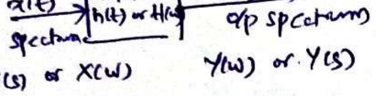
Super position principle states that the response to a weighted sum of input signals be equal to the corresponding weighted sum of the outputs of the systems to each of the input signal.

Impulse response: Impulse response is the output of the system for a unit impulse input. For understanding the behavior of a system, impulse response is very important

If i/p $x(t) = \delta(t)$ then o/p $y(t) = h(t)$

T.F. of an LTI system



The Transfer function of an LTI system is defined as the ratio of Fourier Transform of o/p to the FT of the i/p signal when the initial cond^n's are zero

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

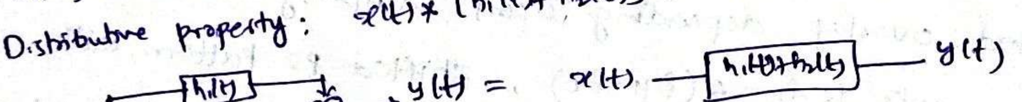
T.F of an LTI system is Fourier Transform of its impulse response

Properties of LTI systems

(1) Commutative : $x(t) * h(t) = h(t) * x(t)$

i.e the o/p of an LTI system with i/p $x(t)$ and imp. res $h(t)$ is identical to the the o/p of an LTI system with i/p $h(t)$ and imp. res. $x(t)$

(2) Distributive property: $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$



Two systems with imp responses $h_1(t)$ & $h_2(t)$ connected in parallel can be replaced by a single system with an impulse response $h(t) = h_1(t) + h_2(t)$

(3) Associative : $x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$

(4) Causality: A causal system is anticipatory and does not produce an o/p before an i/p is applied

For a causal LTI system $h(t) = 0$ for $t < 0$

∴ o/p of a causal LTI system with non-causal i/p is

$$y(t) = \int_0^{\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{t} x(\tau) h(t-\tau) d\tau$$

o/p of a causal LTI system for a causal i/p is

$$y(t) = \int_0^{t} h(\tau) x(t-\tau) d\tau = \int_0^{t} x(\tau) h(t-\tau) d\tau$$

(5) Stability: A system is stable if every bounded i/p produces a bounded o/p. The BIBO stability of an LTI system can be easily determined from its imp. resp.

For a CT LTI system to be BIBO stable, its impulse response must be absolutely integrable

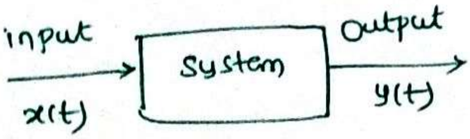
i.e $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

Most of the physically realizable systems are linear and time invariant. For such systems input and output can be related by an impulse response of the system. The convolution of impulse response and input gives output signal.

Fourier Transform of impulse response gives transfer function of the system. Transfer function can be used to obtain the frequency response.

Time - Invariant and Time - Variant systems:

A continuous - time system is time - invariant if the time shift in the input signal results in corresponding time shift in the output signal.



$$y(t) = T[x(t)]$$

if $T[x(t-t_1)] = y(t-t_1)$, it is said to be time - invariant system.

where $T[x(t-t_1)]$ is response of the system due to delayed input.

If the above relation is not satisfied then it is said to be time variant system.

Similarly if the input - output characteristics of a Discrete - Time system do not change w.r.t. time, such systems are called shift invariant or time - invariant systems

let $x(n)$ be the input and $y(n)$ be the output

For Time Invariant systems: $y(n-k) = T[x(n-k)]$

For Time variant systems : $y(n-k) \neq T[x(n-k)]$

4) Given $y(n) = n^2 x(n)$

$$y(n,k) = n^2 x(n-k)$$

$$y(n-k) = (n-k)^2 x(n-k)$$

$$y(n,k) \neq y(n-k)$$

hence the given system is time variant

Linear and Non-linear Systems:

A system is said to be linear if it satisfies the super position principle.

let us consider two systems as follows

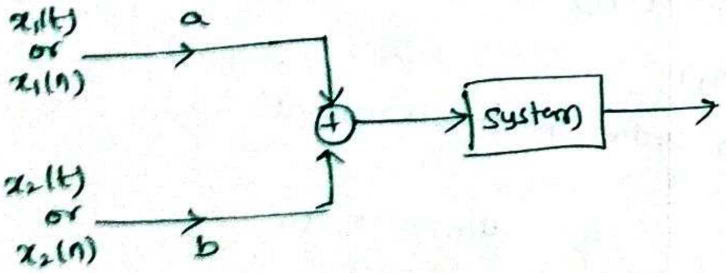
$$y_1(t) = T[x_1(t)] \text{ and } y_2(t) = T[x_2(t)]$$

Then the system is said to be linear if and only if

$$T[a x_1(t) + b x_2(t)] = a y_1(t) + b y_2(t)$$

If the above eqn is not satisfied, then it is said to be non-linear.

$\therefore$ For a linear system, combined response due to $x_1(t)$ and $x_2(t)$ together is same as the sum of individual responses

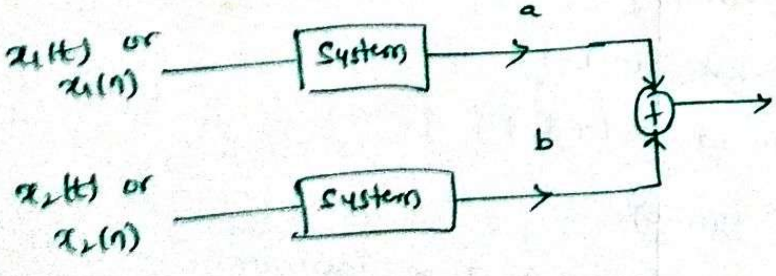


$$y_3(t) = T[a x_1(t) + b x_2(t)]$$

or

$$y_3(n) = T[a x_1(n) + b x_2(n)]$$

$\equiv$



$$y_3'(t) = a y_1(t) + b y_2(t)$$

or

$$y_3'(n) = a y_1(n) + b y_2(n)$$

Steps to determine the time - invariance property

1) Delay the input by t_1 then find the output

i.e, $y(t,t_1) = T[x(t-t_1)]$

2) Delay the output by t_1 ie, $y(t-t_1)$

3) if $y(t-t_1) = y(t,t_1)$, it is time invariant system

if $y(t,t_1) \neq y(t-t_1)$ then it is time variant system.

