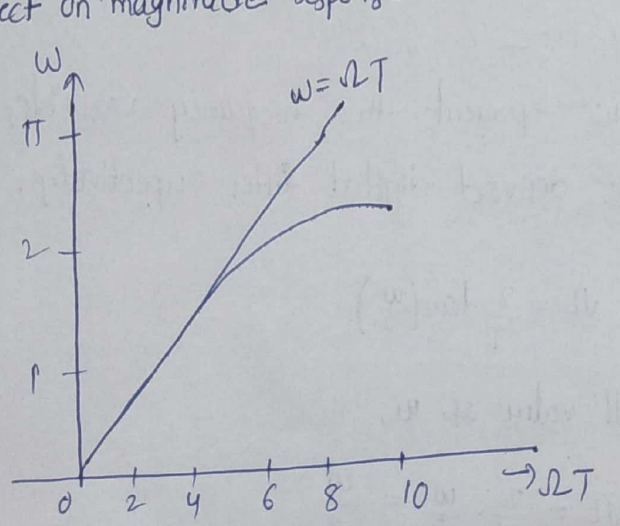

fig: Relationship between Ω & ω.

prewarping :-

The warping effect can be elimated by prewarping the analog filter. This can be done by finding prewarping analog frequencies using the formula.

$$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$$

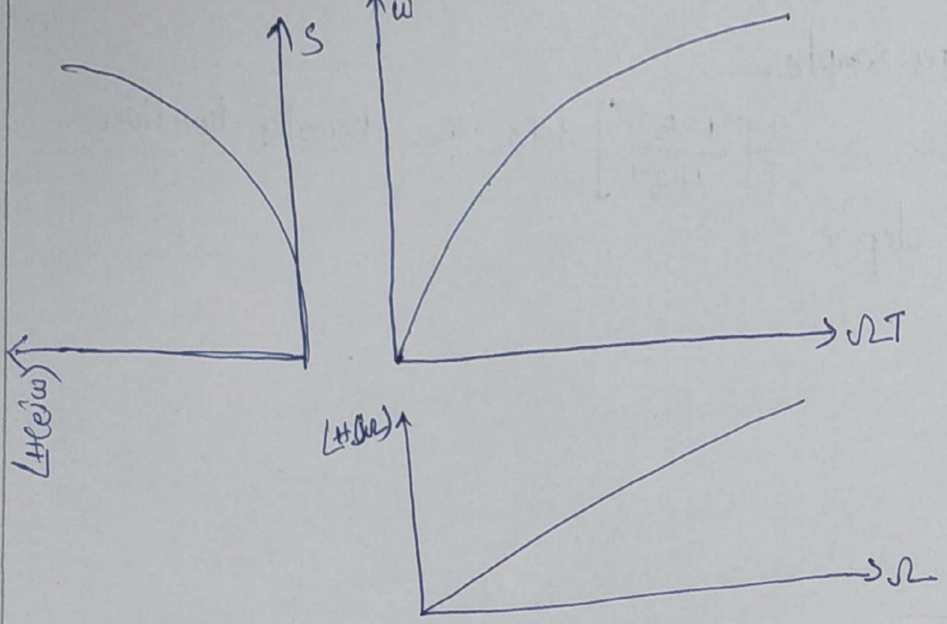

fig: the effect on phase response due to warping effect

therefore, we have

$$\Omega_p = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right)$$

$$\Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right).$$

steps to design digital filter using bilinear transformation technique:-

1. From the given specifications, find prewarping analog frequencies using formula $\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$

2. Using the analog frequencies find $H(s)$ of the analog filter.

3. select the sampling rate of the digital filter, call it T seconds per sample.

4. substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ into the transfer function found in step 2.

Determine $H(z)$ that results when the bilinear transformation is applied to $H_a(s) = \dfrac{s^2+4.525}{s^2+0.692s+0.504}$

sol

In bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

Assume $T = 1\,sec$

then

$$H(z) = \frac{\left[2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right]^2 + 4.525}{4\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2\right] + 0.692\left[2\times\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + 0.504}$$

$$= \frac{4(1-z^{-1})^2 + 4.525(1+z^{-1})^2}{4(1-z^{-1})^2 + 0.692\times2\times(1-z^{-1})(1+z^{-1}) + 0.504(1+z^{-1})^2}$$

$$= \frac{4(1-z^{-1})^2 + 4.525(1+z^{-1})^2}{4(1-z^{-1})^2 + 0.692\times2\times(1-z^{-2}) + 0.504(1+z^{-1})^2}$$

$$H(z) = \frac{1.4479 + 0.1783z^{-1} + 1.4479z^{-2}}{1 - 1.1875z^{-1} + 0.5299z^{-2}}$$

Apply bilinear transformation to $H(s) = \dfrac{2}{(s+1)(s+2)}$ with $T = 1\,sec$ and find $H(z)$.

<u>sol</u>

$$H(s) = \frac{2}{(s+1)(s+2)} \quad ; \; T = 1\,sec$$

substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(s)$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{2}{(s+1)(s+2)}\Bigg|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$H(z) = \frac{2}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right]\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right]}$$

$$H(z) = \frac{2(1+z^{-1})^2}{\left(2(1-z^{-1})+1+z^{-1}\right)\left(2(1-z^{-1})+2(1+z^{-1})\right)}$$

$$H(z) = \frac{2(1+z^{-1})^2}{\left(2-2z^{-1}+1+z^{-1}\right)\left(2-\cancel{2z^{-1}}+2+\cancel{2z^{-1}}\right)}$$

$$H(z) = \frac{\cancel{2}(1+z^{-1})^2}{(3-z^{-1})\cancel{4}\,2}$$

$$H(z) = \frac{(1+z^{-1})^2}{2(3-z^{-1})}$$

$$H(z) = \frac{(1+z^{-1})^2}{6\left[1-\frac{1}{3}z^{-1}\right]} = \frac{0.166\,(1+z^{-1})^2}{(1-0.33z^{-1})}$$

Frequency Transformation in Digital Domain.

