UNIT – III

CONCEPT OF STABILITY & ROOT LOCUS

Definitions of stability:

The term stability refers to the stable working condition of a control system. In a stable system, the response or the output is predictable, finite and stable for a given input

1. A system is stable, if its output is bounded (finite) for bounded input.
2. A system is asymptotically stable, if in the absence of input, the output tends towards zero irrespective of intial conditions.
3. A system is stable if for a bounded disturbing input signal, the output vanishes ultimately as $t \to \infty$
4. A system is unstable for a bounded disturbing input signal, the o/p is infinite amplitude or oscillatory.
5. For a bounded input signal, if the o/p has a constant amplitude oscillations then the system may be stable or unstable under some limited constraints. Such a system is called limitedly stable.
6. If a system output is stable for all variations of its parameters, then the system is called absolutely stable system.
7. If a system is stable for a limited range of variation of its parameters, then the system is called conditionally stable system.

Location of poles on s-plane for stability.

For BIBO Stability the integral of impulse response should be finite. i.e, the impulse response should be finite as $t \to \infty$

The closed loop T.F $M(s) = \frac{C(s)}{R(s)}$ can be expressed as a ratio of two polynomials in s as

$$M(s) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n}$$

$$= \frac{(s+z_1)(s+z_2)(s+z_3) \dots (s+z_m)}{(s+p_1)(s+p_2)(s+p_3) \dots (s+p_n)}$$

roots of numenator → zeros

roots of denominator → poles

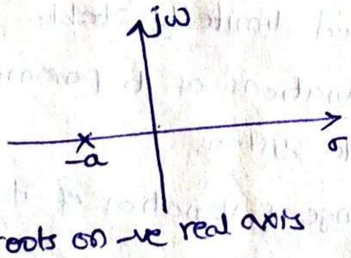
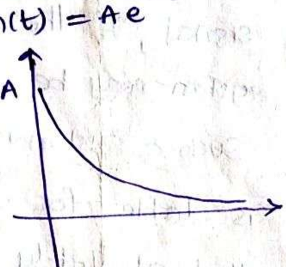
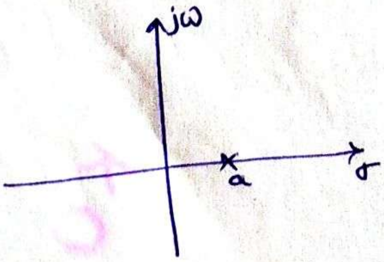
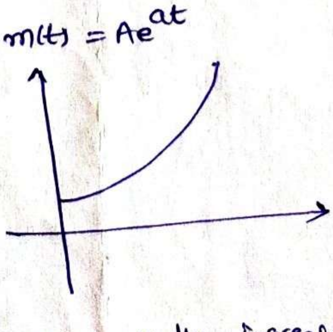
The denominator polynomial is called characteristic eqn and so the poles are roots of characteristic eqn.

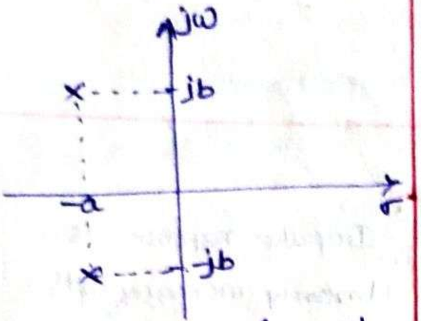
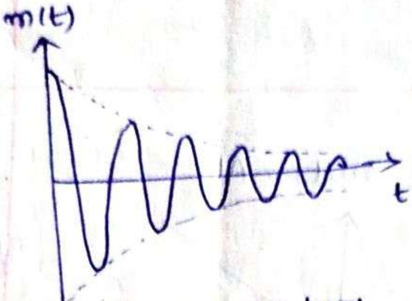
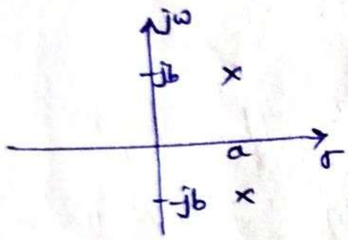
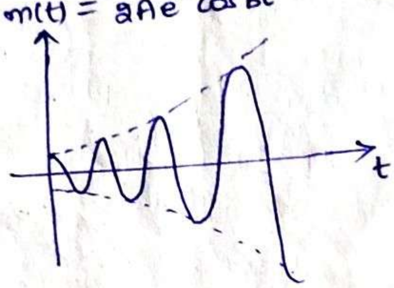
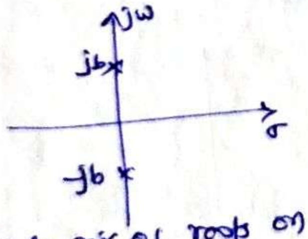
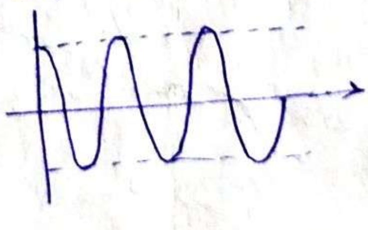
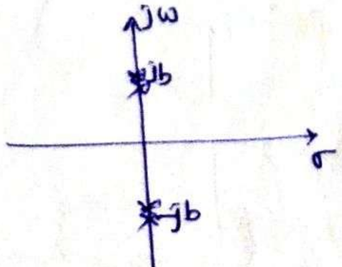
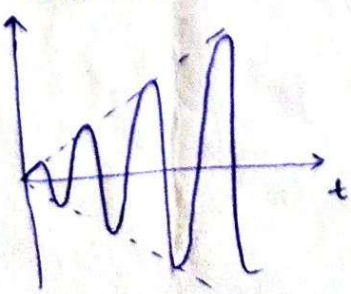
By partial fractions Expansion we can write

