UNIT-I

Exact Differential Equations:

Def: Let M(x,y)dx +N(x,y) dy =0 be a first order and first degree Differential Equation where M & N are real valued functions of x,y . Then the equation Mdx + Ndy =0 is said to be an exact Differential equation if ∃ a function f ∋.

$$d[f(x,y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Condition for Exactness: If M(x,y) & N (x,y) are two real functions which have continuous partial derivatives then the necessary and sufficient condition for the Differential equation Mdx+ Ndy =0 is to be exact is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence solution of the exact equation M(x,y)dx +N(x,y) dy =0. Is

$$\int M dx + \int N dy = c.$$

(y constant) (terms free from x).

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PROBLEMS:

1)Solve $(1+e^{\frac{x}{4}})$ dx + $e^{\frac{x}{y}}(1-\frac{x}{y})$ .dy =0

Sol: Hence M = $1+e^{\frac{x}{y}}$ & N = $e^{\frac{x}{y}}(1-\frac{x}{y})$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \quad \& \quad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-1}{y}\right) + (1-\tfrac{x}{y})e^{\frac{x}{y}}(\frac{1}{y})$$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \quad \& \quad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{equation is exact}$$

General solution is

$$\int M dx + \int N dy = c.$$

(y constant) (terms free from x)

$$\int (1+e^{\frac{x}{y}})\,dx + \int 0\,dy = c.$$

$$\Rightarrow x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$\Rightarrow x + y\,e^{\frac{x}{y}} = C$$

2. $(e^{y}+1)$ .cosx dx + $e^{y}\ sinx\ dy$ =0.

Ans: $(e^{y}+1)$. sinx =c $\qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ = e^{x}cosx

3. (r+sinθ $-$casθ) dr + r (sin$\theta + cos\theta$) d$\theta = 0$.

Ans: r^2+2r(sin$\theta + cos\theta) = 2c$

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta} = \sin\theta + cos\theta.$$

4. Solve [y($1 + \frac{1}{x}$) +cos y] dx+ [x +logx –xsiny]dy =0.

Sol: hence M = y($1 + \frac{1}{x}$) +cos y N = x +logx –xsiny.

$\frac{\partial M}{\partial y}$ = 1+$\frac{1}{x}$ -siny $\qquad \frac{\partial N}{\partial x}$ = 1 +$\frac{1}{x}$ -siny

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the equation is exact

General sol $\int Mdx$ + $\int Ndy$ = c.

(y constant) (terms free from x)

$\int[y + \frac{y}{x} + cosy]dx$ + $\int o\ .dy$ = c.

$\Rightarrow$Y(x+ logx) +x cosy = c.

5. ysin2xdx – (y^2+cosx) .dy =0.

6. (cosx-xcosy)dy – (siny+(ysinx))dx =0

Sol: N = cosx-x cosy & M = -siny-ysinx

$\frac{\partial N}{\partial x}$ = -sinx - cosy $\qquad \frac{\partial M}{\partial y}$ = -cosy - sinx

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow$ the equation is exact.

General sol $\int Mdx$ + $\int Ndy$ = c.

(y constant) (terms free from x)

=> $\int(-siny - ysinx).dx$ + $\int o\ .dy$ = c

=> -xsiny+ ycos x =c

=> ycosx – xsiny =c.

7. (sinx . siny - x e^{y}) dy = (e^{y} +cosx-cosy) dx

Ans: xe^{y} +sinx.cosy =c.

8. $(x^2+y^2-a^2)$ x dx +$(x^2-y^2-b^2)$. y .dy =0

Ans: $x^4+2x^2y^2-2a^2x^2-2b^2y^2$ =c .

REDUCTION OF NON-EXACT DIFFERENTIAL EQUATIONS TO EXACT USING INTEGRATING FACTORS

Definition: If the Differential Equation M(x,y) dx + N (x,y) dy = 0. Can be made exact by multiplying with a suitable function u (x,y) $\neq$ 0. Then this function is called an Integrating factor(I.F).

Note: there may exits several integrating factors.

Some methods to find an I.F to a non-exact Differential Equation Mdx+N dy =0

Case -1: Integrating factor by inspection/ (Grouping of terms).

Some useful exact differentials

1. $d(xy) = x\,dy + y\,dx$
2. $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$
3. $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$
4. $d\left(\frac{x^2+y^2}{2}\right) = x\,dx + y\,dy$
5. $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{xy}$
6. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{xy}$
7. $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{x^2+y^2}$
8. $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$
9. $d(\log(xy)) = \frac{x\,dy + y\,dx}{xy}$
10. $d(\log(x^2+y^2)) = \frac{2(x\,dx + y\,dy)}{x^2+y^2}$
11. $d\left(\frac{e^x}{y}\right) = \frac{y e^x\,dx - e^x\,dy}{y^2}$

PROBLEMS:

1 . Solve $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$.

Sol: Given equation $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

$$d\left(\frac{x^2+y^2}{2}\right) + d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$$

on Integrating

$$\frac{x^2+y^2}{2} + tan^{-1}\left(\frac{y}{x}\right) = c.$$

2 . Solve y(x^3. $e^{xy} - y$) dx + x (y + x^3. e^{xy}) dy = 0.

Sol: Given equation is on Regrouping

We get $yx^3\, e^{xy}\, dx$ - y^2dx+ x^2y dy +x^4 e^{xy} dy =0.

$$X^3 e^{xy}(ydx+ xdy)+ y (x\, dy – ydx) = 0$$

Dividing by x^3

$$e^{xy} (ydx + xdy) +\left(\frac{y}{x}\right) . \left(\frac{xdy-ydx}{x^2}\right) =0$$

$$d (e^{xy}) +\left(\frac{y}{x}\right) .d +\left(\frac{y}{x}\right) = 0$$

on Integrating

$$e^{xy} + ½\left(\frac{y}{x}\right)^2 = C$$ is required G.S.

3. (1+xy) x dy + (1- yx) y dx = 0

Sol: given equation is (1+xy) x dy +(1-yx) y dx =0.

$$(xdy + y\, dx) + xy (xdy – y\, dx) = 0.$$

Divided by x^2y^2 => $\left(\frac{xdy+ydx}{x^2y^2}\right) + \left(\frac{xdy-ydx}{xy}\right) =0$

$$\left(\frac{d(xy)}{x^2y^2}\right) + \frac{1}{y}\, dy - \frac{1}{x}\, dx =0.$$

On integrating => $\frac{1}{xy}$+ log y – log x =log c

$$-\frac{1}{xy} - \log x +\log y =\log c.$$

4. Solve ydx –x dy = a $(x^2 + y^2)$ dx

Ans: $\frac{ydx - x\,dy}{(x^2+y^2)}$ = a dx

$$d \left(tan^{-1} \frac{y}{x}\right) = a\, dx$$

integrating on $tan^{-1} \frac{y}{x}$ = ax +c

Method -2: If M(x,y) dx + N (x,y) dy =0 is a homogeneous differential equation and Mx +Ny $\neq$ 0, then $\frac{1}{Mx+Ny}$ is an integrating factor of Mdx+ Ndy =0.

1 . Solve x^2y dx – (x^3+ y^3) dy = 0

Sol : $x^2y\,dx - (x^3 + y^3)\,dy = 0$-----------------(1)

Where $M = x^2y$ & $N = (-x^3 - y^3)$

Consider $\frac{\partial M}{\partial y} = x^2$ & $\frac{\partial N}{\partial x} = -3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

But given equation(1) is homogeneous D.Equation then

So $Mx + Ny = x(x^2y) - y(x^3 + y^3) = -y^4 \neq 0$.

$$\text{I.F} = \frac{1}{Mx + Ny} = \frac{-1}{y^4}$$

Multiplying equation (1) by $\frac{-1}{y4}$

$$=> \frac{x^2y}{-y^4}\,dx - \frac{x^3 + y^3}{-y^4}\,dy = 0 \quad \text{----------------------(2)}$$

$$=> -\frac{x2}{y3}\,dx - \frac{x3 + y3}{-y4}\,dy = 0$$

This is of the form $M_1dx + N_1dy = 0$

For $M_1 = \frac{-x2}{y3}$ & $N_1 = \frac{x^3 + y^3}{-y^4}$

$$=> \frac{\partial M1}{\partial y} = \frac{3x^2}{y4} \;\&\; \frac{\partial N1}{\partial x} = \frac{3x^2}{-y^4}$$

$=> \frac{\partial M1}{\partial y} = \frac{\partial N1}{\partial x}$ equation (2) is an exact D.equation.

General sol $\int M\,dx + \int N\,dy = c.$

(y constant) (terms free from x in N)

$$=> \int \frac{-x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c.$$

$$=> \frac{-x^3}{3y^3} + \log|y| = c.//$$

2. Solve $y^2\,dx + (x^2 - xy - y^2)\,dy = 0$

Ans: $(x-y) . y^2 = c1^2(x+y)$.

3. Solve $y(y^2 - 2x^2)\,dx + x(2y^2 - x^2)\,dy = 0$ -----------(1)

Sol: it is the form $Mdx + Ndy = 0$

Where $M = y(y^2 - 2x^2)$ $N = x(2y^2 - x^2)$

Consider $\frac{\partial M}{\partial y}$ = $3y^2-2x^2$ & $\frac{\partial N}{\partial x}$ = $2y^2-3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

Since equation(1) is homogeneous D.Equation then

Consider Mx+ N y= $x[y(y^2-2x^2)] + y[x(2y^2-x^2)]$

$= 3xy(y^2-x^2) \neq 0.$

=> I.F. = $\frac{1}{3xy(y^2-x^2)}$

Multiplying equation (1) by $\frac{1}{3xy(y^2-x^2)}$ we get

$=> \frac{y(y^2-x^2)}{3xy(y^2-x^2)} dx + = \frac{x(y^2-x^2)}{3xy(y^2-x^2)}$ dy =0

=> now it is exact (check)

$\frac{(y^2-x^2)-x^2}{3xy(y^2-x^2)}$ dx + $\frac{y^2+(y^2-x^2)}{3xy(y^2-x^2)}$ dy =0.

$\frac{dx}{x} - \frac{xdx}{y^2-x^2} + \frac{ydy}{y^2-x^2} + \frac{dy}{y} = 0.$

$(\frac{dx}{x} + \frac{dy}{y}) + \frac{2ydy}{2(y^2-x^2)} \frac{2xdx}{2(y^2-x^2)} = 0$

Log x +log y $+\frac{1}{2}$ log (y^2-x^2) $-\frac{1}{2}$ log $(y^2-x^2) = c \Rightarrow$ xy = c

4. $r(\theta^2+r^2)\ d\theta - \theta(\theta^2+2r^2)$ dr =0

Ans: $\frac{\theta^2}{2r^2}$ + logθ + log r^2 =c.

Method- 3: If the equation Mdx + N dy =0 is of the form y. f (xy) .dx + x . g (xy) dy = 0 & Mx- Ny $\neq 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor of Mdx+ Ndy =0.

Problems:

1 . solve (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0.

Sol: (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0 -------(1).

=> this is the form y. f(xy) .dx + x . g (xy) dy =0.

=> consider Mx-Ny

Here M =(xy sinxy + cos xy) y

N= (xy sinxy- cos xy) x

Consider Mx-Ny =2xycosxy

Integrating factor = $\frac{1}{2xycosxy}$

So equation (1) x I.F

$\Rightarrow \dfrac{(xy\sin xy+\cos xy)x}{2xy\ \cos xy}dx + \dfrac{(xy\sin xy+\cos xy)y}{2xy\ \cos xy}dy = 0.$

$\Rightarrow$ (y tan xy + $\frac{1}{x}$) dx + (y tan xy - $\frac{1}{y}$) dy =0

$\Rightarrow$ M_1 dx + N_1 dx =0

Now the equation is exact.

General sol $\int$ M_1 dx + $\int$ N_1 dy = c.

(y constant) (terms free from x in N_1)

=> $\int(ytanxy+\frac{1}{x})dx$ + $\int\frac{-1}{y}dy$ = c.

=> $\dfrac{y.\log|secxy|}{y}$ +logx + (-logy) = log c

=> log|sec(xy)| +log$\frac{x}{y}$ =log c.

=> $\frac{x}{y}$. secxy =c.

2. Solve (1+xy) y dx + (1-xy) x dy =0

Sol : I.F =$\frac{1}{2x^2y^2}$

=> $\int\frac{1}{2x^2y}+\frac{1}{2x})dx$ + $\int\frac{-1}{2y}dy$ =c

=> $\frac{-1}{2xy}$ + $\frac{1}{2}$ logx -$\frac{1}{2}$ log y =c.

=> $\frac{-1}{xy}$ +log($\frac{x}{y}$) =c^1 where c^1 = 2c.

3. Solve (2xy+1) y dx + (1+ 2xy-x^3y^3) x dy =0

Ans: log y + $\frac{1}{x^2y^2}$ + $\frac{1}{3x^3y^3}$ =c.

4. solve (x^2y^2 +xy +1) ydx +(x^2y^2- xy+1) xdy =0

Ans: xy -$\frac{1}{xy}$ + log($\frac{x}{y}$) =c .

Method -4: If there exists a single variable function $\int$f(x) dx such that $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ =f(x), then I.F. of Mdx + N dy =0 is e

PROBLEMS:

1 . Solve (3xy – 2ay^2) dx + (x^2-2axy) dy =0

Sol: given equation is the form Mdx+ Ndy = 0

$$=> \quad M = 3xy - 2ay^2 \qquad \& \ N = x^2 - 2axy$$

$$\frac{\partial M}{\partial y} = 3x-4ay \ \& \quad \frac{\partial N}{\partial x} = 2x-2ay$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{equation not exact .}$$

Now consider $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \dfrac{(3x-4ay)-(2x-2ay)}{(2x-2ay)}$

$$=> \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{1}{x} = f(x).$$

$=> \quad e^{\int \frac{1}{x}dx} = x$ is an Integrating factor of (1)

=> equation (1) x I.F = equation (1) X x

$$=> \frac{(3xy-2ay^2)}{1} \ x\,dx + \frac{(x^2-2axy)}{1} \ x\,dy = 0$$

$=> (3x^2y - 2ay^2x)\,dx + (x^3-2ax^2y)\,dy = 0$

It is the form $M_1dx + N_1dy = 0$

General sol $\int M_1dx + \int N_1dy = c.$

$$=> \int (3x^2 - 2ay^2x)dx + \int o\,d\,y = c$$

$=> x^3y - ax^2y^2 = c$ .//

2 . Solve $ydx-xdy+(1+x^2)dx + x^2 \sin y\, dy = 0$

Sol : given equation is $(y+1+x^2)\,dx + (x^2 \sin y - x)\,dy = 0.$

$$M = y+1+x^2 \quad \& \quad N = x^2 \sin y - x$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 2x \sin y - 1$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ => the equation is not exact.

So consider $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M} = \dfrac{(1-2x\sin y-x)}{x^2\sin y - x} = \dfrac{-2x\sin y - x)}{x^2 \sin y - x} = \dfrac{-2}{x}$

$$\text{I.F} = e^{\int g(y)dy} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$$

Equation (1) X I.F $\quad => \dfrac{y+1+x^2}{x^2}\,dx + \dfrac{x^2 \sin y - x}{x^2}\,dy = 0$

It is the form of M1dx+ N1 dy =0.

Gen soln $\quad => \int (\frac{y}{x^2} + \frac{1}{x^2} + 1)dx + \int \sin y dy = 0$

$$=> \frac{-y}{x} - \frac{1}{x} + x - \cos y = c.$$

$$=> x^2 - y - 1 - x\cos y = cx.//$$

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3. Solve $2xy\,dy - (x^2+y^2+1)dx = 0$

Ans: $-x + \frac{y^2}{x} + \frac{1}{x} = c$.

4. Solve $(x^2+y^2)\,dx - 2xy\,dy = 0$

Ans: $x^2-y^2=cx$.

Method -5: For the equation $M dx + N dy = 0$ if $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = g(y)$ (is a function of y alone) then $e^{\int g(y)dy}$ is the Integrating factor of $M\,dx + N\,dy = 0$.

Problems:

1 . Solve $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)\,dy = 0$

Sol: $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)\,dy = 0$ -----------------(1).

Here $M\,dx + N\,dy = 0$.

Where $M = 3x^2y^4+2xy$ & $N = 2x^3y^3-x^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{equation (1) not exact.}$$

So consider $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{-2}{y} = g(y)$

$$\text{I.F} = e^{\int g(y)dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = \frac{1}{y^2}.$$

Equation (1) x I.F => $\left(\frac{3x2y4+2xy}{y^2}\right)dx + \left(\frac{2x3y3-x2}{y^2}\right)dy = 0$

It is the form $M1dx + N1\,dy = 0$

General sol $\int M1dx + \int N1dy = c$.

(y constant) (terms free from x in N1)

$$=> \int\left(3x^2y^2 + \frac{2x}{y}\right)dx + \int o\,dy = c.$$

$$=> \frac{3x^3y^2}{3} + \frac{2x^2}{2y} = c.$$

$$=> x^3y^2 + \frac{x^2}{y} = c.//$$

2 . Solve $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$

Sol: $\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M} = \frac{(4xy^2+2)-(3xy^2+1)}{xy^3+y} = \frac{1}{y} = g(y)$.

$$\text{I.F} = e^{\int g(y)dy} = e^{\int \frac{1}{y}dy} = y.$$

Gen sol: $\int (xy4 + y2)dx + \int (2y5)dy = c$

$\frac{x^2 y^4}{2} + y^2$ x $+ \frac{2y^6}{6}$ =c .

3 . solve $(y^4+2y)dx + (xy^3 + 2y^4 - 4x)\, dy = 0$

Sol: $\frac{(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})}{M} = \frac{(y^3-4)-(4y^3+2)}{y^4+2y} = \frac{-3}{y}$ = g(y).

I.F $= e^{\int g(y)dy} = e^{-3\int \frac{1}{y}dy} = \frac{1}{y^3}$

Gen soln : $\int \left(y + \frac{2}{y^2} \right) dx + \int 2y dy = c$.

$\left(y + \frac{2}{y^2}\right)x + y^2 = c.$ //

4 Solve $(3x^2 y^4 + 2xy)dx + (2x^3y^3 - x^2)\, dy = 0$

Ans : $x^3y^3 + x^2 = cy$

5. Solve $(y + y^2)dx + xy\, dy = 0$

Ans: x + xy =c .

6. Solve $(xy^3+y)\, dx + 2(x^2y^2+x+y^4)dy = 0$.

Ans: $(x^2+y^4-1)\, e^{x^2}$ =c.

LINEAR DIFFERENTIAL EQUATION'S OF FIRST ORDER:

Def: An equation of the form $\frac{dy}{dx} + P(x).y = Q(x)$ is called a linear differential equation of first order in y.

Working Rule: To solve the liner equation $\frac{dy}{dx} + P(x).y = Q(x)$

first find the Integrating factor I.F $= e^{\int p(x)dx}$

General solution is y x I.F = $\int Q(x) \times I.F.dx + c$

Note: An equation of the form $\frac{dx}{dy} + p(y).x =$ Q(y) is called a linear Differential equation of first order in x.

Then Integrating factor $= e^{\int p(y)dy}$

Gen soln is = x X I.F = $\int Q(y) \times I.F.dy + c$

PROBLEMS:

1 . Solve $(1+ y^2)\, dx = (\tan^{-1}y - x)\, dy$

Sol: $(1+ y^2)\frac{dx}{dy} = (\tan^{-1}y - x)$

$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right) . x = \frac{\tan^{-1}}{1+y^2}$

It is the form of $\frac{dx}{dy}$ + p(y).x = Q(y)

I.F $= e^{\int p(x)dx} = e^{\int \frac{1}{1+y^2}dy} = e^{tan^{-1} y}$

=> Gen sol is x. $e^{tan^{-1} y} = \int \frac{tan^{-1}}{1+y^2} \cdot e^{tan^{-1} y}\, dy$+ c.

= > x. $e^{tan^{-1} y} = \int t.\ e^t\, dt + c$

[put $tan^{-1} y = t$

$\Rightarrow \frac{1}{1+y^2} dy = dt$]

$\Rightarrow$x. $e^{tan^{-1} y}$ = t . e^t -e^t +c

= > x. $e^{tan^{-1} y} = tan^{-1} y. e^{tan^{-1} y}$ -$e^{tan^{-1} y}$ +c

= > x $= tan^{-1} y - 1$ + c/ $e^{tan^{-1} y}$ is the required solution

2. Solve (x+y+1) $\frac{dy}{dx}$ = 1.

Sol: g iven equation is (x+y+1) $\frac{dy}{dx}$ = 1.

= > $\frac{dx}{dy} - x$ = y+1.

It is of the form $\frac{dx}{dy}$ + p(y).x = Q(y)

Where p(y) = -1 ; Q(y) = 1+y

= > I.F $= e^{\int p(y)dy} = e^{-\int dy} = e^{-y}$

Gen soln = x X I.F = $\int Q(y) \times I.F.dy + c$

= > x . $e^{-y} = \int (1+y)\, e^{-y} dy + c$

= > x . $e^{-y} = \int e^{-y} dy + \int y e^{-y} dy + c$

= > xe^{-y} = - e^{-y} − y xe^{-y} - e^{-y} +c

= > xe^{-y} = - $e^{-y}(2+y)$ +c .//

3. Solve $y^1 + y = e^{e^x}$

Sol: this is of the form $\frac{dy}{dx} + p(x).y =$ Q(x)

Where p(x) =1 Q(x) = e^{e^x}

=> I.F = $e^{\int p(x)dx} = e^{\int dx} = e^x$

Gen soln is is y x I.F = $\int Q(x) \times I.F.dx + c$

=> y. $e^x = \int e^{e^x} e^x dx + c$

$\Rightarrow$ y. $e^x = \int e^t t\, dt + c$ put $e^x = t$

$\Rightarrow$ y. e^x = t $e^t - e^t$ +c $e^x dx = dt$

$\Rightarrow$ y. $e^x = e^{e^x}(e^x - 1) + c.$

4. Solve x. $\frac{dy}{dx}$ + y =log x

Sol : this is of the form $\frac{dy}{dx}$ +p(x)y = $\emptyset(x)$.

Where p(x) = $\frac{1}{x}$ & $\emptyset(x) = \frac{\log x}{x}$

i.e , $\frac{dy}{dx} + \frac{1}{x}$. y = $\frac{\log x}{x}$

$\Rightarrow$ I.F = $e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x}$ =x.

Gen soln is is y x I.F = $\int Q(y) \times I.F.dy + c$

$\Rightarrow$ y.x = $\int \frac{\log x}{x}\, x\, dx + c$

$\Rightarrow$ y .x = x (logx-1) +c.//

5 . Solve (1+y^2) + (x- $e^{\tan^{-1} y}$) $\frac{dy}{dx}$ = 0.

Sol : Given equation is $\frac{dx}{dy} + \frac{x}{1+y2} = \frac{e^{\tan^{-1} y}}{1+y2}$

It is of the form $\frac{dx}{dy}$ + p(y) .x = Q(y)

Where p (y) = $\frac{1}{1+y2}$ Q(y) = $\frac{e^{\tan^{-1} y}}{1+y2}$.

I.F = $e^{\int p(y)dy} = e^{\int \frac{1}{1+y2} dy} = e^{\tan^{-1} y}$.

General solution is is x x I.F = $\int Q(y) \times I.F.dy + c$.

= > x . $e^{\tan^{-1} y}$. = $\int \frac{e^{\tan^{-1} y}}{1+y2}\, e^{\tan^{-1} y}$. dy +c

= > x . $e^{\tan^{-1} y} = \int e^t\, e^t$. dt +c

[**Note: put** $\tan^{-1} y$ = t

= > $\frac{1}{1+y2}\, dy$ = dt]

= > x . $e^{\tan^{-1} y} = \int e^{2t}$. dt +c

= > x . $e^{\tan^{-1} y} = \frac{e^{2t}}{2}$ + c

= > x . $e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2}$ + c //

6. solve $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{\sin 2x}{\log x}$

Ans: ylogx = $\frac{-\cos 2x}{2}$ + c.

7 . $\frac{dy}{dx}$ + (y-1). Cox = $e^{-\sin y}$ $\cos^2$ x

Ans: y. $e^{-\sin y} = \frac{x}{2} + \frac{\sin 2x}{4} + e^{-\sin y}$ +c //

8. $\frac{dy}{dx} + \frac{2x}{1+x2}$. y= $\frac{1}{(1+x^2)^2}$ given y = 0 , where x= 1.

Ans : y(1 + x2) = $\tan^{-1} x - \frac{\pi}{4}$

9. Solve $\frac{dy}{dx} - \frac{\tan y}{1+x}$ = (1+x) e^x . sec y

Sol : the above equation can be written as

Divided by sec y => cos y $\frac{dy}{dx}$ - $\frac{\sin y}{1+x}$ = (1+x) e^x ------------(1)

Put sin y = u

= > cos y $\frac{dy}{dx}$ =$\frac{du}{dx}$

D. Equation (1) is $\frac{du}{dx} - \frac{1}{1+x}$. u = (1+x) e^x

this is of the form $\frac{du}{dx} + p(x).u =$ Q(x)

Where p(x) =$\frac{-1}{1+x}$ Q(x) =(1+x) e^x

=> I.F = $e^{\int p(x)dx}$ = $e^{\int \frac{-1}{1+x}dx}$ =$e^{-\log(1+x)}$ =$\frac{1}{1+x}$

Gen soln is is u x I.F = $\int Q(y) \times I.F.dy + c$

=> u. $\frac{1}{1+x} = \int (1+x)\, e^x \frac{1}{1+x} dx + c$

=> u. $\frac{1}{1+x} = \int e^x dx + c$

=> (sin y) $\frac{1}{1+x}$ = e^x +c

(Or)

= > sin y = (1+x) e^x +c . (1+x) is required solution.

10. Solve $\frac{dy}{dx}$ - ytan = $\frac{\sin x \,.\, \cos^2 x}{y^2}$

Ans : $y^3 \cos^3 x = \frac{-\cos^6 x}{2}$ +c .

11 .Solve $\frac{dy}{dx}$ – yx = $y^2\, e^{\frac{x^2}{2}}\, .\sin x$

Ans: $\frac{1}{y} \; e^{\frac{-x^2}{2}}$ = cosx + c.

12. $e^{x} . \frac{dy}{dx}$ = 2xy^2 + y e^{x}

Ans : $\frac{1}{y} \; e^{x} = x^2$ + c.

13. $\frac{dy}{dx}$ + ycos x = y^3 sin x

Ans : : $\frac{1}{y^2}$ = (1+ 2 sinx) + c e^{2sinx} (or)

$\frac{-1}{y^2} \; e^{-2sinx}$= -(1+ 2 sinx) e^{-2sinx}+ c .

14. $\frac{dy}{dx}$ + y cot x = $y^2\, sin^2 x\, cos^2 x$

Ans: ysinx (c + $cos^3 x$) =3.

15. Solve $\frac{dy}{dx} = e^{x-y} (e^{x} - e^{y})$

Ans: $e^{x} . e^{e^{x}} = e^{e^{x}} (e^{x} - 1) + c$

BERNOULI'S EQUATION :

(EQUATION'S REDUCIBLE TO LINEAR EQUATION)

Def: An equation of the form $\frac{dy}{dx}$ + p(x) .y = Q(x) y^n **----- --------(1)**

Is called Bernoulli's Equation, where p & Q are function of x and n is a real constant.

Working Rule:

Case -1 : if n=1 then the above equation becomes $\frac{dy}{dx}$ + p. y = Q.

=> Gen soln of $\frac{dy}{dx} + (p - Q)y = 0$ is

$\int \frac{dy}{dx} + \int (p - Q)dx = c$ by variable separation method.

Case -2: if $n \neq 1$ then divide the given equation (1) by y^n

$\Rightarrow \quad y^{-n} . \frac{dy}{dx}$ + p(x) . y^{1-n} = Q ---------(2)

Then take y^{1-n} = u

(1-n) $y^{-n} . \frac{dy}{dx} = \frac{du}{dx}$

$$\Rightarrow \quad y^{-n}.\ \frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

Then equation (2) becomes

$$\frac{1}{1-n}\frac{du}{dx} + p(x)\,.\,u = Q$$

$\frac{du}{dx}$ + (1-n) p.u = (1-n)Q which is linear and hence we can solve it.

Problems:

1 . Solve $x\frac{dy}{dx} + y = x^3 y^6$

Sol: given equation can be written as $\frac{dy}{dx} + \left(\frac{1}{x}\right) y = x^2 + y^6$

Which is of the form $\frac{dy}{dx} + p(x).y = Q\ y^n$

Where p(x) = $\frac{1}{x}$ Q(x) = x^2 & n=6

Divided by y^2 => $\frac{1}{y^6}\ .\ \frac{dy}{dx} + \frac{1}{x}\frac{1}{y^5} = x^2$ ---------------(2)

Take $\frac{1}{y^5}$ = u

$$\Rightarrow \frac{-5}{y^6}\frac{dy}{dx} = \frac{du}{dx} \quad \}\text{----------------(3)}$$

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} = \frac{-1}{5}\frac{du}{dx} \quad \}\text{----------------(3)}$$

(3) in (2) => $\frac{du}{dx} - \frac{5}{x}u = -5x^2$

Which is a L.D equation in u

$$\text{I.F} = e^{\int p(x)dx} = e^{-5\int \frac{1}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

Gensol $\Rightarrow$ u .I.F = $\int Q(y) \times I.F.dy + c$

$$u.\frac{1}{x^5} = \int -5x2.\frac{1}{x^5}\ dx + c$$

$$\frac{1}{y^5x^5} = \frac{5}{2x^2} + c \quad \text{(or)} \ \frac{1}{y^5} = \frac{5x^3}{2} + cx^5$$

2. Solve $\frac{dy}{dx}(x^2y^3 + xy) = 1$

Sol: $\frac{dx}{dy}$ -x .y= x^2y^3 => $\frac{1}{x^2}.\frac{dx}{dy} - \frac{1}{x}.y = y^3$ ---------------------(1)

Put $\frac{1}{x}$ = u

$$\Rightarrow \frac{-1}{x^2}.\frac{dx}{dy} = \frac{du}{dx} \quad \text{---------------(2).}$$

(2) in (1) $\Rightarrow$ $-\frac{du}{dx} -$ u .y = y^3

(Or) $\frac{du}{dx}$ + u .y = -y^3.

Is a L.D Equation in ‘ u’

I.F = $e^{\int P(y)dy} = e^{\int y dy} = e^{-\frac{y^2}{2}}$

Gensol $\Rightarrow$ u .I.F =$\int Q(y)\times I.F.dy + c$

$\Rightarrow$ u . $e^{-\frac{y^2}{2}} = \int y^3 . e^{-\frac{y^2}{2}} dy$ + c

$\Rightarrow \frac{e^{-\frac{y^2}{2}}}{x}$ = -2($\frac{y^2}{2} - 1$) . $e^{-\frac{y^2}{2}}$ +c

(or)

X(2-y^2)+ cx$e^{-\frac{y^2}{2}}$ =1.

3. Solve $\frac{dy}{dx}$ -y tanx = y^2 sec x

Ans: I.F = $e^{-\int tanx dx} = e^{\int \log cox} = cosx$

Gen sol $\frac{1}{y}$ cos x= -x +c .

4. (1-x^2) $\frac{dy}{dx}$ + xy = y^3 sin^{-1}x

Sol: given equation can be written as

$\frac{dy}{dx} + \frac{x}{1-x^2}$ y = $\frac{y^3}{1-x^2} sin^{-1}$x

Which is a Bernoulli’s equation in ‘ y ‘

Divided by y^3 $\Rightarrow \frac{1}{y^3} . \frac{dy}{dx} + \frac{1}{y^2}\frac{x}{1-x^2} = \frac{sin^{-1}x}{1-x^2}$ -----------(1).

Let $\frac{1}{y^2}$ = u

$\Rightarrow \frac{-2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$ => $\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx}$-------------(2)

(2) in (1) $\Rightarrow -\frac{1}{2}\frac{du}{dx} + \frac{x}{1-x^2} . u = \frac{sin^{-1}x}{1-x^2}$

Which is a L.D equation in u

$\Rightarrow$ I.F = $e^{\int p(x)dx} = e^{-\int \frac{2x}{1-x^2}dx} = e^{\log(1-x^2)} = (1-x^2)$

Gensol $\Rightarrow$ u .I.F =$\int Q(x)\times I.F.dx + c$

$\Rightarrow \frac{1}{y^2} (1-x^2)$ = -$\int \frac{2sin^{-1}x}{1-x^2}(1-x^2)dx + c$

=> $\frac{(1-x^2)}{y^2}$ = -2 [xsin^{-1}x +$\sqrt{1-x^2}$] +c

5 . $e^x \frac{dy}{dx}$ = $2xy^2 + y\,.\ e^x$

Ans: $\frac{e^x}{y} = x^2 + c$.

NEWTON'S LAW OF COOLING

STATEMENT: The rate of change of the temp of a body is proportional to the difference of the temp of the body and that of the surroundings medium.

Let $'\theta'$ be the temp of the body at time 't' and θo be the temp of its surroundings medium(usually air). By the Newton's low of cooling , we have

$$\frac{d\theta}{dt} \propto (\theta - \theta o) \Rightarrow -\frac{d\theta}{dt} = k(\theta - \theta o) \quad \text{k is +ve constant}$$

$$\Rightarrow \int \frac{d\theta}{(\theta - \theta o)} . = -k \int dt$$

$$\Rightarrow \log(\theta - \theta o) = -kt + c.$$

If initially $\theta = \theta 1$ is the temp of the body at time t=0 then

$$c = \log(\theta 1 - \theta o) \Rightarrow \log(\theta - \theta o) = -kt + \log(\theta 1 - \theta o)$$

$$\Rightarrow \log\left(\frac{(\theta - \theta o)}{(\theta 1 - \theta o)}\right) = -kt.$$

$$\Rightarrow \frac{(\theta - \theta o)}{(\theta 1 - \theta o)} = e^{-kt}$$

$$\theta = \theta o + (\theta 1 - \theta o).\, e^{-kt}$$

Which gives the temp of the body at time 't' .

