LAPLACE TRANSFORM

* Fourier Transform represents continuous time signal in terms of complex sinusoidals, i.e. $e^{j\omega t}$.
* Laplace Transform represents continuous time signals in terms of complex exponentials, i.e. e^{-st}. Hence Laplace transform can be used to analyze the signals or functions which are not absolutely integrable.

Types of Laplace Transform

1) Bilateral or two sided Laplace transform
2) Unilateral or one sided Laplace transform

Definition of Laplace Transform

Laplace Transform of continuous time signal $x(t)$ is given by

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad \text{[Bilateral Laplace Transform]}$$

where 's' is the complex variable and is given by $s = \sigma + j\omega$

here 'σ' is real part of 's'. It is called attenuation const. and '$j\omega$' is called imaginary part of 's' and is called complex frequency

Unilateral Laplace Transform is given as

$$X(s) = \int_{0}^{\infty} x(t) . e^{-st} dt$$

The inverse Laplace transform is given as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

Relationship between Fourier Transform and Laplace Transform

Fourier Transform is given as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{—— (1)}$$

Fourier Transform can be calculated only if $x(t)$ is absolutely integrable

i.e., $$\int_{-\infty}^{\infty} |x(t)| dt < \infty \quad \text{—— (2)}$$

We know that

$$\cancel{s=j} \quad s = \sigma + j\omega \quad \text{—— (3)}$$

Laplace Transform of $x(t)$ is given as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad \text{--- (4)}$$

From eqn (3) & (4)

$$= \int_{-\infty}^{\infty} x(t) e^{-(\sigma+j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt \quad \text{--- (5)}$$

Comparing eqn (1) & (5), we find that L.T of $x(t)$ is basically the F.T of $x(t)e^{-\sigma t}$. If $s = j\omega$, i.e $\sigma = 0$; then the above eqn becomes.

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}$$

$$X(s) = X(j\omega) \quad \text{when } s = j\omega$$

Convergence of Laplace Transform

The range of values of σ for which the L.T converges is known as Region of convergence (ROC).

The necessary and sufficient condn for the existance of LT is absolute integrability of $x(t)e^{-\sigma t}$.

i.e., $X(s)$ exists if

$$\int_{-\infty}^{\infty} |x(t).e^{-\sigma t}| dt < \infty$$

Find the Laplace Transform and ROC of the following signals.

(1) $e^{-at} u(t)$ (2) $-e^{-at} u(-t)$

Soln: (1) $x(t) = e^{-at} u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$u(t) = 1$ for $t \geq 0$

$= 0$ for $t < 0$

$$= \int_{-\infty}^{\infty} e^{-at} . u(t) e^{-st} dt$$

we know that $s = \sigma + j\omega$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt$$

The above integral converges if $(s+a) > 0$

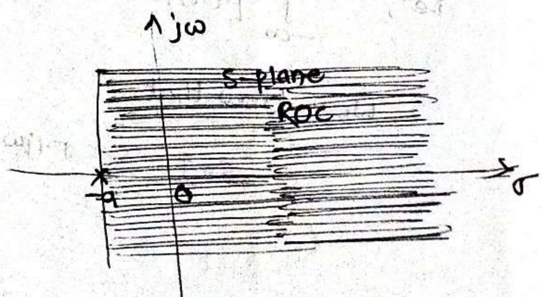
$\therefore$ ROC is $s + a > 0$

i.e $s > -a$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$= -\frac{1}{s+a} [0 - 1]$$

$$= \frac{1}{s+a}$$



② $x(t) = -e^{-at} u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$u(-t) = 1$ for $-t \geq 0$ or $t \leq 0$

$u(t) = 0$ for $t > 0$

$$= -\int_{-\infty}^{\infty} e^{-at} u(-t) e^{-st} dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t} dt = -\left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{-\infty}^{0}$$

$$= \frac{1}{s+a}\left[e^{-(s+a)0} - e^{-(s+a)\cdot\infty}\right]$$

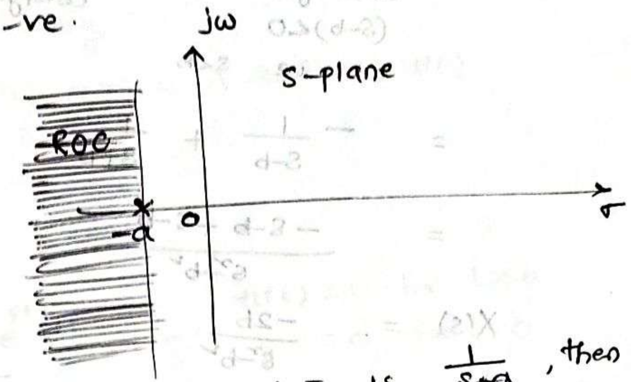
$$= \frac{1}{s+a}$$

$$\boxed{= \int_{0}^{\infty} e^{(s+a)t} dt = \left[\frac{e^{(s+a)t}}{s+a}\right]_{0}^{\infty} = e^{s+a}}$$

In the second term 't' tends to $-\infty$. Hence $s+a$ should be $-ve$ to make overall exponent $-ve$.

i.e ROC is $s+a < 0$

$s < -a$



Note: $e^{at} u(t)$ is causal signal and its L.T is $\frac{1}{s-a}$, then its ROC is $s > a$

$-e^{at} u(-t)$ is non causal signal and its L.T is also $\frac{1}{s-a}$

But its ROC is $s < a$.

⑧ Find the Laplace Transform of the signal $x(t) = e^{-at} u(t) + e^{-bt} u(-t)$

Soln $e^{-at} u(t) \rightarrow$ causal signal and L.T is $\frac{1}{s+a}$

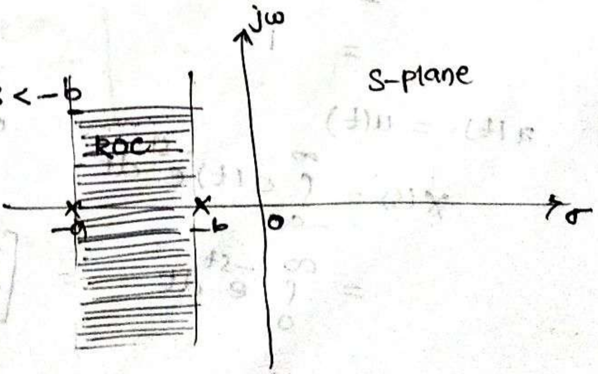
ROC is $s > -a$

$e^{-bt} u(-t) \rightarrow$ non causal signal and L.T is $-\frac{1}{s+b}$

ROC is $s < -b$

$$\therefore X(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

and ROC is $-a < s < -b$



4) Find the Laplace Transform of the signal

$x(t) = e^{-b|t|}$

Soln: $x(t) = e^{-b|t|}$

$$X(s) = \int_{-\infty}^{\infty} x(t).e^{-st}\,dt$$

$$= \int_{-\infty}^{0} x(t)e^{-st}\,dt + \int_{0}^{\infty} x(t)e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{bt}e^{-st}\,dt + \int_{0}^{\infty} e^{-bt}e^{-st}\,dt$$