$$M(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \dots + \frac{A_n}{s+p_n}$$

The roots (poles) may be at origin or lying on imaginary axis or lying on right or lying on left half of s-plane.

The impulse response for various types of poles are shown in table below.

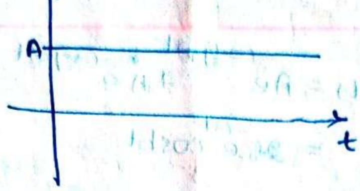
Transfer Function M(s) & location of poles on s-plane	Impulse response	Comment
1. $M(s) = \frac{A}{s+a}$  roots on -ve real axis	$m(t) = Ae^{-at}$  exponentially decaying (stable)	If all the poles are lying on left half of s-plane then the impulse response is bounded.
2. $M(s) = \frac{A}{s-a}$  Roots on +ve real axis	$m(t) = Ae^{at}$  Exponentially increasing (Unstable system)	If the roots of CE has a +ve real part then the impulse response is unbounded

Transfer Function $M(s)$ & location of roots on s-plane	Impulse response $m(t)$	Comment
3. $M(s) = \frac{A}{s+a+jb} + \frac{A^*}{s+a-jb}$  Complex conjugate roots on left half of s-plane	$m(t) = Ae^{-(a+jb)t} + A^* e^{-(a-jb)t}$ $= 2Ae^{-at}\cos bt$  damped oscillatory	Impulse response is exponentially decreasing sinusoidal. ∴ Stable system
4. $M(s) = \frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}$  Complex conjugate roots on right half of s-plane	$m(t) = Ae^{-(-a+jb)t} + A^* e^{-(-a-jb)t}$ $m(t) = 2Ae^{at}\cos bt$ 	Impulse response is exponentially increasing sinusoidal ∴ Unstable system
5. $M(s) = \frac{A}{s+jb} + \frac{A^*}{s-jb}$  Single pair of roots on imaginary axis	$m(t) = Ae^{-jbt} + A^* e^{jbt}$ $m(t) = 2A\cos bt$ 	Impulse response is oscillatory hence it is marginally stable system.
6. $M(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}$  Double pair of roots on imaginary axis	$m(t) = Ate^{-jbt} + A^* te^{jbt}$ $m(t) = 2At\cos bt$ 	Impulse response is linearly increasing sinusoidal hence the system is unstable.

7. $M(s) = \frac{A}{s}$

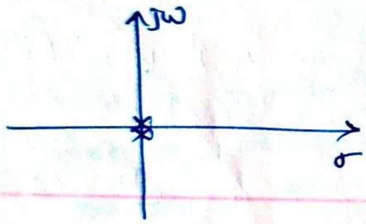
Single root at origin

$m(t) = A$



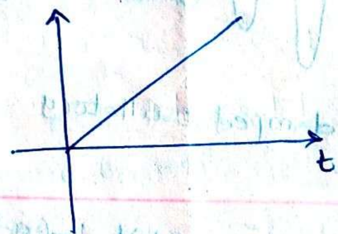
Impulse response is constant. hence it is marginally stable system

8. $M(s) = \frac{A}{s^2}$



Double roots at origin

$m(t) = At$



Impulse response is linearly increases with time hence it is unstable

ROUTH HURWITZ CRITERION:

The Routh Hurwitz stability criterion is an analytical procedure for determining whether all the roots of a polynomial have negative real part or not.