1. Find the O.T of the co focal and coaxial parabolas $r = \frac{2a}{1+\cos\theta}$

Ans: $r = \frac{c}{1-\cos\theta}$

Problems:

1 A body is originally at 80^0 and cools dowm to 60^0 c in 20 min . if the temp of the air is 40^0 c. Find the temp of body after 40 min.

Sol: By Newton's low of cooling we have

$$\frac{d\theta}{dt} = k(\theta - \theta o) \qquad \theta o \text{ is the temp of the air.}$$

$$\Rightarrow \int \frac{d\theta}{(\theta - 40)} . = -k \int dt \qquad \theta o = 40^0 \text{ c}$$

$$\Rightarrow \log(\theta - 40) = -kt + \log c$$

$$\Rightarrow \log\left(\frac{\theta - 40}{c}\right) = -kt$$

$\Rightarrow \frac{\theta - 40}{c} = e^{-kt}$

$\Rightarrow \theta = 40 + c\, e^{-kt}$ --------------(1)

When t=0 , $\theta = 80^0$ c $\Rightarrow$ 80 = 40 +c ------------(2).

When t=20 , $\theta = 60^0$ c $\Rightarrow$ 60 = 40 +ce^{-20k} ------------(3).

Solving (2) & (3) $\Rightarrow$ c$e^{-20k} = 20$

C=40 $\Rightarrow$ 40e^{-2k} =20

=> k = $\frac{1}{20}$ log2

When t= 40^0 c => equation (1) is $\theta = 40 + 40\, e^{-(\frac{1}{20}\log 2)40}$

= 40 +40 $e^{-2\log 2}$

= 40 + (40 x $\frac{1}{4}$)

$\Rightarrow \theta = 50^0$ c

2 . An object when temp is 75^0c cools in an atmosphere of constant temp. 25^0 c, at the rate k θ ,θ being the excess temp of the body over that of the temp. If after 10min , the temp of the object falls to 66^0 c , find its temp after 20 min . also find the time required to cool down to 55^0c .

Sol : we will take one as unit of time.

It is given that $\frac{d\theta}{dt}$ = - kθ

$\Rightarrow$ sol is $\theta =$ c e^{-kt} ------------(1).

Initially when t=0 $\Rightarrow \theta = 75^0 - 25^0 = 50^0$

$\Rightarrow$ c= 50^0 -----------------(2)

When t= 10 min $\Rightarrow \theta = 65^0 - 25^0 = 40^0$

$\Rightarrow$ 40= 50 e^{-10k}

$\Rightarrow e^{-10k} = \frac{4}{5}$ ---------------------(3).

The value of θ when t=20 $\Rightarrow \theta =$ c e^{-kt}

$\theta = 50e^{-20k}$

$\theta = 50(e^{-10k})^2$

$\theta = 50(\frac{4}{5})^2$

when t=20 $\Rightarrow \theta = 32^0$ c.

3. A body kept in air with temp 25^0c cools from 140^0 c to 80^0 in 20 min. Find when the body cools down in 35^0.

Sol : here $\theta o = 25^0c$ $\Rightarrow$ $\frac{d\theta}{(\theta-25)}$ = -k dt

$\Rightarrow$ log $(\theta - 25)$ = -kt +c -------------(1).

When t=0 , $\theta = 140^0$ c $\Rightarrow$ log (115) =c

$\Rightarrow$ c =log (115).

$\Rightarrow$ kt = - log $(\theta - 25)$+ log 115--------(2)

When t=20 , $\theta = 80^0$ c

$\Rightarrow$ log(80^0 c) = -20k + log 115

$\Rightarrow$ 20 k =log (115) - log(55) ---------(3)

(2)/ (3) => $\frac{kt}{20k} = \frac{\log 115 - \log(\theta-25)}{\log 115 - \log 55}$

$\frac{t}{20} = \frac{\log 115 - \log(\theta-25)}{\log 115 - \log 55}$

When $\theta = 35^0$ c $\Rightarrow$ $\frac{t}{20} = \frac{\log 115 - \log(10)}{\log 115 - \log 55}$

$\Rightarrow$ $\frac{t}{20} = \frac{\log(11.5)}{\log(\frac{23}{11})}$ = 3. 31

$\Rightarrow$ t = 20 $\times$ 3.31 = 66.2

The temp will be 35^0 c after 66.2 min.

4 . If the temp of the air is 20^0 c and the temp of the body drops from 100^0 c to 80^0 c in 10 min. What will be the its temp after 20min. When will be the temp 40^0 c .

Sol: log $(\theta - 20)$ = -kt + log c

c =80^0 c and $e^{-10k} = \frac{3}{4}$.

t =$\frac{10 \log(\frac{1}{4})}{\log(\frac{3}{4})}$.

5. the temp of the body drops from 100^0 c to 75^0 c is temp in 10 min. When the surrounding air is at 20^0 c temp. What will be its temp after half an hour, when will the temp be 25^0 c .

Sol : $\frac{d\theta}{dt}$ = -k$(\theta - \theta o)$

log $(\theta - 20)$ = -kt + log c

when t=0 , θ =100^0 => c=80

when t=10 , θ =75^0 = > e^{-10k} = $\frac{11}{16}$.

when t =30min => θ = 20 +80 ($\frac{1331}{4096}$) = 46^{O}C

when θ =25^0c = > t = 10 ($\frac{log5-log80}{log11-log6}$) = 74.86 min

LAW OF NATURAL GROWTH OR DECAY

(STATEMENT: Let x(t) or x be the amount of a substance at time ‘ t’ and let the substance be getting converted chemically . A law of chemical conversion states that the rate of change of amount x(t) of a chemically changed substance is proportional to the amount of the substance available at that time

$$\frac{dx}{dt} \quad \alpha \quad x \quad \text{(or)} \quad \frac{dx}{dt} = -\,kt \;;\; (\,k>0)$$

Where k is a constant of proportionality

Note: In case of Natural growth we take

$$\frac{dx}{dt} = k\,.x\,)$$

PROBLEMS

1 The number N of bacteria in a culture grew at a rate proportional to N . The value of N was initially 100 and increased to 332 in one hour. What was the value of N after 1hrs

Sol: The D. Equation to be solved is $\frac{dN}{dt}$ = kN

$\Rightarrow \quad \frac{dN}{N} = k\,dt$

$\Rightarrow \quad \int \frac{dN}{N} .= \int kdt$

$\Rightarrow$ log N = kt + log e

$\Rightarrow$ N = c e^{kt} ------------(1).

When t= 0sec , N =100 $\Rightarrow$ 100 = c $\Rightarrow$ c =100

When t =3600 sec , N =332 $\Rightarrow$ 332 =100 e^{3600k}

$\Rightarrow \quad e^{3600k} = \frac{332}{100}$

Now when t =$\frac{3}{2}$ hors = 5400 sec then N=?

$\Rightarrow$ N =100 e^{5400k}

$\Rightarrow$ N =100[e^{3600k}] $^{\frac{3}{2}}$

$\Rightarrow$ N = 100 $[\frac{332}{100}]^{\frac{3}{2}}$ = 605.

$\Rightarrow$ N = 605.

2 . In a chemical reaction a given substance is being converted into another at a rate proportional to the amount of substance converted. If $\frac{1}{5}$ of the original amount has been transformed in 4 min, how much time will be required to transform one half.

Ans: t= 13 mins.

3. The temp of cup of coffie is 92^0c . in which freshly period the room temp being 24^0c . in one min it was cooled to 80^0c . how long a period must elspse , before the temp of the cup becomes 65^0c .

Sol: : By Newton's Law of Cooling,

$\frac{d\theta}{dt}$ = -k($\theta - \theta o$) ; k>0

θo = 24^0c $\Rightarrow$ log ($\theta - 24$) = -kt + log c--------------(1).

When t=0 ; θ = 92 $\Rightarrow$ c =68

When t =1 ; $\theta = 80^0$c $\Rightarrow$ $e^k = \frac{68}{56}$

$\Rightarrow$ k = log ($\frac{68}{56}$).

When $\theta = 65^0$c , t =?

Ans: t = $\frac{41}{56}$ min.

RATE OF DECAY OR RADIO ACTIVE MATERIALS STATEMENT:

The disintegration at any instance is propositional to the amount of material present in it.

If u is the amount of the material at any time 't' , then $\frac{du}{dt}$ = - ku , where k is any constant (k >0).

Problems:

1). if 30% of a radioactive substance disappears in 10days flow long will it take for 90% of it to disappear.

Ans: 64.5 days

2). In a chemical reaction a gives substance is being converted into another at a rate proportional to the amount of substance unconverted. If $\frac{1}{5}$ Of the original amount has been transformed to required to transform one-half.

Ans:

3 The radioactive material disintegrator at a rate proportional to its mass. When mass is 10 mgm , the rate of disintegration is 0.051 mg per day . how long will it take for the mass to be reduced from 10 mg to 5 mg.

Ans: 136 days.

4. uranium disintegrates at a rate proportional to the amount present at any instant . if m1 and M2 are grms of uranium that are present at times T1 and T2 respectively find the half=cube of uranium.

Ans: $T = \frac{(T2-T1)\,log2}{\log(\frac{M1}{M2})}$.

5. The rate at which bacteria multiply is proportional to the instance us number present. If the original number double in 2 hrs, in how many hours will it be triple.

Ans: $\frac{2log3}{log2}$ hrs.

6. a) if the air is maintained at 30^0c and the temp of the body cools from 80^0c to 60^0c in 12 min . find the temp of the body after 24 min.

Ans: 48^0c

b) If the air is maintained at 150^0c and the temp of the body cools from 70^0c to 40^0c in 10 min. Find the temp after 30 min.

FIRST-ORDER DIFFERENTIAL EQUATIONS OF HIGHER DEGREE

Equations of the First-order and not of First Degree

First-Order Equations of Higher Degree Solvable for Derivative $\frac{dy}{dx} = p$

Equations Solvable for y

Equations Solvable for x

Equations of the First Degree in x and y - Lagrange and Clairant Equations

Exercises

Equations of the first-Order and not of First Degree

In this Chapter we discuss briefly basic properties of differential equations of first-order and higher degree. In general such equations may not have solutions. We confine ourselves to those cases in which solutions exist.

The most general form of a differential equation of the first order and of higher degree say of nth degree can be written as

$$\left(\frac{dy}{dx}\right)^n + a_1(x,y)\left(\frac{dy}{dx}\right)^{n-1} + a_2(x,y)\left(\frac{dy}{dx}\right)^{n-2} + \ldots \ldots$$

$$\ldots \ldots \ldots + a_{n-1}(x,y)\frac{dy}{dx} + a_n(x,y) = 0$$

or $p^n+a_1p^{n-1}+a_2p^{n-2}+ \ldots\ldots +a_{n-1}\, p+a_n=0$ (1)

where $p = \frac{dy}{dx}$ and $a_1, a_2, \ldots, a_n$ are functions of x and y.

(1) can be written as

$F(x, y, p) = 0$ (2)

First-Order Equations of Higher Degree Solvable for p

Let (2) can be solved for p and can be written as

$(p-q_1(x,y))\ (p-q_2(x,y))\ \ldots\ldots\ (p-q_n(x,y)) = 0$

Equating each factor to zero we get equations of the first order and first degree.

One can find solutions of these equations by the methods discussed in the previous chapter. Let their solution be given as:

$f_i(x,y,c_i)=0,\ i=1,2,3 \ldots\ldots n$ (3)

Therefore the general solution of (3.1) can be expressed in the form

$f_1(x,y,c)\ f_2(x,y,c) \ldots\ldots f_n(x,y,c) = 0$ (4)

where c in any arbitrary constant.

It can be checked that the sets of solutions represented by (3) and (4) are identical because the validity of (4) in equivalent to the validity of (3) for at least one i with a suitable value of c, namely $c=c_i$

Example 1 Solve $xy\left(\frac{dy}{dx}\right)^2 + (x^2+y^2)\frac{dy}{dx} + xy = 0$ (1)

Solution: This is first-order differential equation of degree 2. Let $p = \frac{dy}{dx}$

Equation (1) can be written as

$xy\,p^2+(x^2+y^2)\,p+xy=0$ (2)

$(xp+y)(yp+x)=0$

This implies that

$xp+y=0,\ yp+x=0$ (3)

By solving equations in (3) we get

$xy=c_1$ and

$x^2+y^2=c_2$ respectively

$[x\frac{dy}{dx}+y=0$ or $\frac{dy}{dx}+\frac{1}{x}y=0$, Integrating factor

$I(x)=e^{\int\frac{1}{x}dx}=e^{\log x}$. This gives

$y.x = \int o.x\,dx + c_1$ or $xy=c_1]$

$[y\frac{dy}{dx}+x=0,$ or $ydy+xdx=0$

By integration we get $\frac{1}{2}y^2+\frac{1}{2}x^2=c$

or $x^2+y^2 = c_2,\ c_2>0,\ -\sqrt{c_2}\le x\le\sqrt{c_2}\ ]$

The general solution can be written in the form

$(x^2+y^2-c_2)\,(xy-c_1)=0$

It can be seen that none of the nontrivial solutions belonging to $xy=c_1$ or $x^2+y^2=c_2$ is valid on the whole real line.

Equations Solvable for y

Let the differential equation given by $F(x, y, p) = 0$ be solvable for y. Then y can be expressed as a function x and p, that is,

$$y= f\,(x,p) \qquad (1)$$

Differentiating (1) with respect to x we get

$$\frac{dy}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}.\frac{dp}{dx} \qquad (2)$$

(2) is a first order differential equation of first degree in x and p. It may be solved by

$$\varphi(x,p,c)=0 \qquad (3)$$

The solution of equation (1) is obtained by eliminating p between (1) and (2). If elimination of p is not possible then (1) and (3) together may be considered parametric equations of the solutions of (1) with p as a parameter.

Example 2: Solve y^2-1-p^2=o

Solution: It is clear that the equation is solvable for y, that is

$$y=\sqrt{1+p^2} \qquad (1)$$

By differentiating (1) with respect to x we get

$$\frac{dy}{dx}=\frac{1}{2}\frac{1}{\sqrt{1+p^2}}.2p\frac{dp}{dx}$$

or $$p=\frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}$$

or $$p\left[1-\frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}\right]=0 \qquad (2)$$

(2) gives p=o or $1-\frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}=0$

By solving p=0 in (1) we get

y=1

By $1-\frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}=0$

we get a separable equation in variables p and x.

$$\frac{dp}{dx}=\sqrt{1+p^2}$$

By solving this we get

p=sinh (x+c) (3)

By eliminating p from (1) and (3) we obtain

y=cos h (x+c) (4)

(4) is a general solution.

Solution y=1 of the given equation is a singular solution as it cannot be obtained by giving a particular value to c in (4).

Equations Solvable for x

Let equation $F(x, y, p) = 0$ be solvable for x,

that is x=f(y,p) .. (1)

Then as argued in the previous section for y we get a function Ψ such that

$\Psi(y, p, c) = 0$ (2)

By eliminating p from (1) and (2) we get a general solution of $F(x, y, p) = 0$.If elimination of p with the help of (1) and (1) is combursome then these equations may be considered parametric equations of the solutions of (1) with p as a parameter.

Example 3

Solve $x\left(\frac{dy}{dx}\right)^3 - 12\frac{dy}{dx} - 8 = 0$

Solution: Let $p = \frac{dy}{dx},\ then$

xp^3-12p-8=0

It is solvable for x, that is,

$$x = \frac{12p+8}{p3} = \frac{12}{p^2} + \frac{8}{p^3} \qquad \ldots \qquad (1)$$

Differentiating (1) with respect to y, we get

$$\frac{dx}{dy} = -2\frac{12}{p^3}\frac{dp}{dy} - 3\frac{8}{p^4}\frac{dp}{dy}$$

$$\text{or}\ \frac{1}{p} = -\frac{24}{p^3}\frac{dp}{dy} - \frac{24}{p^4}\frac{dp}{dy}$$

$$or \quad dy = \left(-\frac{24}{p^2} - \frac{24}{p^3}\right)dp$$

$$or \quad y = +\frac{24}{p} + \frac{12}{p^2} + c \qquad \ldots \qquad (2)$$

(1) and (2) constitute parametric equations of solution of the given differential equation.

Equations of the First Degree in x and y – Lagrange's and Clairaut's Equation.

Let Equation $F(x, y, p) = 0$ be of the first degree in x and y, then

$y = x\varphi_1(p) + \varphi_2(p)$ … (1)

Equation (1) is known as Lagrange's equation.

If $\varphi_1(p) = p$ then the equation

$y = xp + \varphi_2(p)$ .. (2)

is known as Clairaut's equation

By differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = \varphi_1(p) + x\varphi_1'(p)\frac{dp}{dx} + \varphi_2'(p)\frac{dp}{dx}$$

or $p - \varphi_1(p) = (x\varphi_1'(p) + \varphi_2'(p))\frac{dp}{dx}$ … (3)

From (3) we get

$(x + \varphi_2'(p))\frac{dp}{dx} = 0$ for $\varphi_1(p)=p$

This gives

$\frac{dp}{dx} = 0$ or $x+\varphi_2'(p) = 0$

$\frac{dp}{dx} = 0$ gives p = c and

by putting this value in (2) we get

$y=cx+\varphi_2(c)$

This is a general solution of Clairaut's equation.

The elimination of p between

$x+\varphi_2'(p) = 0$ and (2) gives a singular solution.

If $\varphi_1(p) \neq p$ for any p, then we observe from (3) that

$\frac{dp}{dx} \neq 0$ everywhere. Division by

$[p - \varphi_1(p)]\frac{dp}{dx}$ in (3) gives

$$\frac{dx}{dp} - \frac{\varphi_1'}{p-\varphi_1(p)}x = \frac{\varphi_2'(p)}{p-\varphi_1(p)}$$

which is a linear equation of first order in x and thus can be solved for x as a function of p, which together with (1) will form a parametric representation of the general solution of (1)

Example 4 Solve $\left(\frac{dy}{dx}-1\right)\left(y-x\frac{dy}{dx}\right)=\frac{dy}{dx}$

Solution: Let $p=\frac{dy}{dx}$ then,

(p-1)(y-xp)=p

This equation can be written as

$$y = xp + \frac{p}{p-1}$$

Differentiating both sides with respect to x we get

$$\frac{dp}{dx}\left[x - \frac{1}{(p-1)^2}\right] = 0$$

Thus either $\frac{dp}{dx} = 0$ or

$$x - \frac{1}{(p-1)^2} = 0$$

$\frac{dp}{dx} = 0$ gives p=c

Putting p=c in the equation we get

$$y = cx + \frac{c}{c-1}$$

(y-cx)(c-1)=c

which is the required solution.

Exercises

Solve the following differential equations

1. $\left(\frac{dy}{dx}\right)^3 = \frac{dy}{dx}e^{2x}$

2. $y(y-2)p^2 - (y-2x+xy)p+x=0$

3. $-\left(\frac{dy}{dx}\right)^2 + 4y - x^2 = 0$

4. $\left(\frac{dy}{dx} + y + x\right)\left(x\frac{dy}{dx} + y + x\right)\left(\frac{dy}{dx} + 2x\right) = 0$

5. $y + x\frac{dy}{dx} - x^4\left(\frac{dy}{dx}\right)^2 = 0$

6. $\left(x\frac{dy}{dx} - y\right)\left(y\frac{dy}{dx} + x\right) = h^2\frac{dy}{dx}$

7. $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} = x$

8. $x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} + ax = 0$

9. $\left(\frac{dy}{dx}\right)^2 = y - x$

10. $xy\left(y - x\frac{dy}{dx}\right) = x + y\frac{dy}{dx}$

Multiple choice

The order of $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - 3y = x$ is

a) 2 b) 3 c) 1 d) None

1) The order of $\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$ is

a) 2 b) 1 c) 3 d)None

2) The degree of Differential Equation $\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$ is

a) 3 b) 2 c) 1 d) 9

3) The degree of Differential Equation $\left(\frac{d^2y}{dx^2}\right)^4 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$ is

a) 4 b) 3 c) 2 d) None

4) The general solution of $\frac{dy}{dx} = e^{x+y}$ is

a) $e^x + e^y = c$ b) $e^x + e^{-y} = c$ c) $e^{-x} + e^y = c$ d) $e^{-x} + e^{-y} = c$

5) Find the differential equation corresponding to $y = a\,e^x + b\,e^{2x} + c\,e^{3x}$

a) $y^{111} - 6y^{11} + 11y^1 - 6y = 0$ b) $y^{111} + y^{11} - 3y^1 = 0$

c) $y^{11} + 2y^1 + y = 0$ d) $y^{111} - 2y^{11} + 3y^1 + y = 0$

7) Find the differential equation of the family of curves $y = e^x (A\cos x + B\sin x)$

a) $y^{11} - 2y^1 + 3y = 0$ b) $y^{11} - 3y^1 + y = 0$

c) $y^{11} - 2y^1 + 3y = 0$ d) None

8) Form the differential equation by eliminating the arbitary constant : $y^2 = (x - c)^2$

a) $(y^1)^2 = 1$ b) $y^{11} + 2y^1 = 2$ c) $(y^1)^2 = 0$ d) None

9) Find the differential equation of the family of parabolas having vertex at the origin and foci on y -axis

a) $xy^1 = 2x$ b) $xy^1 = 2y$ c) $xy^1 = 4y$ d) None

10) Form the differential equation by eliminating the arbitary constant

tanx + tany = c

a) $y_1(\tan y + \sec^2 x) = 0$ b) $y_1(\tan y \sec^2 y) + \tan y \sec^2 x = 0$

c) $y_1(\tan x \sec^2 x) + \tan y \sec^2 y = 0$ d) None

11) Obtain the differential equation of the family of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

a) $xyy^{11} + xy^1 = 0$ b) $xy^{11} + xy = 0$

c) $xyy^{11} + x(y^1)^2 - yy^1 = 0$ d) None

12) The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$ under the condition that y = 1 when x=1 is

a) $4xy = x^3+3$ b) $4xy = x^4+3$ c) $4xy = y^4+3$ d) None

13) The family of straight lines passing through the origin is represented by the differential equation

a) ydx + xdy=0 b) xdy – ydx = 0 c) xdx + ydx = 0 d) ydy – xdx = 0

14) The differential equation of a family of circles having the radius 'r' and centre on the x – axis is

a) $y^2[1 + (\frac{dy}{dx})^2] = r^2$ b) $x^2[1 + (\frac{dy}{dx})^2] = r^2$

c) $(x^2 + y^2)[1 + (\frac{dy}{dx})^2] = r^2$ d) $r^2[1 + (\frac{dy}{dx})^2] = x^2$

15) The differential equation satisfying the relation x = A cos(mt - $\propto$) is

a) $\frac{dx}{dt} = 1 - x^2$ b) $\frac{d^2x}{dt^2} = -\propto^2 x$

c) $\frac{d^2x}{dt^2} = -m^2 x$ d) $\frac{dx}{dt} = -m^2 x$

16) The equation $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$ is

a) Homogeneous b) Variable separable c) Exact d) None

17) Find the differential equation of the family of cardioids $r = a(1+\cos\theta)$

a) $\frac{dr}{d\theta}$ + rsinx = 0   b) $\frac{dr}{d\theta}$ + r tan($\frac{\theta}{2}$) = 0

c) $\frac{dr}{d\theta}$ + r sin($\frac{\theta}{2}$) = 0   d) *None*

18) The equation $\frac{dy}{dx} + \sqrt{\frac{1+y^2}{1+x^2}} = 0$ is

a) Variable separableb) Exact   c) Homogeneous   d) None

19) The solution of the differential equation is $\frac{dy}{dx} = e^{(x-y)} + x^2e^{-y}$

a) $e^y = \frac{x^3}{3} + e^x + c$ b) $e^y = e^x + 3x + c$ c) $e^x = \frac{x^3}{3} + e^y + c$ d) None

20) The general solution of $\frac{dy}{dx} = (4x + y+1)^2$ is

a) $\tan^{-1}(\frac{4x+y+1}{2}) = c$ b) $\frac{1}{2}\tan^{-1}(\frac{4x+y+1}{2}) = y + c$

c) $\frac{1}{2}\tan^{-1}(\frac{4x+y+1}{2}) = x + c$ d) None

21) The solution of of the Differential equation $(x^2+1)\, y_1 + y^2 + 1 = 0$, $y(0) = 1$ is

a) $\frac{\pi}{4}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{2}$ d) $\frac{\pi}{8}$

22) The solution of $\frac{ydx-xdy}{y^2} = 0$ is

a) xy = c   b) y = cx   c) x = cy   d) $x = cy^2$

23) The general solution of $\frac{xdx+ydy}{x^2+y^2} = 0$ is

a) log(x+y) = c   b) $\log(x^2 + y^2) = c$ c)log(xy) = c   d) None

24) The equation of the form $\frac{dy}{dx}$ + p(x)y = q(x) is

a) Homogeneous   b) Exact   c) Linear   d) None

25) Integral factor of $\frac{dy}{dx}$ + p(x)y = q(x) is

a) $e^{\int pdx}$ b) $e^{\int pdy}$ c) $e^{\int qdx}$ d) $e^{\int qdy}$

26) The general solution of $\frac{dy}{dx}$ + ycotx = cosx is

a) $y = \frac{1}{2}$ sinx + c cosx

b) $y = \frac{1}{2}$ cosx + c sinx

c) $y = \frac{1}{2}$ sinx + c cosecx

d) None

27) The form of Bernoulli's Equation is

a) $\frac{dy}{dx}$ + px = Qy^n

b) $\frac{dy}{dx}$ + py = Qx^n

c) $\frac{dy}{dx}$ + Qy^n = px

d) $\frac{dy}{dx}$ + py = Qy^n

28) The equation of the form M(x,y)dx + N(x,y)dy = 0 is called if $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$

a) Linear

b) Bernoulli's

c) Exact

d) Homogeneous

29) Integrating factor of the homogenous de Mdx + Ndy = 0 is

a) $\frac{1}{Mx - Ny}$

b) $\frac{1}{Mx + Ny}$

c) $\frac{1}{Nx - My}$

d) None

30) If $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is a function of x alone say f(x) then the integrating factor of Mdx + Ndy = 0 is

a) $e^{\int f(x)dy}$

b) $e^{\int f(y)dy}$

c) $e^{\int f(x)dx}$

d) $e^{\int f(x)dy}$

31) The integrating factor of $(x^2 - 3xy + 2y^2)dx + x(3x-2y)dy = c$ is

a) $\frac{1}{x^2}$

b) $\frac{1}{x^5}$

c) $\frac{1}{x}$

d) $\frac{1}{x^3}$

32) The given differential equation y(x+y)dx + (x+2y-1)dy = 0 is

a) Exact

b) Not Exact

c) We can't say

d) None

OBJECTIVE

1) The order of $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - 3y = x$ is ___________________.

2) The differential equation $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$ is ___________________.

3) The integrating factor of $x\frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$ is _______________________.

4) The integrating factor of $(1 - x^2)\, y + xy = ax$ is _________________________.

5) The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \tan(\frac{y}{x})$ is _______________.

6) The integrating factor of $(x^2 - 3xy + 2y^2)dx + x(3x-2y)dy = c$ is ________________.

7) The newton law of cooling is ____________________.

8) Mdx+Ndy is exact if ________________________.

9) statement of law of Natural growth or decay is ____________________

10)Solution of linear differential equation of first order in y is (independent variable x) _________________.

11)Bernoulli's equation is ________________.

12))If $\frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$ is a function of y alone then the integrating factor is

_______________________________.

13) The general solution of $(1 + x^2)\, dy - (1+y^2)\, dx = 0$ is __________________.

14) The general solution of $\frac{dy}{dx} + xy = x$ is _______________________.

15) The integrating factor of the equation $y\, f_1(xy)dx + x\, f_2(xy)dy$ is ______________.

UNIT-II

HOMOGENEOUS LINEAR EUATIONS (OR) CAUCHY'S EULAR EUATIONS

Definition: An equation of the form P0 $x^n \frac{d^n y}{dx^n}$ + $P_1(x)$ x^{n-1} . $\frac{d^{n-1}y}{dx^{n-1}}$ + --------+ $P_n(x)$ .y = Q(x)…………(1)

Where P_0 (x) $P_1(x)$, $P_2(x)$, $P_3(x)$… …..$P_n(x)$ are real constant ,

Q(x) (functions of x) continuous eq(1) of operator form is ($x^n D^n + P_1 x^{n-1} D^{n-1}$ +-----------+P_n)y= Q(x) is called a linear differential equation of order n.

LINEAR DIFFERENTIAL EQUN' WITH CONSTANT COEFFICIENTS:

Def: An equation of the form $\frac{d^n y}{dx^n}$ + P_1 . $\frac{d^{n-1}y}{dx^{n-1}}$ + P_2 . $\frac{d^{n-2}y}{dx^{n-2}}$ + --------+ P_n .y = Q(x) where P_1, P_2, P_3,…..P_n, are real constants and Q(x) is a continuous functions of x is called an L.D equation of order ' n' with constant coefficients.

Note:

1. operator D =$\frac{d}{dx}$; D^2 =$\frac{d^2}{dx^2}$; …………………… D^n =$\frac{d^n}{dx^n}$

   Dy =$\frac{dy}{dx}$; D^2 y=$\frac{d^2y}{dx^2}$; …………………… D^n y=$\frac{d^n y}{dx^n}$
2. operator $\frac{1}{D}$Q = $\int Q$ i e D^{-1}Q is called the integral of Q.

<u>To find the general solution of f(D).y = 0</u> :

Where f(D) = $D^n + P_1 D^{n-1} + P_2 D^{n-2}$ +-----------+P_n is a polynomial in D.

Now consider the auxiliary equation : f(m) = 0

i.e f(m) = $m^n + P_1 m^{n-1} + P_2 m^{n-2}$ +-----------+P_n = 0

where p_1,p_2,p_3 ……………p_n are real constants.

Let the roots of f(m) =0 be m_1, m_2, m_3,…..m_n.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

E.no	Roots of A.E f(m) =0	Complementary function(C.F)
1.	$m_1, m_2, ..m_n$ are real and distinct.	$Y_c = c_1e^{m_1x} + c_2e^{m_2x} +\dots+ c_ne^{m_nx}$
2.	$m_1, m_2, ..m_n$ are $\ni$: m_1, m_2 are equal and real(i.e repeated twice) &the rest are real and different.	$Y_c = (c_1+c_2x)e^{m_1x}+ c_3e^{m_3x} +\dots+ c_ne^{m_1x}$
3.	$m_1, m_2, ..m_n$ are $\ni$: m_1, m_2 , m_3 are equal and real(i.e repeated thrice) &the rest are real and different.	$Y_c = (c_1+c_2x+c_3x^2)e^{m_1x} + c_4e^{m_4x}+\dots+ c_ne^{m_nx}$
4.	Two roots of A & B are complex say $\alpha+i\beta$ α -iβ and rest are real and distinct.	$Y_c = e^{\alpha x}$ ($c_1 \cos\beta x + c_2\sin\beta x$)+ $c_3e^{m_3x} +\dots+ c_ne^{mnx}$
5.	If $\alpha\pm i\beta$ are repeated twice & rest are real and distinct	$Y_c = e^{\alpha x}$ $[(c_1+c_2x)\cos\beta x + (c_3+c_4x) \sin\beta x)]+ c_5e^{m_5x} +\dots+ c_ne^{m_nx}$
6.	If $\alpha\pm i\beta$ are repeated thrice & rest are real and distinct	$Y_c = e^{\alpha x}$ $[(c_1+c_2x+ c_3x^2)\cos\beta x + (c_4+c_5x+ c_6x^2) \sin\beta x)]+ c_7e^{m_7x} +\dots\dots\dots + c_ne^{m_nx}$
7.	$\alpha\pm i\beta$	$Y_c = e^{\alpha x}$ ($c_1 \cos\beta x + c_2\sin\beta x$)

Solve the following Differential equations :

1. $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Sol: Given equation is of the form f(D).y = 0

Where $f(D) = (D^3 -3D +2) Y = 0$

Now consider the auxillary equation f(m) = 0

$f(m) = m^3 -3m +2 = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1 , 1 ,-2$

Since m_1 and m_2 are equal and m_3 is -2

We have $Y_c = (c_1+c_2x)e^x + c_3e^{-2x}$

2. $(D^4 -2 D^3 - 3 D^2 + 4D +4)Y = 0$

Sol: Given $f(D) = (D^4 -2 D^3 - 3 D^2 + 4D +4) Y = 0$

⇨ A.equation $f(m) = (m^4 -2 m^3 - 3 m^2 + 4m +4)$

⇨ $(m + 1)^2 (m – 2)^2 = 0$

⇨ m= -1 , -1 , 2 , 2

⇨ $Y_c = (c_1+c_2x)e^{-x} +(c_3+c_4x)e^{2x}$

3. $(D^4 + 8D^2 + 16)\, Y = 0$

Sol: Given $f(D) = (D^4 + 8D^2 + 16)\, Y = 0$

Auxillary equation $f(m) = (m^4 + 8m^2 + 16)\, Y = 0$

⇨ $(m^2 + 4)^2 = 0$

⇨ $(m+2i)^2 (m+2i)^2 = 0$

⇨ $m = 2i, 2i, -2i, -2i$

$Y_c = e^{0x} [(c_1+c_2x)\cos 2x + (c_3+c_4x)\sin 2x)]$

4. $y^{11}+6y^1+9y = 0$; $y(0) = -4$, $y^1(0) = 14$

Sol: $f(D)\, y = 0 \Rightarrow (D^2 + 6D + 9)\, Y = 0$

A.equation $f(m) = 0 \Rightarrow (m^2 + 6m + 9) = 0$

$\Rightarrow m = -3, -3$

$Y_c = (c_1+c_2x)e^{-3x}$ --------------------> (1)

D. of (1) w.r.to x $\Rightarrow y^1 = (c_1+c_2x)(-3e^{-3x}) + c_2(e^{-3x})$

Given $y_1(0) = 14 \Rightarrow c_1 = -4$ & $c_2 = 2$

Hence we get $y = (-4 + 2x)(e^{-3x})$

5. Solve $4y^{111} + 4y^{11} + y^1 = 0$

Sol: equation $f(m) = 0$

$4m^3 + 4m^2 + m = 0$

$m(4m^2 + 4m + 1) = 0$

$m(2m + 1)^2 = 0$

$m = 0, -1/2, -1/2$

$y = c_1 + (c_2 + c_3x)\, e^{-x/2}$

6. $(D^2 - 3D + 4)\, Y = 0$

Sol: equation $f(m) = 0$

$m^2 - 3m + 4 = 0$

$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm i\sqrt{7}}{2}$

$\propto \pm \beta = \frac{3 \pm i\sqrt{7}}{2}$

$y = e^{\frac{3}{2}x} (c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x)$

<u>General solution of f(D) y = Q(x)</u>

Is given by $y = y_c + y_p$

i.e. y = C.F+P.I

Where the P.I consists of no arbitrary constants and P.I of f (D) y = Q(x)

Is evaluated as P.I = $\frac{1}{f(D)}$. Q(x)

Depending on the type of function of Q(x).