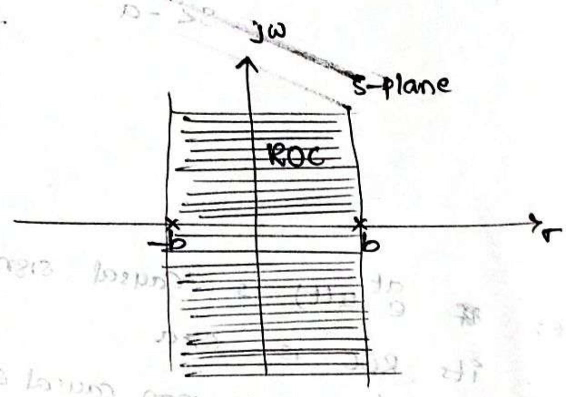
$$= \underbrace{\int_{-\infty}^{0} e^{-(s-b)t}\,dt}_{\text{Converges if } (s-b)<0 \text{ i.e. } s<b} + \underbrace{\int_{0}^{\infty} e^{-(s+b)t}\,dt}_{\text{Coverges if } s+b>0 \text{ i.e. } s>-b}$$

$$= -\frac{1}{s-b} + \frac{1}{s+b}$$

$$= \frac{-s-b+s-b}{s^2-b^2}$$

$$X(s) = \frac{-2b}{s^2-b^2}$$

ROC: $-b<s<b$



⑤ Find the Laplace transform of the following signals

i) $x(t) = \delta(t)$ ii) $x(t) = u(t)$ (iii) $x(t) = r(t)$

Soln: i) $x(t) = \delta(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} \delta(t)e^{-st}\,dt$$

$\delta(t) = 1$ for $t=0$

$= 0$ for $t \neq 0$.

$$= \delta(t)e^{-st}\Big|_{t=0}$$

$$= 1$$

ROC : ~~s>0~~ entire s-plane

ii) $x(t) = u(t)$

$u(t) = 1$ for $t \geq 0$

$= 0$ for $t<0$

$$X(s) = \int_{-\infty}^{\infty} u(t)e^{-st}\,dt$$

$$= \int_{0}^{\infty} e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{-1}{s}[0-1]$$

$$= \frac{1}{s}$$

ROC: $s>0$

iii) $x(t) = r(t)$ $\qquad r(t) = t$ for $t \geq 0$

$\qquad\qquad\qquad\qquad = 0$ for $t < 0$

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

$$= \int_{0}^{\infty} t \cdot e^{-st} dt \qquad \int u \cdot v = u\int v - \int u' \int v$$

$$\frac{t \cdot e^{-st}}{-s} - \int \frac{e^{-st}}{-s} dt$$

$$\left.\frac{te^{-st}}{-s}\right]_0^{\infty} - \left.\frac{e^{-st}}{s^2}\right]_0^{\infty}$$

$$0 - \frac{1}{s^2}[0-1]$$

$$X(s) = \frac{1}{s^2}$$

ROC: $s > 0$

(6) Find the L.T of the following

i) $x(t) = A \sin \omega_0 t \; u(t)$ $\qquad$ ii) $x(t) = A \cos \omega_0 (t) \, u(t)$

Soln: $x(t) = A \sin \omega_0 (t) \, u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$u(t) = 1$ for $t \geq 0$

$= 0$ for $t < 0$

$$= \int_{-\infty}^{\infty} A \sin \omega_0 t \; u(t) \, e^{-st} dt$$

$$= \int_{0}^{\infty} A \sin \omega_0 t \; e^{-st} dt$$

$$= A \int_{0}^{\infty} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \right) e^{-st} dt$$

$$= \frac{A}{2j} \left[\int_{0}^{\infty} e^{j\omega_0 t} e^{-st} dt - \int_{0}^{\infty} e^{-j\omega_0 t} e^{-st} dt \right]$$

We know that $L[e^{at} u(t)] = \frac{1}{s-a}$

$$\therefore = \frac{A}{2j} \left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0} \right]$$

$$= \frac{A}{2j} \left[\frac{\cancel{s} + j\omega_0 - \cancel{s} + j\omega_0}{s^2 + \omega_0^2} \right]$$

$$= \frac{A \cdot \cancel{2j}\omega_0}{\cancel{2j} \; s^2 + \omega_0^2}$$

$$X(s) = \frac{A\omega_0}{s^2 + \omega_0^2}$$

6) $x(t) = A\cos\omega_0 t\, u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$u(t) = 1$ for $t \geq 0$
$= 0$ for $t < 0$

$$= \int_{-\infty}^{\infty} A\cos\omega_0 t\, u(t) e^{-st} dt$$

$$= A\int_{0}^{\infty} \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] e^{-st} dt$$

$$= \frac{A}{2}\int_{0}^{\infty} e^{j\omega_0 t} e^{-st} dt + \frac{A}{2}\int_{0}^{\infty} e^{-j\omega_0 t} e^{-st} dt$$

$$= \frac{A}{2}\int_{0}^{\infty} e^{-(s-j\omega_0)t} + \frac{A}{2}\int_{0}^{\infty} e^{-(s+j\omega_0)t} dt$$

$$= \frac{A}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right]$$

$$= \frac{A}{2}\left[\frac{s+\cancel{j\omega_0} + s-\cancel{j\omega_0}}{s^2+\omega_0^2}\right]$$

$$X(s) = \frac{A \cdot s}{s^2+\omega_0^2}$$

7) Find the L.T of $x(t) = e^{-at}\sin\omega t$

Soln:
$$x(t) = e^{-at}\sin\omega t$$

$$= e^{-at} \cdot \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$x(t) = \frac{1}{2j}\left[e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right]$$

$$LT\{x(t)\} = X(s) = \frac{1}{2j} L\{e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\}$$

We know that $L\{e^{at}u(t)\} = \frac{1}{s-a}$
& $L\{e^{-at}u(t)\} = \frac{1}{s+a}$

$$\therefore L\{e^{-(a-j\omega)t}\} = \frac{1}{s+(a-j\omega)} = \frac{1}{(s+a)-j\omega}$$

lly $$L\{e^{-(a+j\omega)t}\} = \frac{1}{s+(a+j\omega)} = \frac{1}{(s+a)+j\omega}$$

$$\therefore X(s) = \frac{1}{2j}\left[\frac{1}{(s+a)-j\omega} - \frac{1}{(s+a)+j\omega}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{(s+a)} + j\omega - \cancel{(s+a)} + j\omega}{(s+a)^2 + \omega^2}\right]$$

$$= \frac{1}{2j}\left[\frac{+2j\omega}{(s+a)^2+\omega^2}\right]$$

$$X(s) = \frac{\omega}{(s+a)^2+\omega^2} \qquad \text{ROC: } s > -a$$

Ily. $$L\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$

Properties of Laplace Transform

① Linearity

If $L\{x_1(t)\} = X_1(s)$ & $L\{x_2(t)\} = X_2(s)$

Then $L\{a_1x_1(t) + a_2x_2(t)\} = a_1X_1(s) + a_2X_2(s)$

Proof: $$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$L\{a_1x_1(t) + a_2x_2(t)\} = \int_{-\infty}^{\infty}[a_1x_1(t) + a_2x_2(t)]e^{-st}\,dt$$

$$= a_1\int_{-\infty}^{\infty} x_1(t)e^{-st}\,dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-st}\,dt$$

$$= a_1X_1(s) + a_2X_2(s)$$

② Time shifting

Stmt: If $L\{x(t)\} = X(s)$

Then $L\{x(t-t_0)\} = e^{-st_0}X(s)$

Proof: $$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$L\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)e^{-st}\,dt$$