Problem:

Check whether the following systems are time variant or time invariant

1) $y(t) = \sin x(t)$ 2) $y(t) = t \cdot x(t)$

Soln: i) Given $y(t) = \sin x(t)$

$$y(t,t_1) = \sin x(t-t_1)$$
$$y(t-t_1) = \sin x(t-t_1)$$
$$y(t,t_1) = y(t-t_1)$$

hence the given system is time - invariant

ii) Given $y(t) = t \cdot x(t)$

$$y(t,t_1) = t\, x(t-t_1)$$
$$y(t-t_1) = (t-t_1)\, x(t-t_1)$$

$$y(t,t_1) \neq y(t-t_1)$$

hence the given system is time variant

3) $y(n) = x(n) - x(n-1)$ 4) $y(n) = n^2 x(n)$

Given $y(n) = x(n) - x(n-1)$

$$y(n,k) = x(n-k) - x(n-k-1)$$
$$y(n-k) = x(n-k) - x(n-k-1)$$

$$y(n,k) = y(n-k)$$

hence the given system is time - invariant

Problems:

① Let the system function of an LTI system be $\frac{1}{j\omega+2}$. what is the output of the system for an input $(0.8)^t u(t)$?

Soln: Given the system function

$$H(\omega) = \frac{1}{j\omega+2}$$

input $x(t) = (0.8)^t u(t)$

impulse response $h(t) = IFT\,[H(\omega)]$

$$h(t) = IFT\left[\frac{1}{j\omega+2}\right]$$

$$h(t) = e^{-2t}u(t)$$

output $y(t)$ is the convolution of $x(t)$ and $h(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau)\, (0.8)^{t-\tau} u(t-\tau)\, d\tau$$

$u(\tau) = 1$ for $\tau \geq 0$ $\qquad$ $u(t-\tau) = 1$ for $t-\tau \geq 0$

$= 0$ for $\tau < 0$ $\qquad$ $t \geq \tau$

$\tau \leq t$

$= 0$ fo $\tau \geq t$

$$\therefore y(t) = \int_0^t e^{-2\tau} (0.8)^{t-\tau}\, d\tau$$

$$= (0.8)^t \int_0^t (0.8e^2)^{-\tau}\, d\tau$$

let $(0.8e^2)^{-1} = a$

$$y(t) = (0.8)^t \int_0^t a^\tau\, d\tau$$

$$= (0.8)^t \left.\frac{a^\tau}{\log a}\right|_0^t$$

$$= (0.8)^t \left[\frac{a^t - a^0}{\log a}\right]$$

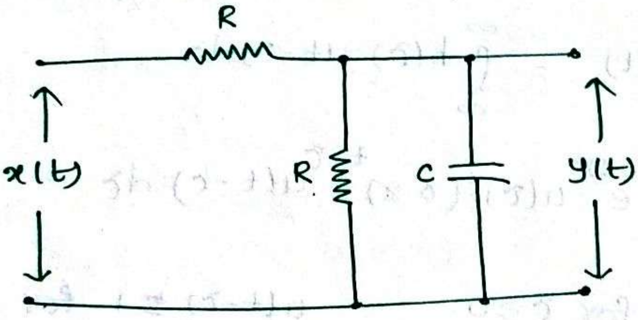
$$\log a = \log (0.8e^{2})^{-1}$$
$$= -\log(0.8e^{2})$$
$$= -\left[\log 0.8 + \log e^{2}\right]$$
$$= -\left[\log 0.8 + 2\right]$$

$$= \frac{(0.8)^t (0.8e^{2})^{-1} - (0.8)^t}{-(\log 0.8 + 2)}$$

$$y(t) = \frac{-(0.8)^t \left((0.8e^{2})^{-1} - 1\right)}{\log 0.8 + 2}$$

Problem 2:

Consider the filter circuit shown below

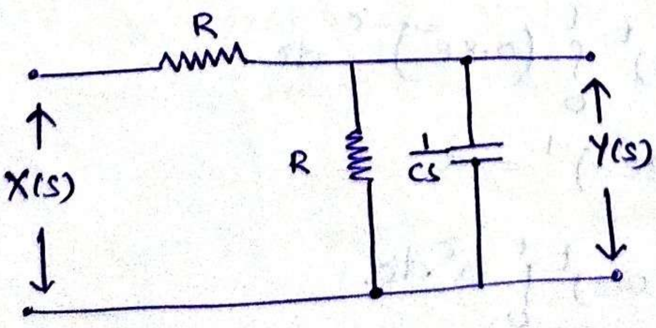


a) Write the input/output relationship

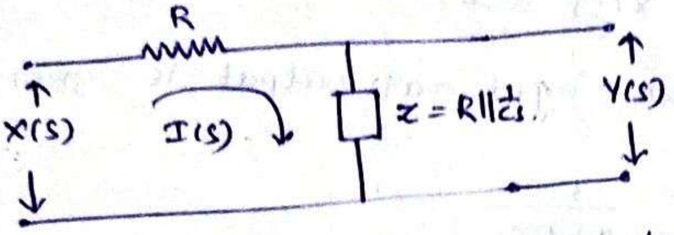
b) Obtain its impulse response

c) Find the step response

Soln. Laplace transformed n/w for the above n/w shown below.