P.I is evaluated as follows:

1. P.I of f (D) y = Q(x) where Q(x) =e^{ax} for (a) $\neq$ 0

Case1: P.I = $\frac{1}{f(D)}$. Q(x) = $\frac{1}{f(D)} e^{ax}$ = $\frac{1}{f(a)} e^{ax}$

Provided f(a) $\neq$ 0

Case 2: If f(a) = 0 then the above method fails. Then

if f(D) = $(D-a)^k \emptyset(D)$

(i.e ' a' is a repeated root k times).

Then P.I = $\frac{1}{\emptyset(a)} e^{ax} \cdot \frac{1}{k!} x^k$ provided $\emptyset$ (a) $\neq$ 0

2) P.I of f(D) y =Q(x) where Q(x) = sin ax or Q(x) = cos ax where ' a ' is constant then P.I = $\frac{1}{f(D)}$. Q(x).

Case 1: In f(D) put $D^2 = -a^2 \ni f(-a^2) \neq 0$ then P.I = $\frac{1}{f(D)}$ sin ax

Case 2: If $f(-a^2) = 0$ then $D^2 + a^2$ is a factor of $\emptyset(D^2)$ and hence it is a factor of f(D).

Then let $f(D) = (D^2 + a^2) .f(D^2)$.

Then $\frac{1}{(D2+a2)}$ (sin ax) = $\frac{-x\, cosax}{2a}$

& $\frac{1}{(D2+a2)}$ (cos ax) = $\frac{x\, sinax}{2a}$

1) P.I for f(D) y = Q(x) where Q(x) = x^k where k is a positive integer

Then express f(D) =[1± $\emptyset(D)$]

Express $\frac{1}{f(D)} = \frac{1}{1\pm\emptyset(D)}$ = $[1\pm \emptyset(D)]^{-1}$

Hence P.I = $\frac{1}{1\pm\emptyset(D)}$ Q(x).

$= [1\pm \emptyset(D)]^{-1} .x^k$

2) P.I of f(D) y = Q(x) when Q(x) = e^{ax} V where 'a' is a constant and V is function of x. where V =sin ax or cos ax or x^k

Then P.I =$\frac{1}{f(D)}$ Q(x)

$= \frac{1}{f(D)} e^{ax}$ V

$$= e^{ax}\left[\frac{1}{f(D+a)}(V)\right]$$

& $\frac{1}{f(D+a)}$ V is evaluated depending on V.

3) P.I of f(D) y = Q(x) when Q(x) = x V where V is function of x.

Then P.I = $\frac{1}{f(D)}$ **Q(x)**

$$= \frac{1}{f(D)}\ x\,V$$

$$= \left[x - \frac{1}{f(D)} f^1(D)\right]\frac{1}{f(D)} V$$

Formulae

1. $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 +$ ------------------
2. $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 +$ ------------------
3. $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 +$ ------------------
4. $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 +$ ------------------
5. $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 +$ ------------------
6. $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 +$ ------------------

I. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS:

1. Find the Particular integral of f(D) y = e^{ax} when f(a) $\neq 0$
2. Solve the D.E $(D^2 + 5D + 6)\ Y = e^x$
3. Solve $y^{11} + 4y^1 + 4y = 4e^{3x}$; $y(0) = -1$, $y^1(0) = 3$
4. Solve $y^{11} + 4y^1 + 4y = 4\cos x + 3\sin x$, $y(0) = 1$, $y^1(0) = 0$
5. Solve $(D^2+9)\ y = \cos 3x$
6. Solve $y^{111} + 2y^{11} - y^1 - 2y = 1 - 4x^3$
7. Solve the D.E $(D^3 - 7D^2 + 14D - 8)\ Y = e^x \cos 2x$
8. Solve the D.E $(D^3 - 4D^2 - D + 4)\ Y = e^{3x} \cos 2x$
9. Solve $(D^2 - 4D + 4)\ Y = x^2 \sin x + e^{2x} + 3$
10. Solve $x^2D^2 - xD + y = \log x$
11. Solve the D.E $(x^2D^2 - 3xD + 1)\ y = \frac{\log x.\ \sin(\log x) + 1}{x}$

12. Apply the method of variation parameters to solve $\frac{d^2y}{dx^2}$ + y = cosecx

13. Solve $\frac{dx}{dt}$ = 3x + 2y , $\frac{dy}{dt}$ + 5x + 3y =0

14. Solve $(D^2 + D - 3)\, Y = x^2e^{-3x}$

15. Solve $(D^2 - D - 2)\, Y = 3e^{2x}$,y(0) = 0 , y^1 (0) = -2

SOLUTIONS:

1) Particular integral of f(D) y = e^{ax} when f(a) ≠0

Working rule:

Case (i):

In f(D), put D=a and Particular integral will be calculated.

Particular integral= $\frac{1}{f(D)}\ e^{ax} = \frac{1}{f(a)}\ e^{ax}$ provided f(a) ≠0

Case (ii) :

If f(a)= 0 , then above method fails. Now proceed as below.

If f(D)= $(D-a)^K\ \phi(D)$

i.e. 'a' is a repeated root k times, then

Particular integral= $\frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$ provided ϕ(a) ≠0

2. Solve the Differential equation$(D^2+5D+6)y=e^x$

Given equation is $(D^2+5D+6)y=e^x$

Here Q(x) =e^x

f(m) = (m^2+5m+6)

Auxiliary equation is f(m) = $m^2+5m+6=0$

$m^2+3m+2m+6=0$

m(m+3)+2(m+3)=0

m=-2 or m=-3

the roots are real and distinct

C.F = $y_c = c_1e^{-2x} + c_2\, e^{-3x}$

Particular Integral = $y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{D2+5D+6}\ e^x \qquad = \frac{1}{(D+2)(D+3)}\ e^x$

Put D = 1 in f(D)

P.I. = $\frac{1}{(3)(4)}$ e^x

Particular Integral = $y_p = \frac{1}{12} \cdot e^x$

General equation is y=yc+yp

$Y=c_1e^{-3x}+c_2\, e^{-2x} + \frac{e^x}{12}$

3). Solve $y^{11}-4y^1+3y=4e^{3x}$, y(0) = -1, $y^1(0)$ = 3

Given equation is $y^{11}-4y^1+3y=4e^{3x}$

i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4e^{3x}$

it can be expressed as

$D^2y-4Dy+3y=4e^{3x}$

$(D^2-4D+3)y=4e^{3x}$

Here $Q(x)=4e^{3x}$; $f(D)= D^2-4D+3$

Auxiliary equation is $f(m)=m^2-4m+3 = 0$

$m^2-3m-m+3 = 0$

m(m-3) -1(m-3)=0 => m=3 or 1

The roots are real and distinct.

C.F= $y_c=c_1e^{3x}+c_2e^x$ ----→ (2)

P.I.= $y_p = \frac{1}{f(D)} \cdot Q(x)$

= $y_p = \frac{1}{D^2-4D+3} \cdot 4e^{3x}$

= $y_p = \frac{1}{(D-1)(D-3)} \cdot 4e^{3x}$

Put D=3

$y_p = \frac{4xe3x}{(3-1)} = 2xe^{3x}$

General solution is $y = y_c+y_p$

$y=c_1e^{3x} + c_2\, e^x + 2xe^{3x}$ -------------------→ (3)

Differentiating with respect to ‘x’

$y^1=3c_1e^{3x}+c_2e^x+2e^{3x}+6xe^{3x}$ -----------→ (4)

By data, y(0) = -1 , $y^1(0)$=3

From (3), $-1=c_1+c_2$ -------------------→ (5)

From (4), $3=3c_1+c_2+2$

$3c_1+c_2=1$ -------------------→ (6)

Solving (5) and (6) we get $c_1=1$ and $c_2 = -2$

$y=-2e^{x} +(1+2x)\, e^{3x}$

(4). Solve $y^{11}+4y^{1}+4y= 4\cos x + 3\sin x$, $y(0) = 0$, $y^{1}(0) = 0$

Sol: Given differential equation in operator form

$(D^2 + 4D + 4)y = 4\cos x + 3\sin x$

A.E is $m^2+4m+4 = 0$

$(m+2)^2=0$ then $m=-2, -2$

$\therefore$ C.F is $y_c= (c_1 + c_2x)e^{-2x}$

P.I is $= y_p= \frac{4\cos x+3\sin x}{(D^2+4D+4)}$ put $D^2 = -1$

$y_p= \frac{4\cos x+3\sin x}{(4D+3)} = \frac{(4D-3)(4\cos x+3\sin x)}{(4D-3)(4D+3)}$

$= \frac{(4D-3)(4\cos x+3\sin x)}{16D^2-9}$

Put $D^2 = -1$

$\therefore y_p= \frac{(4D-3)(4\cos x+3\sin x)}{-16-9}$

$= \frac{-16\sin x+12\cos x-12\cos x-9\sin x)}{-25} = \frac{-25\sin x}{-25} = \sin x$

$\therefore$general equation is $y = y_c+ y_p$

$Y = (c_1 + c_2x)e^{-2x} + \sin x$ ------------ (1)

By given data, $y(0) = 0 \therefore c_1 = 0$ and

Diff (1) w.r.. t. $y^1 = (c_1 + c_2x)(-2)e^{-2x} + e^{-2x}(c_2) +\cos x$ ------------ (2)

given $y^1(0) = 0$

(2) $\Rightarrow -2c_1 + c_2+1=0$ $\therefore c_2 = -1$

$\therefore$required solution is $y = -xe^{-2x}+\sin x$

5. Solve $(D^2+9)y = \cos 3x$

Sol:Given equation is $(D^2+9)y = \cos 3x$

A.E is $m^2+9 = 0$

$\therefore m = \pm 3i$

$Y_c = C.F = c_1 \cos 3x+ c_2\sin 3x$

$Y_c =P.I = \frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$

$= \frac{x}{2(3)} \sin 3x = \frac{x}{6} \sin 3x$

General equation is y = y_c+ y_p

Y = $c_1$cos3x + $c_2$cos3x + $\frac{x}{6}$ sin3x

6. $y^{111}+2y^{11} - y^1-2y= 1-4x^3$

Sol:Given equation can be written as

$(D^3 + 2D^2 - D - 2)y = 1-4x^3$

A.E is $(m^3 + 2m^2 - m - 2) = 0$

$(m^2 - 1)(m+2) = 0$

$m^2 = 1$ *or* m=- 2

m = 1, -1, -2

C.F =$c_1e^x + c_2e^{-x} + c_3e^{-2x}$

P.I = $\frac{1}{(D^3+2D^2-D-2)}(1-4x^3)$

$= \frac{-1}{2[1-\frac{(D^3+2D^2-D)}{2}]}(1-4x^3)$

$= \frac{-1}{2}[1 - \frac{(D^3+2D^2-D)}{2}]^{-1}(1-4x^3)$

$= \frac{-1}{2}[1 + \frac{(D^3+2D^2-D)}{2} + \frac{(D^3+2D^2-D)^2}{4} + \frac{(D^3+2D^2-D)^3}{8} + \ldots..](1-4x^3)$

$= \frac{-1}{2}[1+\frac{1}{2}(D^3 + 2D^2 - D) + \frac{1}{4}(D^2 - 4D^3)+\frac{1}{8}(-D^3)(1-4x^3)$

$= \frac{-1}{2}[1 - \frac{5}{8}(D^3) + \frac{5}{4}(D^2) - \frac{1}{2}D](1-4x^3)$

$= \frac{-1}{2}[(1-4x^3) - \frac{5}{8}(-24) + \frac{5}{4}(-24x) - \frac{1}{2}(-12x^2)$

$= \frac{-1}{2}[-4x^3+6x^2 -30x +16] =$

$= [2x^3-3x^2 +15x -8]$

The general solution is

y= C.F + P.I

y= $c_1e^x + c_2e^{-x} + c_3e^{-2x} + [2x^3-3x^2 +15x -8]$

7. Solve $(D^3 - 7D^2 + 14D -8)y = e^x$ cos2x

Given equation is

$(D^3 - 7D^2 + 14D -8)y = e^x$ cos2x

A.E is $(m^3 - 7m^2 + 14m - 8) = 0$

(m-1) (m-2)(m-4) = 0

Then m = 1,2,4

C.F = $c_1e^{x} + c_2e^{2x} + c_3e^{4x}$

P.I = $\frac{e^{x}cos2x}{(D^3-7D^2+14D-8)}$

$= e^{x}.\frac{1}{(D+1)^3-7(D+1)^2+14(D+1)-8}$. Cos2x

$= e^{x}.\frac{1}{(D^3-4D^2+3D)}$ .cos2x

$= e^{x}.\frac{1}{(-4D+3D+16)}$ .cos2x

$= e^{x}.\frac{1}{(16-D)}$ .cos2x

$= e^{x}.\frac{16+D}{(16-D)(16+D)}$ .cos2x

$= e^{x}.\frac{16+D}{256-D^2}$ .cos2x

$= e^{x}.\frac{16+D}{256-(-4)}$ .cos2x

$= \frac{e^{x}}{260}$ (16cos2x – 2sin2x)   G.S. is y = $y_c + y_p$

8. Solve $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

Sol:

Given $(D^2 - 4D + 4)y = x^2\ sinx + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m-2)^2 = 0$ then m=2,2

C.F. = $(c_1 + c_2x)e^{2x}$

P.I = $\frac{x^2 sinx + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2 sinx) + \frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}(3)$

Now $\frac{1}{(D-2)^2}(x^2 sinx) = \frac{1}{(D-2)^2}(x^2)$ (I.P of e^{ix})

= I.P of $\frac{1}{(D-2)^2}(x^2)(e^{ix})$

= I.P of $(e^{ix}).\frac{1}{(D+i-2)^2}(x^2)$

On simplification, we get

$\frac{1}{(D+i-2)^2}(x^2\ sinx) = \frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx]

and $\frac{1}{(D-2)^2}(e^{2x}) = \frac{x^2}{2}(e^{2x})$,

$\frac{1}{(D-2)^2}(3) = \frac{3}{4}$

P.I = $\frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx] + $\frac{x^2}{2}(e^{2x}) + \frac{3}{4}$

y = $y_c + y_p$

y= $(c_1 + c_2x)e^{2x} + \frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx] + $\frac{x^2}{2}(e^{2x}) + \frac{3}{4}$

10. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}$ + y = cosecx

Sol: Given equation in the operator form is $(D^2+1)y = cosecx$----------------(1)

A.E is (m^2+1) = 0

The roots are complex conjugate numbers.

∴ C.F. is $y_c=c_1\cos x + c_2\sin x$

Let y_p = Acosx + Bsinx be P.I. of (1)

$u\frac{dv}{dx} - v\frac{du}{dx} = \cos^2 x + \sin^2 x$ =1

A and B are given by

A= - $\int \frac{vR}{uv^1-vu^1}$ = - $\int \frac{\sin x \ cosec\, x}{1}$ dx = - $\int dx$ = - x

B = $\int \frac{vR}{uv^1-vu^1}$ = $\int \cos x.cosecx\, dx = \int \cot x\, dx$ = log(sinx)

∴y_p= -xcosx +sinx. log(sinx)

∴ General solution is y = y_c+ y_p.

y = $c_1$cosx + $c_2$sinx-xcosx +sinx. log(sinx)

11. Solve $(4D^2 - 4D$ +1)y = 100

Sol: A.E is $(4m^2 - 4m + 1)$ = 0

$(2m-1)^2 = 0$then m = $\frac{1}{2},\frac{1}{2}$

C.F = (c_1+c_2x) $e^{\frac{x}{2}}$

P.I = $\frac{100}{(4D^2-4D+1)} = \frac{100\, e^{0.x}}{(2D-1)^2} = \frac{100}{(0-1)^2}$ = 100

Hence the general solution is y = C.F +P.I

y= (c_1+c_2x) $e^{\frac{x}{2}}$ + 100

HOMOGENEOUS L.E (OR) CAUCHY'S-EULAR EQ'S:-An equation of the form

$p_0.x^n\frac{d^2y}{dx^2} + p_0.x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ---+p_n.y = Q(x)$—(1)

Where $P_0,P_1,P_2,\ldots\ldots\ldots P_n$ are real constants. Q(x) is a function of 'x' is called C-E Eq-(1) of the operator form is

$$(x^nD^n + p_1x^{n-1}D^{n-1} + ---+p_n)y = Q(x)\text{-(2)}$$

Cauchy's linear differential equation can be transformed in to L.D.E. with constant co-efficients by the change of independent variable with the substitution

Let $x=e^z$ so that Z=logX----(a)

$\frac{dz}{dx} = \frac{1}{x}$-------------(b)

Now

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$$

$$\therefore \ \frac{dy}{dx} = \frac{1}{x}.\frac{dy}{dz} - -(c) \qquad i.e., x.\frac{dy}{dx} = \frac{dy}{dz} - - - (c)$$

Again

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dz}\right)$$

$$\frac{d^2x}{dx^2} = \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\cdot\frac{-1}{x^2}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dx}\right) - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right) - \frac{1}{x^2}\cdot\frac{dy}{dz}$$

$$= \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz}$$

$$\frac{1}{x}\cdot\left(\frac{d^2y}{dz^2} - \frac{dy}{dx}\right)$$

$\therefore \ x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$-------(d)

Let us denote $\frac{d}{dx} = D$ & $\frac{d}{dz} = \theta$

(c) &(d) can be written as

XD= θ; $x^2 D^2 = \theta(\theta-1)$

Llly, $x^3D^3 = \theta(\theta-1)(\theta-2)$; $x^4D^4=\theta(\theta-1)(\theta-2)(\theta-3)$

& soon

Formula's XD=θ

$X^2D^2 =(\theta-1)$

$X^3D^3=\theta(\theta-1)(\theta-2)$ & Soon

Problem:

1. Solve

G.T ($x^2D^2 - 4XD+6)y = (\log x)^2$--------(1)

This is a homogenous D.E

Let $x=e^z$ (or) z= log x then we have

$X^2D^2=\theta(\theta-1)$

$XD=\theta$------(2)

Now from (1),(2) we have

$= (\theta(\theta-1) - 4\theta + 6)y = (\log x)^2$

$=(\theta^2-\theta - 4\theta + 6)y=(\log x)^2$

$=(\theta^2-5\theta + 6)y=(\log x)^2$

$(\theta^2-5\theta + 6)y = Z^2$

This is in the form of $f(\theta)y = Q(z)$

: . The general solution is $Y=Y_c+Y_p$

To find Y_c :-

Take A.E f(m)=0

= m^2-5m+6=0

=m^2-2m-3m+6=0

=m(m-2)-3(m-2)=0

=(m-2)(m-3)=0

: . m=2,3

: . The complementary function is

$Y=Y_c=C_1\, e^{2z}=C_2\, e^{3z}$-------------(a)

To find Yp :-

Let

$(\theta^2-5\theta+6)y=Z^2$

Then

$$Y = \frac{1}{\theta^2-5\theta+6}\,.Z^2$$

$$= \frac{1}{6\left(1+\frac{\theta^2-5\theta}{6}\right)}\,.Z^2$$

$$=\frac{1}{6}\left(1-\left(\frac{\theta^2-5\theta}{6}\right)+\frac{(\theta^2-5\theta)^2}{36}\right)Z^2$$

$$= \frac{1}{6}\left(1-\frac{1}{6}(2-5.2z)+\frac{1}{36}(0+25.2-0)\right)$$

$$= \frac{1}{6}\left(Z^2-\frac{2}{6}+\frac{10Z}{6}+\frac{1}{36}.50\right)$$

$$=\frac{1}{6}\left(Z^2-\frac{1}{3}+\frac{5}{3}Z+\frac{25}{18}\right)$$

$$=\frac{1}{6}\left(Z^2+\frac{5}{3}Z+\frac{19}{18}\right)$$

: . The particular integral $Y_p = \frac{1}{6}\left(Z^2+\frac{5}{3}Z+\frac{19}{18}\right)$-------(b)

: . The general solution is

$$Y=C_1\, e^{2z}+\ C_1 e^{3z}\,\frac{1}{6}\left(Z^2+\frac{5}{3}Z+\frac{19}{18}\right)$$

(: . from (a) (b))

: . $Y = C_1\, e^{2\log x}+C_2\, e^{3\log x}+\frac{1}{6}((\log x)^2+\frac{5}{3}\log x+\frac{19}{18}$

$$= C_1 e^{\log x^2}+\ C_2 e^{logx^2}+\frac{1}{6}\left((\log x)^2+\frac{5}{3}\log x+\frac{19}{18}\right)$$

$$= C1\ x2 + C2x3 + \frac{1}{6}((\log x\ )^2 + \frac{5}{3}\log x + \frac{19}{18}$$

Which is the required solution

2. G.T $(x^2D^2-3XD+1) = \frac{\log x\ .\sin(\log x\ )+1}{x}$ -------(1)

   This is a homoheneous L.D.E

   Let $x=e^z$ (or) Z = logx Then we have

   $X^2D^2 = \theta(\theta-1)$

   $XD= \theta$-------------(a)

   Now substituting (a) in (1) we get

   $= (\theta(\theta-1) - 3\theta + 1)y = \frac{\log x\ .\sin(\log x)+1}{x}$

   $= (\theta^2 - \theta - 3\theta + 1)y = \frac{Z.sinZ+1}{e^z}$

   $= (\theta^2 - 4\theta + 1)y = e^{-z}.(Z.\sin Z+1)$

   This is in the form of $F(\theta)y = Q(Z)$

   :. The general solution is $Y=Y_c + Y_p$

   To find Y_c :-

   Take A.E f(m) =0

$$m^2 - 4m + 1 = 0$$

$$m = \frac{4\ \pm\sqrt{16 - 4.1.1}}{2.1}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2} = \frac{2(2 \pm \sqrt{3})}{2}$$

$$= 2 + \sqrt{3}, 2 - \sqrt{3}$$

:. The complementary function is $Y_c=e^{2z}\ (C_1 \cos h\ \sqrt{3}x+C_2 \sin h\ \sqrt{3}x)$

(or)

$Y_c = C_1e^{(2+\sqrt{3})Z} + C_2e^{(2-\sqrt{3})Z}$-----------$(a_1)$

To find Y_p :-

Let $(\theta^2-4\theta+1)Y = e^{-z}(Z \sin Z+1)$

$Y= \frac{1}{\theta^2-4\theta+1} \, . e^{-Z} \, (Z \sin Z + 1)$

Then

$Y_p= e^{-Z} \frac{1}{(\theta-1)^2-4(\theta-1)+1} \, (Z. sinZ + 1)$

$=e^{-z} \frac{1}{\theta^2+1-2\theta-4\theta+4+1} \, Z \, sinZ + 1$

$$= e^{-z}. \frac{1}{\theta^2 - 6\theta + 6}. Z \sin Z + \frac{1}{\theta^2 - 6\theta + 6}. e^{0.Z}$$

$= e^{-z}\left\{\frac{1}{\theta^2-6\theta+6} \, . \, Z \, im \, e^{iz} + \frac{1}{0-0+6} \, .1\right\}$

$=e^{-z}\left\{im \, e^{iz} \, \frac{1}{(\theta-1^2)-6(\theta+i)+6} \, .Z + \frac{1}{6}\right\}$

$= e^{-z}\left\{im \, e^{iz} \, \frac{1}{\theta^2-1+2\theta i-6\theta-6i+6}.Z + -1/6\right\}$

$=e^{-z}\left\{im \, e^{iz} \, .\frac{1}{\theta^2-1+2\theta i-6\theta-6i+} \, .Z + \frac{1}{6}\right\}$

$=e^{-z}\left\{im \, e^{iz} \, .\frac{1}{(5-6i)\left(1+\frac{\theta^2+2\theta i-6\theta}{5-6i}\right)} \, .Z + \frac{1}{6}\right\}$

$=e^{-z}\left\{im \, e^{iz} \frac{1}{5-6i}\left(1 + \frac{\theta^2+2\theta i-6\theta}{5-6i}\right) \, Z + \frac{1}{6}\right\}$

$= e^{-z}\left\{im \, e^{iz} \frac{1}{5-6i}\left(1 - \frac{\theta^2+2\theta i-6\theta}{5-6i}\right) \, Z + \frac{1}{6}\right\}$

$=e^{-z}\left\{im \, e^{iz} \frac{1}{5-6i}\left(\frac{5-6i-\theta^2-2\theta i+6\theta}{5-6i}\right) \, Z + \frac{1}{6}\right\}$

$=e^{-z}\left\{im \, e^{iz} \frac{1}{(5-6i)^2}\{5z - i6z - 0 - 2i + 6\} + \frac{1}{6}\right\}$

$=e^{-z}\left\{im \, e^{iz} \left(\frac{z(5-6i)}{5-6i^2} + \frac{6-2i}{(5-6i)^2}\right) + \frac{1}{6}\right\}$

$=e^{-z}\left\{im \, e^{iz} \left(\frac{z}{5-6i} * \frac{5+6i}{5+6i} + \frac{6-2i}{25-35-60i}\right) + \frac{1}{6}\right\}$

$=e^{-z}\left\{ime^{iz} \left(\frac{(5+6i)z}{25+36} + \frac{6-2i}{-11-60i}\right) + \frac{1}{6}\right\}$

$=e^{-z}\left\{im \, e^{iz} \left(\frac{5z+i6z}{6i} + \frac{6-2i}{-(11+60i)} * \frac{11-60i}{11-60i}\right) + \frac{1}{6}\right\}$

=$e^{-z}\left\{(cosZ + i\sin Z)\left(\frac{5Z}{61} + i.\frac{6z}{61} - \left(\frac{66-360i-22i-120}{3721}\right)\right) + \frac{1}{6}\right\}$

=$e^{-z}\left\{(\cos Z + i\ sinZ)\left(\frac{5Z}{61} + i.\frac{6Z}{61} - \left(\frac{-54-382i}{3721}\right)\right) + \frac{1}{6}\right\}$

Compare in part we get

$$=e^{-z}\left\{\frac{5}{61}Z\sin Z + \frac{6}{61}Zcos\ Z + \frac{54}{3721}sinZ + \frac{382}{3721}cosz + \frac{1}{6}\right\}382$$

=$\frac{e^{-z}}{61}\left\{5Z\ sinz + 6Z\cos Z + \frac{54}{61}sinZ + \frac{382}{61}cosZ + \frac{1}{6}\right\}$

: . The general solution is

Y=$C_1e^{(2+\sqrt{3})logx} + C_2e^{(2-\sqrt{3})logx} + \frac{e^{-logx}}{61}\left\{5logx.sin(logx) + 6logx.cos(logx) + \right.$

$\left.\frac{54}{61}sin(logx) + \frac{382}{61}cos(logx) + \frac{1}{6}\right\}$

3. Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx}$ +4y =$(1+x)^2$----------------(1)

Sol: This is a homogeneous L.D.E.

Given equation of operator from is

$(x^2D^2 - 3xD + 4)y = (1+x)^2$---------(2)

Let x=e^z => Z=logx Then we have

XD = ; x^2 D^2 =$\theta(\theta-1)$------(a)

Now substituting (a) in (2) we get

- ⇨ $(\theta(\theta-1) - 3\theta + 4\theta)y = (1+e^z)^2$
- ⇨ $(\theta^2 - \theta - 3\theta + 4)y = 1 + e^{2z} + 2e^z$
- ⇨ $(\theta^2 - 4\theta + 4)y = 1 + e^{2z} + 2e^z$

This is in the form of $F(\theta)y = Q(z)$

: . The general solution is Y=Y_c + Y_p

To find Y_c :-

Take A.E f(m)=0

⇨ $m^2 - 4m + 4 = 0$

⇨ $m^2 - 2m - 2m + 4$=0

⇨ $m(m-2) - 2(m-2) = 0$

⇨ $(m-2)(m-2) = 0$

⇨ M=2,2

: . The complementary function is

$Y_c = (C_1 + C_2 Z)e^{2z}$

$=(C_1 + C_2\ logx)e^{2logx}$

: . $Y_c = \left(C_1 + C_{2\ logx}\right)x^2$

To find Y_p :-

Let $(\theta^2 - 4\theta + 4)y = 1 + e^{2z} + 2e^z$

⇨ $Y = \frac{1}{\theta^2-4\theta+4}. 1 + e^{2z} + 2e^z$

Then

$Y_p = \frac{1}{\theta^2-4\theta+4} e^{o.z} + \frac{1}{\theta^2-4\theta+4} e^{2z} + \frac{1}{\theta^2-4\theta+4}. 2e^{1.z}$

$= \frac{1}{0-0+4}. 1 + \frac{1}{4-8+4} e^{2z} + \frac{1}{1-4+4} 2e^z$

$= \frac{1}{4} + \frac{ze^{2z}}{2\theta-4} + 2e^z$

$= \frac{1}{4} + z\frac{e^{2z}}{4-4} + 2e^z$

$= \frac{1}{4} + z^2\frac{e^{2z}}{2} + 2e^z$

$= \frac{1}{4} + (logx)^2. \frac{1}{2}. e^{2logx} + 2e^{logx}$

$= \frac{1}{4} + (logx)^2. \frac{1}{2.}. x^2 + 2. x$

The general solution of (1) is $Y = Y_c + Y_p$

4. G.T

$(x^2D^2+4xD+2)y = e^x$--------(1)

This is a H.L.D.E.

Let x=e^z => Z=logx& $\frac{d}{dx} = D$ & $\frac{d}{dz} = \theta$ Then

We have

xD= θ ; $x^2D^2 = \theta(\theta-1)$----------------(a)

From (1) (a) we have

⇨ $(\theta(\theta-1)+4\theta+2)y = e^{e^z}$

⇨ $(\theta^2-\theta+4\theta+2)y = e^{e^z}$

⇨ $(\theta^2+3\theta+2)y = e^{e^z}$

This is in the form of F(θ)y=Q(Z)

: . The general solution is $Y=Y_c+Y_p$

To find Y_c:-

Take A.E. F(m)=0

⇨ $m^2+3m+2=0$

⇨ $m^2+2m+m+2=0$

⇨ $m(m+2)+1(m+2)=0$

⇨ $(m+2)(m+1)=0$

⇨ m=-1,-2

The complementary function is

$Y_c = C_1e^{-z} + C_2e^{-2z}$

$= C_1e^{-logx} + C_2e^{-2logx}$

: . $Y_c = C_1x^{-1} + C_2x^{-2}$----------------(2)

To find Y_p :-

Let $(\theta^2+3\theta+2)y = e^{e^z}$

$Y = \frac{1}{\theta^2+3\theta+2} \cdot e^{e^z}$

Then $Y_p = \frac{1}{(\theta+2)(\theta+1)} \cdot e^{e^z}$

$= \frac{1}{\theta+2}\left[\frac{1}{\theta+1} \cdot e^{e^z}\right]$

$=\frac{1}{\theta+2}\left[e^{-z}\int e^{e^z}.e^z.dz\right]$

$=\frac{1}{\theta+2}\left[e^{-z}.e^{e^z}\right]$

$$\left(\because \int e^{f(x)}.\ f^{|}(x).dx\ =\ e^{f(x)}\right)$$

$=e^{-2z}\int e^{-z}e^{e^z}e^{2z}\ dz$

$=e^{-2z}\int e^{e^z}e^z\ dz$

$=e^{-2z}e^{e^z}$

$=e^{-2logx}.e^x$

$=\frac{1}{x^2}.e^x$

$$\therefore Y_p=\frac{e^x}{x^2}-----(c)$$

The general solution is $Y=Y_c+Y_p$(from (b) &(c))

Home work :

Solve 1) $(x^2D^2-4xD+6)y=x^2$

2) $(x^2D^2-xD+1)y = \log x$

3) $(x^3D^3+2x^2D^2+2)y = 10\ (x+\frac{1}{x})$ $\rightarrow \left(use\ \ x^3D^3=\theta(\theta-1)(\theta-2)\right)$

4) $(x^3D^3+3x^2D^2+xD+8)y=65\cos(\log x)$

5) $(x^2D^2-xD+2)y = x\log x$

6) $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}\ =\ \frac{12\log x}{x^2}$

Ans: $x^2.\frac{d^2y}{dx^2}+x.\frac{dy}{dx}= 12\log x$

i.e., $(x^2D^2+xD)y = 12\log x$

7) $(x^2D^2+xD+4)y= \log x\ .\cos(2\log x)$

8) $(x^2D^2-3xD+1)y= \log x\ .\left(\frac{\sin(\log x)+1}{x}\right)$

i.e., $(x^2D^2-3xD+1)y = \frac{\log x\ .\sin(\log x)}{x}+\ \frac{\log x}{x}$

LEGENDRE'S LINEAR EQUATION :-

An equation of the form

$$p_0(a+bx)^n \frac{d^n y}{dx^n} + (a+bx)^{n-1} p_1 \frac{d^{n-1}y}{dx^{n-1}} + - - + p_n y = Q(x)$$

Where $P_0, P_1, P_2 \ldots\ldots P_n$ are constant & Q(x) is function of 'x' is called LEGENDRE'S LINEAR EQUATION .