let $t - t_0 = m$

$t = m + t_0$

$dt = dm$

$$= \int_{-\infty}^{\infty} x(m)e^{-s(m+t_0)}\,dm$$

$$= \int_{-\infty}^{\infty} x(m)e^{-sm}\cdot e^{-st_0}\,dm$$

$$= e^{-st_0}\int_{-\infty}^{\infty} x(m)e^{-sm}\,dm$$

$$= e^{-st_0}X(s)$$

3) Shifting in s-domain

Stmt: If $L\{x(t)\} = X(s)$

Then $L\{e^{s_0 t} x(t)\} = X(s-s_0)$

Proof: $L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$L\{e^{s_0 t} x(t)\} = \int_{-\infty}^{\infty} e^{s_0 t} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(s-s_0)t} dt$$

$$= X(s-s_0)$$

4) Time scaling

Time Reversal

Stmt: If $L\{x(t)\} = X(s)$

Then $L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$

Proof: $L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$L\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-st} dt$$

let $at = m$

$t = \frac{m}{a}$ $\quad dt = \frac{dm}{a}$

$$= \int_{-\infty}^{\infty} x(m) e^{-s\frac{m}{a}} \frac{dm}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(m) e^{-\left(\frac{s}{a}\right)m} dm$$

$$= \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

5) Differentiation in Time domain

Stmt: If $L\{x(t)\} = X(s)$

Then $L\left\{\frac{d}{dt} x(t)\right\} = s.X(s)$

Proof: $x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$

Differentiate both sides w.r.t time

$$\frac{d}{dt} x(t) = \frac{1}{2\pi j} \frac{d}{dt} \left\{ \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds \right\}$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} s X(s) e^{st} ds$$

$$L\left\{\frac{d}{dt} x(t)\right\} = s.X(s)$$

If ~~all~~ initial conditions are given

Then $L\{\frac{d}{dt}x(t)\} = sX(s) - x(0)$

lly $L\{\frac{d^n}{dt^n}x(t)\} = s^n X(s) - s^{n-1}x(0) - s^{n-2}\frac{dx(0)}{dt} \cdots\cdots - \frac{d^{n-1}}{dt^{n-1}}x(0)$

for $n=2$

$$L\{\frac{d^2}{dt^2}x(t)\} = s^2 X(s) - s\,x(0) - \frac{d}{dt}x(0)$$

(6) Differentiation in s-domain

Stmt: If $L\{x(t)\} = X(s)$

Then $L\{-t\,x(t)\} = \frac{d}{ds}X(s)$

Proof: $L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

Diff above eqn w.r.t s

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left\{\int_{-\infty}^{\infty} x(t)e^{-st}\,dt\right\}$$

$$= \int_{-\infty}^{\infty} x(t)e^{-st}(-t)\,dt$$

$$\frac{d}{ds}X(s) = \int_{-\infty}^{\infty} [-t.x(t)]e^{-st}\,dt$$

$$\cancel{L\{\frac{d}{ds}X(s)\}} =$$

$$L\{-t.x(t)\} = \frac{d}{ds}X(s)$$

lly $L\{(-t)^n x(t)\} = \frac{d^n}{ds^n}X(s)$

(7) Convolution in Time domain

Stmt: If $L\{x_1(t)\} = X_1(s)$

& $L\{x_2(t)\} = X_2(s)$

Then $L\{x_1(t) * x_2(t)\} = X_1(s).X_2(s)$

Proof: $L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

$$L\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}[x_1(t) * x_2(t)]e^{-st}\,dt$$

But we know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$$

$$\therefore L\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau) e^{-st} d\tau\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(t-\tau) e^{-st} d\tau\, dt$$

let $t-\tau = m$

$t = m+\tau$

$dt = dm$

$$= \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(m)\, e^{-s(m+\tau)} d\tau\, dm$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-s\tau} d\tau \int_{-\infty}^{\infty} x_2(m)\, e^{-sm} dm$$

$$= X_1(s).\, X_2(s)$$

(8) Integration in Time domain

Stmt: If $L\{x(t)\} = X(s)$

Then $L\left\{\int_{-\infty}^{t} x(\tau) d\tau\right\} = \frac{X(s)}{s}$

Proof: Let us consider the convolution of $x(t)$ and $u(t)$

$$x(t) * u(t) = \int_{-\infty}^{\infty} u(t-\tau)\, x(\tau)\, d\tau$$

$$u(t-\tau) = \begin{matrix} 1 & \text{for } t \geq \tau & \text{or } \tau \leq t \\ 0 & \text{for } t < 0 & \end{matrix}$$

$$\therefore x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

Take L.T on both sides

$$L\{x(t) * u(t)\} = L\left\{\int_{-\infty}^{t} x(\tau) d\tau\right\}$$

$$L\left\{\int_{-\infty}^{t} x(\tau) d\tau\right\} = L\{x(t)\}\, L\{u(t)\}$$

$$= \frac{X(s)}{s}.$$

(9) Integration in s-domain

Stmt: If $L\{x(t)\} = X(s)$

Then $L\left\{\frac{x(t)}{t}\right\} = \int_{s}^{\infty} X(s) ds$

Proof: $\int_s^{\infty} X(s)\,ds = \int_s^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-st}\,dt\right]ds$

changing the order of integration and rearranging the terms

$$\int_s^{\infty} X(s)\,ds = \int_{-\infty}^{\infty} x(t)\left[\int_s^{\infty} e^{-st}\,ds\right]dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\lim_{s\to\infty}\frac{e^{-st}}{-t} - \frac{e^{-st}}{-t}\right]dt$$

$$= \int_{-\infty}^{\infty} \frac{x(t)}{-t}\left[0 - e^{-st}\right]dt$$

$$= \int_{-\infty}^{\infty} \frac{x(t)}{t}e^{-st}\,dt$$

$$L\left\{\frac{x(t)}{t}\right\} = \int_s^{\infty} X(s)\,ds$$

Properties of ROC

1. The ROC consists of strips of area parallel to $j\omega$-axis in s-plane.
2. ROC does not contain any poles.
3. ROC is entire s-plane for absolutely integrable and finite duration signals.
4. Let $x(t)$ is right sided signal, and the line $Re(s) = \sigma_0$ is in ROC. Then all values of s for which $Re\{s\} > \sigma_0$ will also be in ROC.
5. Let $x(t)$ is left sided signal, and the line $Re\{s\} = \sigma_0$ is in ROC, Then all values of s for which $Re\{s\} < \sigma_0$ will also be in ROC.
6. Let $x(t)$ is two sided signal and $Re\{s\} = \sigma_0$ is in the ROC. Then the ROC ~~is the strip parallel~~ will be a strip in the s-plane that includes line $Re\{s\} = \sigma_0$.
7. ROC is bounded by poles or extends to infinity.
8. If $x(t)$ is right sided, then ROC is region to the right side of right most pole. Similarly if $x(t)$ is left sided, then ROC is region in s-plane to left side of leftmost pole.

Determine the Laplace Transform of $x(t) = t.e^{-at}u(t)$

Soln: we know that $L\{e^{-at}u(t)\} = \frac{1}{s+a}$.

Using differentiation in S-domain

$$L\{-t.x(t)\} = \frac{d}{ds}X(s)$$

$$L\{-t.e^{-at}u(t)\} = \frac{d}{ds}\left[\frac{1}{s+a}\right]$$

$$= \frac{-1}{(s+a)^2}$$

$$\therefore L\{t.e^{-at}u(t)\} = \frac{1}{(s+a)^2}$$

lly $\frac{t^2}{2}e^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{(s+a)^3}$

$$\frac{t^3.e^{-at}u(t)}{3\times 2} \overset{L}{\longleftrightarrow} \frac{1}{(s+a)^4}$$

In General $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n}$

★ Find Laplace Transform for the following

i) $x(t) = \sinh(at)$ $\qquad \sinh(at) = \frac{e^{at}-e^{-at}}{2}$

ii) $x(t) = \cosh(at)$ $\qquad \cosh(at) = \frac{e^{at}+e^{-at}}{2}$

Intial Value Theorem

if $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ Then intial value of $x(t)$ is given as

$$x(0+) = \underset{t\to 0^+}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\left[sX(s)\right]$$

provided that the first derivative of $x(t)$ should be laplace transfomable.