The equivalent circuit is shown below



let us consider the input and outputs are as voltages.

$$\therefore \quad Y(S) = I(S)\, z \qquad \text{—(1)}$$

$$\text{wher} \quad z = R \,||\, \frac{1}{Cs} = \frac{R\cdot\frac{1}{Cs}}{R+\frac{1}{Cs}} = \frac{\frac{R}{Cs}}{\frac{1+RCs}{Cs}} = \frac{R}{1+RCs} \qquad \text{—(2)}$$

$$I(S) = \frac{X(S)}{R+z} = \frac{X(S)}{R+\frac{R}{1+RCs}} = \frac{X(S)}{\frac{(1+RCs)R+R}{1+RCs}} \qquad \text{—(3)}$$

From eqn (1), (2) & (3)

$$Y(s) = \frac{X(S)}{\frac{R[1+RCs+1]}{\cancel{1+RCs}}} \times \frac{\cancel{R}}{(\cancel{1+RCs})} = \frac{X(S)}{2+RCs}$$

$$H(S) = \frac{Y(S)}{X(S)} = \frac{1}{2+RCs}$$

$$\therefore \quad H(\omega) = \frac{1}{2+j\omega RC}$$

Impulse response $\quad h(t) = IFT\left[\frac{1}{2+RCs}\right]$

$$= IFT\left[\frac{1}{RC(s+2/RC)}\right]$$

$$= \frac{1}{RC}\, IFT\left[\frac{1}{s+2/RC}\right]$$

$$h(t) = \frac{1}{RC}\, e^{-2t/RC}\, u(t)$$

For a step unit $x(t) = u(t)$

$$X(s) = \frac{1}{s}.$$

The relationship b/w input and output is given by

$$\frac{Y(s)}{X(s)} = \frac{1}{2+RCs}$$

$$Y(s) = X(s) \cdot \frac{1}{2+RCs}$$

$$= \frac{1}{s} \cdot \frac{1}{RC(s+2/RC)}$$

$$Y(s) = \frac{1}{RC}\left[\frac{1}{s(s+2/RC)}\right]$$

By taking partial fractions

$$Y(s) = \frac{1}{RC} \cdot \left[\frac{A}{s} + \frac{B}{s+2/RC}\right]$$

$$A = s.Y(s)|_{s=0} = RC/2$$

$$B = (s+2/RC)\,Y(s)|_{s=-2/RC} = \frac{1}{-2/RC} = -RC/2$$

$$\therefore \quad Y(s) = \frac{1}{RC}\left[\frac{RC/2}{s} + \frac{-RC/2}{s+2/RC}\right]$$

$$Y(s) = \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2/RC}\right]$$

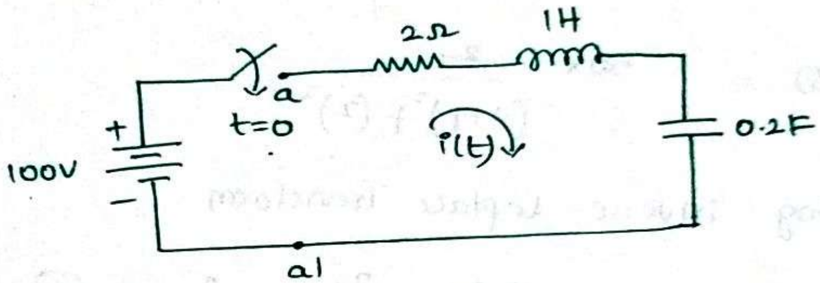
By taking inverse laplace Transform

$$y(t) = \frac{1}{2}\left[u(t) - e^{-(2/RC)t}u(t)\right]$$

$$\therefore \; y(t) = \frac{1}{2}\left[1 - e^{-(2/RC)t}\right]u(t)$$

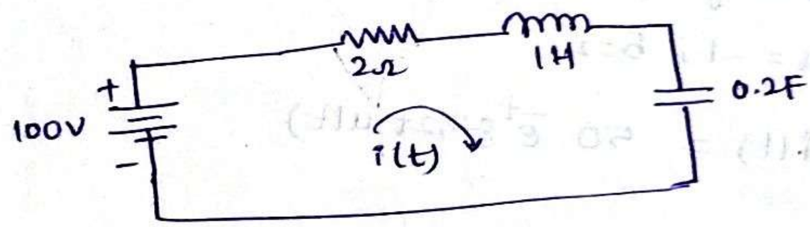
Problem 3:

Find the current $i(t)$ in a series RLC circuit shown in Figure below. When a voltage of 100 volts is switched on across the terminals aa' at $t=0$



Soln: when the switch is closed at $t=0$

i.e, after $t>0$, the equivalent circuit is shown below



By writting loop equation.

$$100 = 2i(t) + (1)\frac{di(t)}{dt} + \frac{1}{0.2}\int i(t)\,dt$$

Taking laplace Transform on both sides

$$\frac{100}{S} = 2I(S) + SI(S) + \frac{1}{0.2}\frac{I(S)}{S}$$

$$= I(S)\left[2 + S + \frac{1}{0.2S}\right]$$

$$= I(S)\left[2 + S + \frac{5}{S}\right]$$

$$\frac{100}{S} = I(S)\left(\frac{S^2+2S+5}{S}\right)$$

$$I(S) = \frac{100}{S^2+2S+5}$$

$$I(s) = \frac{100}{(s+1)^2 + 2^2}$$

$$L[e^{at} \sin bt\, u(t)] = \frac{b}{(s-a)^2+b^2} \quad \text{— (1)}$$

$$\therefore I(s) = 50 \times \frac{2}{(s+1)^2 + (2)^2}$$

By taking inverse Laplace Transform

$$i(t) = 50 . L^{-1}\left\{ \frac{2}{(s+1)^2+(2)^2} \right\} \quad \text{— (2)}$$

By comparing eqn (1) & (2)

$$a = -1, \; b = 2$$

$$\therefore i(t) = 50\, e^{-t} \sin 2t\, u(t)$$

Problem 4:

Consider a stable LTI system that is characterized by the differential eqn

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{d}{dt}x(t) + 2x(t)$$

Find its response for input $x(t) = e^{-t} u(t)$

Soln: The given D.E is

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{d}{dt}x(t) + 2x(t)$$

Taking Laplace Transform on both sides

$$s^2Y(s) + 4sY(s) + 3Y(s) = sX(s) + 2X(s)$$

$$Y(s)\left[s^2+4s+3\right] = X(s)(s+2)$$

$$Y(s) = \frac{s+2}{s^2+4s+3} X(s)$$

Given $x(t) = e^{-t}u(t)$

$$\therefore X(s) = \frac{1}{s+1}$$

$$\therefore Y(s) = \frac{s+2}{s^2+4s+3} \times \frac{1}{(s+1)}$$

$$Y(s) = \frac{s+2}{(s+1)(s+3)} \times \frac{1}{(s+1)}$$

$$Y(s) = \frac{s+2}{(s+1)^2(s+3)}$$

By taking partial fractions

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$A = \frac{d}{ds}\left[(s+1)^2 Y(s)\right]\Big|_{s=-1} = \frac{d}{ds}\left[\cancel{(s+1)^2}\frac{(s+2)}{\cancel{(s+1)^2}(s+3)}\right]\Big|_{s=-1}$$

$$= \frac{(s+3)-(s+2)}{(s+3)^2}\Big|_{s=-1} = \frac{1}{4}$$

$$B = (s+1)^2 Y(s)\Big|_{s=-1} = \frac{\cancel{(s+1)^2}(s+2)}{\cancel{(s+1)^2}(s+3)}\Big|_{s=-1} = \frac{-1+2}{-1+3} = \frac{1}{2}$$

$$C = (s+3)Y(s)\Big|_{s=-3}$$

$$= \cancel{(s+3)}\frac{(s+2)}{(s+1)^2\cancel{(s+3)}}\Big|_{s=-3} = \frac{-3+2}{(-3+1)^2} = \frac{-1}{4}$$

$$Y(s) = \frac{(1/4)}{s+1} + \frac{(1/2)}{(s+1)^2} - \frac{(1/4)}{s+3}$$

$$\boxed{\text{we know } \quad L\{e^{-t}u(t)\} = \frac{1}{s+1} \qquad L\{te^{-t}u(t)\} = \frac{1}{(s+1)^2}}$$

By taking inverse laplace Transform

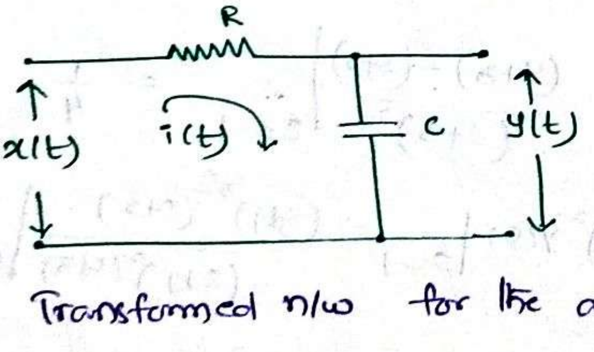
$$y(t) = \left(\tfrac{1}{4}\right) L^{-1}\left\{\frac{1}{s+1}\right\} + \left(\tfrac{1}{2}\right) L^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \left(\tfrac{1}{4}\right) L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\therefore\ y(t) = \frac{1}{4}e^{-t}u(t) + \frac{1}{2}te^{-t}u(t) - \frac{1}{4}e^{-3t}u(t)$$

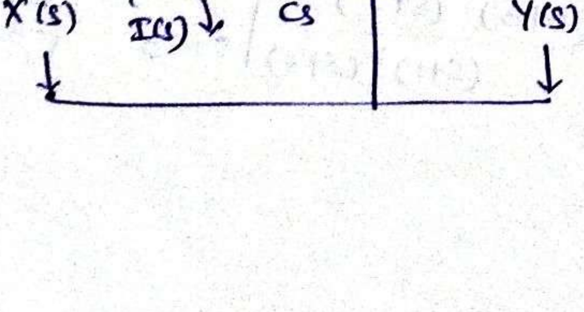
$$y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t)$$

Problem 5:

Find the output voltage of the RC low pass filter shown below for an input voltage of $t.e^{-t/RC}$.



Soln:- Laplace Transformed n/w for the above ckt is shown below.



$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1+RCs}$$

$$\therefore\ Y(s) = \frac{1}{1+RCs} X(s)$$

Given $x(t) = t.e^{-t/RC}$

$$\therefore\ X(s) = \frac{1}{s+\left(\frac{1}{RC}\right)^2}$$

$$\therefore\ Y(s) = \frac{1}{1+RCs} \frac{1}{s+\left(\frac{1}{RC}\right)^2}$$

$$Y(s) = \frac{1}{RC(s+1/RC)} \cdot \frac{1}{\left(s+\frac{1}{RC}\right)^2}$$

$$Y(s) = \frac{1}{RC} \cdot \frac{1}{\left(s+1/RC\right)^3}$$

Taking Inverse L.T on both sides

$$L\{t^n\} = \frac{(n!)}{s^{n+1}}$$

$$y(t) = \frac{1}{RC} L^{-1}\left\{\frac{1}{\left(s+1/RC\right)^3}\right\}$$

$$L\{e^{-t}u(t)\} = \frac{1}{s+1}$$

$$L\{e^{-t/RC}u(t)\} = \frac{1}{s+1/RC}$$

$$L\{t.e^{-t/RC}u(t)\} = \frac{1}{\left(s+1/RC\right)^2}$$

$$L\{t^2 e^{-t/RC}u(t)\} = \frac{2!}{\left(s+\frac{1}{RC}\right)^3}$$

$$\therefore\ y(t) = \frac{1}{RC}\ \frac{t^2}{2} e^{-t/RC}u(t)$$

$$y(t) = \frac{t^2}{2RC} e^{-t/RC} u(t)$$

Problem 6:

A system produces an output of $y(t) = e^{-t}u(t)$ for an input of $x(t) = e^{-2t}u(t)$. Determine the impulse response and frequency response of the system.