The Routh stability criterion is based on ordering the coefficients of the characteristic equation into a schedule, called Routh array as shown below.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0 \quad \text{where } a_0 > 0$$

$$\begin{array}{lllll}
s^n: & a_0 & a_2 & a_4 & a_6 \cdots \\
s^{n-1}: & a_1 & a_3 & a_5 & a_7 \cdots \\
s^{n-2}: & b_0 & b_1 & b_2 & b_3 \cdots \\
s^{n-3}: & c_0 & c_1 & c_2 & c_3 \cdots \\
\vdots & & & & \\
s^1: & g_0 & & & \\
s^0: & h_0 & & &
\end{array}$$

The necessary and sufficient condition for stability is that all of the elements in the 1st column of the Routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the 1st column of the Routh array corresponds to the number of roots of the characteristic eqn in the right half of the s-plane.

Let the characteristic polynomial be

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s^1 + a_n s^0$$

The coefficients of the polynomial are arranged in two rows as

$$\begin{array}{lllll}
s^n: & a_0 & a_2 & a_4 & a_6 \cdots \\
s^{n-1}: & a_1 & a_3 & a_5 & a_7 \cdots
\end{array}$$

* when 'n' is even, s^n row is formed by coefficents of even order terms and s^{n-1} row is formed by coefficients of odd order terms
* when 'n' is odd, s^n row is formed by coefficents of odd order terms and s^{n-1} row is formed by coefficents of even order terms.

The other rows of Routh array upto s^0 can be formed by the following procedure.

let us consider the two consecutive rows of Routh array as shown below

$$s^n : \quad x_0 \quad x_1 \quad x_2 \quad x_3 \cdots$$

$$s^{n-1} : \quad y_0 \quad y_1 \quad y_2 \quad y_3 \cdots$$

Then next row will be

$$s^{n-2} : \quad z_0 \quad z_1 \quad z_2 \quad z_3$$

where $z_0 = \dfrac{y_0x_1 - x_0y_1}{y_0}$, $z_1 = \dfrac{y_0x_2 - x_0y_2}{y_0}$

$$z_2 = \frac{y_0x_3 - x_0y_3}{y_0} \cdots\cdots$$

Now the ~~last~~ previous two rows are

$$s^{n-1} : \quad y_0 \quad y_1 \quad y_2 \quad y_3 \cdots$$

$$s^{n-2} : \quad z_0 \quad z_1 \quad z_2 \quad z_3 \cdots$$

Then $s^{n-3} : \quad P_0 \quad P_1 \quad P_2 \quad P_3$

where $P_0 = \dfrac{y_1z_0 - y_0z_1}{z_0}$ $\quad P_1 = \dfrac{z_0y_2 - y_0z_2}{z_0}$

$$P_2 = \frac{z_0y_3 - y_0z_3}{z_0} \cdots$$

$\vdots$

and so on

Routh's stability criterion states that the number of roots of the characteristic eqn with +ve real part is equal to the number of sign changes in the first column of the Routh array

According to Routh stability criterion, If there are no sign changes in the 1st column of Routh array, it is said to be stable.

If any change in the sign in the 1st column of the Routh array indicates that the system is unstable and the number of sign changes are equal to the number of roots lying on the right half of s-plane.

Routh criterion can be applied to the denominator of a transfer function to determine whether the system is stable or not.

Special Cases of Routh - Hurwitz Criterion

1. First element of a row is zero but some other elements are non-zero
2. A row of all zeros.

Case 1: First element of a row is zero

If the 1st element of a term is zero but the remaing terms are nonzero Then two problems may arise

i) The sign of this term is questionable
ii) The subsequent row will be indeterminate since it is obtained by this zero.

The following methods can be used to remedy this solution.

Method 1:

If only the 1st element in one of the row is zero, then we replace this zero with a small +ve constant $\epsilon > 0$ and proceed to complete the rest of the array. Then we apply the stability criterion by taking the limit as $\epsilon \to 0$. This method is called Epsilon method.

Method 2:

Put $s = \frac{1}{x}$ in the original characteristic polynomial. Rearrange the polynomial in descending ~~order~~ power of x. Develop a new Routh array. Examine the terms of the 1st column for the number of sign changes. This method is called reverse coefficient method.