Proof: From Time differentiation property of L.T,

$$L\{\tfrac{d}{dt}x(t)\} = s.X(s) - x(0^-)$$

Now take the limits as $s\to\infty$

$$\underset{s\to\infty}{Lt}\, L\{\tfrac{d}{dt}x(t)\} = \underset{s\to\infty}{Lt}\left[sX(s) - x(0^-)\right]$$

$$\underset{s\to\infty}{Lt}\int_0^\infty \frac{d}{dt}x(t).e^{-st}dt = \underset{s\to\infty}{lt}\left[x(t)\frac{e^{-st}}{s}\right]_0^\infty$$

$$= 0.$$

$$\therefore \lim_{s\to\infty}\left[sX(s) - x(0^-)\right] = 0$$

$$x(0^-) = \lim_{s\to\infty}\left[sX(s)\right]$$

$x(0^-) \rightarrow$ value of $x(t)$ just before $t=0$

$x(0^+) \rightarrow$ value of $x(t)$ just after $t=0$

If $x(t)$ is continuous at $t=0$, then its value just before and after $t=0$ is same

i.e., $x(0^-) = x(0^+)$

$$\therefore x(0^+) = \lim_{s\to\infty}\left[sX(s)\right]$$

Final Value Theorem

If $x(t) \xleftrightarrow{L} X(s)$ Then final value of $x(t)$ is given as

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0}\left[s.X(s)\right]$$

Proof: From the Time differentiation property

$$L\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0^-)$$

Now take the limits as $s\to 0$

$$\lim_{s\to 0} L\left\{\frac{d}{dt}x(t)\right\} = \lim_{s\to 0}\left[s.X(s) - x(0^-)\right]$$

$$= \lim_{s\to 0}\left[sX(s)\right] - x(0^-) \quad \text{—(1)}$$

LHS term is

$$\lim_{s\to 0} L\left\{\frac{d}{dt}x(t)\right\} = \lim_{s\to 0}\int_{0^-}^{\infty}\frac{d}{dt}x(t).\,e^{-st}\,dt$$

$$= \int_{0^-}^{\infty}\frac{d}{dt}x(t)\,dt$$

$$= \left[x(t)\right]_{0^-}^{\infty}$$

$$= x(\infty) - x(0^-) \quad \text{—(2)}$$

From (1) & (2)

$$\therefore \lim_{t\to\infty} x(t) - \cancel{x(0^-)} = \lim_{s\to 0}\left[s.X(s)\right] - \cancel{x(0^-)}$$

$$\therefore \lim_{t\to\infty} x(t) = \lim_{s\to 0}\left[s.X(s)\right]$$

* Determine the intial and final values of the function whose Laplace Transform is given as

$$X(s) = \frac{5s+50}{s(s+5)}$$

Soln: Intial value is given by

$$x(0^+) = \lim_{s\to\infty} s.X(s)$$

$$= \lim_{s\to\infty} \cancel{s}. \frac{5s+50}{\cancel{s}(s+5)} \qquad = \lim_{s\to\infty} \frac{\cancel{s}\left[5+\frac{50}{s}\right]}{\cancel{s}\left[1+\frac{5}{s}\right]}$$

$$= 5.$$

The final value is given by

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s.X(s)$$

$$= \lim_{s\to 0} \cancel{s}. \frac{5s+50}{\cancel{s}(s+5))}$$

$$= \frac{50}{5} = 10.$$

* Find the intial and final values of the function

$$F(s) = \frac{17s^3+7s^2+s+6}{s^5+3s^4+5s^3+4s^2+2s}$$

Soln: $$F(s) = \frac{17s^3+7s^2+s+6}{s(s^4+3s^3+5s^2+4s+2)}$$

Ans: inital value 0

Final value 3

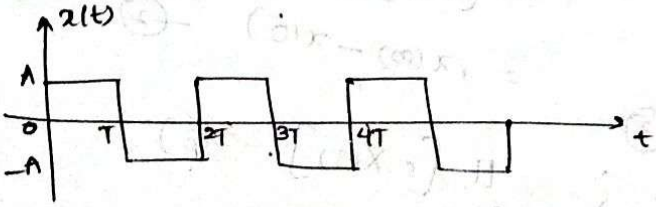
Laplace Transform of a periodic signal

let the L.T of the first cycle of Periodic function is $F_1(s)$. Then the L.T of the periodic function with period T is given as

$$F(s) = \frac{1}{1-e^{-sT}} F_1(s)$$

Problem: Find the Laplace Transform of the periodic square wave of amplitude range [-A, A] and time period 2T

Soln:



periodic square wave

$$x(t) = A \qquad 0<t<T$$

$$= -A \qquad T<t<2T$$

$$LT[x(t)] = \frac{1}{1-e^{-s(2T)}} \int_0^{2T} x(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-2sT}}\left[\int_0^T x(t)e^{-st}dt + \int_T^{2T} x(t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2sT}}\left[A\int_0^T e^{-st}dt + (-A)\int_T^{2T} e^{-st}dt\right]$$

$$= \frac{A}{1-e^{-2sT}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^T - \left[\frac{e^{-st}}{-s}\right]_T^{2T}\right\}$$

$$= \frac{A}{1-e^{-2sT}}\left\{\left(-\frac{1}{s}\right)\left[e^{-sT}-1\right] + \frac{1}{s}\left[e^{-2sT} - e^{-sT}\right]\right\}$$

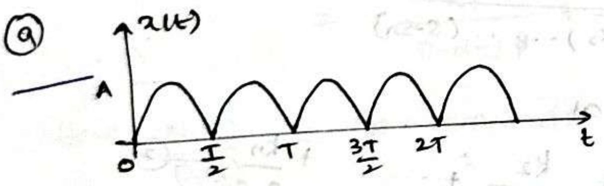
$$= \frac{A}{s}\cdot\frac{1}{1-e^{-2sT}}\left[e^{-2sT} - e^{-sT} - e^{-sT} + 1\right]$$

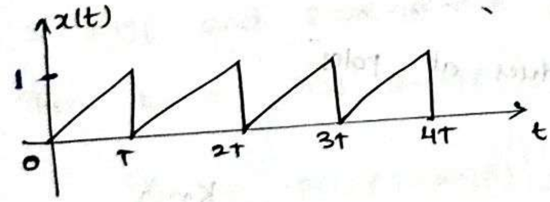
$$= \frac{A}{s}\cdot\frac{1}{1-e^{-2sT}}\left[e^{-2sT} - 2e^{-sT} + 1\right]$$

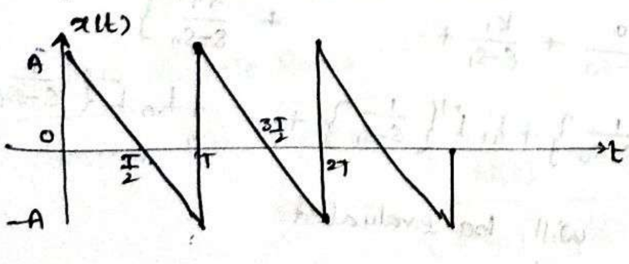
$$= \frac{A}{s}\cdot\frac{(-e^{-sT}+1)^2}{(1-e^{-sT})(1+e^{-sT})}$$

$$= \frac{A}{s}\cdot\left[\frac{1-e^{-sT}}{1+e^{-sT}}\right]$$

* Find the L.T of the wave form shown in Fig. below

(a) 

(b) 

(c) 

The time shift theorem is useful in determining the transform of periodic time functions. Let the f^n $x(t)$ be a causal periodic waveform which satisfies the $cond^n$ $x(t) = x(t+nT)$ for all $t>0$, where T is the period of the function and $n = 0, 1, 2, \ldots$

Inverse Laplace Transform can be found by using

1) partial fraction expansion method
2) Convolution integral method.