Soln: Given $y(t) = e^{-t}u(t) \Rightarrow Y(s) = \frac{1}{s+1}$

& $x(t) = e^{-2t}u(t) \Rightarrow X(s) = \frac{1}{s+2}$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s+1}}{\frac{1}{s+2}} = \frac{s+2}{s+1}$$

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{\frac{s+2}{s+1}\right\} = L^{-1}\left\{\frac{s+1+1}{s+1}\right\}$$

$$= L^{-1}\left\{1 + \frac{1}{s+1}\right\}$$

$h(t) = \delta(t) + e^{-t}u(t)$ impulse response

Frequency response is obtained by substituting $s = j\omega$ in $H(s)$

$$\therefore \quad H(\omega) = H(s)\Big|_{s=j\omega} = \frac{j\omega+2}{j\omega+1}$$

Problem 7:

The input voltage to an RC circuit is given as $x(t) = te^{-3t}u(t)$ and the impulse response of this circuit is given as $2e^{-4t}u(t)$. Determine the output.

soln: Given $x(t) = te^{-3t}u(t)$

$$X(s) = \frac{1}{(s+3)^2}$$

$$h(t) = 2e^{-4t}u(t)$$

$$H(s) = \frac{2}{s+4}$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s)\,X(s)$$

$$Y(s) = \frac{2}{s+4} \cdot \frac{1}{(s+3)^2}$$

$$Y(s) = \frac{2}{(s+3)^2(s+4)}$$

By taking partial fractions

$$Y(s) = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+4}$$

$$A = \frac{d}{ds}\left[(s+3)^2 Y(s)\right]\Big|_{s=-3}$$

$$= \frac{d}{ds}\,\cancel{(s+3)^2}\frac{2}{\cancel{(s+3)^2}(s+4)}\Big|_{s=-3} = \frac{-2}{(s+4)^2}\Big|_{s=-3} = -2$$

$$B = (s+3)^2\,Y(s)\big|_{s=-3}$$

$$= \cancel{(s+3)^2}\frac{2}{\cancel{(s+3)^2}(s+4)}\Big|_{s=-3} = 2$$

$$C = (s+4)\,Y(s)\big|_{s=-4}$$

$$= \cancel{(s+4)}\frac{2}{(s+3)^2\cancel{(s+4)}}\Big|_{s=-4} = 2$$

$$\therefore \quad Y(s) = \frac{-2}{s+3} + \frac{2}{(s+3)^2} + \frac{2}{s+4}.$$

By Taking Inverse Laplace Transform on both sides

$$y(t) = -2\,L^{-1}\left\{\frac{1}{s+3}\right\} + 2\,L^{-1}\left\{\frac{1}{(s+3)^2}\right\} + 2\,L^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\therefore\ y(t) = -2e^{-3t}u(t) + 2te^{-3t}u(t) + 2e^{-4t}u(t)$$

$$y(t) = 2\left[e^{-4t} - e^{-3t} + te^{-3t}\right]u(t)$$

Problem 8:

For a system excited by $x(t) = e^{-3t}u(t)$ and the impulse response $h(t) = e^{-2t}u(t) + e^{2t}u(-t)$. Find the output.

Ans: $y(t) = -\frac{4}{5}e^{-3t}u(t) + e^{-2t}u(t) + \frac{1}{5}e^{2t}u(-t)$

Problem 9:

Consider a stable LTI system characterized by the differential equation $\frac{dy(t)}{dt} + 2y(t) = x(t)$

Find its impulse response.

Ans: $h(t) = e^{-2t}u(t)$

Problem 10:

Consider a causal LTI system with frequency response $H(\omega) = \frac{1}{4+j\omega}$ for a particular input $x(t)$. The system is observed to produce the output $y(t) = e^{-2t}u(t) - e^{-4t}u(t)$

Find the input $x(t)$

Ans: $x(t) = 2e^{-2t}u(t)$

Distortionless Transmission through a system :

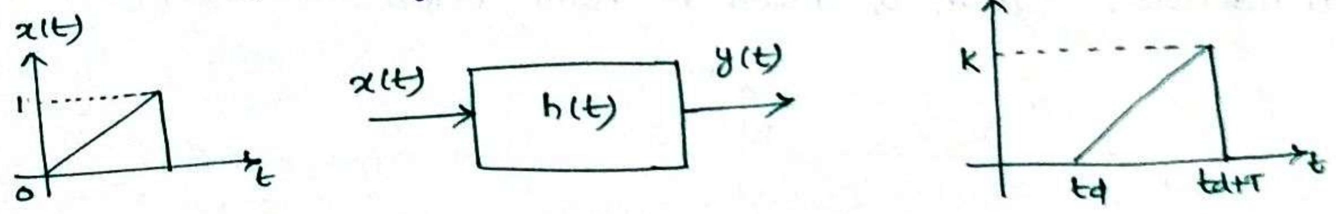
The change of shape of the signal when it is transmitted through a system is called distortion. Transmission of a signal through a system is said to be distortionless if the output is an exact replica of the input signal. This replica may have different amplitude and also it may have different time delay. A constant change in amplitude and a ~~the~~ constant change in time delay are not considered as distortion. Only the shape of the signal is ~~different~~ important.

Mathematically, we can write the equation for the ouput when a signal is transmitted through a system without distortion as

$$y(t) = k\,x(t-t_d) \quad \text{---} ①$$

where k is a constant representing change in amplitude

t_d - delay time



Distortionless system

Apply F.T to the eqn ①

$$FT[y(t)] = k\,FT[x(t-t_d)] \quad \text{---} ②$$

Using Time shifting property $FT[x(t-t_d)] = e^{-j\omega t_d} X(\omega)$ --- ③

From eqns ② & ③

$$Y(\omega) = k e^{-j\omega t_d} X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = k e^{-j\omega t_d} \quad \text{---} ④$$

$H(\omega)$ - Transfer function of the system

Transfer function can be represented by using its magnitude and phase function as

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)} \quad \text{——} (5)$$

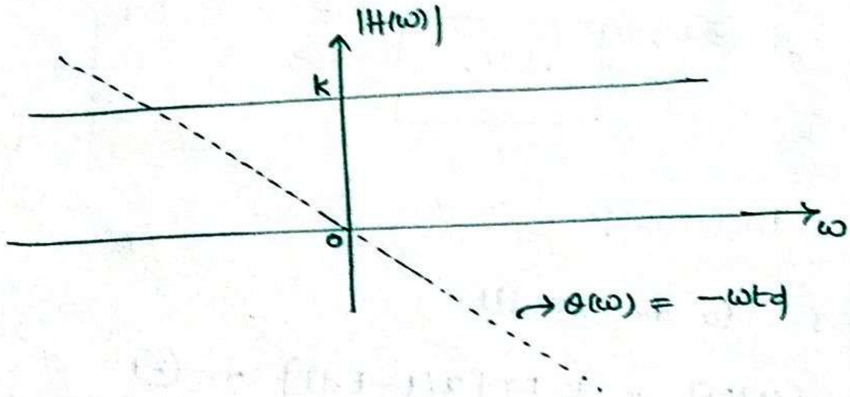
From eqn (4) $|H(\omega)| = k$

& $\theta(\omega) = -\omega t_d$

In general $\theta(\omega) = n\pi - \omega t_d$

So for a distortionless transmission of a signal through a system, the magnitude should be constant. i.e, system bandwidth is infinite and the phase spectrum should be proportional to the frequency. But in practice, no system can have infinite bandwidth and hence distortionless conditions are never met exactly.