Case 2: A row of all zeros

This situation occurs whenver the preceeding two are proportional to each other. Here we use the following procedure.

The polynomial where the coefficients are the elements of the row just above the rows of zeros in Routh array is called Auxiliary polynomial.

i) Determine the auxilliary polynomial $A(s)$

ii) Differentiate the auxiliary polynomial w.r.t s to get $\frac{dA(s)}{ds}$

iii) The row of zeros is replaced with the coefficients of $\frac{dA(s)}{ds}$

iv) Continue the construction of the array in usual manner

a) If there are sign changes in the 1st column of routh array then the system is unstable. The number of roots lying on the right half of s-plane is equal to the no. of sign changes. The no. of roots on imaginary axis can be estimated from the roots of auxiliary polynomial and the remaining roots are lying on the left half of s-plane.

b) If there are no sign changes in the 1st column of routh array then the all zeros row indicate the existence of purely imaginary roots and so the system is limitedly or marginally stable. The roots of auxillary polynomial lies on img. axis and the remaining roots lies on left half of s-plane.

Application of Routh stability criterion:

let the closed loop T.F is

$$\frac{C(s)}{R(s)} = \frac{K\,G(s)}{1+K\,G(s)\,H(s)}$$

Hence the characteristic eqn is

$$1+K\,G(s)H(s) = 0$$

∴ the roots of the above eqn depend upon the proper selection of K. The range of values of K for which it will not produce any sign change in the 1st column of Routh array is obtained to find the stability of the system. Such a system in which stability depends on the condition of parameter 'K' is called Conditionally Stable system.

Problems

1. Using Routh criterion, determine the stability of the system represented by the characteristic equation $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of roots of characteristic eqn.

Soln: Given characteristic eqn is

$$s^4+8s^3+18s^2+16s+5=0$$

s^4 : 1 18 5 → row 1

s^3 : 8 16 → row 2

$\therefore s^3$: 1 2 → row 2

The elements of s^3 can be divided 8 to simplify the computations

s^2 : $\frac{1\times18-2\times1}{1}$ $\frac{1\times5-0\times1}{1}$

s^1 : $\frac{16\times2-1\times5}{16} = 1.6875 \simeq 1.7$

$s^0 = \frac{1.7\times5-16\times0}{1.7} = 5$

Routh's Array

s^4 :	1	18	5	row 1
s^3 :	1	2		row 2
s^2 :	16	5		row 3
s^1 :	1.7			row 4
s^0 :	5			Row 5

Column-1

All the elements of the 1st column in Routh array are +ve and there is no sign change. Hence all the roots are lying on left half of s-plane and the system is stable.

2. $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

6th order eqn and so it has 6 roots

s^6 : 1 8 20 16 row 1

s^5 : 2 12 16 row 2

s^5 : 1 6 8 divided by 2

s^6 : 1 8 20 16 → row 1

s^5 : 1 6 8 → row 2

s^4 : $\frac{1\times8-1\times6}{1}$ $\frac{1\times20-1\times8}{1}$ $\frac{1\times16-1\times0}{1}$

s^4 : 2 12 16

s^4 : 1 6 8

s^5 : 1 6 8 → row 2

s^4 : 1 6 8 → row 3

s^3 : $\frac{1\times6-1\times6}{1}$ $\frac{1\times8-1\times8}{1}$

s^3 : 0 0 → row 4

All the elements are zero's in row 4

∴ auxiliary eqn's $A = s^4+6s^2+8$

$$\frac{dA}{ds} = 4s^3+12s$$

Now the coefficients of $\frac{dA}{ds}$ are used to form s^3 row

∴ s^3 : 4 12 ⇒ 1 3

s^4 : 1 6 8

s^3 : 1 3

∵ s^3 row is divided by 3

s^2 : $\frac{1\times6-1\times3}{1}$ $\frac{1\times8-1\times0}{1}$

s^2 : 3 8 row 5

s^3 : 1 3

s^2 : 3 8

s^1 : $\frac{3\times3-1\times8}{3} = \frac{1}{3} = 0.33$ row 6

s^2 : 3 8

s^1 : 0.33

s^0 : $\frac{0.33\times8-3\times0}{0.33} = 8$ row 7

∴ Routh Array is Given below

s^6 :	1	8	20	16
s^5 :	1	6	8	
s^4 :	1	6	8	
s^3 :	0	0		
s^3 :	1	3		
s^2 :	3	8		
s^1 :	0.33			
s^0 :	8			

Column-1

There is no sign change in the 1st column. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

Auxiliary Polynomial is

$$s^4 + 6s^2 + 8 = 0$$

let $s^2 = x$

$$x^2 + 6x + 8 = 0$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4\times1\times8}}{2}$$

$$x = -3 \pm 1$$

$$= -2 \text{ or } -4$$

The roots of auxiliary polynomial is

$$s = \pm\sqrt{x} = \pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$= \pm j\sqrt{2},\ \pm j2$$

The roots of auxiliary polynomial are also roots of characteristic eqn. Hence 4 roots are lying on imaginary axis and the remaining roots are lying on the left half of s-plane.