Any Laplace Transform can be expressed as rational function.

So i.e., $F(s) = \frac{N(s)}{D(s)}$ $N(s)$ - Numerator polynomial, $D(s)$ - Denominator polynomial

If degree of $N(s)$ is higher than $D(s)$, we should divide N(s) to obtain quotient and remainder term

For Ex: $F(s) = \frac{N(s)}{D(s)} = \frac{s^3+3s^2+3s+2}{s^2+2s+2}$

$$\therefore \quad F(s) = s+1 - \frac{s}{s^2+2s+2}$$

$$F(s) = (s+1) - F_1(s)$$

where $F_1(s) = \frac{s}{s^2+2s+2}$

Then $F_1(s)$ can be easily expanded into partial fractions. The Inverse L.T can be taken from standard transform pairs.

Use of pole zeros and ROCs for Inverse transform

let the roots of denominator polynomial be $s_0, s_1, s_2, \ldots s_n$. Then the T.F $F(s)$ can be expressed as

$$F(s) = \frac{N(s)}{(s-s_0)(s-s_1)(s-s_2)\ldots\ldots(s-s_n)} \quad \text{—— (1)}$$

eqn (1) can also be expressed as

$$F(s) = \frac{k_0}{s-s_0} + \frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \cdots + \frac{k_n}{s-s_n} \quad \text{—— (2)}$$

$k_0, k_1, k_2, \ldots k_n$ are residues at poles.

$$\therefore \quad f(t) = L^{-1}\{F(s)\}$$

$$= L^{-1}\left\{\frac{k_0}{s-s_0} + \frac{k_1}{s-s_1} + \cdots + \frac{k_n}{s-s_n}\right\}$$

$$= k_0 . L^{-1}\left\{\frac{1}{s-s_0}\right\} + k_1 L^{-1}\left\{\frac{1}{s-s_1}\right\} + \cdots + k_n L^{-1}\left\{\frac{1}{s-s_n}\right\}$$

Based on ROC $L^{-1}\left\{\frac{1}{s-s_0}\right\}$ will be evaluated.

If $Re\{s\} > a$ $e^{at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s-a}$

$$-e^{at}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{s-a}$$

Inverse Laplace Transform using Partial Fraction Expansion Method

F(S) can be expanded in partial fractions. Inverse Laplace Transform can be taken according to location of poles and ROC of F(S). The roots of D(S), i.e., poles can be simple and real, complex or multiple.

Case - I Simple and real roots

F(S) is expanded in partial fractions as

$$F(S) = \frac{k_0}{S-S_0} + \frac{k_1}{S-S_1} + \frac{k_2}{S-S_2} + \cdots + \frac{k_n}{S-S_n}$$

let $S_0, S_1, S_2, \ldots S_n$ are simple and real roots.

Then $k_0 = (S-S_0) \cdot F(S) \big|_{S=S_0}$

$k_1 = (S-S_1)\, F(S) \big|_{S=S_1}$

$k_2 = (S-S_2)\, F(S) \big|_{S=S_2}$

$\vdots$

$k_n = (S-S_n)\, F(S) \big|_{S=S_n}$

Case - II : Complex roots

If the roots are complex, then F(S) can be written as

$$F(S) = \frac{N(S)}{(S-\alpha-j\beta)(S-\alpha+j\beta)\, D_1(S)}$$

$$= \frac{k_1}{S-\alpha-j\beta} + \frac{k_2}{S-\alpha+j\beta} + \frac{N_1(S)}{D_1(S)}$$

Here $\frac{N_1(S)}{D_1(S)}$ is the remainder term other than complex.

$S=\alpha+j\beta$ and $S=\alpha-j\beta$ are complex roots

Then $k_1 = F(S) \cdot (S-\alpha-j\beta) \big|_{S=\alpha+j\beta}$

$k_2 = F(S)\, (S-\alpha+j\beta) \big|_{S=\alpha-j\beta}$

Case - III : Multiple Roots

Consider the function

$$F(S) = \frac{N(S)}{(S-S_0)^n\, D_1(S)}$$

$$* \quad F(s) = \frac{k_0}{(s-s_0)^n} + \frac{k_1}{(s-s_0)^{n-1}} + \frac{k_2}{(s-s_0)^{n-2}} + \cdots + \frac{k_{n-1}}{(s-s_0)^{1}} + \frac{N_1(s)}{D_1(s)}$$

where $k_0 = (s-s_0)^n . F(s) \Big|_{s=s_0}$

$$k_1 = \frac{d(s-s_0)^n F(s)}{ds} \Big|_{s=s_0}$$

$$k_2 = \frac{1}{2!} \frac{d^2}{ds^2} (s-s_0)^n F(s) \Big|_{s=s_0}$$

$$k_3 = \frac{1}{3!} \frac{d^3}{ds^3} (s-s_0)^n F(s) \Big|_{s=s_0}$$

$$\vdots$$

$$k_n = \frac{1}{n!} \frac{d^n}{ds^n} (s-s_0)^n F(s) \Big|_{s=s_0}$$

Inverse Laplace Transform using convolution Integral

If $L[f_1(t)] = F_1(s)$ & $L[f_2(t)] = F_2(s)$

Then Convolution theorem states that

$$L\{f_1(t) * f_2(t)\} = F_1(s) . F_2(s)$$

let $F(s) = F_1(s) . F_2(s)$

$$F(s) = L\{f_1(t) * f_2(t)\}$$

$$L^{-1}\{F(s)\} = f_1(t) * f_2(t)$$

Problems

① Find the inverse Laplace Transform using partial fraction expansion method for the following function.

$$X(s) = \frac{s^2+2s+2}{s(s+2)(s-3)} \quad ; \quad ROC : Re(s) > 3.$$

Soln:

$$X(s) = \frac{s^2+2s-2}{s(s+2)(s-3)}$$

$$X(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$A = s . X(s) \Big|_{s=0}$$

$$A = \frac{\cancel{s}(s^2+2s-2)}{\cancel{s}(s+2)(s-3)} \Big|_{s=0} = \frac{-2}{2(-3)} = \frac{1}{3}$$

$$B = (s+2) X(s) \Big|_{s=-2}$$

$$\frac{\cancel{(s+2)} . s^2+2s-2}{s\cancel{(s+2)}(s-3)} \Big|_{s=-2} = \frac{\cancel{4}-\cancel{4}-2}{-2(-5)} = \frac{-1}{5}$$

$$C = (s-3)\,X(s)\big|_{s=3}$$

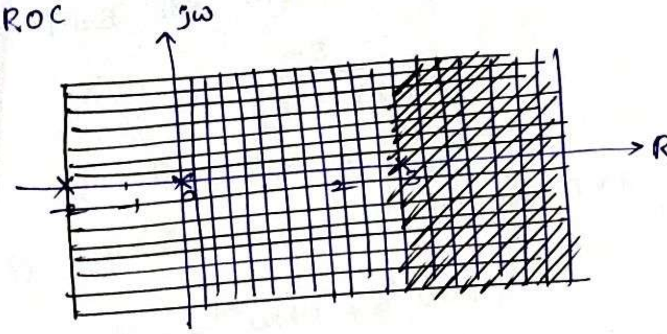
$$= \cancel{(s-3)}\,\frac{s^2+2s-2}{s(s+2)\cancel{(s-3)}}\Bigg|_{s=3} = \frac{9+6-2}{3(5)} = \frac{13}{15}$$