The magnitude and phase characteristics of a distortionless transmission system is shown in figure below.



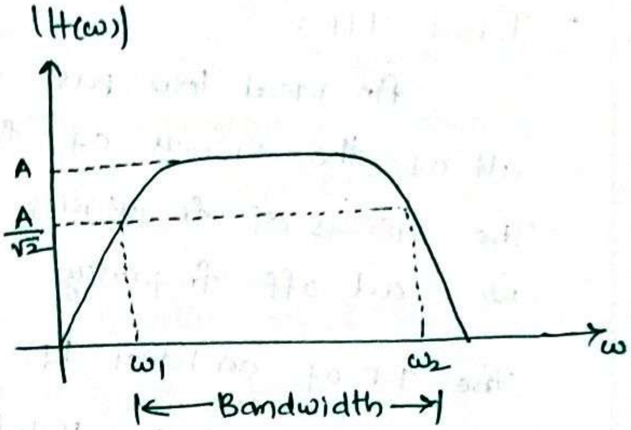
Signal Bandwidth:

The spectral components of a signal extend from $-\infty$ to ∞. Any practical signal has finite energy. As a result, the spectral components approach zero as $\omega \to \infty$. Therefore we can neglect the spectral components which have negligible energy and select only a band of frequency components which have most of the signal energy. This band of frequencies which contain most of the signal energy (95% of total energy) is known as the bandwidth of the signal.

System Bandwidth:

For distortionless transmission, we need a system with infinite bandwidth. Due to physical limitations, it is impossible to design a system with infinite bandwidth. But it is possible to design a distortionless system over a certain range of frequencies. This range of frequencies is known as the system bandwidth.

The bandwidth of a system is defined as the range of frequencies over which the magnitude $|H(\omega)|$ remains within $\frac{1}{\sqrt{2}}$ times of its value at midband.



Here the Bandwidth is $\omega_2 - \omega_1$

ω_2 - upper cut-off (or) upper 3dB frequency

ω_1 - lower cut-off (or) lower 3dB d frequency

Ideal Filter characteristics:

A filter is a frequency selective network. It allows the transmission of signals of certain frequencies without attenuation or with very little attenuation and it rejects or heavily attenuates signals of all other frequencies.

An ideal filter has very sharp cut-off frequencies characteristics, so it passes signals of certain specified band of frequencies exactly and totally rejects signals of frequencies outside the specified band.

Based on the frequency response characteristics Filters are classified into

① low pass filter (LPF)

② high pass filter (HPF)

③ Band pass filter (BPF)

④ Band stop or Band eliminate or Band Reject filter (BSF or BEF or BRF)

Ideal LPF:

An ideal low pass filter transmits without any distortion, all of the signals of frequencies below a certain frequency ω_c rad/sec. The signals of frequencies above ω_c rad/sec are completely attenuated.

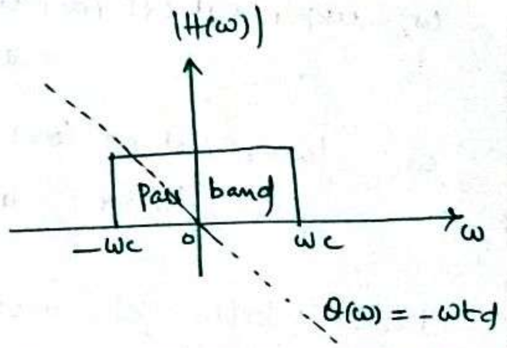
ω_c - cut-off frequency

The T.F of an ideal LPF is given by

$$|H(\omega)| = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

and phase function is

$$\theta(\omega) = \angle H(\omega) = -\omega t_d$$



Freq. response characteristics of an Ideal LPF

Ideal HPF:

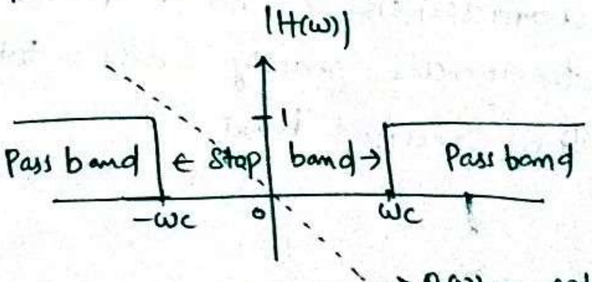
An Ideal high pass filter transmits without any distortion, all of the signals above a certain frequency ω_c rad/sec and attenuates completely the signals of frequencies below ω_c rad/sec.

ω_c - cut-off frequency

The Transfer Function of an Ideal HPF is given by

$$|H(\omega)| = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$

and $\theta(\omega) = \angle H(\omega) = -\omega t_d$



Freq. response characteristics of an Ideal HPF

Ideal BPF :

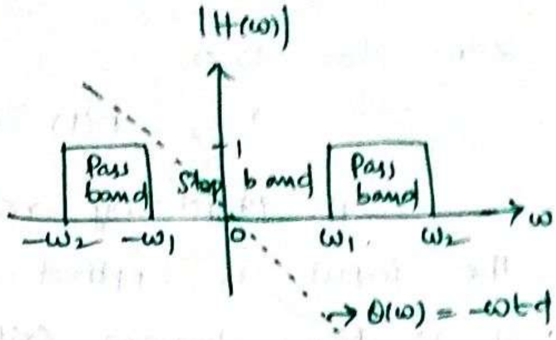
An ideal band pass filter transmits without any distortion, all of the signals of frequencies within a specified frequency band and attenuates completely the signals of frequencies outside this band