This can be solved by the substitution $a+bx = e^z$ (or) $\log(a+bx) = z$

1. G.T.

   $(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2 + x + 1$------(1)

   This is a legendre's L.D.E

   Put x+1 = u

- ⇨ X = u – 1
- ⇨ dx = du
- ⇨ $\frac{du}{dx} = 1$

  $\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot 1 = \frac{dy}{du}$

  $from\ (1) we\ have$

  $$u^2 \frac{d^2y}{du^2} - 3u \frac{dy}{du} + 4y = (u-1)^2 + 4$$

- ⇨ $(u^2D^2 - 3uD + 4)y = u^2 + 1 - 2u + 4$
- ⇨ $(u^2D^2 - 3uD + 4)y = u^2 - 4 + 1$

Which is a homogeneous L.D.E.----(2)

Let u= e^z

Z = log u & also $\frac{d}{du} = D$; $\frac{d}{dz} = \theta$

Then we have $UD = \theta$

$$u^2D^2 = \theta(\theta - 1)$$

From (2) we have

⇨ $(\theta(\theta-1) - 3\theta + 4)y = e^{2z} - e^z + 1$

⇨ $(\theta^2-\theta-3\theta+4)y=e^{2z}-e^z+1$

⇨ $(\theta^2-4\theta+4)y= e^{2z}-e^z+1$

This is in the form of F$(\theta)y=Q(Z)$

To find Y_c :

Take A.E f(m) =0

$\rightarrow m^2-4m+4=0$

$\rightarrow \quad (m-2)^2=0$

$\rightarrow \quad m=2\ ,2$

$\therefore \ C.F\ (y_c)=(C_1+\ C_2\ \ Z\ )^{e^{2z}}$

$\therefore \ y_c=\ (C_1+C_2\log u)e^{2\log u}$

$$=\ \big(C_1+\ C_2\ log(x+1)\big)(x+1)^2$$

$$=(C_1+\ C_2\ \log(x+1))(x+1)^2$$

To find Y_p:

Let $(\theta^2-4\theta+4)y=e^{2z}-e^z+1$

Y= $\frac{1}{\theta^2-4\theta+4}\ .\ e^{2z}-e^z+1$

Then Y_p :$\frac{1}{\theta^2-4\theta+4}e^{2z}\ -\frac{1}{\theta^2-4\theta+4}.\ e^z+\ \frac{1}{\theta^2-4\theta+4.}e^{0.z}$

$$=\frac{1}{4-8+4}e^{2z}-\frac{1}{1-4+4}e^z+\frac{1}{0-0+4}.1$$

$$=\frac{ze^{2z}}{2\theta-4}-e^z+\frac{1}{4}$$

$$=\ \frac{ze^{2z}}{4-4}-\ e^z+\frac{1}{4}$$

$$=\frac{z^2e^{2z}}{2}-\ e^z+\frac{1}{4}$$

$$=\ \frac{(\log u\ )^2e^{2\log u}}{2}-\ e^{\log u}+\ \frac{1}{4}$$

$$=\ \frac{\big(log(u)\big)^2.\ e^{logu^2}}{2}-u+\frac{1}{4}$$

$\therefore Y_p = \dfrac{(logu)^2 . e^{\log u^2}}{2} - u + \dfrac{1}{4}$

$\therefore$ $The\ general\ solution\ is\ Y = Y_c + Y_p$

2) G.T

$(2x-1)^3 \frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx}$ -2y = x -----(1)

Eq (1) can be written as

$2^3\left(x - \frac{1}{2}\right)^3 \frac{d^3y}{dx^3} + 2\left(x - \frac{1}{2}\right)\frac{dy}{dx} - 2y = x$-------(2)

This is in the form of $p_{0.}(ax+b)^n \frac{d^ny}{dx^2} + p_{1.}(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}$ + ------+$p_n . y = Q(x)$

(or)

Which is a legendre's L.D.E.

Put $x - \frac{1}{2} = u$ => $x = u + \frac{1}{2}$

$dx = du$

From (2) we have

⇨ $8u^3 \frac{d^3y}{du^3} + 2u\frac{dy}{du} - 2y = u + \frac{1}{2}$----------(3)

Which is a homogeneous L.D.E.

Put $\frac{d}{du} = D$; $\frac{d}{dz} = \theta; u = e^z$ $=> \theta Z = \log u\ \ Then$

We have $u^3D^3 = \theta(\theta-1)(\theta-2)$

$uD = \theta$--------(a)

$from\ (3)\&(4) we\ have$

⇨ $(8u^3D^3 + 2uD - 2)y = u + \frac{1}{2}$

⇨ $(8\theta(\theta-1)(\theta-2) + 2\theta - 2)y = e^z + \frac{1}{2}$

⇨ $(8\theta(\theta^2 - 3\theta + 2) + 2\theta - 2)y = e^z + \frac{1}{2}$

⇨ $(8\theta^3 - 24\theta^2 + 16\theta + 2\theta - 2)y = e^z + \frac{1}{2}$

⇨ $(8\theta^3 - 24\theta^2 + 18\theta - 2)y = e^z + \frac{1}{2}$

This is in the form of $F(\theta)y = Q(Z)$

$\therefore The\ general\ sol\ is\ Y = Y_c + Y_p$

To Find Y_c : Take A.E F(m)=0

⇨ $8m^3 - 24m^2 + 18m - 2 = 0$

⇨ $m = 1\ (or)\ 8m^2 - 16m + 2 = 0$

⇨ $m = 1\ (or) 8m^2 - 16m + 2 = 0$

⇨ $m = 1\ (or) m = \frac{8 \pm \sqrt{64 - 4.4.1}}{2.4}$

$$= \frac{8 \pm \sqrt{48}}{8} = \frac{8 \pm \sqrt{4 \times 2 \times 2 \times 3}}{8}$$

$$= \frac{8 \pm 4\sqrt{3}}{8}$$

$$= 4\frac{(2 \pm \sqrt{3})}{8}$$

$$= 1 \pm \frac{\sqrt{3}}{2}$$

$\therefore Y_c = c_1 e^x + e^x (C_2 \cosh \frac{\sqrt{3}}{2} x + C_3 \sin h\ \frac{\sqrt{3}}{2} x)$

To Find Yp :

Let $(8\theta^3 - 24\theta^2 - 18\theta - 2)y = e^z + \frac{1}{2}$

⇨ $Y = \frac{1}{8\theta^3 - 24\theta^2 - 18\theta - 2} . e^z + \frac{1}{2} . e^{0.z}$

Then $Y_p = \frac{1}{8.1 - 24.1 - 18.1 - 2} e^z + \frac{1}{2} . \frac{1}{0 - 0 - 0 - 2}$

⇨ $\frac{1}{-36} . e^z - \frac{1}{4}$

⇨ $\frac{-1}{36} . e^{log\left(x - \frac{1}{2}\right)} - \frac{1}{4}$

⇨ $\frac{-1}{36}\left(x - \frac{1}{2}\right) - \frac{1}{4}$

$\therefore The\ general\ solution\ is\ Y = Y_c + Y_p$

FILL IN THE BLANKS;

1 .The general solution of $(4D^2+4D+1)y=0$ is ……………………
2.The C.F of $(D+1)(D-2)^2 y=e^{3x}$ is……………………
3.The P.I of $\frac{d^3y}{dx^3}+y=e^{-x}$ is……………………..
4.The P.I of $(D^2+a^2)y=\cos ax$ is……………………
5. The P.I of $(D^2-5D+6)y=e^{2x}$ is…………….
6. The P.I of $(D+1)^2 y=x$ is…………….
7.$\frac{1}{D^2+D+1}\sin x$ = ……………….
8. The P.I of $(D-1)^4 y=e^x$ is……………….
9.The value of $\frac{1}{D-2}\sin x$ is ……………….
10. The value of $\frac{1}{D^2+4}\sin 2x$ is………………..
11.$\frac{1}{D^2-1}e^x$ =………………….
12.$\frac{1}{(D+2)}(x+e^x)$=………………………..
13. The C.F of the equation $(D^3-D)y=x$ is…………………….
14. The C.F of the equation $(D^2+4D+5)y=13e^x$ is………………………
15.C.F of $(D-1)^2 y=\sin 2x$ is………………………….
16.The equation $e^4dx+(xe^y+2y)dy=0$ is………….
a.Homogeneous b.Variable Separable c. Exact d.Non homogeneous
17.P.I of $(D^2-2D+1)y=\cosh x$ is………………

MULTIPLE CHOICE QUESTIONS;

1.The general solution of $(4D^2+4D+1)y=0$ is………….
a. $y=c_1e^{\frac{-x}{2}}+c_2e^{\frac{-x}{2}}$ b. $y=(c_1x+c_2)\,e^{\frac{-x}{2}}$
c. $y=c_1e^{\frac{x}{2}}+c_2e^{\frac{x}{2}}$ d. $y=(c_1+c_2x)\,e^{\frac{x}{2}}$

2.The C.F of $(D+1)(D-2)^2 y=e^{3x}$ is……………………
a. $(c_1+c_2x)e^{-x}+c_3e^{3x}$ b. $(c_1+c_2x)e^{2x}+c_3e^{-x}$
c. $c_1e^{-x}+c_2e^{2x}$ d.None
3.P.I of $(D^3+1)y=e^{-x}$ is…………………..
a. $xe^{\frac{-x}{3}}$ b. $e^{\frac{-x}{3}}$ c. $-xe^{\frac{-x}{3}}$ d.None
4.The P.I of $(D^2+a^2)y=\cos ax$ is……………..
a. $-\frac{x}{2a}\cos ax$ b. $\frac{x}{2a}\sin ax$ c. $x\cos ax$ d. $x\sin ax$
5.The P.I of $(D^2-5D+6)y=e^{2x}$ is………………
a. $-xe^{2x}$ b. xe^{2x} c. e^{2x} d.0
6.P.I of $(D+1)^2 y=x$ is………………………
a. x b. x-2 c. $(x+1)^2$ d. $(x+2)^2$
7. $\frac{1}{D^2+D+1}\sin x$=……………….
a. sin x b. cos x c. $\frac{1}{3}\sin x$ d. 1-cos x
8. P.I of $(D-1)^4 y=e^x$ is…………………..
a. $\frac{x^4}{4}e^x$ b. x^4e^x c. e^x d. $\frac{e^x}{4}$

9. The value of $\frac{1}{D-2}$ sin x is …………….

a. $\frac{-1}{5}$(cos x+sin x) b. $\frac{1}{5}$cos x c. $\frac{1}{5}$sin x d. $\frac{1}{5}$(cos x+sin x)

10. The value of $\frac{1}{D^2+4}$ sin 2x is……………….

a. $\frac{1}{5}$sin 2x b. $\frac{-1}{5}$ sin^2x c. $\frac{1}{5}$cos 2x d. $\frac{-1}{5}$cos 2x

11. $\frac{1}{D^2-1}e^x$=…………………………….

a. $\frac{1}{2}xe^x$ b. $\frac{-1}{2}xe^x$ c. $\frac{x^2}{2}e^x$ d. None

12. $\frac{1}{D+2}(x+e^x)$=……………………

a. $\frac{-x}{4}-\frac{1}{16}+\frac{e^x}{3}$ b. $\frac{x}{4}+\frac{1}{16}-\frac{e^x}{3}$ c. $\frac{x}{4}-\frac{1}{16}+e^x$ d. None

13. The C.F of the equation $(D^3-D)y=x$ is …………………………….

a. $c_1+c_2x+c_3e^x$ b. $c_1+c_2e^x+c_3e^{-x}$ c. $(c_1+c_2x)e^x+c_3e^{-x}$ d. None

14. The C.F of $(D^2+4D+5)y=13e^x$ is…………………………

a. $e^{-2x}(c_1cosx+c_2sinx)$ b. $e^{2x}(c_1cosx+c_2sinx)$ c. $e^x(c_1cos2x+c_2sin2x)$ d. None

15. C.F of $(D-1)^2y$=Sin 2x is……………….

a. $(c_1+c_2x)e^x$ b. $(c_1+c_2x)e^{-x}$ c. $c_1x+c_2e^x$ d. None

16. The substitution to transform homogeneous linear equation into a linear equation with constant coefficient is…………..

a. $x=e^z$ b. $z=e^x$ c. x=logz d. x=y

17. By eliminating y from the simultaneous equation (D-1)x+2y=0, (D-3)y-5x=0 where $D=\frac{d}{dt}$ the differential equation obtained is ……………

a. $(D^2+4D-13)x=0$ b. $(D^2-4D+13)x=0$

c. $(D^2-4D-13)x=0$ d. $(D^2+4D+13)x=0$

18) If m_1, m_2, m_3 are real and distinct roots then the complementary function is

(a) $c_1e^{(m1x+m2x+m3x)}$ (b) $c_1e^{m1x}+c_1e^{m2x}+c_3e^{m3x}$

(b) (c) $c_1e^{m1x}+c_2e^{m2x}+c_3e^{m3x}$ (d) None

19) If m_1, m_2, m_3 are roots are real & equal and m_4, m_5 are real and different

Then complementary function is

(a) $c_1e^{m1x}+c_2e^{m2x}+(c_3+c_4)\ e^{m3x}$ (b) $(c_1+c_2x)e^{m1x}+c_4e^{m3x}+c_5e^{m4x}$

(c) $(c_1+c_2x+c_3x^2+c_4x^3)\ e^{m1x}$ (d) none

20) If two roots of auxiliary equation are complex say $\alpha+i\beta$, $\alpha-i\beta$ then the complementary function is

(a) $(c_1\cos\alpha x+c_2\sin\beta x)$ (b) $e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$

(c). $e^{\alpha x}[(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x)$ (d) None

UNIT-III

Multiple Integrals:

Definite Integrals: Let y =f(x) be a function of one variable define and bounded on [a,b] consider the sum $\sum_{i=1}^{n} f(x_i)\ \delta x_i$ of this sum tends to a finite limit as n =>∞ such that length of δx_i tends to O for arbitrary choice of the t_i's. The limit is define to be the definite integral $\int_a^b f(x)dx$.

The generalization of this definition to two dimensions is called a double integral and to three dimensions is called a triple integral.

Double Integral: An expression of the form $\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)dy\,dx$ or $\int_a^b \int_{x_1(y)}^{x_2(y)} f(x,y)dx\,dy$ is called an iterated integral or double integral.

4) Evaluate $\int_0^1 \int_0^x e^{x+y}$ dy dx

Ans: = $\frac{1}{2}(e-1)^2$

5) **Evaluate** $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)$ **dxdy**

$$= \int_{x=0}^{1} dx\ \int_{y=x}^{\sqrt{x}} (x^2+y^2)dy$$

$$= \int_{x=0}^{1} dx\ [x^2y+\frac{y^3}{3}]_x^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} [x^2\sqrt{x}+\frac{(\sqrt{x})^3}{3}] - [x^3+\frac{x^3}{3}]]\ dx$$

$$= \int_{x=0}^{1} [x^{\frac{5}{2}}+\frac{(x)^{\frac{3}{2}}}{3} - \frac{4x^3}{3}]\ dx$$

$$= [\frac{(x)^{\frac{7}{2}}}{\frac{7}{2}} + \frac{(x)^{\frac{5}{2}}}{3.\frac{5}{2}} - \frac{4x^4}{3\ .4}]_0^1$$

$$= \frac{2}{7}+\frac{2}{15}-\frac{1}{3} \quad = \frac{30+14-35}{105} = \frac{9}{105} = \frac{3}{35}$$

P.T $\int_1^2 \int_3^4 (xy+\ e^y)$ dx dy

$= \int_3^4 \int_1^2 (xy+\ e^y)$ dy dx

L.H.S = $\int_1^2 [\int_3^4 (xy+\ e^y)$ dx]dy

$= \int_1^2 [y \cdot \frac{x^2}{2} + e^y \cdot x]_3^4 dy$

$= \int_1^2 \{[y \cdot \frac{16}{2} + 4e^y] - [y \cdot \frac{9}{2} + 3e^y]\} dy$

$= \int_1^2 [y \cdot \frac{7}{2} + e^y] dy$

$= [y \cdot \frac{7}{2} + e^y]_1^2$

$= (\frac{7}{4} * 4 + e^2) - (\frac{7}{4} + e)$

$= 7 - \frac{7}{4} + e^2 - e$

$= \frac{21}{4} + e^2 - e$

R.H.S $= \int_3^4 [\int_1^2 (xy + e^y) dx] dy$

$= \int_3^4 [x \cdot \frac{y^2}{2} + e^y]_1^2 dx$

$= \int_3^4 \{[x \cdot \frac{4}{2} + e^2] - [\frac{x}{2} + e]\} dx$

$= \int_3^4 [\frac{3x}{2} + e^2 - e] dx$

$= [\frac{3}{2}\frac{x^2}{2} + e^2 x - ex]_3^4$

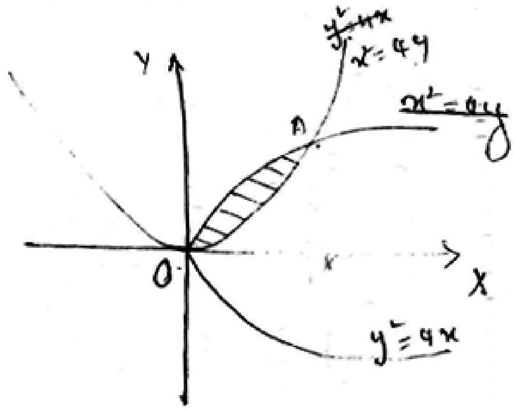
$= (\frac{3}{4} * 9 + 3e^2 - 3e) - (\frac{3}{4} * 16 + 4e^2 - 4e)$

$= \frac{21}{4} + e^2 - e$

$\therefore$ L.H.S = R.H.S

3. Evaluate $\int\int y\, dx\, dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$

The co-ordinates of points O & A are (0,0) and (4,4).



$$\int\int y\,dx\,dy = \int_{x=0}^{4} dx \int_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} y\,dy$$

$$= \int_{x=0}^{4} dx \left(\frac{y^2}{4}\right)_{\frac{x^2}{4}}^{2\sqrt{x}}$$

$$= \int_{x=0}^{4}\left[\frac{4x}{2} - \frac{x^4}{2.16}\right] dx$$

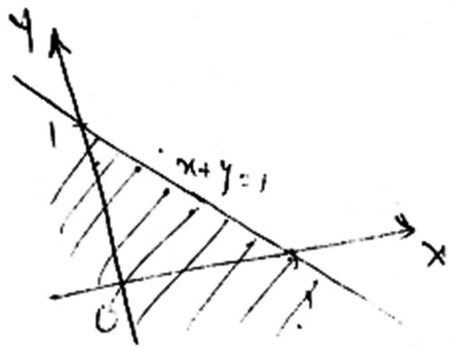
$$= \frac{1}{2}\left[4 . \frac{x^2}{2} - \frac{x^2}{16.\,5}\right]_0^4$$

$$= \frac{1}{2}\left[32 - \frac{64}{16}\right]$$

$$= \frac{1}{2}\left[\frac{160-64}{5}\right] = \frac{1}{2}\left[\frac{96}{5}\right] = \frac{48}{5}$$

$$\therefore \int\int y\,dx\,dy = \frac{48}{5}$$

4. Evaluate $\int\int x^2 + y^2$ dx dy in positive quadrant for which x+y≤1.



$\int\int x^2 + y^2$dxdy =

$$= \int_{x=0}^{1} dx \int_{y=0}^{1-x} (x^2 + y^2)\, dy$$

$$= \int_{x=0}^{1} (x^2 y + \frac{y^3}{3})_0^{1-x} \ \mathrm{dx}$$

$$= \int_0^1 [\, x^2(1-x) + \frac{(1-x)^3}{3}]\, dx$$

$$= \int_0^1 [\, x^2 - x^3 + \frac{1}{3}\,(1 - 3\mathrm{x} + 3x^2 - x^3)]\ \mathrm{dx}$$

$$= (\frac{2x^3}{3} - \frac{4}{3} \cdot \frac{x^4}{4} + \frac{1}{3}x - \frac{x^2}{2})_0^1$$

$$= \frac{2}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$$

$$\therefore \int\int x^2 + y^2 \mathrm{dxdy} = \frac{1}{6}$$

Change of order of integration:

5. Evaluate the following integrals by changing the order of integration.

Sol:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dx\, dy$$

The area of integration lies between y=0 which is x-axis and

$$\mathrm{y} = \sqrt{1 - x^2} \Rightarrow x^2 + y^2 = 1$$

Which is a circle. Also limits of x are 0 to 1.

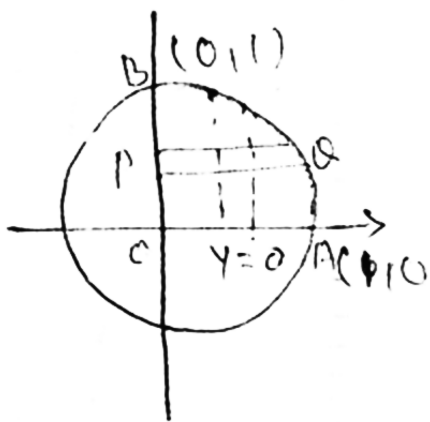
Hence the region of integration is OAB and is divided into vertical strip for changing the order of integration, we shall divide the region of integration into horizontal strips.

The new limits of integration become x = 0 to x= $\sqrt{1 - y^2}$ and those for ‘y’ will be y=0 to y = 1.

Hence

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dx\, dy \ = \int_{y=0}^{1} dy \int_{x=0}^{\sqrt{1-y^2}} y^2\, dx$$

$$= \int_{y=0}^{1} y^2 [\, (x\,)_0^{\sqrt{1-y^2}}\,] dy$$



$$= \int_{y=0}^{1} y^2 \sqrt{1 - y^2}\, dy$$

Put y = sinθ dy=cos$\theta\, d\theta$

y=0 =>$\theta = 0$,

y=1 =>$\theta = \frac{\pi}{2}$

hence I= $\int_{\theta=0}^{\frac{\pi}{2}} sin^2\theta cos^2\theta d\theta$

$= \frac{1}{4} . \frac{\pi}{2} . \frac{1}{2} \qquad = \frac{\pi}{16}$

6. $\int_0^a \int_y^a \frac{x}{x^2+y^2}$dy dx

Sol. The region of integration lies between x=y a straight line and passing through the origin x=a and y=0. Also the limits for y are 0 to a, which is ΔOAB and the region is divided by horizontal strips.

By changing the order of integration take a vertical strip PQ so that the new limits become y=0 to y=x and x varies from 0 to a.

Hence I = $\int_0^a \int_y^a \frac{x}{x^2+y^2}$dy dx

$= \int_{x=0}^{a} dx \int_{y=0}^{x} \frac{x}{x^2+y^2}$

dy

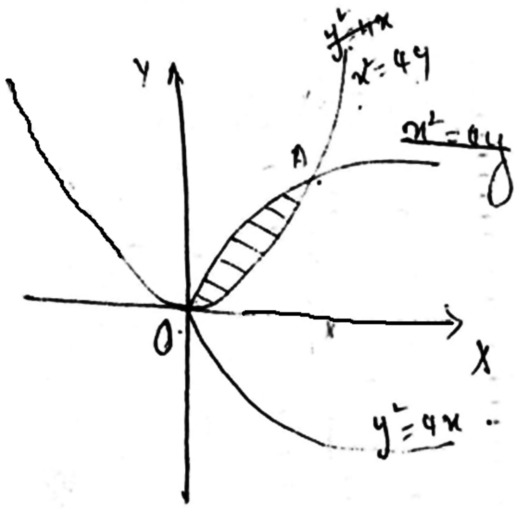
=

$\int_{x=0}^{a} x. \left(\frac{1}{x} tan^{-1}\frac{y}{x}\right)_0^x dx$

$= \int_{x=0}^{a} tan^{-1}(1) dx$

$= \frac{\pi}{4} (x)_0^a = \frac{\pi a}{4}$

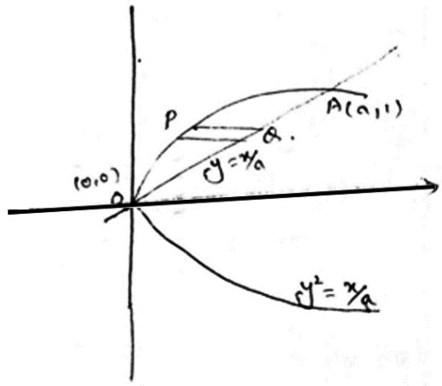
$\therefore \int_0^a \int_y^a \frac{x}{x^2+y^2}$dy dx = $\frac{\pi a}{4}$



7. $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy$

Sol. The region of integration is defined by y= $\sqrt{\frac{x}{a}}$ => $y^2 = \frac{x}{a}$ which is a parabola and y= $\frac{x}{a}$ => x=ay is a straight line passing through the origin. The points of intersection are O(0,0) and A(a,1). The limits for x are 0 to a.

Integration is done by taking strip parallel to y-axis. By changing the order of integration take a strip PQ parallel to x-axis. The limits for x in this case will be x=a y^2 to x=ay and that for y will be y=0 to y=1.



$$\therefore \text{I} = \int_0^a \int_{y=\frac{x}{a}}^{y=\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy$$

$$= \int_{y=0}^1 \int_{x=ay^2}^{x=ay} (x^2 + y^2)\, dx\, dy$$

$$= \int_{y=0}^1 \left(\frac{x^3}{3} + y^2 x\right)_{ay^2}^{ay} dy$$

$$= \int_0^1 \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right) dy$$

$$= \left(\frac{a^3}{3}\frac{y^4}{4} + a\frac{y^4}{4} - \frac{a^3}{3}\frac{y^7}{7} - a\frac{y^5}{5}\right)_0^1$$

$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} \qquad = \frac{a^3}{28} + \frac{a}{20}$$

$$\therefore \int_0^a \int_{y=\frac{x}{a}}^{y=\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy = \frac{a^3}{28} + \frac{a}{20}$$

Change of variables:

Let x and y be functions of u and v and let x = $\emptyset(u,v)$ and y = x(u,v) then

$\int_R \int f(x,y)dx\,dy$ is transformed into $\int_{R^1} \int f\{\emptyset(u,v), \text{x}(u,v)\} |\mathcal{J}| du\, dv$

Where $\mathcal{J} = \frac{\partial(x,y)}{\partial(u,v)}$ is the jacobian of transformation from (x,y) to(u,v) co-ordinates and $R^1$is the region in the uv plane corresponding to R in the xy plane.

In polar co-ordinates x=rcosθ , y= rsinθ

$$\mathcal{J} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = \text{r}$$

$$\therefore \int_R \int f(x,y)dx\,dy = \int_{R^1} \int f\{\text{rcos}\theta, \text{rsin}\theta\}\, r d\theta$$

8. Evaluate the following integrals by changing to polar co-ordinates.

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \text{dxdy}$$

Since both x and y vary from 0 to ∞, the region of integration is the xoy plane, change to polar co-ordinates, x= $r\cos\theta$, y = $r\sin\theta$ dx dy = r dr dθ and $(x^2 + y^2)$ = r^2. In the region of integration 'r' varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

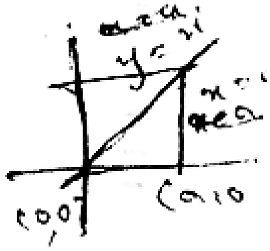
$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2}\ r\,dr\,d\theta$$

Put t =r^2

$\therefore$dt = 2r dr

r =0 => t=0

r=∞ => t = ∞



$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_0^\infty e^{-t}\frac{dt}{2}d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [-e^{-t}]_0^\infty d\theta$$

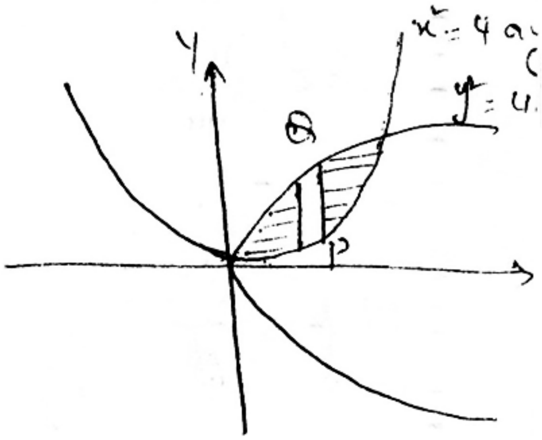
$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} (0-1)d\theta$$

$$= \frac{1}{2}\ [\theta]_0^{\frac{\pi}{2}} = \frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}$$

9. Show by double integration, the area between the parabolas y^2 = 4ax and x^2 = 4ay is $\frac{16}{3}a^2$

Sol:

The P OI of given curves is A(0,0) and B(4a,4a). by taking a vertical strip parallel to y-axis. We get the area between the two parabolas as:



$$A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx$$

$$= \int_{x=0}^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} \text{dy}$$

$$= \int_{x=0}^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) \text{dx}$$

$$= \left[2\sqrt{a}.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

$$\therefore \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx = \frac{16}{3}a^2$$

Triple integrals:

Let f(x,y,z) be a function which is defined at all points in a finite region v in space. Let δx, δy,

δzbe an elementary volume v enclosing of the point (x,y,z) thus the triple summation.

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}} \sum\sum\sum f(x,y,z)\, \delta x, \delta y, \delta z$$

If it exists is written as $\iiint f(x,y,z)$ dx dydz which is called the triple integral of f(x,y,z) over the region v.

If the region v is bounded by the surfaces $x=x_1$, $x=x_2$, $y=y_1$, $y=y_2$, $z=z_1$, $z=z_2$ then

$$\iiint f(x,y,z)\text{dxdydz} = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$$

Note:

(i) If x_1, x_2; y_1, y_2; z_1, z_2 are all constants then the order of integration is immaterial provide the limits of integration are changed accordingly.

i.e.

$$= \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y, z) dx\ dy\ dz$$

$$= \int_{y_1}^{y_2} \int_{z_1}^{z_2} \int_{x_1}^{x_2} f(x, y, z) dx\ dz\ dy$$

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z) dz\ dy\ dx$$

(ii) If, however z_1, z_2 are functions of x and y and y_1, y_2 are functions of x while x_1 and x_2 are constants then the integration must be performed first w.r.to 'z' then w.r.to 'y' and finally w.r.to 'x'.

i.e.

$\iiint f(x, y, z)$dxdydz =

$$= \int_{x=a}^{x=b} \int_{y=\emptyset_1(x)}^{y=\emptyset_2(x)} \int_{z=\delta_1(x,y)}^{z=(x,y)} f(x, y, z) dz\ dy\ dx$$

10. Evaluate the following integrals:

(i) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\ dz\ dy\ dx$

Sol $\int_{x=0}^{1} \left\{ \int_{y=0}^{\sqrt{1-x^2}} \left[\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\ dz \right] dy \right\} dx$

$$= \int_{x=0}^{1} \left\{ \int_{y=0}^{\sqrt{1-x^2}} \left[xy\frac{z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} \right\} dy\ dx$$

$$= \int_{x=0}^{1} \left\{ \int_{y=0}^{\sqrt{1-x^2}} \left[\frac{xy\,(1-x^2-y^2)}{2} \right] \right\} dy\ dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left\{ \int_{y=0}^{\sqrt{1-x^2}} [\,xy - x^3y - xy^3] \right\} dy\ dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4} dx$$

$= \frac{1}{2}\int_0^1 \frac{x-x^3-x^3+x^5)}{2} - \frac{x(1-2x^2+x^4)}{4}\, dx$

$= \frac{1}{2}\int_0^1 \frac{x-2x^3+x^5)}{2} - \frac{2x^3-x-x^5)}{4}\, dx$

$= \frac{1}{8}\int_0^1 x - 2x^3 + x^5 dx$

$= \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} - \frac{x^6}{6}\right]_0^1$

$= \frac{1}{8}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{1}{48}$

$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\ dz\ dy\ dx = \frac{1}{48}$

(ii) $\int_1^e \int_1^{logy} \int_1^{e^x} \log y\ dz\ dx\ dy$

Sol. $I = \int_{y=1}^e \int_{x=1}^{logy} \int_{z=1}^{e^x} \log z\ dz\ dx\ dy$

Consider $\int_{z=1}^{e^x} \log z\ dz = [\text{zlogz} - \text{z}]_1^{e^x}$

$= e^x \log e^x{}_e - e^x + 1$

$= xe^x - e^x + 1$

$= e^x (x-1) + 1$

$I = \int_{y=1}^e \int_{x=1}^{logy} \{(x-1)\, e^x + 1\}\ dx$

Consider $\int_{x=1}^{x=logy} \{(x)\, e^x - e^x + 1\}\ dx$

$= [xe^x - e^x - e^x + 1]_{x=1}^{logy}$

$= [xe^x - 2e^x + 1]_1^{logy}$

$= [\text{ylogy} - 2\text{y} + \text{logy}] - [e-2e+1]$

$= (\text{y}+1)\text{logy} - 2\text{y} + (\text{e}-1)$

$\therefore I = \int_{y=1}^e ylogy + logy - 2y + (e-1)dy$

$= \left[\frac{y^2}{2}logy - \frac{y^2}{4} + ylogy - y - y^2 + (e-1)y\right]_1^e$

$= [\frac{e^2}{2}loge - \frac{e^2}{4} + eloge - e - e^2 + (e-1)e] -$

$$- \quad [\tfrac{1}{2} log1 \quad - \tfrac{1}{4} + \ log1 - 1 - 1 + e - 1]$$

$$= (\frac{e^2}{2} \quad - \frac{e^2}{4} + e - 2e) - (-\frac{1}{4} - 3 - e)$$

$$= \frac{2e^2 - e^2 - 8e + 1 + 12}{4} = \frac{1}{4} [e^2 \ -8e+13]$$

$$\int_1^e \int_1^{logy} \int_1^{e^x} \log y \ dz \ dx \ dy = \frac{1}{4} [e^2 \ -8e+13]$$

APPLICATIONS OF INTEGRATION

Length of curves (Rectification):

The process of finding the length of an arc of the curve is called rectification .