$$X(s) = \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{5}\cdot\frac{1}{s+2} + \frac{13}{15}\cdot\frac{1}{s-3}$$

$$x(t) = \frac{1}{3}\underset{Re(s)>0}{L^{-1}\left\{\frac{1}{s}\right\}} - \frac{1}{5}\underset{Re(s)>-2}{L^{-1}\left\{\frac{1}{s+2}\right\}} + \frac{13}{15}\underset{Re(s)>3}{L^{-1}\left\{\frac{1}{s-3}\right\}}$$

$$x(t) = \frac{1}{3}u(t) - \frac{1}{5}e^{-2t}u(t) + \frac{13}{15}e^{3t}u(t)$$

ROC



Common ROC is $Re(s) > 3$

Hence

$$x(t) = \frac{1}{3}u(t) - \frac{1}{5}e^{-2t}u(t) + \frac{13}{15}e^{3t}u(t)$$

② Determine the inverse Laplace Transform of the function

$$X(s) = \frac{1}{s^2+3s+2} \qquad ROC: -2 < Re(s) < -1$$

$-2 < s$

$s > -2$

Soln:

$$X(s) = \frac{1}{(s+1)(s+2)}$$

$$X(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

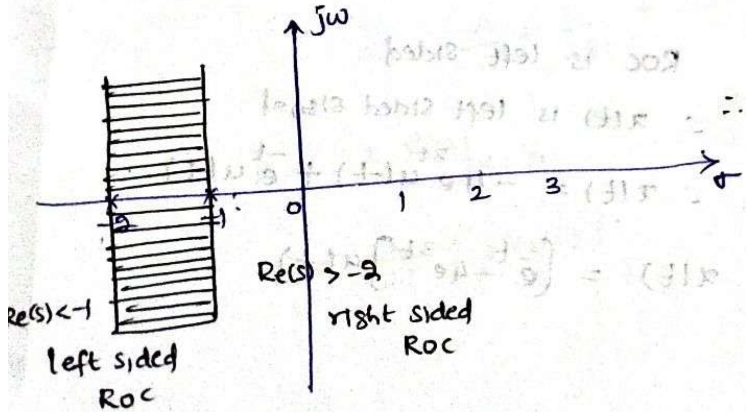
$$A = (s+1).\,X(s)\big|_{s=-1} = (s+1).\frac{1}{(s+1)(s+2)}\Bigg|_{s=-1} = 1$$

$$B = (s+2)\,X(s)\big|_{s=-2} = \cancel{(s+2)}\frac{1}{(s+1)\cancel{(s+2)}}\Bigg|_{s=-2} = -1$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$x(t) = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

Given ROC is $-2 < Re(s) < -1$



From the ROC

$Re(s) > -2$ and $Re(s) < -1$

∴ For $Re(s) > -2$ $x(t)$ is right sided i.e, causal

$$L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}u(t)$$

For $Re(s) < -1$ ROC is left sided

i.e., $x(t)$ non causal

$$\therefore L^{-1}\left\{\frac{1}{s+1}\right\} = -e^{-t}u(-t)$$

$$\therefore x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$$

(3) Find inverse Laplace Transform of the function

$$X(s) = \frac{3s+7}{(s^2-2s-3)}$$

For i) $Re(s) > 3$

ii) $Re(s) < -1$

iii) $-1 < Re(s) < 3$

Soln: $$X(s) = \frac{3s+7}{s^2-2s-3}$$

$$X(s) = \frac{3s+7}{(s-3)(s+1)}$$

$$X(s) = \frac{A}{s-3} + \frac{B}{s+1}$$

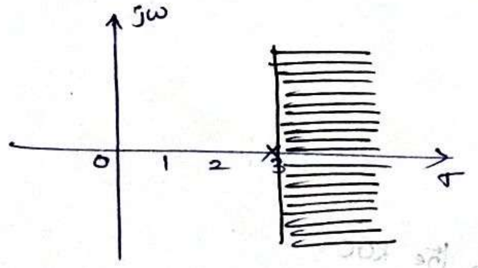
$$A = (s-3)\cdot X(s)\Big|_{s=3} = \cancel{(s-3)}\frac{3s+7}{\cancel{(s-3)}(s+1)}\Bigg|_{s=3} = 4$$

$$B = (s+1)\,X(s)\Big|_{s=-1} = \frac{\cancel{s+1)}\,(3s+7)}{\cancel{(s+1)}\,(s-3)} = \frac{4}{-4} = -1$$

$$X(s) = \frac{4}{s-3} - \frac{1}{s+1}$$

$$x(t) = 4.\,L^{-1}\left\{\frac{1}{s-3}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

i) $Re(s) > 3$

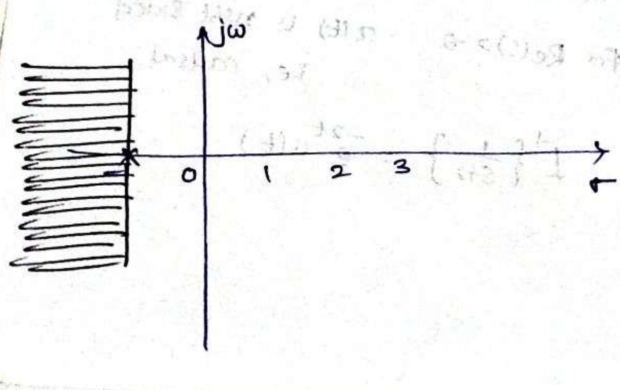


From the figure it is clear that Given ROC is right sided ROC

So $x(t)$ is right sided for two poles

$$\therefore x(t) = 4e^{3t}u(t) - e^{-t}u(t)$$

ii) $Re(s) < -1$



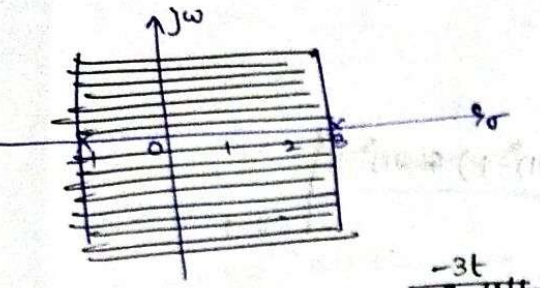
ROC is left sided

$\therefore x(t)$ is left sided signal

$$\therefore x(t) = -4e^{3t}u(-t) + e^{-t}u(-t)$$

$$x(t) = \left[e^{-t} - 4e^{3t}\right]u(-t)$$

iii) $-1 < Re(s) < 3$



For pole $s = -1$
ROC is right sided

For pole $s = 3$
ROC is left sided.