An ideal BPF is specified by

$$|H(\omega)| = \begin{cases} 1, & |\omega_1| < \omega < |\omega_2| \\ 0 & \omega < |\omega_1| \text{ \& } \omega > |\omega_2| \end{cases}$$

and $\theta(\omega) = \angle H(\omega) = -\omega t_d$

Bandwidth $= \omega_2 - \omega_1$



Freq. Response characteristics of an ideal BPF

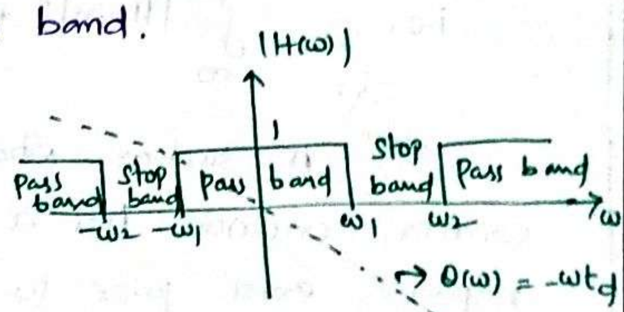
Ideal BRF :

An ideal band rejection filter rejects totally all of the signals of frequencies within a specified frequency band $(\omega_2 - \omega_1)$ rad/sec and transmits without any distortion all the signals of frequencies outside this band. $(\omega_2 - \omega_1)$ is rejection band.

An ideal BRF is specified by

$$|H(\omega)| = \begin{cases} 0, & |\omega_1| < \omega < |\omega_2| \\ 1 & \omega < |\omega_1| \text{ \& } \omega > |\omega_2| \end{cases}$$

and $\angle H(\omega) = -\omega t_d$

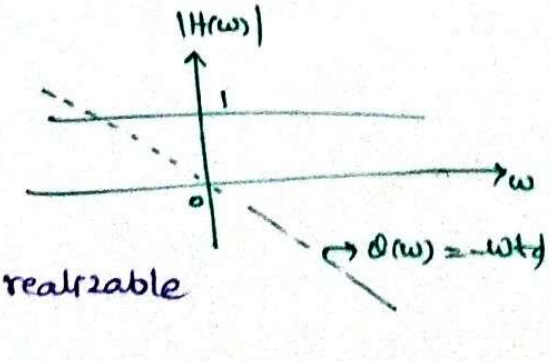


Freq. Response characteristics of an ideal BRF

All Pass Filters :

All pass filter transmits signals of all frequencies without any distortion. Its bandwidth is ∞

$|H(\omega)| = 1 \quad \forall\, \omega$ & $\angle H(\omega) = -\omega t_d$



All ideal filters are non-causal systems. Hence these are not physically realizable

Causality and Paley - Wiener Criterion:

A system is said to be causal if does not produce an o/p before the input is applied. For an LTI system to be causal, the condition to be satisfied is its impulse response must be zero for $t<0$

i.e; $h(t)=0$ for $t<0$

A physically realizable system cannot have a response before the input is applied. This is known as causality condition. And it is time domain criterion of physically realizability.

In the frequency domain, causality condition for a magnitude function $H(\omega)$ to be physically realizable is

$$\int_{-\infty}^{\infty} \frac{\ln|H(\omega)|}{1+\omega^2} d\omega < \infty$$

For above eqn to be valid, $|H(\omega)|$ must be square-integrable

i.e. $$\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega < \infty$$

A system whose magnitude function violates the Paley-Wiener criterion has a non-causal impulse response i.e, the response exist prior to the application of the input. This type of systems are not physically realizable.

Relationship between Bandwidth and Rise time :

The transfer function of an ideal LPF is given by

$$H(\omega) = |H(\omega)| e^{-j\omega t_d}$$

where $$|H(\omega)| = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

ω_c - cut-off frequency

$$\therefore \quad H(\omega) = e^{-j\omega t_d} \quad \text{for } |\omega| < \omega_c \quad \text{i.e } -\omega_c < \omega < \omega_c$$

$$= 0 \quad \text{for } \omega > |\omega_c|$$

The impulse response $h(t)$ of the LPF is obtained by taking the inverse Fourier Transform of $H(\omega)$

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega$$

$$h(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega t_d} \cdot e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_d)} d\omega$$

$$= \frac{1}{2\pi} \left. \frac{e^{j\omega(t-t_d)}}{j(t-t_d)} \right|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi j (t-t_d)} \left(e^{j\omega_c(t-t_d)} - e^{-j\omega_c(t-t_d)} \right)$$

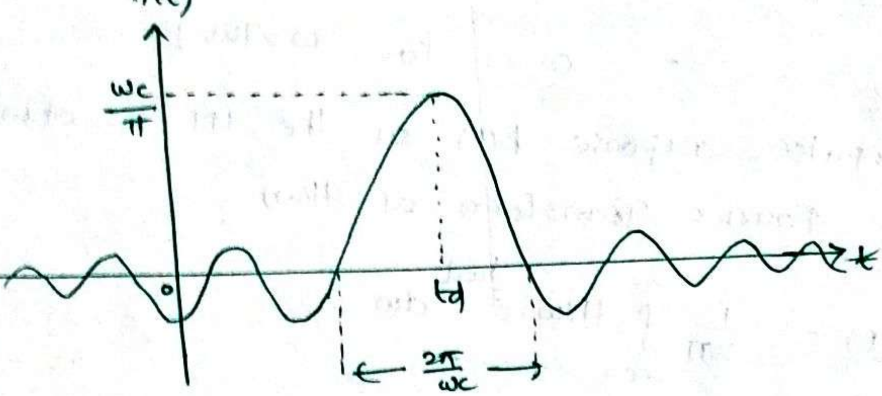
$$= \frac{1}{\pi(t-t_d)} \times \frac{e^{j\omega_c(t-t_d)} - e^{-j\omega_c(t-t_d)}}{2j}$$

$$h(t) = \frac{1}{\pi(t-t_d)} \sin w_c(t-t_d)$$

$$h(t) = \frac{\sin w_c(t-t_d)}{w_c(t-t_d)} \cdot \left(\frac{w_c}{\pi}\right)$$

$$h(t) = \frac{w_c}{\pi}\left[\frac{\sin w_c(t-t_d)}{w_c(t-t_d)}\right] \quad \text{—— (1)}$$

$$h(t) = \frac{w_c}{\pi} \operatorname{Sinc}(w_c(t-t_d)) \quad \text{—— (2)}$$



Impulse response of an ideal LPF

impulse response $h(t)$ is non zero for $t<0$, even the input $\delta(t)$ is applied at $t=0$, the impulse response begins befor the input is applied. So this type of systems are not physically realizable. i.e, Ideal LPF is not physically realizable.