Equation of curve	**Length of arc**

Cartesion form:
i) y=f(x) & x= a, x=b;
ii) x=f(y) & y=a, y=b;
	$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\ dx.$
	$S = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2}\ dy.$

Parametric equation: $x=x(\theta)$, $y=y(\theta)$ & $\theta = \theta_1$, $\theta = \theta_2$	$S = \int_{\theta 1}^{\theta 2} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\ d\theta.$
Polar form : i) $r = f(\theta)$ & $\theta = \alpha, \theta = \beta$ ii) $\theta = \emptyset(r)$ & $r = r_1$, $r = r_2$	$S = \int_\alpha^\beta \sqrt{(r)^2 + (\frac{dr}{d\theta})^2}\ d\theta.$ $S = \int_{r1}^{r2} \sqrt{1 + (r)^2 (\frac{d\theta}{dr})^2}\ dr.$

1. **Find the length of the arc of the curve y = log $(\frac{e^x - 1}{e^x + 1})$ from x =1 to x =2**

Sol: Given y $= \log(e^x - 1) - \log(e^x + 1)$

$$\frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = e^x [\frac{e^x - 1 + e^x + 1}{e^{2x} - 1}] = \frac{2e^x}{e^{2x} - 1}$$

Hence required length $= \int_1^2 \sqrt{1 + (\frac{dy}{dx})^2}\ dx.$

$$= \int_1^2 \sqrt{1 + \frac{4e^{2x}}{(e^{2x}-1)^2}}\, dx.$$

$$= \int_1^2 \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\, dx. \quad = \int_1^2 \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\, dx.$$

$$= \int_1^2 \sqrt{\frac{\frac{(e^{2x}+1)}{e^x}}{\frac{(e^{2x}-1)}{e^x}}}\, dx. \qquad = \int_1^2 \frac{e^x+e^{-x}}{e^x-e^{-x}}\, dx.$$

$$= \log\left[(e^x - e^{-x})\right]_1^2$$

$$= \log\left(\frac{e^2-e^{-2}}{e^1-e^{-1}}\right)$$

$$= \log\left(\frac{(e^1+e^{-1})(e^1-e^{-1})}{e^1-e^{-1}}\right)$$

$$= \log\left(e + \frac{1}{e}\right)$$

2. Find length of the loop of the curve $x = t^2$, $y = t - t^3/2$

Sol : Here 'x' is even function of 't'. & 'y' is an odd func of t , so the curve is Symmetrical about x- axis.

Intersection with the co-ordinate axis:

Put y=0 => $t - \frac{t^3}{3} = 0$ => $t\left[1 - \frac{t^2}{3}\right] = 0$

⇨ t= 0 or $3 = t^2$ => $t = \sqrt{3}$

⇨ ∴ x = 0 or $x = \sqrt{3} - \frac{3\sqrt{3}}{3} = 0$

or x =3

Thus the curve cuts x-axis at A(0,3) and O(0,0) due to symmetry a loop is formed in between these two points.

∴ Required length $= 2\int_{t=0}^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= 2\int_{t=0}^{\sqrt{3}} \sqrt{4t^2 + (1-t^2)^2}\, dt$

$= 2\int_0^{\sqrt{3}} \sqrt{(1+t^2)^2}\, dt$

$= 2\left[t + \frac{t^3}{3}\right]_0^{\sqrt{3}} = 2\left[\sqrt{3} + \frac{3\sqrt{3}}{3}\right] = 2\sqrt{3}\ [2] = 4\sqrt{3}$

3. **Find the perimeter of the cardiod r=a(1-cosθ).**

Sol: This cardiod is symmetrical

about the initial line and the maximum value of r is 2a when $\theta = \pi$ and minimum value of r is 0 when $\theta = 0$

∴ It lies with in the circle r=2a

∴ perimeter = 2 × length of curve

$= 2\int_{\pi}^{0} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$

$= -2\int_{0}^{\pi} \sqrt{a^2(1 - cos\theta)^2 + a^2sin^2\theta}\ d\theta$

$= -2a\int_{0}^{\pi} \sqrt{(2 - 2cos\theta)^2}\ d\theta$

$= -2a\int_{0}^{\pi} \sqrt{2[1 - (1 - 2sin^2(\frac{\theta}{2})}\ d\theta$

$= -2a\ (2)\int_{0}^{\pi} sin\ (\frac{\theta}{2})\ d\theta$

$= -4a\ [\frac{-cos\frac{\theta}{2}}{\frac{1}{2}}]_0^{\pi}$

$= -8a\ [-cos\frac{\pi}{2} + cos0]$

$= -8a$

∴ perimeter = 8a

Volumes of solids of Revolution:

Region	**Volume of solid generated**
Cartesian form:	
(i) y = f(x) the x-axis and the lines x =a, x=b	$V = \pi\int_a^b y^2dx$
(ii) x = g(y) the y-axis and the lines y =c, y=d	$V = \pi\int_c^d x^2dy$
(iii) $y=y_1(x)$, $y = y_2(x)$ the x-axis and ordinates x=a,x=b	$V = \pi\int_a^b (y_2^2 - y_1^2)dx$

(iv) $x=x_1(y)$, $x = x_2(y)$ the y-axis and ordinates y=a,y=b	$V = \pi \int_a^b (x_2^2 - x_1^2)dy$
Parametric form:	
(i) $x = \emptyset(t)$, $y = \varphi(t)$ the x-axis and ordinates $t=t_1$, $t=t_2$	$V = \pi \int_{t1}^{t2} y^2 \frac{dx}{dt} dt$
(ii) $x = \emptyset(t)$, $y = \varphi(t)$ the y-axis and abscissae $t=t_1$, $t=t_2$	$V = \pi \int_{t1}^{t2} x^2 \frac{dy}{dt} dt$
Polar form:	
(i) $r =f(\theta)$ the initial line $\theta = 0$ and the radii vectors $\theta =\alpha$, $\theta=\beta$.	$V=\frac{2\pi}{3} \int_\alpha^\beta r^3\ sin\theta\ d\theta$
(ii) $r =f(\theta)$ the initial line $\theta = \frac{\pi}{2}$ perpendicular to the initial line and the radii vectors $\theta =\alpha$, $\theta=\beta$.	$V=\frac{2\pi}{3} \int_\alpha^\beta r^3\ cos\theta\ d\theta$
(iii) $r =f(\theta)$ the initial line $\theta = r$ and the radii vectors $\theta =\alpha$, $\theta=\beta$.	$V=\frac{2\pi}{3} \int_\alpha^\beta r^3 \sin\ (\theta - r)\ d\theta$

1) Find the volume of a sphere of radius 'a'.

Sol: Sphere is formed by the revolution of the area enclosed by a semi-circle about its diameter.

Equation to circle of radius 'a' is $x^2 + y^2 = a^2$-------(1)

In Semi-circle 'x' varies from –a to a.

Required volume = $\int_{-a}^{a} \pi\ y^2$ dx

$= \pi \int_{-a}^{a} (a^2 - x^2)$ dx

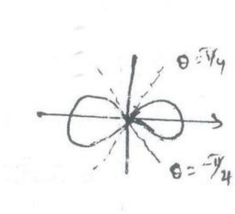
$= \pi [a^2 x - \frac{x^3}{3}]_{-a}^{a}$

$= \pi\, [a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3}]$

$= \pi\, [2a^3 - \frac{2a^3}{3}] = \frac{4\pi a^3}{3}$

2) **Find the volume of the solid generated by revolving the lemniscate**

$r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$



Sol: Given $r^2 = a^2\cos 2\theta$ the upper half of the loop. 'θ' varies from 0 to $\frac{\pi}{4}$

Required volume obtained by revolution of the loop about the line

$\theta = \frac{\pi}{2}$

Volume = $2 * \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} r^2 \cos\theta \, d\theta$

$= \frac{4\pi}{3} \int_0^{\frac{\pi}{4}} (a^2 \cos\theta)^{\frac{3}{2}} \cos\theta \, d\theta$

$= \frac{4\pi}{3} \int_0^{\frac{\pi}{4}} a^3 (1 - 2sin^2\theta)^{\frac{3}{2}} \cos\theta \, d\theta$

Let $\sqrt{2}\sin\theta = \sin\emptyset$

$\sqrt{2}\cos\theta \, d\theta = \cos\emptyset \, d\emptyset$

When $\theta = 0 \Rightarrow \sin\emptyset = 0 \Rightarrow \emptyset = 0$

$\theta = \frac{\pi}{4} \Rightarrow \sin\emptyset = \sqrt{2}.\frac{1}{\sqrt{2}} \Rightarrow \emptyset = \frac{\pi}{2}$

$= \frac{4\pi}{3} a^3 \int_0^{\frac{\pi}{2}} (1 - 2sin^2\emptyset)^{\frac{3}{2}} \frac{1}{\sqrt{2}} \cos\emptyset \, d\emptyset$

$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^{\frac{\pi}{2}} (cos^4\emptyset) \, d\emptyset$

since $\int cos^4\theta \, d\theta = \frac{(n-1)}{n} . \frac{(n-3)}{(n-2)}$ --------- $\frac{1}{2} * \frac{\pi}{2}$ (if n is even)

$= \frac{2\sqrt{2}\pi a^3}{3} . \frac{3}{4} * \frac{1}{2} * \frac{\pi}{2} = \frac{\sqrt{2}\pi^2 a^3}{8}$

Surface areas of Revolution:

Equation of curve	Axis of revolution	Surface Area
Cartesian Form:		

(i) $y = f(x)$ (ii) $x = \phi(y)$	X – axis Y – axis	$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ $S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$
Parametric Form:		
(i) $x = x(t)$, $y = y(t)$ (ii) $x = x(t)$, $y = y(t)$	X – axis Y – axis	$S = 2\pi \int_{t1}^{t2} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $S = 2\pi \int_{t1}^{t2} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
Polar Form:		
(i) $r = f(\theta)$ (ii) $r = f(\theta)$	$\theta = 0$ (x –axis) $\theta = \frac{\pi}{2}$ (y –axis)	$S = 2\pi \int_{\theta 1}^{\theta 2} r\sin\theta \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ $S = 2\pi \int_{\theta 1}^{\theta 2} r\cos\theta \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

1) Find the surface area of the solid generated by the revolution of the cycloid

$x=a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.

Sol : Given $x=a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$

$\frac{dx}{d\theta} = a(1 + \cos\theta)$, $\frac{dy}{d\theta} = -a\sin\theta$

For cycloid θ varies from 0 to 2π

$$\text{Surface area} = 2\pi \int_0^{2\pi} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2\pi \int_0^{2\pi} a(1 + \cos\theta)\ 2a.\cos\left(\frac{\theta}{2}\right) d\theta$$

$$= 2\pi\ 2a^2 \int_0^{2\pi} 2\cos^3\left(\frac{\theta}{2}\right) d\theta$$

$$= 8\pi a^2 \int_0^{2\pi} \cos^3\left(\frac{\theta}{2}\right) d\theta$$

Let $t = \frac{\theta}{2}$ $\qquad 2dt = d\theta$

$\theta = 0 \Rightarrow t = 0$

$\theta = 2\pi$ => t = π

= $8\pi a^2 \int_0^{\pi} cos^3 t\ 2\ dt$

= $16\ \pi a^2 2 \int_0^{\frac{\pi}{2}} cos^3 t\ \ dt$

= $32\ \pi\ a^2 . \frac{2}{3}$. 1

= $\frac{64\pi a^2}{3}$

2) Find the area of the surface of the revolution generated by revolving about the x-axis of the arc of the parabola y^2 = 12x from x = 0 to x = 3

Sol: y = $2\sqrt{3}\sqrt{x}$ =>$\frac{dy}{dx}$= $2\sqrt{3} . \frac{1}{2\sqrt{x}}$

surface area = $2\pi \int_0^3 y \sqrt{1 + (\frac{dy}{dx})^2}$ dx

= $2\pi \int_0^3 2\sqrt{3}\sqrt{x} \sqrt{1 + \frac{3}{x}}$ dx

= $4\pi\sqrt{3} \int_0^3 \sqrt{x} \sqrt{\frac{x+3}{x}}$ dx

= $4\pi\sqrt{3}[\frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}}]_0^3$

= $\frac{8\pi}{\sqrt{3}}$ $[(6)^{\frac{3}{2}} - (3)^{\frac{3}{2}}]$

= $\frac{8\pi}{\sqrt{3}} (3)^{\frac{3}{2}} [(2)^{\frac{3}{2}} - 1]$

= $24\pi\ [2\sqrt{2}\ -1]$

3) Find the length of arc of the curve x = 2$\theta - sin2\theta$, y= 2 $sin^2\theta$ as θ varies from 0 to π

Sol: $\frac{dx}{d\theta}$ = 2- $2cos2\theta$ = 2(1-cos2θ)

$\frac{dy}{d\theta}$ =4$sin\theta$ cosθ = 2 $sin2\theta$

$\frac{dx}{d\theta}$ = 2(1- $cos2\theta$)

$$\frac{ds}{d\theta} = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}$$

$= \sqrt{4\ (1 - cos2\theta)^2 + \ 4}\ sin^2 2\theta$

$= 2\sqrt{(1 - 2cos2\theta)^2 + \ cos^2 2\theta}\ + sin^2 2\theta$

$= 2\sqrt{(1 - 2cos2\theta)}+1$

$= 2\sqrt{2(1 - cos2\theta)}$

$= 2\sqrt{2.2\ \ sin^2\theta} = 2.2\ sin\theta = 4\ sin\theta$

Length of arc = $\int_0^{\pi} \frac{ds}{d\theta} d\theta = \int_0^{\pi} 4\ sin\theta\, d\theta$

$= -4\ [cos\theta]_0^{\pi}$ $\qquad = -4\ [\cos\pi - \cos 0]$

$= -4[-1-1]$

$=-4[-2] = 8$

4) Find the length of the curve $9x^2 = 4\ (1 + \ y^2)^{\ 3}$ from the point ($\frac{2}{3}$, 0) to the point ($\frac{10\sqrt{5}}{3}$,2)

Sol : Given curve is $9\mathbf{x}^2 = 4\ (1 + \ y^2)^{\ 3}$

$x^2 = \frac{4}{9}(1 + \ y^2)^{\ 3}$

$x = \frac{2}{3}(1 + \ y^2)^{\ \frac{3}{2}}$

$\frac{dx}{dy} = \frac{2}{3} * \frac{3}{2}(1 + \ y^2)^{\ \frac{1}{2}}.2y$

$= 2y\ (1 + \ y^2)^{\ \frac{1}{2}}$

$\frac{ds}{dy} = \sqrt{1 + (\frac{dx}{dy})^2} = \sqrt{1 + 4y^2(1 + \ y^2)}$

$= \sqrt{1 + 4y^2 + \ 4y^4)}$

$= \sqrt{(2y^2)^2 + 2.2\ y^2.1 + \ 1^2)}$

$= (2y^2 + 1)$

Length of curve $\qquad = \int_0^2 (2y^2 + 1)dy$

$= (2.\frac{y^3}{3} + y)_0^2$

$= (2.\frac{8}{3} + 2) - 0$

$= \frac{16+6}{3} = \frac{22}{3}$

PROBLEMS

1). Show that the volume of the solid generated by the revolution of the loop of the curve $y^2(a-x) = x^2(a+x)$ about the x-axis is 2π [log2 - $\frac{2}{3}$] a^3

Sol: Given curve is $y^2(a-x) = x^2(a+x)$

It contains only even powers of y hence it is symmetric about x-axis for x=0 => y = 0 and for y = 0 => x =0 or x=-a

∴it passes through (0,0) and (-a,0)

$\therefore V = \pi \int_{-a}^{0} y^2 dx$

$V = \pi \int_{-a}^{0} \frac{x^2(a+x)}{(a-x)} dx$

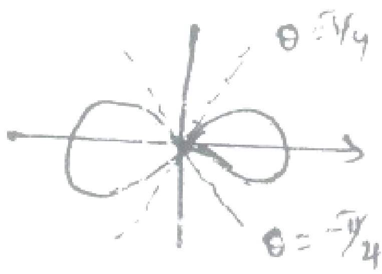
$V = \pi \int_{-a}^{0} \frac{(ax^2+x^3)}{(a-x)} dx$

$V = \pi \int_{-a}^{0} -x^2 - 2ax - 2a^2 + \frac{2a^3}{(a-x)}\, dx$

$V = \pi [\frac{x^3}{3} - \frac{2ax^2}{2} - 2a^2x - 2a^3\log|a-x|]_{-a}^{0}$

$V = 2\pi a^3$ [log2 - $\frac{2}{3}$]

2). Find the volume of the solid generated by revolving the lemniscates $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$.



Given

curve is $r^2 = a^2 \cos 2\theta$ the upper half of the loop θ varies from $\theta\ to\ \frac{\pi}{4}$

Required volume obtained by revolution of the loop about the line OY i.e. $\theta = \frac{\pi}{2}$

Volume = $2.\int_0^{\frac{\pi}{4}} \frac{2\pi}{3} r^3\ cos\theta\ d\theta$

$= 2.\int_0^{\frac{\pi}{4}} \frac{2\pi}{3} (a^2\ \cos 2\theta)^{\frac{3}{2}}\ cos\theta\ d\theta$

$= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}}\ (\cos 2\theta)^{\frac{3}{2}}\ cos\theta\ d\theta$

$= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}}\ (1 - 2sin^2(\theta))^{\frac{3}{2}}\ cos\theta\ d\theta$

Let $\sqrt{2}\ sin\theta = \sin\emptyset$

$=> \sqrt{2}\ cos\theta\ d\theta = \cos\emptyset\ d\emptyset$

When $\theta = 0 => \sin\emptyset = 0 => \emptyset = 0$

$\theta = \frac{\pi}{4} => \sin\emptyset = \sqrt{2}\ .\frac{1}{\sqrt{2}} => \emptyset = \frac{\pi}{2}$

$= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{2}}\ (1 - 2sin^2(\emptyset))^{\frac{3}{2}} \frac{1}{\sqrt{2}}\ cos\emptyset\ d\emptyset$

$= \frac{2\sqrt{2}\,\pi a^3}{3} \int_0^{\frac{\pi}{2}} cos^2(\emptyset) d\emptyset$

$= \frac{2\sqrt{2}\,\pi a^3}{3} * \frac{3}{4} * \frac{1}{2} * \frac{\pi}{2} = = \frac{\sqrt{2}\,\pi^2 a^3}{8}$

$V = \frac{\sqrt{2}\,\pi^2 a^3}{8}$

3). The part of the parabola cut off by the Latusrectum is rotated

(i) about the Latus rectum (ii) about the axis.

Show that the volumes generated are in the ratio 16:15

Sol: Equation of the parabola is $y^2 = 4ax$

Let v_1 be the volume generated when rotated about the Latus rectum and v_2 be the volume generated when rotated about the axis equation to Latus rectum is x=a.

When the area of the parabola cut off by the latus rectum revolves about the latus rectum any point(x,y)

On the parabola describes a circle radius a-x

$$\therefore v_1 = \pi \int_{-2a}^{2a} (a-x)^2 dy$$

$$= \pi \int_{-2a}^{2a} (a - \frac{y^2}{4a})^2 dy$$

$$= 2\pi \int_0^{2a} a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2} dy$$

$$v_1 = 2\pi[a^2 y - \frac{y^3}{6} + \frac{y^5}{80a^2}]_0^{2a}$$

$$= 2\pi[2a^3 - \frac{8a^3}{6} + \frac{32a^5}{80a^2}]$$

$$= 2\pi[2a^3 - \frac{4a^3}{3} + \frac{2a^3}{5}]$$

$$= 2\pi a^3 [\frac{30-20+6}{15}] = 2\pi a^3 \frac{16}{15} = \frac{32\pi a^3}{15}$$

$$V_2 = \pi \int_0^a (y)^2 dx$$

$$= \pi \int_0^a 4ax \, dx$$

$$= \pi[\frac{4ax^3}{2}]_0^a = 2\pi [a^3 - 0] = 2\pi a^3$$

$$\therefore \frac{v1}{V2} = \frac{32\pi a^3}{15*2\pi a^3} = \frac{16}{15}$$

4) Find the surface area of the solid generated by the revolution of the cycloid x=a(θ +sinθ), y = a(1+cosθ) about its base is $\frac{64}{3}\pi a^2$.

Sol: Given equation of cycloid is x=a(θ +asinθ), y = a(1+cosθ)

$$\frac{dx}{d\theta} = a(1+\cos\theta)$$

$$\frac{dy}{d\theta} = -a\sin\theta$$

$$\frac{ds}{d\theta} = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}$$

$$= \sqrt{a^2(1 + 2\cos\theta + cos^2\theta + sin^2\theta)}$$

$$= a\sqrt{2(1 + \cos\theta)}$$

$$= a\sqrt{2.2.\,cos^2(\frac{\theta}{2})}$$

$$= 2.a\,cos(\frac{\theta}{2})$$

For the arc of cycloid θ varies from θ= 0 to θ= 2π

$$\therefore \text{surface area} = \int_0^{2\pi} 2\,\pi y \frac{ds}{d\theta}\, d\theta$$

$$= 2\pi \int_0^{2\pi} a(1 + \cos\theta)\, 2a\,cos(\frac{\theta}{2})\, d\theta$$

$$= 4\pi a^2 \int_0^{2\pi} 2.\,cos^2\left(\frac{\theta}{2}\right).\,cos(\frac{\theta}{2})\, d\theta$$

$$= 4\pi a^3 \int_0^{2\pi} 2.\,cos^3\left(\frac{\theta}{2}\right) d\theta$$

$$= 8\pi a^2 \int_0^{\pi} cos^3 t\;(2dt)$$

Let $t = \frac{\theta}{2}$ | $\theta = 0$ $t = 0$

$d\theta = 2dt$ | $\theta = 2\pi$ $t = \pi$

$$= 16\pi a^2 * 2\int_0^{\frac{\pi}{2}} cos^3 t\;(dt) \quad [\text{since} \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx]$$

$$= 32\,\pi a^2 (\frac{2}{3})\,.1 = (\frac{64\,\pi a^2}{3})$$

5). Find the area of the surface of the revolution generated by revolving about the x-axis of the arc of the parabola y^2 = 12x from x=0 to x=3

Sol: Given parabola y^2 = 12x

⇨ $y = 2\sqrt{3}\,.\sqrt{x}$

⇨ $\frac{dy}{dx} = 2\sqrt{3}\,.\frac{1}{2\sqrt{x}} = \frac{\sqrt{3}}{\sqrt{x}}$

⇨ $\frac{ds}{dx} = \sqrt{1 + (\frac{dy}{dx})^2}$

$= \sqrt{1+\frac{3}{x}}$ $= \sqrt{\frac{x+3}{x}}$

Surface area = $\int_0^3 2\pi y \frac{ds}{dx}\,dx$

$= 2\pi \int_0^3 2\sqrt{3}\sqrt{x}\sqrt{\frac{x+3}{x}}\,dx$

$= 4\pi\sqrt{3}\int_0^3 (x+3)^{\frac{1}{2}}dx$

$= 4\pi\sqrt{3}\left[\frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$

$= \frac{8\pi}{\sqrt{3}}[6^{\frac{3}{2}} - 3^{\frac{3}{2}}]$

$= \frac{8\pi}{\sqrt{3}} 3^{\frac{3}{2}}[2^{\frac{3}{2}} - 1]$

$= 8\pi \,.\, 3\,[2^{\frac{3}{2}} - 1]$

$= 24\pi\,[2^{\frac{3}{2}} - 1]$

6) The lemniscate $r^2 = a^2\cos 2\theta$ revolves about the initial line find the surface area of the solid generated.

Sol: Given curve is $r^2 = a^2\cos 2\theta$

Differentiating w.r.to 'θ'

$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta$

$\frac{dr}{d\theta} = \frac{-a^2\sin 2\theta}{r}$

$\frac{ds}{d\theta} = \sqrt{r^2 + (\frac{dr}{d\theta})^2} = \sqrt{a^2\cos 2\theta + \frac{a^4\sin^2 2\theta}{r^2}}$

$= \sqrt{a^2\cos 2\theta + \frac{a^4\sin^2 2\theta}{a^2\cos 2\theta}}$

$= a\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}$

$= \frac{a}{\sqrt{\cos 2\theta}}$

If r = 0 => $\cos 2\theta = 0$ i.e $2\theta = \pm\frac{\pi}{2}$

⇨ $\theta = \frac{-\pi}{4}, \frac{\pi}{4}$

The curve consists of two equal loops.

∴ Required surface area = 2 $\int_0^{\frac{\pi}{4}} 2\pi y \frac{dr}{d\theta}\ d\theta$

$= 4\pi \int_0^{\frac{\pi}{4}} r \sin\theta \frac{a}{\sqrt{cos2\theta}}\ d\theta$

$= 4\pi \int_0^{\frac{\pi}{4}} a\sqrt{cos2\theta}\ \sin\theta \frac{a}{\sqrt{cos2\theta}}\ d\theta$

$= 4\pi a^2 \int_0^{\frac{\pi}{4}} \sin\theta\ d\theta$

$= 4\pi a^2 [-\cos\theta]_0^{\frac{\pi}{4}}$

$= 4\pi a^2 [\frac{-1}{\sqrt{2}} + 1] = 4\pi a^2 [1 - \frac{1}{\sqrt{2}}]$

MULTIVARIABLE CALCULUS(INTEGRATION)

1) The length of the curve y = $\frac{2}{3}x^{\frac{3}{2}}$ from x =1 to x = 4 is

(a) $\frac{2}{3}5^{\frac{3}{2}}$ (b) $\frac{2}{3}2^{\frac{3}{2}}$ (c) $\frac{2}{3}(5^{\frac{3}{2}} - 2^{\frac{3}{2}})$ (d) $\frac{2}{3}(5^{\frac{3}{2}} + 2^{\frac{3}{2}})$

2) The length of the curve y = $\frac{4}{3}x^{\frac{3}{2}}$ from x =0 to x = 20 is

(a) |2| (b) $|2|\frac{1}{3}$ (c) $|2|\frac{2}{3}$ (d) None

3) The length of the curve $x=t^2 - 3t$, $y=3t^2$ from t=0 to t=1 is

(a) 4 (b) 8 (c) 6 (d) 2

4) The length of the curve $x=e^t \sin t$, $y=e^t \cos t$ from t=0 to t=$\frac{\pi}{2}$ is

(a) $2\,e^{\frac{\pi}{2}}$ (b) $e^{\frac{\pi}{2}}-1$ (c) $2(e^{\frac{\pi}{2}}-1)$ (d) $\sqrt{2}\,e^{\frac{\pi}{2}}-1$

5) The perimeter of the asteroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is

(a) 4a (b) 6a (c) 8a (d) 12a

6) The perimeter of the loop of curve $9y^2 = (x-2)(x-5)^2$ is

(a) $2\sqrt{3}$ (b) $\frac{\sqrt{3}}{2}$ (c) $4\sqrt{3}$ (d) $\frac{\sqrt{3}}{4}$

7) The perimeter of the cardioids r=a(1-cos θ) is

(a)4a (b)2a (c) 6a (d) 8a

8) The upper half of the cardioids r=a(1+cos θ) is bisected by the line

(a)$\theta=\frac{\pi}{3}$ (b) $\theta=\frac{\pi}{4}$ (c) $\theta=\frac{\pi}{6}$ (d) None

9. The volume generated by revolution of r=2acos θ between

$\theta = 0 \ to \ \theta = \frac{\pi}{2}$ is

(a) $\frac{1\pi a^3}{3}$ (b) $\frac{2\pi a^3}{3}$ (c) $\frac{4\pi a^3}{3}$ (d) $\frac{5\pi a^3}{3}$

10. $\int_0^2 \int_0^x y\,dy\,dx$

(a) $\frac{4}{3}$ (b) $\frac{4}{5}$ (c) $\frac{2}{3}$ (d) $\frac{2}{5}$

11. $\int_{x=0}^{a} \int_{y=0}^{b} (x^2+y^2)$ dx dy

(a) $\frac{1}{3}(a^2+b^2)$ (b) $\frac{a}{3}(a^2+b^2)$ (c) $\frac{b}{3}(a^2+b^2)$ (d) $\frac{ab}{3}(a^2+b^2)$

12. $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi^2}{2}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi}{4}$

13. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}$dxdy

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$

14. $\int_0^\pi \int_0^{a\sin\theta} r\,dr\,d\theta$

(a) $\frac{\pi a^2}{4}$ (b) $\frac{\pi a}{4}$ (c) $\frac{\pi a^2}{2}$ (d) $\frac{\pi a}{2}$

15. The iterated integral for $\int_{-1}^{1}\int_{0}^{1-x^2} f(x,y)dx\,dy$ after changing order of integration is-------

Ans: $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx\,dy$

FILL IN THE BLANKS;

16. $\int_0^a \int_0^{\sqrt{x^2+y^2}} dx\,dy$after changing to polar co-ordinates is…………..

17. $\int_0^a \int_0^{\frac{a\sqrt{b^2-y^2}}{b}} xy\,dx\,dy$after changing the order of integration is…………..

18. $\int_0^1 \int_1^2 \int_2^3 x\,y\,z\,dx\,dy\,dz$…………

19. The area enclosed by the parabolas x^2 = y and y^2=x is……………..

20. The area of the region bounded by y^2 = 4ax and x^2 = 4ay is…………………

21. the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is…………….

22. $\int_0^1 \int_0^1 \int_0^1 x\,y\,x\,dx\,dy\,dz$…………………..

23. $\int_{-1}^1 \int_{-2}^2 \int_{-3}^3 dx\,dy\,dz$

(a) 12 (b) 24 (c) 48 (d) 36

24. The volume of tetrahedron formed by the surfaces x=0, y=0,z=0 and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$ = 1 is……………..

25. The volume of tetrahedron bounded by the co-ordinate planes and the plane x+y+z=1 is………………..

INTEGRATION AND ITS APPLICATIONS

(Assignment Problems)

1) (i) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$

(i) Evaluate $\int_0^5 \int_0^{x^2} x(x^2 + y^2)$ dx dy

2) (i) Evaluate $\iint (x^2 + y^2)$ dx dy in the positive quadratic for which

$x + y \leq 1$

(ii) Evaluate $\iint (x^2 + y^2)$ dx dy over the area bounded by the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

3) Evaluate $\int_0^{\frac{\Pi}{4}} \int_0^{a\sin\theta} \frac{r\, dr\, d\theta}{\sqrt{a^2 - b^2}}$

4) Evaluate $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2+y^2}$ dx dy by changing into polar co-ordinates.

5) By changing the order of integration, evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$

6) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} x\ xyz\, dz\, dy\, dx$

UNIT-IV

vector Calculus and Vector Operators

INTRODUCTION

In this chapter, vector differential calculus is considered, which extends the basic concepts of differential calculus, such as, continuity and differentiability to vector functions in a simple and natural way. Also, the new concepts of gradient, divergence and curl are introduced.

DIFFERENTIATION OF A VECTOR FUNCTION

Let S be a set of real numbers. Corresponding to each scalar t ε S, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. S is called the domain of $\bar{f}$. We write $\bar{f} = \bar{f}$ (t).

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional space. We can write $\bar{f} = \bar{f}$ (t)= $f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, where $f_1(t), f_2(t), f_3(t)$ are real valued functions (which are called components of $\bar{f}$). (we shall assume that $\bar{i}, \bar{j}, \bar{k}$ are constant vectors).

1. Derivative:

Let $\bar{f}$ be a vector function on an interval *I* and a ϵ *I*. Then $Lt_{t\to a} \dfrac{\bar{f}(t) - \bar{f}(a)}{t-a}$, if exists, is called the derivative of $\bar{f}$ at a and is denoted by $\bar{f}^{\,1}$(a) or $\left(\dfrac{d\bar{f}}{dt}\right)$ at t = a. We also say that $\bar{f}$ is differentiable at t =a if $\bar{f}^{\,1}$(a) exists.

2. Higher order derivatives

Let $\bar{f}$ be differentiable on an interval *I* and $\bar{f}^{\,1} = \dfrac{d\bar{f}}{dt}$ be the derivative of $\bar{f}$. If $Lt_{t\to a} \dfrac{\bar{f}^1(t) - \bar{f}^1(a)}{t-a}$ exists for every a Є $I_1 \subset I$. It is denoted by $\bar{f}^{\,11} = \dfrac{d^2\bar{f}}{dt^2}$.
Similarly we can define $\bar{f}^{\,111}(t)$ etc.

We now state some properties of differentiable functions (without proof)

(1) Derivative of a constant vector is $\bar{a}$.

If $\bar{a}$ and $\bar{b}$ are differentiable vector functions, then

(2). $\dfrac{d}{dt}(\bar{a} \pm \bar{b}) = \dfrac{d\bar{a}}{dt} \pm \dfrac{d\bar{b}}{dt}$

(3). $\frac{d}{dt}(\bar{a}.\bar{b}) = \frac{d\bar{a}}{dt}.\bar{b} + \bar{a}.\frac{d\bar{b}}{dt}$

(4). $\frac{d}{dt}(\bar{a}\times\bar{b}) = \frac{d\bar{a}}{dt}\times\bar{b} + \bar{a}\times\frac{d\bar{b}}{dt}$

(5). If $\bar{f}$ is a differentiable vector function and ϕ is a scalar differential function, then $\frac{d}{dt}(\phi\,\bar{f}) = \phi\frac{d\bar{f}}{dt} + \frac{d\phi}{dt}\bar{f}$

(6). If $\bar{f} = f_1(t)\bar{i} + f_2(t)\,\bar{j} + f_3(t)\bar{k}$ where $f_1(t)$, $f_2(t)$, $f_3(t)$ are cartesian components of the vector $\bar{f}$, then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\bar{j} + \frac{df_3}{dt}\bar{k}$

(7). The necessary and sufficient condition for $\bar{f}$ (t) to be constant vector function is $\frac{d\bar{f}}{dt} = \bar{0}$.