$\therefore \cancel{x(t)} = \cancel{4e^{-3t}u(t)} + \cancel{e^{-t}u(-t)}$

$$x(t) = -4e^{-3t}u(-t) - e^{-t}u(t)$$

(4) Find the inverse Laplace Transform of

$$X(s) = \frac{-3}{(s+2)(s-1)}$$

if the ROC is

i) $-2 < Re(s) < 1$ (ii) $Re(s) > 1$ (iii) $Re(s) < -2$

Ans. i) $e^{-2t}u(t) + e^{t}u(-t)$

ii) $e^{-2t}u(t) + e^{t}u(t)$

iii) $(e^{t} - e^{-2t})u(-t)$

(5) Obtain right sided time domain signal for the following function

$$X(s) = \frac{s^2+2s+1}{(s+2)(s^2+4)}$$

Hint $X(s) = \frac{A}{s+2} + \frac{B}{s+j2} + \frac{C}{s-j2}$

$A = \frac{1}{8}$, $B = 0.437 + j0.0625$

$B = 0.437 - j0.0625$

$\frac{1}{8}e^{-2t} + (0.437 + j0.0625)e^{-j2t} + (0.437 - j0.0625)e^{j2t}$

$= \frac{1}{8}e^{-2t} + (0.437 + j0.0625)(\cos 2t - j\sin 2t) + (0.437 - j0.0625)(\cos 2t + j\sin 2t)$

$= \frac{1}{8}e^{-2t} + 0.437\cos 2t - j\cancel{0.437\sin 2t} + j\cancel{0.0625\cos 2t} + 0.0625\sin 2t + 0.437\cos 2t + j\cancel{0.437\sin 2t} - j\cancel{0.0625}\cos 2t + 0.0625\sin 2t$

Ans: $\left[\frac{1}{8}e^{-2t} + 0.874\cos 2t + 0.25\sin 2t\right]u(t)$

(6) Find the inverse Laplace Transform of $X(s) = \frac{s-2}{s(s+1)^3}$ For ROC $Re(s) < -1$

Soln: $X(s) = \frac{s-2}{s(s+1)^3}$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$A = s \cdot X(s)\big|_{s=0} = \frac{s-2}{(s+1)^3} = -2$

$B = \frac{1}{2!}\frac{d^2}{ds^2} s\,X(s)\Big|_{s=-1}$

$$\frac{d}{ds}\, s.X(s) = \frac{d}{ds}\, s.\ \frac{s-2}{s(s+1)^3}$$

$$\frac{d}{ds} = \frac{(s+1)^3 - (1)}{(s+1)^6}$$

$$\frac{d^2}{ds^2} = \frac{3(s+1)^6\,(s+1)^2 - \left[(s+1)^3 - 1\right] 6(s+1)^5}{(s+1)^{12}}\Bigg|_{s=-1}$$

$$= 0 -$$

$$\frac{d}{ds}(s+1)^3 . \frac{s-2}{s(s+1)^3} = \frac{d}{ds}\ \frac{s-2}{s.(s+1)^2}$$

$$\frac{s(s+1)^2(1) - (s-2)\left[s\,2(s+1) + (s+1)^2\right]}{s^2(s+1)^4}$$

$$\frac{d}{ds}(s+1)^3 . X(s)$$

$$= \frac{d}{ds} . \frac{s-2}{s} = \frac{s(1) - (s-2)}{s^2} = \frac{2}{s^2}$$

$$= \frac{d^2}{ds^2} = \frac{s^2[0] - 2[2s]}{s^4} = \frac{-4}{s^3}$$

$$B = \frac{1}{2!}\frac{d^2}{ds^2}(s+1)^3 . X(s)\Big|_{s=-1} = \frac{1}{2!}\left(\frac{-4}{s^3}\right)_{s=-1}$$

$$= 2$$

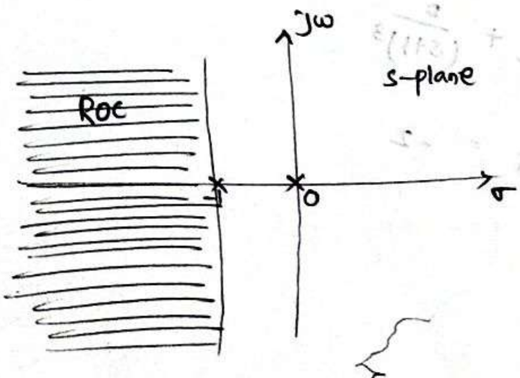
$$C = \frac{1}{1!}\frac{d}{ds}(s+1)^3 . X(s)\Big|_{s=-1}$$

$$= \frac{d}{ds}(s+1)^3 . \frac{s-2}{s(s+1)^3} = \frac{2}{s^2}\Big|_{s=-1} = 2$$

$$D = \frac{1}{0!}(s+1)^3 . X(s)\Big|_{s=-1} = \frac{s-2}{s} = \frac{-3}{-1} = 3.$$

$$X(s) = \frac{-2}{s} + \frac{2}{s+1} + \frac{2}{(s+1)^2} + \frac{3}{(s+1)^3}$$

Given ROC is $Re(s) < -1$



From the fig. It is clear the Time domain signal will be left sided.

we know that

$$-\frac{t^{n-1}}{(n-1)!} e^{-at} u(-t) \xleftrightarrow{L} \frac{1}{(s+a)^n} \quad \therefore \text{For } Re(s) < -a$$

$$-e^{-t} u(-t) \xleftrightarrow{L} \frac{1}{s+1}$$

$$-t e^{-at} u(-t) \xleftrightarrow{L} \frac{1}{(s+1)^2}$$

$$-\frac{t^2}{2} e^{-at} u(-t) \xleftrightarrow{L} \frac{1}{(s+1)^3}$$

$$-u(-t) \xleftrightarrow{L} \frac{1}{s}$$

$$\therefore x(t) = -2 L^{-1}\left[\frac{1}{s}\right] + 2 L^{-1}\left[\frac{1}{s+1}\right] + 2 L^{-1}\left[\frac{1}{(s+1)^2}\right] + 3 \cdot L^{-1}\left[\frac{1}{(s+1)^3}\right]$$

$$= +2u(-t) - 2e^{-t}u(-t) - 2te^{-t}u(-t) - \frac{3}{2}t^2e^{-t}u(-t)$$

$$= \left[2 - \frac{3t^2}{2}e^{-t} - 2te^{-t} - 2e^{-t}\right] u(-t)$$

(7) Using convolution integral find out $L^{-1} \frac{1}{s^2(s+2)}$

Soln: $F(s) = \frac{1}{s^2(s+2)}$

let $F_2(s) = \frac{1}{s^2}$ & $F_1(s) = \frac{1}{s+2}$

Then $F(s) = F_1(s) \cdot F_2(s)$

& $f(t) = f_1(t) * f_2(t)$

where $f_1(t) = L^{-1}\left[\frac{1}{s+2}\right]$ & $f_2(t) = L^{-1}\left\{\frac{1}{s^2}\right\}$

$f_1(t) = e^{-2t}$ $\qquad f_2(t) = t$

$$f(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau$$

$$= \int_0^t e^{-2(t-\tau)} \tau\, d\tau = \int_0^t e^{-2t} \cdot e^{2\tau} \tau\, d\tau$$

$$= e^{-2t} \int_0^t e^{2\tau} \tau\, d\tau$$

$$= e^{-2t} \left[\tau \frac{e^{2\tau}}{2} - 1 \cdot \frac{e^{2\tau}}{4}\right]_0^t$$

$$= e^{-2t} \left[\frac{1}{2}[te^{2t} - 0) - \frac{1}{4}(e^{2t} - e^0) \right)$$

$$= \frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{-2t}$$

Solution of Differential equations

The unilateral L.T. is used to solve differential equations with initial conditions.