If the impulse response is known, the step response can be obtained by the convolution.

here input $x(t) = u(t)$

$$\therefore \quad y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$

$$u(t-\tau) = 1 \quad \text{for} \quad t-\tau \geq 0$$
$$t \geq \tau$$
$$\tau \leq t$$

$$\therefore\ y(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \quad \text{——} \ (3)$$

From eqn (2) $\quad h(t) = \frac{\omega_c}{\pi} \frac{\sin \omega_c (t-t_d)}{\omega_c (t-t_d)}$

$$y(t) = \int_{-\infty}^{t} \frac{\omega_c}{\pi} \frac{\sin \omega_c (\tau - t_d)}{\omega_c (\tau - t_d)}\, d\tau$$

let $\quad x = \omega_c (\tau - t_d)$

$\quad dx = \omega_c\, d\tau \quad \Rightarrow \quad d\tau = \frac{dx}{\omega_c}$

$$y(t) = \int_{-\infty}^{\omega_c (t-t_d)} \frac{\omega_c}{\pi} \frac{\sin x}{x} \frac{dx}{\omega_c}$$

$$y(t) = \frac{1}{\pi} \int_{-\infty}^{\omega_c (t-t_d)} \frac{\sin x}{x}\, dx$$

$$y(t) = \frac{1}{\pi} \left[Si \right]_{-\infty}^{\omega_c (t-t_d)}$$

$\int \frac{\sin x}{x}\, dx = Si[x]$

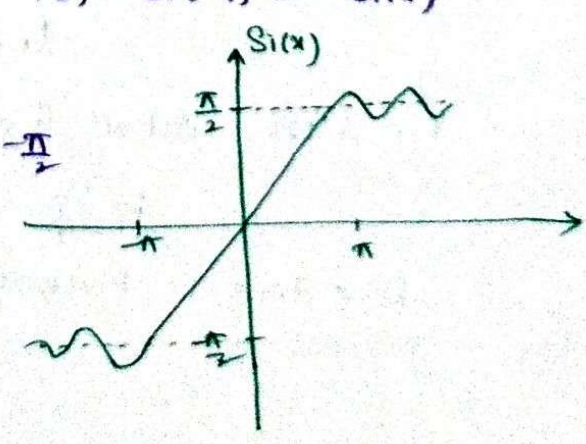
Si - Sine-integral function

The properties of sine integral functions are

i) $Si(x)$ is an odd function i.e, $Si(-x) = -Si(x)$

ii) $Si(0) = 0$

iii) $Si(\infty) = \frac{\pi}{2}$ and $Si(-\infty) = -\frac{\pi}{2}$



$$y(t) = \frac{1}{\pi} \, Si(x) \Big]_{-\infty}^{\omega_c(t-t_d)}$$

$$\therefore \; y(t) = \frac{1}{\pi}\left\{ Si[\omega_c(t-t_d)] - Si(-\infty)\right]$$

$$y(t) = \frac{1}{\pi}\left\{ Si[\omega_c(t-t_d)] + Si(\infty)\right]$$

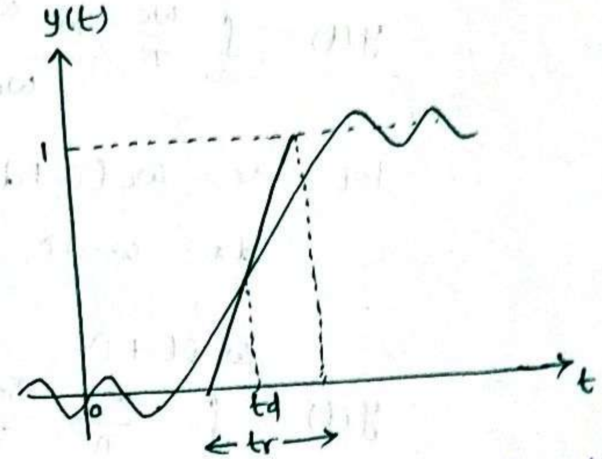
$$y(t) = \frac{1}{\pi}\left\{ Si[\omega_c(t-t_d)] + \frac{\pi}{2}\right] = \frac{1}{2} + \frac{1}{\pi} Si[\omega_c(t-t_d)]$$

if $\omega_c \to \infty$ then $y(t) = \frac{1}{2} + \frac{1}{\pi} Si(\infty) = \frac{1}{2} + \frac{1}{\pi}\cdot\frac{\pi}{2} = 1$

if $\omega_c \to -\infty$ then $y(t) = \frac{1}{2} + \frac{1}{\pi} Si(-\infty) = \frac{1}{2} + \frac{1}{\pi}\left(\frac{-\pi}{2}\right) = 0$

The step response of LPF is shown in Figure below

From the fig. it is clear that $y(t)$ approaches a delayed unit step $u(t-t_d)$



Rise time t_r is defined as the time required for the response to reach from 0% to 100% of the final value. To find it draw a tangent at $t = t_d$ with the line $y(t)=0$ and $y(t)=1$

$$\therefore \; \frac{dy(t)}{dt}\Big|_{t=t_d} = \frac{1-0}{t_r} = \frac{\omega_c}{\pi} \, \frac{\sin \omega_c(t-t_d)}{\omega_c(t-t_d)}\Big|_{t=t_d} = \frac{\omega_c}{\pi}$$

$$\therefore \; t_r = \frac{\pi}{\omega_c}$$

For a LPF cutoff frequency = Bandwidth

$$\therefore \; t_r = \frac{\pi}{B}$$

Rise time is inversely proportional to the bandwidth.