3. Partial Derivatives

Partial differentiation for vector valued functions can be introduced as was done in the case of functions of real variables. Let $\bar{f}$ be a vector function of scalar variables *p, q, t*. Then we write $\bar{f} = \bar{f}\,(p,q,t)$. Treating *t* as a variable and *p,q* as constants, we define

$$Lt_{\delta t\to 0}\ \frac{\bar{f}(p,q,t+\delta t) - \bar{f}(p,q,t)}{\delta t}$$

if exists, as partial derivative of $\bar{f}$ w.r.t. *t* and is denot by $\frac{\partial\bar{f}}{\partial t}$

Similarly, we can define $\frac{\partial\bar{f}}{\partial p}, \frac{\partial\bar{f}}{\partial q}$ also. The following are some useful results on partial differentiation.

4. Properties

1) $\frac{\partial}{\partial t}(\phi\bar{a}) = \frac{\partial\phi}{\partial t}\bar{a} + \phi\frac{\partial\bar{a}}{\partial t}$

2). If λ is a constant, then $\frac{\partial}{\partial t}(\lambda\bar{a}) = \lambda\frac{\partial\bar{a}}{\partial t}$

3). If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi\bar{c}) = \bar{c}\,\frac{\partial\phi}{\partial t}$

4). $\frac{\partial}{\partial t}(\bar{a}\pm\bar{b}) = \frac{\partial\bar{a}}{\partial t} \pm \frac{\partial\bar{b}}{\partial t}$

5). $\frac{\partial}{\partial t}(\bar{a}.\bar{b}) = \frac{\partial\bar{a}}{\partial t}.\bar{b} + \bar{a}.\frac{\partial\bar{b}}{\partial t}$

6). $\frac{\partial}{\partial t}(\bar{a}\times\bar{b}) = \frac{\partial\bar{a}}{\partial t}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial t}$

7). Let $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$, where f_1, f_2, $f_3$are differential scalar functions of more then one variable, Then $\frac{\partial \bar{f}}{\partial t} = \bar{i}\frac{\partial f_1}{\partial t} + \bar{j}\frac{\partial f_2}{\partial t} + \bar{k}\frac{\partial f_3}{\partial t}$(treating $\bar{i}, \bar{j}, \bar{k}$ as fixed directions)

5. Higher order partial derivatives

Let $\bar{f} = \bar{f}(p,q,t)$. Then $\frac{\partial^2 \bar{f}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \bar{f}}{\partial t}\right), \frac{\partial^2 \bar{f}}{\partial p \partial t} = \frac{\partial}{\partial p}\left(\frac{\partial \bar{f}}{\partial t}\right) etc.$

6.Scalar and vector point functions: Consider a region in three dimensional space. To each point $p(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$we associate a unique vector $\bar{f}(x,y,z)$, $\bar{f}$ is called a ***vector point function.***

Examples:

For example take a heated solid. At each point p(x,y,z)of the solid, there will be temperature T(x,y,z). This T is a scalar point function.

Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position p(x,y,z) in space, it will be having some speed, say, v. This **speed**v is a scalar point function.

Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.

In a magnetic field, at any point P(x,y,z) there will be a magnetic force $\bar{f}(x,y,z)$. This is called magnetic force field. This is also an example of a vector point function.

7. Tangent vector to a curve in space.

Consider an interval [a,b].

Let x = x(t),y=y(t),z=z(t)be continuous and derivable for a≤ t ≤b.

Then the set of all points (x(t),y(t),z(t)) is called a curve in a space.

Let A = (x(a),y(a),z(a)) and B = (x(b),y(b),z(b)). These A,B are called the end points of the curve. If A =B, the curve in said to be a closed curve.

Let P and Q be two neighbouring points on the curve.

Let $\overline{OP} = \bar{r}(t), \overline{OQ} = \bar{r}(t+\delta t) = \bar{r} + \delta\bar{r}. Then\ \delta\bar{r} = \overline{OQ} - \overline{OP} = \overline{PQ}$

Then $\frac{\delta\bar{r}}{\delta t}$ is along the vector $\overline{PQ}$. As Q→P, $\overline{PQ}$ and hence $\frac{\overline{PQ}}{\delta t}$ tends to be along the tangent to the curve at P.

Hence $\underset{\delta t \to 0}{lt} \frac{\delta \bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$ will be a tangent vector to the curve at P. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector)

Suppose arc length AP = s. If we take the parameter as the arc length parameter, we can observe that $\frac{d\bar{r}}{ds}$ is unit tangent vector at P to the curve.

VECTOR DIFFERENTIAL OPERATOR

Def. The vector differential operator ∇(read as del) is defined as

$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$. This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator. We will define now some quantities known as "**gradient", "divergence"** and **"curl"** involving this operator ∇. We must note that this operator has no meaning by itself unless it operates on some function suitably.

GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ or $\nabla\phi$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

Properties:

(1) If f and g are two scalar functions then grad(f ±g)= grad f ± grad g

(2) The necessary and sufficient condition for a scalar point function to be constant is that ∇f = $\bar{0}$

(3) grad(fg) = f(grad g)+g(grad f)

(4) If c is a constant, grad (cf) = c(grad f)

(5) grad $\left(\frac{f}{g}\right) = \frac{g(grad\ f) - f(grad\ g)}{g^2}$, $(g \neq 0)$

(6) Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$. Then $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$ if ϕ is any scalar point function, then

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \left(\bar{i}\frac{\partial \Phi}{\partial x} + \bar{j}\frac{\partial \Phi}{\partial y} + \bar{k}\frac{\partial \Phi}{\partial z}\right).\left(\bar{i}dx + \bar{j}dy + \bar{k}dz\right) = \nabla\Phi . d\bar{r}$$

DIRECTIONAL DERIVATIVE

Let ϕ(x,y,z) be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is $\overline{OP} = \bar{r}$. Let $\phi+\Delta\phi$ be the value of the function at neighbouring point Q. If $\overline{OQ} = \bar{r} + \Delta\bar{r}$. Let Δr be the length of $\Delta\bar{r}$

$\dfrac{\Delta\phi}{\Delta r}$ gives a measure of the rate at which ϕ change when we move from P to Q. The limiting value of $\frac{\Delta\phi}{\Delta r}$ as $\Delta r \to 0$ is called the derivative of ϕ in the direction of $\overline{PQ}$ or simply directional derivative of ϕ at P and is denoted by $d\phi/dr$.

Theorem 1: The directional derivative of a scalar point function ϕ at a point P(x,y,z) in the direction of a unit vector $\bar{e}$ is equal to $\bar{e}$. grad $\phi = \bar{e} \cdot \nabla\phi$.

Level Surface

If a surface $\phi(x,y,z) = c$ be drawn through any point $P(\bar{r})$, such that at each point on it, function has the same value as at P, then such a surface is called a level surface of the function ϕ through *P*.

e.g : equipotential or isothermal surface.

Theorem 2: $\nabla\phi$ at any point is a vector normal to the level surface $\phi(x,y,z)=c$ through that point, where *c* is a constant.

The physical interpretation of $\nabla\phi$

The gradient of a scalar function ϕ(x,y,z) at a point *P*(x,y,z) is a vector along the normal to the level surface $\phi(x,y,z) = c$ at *P* and is in increasing direction. Its magnitude is equal to the greatest rate of increase of ϕ. Greatest value of directional derivative of $\bar{\Phi}$ **at a point P = |grad ϕ|** at that point.

SOLVED PROBLEMS

1: If a=x+y+z, b= $x^2+y^2+z^2$, c = xy+yz+zx, prove that [grad a, grad b, grad c] = 0.

Sol:- Given a=x+y+z

There fore $\dfrac{\partial a}{\partial x} = 1,\ \dfrac{\partial a}{\partial y} = 1, \dfrac{\partial a}{\partial z} = 1$

Grad a = ∇a = $\sum \bar{i}\dfrac{\partial a}{\partial x} = \bar{i} + \bar{j} + \bar{k}$

Given b= $x^2+y^2+z^2$

Therefore $\dfrac{\partial b}{\partial x} = 2x,\ \dfrac{\partial b}{\partial y} = 2y, \dfrac{\partial b}{\partial z} = 2z$

Grad b = ∇b $= \bar{i}\dfrac{\partial b}{\partial x} + \bar{j}\dfrac{\partial b}{\partial y} + \bar{z}\dfrac{\partial b}{\partial z} = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

Again c = xy+yz+zx

Therefore $\frac{\partial c}{\partial x} = y + z,\ \frac{\partial c}{\partial y} = z + x, \frac{\partial c}{\partial z} = y + x$

Grad c = $\bar{i}\,\frac{\partial c}{\partial x} + \bar{j}\,\frac{\partial c}{\partial y} + \bar{z}\,\frac{\partial c}{\partial z} = (y+z)\bar{i} + (z+x)\,\bar{j} + (x+y)\bar{k}$

[grad a, grad b, grad c] = $\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0, (on\ simplification)$

[grad a, grad b, grad c] =0

2: Show that ∇[f(r)] = $\frac{f^{i}(r)}{r}\,\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol:- Since $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, we have $r^2 = x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly\ \frac{\partial r}{\partial y} = \frac{y}{r}\ ,\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla[f(r)] = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right) f(r) = \sum \bar{i} f^{1}(r)\frac{\partial r}{\partial x} = \sum \bar{i} f^{1}(r)\frac{x}{r}$

$= \frac{f^{1}(r)}{r}\sum \bar{i}x = \frac{f^{1}(r)}{r}.\bar{r}$

Note : From the above result, $\nabla(\log r) = \frac{1}{r^2}\bar{r}$

3: Prove that $\nabla(r^n) = nr^{n-2}\,\bar{r}$.

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$. Then we have $r^2 = x^2+y^2+z^2$ Differentiating w.r.t. x partially, we have

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly = \frac{\partial r}{\partial y} = \frac{y}{r}\ and\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla(r^n) = \sum \bar{i}\,\frac{\partial}{\partial x}(r^n) = \sum \bar{i}\,nr^{n-1}\frac{\partial r}{\partial x} = \sum \bar{i}\,nr^{n-1}\frac{x}{r} = n\,r^{n-2}\sum \bar{i}x = n\,r^{n-2}(\bar{r})$

Note : From the above result, we can have

(1). $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$, taking n = -1 (2) grad r = $\frac{\bar{r}}{r}$, taking n = 1

4: Find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).

Sol:- Given $f = xy+yz+zx$.

Grad f = $\bar{i}\,\frac{\partial f}{\partial x} + \bar{j}\,\frac{\partial f}{\partial y} + \bar{z}\,\frac{\partial f}{\partial z} = (y+z)\bar{i} + (z+x)\,\bar{j} + (x+y)\bar{k}$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$, then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

Directional derivative of f along the given direction $= \bar{e}.\nabla f$

$$= \frac{1}{3}\left(\bar{i} + 2\bar{j} + 2\bar{k}\right).\left[(y+z)\bar{i} + (z+x)\bar{j} + (x+y\bar{k})\right] at\,(1,2,0)$$

$$= \frac{1}{3}\left[(y+z) + 2(z+x) + 2(x+y)\right](1,2,0) = \frac{10}{3}$$

5: Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at the point (1,1,1).

Sol: - Here $f = xy^2+yz^2+zx^2$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \left(y^2 + 2xz\right)\bar{i} + \left(z^2 + 2xy\right)\bar{j} + \left(x^2 + 2yz\right)\bar{k}$$

At (1,1,1), $\nabla f = 3\bar{i} + 3\bar{j} + 3\bar{k}$

Let $\bar{r}$ be the position vector of any point on the curve $x = t$, $y = t^2$, $z = t^3$. then

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$$

$$\frac{\partial \bar{r}}{\partial t} = \bar{i} + 2t\bar{j} + 3t^2\bar{k} = (\bar{i} + 2\bar{j} + 3\bar{k}) \text{ at } (1,1,1)$$

We know that $\frac{\partial \bar{r}}{\partial t}$ is the vector along the tangent to the curve.

Unit vector along the tangent $= \bar{e} = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{1 + 2^2 + 3^2}} = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{14}}$

Directional derivative along the tangent $= \nabla f.\bar{e}$

$$= \frac{1}{\sqrt{14}}\,(\bar{i} + 2\bar{j} + 3\bar{k}).3(\bar{i} + \bar{j} + \bar{k})\;\frac{3}{\sqrt{14}}(1 + 2 + 3) = \frac{18}{\sqrt{14}}$$

6: Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).

Sol:- The position vectors of P and Q with respect to the origin are $\overline{OP} = \bar{i} + 2\bar{j} + 3\bar{k}$ and $\overline{OQ} = 5\bar{i} + 4\bar{k}$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i} - 2\bar{j} + \bar{k}$$

Let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$. Then $\bar{e} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

grad $f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$

The directional derivative of $\bar{f}$ at P (1,2,3) in the direction of $\overline{PQ} = \bar{e}.\nabla f$

$$= \frac{1}{\sqrt{21}}\,(4\bar{i} - 2\bar{j} + \bar{k}).(2x\bar{i} - 2y\bar{j} + 4z\bar{k})\;\frac{1}{\sqrt{21}}(8x + 4y + 4z)_{at(1,2,3)} = \frac{1}{\sqrt{21}}(28)$$

7: Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at (2,1,-1).

Sol: we have

grad f = $\bar{i}\,\frac{\partial f}{\partial x} + \bar{j}\,\frac{\partial f}{\partial y} + \bar{k}\,\frac{\partial f}{\partial z} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = -4\bar{i} - 4\bar{j} + 12\bar{k}$ at (2,1,-1).

Greatest value of the directional derivative of $f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$

8: Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a direction of the normal to the surface $3xy^2+y= z$ at (0,1,1).

Sol:- Let $f(x, y, z) \equiv 3xy^2+y- z = 0$

Let us find the unit normal e to this surface at (0,1,1). Then

$$\frac{\partial f}{\partial x} = 3y^2,\ \frac{\partial f}{\partial y} = 6xy+1, \frac{\partial f}{\partial z} = -1.$$

$\nabla f = 3y^2i+(6xy+1)j-k$

$(\nabla f)_{(0,1,1)} = 3i+j-k = \bar{n}$

$$\bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{3i+j-k}{\sqrt{9+1+1}} = \frac{3i+j-k}{\sqrt{11}}$$

Let $g(x,y,z) = xyz^2+xz$,then

$$\frac{\partial g}{\partial x} = yz^2 + z,\ \frac{\partial g}{\partial y} = xz^2, \frac{\partial g}{\partial z} = 2xy + x$$

$\nabla g=(yz^2+z)i+xz^2j+(2xyz+x)k$

And $[\nabla g]_{(1,1,1)} = 2i+j+3k$

Directional derivative of the given function in the direction of $\bar{e}$ at (1,1,1) = $\nabla g.\bar{e}$

$$=(2i+j+3k).\left(\frac{3i+j-k}{\sqrt{11}}\right) = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

9: Find the directional derivative of $2xy+z^2$ at (1,-1,3) in the direction of $\bar{i} + 2\bar{j} + 3\bar{k}$.

Sol: Let $f = 2xy+z^2$then $\frac{\partial f}{\partial x} = 2y,\ \frac{\partial f}{\partial y} = 2x, \frac{\partial f}{\partial z} = 2z.$

grad f = $\sum \bar{i}\,\frac{\partial f}{\partial x} = 2y\bar{i} + 2x\bar{j} + 2z\bar{k}$ and (grad f)at (1,-1,3)= $-2\bar{i} + 2\bar{j} + 6\bar{k}$

given vector is $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k} \Rightarrow |\bar{a}| = \sqrt{1+4+9} = \sqrt{14}$

Directional derivative of f in the direction of $\bar{a}$ is

$$\frac{\bar{a}.\nabla f}{|\bar{a}|} = \frac{(\bar{i}+2\bar{j}+3\bar{k})(-2\bar{i}+2\bar{j}+6\bar{k}).}{\sqrt{14}} = \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

10: Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.

Sol:- Given $\phi = x^2yz+4xz^2$

$$\frac{\partial \phi}{\partial x} = 2xyz + 4z^2,\ \frac{\partial \phi}{\partial y} = x^2z, \frac{\partial \phi}{\partial z} = x^2y + 8xz.$$

Hence $\nabla\phi = \sum \bar{i}\,\frac{\partial\phi}{\partial x} = \bar{i}\,(2xyz + 4z^2) + \bar{j}x^2 z + \bar{k}\,(x^2 y + 8xz)$

$\nabla\phi$ at (1,-2,-1) = i(4+4)+j(-1)+k(-2-8)= 8i-j-10k.

The unit vector in the direction 2i-j-2k is

$$\bar{a} = \frac{2i - j - 2k.}{\sqrt{4+1+4}} = \frac{1}{3}(2i - j - 2k)$$

Required directional derivative along the given direction = $\nabla\phi.\ \bar{a}$

= (8i-j-10k). 1/3 (2i-j-2k)

= 1/3(16+1+20) = 37/3.

11: If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change.

Sol:- The greatest rate of increase of t at any point is given in magnitude and direction by ∇t.

We have $\nabla t = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right)(xy + yz + zx)$

$= \bar{i}\,(y+z) + \bar{j}(z+x) + \bar{k}\,(x+y) = 2\bar{i} + 2\bar{j} + 2\bar{k}$ at (1,1,1)

Magnitude of this vector is $\sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

Hence at the point (1,1,1) the temperature changes most rapidly in the direction given by the vector $2\bar{i} + 2\bar{j} + 2\bar{k}$ and greatest rate of increase = $2\sqrt{3}$.

12: Findthe directional derivative of ϕ(x,y,z) = $x^2yz+4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface *f*(x,y,z) = x log z-y^2 at (-1,2,1).

Sol:- Given ϕ(x,y,z) = $x^2yz+4xz^2$ at (1,-2,-1) and *f*(x,y,z) = x log z-y^2 at (-1,2,1)

Now $\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$= (2xyz + 4z^2)\bar{i} + (x^2 z)\bar{j} + (x^2 y + 8xz)\bar{k}$

$(\nabla\phi)_{(1,-2,-1)} = [2(1)(-2)(-1) + 4(-1)^2]\bar{i} + [(1)^2(-1)\bar{j}] + [(1^2)(-2) + 8(-1)]\bar{k} ----(1)$

$= 8\bar{i} - \bar{j} - 10\bar{k}$

Unit normal to the surface

f(x,y,z)= x log z- y^2 is $\frac{\nabla f}{|\nabla f|}$

Now $\nabla f = \bar{i}\,\frac{\partial f}{\partial x} + \bar{j}\,\frac{\partial f}{\partial y} + \bar{k}\,\frac{\partial f}{\partial z} = \log z\,\bar{i} + (-2y)\,\bar{j} + \frac{x}{z}\bar{k}$

At (-1,2,1), $\nabla f = \log(1)\,\bar{i} - 2(2)\,\bar{j} + \frac{-1}{1}\bar{k} = -4\bar{j} - \bar{k}$

$$\frac{\nabla f}{|\nabla f|} = \frac{-4\bar{j}-\bar{k}.}{\sqrt{16+1}} = \frac{-4\bar{j}-\bar{k}.}{\sqrt{17}}$$

Directional derivative = $\nabla\phi . \frac{\nabla f}{|\nabla f|}$

$$= (8\bar{i} - \bar{j} - 10\bar{k}).\frac{-4\bar{j}-\bar{k}.}{\sqrt{17}} = \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}.$$

13: Find a unit normal vector to the given surface $x^2y+2xz = 4$ at the point (2,-2,3).

Sol:- Let the given surface be $f = x^2y+2xz - 4$

On differentiating,

$$\frac{\partial f}{\partial x} = 2xy + 2z,\ \frac{\partial f}{\partial y} = x^2, \frac{\partial f}{\partial z} = 2x.$$

grad $f = \sum \bar{i}\frac{\partial f}{\partial x} = \bar{i}(2xy+2z) + \bar{j}x^2 + 2x\bar{k}$

(grad f) at (2,-2,3) $= \bar{i}(-8+6) + 4\bar{j} + 4\bar{k} = 2\bar{i} + 4\bar{j} + 4\bar{k}$

grad (f) is the normal vector to the given surface at the given point.

Hence the required unit normal vector $\frac{\nabla f}{|\nabla f|} = \frac{2(-\bar{i}+2\bar{j}+2\bar{k}).}{2\sqrt{1+2^2+2^2}} = \frac{-\bar{i}+2\bar{j}+2\bar{k}}{3}$

14: Evaluate the angle between the normal to the surface $xy= z^2$ at the points (4,1,2) and (3,3,-3).

Sol:- Given surface is $f(x,y,z) = xy- z^2$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normal to this surface at (4,1,2) and (3,3,-3) respectively.

Differentiating partially, we get

$$\frac{\partial f}{\partial x} = y,\ \frac{\partial f}{\partial y} = x, \frac{\partial f}{\partial z} = -2z.$$

grad f = $y\bar{i} + x\bar{j} - 2z\bar{k}$

$\bar{n}_1$ = (grad f) at (4,1,2) $= \bar{i} + 4\bar{j} - 4\bar{k}$

$\bar{n}_2$ = (grad f) at (3,3,-3) = $3\bar{i} + 3\bar{j} + 6\bar{k}$

Let θ be the angle between the two normal.

$$\cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|} = \frac{(i+4j-4k)}{\sqrt{1+16+16}}.\frac{(3i+3j+6k)}{\sqrt{9+9+36}}$$

$$\frac{(3+12-24)}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

15: Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3).

Sol:- Let the given surface be $f(x,y,z) \equiv x^2+y^2+2z^2 - 26=0$. Then

$$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 4z.$$

grad f = $\sum \bar{i}\,\frac{\partial f}{\partial x} = 2xi+2yj+4zk$

Normal vector at(2,2,3) = $[\nabla f\,]_{(2,2,3)} = 4\bar{i} +4\,\bar{j} +12\,\bar{k}$

Unit normal vector = $\dfrac{\nabla f}{|\nabla f|} = \dfrac{4(\bar{i}+\bar{j}+3\bar{k})}{4\sqrt{11}} = \dfrac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}$

16: Find the values of a and b so that the surfaces $ax^2-byz = (a+2)x$ and $4x^2y+z^3= 4$ may intersect orthogonally at the point (1, -1,2).

(or) Find the constants a and b so that surface $ax^2-byz=(a+2)x$ will orthogonal to $4x^2y+z^3=4$ at the point (1,-1,2).

Sol:- Let the given surfaces be $f(x,y,z) = ax^2-byz - (a+2)x$-------------(1)

And $g(x,y,z) = 4x^2y+z^3- 4$------------(2)

Given the two surfaces meet at the point (1,-1,2).

Substituting the point in (1), we get

$a+2b-(a+2) = 0 \Rightarrow b=1$

Now $\dfrac{\partial f}{\partial x} = 2ax-(a+2),\ \dfrac{\partial f}{\partial y} = -bz\ and\ \dfrac{\partial f}{\partial z} = -by.$

$\nabla f = \sum \bar{i}\,\dfrac{\partial f}{\partial x} = [(2ax-(a+2)]i-bz+bk = (a-2)i-2bj+bk$

$= (a-2)i-2j+k = \bar{n}_1$, normal vector to surface 1.

Also $\dfrac{\partial g}{\partial x} = 8xy,\ \dfrac{\partial g}{\partial y} = 4x^2, \dfrac{\partial g}{\partial z} = 3z^2.$

$\nabla g = \sum \bar{i}\,\dfrac{\partial g}{\partial x} = 8xyi+4x^2j+3z^2k$

$(\nabla g)_{(1,-1,2)} = -8i+4j+12k = \bar{n}_2$, normal vector to surface 2.

Given the surfaces f(x,y,z), g(x,y,z) are orthogonal at the point (1,-1,2).

$[\nabla f].[\nabla g] = 0 \Rightarrow ((a-2)i-2j+k). (-8i+4j+12k)=0$

$\Rightarrow -8a+16-8+12 \Rightarrow a =5/2$

Hence a = 5/2 and b=1.

17: Find a unit normal vector to the surface z= x^2+y^2 at (-1,-2,5)

Sol:- Let the given surface be f = x^2+y^2-z

$$\frac{\partial f}{\partial x}=2x,\ \frac{\partial f}{\partial y}=2y, \frac{\partial f}{\partial z}=-1.$$

grad f = $\nabla f = \sum \bar{i}\,\frac{\partial f}{\partial x}$ = 2xi+2yj-k

(∇f) at (-1,-2,5)= -2i-4j-k

∇f is the normal vector to the given surface.

Hence the required unit normal vector = $\dfrac{\nabla f}{|\nabla f|}$ =

$$\frac{-2i-4j-k}{\sqrt{(-2)^2+(-4)^2+(-1)^2}}=\frac{-2i-4j-k}{\sqrt{21}} = -\frac{1}{\sqrt{21}}(2i+4j+k)$$

18: Find the angle of intersection of the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point (4,-3,2).

Sol:- Let f = $x^2+y^2+z^2-29$ and g = $x^2+y^2+z^2+4x-6y-8z-47$

Then grad f= $\bar{i}\,\frac{\partial f}{\partial x}+\bar{j}\,\frac{\partial f}{\partial y}+\bar{k}\,\frac{\partial f}{\partial z} = 2x\bar{i}+2y\bar{j}+2z\bar{k}$ and

grad g = $(2x+4)\bar{i}+(2y-6)\bar{j}+(2z-8)\bar{k}$

The angle between two surfaces at a point is the angle between the normal to the surfaces at that point.

Let $\bar{n}_1$= (grad *f*) at (4,-3,2) =$8\bar{i}-6\bar{j}+4\bar{k}$

$\bar{n}_2$= (grad *f*) at (4,-3,2) = $12\bar{i}-12\bar{j}-4\bar{k}$

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normal to the two surfaces at (4,-3,2). Let θ be the angle between the surfaces. Then

$$\text{Cos }\theta=\frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|}=.\frac{152}{\sqrt{116}\sqrt{304}}$$

$$\therefore \theta=\cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

19: Find the angle between the surfaces $x^2+y^2+z^2=9$, and z = x^2+y^2-3 at point (2,-1,2).

Sol:- Let $\phi_1 = x^2+y^2+z^2-9=0$ and $\phi_2= x^2+y^2-z-3=0$ be the given surfaces. Then

$\nabla\phi_1$= 2xi+2yj+2zk and $\nabla\phi_2$ = 2xi+2yj-k

Let $\bar{n}_1 = \nabla\phi_1$ at(2,-1,2)= 4i-2j+4k and

$\bar{n}_2 = \nabla\phi_2$ at (2,-1,2) = 4i-2j-k

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces at the point (2,-1,2). Let θ be the angle between the surfaces. Then

$$\text{Cos}\ \theta = \frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|} = \frac{(4i-2j+4k)}{\sqrt{16+4+16}}\cdot\frac{(4i-2j-k)}{\sqrt{16+4+16}} = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

20: If $\bar{a}$ is constant vector then prove that grad ($\bar{a}\,.\bar{r}$)= $\bar{a}$

Sol: Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, where a_1,a_2,a_3 are constants.

$\bar{a}\,.\bar{r} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}).(x\bar{i} + y\bar{j} + z\bar{k}) = a_1x + a_2y + a_3z$

$$\frac{\partial}{\partial x}(\bar{a}.\bar{r}) = a_1,\ \frac{\partial}{\partial y}(\bar{a}.\bar{r}) = a_2,\ \frac{\partial}{\partial z}(\bar{a}.\bar{r}) = a_3$$

grad ($\bar{a}\,.\bar{r}$)= $a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

21: If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find ϕ.

Sol:- We know that $\nabla\phi = \bar{i}\,\frac{\partial f}{\partial x} + \bar{j}\,\frac{\partial f}{\partial y} + \bar{k}\,\frac{\partial f}{\partial z}$

Given that $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$

Comparing the corresponding coefficients, we have $\frac{\partial\phi}{\partial x} = yz,\ \frac{\partial\phi}{\partial y} = zx,\ \frac{\partial\phi}{\partial z} = xy$

Integrating partially w.r.t. x,y,z, respectively, we get

ϕ= xyz + a constant independent of x.

ϕ= xyz + a constant independent of y.

ϕ= xyz + a constant independent of z.

Here a possible form of ϕ is ϕ= xyz+a constant.

DIVERGENCE OF A VECTOR

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

$$\text{i.e., div}\ \bar{f} = \bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z} = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right).\bar{f}$$

Hence we can write div $\bar{f}$ as

div $\bar{f} = \nabla . \bar{f}$

This is a scalar point function.

Theorem 1: If the vector $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$, then div $\bar{f} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

Prof: Given $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$

$$\frac{\partial \bar{f}}{\partial x} = \bar{i}\,\frac{\partial f_1}{\partial x} + \bar{j}\,\frac{\partial f_2}{\partial x} + \bar{k}\,\frac{\partial f_3}{\partial x}$$

Also $\bar{i}.\dfrac{\partial \bar{f}}{\partial x} = \dfrac{\partial f_1}{\partial x}$. Similarly $\bar{j}.\dfrac{\partial \bar{f}}{\partial y} = \dfrac{\partial f_2}{\partial y}$ and $\bar{k}.\dfrac{\partial \bar{f}}{\partial z} = \dfrac{\partial f_3}{\partial z}$

We have div $\bar{f} = \sum \bar{i}.\left(\dfrac{\partial \bar{f}}{\partial x}\right) = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

Note : If $\bar{f}$ is a constant vector then $\dfrac{\partial f_1}{\partial x}, \dfrac{\partial f_2}{\partial y}, \dfrac{\partial f_3}{\partial z}$ are zeros.

$\therefore$ div $\bar{f}$ =0 for a constant vector $\bar{f}$.

Theorem 2: div ($\bar{f} \pm \bar{g}$) = $div\,\bar{f} \pm div\,\bar{g}$

Proof: div ($\bar{f} \pm \bar{g}$)= $\sum \bar{i}.\dfrac{\partial}{\partial x}\left(\bar{f} \pm \bar{g}\right) = \sum \bar{i}.\dfrac{\partial}{\partial x}\left(\bar{f}\right) \pm \sum \bar{i}.\dfrac{\partial}{\partial x}(\bar{g}) = div\,\bar{f} \pm div\,\bar{g}$.

Note: If ϕ is a scalar function and $\bar{f}$ is a vector function, then

(i). $(\bar{a}.\nabla)\phi = \left[\bar{a}.\left(\bar{i}\,\dfrac{\partial}{\partial x} + \bar{j}\,\dfrac{\partial}{\partial y} + \bar{k}\,\dfrac{\partial}{\partial z}\right)\right]\phi$

$$= \left[(\bar{a}.\bar{i})\frac{\partial}{\partial x} + (\bar{a}.\bar{j})\frac{\partial}{\partial y} + (\bar{a}.\bar{k})\frac{\partial}{\partial z}\right]\phi$$

$$= \left[(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \phi}{\partial z}\right]$$

$= \sum (\bar{a}.\bar{i})\dfrac{\partial \phi}{\partial x}$. and

(ii). $(\bar{a}.\nabla)\,\bar{f} = \sum (\bar{a}.\bar{i})\dfrac{\partial \bar{f}}{\partial x}$. by proceeding as in (i) [simply replace ϕ by $\bar{f}$ in (i)].

SOLENOIDAL VECTOR

A vector point function $\bar{f}$ is said to be solenoidal if div $\bar{f}$ =0.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem, we can interpret div $\bar{f}$ (= $\nabla . \bar{f}$).

Suppose $\overline{F}$ (x,y,z,t) is the velocity of a fluid at a point(x,y,z) and time ‘t’. Though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagine a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of $\bar{F}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

Similar meanings are to be understood with respect to divergence of vectors $\bar{f}$ from other branches. A detailed elementary interpretation can be seen in standard books on fluid dynamics, electricity and magnetism etc.

SOLVED PROBLEMS

1: If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

Sol:- Given $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$.

Thendiv $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2) =$ y^2+2x^2z-6yz

(div $\bar{f}$) at (1, -1, 1) = 1+2+6 =9

2: Find div $\bar{f}$ when grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$.

Then $\frac{\partial \phi}{\partial x} = 3x^2 - 3yz, \frac{\partial \phi}{\partial y} = 3y^2 - 3zx, \frac{\partial \phi}{\partial z} = 3z^2 - 3xy$

grad $\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)] + \frac{\partial}{\partial y}[3(y^2 - zx)] + \frac{\partial}{\partial z}[3(z^2 - xy)]$

$= 3(2x)+3(2y)+3(2z) = 6(x+y+z)$

3: If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal, find P.

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1, \frac{\partial f_2}{\partial y} = 1, \frac{\partial f_3}{\partial z} = p$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ = 1+1+p =2+p

since $\bar{f}$ is solenoidal, we have div $\bar{f}$ = 0 $\Rightarrow 2 + p = 0 \Rightarrow p = -2$

4: Find div $\bar{f} = r^n\bar{r}$. Find n if it is solenoidal?

Sol: Given $\bar{f} = r^n\bar{r}$. where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ *and* $r = |\bar{r}|$

We have $r^2 = x^2+y^2+z^2$
Differentiating partially w.r.t. x , we get

$$2r\,\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}\, and\, \frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f} = r^n\,(x\bar{i} + y\bar{j} + z\bar{k})$

$$\text{div } \bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= nr^{n-1}\frac{\partial r}{\partial x}x + r^n + nr^{n-1}\frac{\partial r}{\partial y}y + r^n + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$$

$$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$$

Let $\bar{f} = r^n\bar{r}$ be solenoidal. Then div $\bar{f} = 0$

$(n+3)r^n = 0 \Rightarrow n = -3$

5: Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk\ and\ r = |\bar{r}|$.