$$L\left\{\frac{d}{dt}x(t)\right\} = s.X(s) - x(0^-)$$

* Solve the differential equation,

$$\frac{dy(t)}{dt} + 5y(t) = x(t)$$

with initial conditions $y(0^+) = -2$ and input $x(t) = 3.e^{-2t}u(t)$

Soln: The Given D.E is

$$\frac{dy(t)}{dt} + 5y(t) = x(t)$$

Taking Laplace Transform to both sides

$$s.Y(s) - y(0^-) + 5Y(s) = X(s)$$

initial conditions given as $y(0^+) = y(0^-) = -2$

$$x(t) = 3.e^{-2t}u(t)$$

$$X(s) = 3.\frac{1}{s+2}$$

$$sY(s) + 2 + 5Y(s) = \frac{3}{s+2}$$

$$Y(s)[s+5] = \frac{3}{s+2} - 2$$

$$Y(s) = \frac{3}{(s+2)(s+5)} - \frac{2}{s+5}$$

$$Y(s) = \frac{1}{s+2} - \frac{1}{s+5} - \frac{2}{s+5}$$

$$Y(s) = \frac{1}{s+2} - \frac{3}{s+5}$$

$$y(t) = e^{-2t}u(t) - 3e^{-5t}u(t)$$

* Solve the following D.E

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = 5x(t)$$

with $y(0) = 1$ and $\left.\frac{dy(t)}{dt}\right|_{0^-} = 2$ and $x(t) = u(t)$

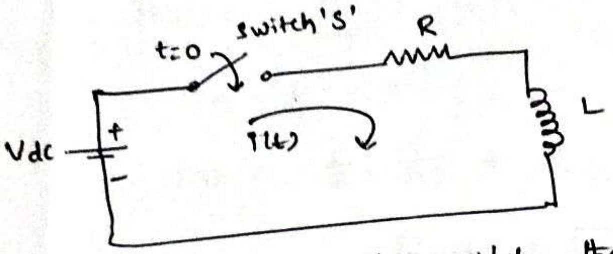
$$s^2Y(s) - y(0) - \frac{d}{dt}y(0) + 4[sY(s) - y(0)] + 5Y(s) = 5\left(\frac{1}{s}\right)$$

$$Y(s)[s^2+4s+5] - 1 - 2 - 4 = \frac{5}{s}$$

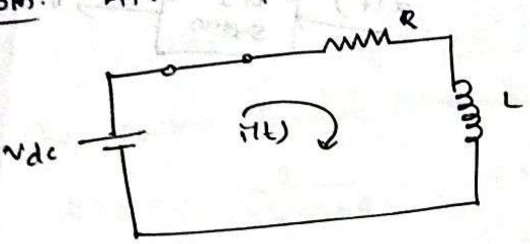
$$Y(s) = \frac{5}{s(s^2+4s+5)} - \frac{7}{s^2+4s+5}$$

$$Y(s)(s^2+4s+5) = \frac{5}{s} - 7$$

* A Series RL circuit is shown in Fig. below. Let the switch is closed at $t=0$. Then find out the expression for current in the ckt using Laplace Transform



<u>Soln:</u> After closing the switch, the ckt will be as follows



Apply KVL

$$V_{dc} = L\frac{di(t)}{dt} + Ri(t)$$

Taking L.T on both sides

$$\frac{V_{ds}}{s} = LsI(s) - i(0^-) + R.I(s) \qquad i(0^-) = 0 \quad \because \text{switch is open}$$

$$\frac{V_{ds}}{s} = I(s)\left[sL + R\right]$$

$$I(s) = \frac{V_{ds}}{s(sL+R)} = \frac{V_{ds}/L}{s(s+\frac{R}{L})}$$

$$I(s) = \frac{V_{dc}}{L}\left[\frac{A}{s} + \frac{B}{s+\frac{R}{L}}\right]$$

$$A = \frac{L}{R} \quad \& \quad B = \frac{-L}{R}$$

$$\therefore \quad I(s) = \frac{V_{dc}}{L}\left[\frac{L}{R}\cdot\frac{1}{s} - \frac{L}{R}\cdot\frac{1}{s+\frac{R}{L}}\right]$$

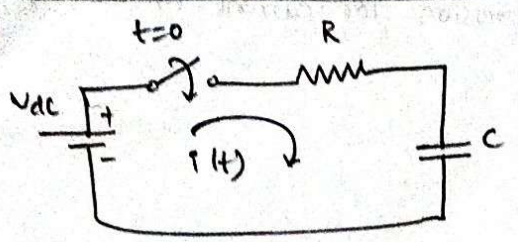
$$I(s) = \frac{V_{dc}}{R}\left[\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right]$$

$$\therefore \quad i(t) = \frac{V_{dc}}{R}\left[u(t) - e^{-\frac{R}{L}t}u(t)\right]$$

$$i(t) = \frac{V_{dc}}{R}\left[\left(1 - e^{-\frac{R}{L}t}\right]u(t)\right]$$

s^2+4s-5
$s^2+5s-s-5$
$s(s+5)-1(5$

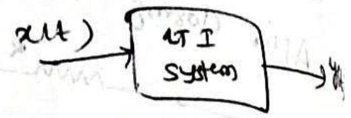
Similarly For series RC circuit



Ans: $\frac{V_{dc}}{R} e^{-t/RC}$

Transfer function of the system.

The o/p of an LTI CT system is given as



$$y(t) = h(t) * x(t)$$

Apply L.T

$$Y(s) = H(s) . X(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

↳ System function or T.F

Impulse response of the system can be obtained by taking ILT of H(s).

Frequency response of the system

By substituting $s=j\omega$ in H(s) we will get the freq response of the system.

i.e; $H(j\omega) = H(s)|_{s=j\omega}$

Problem

For the following differential eqn find system function, freq response and impulse response.

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Soln: Apply L.T to Given D.E

$$s^2Y(s) + 4sY(s) + 3Y(s) = sX(s) + 2X(s)$$

$$Y(s)[s^2+4s+3] = [s+2]X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2+4s+3}$$

Freq. response $H(\omega) = H(s)|_{s=j\omega} = \dfrac{j\omega+2}{(j\omega)^2+4(j\omega)+3}$

Impulse response

$$H(s) = \frac{s+2}{s^2+4s+3} = \frac{s+2}{(s+3)(s+1)}$$

$$H(s) = \frac{A}{s+3} + \frac{B}{s+1}$$

$$A = \frac{1}{2} \qquad B = \frac{1}{2}$$

$$H(s) = \frac{1}{2}\cdot\frac{1}{s+3} + \frac{1}{2}\cdot\frac{1}{s+1}$$

$$h(t) = \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)$$

$$h(t) = \frac{1}{2}\left[e^{-3t}+e^{-t}\right]u(t)$$

* The system function is given as

$$H(s) = \frac{s}{s^2+5s+6} \quad \text{and} \quad x(t) = e^{-t}u(t) \quad \text{Find output}$$

Ans: $\left[2e^{-2t} - \frac{3}{2}e^{-3t} - \frac{1}{2}e^{-t}\right]u(t)$

Causality and stability

Causality refers to the right sided time domain signal. If the impulse response of the system is right sided, then it is called causal system. i.e.,

$$h(t) = 0 \quad \text{for } t<0$$

An LTI system is said to be ~~causal~~ stable if all the poles of its system function lie on left side of the ROC.

The system is said to stable, if its impulse response is absolutely integrable. i.e.,

$$\sum_{t=-\infty}^{\infty} |h(t)| < \infty$$

An LTI system is said to be stable if and only if the ROC of its system function includes $Re(s)=0$, i.e, $j\omega$ axis of the s-plane.

A system will be causal and stable simultaneously if and only if all the poles of $H(s)$ lie on left side of $j\omega$-axis or left half of s-plane.