Problem 3: $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

Soln: 5th order eqn and so it has 5 roots

s^5 :	1	2	3
s^4 :	1	2	5
s^3 :	$\frac{1\times2 - 1\times2}{1}$	$\frac{1\times3 - 1\times5}{1}$	
s^3 :	0	-2	

First element is row 3 is zero

So replace 0 by e

$s^4 : \quad 1 \quad 2 \quad 5$

$s^3 : \quad \epsilon \quad -2$

$s^2 : \quad \dfrac{\epsilon\times 2 - (-2\times 1)}{\epsilon} \quad \dfrac{\epsilon\times 5 - 1\times 0}{\epsilon}$

$s^2 : \quad \dfrac{2\epsilon + 2}{\epsilon} \quad 5$

$s^3 : \quad \epsilon \quad -2$

$s^2 : \quad \dfrac{2\epsilon+2}{\epsilon} \quad 5$

$s^1 : \quad \dfrac{\left(\dfrac{2\epsilon+2}{\epsilon}\right)(-2) - 5\times\epsilon}{\dfrac{2\epsilon+2}{\epsilon}} = \dfrac{\dfrac{-4\epsilon-4-5\epsilon^2}{\epsilon}}{\dfrac{2\epsilon+2}{\epsilon}}$

$s^1 : \quad -\dfrac{(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}$

$s^2 : \quad \dfrac{2\epsilon+2}{\epsilon} \quad 5$

$s^1 : \quad -\dfrac{(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}$

$s^0 : \quad \dfrac{-5\dfrac{(5\epsilon^2+4\epsilon+4)}{2\epsilon+2} - \left(\dfrac{2\epsilon+2}{\epsilon}\right)\times 0}{-\dfrac{(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}} = 5$

$\therefore$ Routh array will be

s^5:	1	2	3
s^4:	1	2	5
s^3:	ϵ	-2	
s^2:	$\dfrac{2\epsilon+2}{\epsilon}$	5	
s^1:	$-\left(\dfrac{5\epsilon^2+4\epsilon+4}{2\epsilon+2}\right)$		
s^0	5		

Substituting $\epsilon \to 0$

s^5:	1	2	3
s^4:	1	2	5
s^3:	0	-2	
s^2:	∞	5	
s^1:	-2		
s^0:	5		

Coulmn 1

There are two sign changes in the 1st column. Hence two roots are lying on right half of s-plane Hence the system is unstable
The remaining three roots are lying on left half of s-plane.

Problem 4:

$$9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$$

Soln: Some of the coefficients of characteristic eqn are -ve and so some roots will lie on right half of s-plane. Hence the system is unstable.

Since the characteristic eqn is of order 5, it has 5 roots.

s^5 :	9	10	-9
s^4 :	-20	-1	-10
s^3 :	9.55	-13.5	
s^2 :	-29.3	-10	
s^1 :	-16.8		
s^0 :	-10		

Column-1

$$s^3 : \frac{-20\times 10 - 9(-1)}{-20} \qquad \frac{-20(-9) - 9(-10)}{-20}$$

$$s^3 : \quad 9.55 \qquad -13.5$$

$$s^2 : \frac{9.55(-1) - (-20)(-13.5)}{9.55} \qquad \frac{9.55(-10) - (-20)(0)}{9.55}$$

$$s^2 : \quad -29.3 \qquad -10$$

$$s^1 : \frac{-29.3(-13.5) - 9.55(-10)}{-29.3}$$

$$s^1 : \quad -16.8$$

$$s^0 : \frac{-16.8(-10) - 29.3(0)}{-16.8}$$

There are three sign changes in the 1st column and so three roots will lie on right half of s-plane and the remaining two roots will lie on left half of s-plane.