Sol:- We have

$\bar{r}$ = xi+yj+zk and r = $\sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, and\, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$\therefore \frac{\bar{r}}{r^3} = \bar{r}.\ r^{-3} = r^{-3}xi + r^{-3}yj + r^{-3}zk = f_1 i + f_2 j + f_3 k$

Hence $\nabla.\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

We have $f_1 = r^{-3}x \Rightarrow \frac{\partial f_1}{\partial x} = r^{-3}.1 + x(-3)r^{-4}.\frac{\partial r}{\partial x}$

$$\therefore \frac{\partial f_1}{\partial x} = r^{-3} - 3xr^{-4}\frac{x}{y} = r^{-3} - 3x^2 r^{-5}$$

$$\nabla.\left(\frac{\bar{r}}{r^3}\right) = \sum \frac{\partial f_1}{\partial x} = 3r^{-3} - 3r^{-5}\sum x^2$$

$= 3r^{-3} - 3r^{-5}r^2 = 3r^{-3} - 3r^{-3} = 0$

6: Find div $\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$

$$\text{div } \bar{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1=3$$

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x $\bar{f}$).

$$\text{Curl } \bar{f} = \bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z} = \sum\left(\bar{i}\times\frac{\partial \bar{f}}{\partial x}\right)$$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$ then

$$\text{curl } \bar{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

Proof : $\text{curl } \bar{f} = \sum \bar{i}\times\frac{\partial}{\partial x}(\bar{f}) = \sum \bar{i}\times\frac{\partial}{\partial x}(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}) = \sum\left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j}\right)$

$$= \left(\frac{\partial f_2}{\partial x}\bar{k} - \frac{\partial f_3}{\partial x}\bar{j}\right) + \left(\frac{\partial f_3}{\partial y}\bar{i} - \frac{\partial f_1}{\partial y}\bar{k}\right) + \left(\frac{\partial f_1}{\partial z}\bar{j} - \frac{\partial f_2}{\partial z}\bar{i}\right)$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \bar{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Note : (1) The above expression for curl $\bar{f}$ can be remembered easily through the representation.

$$\textbf{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \mathbf{x}\, \bar{f}$$

Note (2) : If $\bar{f}$ is a constant vector then curl $\bar{f} = \bar{o}$.

Theorem 2: curl $\left(\bar{a} \pm \bar{b}\right) = curl\,\bar{a} \pm curl\,\bar{b}$

Proof: $\text{curl}\left(\bar{a} \pm \bar{b}\right) = \sum \bar{i}\times\frac{\partial}{\partial x}\left(\bar{a} \pm \bar{b}\right)$

$$= \sum i\times\left(\frac{\partial \bar{a}}{\partial x} \pm \frac{\partial \bar{b}}{\partial x}\right) = \sum i\times\frac{\partial \bar{a}}{\partial x} \pm \sum ix\frac{\partial \bar{b}}{\partial x}$$

$$= curl\,\bar{a} \pm curl\,\bar{b}$$

1. Physical Interpretation of curl

If $\overline{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point P(x,y,z) on the body, then $\overline{w}$ = ½ curl $\bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word "curl of a vector".

2. Irrotational Motion, Irrotational Vector

Any motion in which curl of the velocity vector is a null vector i.e curl $\bar{v} = \bar{0}$ is said to be Irrotational.

Def: A vector $\bar{f}$ is said to be Irrotational if curl $\bar{f} = \bar{0}$.

If $\bar{f}$ is Irrotational, there will always exist a scalar function φ(x,y,z) such that $\bar{f}$ =grad ϕ. Thisϕ is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f}$ = grad ϕ, then curl $\bar{f}$ = 0.

Hence ∇x $\bar{f}$ = 0 ⇔ there exists a scalar function ϕ such that $\bar{f} = \nabla\phi$.

This idea is useful when we study the "work done by a force" later.

SOLVED PROBLEMS

1: If $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$. Then

$$\text{curl } \bar{f} = \nabla \text{x } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right) + \bar{j}\left(\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right) + \bar{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)$$

$$= \bar{i}\left(-3z^2 - 2x^2z\right) + \bar{j}(0-0) + \bar{k}(4xyz - 2xy) = -\left(3z^2 + 2x^2y\right)\bar{i} + (4xyz - 2xy)\bar{k}$$

$= \text{curl } \bar{f}$ at (1,-1,1) = $-\bar{i} - 2\bar{k}$.

2: Find curl $\bar{f}$ where $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let ϕ= $x^3+y^3+z^3-3xyz$. Then

$$\text{grad } \phi = \sum \bar{i}\,\frac{\partial\phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\,\bar{j} + 3(z^2 - xy)\bar{k}$$

$$\text{curl grad } \phi = \nabla \text{x grad } \phi = 3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$=3[\bar{i}(-x+x)-\bar{j}(-y+y)+\bar{k}(-z+z)]=\bar{0}$$

$\therefore$ curl $\bar{f}=\bar{0}$.

Note: We can prove in general that curl (grad ϕ)=$\bar{0}$.(i.e) grad ϕ is always irrotational.

3: Prove that if $\bar{r}$ is the position vector of an point in space, then $r^n\bar{r}$ is Irrotational. (or) Show that

curl$(r^n\bar{r}) = 0$

Sol:- Let $\bar{r}= x\bar{i}+y\bar{j}+z\bar{k}$ and r = $|\bar{r}|$ $\quad\therefore r^2= x^2+y^2+z^2$.

Differentiating partially w.r.t. 'x', we get

$$2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}, and \frac{\partial r}{\partial z}=\frac{z}{r}$

We have $r^n\bar{r}=r^n(x\bar{i}+y\bar{j}+z\bar{k})$

$$\nabla\times(r^n\bar{r})=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xr^n & yr^n & zr^n\end{vmatrix}$$

$$=\bar{i}\left(\frac{\partial}{\partial y}(r^nz)-\frac{\partial}{\partial z}(r^ny)\right)+\bar{j}\left(\frac{\partial}{\partial z}(r^nx)-\frac{\partial}{\partial x}(r^nz)\right)+\bar{k}\left(\frac{\partial}{\partial x}(r^ny)-\frac{\partial}{\partial y}(r^nx)\right)$$

$$=\sum\bar{i}\left\{znr^{n-1}\frac{\partial r}{\partial y}-ynr^{n-1}\frac{\partial r}{\partial z}\right\}=nr^{n-1}\sum\bar{i}\left\{z\left(\frac{y}{r}\right)-y\left(\frac{z}{r}\right)\right\}$$

$$=nr^{n-2}[(zy-yz)\bar{i}+(xz-zx)\bar{j}+(xy-yz)\bar{k}]$$

$$=nr^{n-2}[0\bar{i}+0j+0\bar{k}]=nr^{n-2}[\bar{0}]=\bar{0}$$

Hence $r^n\bar{r}$ is Irrotational.

4: Prove that curl $\bar{r}=\bar{0}$

Sol:- Let $\bar{r}= x\bar{i}+y\bar{j}+z\bar{k}$

$$\text{curl }\bar{r}=\sum\bar{i}\times\frac{\partial}{\partial x}(\bar{r})=\sum(\bar{i}x\bar{i})=\bar{0}+\bar{0}+\bar{0}=\bar{0}$$

$\therefore\ \bar{r}$ is Irrotational vector.

5: If $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a}x\bar{r}}{r^3}\right)=-\frac{\bar{a}}{r^3}+\frac{3\bar{r}}{r^5}(\bar{a}.\bar{r})$.

Sol:- We have $\bar{r}= x\bar{i}+y\bar{j}+z\bar{k}$

$$\frac{\partial \bar{r}}{\partial x} = \bar{i}\,, \frac{\partial \bar{r}}{\partial y} = \bar{j}\,, \frac{\partial \bar{r}}{\partial z} = \bar{k}$$

If $|\bar{r}|$ = r then $r^2 = x^2+y^2+z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, and \frac{\partial r}{\partial z} = \frac{z}{r}$$

curl $\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum \bar{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)$

Now $\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \bar{a}\times\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^3}\right) = \bar{a}\times\left[\frac{1}{r^3}\frac{\partial \bar{r}}{\partial x} - \frac{3}{r^4}\frac{\partial r}{\partial x}\bar{r}\right]$

$$= \bar{a}\times\left[\frac{1}{r^3}\bar{i} - \frac{3}{r^5}x\bar{r}\right] = \frac{\bar{a}\times\bar{i}}{r^3} - \frac{3x(\bar{a}.\times\bar{r})}{r^5}.$$

$$\therefore\ i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \bar{i}\times\left[\frac{\bar{a}\times\bar{i}}{r^3} - \frac{3x}{r^5}(\bar{a}\times\bar{r})\right] = \frac{\bar{i}\times(\bar{a}\times\bar{i})}{r^3} - \frac{3x}{r^5}\bar{i}\times(\bar{a}\times\bar{r})$$

$$= \frac{(\bar{i}.\bar{i})\bar{a} - (\bar{i}.\bar{a})i}{r^3} - \frac{3x}{r^5}[(\bar{i}.\bar{r})\bar{a} - (i.a)\bar{r}]$$

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then $\bar{i}\,.\,\bar{a} = a_1$, etc.

$$\therefore\ i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum\frac{(\bar{a} - a_1 i)}{r^3} - \frac{3x}{r^3}(x\bar{a} - a_1\bar{r})$$

$$\therefore\ \sum i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum\frac{\bar{a} - a_1\bar{i}}{r^3} - \frac{3}{r^5}\sum(x^2\bar{a} - a_1 x\bar{r})$$

$$= \frac{3\bar{a} - \bar{a}}{r^3} - \frac{3\bar{a}}{r^5}(r^2) + \frac{3\bar{r}}{r^5}(a_1 x + a_2 y + a_3 z)$$

$$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a}) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a})$$

6: Show that the vector $(x^2 - yz)\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$

Then curl $\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = \sum \bar{i}(-x + x) = \bar{0}$

$\therefore\ \bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f} = \nabla\phi$.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (x^2 - yz)\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int(x^2 - yz)dx = \frac{x^3}{3} - xyz + f_1(y,z)......(1)$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(z,x)......(2)$$
$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x,y)......(3)$$

From (1), (2),(3), $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz$

$\therefore \quad \phi = \dfrac{1}{3}(x^3 + y^3 + z^3) - xyz + constant$

Which is the required scalar potential.

7: Find constants a,b and c if the vector $\bar{f}$ = $(2x+3y+az)\bar{i} + (bx+2y+3z)\,\bar{j} + (2x+cy+3z)\,\bar{k}$ is Irrotational.

Sol:- Given $\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\,\bar{j} + (2x+cy+3z)\,\bar{k}$

$$\text{Curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} =$$

$(c-3)\bar{i} - (2-a)\,\bar{j} + (b-3)\,\bar{k}$

If the vector is Irrotational then curl $\bar{f} = \bar{0}$

$\therefore 2 - a = 0 \Rightarrow a = 2, b - 3 = 0 \Rightarrow b = 3, c - 3 = 0 \Rightarrow c = 3$

8: If f(r) is differentiable, show that curl { $\bar{r}$ f(r)} = $\bar{0}$ where $\bar{r}$ = $x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: r = $\bar{r} = \sqrt{x^2 + y^2 + z^2}$

$\therefore r^2 = x^2+y^2+z^2$

$\Rightarrow 2r\,\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}, and\ \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

curl{ $\bar{r}$ f(r)}= curl{f(r)($x\bar{i} + y\bar{j} + z\bar{k}$)}= curl $(x.f(r)\bar{i} + y.f(r)\,\bar{j} + z.f(r)\,\bar{k})$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \sum \bar{i}\left[\frac{\partial}{\partial y}[zf(r)] - \frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum \bar{i}\left[zf^1(r)\frac{\partial r}{\partial y} - yf^1(r)\frac{\partial r}{\partial z}\right] = \sum \bar{i}\left[zf^1(r)\frac{y}{r} - yf^1(r)\frac{z}{r}\right]$$

$= \bar{0}$.

9: If $\bar{A}$ is irrotational vector, evaluate div($\bar{A}$ x $\bar{r}$) where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Given $\bar{A}$ is an irrotational vector

∇x $\bar{A} = \bar{0}$

div ($\bar{A}$ x $\bar{r}$) = ∇.($\bar{A}$ x $\bar{r}$)

= $\bar{r}$.(∇x $\bar{A}$)- $\bar{A}$.(∇x $\bar{r}$)

= $\bar{r}$.($\bar{0}$)- $\bar{A}$.(∇x $\bar{r}$)   [using (1)]

= - $\bar{A}$.(∇x $\bar{r}$)…..(2)

Now ∇x $\bar{r}$ = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$ =

$$\bar{i}\left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right) - \bar{j}\left(\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right) + \bar{k}\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) = \bar{0}$$

$\therefore$ $\bar{A}$.(∇x $\bar{r}$)=0 …(3)

Hence div ($\bar{A}$ x $\bar{r}$)=0. [using (2) and (3)]

10: Find constants a,b,c so that the vector $\bar{A}$ = $(x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\bar{k}$ is Irrotational. Also find ϕ such that $\bar{A}$ = $\nabla\phi$.

Sol: Given vector is $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\bar{k}$

Vector $\bar{A}$ is Irrotational $\Rightarrow$ curl $\bar{A} = \bar{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\bar{k} = \bar{0}$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\bar{k} = 0\bar{i} + 0\,\bar{j} + 0\,\bar{k}$

Comparing both sides,

c+1=0, a-4=0, b-2=0

c= -1, a=4,b=2

Now $\bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\,\bar{j} + (4x-y+2z)\bar{k}$, on substituting the values of a,b,c we have $\bar{A} = \nabla\phi$.

$$\Rightarrow \bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\,\bar{j} + (4x-y+2z)\,\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

Comparing both sides, we have

$\frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \phi = x^2/2+2xy+4zx+f_1(y,z)$

$\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \phi = 2xy-3y^2/2-yz+f_2(z,x)$

$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = 4xz-yz+z^2+f_3(x,y)$

Hence $\phi = x^2/2 - 3y^2/2+z^2+2xy+4zx-yz+c$

11: If ω is a constant vector, evaluate curl V where V = ωx$\bar{r}$.

Sol: curl (ωx$\bar{r}$) = $\sum \bar{i}\times\frac{\partial}{\partial x}(\varpi\times\bar{r}) = \sum \bar{i}\times\left[\frac{\partial\varpi}{\partial x}\times\bar{r} + \varpi\times\frac{\partial\bar{r}}{\partial x}\right]$

$$= \sum \bar{i}\times[\bar{0}+\omega\times\bar{i}] \quad [\because \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b}).\bar{c}]$$
$$= \sum \bar{i}\times(\omega\times\bar{i}) = \sum[(\bar{i}.\bar{i})\omega - (\bar{i}.\omega)\bar{i}] = \sum\omega - \sum(\bar{i}.\omega)\bar{i} = 3\omega-\omega = 2\omega$$

Assignments

1. If $\bar{f}$ = $e^{x+y+z}(\bar{i}+\bar{j}+\bar{k})$ find curl $\bar{f}$.

2. Prove that $\bar{f}$ = $(y+z)\bar{i} + (z+x)\,\bar{j} + (x+y)\,\bar{k}$ is irrotational.

3. Prove that $\nabla.(\bar{a}\times\bar{f}) = -\bar{a}$. curl $\bar{f}$ where $\bar{a}$ is a constant vector.

4. Prove that curl $(\bar{a}\times\bar{r})=2\bar{a}$ where $\bar{a}$ is a constant vector.

5. If $\bar{f} = x^2y\bar{i} - 2zx\,\bar{j} + 2yz\,\bar{k}$ find (i) curl $\bar{f}$ (ii) curl curl $\bar{f}$.

OPERATORS

Vector differential operator ∇

The operator ∇ = $\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ is defined such that ∇φ= $\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$ where φ is a scalar point function.

Note: If φ is a scalar point function then ∇φ= grad φ= $\sum i\frac{\partial\phi}{\partial x}$

(2) Scalar differential operator $\bar{a}$.∇

The operator $\bar{a}.\nabla = (\bar{a}.\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$ is defined such that

$(\bar{a}.\nabla)\phi = (\bar{a}.\bar{i})\frac{\partial \phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \phi}{\partial z}$

And $(\bar{a}.\nabla)\bar{f} = (\bar{a}.\bar{i})\frac{\partial \bar{f}}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \bar{f}}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \bar{f}}{\partial z}$

(3). Vector differential operator $\bar{a}\,x\nabla$

The operator $\bar{a}\,x\nabla = (\bar{a}\times\bar{i})\frac{\partial}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial}{\partial z}$ is defined such that

(i). $(\bar{a}\,x\nabla)\phi = (\bar{a}\times\bar{i})\frac{\partial \phi}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial \phi}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial \phi}{\partial z}$

(ii). $(\bar{a}\,x\nabla).\bar{f} = (\bar{a}\times\bar{i}).\frac{\partial \bar{f}}{\partial x} + (\bar{a}\times\bar{j}).\frac{\partial \bar{f}}{\partial y} + (\bar{a}\times\bar{k}).\frac{\partial \bar{f}}{\partial z}$

(iii). $(\bar{a}\,x\nabla)x\,\bar{f} = (\bar{a}\times\bar{i})\times\frac{\partial \bar{f}}{\partial x} + (\bar{a}\times\bar{j})\times\frac{\partial \bar{f}}{\partial y} + (\bar{a}\times\bar{k})\times\frac{\partial \bar{f}}{\partial z}$

(4). Scalar differential operator ∇.

The operator $\nabla = \bar{i}.\frac{\partial}{\partial x} + \bar{j}.\frac{\partial}{\partial y} + \bar{k}.\frac{\partial}{\partial z}$ is defined such that $\nabla.\bar{f} = \bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z}$

Note: $\nabla.\bar{f}$ is defined as div $\bar{f}$. It is a scalar point function.

(5). Vector differential operator ∇ x

The operator $\nabla\,x = \bar{i}\times\frac{\partial}{\partial x} + \bar{j}\times\frac{\partial}{\partial y} + \bar{k}\times\frac{\partial}{\partial z}$ is defined such that

$$\nabla x\,\bar{f} = \bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z}$$

Note : $\nabla x\,\bar{f}$ is defined as curl $\bar{f}$. It is a vector point function.

(6). Laplacian Operator ∇^2

$$\nabla.\nabla\phi = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}\right) = \sum \frac{\partial^2 \phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Thus the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). If $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

SOLVED PROBLEMS

1: Prove that div.(grad r^m)= $m(m+1)r^{m-2}$ (or) $\nabla^2(r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ then $r^2 = x^2+y^2+z^2$.

Differentiating w.r.t. 'x' partially, wet get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now grad(r^m) = $\sum \bar{i}\frac{\partial}{\partial x}(r^m) = \sum \bar{i} m r^{m-1}\frac{\partial r}{\partial x} = \sum \bar{i} m r^{m-1}\frac{x}{r} = \sum \bar{i} m r^{m-2} x$

$\therefore$ div (grad r^m) = $\sum \frac{\partial}{\partial x}[m r^{m-2} x]$=m$\sum\left[(m-2)r^{m-3}\frac{\partial r}{\partial x}x + r^{m-2}\right]$

$=m\sum\left[(m-2)r^{m-4}x^2 + r^{m-2}\right] = m\left[(m-2)r^{m-4}\sum x^2 + \sum r^{m-2}\right]$

$= m[(m-2)r^{m-4}(r^2)+3r^{m-2}]$

$= m[(m-2)\ r^{m-2}+3r^{m-2}] = m[(m-2+3)r^{m-2}] = m(m+1)r^{m-2}$.

Hence $\nabla^2(r^m) = m(m+1)r^{m-2}$

2: Show that $\nabla^2[f(r)] = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f^{11}(r) + \frac{2}{r} f^1(r)$ where r = $|\bar{r}|$.

Sol: grad [f(r)] = $\nabla f(r) = \sum i\frac{\partial}{\partial x}[f(r)] = \sum i\, f^1(r)\frac{\partial r}{\partial x} = \sum i\, f^1(r)\frac{x}{r}$

$\therefore$ div [grad f(r)] = $\nabla^2[f(r)] = \nabla.\nabla f(r) = \sum \frac{\partial}{\partial x}\left[f^1(r)\frac{x}{r}\right]$

$$= \sum \frac{r\frac{\partial}{\partial x}[f^1(r)x] - f^1(r)x\frac{\partial}{\partial x}(r)}{r^2}$$

$$= \sum \frac{r\left(f^{11}(r)\frac{\partial r}{\partial x}x + f^1(r)\right) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \sum \frac{r f^{11}(r)\frac{x}{r}x + r f^1(r) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \frac{\sum r f^{11}(r)\frac{x}{r}x + r f^1(r) - x^2}{r^2}.\frac{f^1(r)}{r}$$

$$= \frac{f^{11}(r)}{r^2}\sum x^2 + \frac{1}{r}\sum f^1(r) - \frac{1}{r^3}f^1(r)\sum x^2$$

$$= \frac{f^{11}(r)}{r^2}(r^2) + \frac{3}{r}f^1(r) - \frac{1}{r^3}f^1(r)r^2$$

$$= f^{11}(r) + \frac{2}{r}f^1(r)$$

3: If ϕ satisfies Laplacian equation, show that $\nabla\phi$ is both solenoidal and irrotational.

Sol: Given $\nabla^2\phi = 0 \Rightarrow$div(grad ϕ)= 0 $\Rightarrow$ grad ϕ is solenoidal

We know that curl (grad ϕ) = $\bar{0} \Rightarrow$grad ϕ is always irrotational.

4:Show that (i) $(\bar{a}.\nabla)\phi = \bar{a}.\nabla\phi$ (ii) $(\bar{a}.\nabla)\bar{r} = \bar{a}$.

Sol: (i). Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then

$$\bar{a}.\nabla = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}).(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}) = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$\therefore (\bar{a}.\nabla)\phi = a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}$$

Hence $(\bar{a}.\nabla)\phi = \bar{a}.\nabla\phi$

(ii). $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \frac{\partial\bar{r}}{\partial x} = \bar{i}, \ \frac{\partial\bar{r}}{\partial y} = \bar{j}, \ \frac{\partial\bar{r}}{\partial z} = \bar{k}$$

$$(\bar{a}.\nabla)\bar{r} = \sum a_1\frac{\partial}{\partial x}(\bar{r}) = \sum a_1\frac{\partial\bar{r}}{\partial x} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

5: Prove that (i) $(\bar{f}\,\mathrm{x}\nabla).\bar{r} = 0$ (ii). $(\bar{f}\,\mathrm{x}\nabla)\mathrm{x}\bar{r} = -2\bar{f}$

Sol: (i) $(\bar{f}\,\mathrm{x}\nabla).\bar{r} = \sum(\bar{f}\times\bar{i}).\frac{\partial\bar{r}}{\partial x} = \sum(\bar{f}\times\bar{i}).\bar{i} = 0$

(ii) $(\bar{f}\,\mathrm{x}\nabla) = (\bar{f}\times\bar{i})\frac{\partial}{\partial x}\times(\bar{f}\times\bar{j})\frac{\partial}{\partial y}\times(\bar{f}\times\bar{k})\frac{\partial}{\partial z}$

$$(\bar{f}\,\mathrm{x}\nabla)\mathrm{x}\bar{r} = (\bar{f}\times\bar{i})\times\frac{\partial\bar{r}}{\partial x} + (\bar{f}\times\bar{j})\times\frac{\partial\bar{r}}{\partial y} + (\bar{f}\times\bar{k})\times\frac{\partial\bar{r}}{\partial z} = \sum(\bar{f}\times\bar{i})\times\bar{i} = \sum\left[(\bar{f}.i)i - \bar{f}\right]$$

$$= (\bar{f}.i)\bar{i} + (\bar{f}.\bar{j})\bar{j} + (\bar{f}.\bar{k})\bar{k} - 3\bar{f} = \overline{f} - 3\bar{f} = -2\overline{f}$$

6: Find div $\overline{F}$, where $\overline{F}$ = grad $(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi = x^3+y^3+z^3-3xyz$. Then

$\overline{F}$ = grad ϕ

$$= \sum i\frac{\partial\phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\bar{j} + 3(x^2 - xy)\bar{k} = F_1 i + F_2 j + F_3 k \ (say)$$

$$\therefore \text{div } \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 6x+6y+6z = 6(x+y+z)$$

i.e div[grad$(x^3+y^3+z^3-3xyz)$] $= \nabla^2(x^3+y^3+z^3-3xyz) = 6(x+y+z)$.

7: If $f = (x^2+y^2+z^2)^{-n}$ then find div grad f and determine n if div grad $f = 0$.

Sol: Let $f = (x^2+y^2+z^2)^{-n}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| \Rightarrow r^2 = x^2+y^2+z^2$

$\Rightarrow f(r) = (r^2)^{-n} = r^{-2n}$

$\therefore f^1(r) = -2n\, r^{-2n-1}$

and $\quad f^{11}(r) = (-2n)(-2n-1)r^{-2n-2} = 2n(2n+1)r^{-2n-2}$

We have div grad f = $\nabla^2 f(r) = f^{11}(r) + {}^2/_r f^1(r) = (2n)(2n+1)r^{-2n-2} - 4n\, r^{-2n-2}$

$= r^{-2n-2}[2n(2n+1-2)] = (2n)(2n-1)r^{-2n-2}$

If div grad $f(r)$ is zero, we get n = 0 or n = ½ .

8: Prove that $\nabla x\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}.\bar{A})\bar{r}}{r^{n+2}}$.

Sol: We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}| = \sqrt{x^2+y^2+z^2}$

$\therefore \frac{\partial\bar{r}}{\partial x} = \bar{i},\ \frac{\partial\bar{r}}{\partial y} = \bar{j},\ \frac{\partial\bar{r}}{\partial z} = \bar{k}$ and

$r^2 = x^2+y^2+z^2$…..(1)

Diff. (1) partially,

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\ \text{similarly}\ \frac{\partial r}{\partial y} = \frac{y}{r}\ and\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla x\left(\frac{\bar{A}\times\bar{r}}{r^n}\right) = \sum \bar{i}\times\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right)$$

Now $\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right) = \bar{A}\times\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^n}\right) = \bar{A}\times\left[\frac{r^n\bar{i} - \bar{r}\,n\,r^{n-1}}{r^{2n}}\right]\frac{\partial r}{\partial x}$

$$= \bar{A}\times\left[\frac{r^n\bar{i} - n\,r^{n-2}x\bar{r}}{r^{2n}}\right] = \bar{A}\times\left[\frac{1}{r^n}\bar{i} - \frac{n}{r^{n+2}}.x\bar{r}\right]$$

$$= \frac{\bar{A}\times\bar{i}}{r^n} - \frac{n}{r^{n+2}}.x(\bar{A}\times\bar{r})$$

$$\therefore \bar{i}\times\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right) = \frac{\bar{i}\times(\bar{A}\times\bar{i})}{r^n} - \frac{nx}{r^{n+2}}.\bar{i}\times(\bar{A}\times\bar{r})$$

$$= \frac{(\bar{i}.\bar{i})\bar{A} - (\bar{i}.\bar{A})\bar{i}}{r^n} - \frac{nx}{r^{n+2}}[(\bar{i}.\bar{r})\bar{A} - (\bar{i}.\bar{A})\bar{r}]$$

Let $A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$. Then $\bar{i}.\bar{A} = A_1$

$$\therefore \bar{i}\times\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right) = \left(\frac{\bar{A} - A_1\bar{i}}{r^n}\right) - \frac{nx}{r^{n+2}}[x\bar{A} - A_1\bar{r}]$$

and $\sum \bar{i}\times\frac{\partial}{\partial x}\left(\frac{(\bar{A}\times\bar{r})}{r^n}\right) = \sum\left(\frac{\bar{A} - A_1\bar{i}}{r^n}\right) - \frac{nx}{r^{n+2}}[x\bar{A} - A_1\bar{r}]$

$$= \frac{3\overline{A}-\overline{A}}{r^n} - \frac{n}{r^{n+2}}[r^2\overline{A}] + \frac{n\overline{r}}{r^{n+2}}(A_1x + A_2y + A_3z)$$

$$= \frac{2\overline{A}}{r^n} - \frac{n}{r^n}\overline{A} + \frac{n\overline{r}}{r^{n+2}}(\overline{A}.\overline{r}) = \frac{(2-n)\overline{A}}{r^n} + \frac{n\overline{r}}{r^{n+2}}(\overline{A}.\overline{r})$$

Hence the result.

VECTOR IDENTITIES

Theorem 1: If $\overline{a}$ is a differentiable function and ϕ is a differentiable scalar function, then prove that div(ϕ $\overline{a}$)= (grad ϕ). $\overline{a}$ +ϕ div $\overline{a}$ or ∇.(ϕ $\overline{a}$)= (∇ϕ). $\overline{a}$ +ϕ(∇. $\overline{a}$)

Proof: div(ϕ $\overline{a}$)=∇.(ϕ $\overline{a}$)= $\sum i.\frac{\partial}{\partial x}(\phi\overline{a})$

$$= \sum \overline{i}.\left(\frac{\partial\phi}{\partial x}\overline{a} + \phi\frac{\partial\overline{a}}{\partial x}\right) = \sum\left(i.\frac{\partial\phi}{\partial x}\overline{a}\right) + \sum\left(i\frac{\partial\overline{a}}{\partial x}\right)\phi$$

$$= \sum\left(\overline{i}\frac{\partial\phi}{\partial x}\right).\overline{a} + \left(\sum\overline{i}\frac{\partial a}{\partial x}\right)\phi = (\nabla\phi).\overline{a} + \phi(\nabla.\overline{a})$$

Theorem 2: Prove that curl (ϕ $\overline{a}$)= (grad ϕ)x $\overline{a}$ +ϕ curl $\overline{a}$

Proof : curl (ϕ $\overline{a}$)=∇x(ϕ $\overline{a}$)= $\sum i\times\frac{\partial}{\partial x}(\phi\overline{a})$

$$= \sum\overline{i}\times\left(\frac{\partial\phi}{\partial x}\overline{a} + \phi\frac{\partial\overline{a}}{\partial x}\right) = \sum\left(\overline{i}\frac{\partial\phi}{\partial x}\right)\times\overline{a} + \sum\left(\overline{i}\times\frac{\partial\overline{a}}{\partial x}\right)\phi$$

= ∇ϕx $\overline{a}$ +(∇x $\overline{a}$)ϕ=(grad ϕ)x $\overline{a}$ +ϕ curl $\overline{a}$

Theorem 3: Prove that grad ($\overline{a}$. $\overline{b}$)= $(\overline{b}.\nabla)\overline{a} + (\overline{a}.\nabla)\overline{b} + \overline{b}\times curl\,\overline{a} + \overline{a}\times curl\,\overline{b}$

Proof: Consider

$$\overline{a}\times curl(\overline{b}) = \overline{a}\times(\nabla\times\overline{b}) = a\times\sum\left(\overline{i}\times\frac{\partial\overline{b}}{\partial x}\right)$$

$$= \sum\overline{a}\times\left(\overline{i}\times\frac{\partial\overline{b}}{\partial x}\right)$$

$$= \sum\left\{\left(\overline{a}.\frac{\partial\overline{b}}{\partial x}\right)\overline{i} - (\overline{a}.\overline{i})\frac{\partial\overline{b}}{\partial x}\right\} = \sum\overline{i}\left\{\overline{a}.\frac{\partial\overline{b}}{\partial x}\right\} - \left\{\overline{a}.\sum i\frac{\partial}{\partial x}\right\}\overline{b}$$

$$\therefore \overline{a}\times curl\,\overline{b} = \sum\overline{i}\left(\overline{a}.\frac{\partial\overline{b}}{\partial x}\right) - (\overline{a}.\nabla)\overline{b} \quad ....(1)$$

Similarly, $\overline{b}\times curl\,\overline{b} = \sum\overline{i}\left(\overline{b}.\frac{\partial\overline{a}}{\partial x}\right) - (\overline{b}.\nabla)\overline{a}$ ……………………….(2)

(1)+(2) gives

$$\bar{a}\times curl\,\bar{b}+\bar{b}\times curl\,\bar{a}=\sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}\right)-(\bar{a}.\nabla)\bar{b}+\sum\bar{i}\left(\bar{b}.\frac{\partial\bar{a}}{\partial x}\right)-(\bar{b}.\nabla)\bar{a}$$

$$\Rightarrow\bar{a}\times curl\,\bar{b}+\bar{b}\times curl\,\bar{a}+(\bar{a}.\nabla)\bar{b}+(\bar{b}.\nabla)\bar{a}=\sum\bar{i}\left(\bar{a}.\frac{\partial\bar{b}}{\partial x}+\bar{b}.\frac{\partial\bar{a}}{\partial x}\right)$$

$$=\sum\bar{i}\,\frac{\partial}{\partial x}(\bar{a}.\bar{b})$$

$$=\nabla(\bar{a}\,.\,\bar{b})=\text{grad}\,(\bar{a}\,.\,\bar{b})$$

Theorem 4: Prove that $\text{div}\,(\bar{a}\times\bar{b})=\bar{b}.curl\,\bar{a}-\bar{a}.curl\,\bar{b}$

Proof: div $(\bar{a}\times\bar{b})=\sum\bar{i}.\frac{\partial}{\partial x}(\bar{a}\times\bar{b})=\sum\bar{i}.\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}+\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$

$$=\sum\bar{i}.\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right)+\sum\bar{i}.\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)=\sum\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right).\bar{b}-\sum\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right).\bar{a}$$

$$=(\nabla\times\bar{a}).\bar{b}-(\nabla\times\bar{b}).\bar{a}=\bar{b}.curl\,\bar{a}-\bar{a}.curl\,\bar{b}$$

Theorem 5 : Prove that $curl\,(\bar{a}\times\bar{b})=\bar{a}\,div\,\bar{b}-\bar{b}div\,\bar{a}+(\bar{b}.\nabla)\bar{a}-(\bar{a}.\nabla)\bar{b}$

$$\text{Pr}oof:curl\,(\bar{a}\times\bar{b})=\sum\bar{i}\times\frac{\partial}{\partial x}(\bar{a}\times\bar{b})=\sum\bar{i}\times\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b}+\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$\sum\bar{i}\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right)+\sum\bar{i}\times\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$=\sum\left\{(\bar{i}.\bar{b})\frac{\partial\bar{a}}{\partial x}-\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b}\right\}+\sum\left\{\left(\bar{i}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a}-(\bar{i}.\bar{a})\frac{\partial\bar{b}}{\partial x}\right\}$$

$$=\sum(\bar{b}.\bar{i})\frac{\partial\bar{a}}{\partial x}-\sum\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b}+\sum\left(\bar{i}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a}-\left(\bar{a}.\sum\bar{i}\frac{\partial}{\partial x}\right)\bar{b}$$

$$=(\bar{b}.\nabla)\bar{a}-(\nabla.\bar{a})\bar{b}+(\nabla.\bar{b})\bar{a}-(\bar{a}.\nabla)\bar{b}$$

$$=(\nabla.\bar{b})\bar{a}-(\nabla.\bar{a})\bar{b}+(\bar{b}.\nabla)\bar{a}-(\bar{a}.\nabla)\bar{b}$$

$$=\bar{a}\,div\,\bar{b}-\bar{b}\,div\,\bar{a}+(\bar{b}.\nabla)\bar{a}-(\bar{a}.\nabla)\bar{b}$$

Theorem 6: Prove that curl grad $\phi=0$.