A digital lowpass filter can be converted into a digital highpass, bandstop, bandpass or another digital filter. these transformations are given below.

lowpass to lowpass :-

$$z^{-1} \longrightarrow \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$$

$\omega_p \rightarrow$ passband frequency of LPF.

$\omega_p' \rightarrow$ pass band frequency of new LPF.

where $\alpha = \dfrac{\sin[(\omega_p-\omega_p')/2]}{\sin[(\omega_p+\omega_p')/2]}$

lowpass to Highpass :-

$$z^{-1} = -\left[\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right]$$

where $\alpha = -\dfrac{\cos[(\omega_p'+\omega_p)/2]}{\cos[(\omega_p'-\omega_p)/2]}$

$\omega_p \rightarrow$ passband freq of LPF.

$\omega_p' \rightarrow$ passband freq of HPF.

lowpass to Bandpass :-

$$z^{-1} \longrightarrow \frac{-\left[z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + \frac{k-1}{k+1}\right]}{\left[\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1\right]}$$

where $\alpha = \dfrac{\cos[(\omega_u-\omega_l)/2]}{\cos[(\omega_u-\omega_l)/2]}$

$$k = \cot\left[\frac{\omega_u-\omega_l}{2}\right]\tan\left(\frac{\omega_p}{2}\right)$$

$\omega_u \rightarrow$ upper cut-off frequency

$\omega_l \rightarrow$ lower cut-off frequency.

lowpass to Bandstop:-

$$z^{-1} \longrightarrow \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$$

where $\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$

$$k = \tan[(\omega_u - \omega_l)/2] \tan\left(\frac{\omega_p}{2}\right).$$

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Design a chebyshev filter for the following specifications using (a) bilinear transformation (b) Impulse invariance method.

$$0.8 \le |H(e^{jw})| \le 1 \qquad 0 \le w \le 0.2\pi$$

$$|H(e^{jw})| \le 0.2 \qquad 0.6\pi \le w \le \pi$$

Sol

(a) Bilinear transformation :-

$w_s = 0.6\pi$, $w_p = 0.2\pi$

$$\alpha_s = \frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

$$\alpha_p = \frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75$$

$$\Omega_p = \frac{2}{T}\tan\left(\frac{w_p}{2}\right) = 0.6498 \quad (\because T = 1\,sec)$$

$$\Omega_s = \frac{2}{T}\tan\left(\frac{w_s}{2}\right) = 2.752$$

$$N = \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}(1/k)} = 1.208$$

$$\underline{N = 2}$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.6498\left[\frac{3^{1/2} - 3^{-1/2}}{2}\right] = 0.3752$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.6498\left[\frac{3^{1/2} + 3^{-1/2}}{2}\right] = \underline{0.75}$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2$$

$$S_k = a\cos\phi_k + jb\sin\phi_k$$

$$S_1 = -0.2653 + j0.53$$

$$S_2 = -0.2653 - j0.53$$

$$H(s) = (s+0.2653)^2 + (0.53)^2$$

$$H(s) = s^2 + 0.5306s + 0.3516$$

For N even, Numerator of $H(s) = \dfrac{0.3516}{[1+(0.75)^2]^{1/2}} = 0.28$

$$H(s) = \frac{0.28}{s^2 + 0.5306s + 0.3516}$$

Using Bilinear Transformation

$$H(z) = H(z)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \qquad (\because T = 1\,\text{sec})$$

$$H(z) = \frac{0.28(1+z^{-1})^2}{5.4128 - 7.298z^{-1} + 3.29z^{-2}}$$

$$H(z) = \frac{0.052(1+z^{-1})^2}{1 - 1.3480z^{-1} + 0.608z^{-2}}$$

(b) By using impulse invariance method :-

$$\omega = \Omega T \Rightarrow \omega_p = \Omega_p T \quad \& \quad \omega_s = \Omega_s T.$$

for T = 1sec.

$$k = \frac{\omega_s}{\omega_p} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N = \frac{\cosh^{-1}(d/\varepsilon)}{\cosh^{-1}(1/k)} = \frac{\cosh^{-1}\left(\frac{4.899}{0.75}\right)}{\cosh^{-1}(3)} = 1.45$$

$\therefore$ N = 2 & $\mu = 3$.

$$a = \Omega_p \left(\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right)$$

$$a = 0.3627$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

$$b = 0.7255$$

$\phi_1 = 135^0$ & $\phi_2 = 225^0$

$$S_1 = -0.2564 + j\,0.513$$

$$S_2 = -0.2564 - j\,0.513$$

Numerator of $H(s) = 0.264$

$$H(s) = \frac{0.264}{s^2 + 0.513\,s + 0.33}$$

$$H(s) = \frac{(0.5146)(0.513)}{(s+0.2564)^2 + (0.513)^2}$$

Taking Inverse laplace transform,

$$h(t) = 0.5146\, e^{-0.2564} \sin(0.513t)$$

let $t = nT$,

$$h(nT) = 0.5146\, e^{-0.2564} \sin(0.513nT)$$

the z-transform

$$H(z) = \frac{0.5146\, e^{-0.2564T} z^{-1} \sin(0.513T)}{1 - 2e^{-0.2564T} z^{-1} \cos(0.513T) + e^{-0.513T} z^{-2}}$$