Excercise

5. $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$

6. $s^7 + 5s^6 + 9s^5 + 9s^4 + 4s^3 + 20s^2 + 36s + 36 = 0$

7. $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$

8. $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$

Problem 9:

Determine the range of K for stability of unity feedback system whose open loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Soln: The closed loop T.F is $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$ $\because H(s)=1$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{\frac{K}{s(s+1)(s+2)}}{\frac{s(s+1)(s+2)+K}{s(s+1)(s+2)}}$$

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+3s+2)+K} = \frac{K}{s^3+3s^2+2s+K}$$

characteristic eqn is

$$s^3+3s^2+2s+K=0$$

s^3:	1	2
s^2:	3	K
s^1:	$\frac{6-K}{3}$	
s^0:	K	

Column 1

s^1: $\frac{3\times2-1\times K}{3} = \frac{6-K}{3}$

$s^0 = \frac{\left(\frac{6-K}{3}\right)(K)-3\times0}{\left(\frac{6-K}{3}\right)}$

For the system to be stable there should be no sign change in the 1st column.

Hence all the elements should be +ve

$\frac{6-K}{3}>0$ $\quad K>0$

$6-K>0$

$6>K$

$K<6$ $\quad \therefore 0<K<6$

$\therefore$ For the system to be stable the range of K is

$0<K<6$

Problem 10:

The open loop T.F of a unity feedback control system is given by

$$G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying the Routh criterion, discuss the stability of the closed loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed loop system. What are the corresponding oscillating frequencies.

Soln: The closed loop TF is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \quad \because H(s)=1$$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}}{1+\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}}$$

$$\frac{C(s)}{R(s)} = \frac{K}{(s+2)(s+4)(s^2+6s+25)+K}$$

characteristic eqn is

$$(s+2)(s+4)(s^2+6s+25)+K=0$$

$$(s^2+6s+8)(s^2+6s+25)+K$$

$$s^4+12s^3+69s^2+198s+200+K=0$$

Routh Array

s^4: 1 69 200+K

s^3: 12 198

Divide s^3 row by 12

$$s^2: \quad \frac{1\times 69 - 1\times 16.5}{1} \qquad \frac{1(200+K)-1\times 0}{1}$$

s^2: 52.5 200+K

$$s^1: \frac{(52.5)(16.5)-(200+K)}{52.5}$$

$$: \frac{666.25-K}{52.5}$$

Column 1

s^4:	1	69	200+K
s^3:	1	16.5	
s^2:	52.5	200+K	
s^1:	$\frac{666.25-K}{52.5}$		
s^0:	200+K		

$$s^0: \frac{\frac{666.25-K}{52.5}(200+K)-52.5(0)}{\frac{666.25-K}{52.5}}$$

For the system to be stable, all the elements in the 1st column should be +ve

$\therefore$ i) $\frac{666.25-K}{52.5} > 0$

$666.25 - K > 0 \Rightarrow 666.25 > K$

$K < 666.25$

ii) $200 + K > 0$

$K > -200$

but practical values of K start from 0 $\therefore K > 0$

$\therefore \quad 0 < K < 666.25$

When $K = 666.25$ the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis. A system will oscillate if it has roots on imaginary axis and no roots on right half of s-plane.

When $K = 666.25$ auxiliary eqn is

$$52.5 s^2 + 200 + 666.25 = 0$$

$$s^2 = \frac{-200-666.25}{52.5} = -16.5$$

$$s = \pm\sqrt{-16.5}$$

$$s = \pm j4.06$$

The freq of oscillation is given by the value of root on img axis

Problem 11: $G(s) = \frac{K(s+13)}{s(s+3)(s+7)}$

Problem 12: $s^4 + s^3 + 3Ks^2 + (K+2)s + 4 = 0$

Problem 13: $\frac{K(s+2)}{s(s-2)(s^2+5s+16)}$ Determine the value of K which will cause sustained oscillations and what is the corresponding oscillation freq

Advantages of Routh's stability Criterion.

1) stability of the system can be judged without actually solving the characteristic eqn
2) No evaluation of determinants, which saves calculation time
3) For unstable system it gives number of roots of characteristic eqn having +ve real part
4) Relative Stability of the system can be easily judged
5) Critical value of system gain can be determined hence frequency of sustained oscillations can be determined
6) It helps in finding out range of value of k for system stability
7) It helps in find out intersection points of root locus with imaginary axis

Limitations of Routh's stability criterion

1) It is valid only for real coefficients of the characteristic eqn.
2) It does not provide exact location of closed loop poles in left or right half of s-plane
3) It does not suggest methods of stabilyzing an unstable system
4) Applicable only to linear systems