Proof: Let ϕ be any scalar point function. Then

$$.grad\ \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$curl(grad\phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) - \bar{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) - \bar{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right) = \bar{0}$$

Note : Since $Curl(grad\phi) = \bar{0}$, we have $grad\ \phi$ is always irrotational.

7. Prove that div $curl\,\overline{f} = 0$

$\Pr oof$: $Let\ \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore curl\ \overline{f} = \nabla \times \overline{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore\quad div\ \ curl\ \overline{f} = \nabla.(\nabla \times \overline{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} = 0$$

Note : Since $div(curl\,\overline{f}) = 0,$ we have $curl\,\overline{f}$ is always solenoidal.

Theorem 8: If *f* and g are two scalar point functions, prove that *div*($f\nabla$g)= $f\nabla^2$g+∇f. ∇g

Sol: Let *f* and g be two scalar point functions. Then

$$\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$$

Now $\qquad f\nabla \text{g} = \bar{i}f\frac{\partial g}{\partial x} + \bar{j}f\frac{\partial g}{\partial y} + \bar{k}f\frac{\partial g}{\partial z}$

$$\therefore \nabla.(f\nabla \text{g}) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial x^2}\right)+\left(\frac{\partial f}{\partial x}.\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}.\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}.\frac{\partial g}{\partial z}\right)$$

$$=f\nabla^2 g+\left(\bar{i}\,\frac{\partial f}{\partial x}+\bar{j}\,\frac{\partial f}{\partial y}+\bar{k}\,\frac{\partial f}{\partial z}\right).\left(\bar{i}\,\frac{\partial g}{\partial x}+\bar{j}\,\frac{\partial g}{\partial y}+\bar{k}\,\frac{\partial g}{\partial z}\right).$$

$$= f\nabla^2 g+\nabla f.\,\nabla g$$

Theorem 9: Prove that $\nabla x(\nabla x\,\bar{a}\,)=\nabla(\nabla.\,\bar{a}\,)-\nabla^2\,\bar{a}$.

Proof: $\nabla x(\nabla x\,\bar{a}\,) = \sum \bar{i}\times\frac{\partial}{\partial x}(\nabla\times\bar{a})$

Now $\bar{i}\times\frac{\partial}{\partial x}(\nabla\times\bar{a}) = i\times\frac{\partial}{\partial x}\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}+\bar{j}\times\frac{\partial\bar{a}}{\partial y}+\bar{k}\times\frac{\partial\bar{a}}{\partial z}\right)$

$$=\bar{i}\times\left(\bar{i}\times\frac{\partial^2\bar{a}}{\partial x^2}+\bar{j}\times\frac{\partial^2\bar{a}}{\partial x\partial y}+\bar{k}\times\frac{\partial^2\bar{a}}{\partial x\partial z}\right)$$

$$=\bar{i}\times\left(\bar{i}\times\frac{\partial^2\bar{a}}{\partial x^2}\right)+\bar{i}\times\left(\bar{j}\times\frac{\partial^2\bar{a}}{\partial x\partial y}\right)+\bar{i}\times\left(\bar{k}\times\frac{\partial^2\bar{a}}{\partial x\partial z}\right)$$

$$=\left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x^2}\right)\bar{i}-\frac{\partial^2\bar{a}}{\partial x^2}+\left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x\partial y}\right)\bar{j}+\left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\bar{k}\qquad[\because i.i=1, i.j=i.k=0]$$

$$=\bar{i}\,\frac{\partial}{\partial x}\left(\bar{i}\,.\frac{\partial\bar{a}}{\partial x}\right)+j\,\frac{\partial}{\partial y}\left(\bar{i}\,.\frac{\partial\bar{a}}{\partial y}\right)+k\,\frac{\partial}{\partial z}\left(\bar{i}\,.\frac{\partial\bar{a}}{\partial x}\right)-\frac{\partial^2\bar{a}}{\partial x^2}=\nabla\left(\bar{i}\,.\frac{\partial\bar{a}}{\partial x}\right)-\frac{\partial^2\bar{a}}{\partial x^2}$$

$$\therefore\sum\bar{i}\times\frac{\partial}{\partial x}(\nabla\times\bar{a})=\nabla\sum\bar{i}\,.\frac{\partial\bar{a}}{\partial x}-\sum\frac{\partial^2\bar{a}}{\partial x^2}=\nabla(\nabla.\bar{a})-\left(\frac{\partial^2\bar{a}}{\partial x^2}+\frac{\partial^2\bar{a}}{\partial y^2}+\frac{\partial^2\bar{a}}{\partial z^2}\right)$$

$$\therefore \nabla x(\nabla x\,\bar{a}\,)=\nabla(\nabla.\,\bar{a}\,)-\nabla^2\,\bar{a}$$

i.e., $curl curl\bar{a} = grad\ \ div\bar{a}-\nabla^2\bar{a}$

SOLVED PROBLEMS

1: Prove that $(\nabla f\, x\nabla g)$is solenoidal.

Sol: We know that div ($\bar{a}\times\bar{b}$) = $\bar{b}.curl\,\bar{a}-\bar{a}.curl\,\bar{b}$

Take $\overline{a}=\nabla f$ and $\overline{b}=\nabla g$

Then div (∇f x ∇g) = ∇g. curl (∇f) - ∇f. curl (∇g)=0 $\left[\because curl(\nabla f)=\bar{0}=curl(\nabla g)\right]$

$\therefore \nabla f\times\nabla g$ is solenoidal.

2:Prove that (i) $div\left\{(\bar{r}\times\bar{a})\bar{b}\right\}=-2(\bar{b}.\bar{a})$ (ii) $curl\left\{(\bar{r}.\bar{a})\times b\right\}=\bar{b}\times\bar{a}$ where $\bar{a}$ and $\bar{b}$ are constant vectors.

Sol: (i)

$$div\left\{(\bar{r}\times\bar{a})\times\bar{b}\right\}=div[(\bar{r}.\bar{b})\bar{a}-(\bar{a}..\bar{b})\bar{r}]$$

$$=div(\bar{r}.\bar{b})\bar{a}-(\bar{a}..\bar{b})\bar{r}$$

$$=\left[\left(\bar{r}.\bar{b}\right)div\,\bar{a}+\bar{a}.grad\left(\bar{r}.\bar{b}\right)\right]-\left[\left(\bar{a}.\bar{b}\right)div\,\bar{r}+\bar{r}.grad\left(\bar{a}.\bar{b}\right)\right]$$

$We\,have\,div\;\bar{a}=0, div\,\bar{r}=3, grad\;(\bar{a}.\bar{b})=0$

$div\left\{\left(\bar{r}\times\bar{a}\right)\times\bar{b}\right\}=0+\bar{a}.grad\left(\bar{r}.\bar{a}\right)-3\left(\bar{a}.\bar{a}\right)$

$=\bar{a}.\sum\frac{i\partial}{\partial x}\left(\bar{r}.\bar{b}\right)-3\left(\bar{a}.\bar{b}\right)$

$=\bar{a}.\sum\bar{i}\frac{\partial\bar{r}}{\partial x}.\bar{b}-3\left(\bar{a}.\bar{b}\right)$

$=\bar{a}.\sum\bar{i}\left(\bar{i}.\bar{b}\right)-3\left(\bar{a}.\bar{b}\right)$

$=\bar{a}.\bar{b}-3\left(\bar{a}.\bar{b}\right)=-2\left(\bar{a}.\bar{b}\right)$

$=-2\left(\bar{b}.\bar{a}\right)$

(ii) $curl\left\{\left(\bar{r}\times\bar{a}\right)\times\bar{b}\right\}=curl\left[\left(\bar{r}.\bar{b}\right)\bar{a}-\left(\bar{a}.\bar{b}\right)\bar{r}\right]$

$=curl\left(\bar{r}.\bar{b}\right)\bar{a}-curl\left(\bar{a}.\bar{b}\right)\bar{r}$

$=\left(\bar{r}.\bar{b}\right)curl\,\bar{a}+grad\left(\bar{r}.\bar{b}\right)\times\bar{a}$

$=\bar{0}+\nabla\left(\bar{r}.\bar{b}\right)\times\bar{a}\left(\because curl\,\bar{a}=\bar{0}\right)$

$=\bar{b}\times\bar{a}$ Since $grad\left(\bar{r}.\bar{b}\right)=\bar{b}$

3: Prove that $\nabla\left[\nabla.\frac{\bar{r}}{r}\right]=\frac{-2}{r^3}\bar{r}.$

Sol: We have $\nabla.\left(\frac{\bar{r}}{r}\right)=\sum i.\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r}\right)$

$$=\sum i.\left[\frac{1}{r}\frac{\partial\bar{r}}{\partial x}+\bar{r}\left(\frac{-1}{r^2}\right)\left(\frac{x}{r}\right)\right]=\sum i.\left(\frac{1}{r}i-\frac{\bar{r}}{r^3}x\right)$$

$$=\frac{1}{r}\sum i.i-\frac{1}{r^3}r^2=\frac{3}{r}-\frac{1}{r}=\frac{2}{r}$$

$$\therefore\nabla\left[\nabla.\left(\frac{\bar{r}}{r}\right)\right]=\sum i\left(\frac{\partial}{\partial x}\left(\frac{2}{r}\right)\right)=\sum i\left(\frac{-2}{r^2}\right)\left(\frac{x}{r}\right)=\frac{-2}{r^3}\sum xi=\frac{-2\bar{r}}{r^3}.$$

4: Find $(A\times\nabla)\phi$, if $A = yz^2\bar{i} - 3xz^2\bar{j} + 2xyz\,\bar{k}$ and $\phi = xyz$.

Sol : We have

$$A\times\nabla = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ yz^2 & -3xz^2 & 2xyz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial x}(-3xz^2) - \frac{\partial}{\partial y}(2xyz)\right] - \bar{j}\left[\frac{\partial}{\partial z}(yz^2) - \frac{\partial}{\partial x}(2xyz)\right] + \bar{k}\left[\frac{\partial}{\partial y}(yz^2) - \frac{\partial}{\partial x}(-3xz^2)\right]$$

$= \bar{i}(-6xz-2xz) - \bar{j}(2yz-2yz) + \bar{k}(z^2+3z^2) = -8xz\,\bar{i} - 0\,\bar{j} + 4z^2\bar{k}$

$(A\times\nabla)\phi, = (-8xz\,\bar{i} + 4z^2\bar{k})xyz = -8x^2yz^2\bar{i} + 4xyz^3\bar{k}$

Objective questions

1. $\nabla(r^n) =$ ---------------------
2. $\nabla(\frac{1}{r}) =$ ……………………………….

3. the greatest value of the directional derivative of the function $f = x^2yz^3$ at (2,1,-1)is …………….
4. a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3)is………….

5. a unit normal vector to the surface $z= x^2+y^2$ at (-1,-2,5) is…………..

6 The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces .Let θ be the angle between the surfaces. ThenCos θ=…………

7. If the vector $\bar{f} = f_1\bar{i} + f_2\,\bar{j} + f_3\,\bar{k}$, then div $\bar{f} =$ …….

8. A vector point function $\bar{f}$ is said to be solenoidal if div $\bar{f}$ =…………
9. if $\bar{r}$ is the position vector of an point in space, then $r^n\bar{r}$ is Irrotational then curl($r^n\bar{r}$)=……….

Multiple choice questions

∴ 1. **: If** $f = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

∴ a. $-\bar{i} - 2\bar{k}.$ b. $-\bar{i} - 2\bar{k}.$ c. $-\bar{i} - 2\bar{k}.$ d. $-\bar{i} - 2\bar{k}.$

∴ 2. If $f = (x^2+y^2+z^2)^{-n}$ then find div grad f and determine n if div grad f= 0.

∴ a. n = 0 or n = ½ . b. n = 0 or n = ½ . c. n = 0 or n = ½ . d. n = 0 or n = ½ .

∴ 3. **:** Find div $\overline{F}$, where $\overline{F}$ = grad ($x^3+y^3+z^3-3xyz$)

∴ a. 6(*x*+*y*+*z*) b. 6(*x*+*y*+*z*) c. 6(*x*+*y*+*z*) d. 6(*x*+*y*+*z*)

∴ 4. ($\bar{f}$ x∇). $\bar{r}$ =

∴ a. 0 b.1 c.2 d. 3

∴ 5. Find constants a,b and c if the vector $\bar{f}$ =

$(2x+3y+az)\bar{i}+(bx+2y+3z)\,\bar{j}+(2x+cy+3z)\,\bar{k}$ is Irrotational

∴ a.a=2 b=3,c=3 b.a=1,b=2,c=4 c.a=0,b=1,c=4 d.a=1,b=3,c=2

∴ 6. If $\bar{f}$ = $(x+3y)\bar{i}+(y-2z)\,\bar{j}+(x+pz)\bar{k}$ is solenoidal, find *P*.

∴ a.p=4 b.p=-2 c.p=3 d.p=-3

∴ 7. **If** $\bar{f}$ **=** $xy^2\bar{i}+2x^2yz\bar{j}-3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

∴ a.6 b.7 c.8 d.9

∴ 8. Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.

∴ a. 37/3. B. 47/3. C. 27/3. D. 17/3.

∴

9.

∴ If If $\overline{a}$ is constant vector then prove that grad ($\overline{a}$. $\bar{r}$)= $\overline{a}$ $\overline{a}$ is constant vector then grad ($\overline{a}$. $\bar{r}$)=

∴ a. $\overline{a}$ b.0 c. r d. 1

∴ 10.

∴ **:** Find the values of a and b so that the surfaces $ax^2-byz = (a+2)x$ and $4x^2y+z^3= 4$ may intersect orthogonally at the point (1, -1,2).

∴ a.a-3.5b=1 b.a=2.5,b=1 c.a=1,b=1 d. a=1 ,b=0

UNIT-V

Vector Integration

Line integral:- (i) $\int_c \bar{F}.d\bar{r}$ is called Line integral of $\bar{F}$ along c

Note : Work done by $\bar{F}$ along a curve c is $\int_c \bar{F}.d\bar{r}$

PROBLEMS

1. If $\bar{F}$ $(x^2-27)\bar{i}-6yz\bar{j}+8xz^2\bar{k}$, evaluate $\int \bar{F}.d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Solution : Given $\bar{F} = (x^2-27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$

Now $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \quad \bar{F}.d\bar{r} = (x^2-27)dx - (6yz)dy + 8xz^2dz$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here y =0 =z and dy=dz=0. Also x changes from 0 to 1.

$$\therefore \int_{OA} \bar{F}.d\bar{r} = \int_o^1 (x^2-27)dx = \left[\frac{x^3}{3} - 27x\right]_0^1 = \frac{1}{3} - 27 = \frac{-80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here x =1, z=0 $\Rightarrow$ dx=0, dz=0. y changes from 0 to 1.

$$\therefore \int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^1 (-6yz)dy = 0$$

(iii) Along the straight line from B = (1,1,0) to C = (1,1,1)

x =1 =y $\Rightarrow$ dx=dy=0 and z changes from 0 to 1.

$$\therefore \int_{BC} \bar{F}.d\bar{r} = \int_{z=0}^1 8xz^2dz = \int_{z=0}^1 8xz^2dz = \left[\frac{8z^3}{3}\right]_0^1 = \frac{8}{3}$$

$$(i)+(ii)+(iii) \Rightarrow \int_C \bar{F}.d\bar{r} = \frac{88}{3}$$

2. If $\bar{F} = (5xy-6x^2)\bar{i} + (2y-4x)\bar{j}$, evaluate $\int_C \bar{F}.d\bar{r}$ along the curve C in xy-plane $y=x^3$ from (1,1) to (2,8).

Solution : Given $\bar{F}$ =(5xy-6x^2)$\bar{i}$ +(2y-4x) $\bar{j}$,-------(1)

Along the curve y=x^3, dy =3x^2 dx

$\therefore$ $\bar{F}$ =(5x^4-6x^2)$\bar{i}$ +(2x^3-4x) $\bar{j}$, [Putting y=x^3 in (1)]

$d\bar{r} = dx\bar{i} + dy\bar{j} = dx\bar{i} + 3x^2 dx\ \bar{j}$

$\therefore$ $\bar{F}.d\bar{r}$ = [(5x^4-6x^2)$\bar{i}$ +(2x^3-4x) $\bar{j}$].$\left[dx\,\bar{i} + 3x^2 dx\,\bar{j}\right]$

= (5x^4 – 6x^2) dx+(2x^3 – 4x)3x^{3}dx

= (6x^5+5x^4-12x^3 -6x^2)dx

Hence $\int\limits_{y=x^3} \bar{F}.d\bar{r} = \int\limits_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$

$$= \left(6.\frac{x^6}{6} + 5.\frac{x^5}{5} - 12.\frac{x^4}{4} - 6.\frac{x^3}{4}\right) = \left(x^6 + x^5 - 3x^4 - 2x^3\right)_1^2$$

= 16(4+2-3-1) – (1+1-3-2) = 32+3 = 35

3. Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r}$ = cost $\bar{i}$ + sint $\bar{j}$ -t $\bar{k}$ from t = 0 to t = 2π

Solution : Given force $\bar{F}$ = z$\bar{i}$+ x$\bar{j}$ +y $\bar{k}$ and the arc is $\bar{r}$ = cost $\bar{i}$ + sin t $\bar{j}$ -t$\bar{k}$

i.e., x = cost, y= sin t, z = -t

$\therefore$ d$\bar{r}$ = (-sin t $\bar{i}$ +cost $\bar{j}$ - $\bar{k}$)dt

$\therefore$ $\bar{F}.d\bar{r}$ = (-t $\bar{i}$+cost $\bar{j}$ +sin t $\bar{k}$). (-sin t $\bar{i}$ + cost $\bar{j}$ - $\bar{k}$)dt = (t sin t + cos^2 t – sin t)dt

Hence work done = $\int\limits_0^{2\pi} \bar{F}.d\bar{r} = \int\limits_0^{2\pi}$ (t sin t + cos^2 t – sin t) dt

$$= \left[t(-\cos t)\right]_0^{2\pi} - \int\limits_0^{2\pi} (-\sin t)dt + \int\limits_0^{2\pi} \frac{1+\cos 2t}{2} dt - \int\limits_0^{2\pi} \sin t\, dt$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

$$= -2\pi - (1-1) + \frac{1}{2}(2\pi) + (1-1) = -2\pi + \pi = -\pi$$

<u>**Assignment**</u>

1. Find $\int_c \bar{F}.d\bar{r}$ where $\bar{F} = x^2y^2\bar{i} + y\bar{j}$ and the curve $y^2=4x$ in the xy-plane from (0,0) to (4,4).
2. If $\bar{F} = 3xy\bar{i} - 5z\bar{j} + 10x\bar{k}$ evaluate $\int_c \bar{F}.d\bar{r}$ along the curve $x=t^2+1, y=2t^2, z=t^3$ from t = 1 to t= 2.
3. If $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$, find the circulation of $\bar{F}$ round the curve c where c is the circle $x^2 + y^2 = 1$, z=0.
4. (i) If $\phi = x^2yz^3, evaluate \int_c \phi d\bar{r}$ along the curve x= t, y =2t, z=3t from t = 0 to t=1.

   (ii) If $\phi = 2xy^2z + x^2y, evaluate \int_c \phi d\bar{r}$ where c is the curve x=t, $y=t^2$, $z= t^3$ from t=0 to t=1.
5. (i) Find the work done by the force $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ in taking particle from (1,1,1) to (3,-5,7).

   (ii) Find the work done by the force $\bar{F} = (2y+3)\bar{i} + (zx)\bar{j} + (yz - x)\bar{k}$ when it moves a particle from the point (0,0,0) to (2,1,1) along the curve $x = 2t^2$, y = t, $z=t^3$

Surface integral: $\int_S \bar{F}.\bar{n}\,ds$ is called surface integral.

PROBLEMS

1 : Evaluate $\int \bar{F}.ndS$ where $\bar{F} = zi + xj - 3y^2zk$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Sol. The surface S is $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Let $\phi = x^2 + y^2 = 16$

Then $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2x\bar{i} + 2y\bar{j}$

$\therefore$ unit normal $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{x\bar{i} + y\bar{j}}{4}$ $(\because x^2 + y^2 = 16)$

Let R be the projection of S on yz-plane

Then $\int_S \bar{F}.ndS = \iint_R \bar{F}.\bar{n}\frac{dydz}{|\bar{n}\,.\,\bar{i}|}$ ……………. *

Given $\bar{F} = zi + xj - 3y^2zk$

$\therefore \qquad \bar{F} \cdot \bar{n} = \frac{1}{4}(xz + xy)$

and $\qquad \bar{n} \cdot \bar{i} = \frac{x}{4}$

In yz-plane, x = 0, y = 4

In first octant, y varies from 0 to 4 and z varies from 0 to 5.

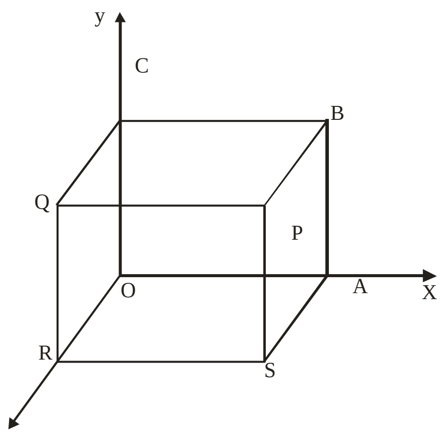
$$\int_S \bar{F}.ndS = \int_{y=0}^{4} \int_{z=0}^{5} \left(\frac{xz + xy}{4}\right) \frac{dydz}{\left|\frac{x}{4}\right|}$$

$$= \int_{y=0}^{4} \int_{z=0}^{5} (y + z)dz\ dy$$

$$= 90.$$

2 : If $\bar{F}$ = zi + xj − $3y^2$zk, evaluate $\int_S \bar{F}.\bar{n}dS$ where S is the surface of the cube bounded by x = 0, x = a, y = 0, y= a, z = 0, z = a.

Sol. Given that S is the surface of the x = 0, x = a, y = 0, y = a, z = 0, z = a, and $\bar{F}$ = zi + xj − $3y^2$zk we need to evaluate $\int_S \bar{F}.\bar{n}dS$.



(i) For OABC

Eqn is z = 0 and dS = dxdy

$$\bar{n} = -\bar{k}$$

$$\int_{S_1} \bar{F}.\bar{n}dS = -\int_{x=0}^{a} -\int_{y=0}^{a} (yz)\ dxdy = 0$$

(ii) For PQRS

Eqn is z = a and dS = dxdy

$$\bar{n} = \bar{k}$$

$$\int_{S_2} \bar{F}.ndS = \int_{x=0}^{a}\left(\int_{y=0}^{a} y(a)dy\right)dx = \frac{a^4}{2}$$

(iii) For OCQR

Eqn is x = 0, and $\bar{n} = -\bar{i}$, dS = dydz

$$\int_{S_3} \bar{F}.ndS = \int_{y=0}^{a}\int_{z=0}^{a} 4xzdydz = 0$$

(iv) For ABPS

Eqn is x = a, and $\bar{n} = -\bar{i}$, dS = dydz

$$\int_{S_3} \bar{F}.\bar{n}dS = \int_{y=0}^{a}\left(\int_{z=0}^{a} 4azdz\right)dy = 2a^4$$

(v) For OASR

Eqn is y = 0, and $\bar{n} = -\bar{j}$, dS = dxdz

$$\int_{S_5} \bar{F}.ndS = \int_{y=0}^{a}\int_{z=0}^{a} y^2dzdx = 0$$

(vi) For PBCQ

Eqn is y = a, and $\bar{n} = -\bar{j}$, dS = dxdz

$$\int_{S_6} \bar{F}.\bar{n}dS = -\int_{y=0}^{a}\int_{z=0}^{a} y^2dzdx = 0$$

From (i) – (vi) we get

$$\int_{S_6} \bar{F}.\bar{n}dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a4 = \frac{3a^4}{2}$$

VOLUME INTEGRALS

Let V be the volume bounded by a surface $\bar{r} = \bar{f}$ (u,v). Let $\bar{F}(\bar{r})$ be a vector point function define over V. Divide V into m sub-regions of volumes $\delta V_1, \delta V_2, \ldots.\delta V_p \ldots.\delta V_m$

Let P_i ($\bar{r}_i$) be a point in δV_i .Then form the sum $I_m = \sum_{i=1}^{m} \bar{F}(r_i)\delta V_i$. Let m $\to \infty$ in such a way that δV_i shrinks to a point,. The limit of I_m if it exists, is called the volume integral of $\bar{F}(\bar{r})$ in the region V is denoted by $\int_V \bar{F}(\bar{r})\,dv$ or $\int_V \bar{F}dv$.

Cartesian form : Let $\bar{F}(r) = F_1\bar{i} + F_2\bar{i} + F_3\bar{k}$ where F_1, F_2, F_3 are functions of x,y,z. We know that

dv = dx dy dz. The volume integral given by

$$\int_v \bar{F}dv = \iiint (F_1\bar{i} + F_2\bar{i} + F_3\bar{k})\,\text{dx dy dz} = \bar{i}\iiint F_1\,\text{dxdydz} + \bar{j}\iiint F_2\,\text{dxdydz} + \bar{k}\iiint F_3\,\text{dxdydz}$$

SOLVED EXAMPLES

Example 1 : If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ evaluate $\int_V \bar{F}dv$ where V is the region bounded by the surfaces $x = 0, x = 2, y = 0, y = 6, z = x^2, z = 4$.

Solution : Given $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$. $\therefore$ The volume integral is

$$\int_V \bar{F}dv = \iiint (2xz\bar{i} - x\bar{j} + y^2\bar{k})dx\,dy\,dz$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\,dx\,dy\,dz - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6}[xz^2]_{x^2}^{4}\,dx\,dy - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}(xz)_{x^2}^{4}\,dx\,dy + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6} y^2(z)_{x^2}^{4}\,dx\,dy$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} x(16 - x^4)dx\,dy - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} x(4 - x^2)dx\,dy - \bar{k}\int_{x=0}^{2}\int_{y=0}^{2} y^2(x^2 - 4)dx\,dy$$

$$= \bar{i}\int_{x=0}^{2}(16x - x^5)(y)_0^6dx - \bar{j}\int_{x=0}^{2}(4x - x^3)(y)_0^6dx - \bar{k}\int_{x=0}^{2}(x^2 - 4)\left(\frac{y^3}{3}\right)_0^6dx$$

$$= \bar{i}\left(8x^2 - \frac{x^6}{6}\right)_0^2(6) - \bar{j}\left(2x^2 - \frac{x^4}{4}\right)_0^2(6) - \bar{k}\left(4x - \frac{x^3}{3}\right)_0^2\left(\frac{211}{3}\right)$$

$$= 128\bar{i} - 24\bar{j} - 384\bar{k}$$

Example 2 : If $\overline{F}=(2x^2-3z)\overline{i}-2xy\overline{j}-4x\overline{k}$ then evaluate (*i*) $\int_V \nabla.\overline{F}\,dv$ and (*ii*) $\int_V \nabla\times\overline{F}dv$

, V is the closed region bounded by $x=0$, $y=0$, $z=0$, $2x+2y+z=4$.

Solution : (*i*) $\nabla\cdot\overline{F}=\overline{i}\cdot\dfrac{\partial\overline{F}}{\partial x}+\overline{j}\cdot\dfrac{\partial\overline{F}}{\partial y}+\overline{k}\cdot\dfrac{\partial\overline{F}}{\partial z}=4x-2x=2x.$

The limits are : $z=0$ to $z=4-2x-2y$, $y=0$ to $\dfrac{4-2x}{2}$ (*i.e.*) $2-x$ and $x=0$ to $\dfrac{4}{2}$ (*i.e.*) 2

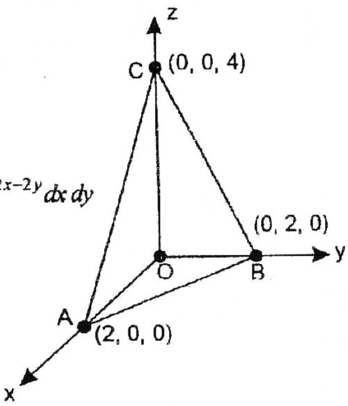
$$\therefore \int_V \nabla.\overline{F}dv=\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}2x\,dx\,dy\,dz=\int_{x=0}^{2}\int_{y=0}^{2-x}(2x)(z)_0^{4-2x-2y}dx\,dy$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}2x(4-2x-2y)dx\,dy=4\int_{x=0}^{2}\int_{y=0}^{2-x}(2x-x^2-xy)dx\,dy$$

$$=4\int_0^2\left(2xy-x^2y-\frac{xy^2}{2}\right)_0^{2-x}dx=4\int_0^2\left[(2x-x^2)(2-x)-\frac{x}{2}(2-x)^2\right]dx$$

$$=\int_0^2(2x^3-8x^2+8x)dx=\left[\frac{x^4}{2}-\frac{8x^3}{2}+4x^2\right]_0^2=\frac{8}{3}$$

$$(ii)\ \nabla\times\overline{F}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x^2-3z & -2xy & -4x\end{vmatrix}=\overline{j}-2y\overline{k}$$



$$\therefore \int_V \nabla\times\overline{F}\,dv=\iiint_V(\overline{j}-2y\overline{k})dx\,dy\,dz=\int_{x=0}^{2}\int_{y=0}^{2-x}(\overline{j}-2y\overline{k})(z)_0^{4-2x-2y}dx\,dy$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}(\overline{j}-2y\overline{k})(4-2x-2y)dx\,dy$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}\left\{\overline{j}[(4-2x)-2y]-\overline{k}\left[(4-2x)\cdot2y-4y^2\right]\right\}dx\,dy$$

$$=\int_{x=0}^{2}\overline{j}\left[(4-2x)y-y^2\right]_0^{2-x}dx-\overline{k}\int_{x=0}^{2}\left[(4-2x)y^2-\frac{4y^3}{3}\right]_0^{2-x}dx$$

$$= \bar{j}\int_0^2 (2-x)^2 dx - \bar{k}\int_0^2 \frac{2}{3}(2-x)^3 dx$$

$$= \bar{j}\left[\frac{(2-x)^3}{-3}\right]_0^2 - \frac{2\bar{k}}{3}\left[\frac{(2-x)^4}{-4}\right]_0^2 = \frac{8}{3}(\bar{j}-\bar{k})$$

EXERCISE

(1) Evaluate $\iiint (2x+y)dv$ where V is the closed region bounded by the cylinder $z=4-x^2$, and planes $x=0,\ y=0,\ y=2,$ and $z=0$.

(2) If $\phi = 45x^2y$ evaluate $\iiint_V \phi dv$ where V is the closed region bounded by the planes $4x+2y+z=8, y=0, z=0$.

(3) Evaluate $\int_V \bar{F} dv$ when $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ and V is the region bounded by $x=0, y=0, y=6, z=4, z=x^2$.

ANSWERS

(1) $\frac{80}{3}$ (2) 128 (3) $24\bar{i} + 96\bar{j} + \frac{384}{5}\bar{k}$

Vector Integral Theorems

Introduction

In this chapter we discuss three important vector integral theorems: (i) Gauss divergence theorem, (ii) Green's theorem in plane and (iii) Stokes theorem. These theorems deal with conversion of

(i) $\int_S \bar{F}.\bar{n}$ ds into a volume integral where S is a closed surface.

(ii) $\int_C \bar{F}.d\bar{r}$ into a double integral over a region in a plane when C is a closed curve in the plane and.

(iii) $\int_S (\nabla\times\bar{A}).\bar{n}$ ds into a line integral around the boundary of an open two sided surface.

I. GAUSS'S DIVERGENCE THEOREM
(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V div\, F dv = \int_S \bar{F}.\bar{n}\ dS$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

SOLVED PROBLEMS

1) Verify Gauss Divergence theorem for $\boldsymbol{\overline{F} = (x^3 - yz)\bar{\imath} - 2x^2 y\bar{\jmath} + z\bar{k}}$ taken over the surface of the cube bounded by the planes x = y = z = a and coordinate planes.

Sol: By Gauss Divergence theorem we have

$$\int_S \overline{F}.\overline{n}dS = \int_V div\overline{F}dv$$

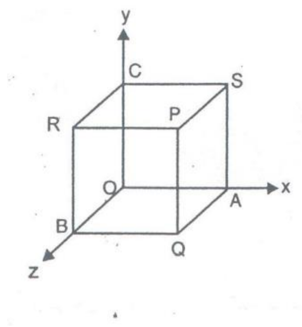
$$RHS = \int_0^a \int_0^a \int_0^a (3x^2 - 2x^2 + 1)dx\,dy\,dz = \int_0^a \int_0^a \int_0^a (x^2 + 1)\,dx\,dy\,dz = \int_0^a \int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\,dz$$

$$\int_0^a \int_0^a \left[\frac{a^3}{3} + a\right]dydz = \int_0^a \left[\frac{a^3}{3} + a\right](y)_0^a dz = \left(\frac{a^3}{3} + a\right)a\int_0^a dz = \left(\frac{a^3}{3} + a\right)\left(a^2\right) = \frac{a^5}{3} + a^3 \text{......(1)}$$

Verification: We will calculate the value of $\int_S \overline{F}.\overline{n}dS$ over the six faces of the cube.

(i) For S_1 = PQAS; unit outward drawn normal $\bar{n} = \bar{\imath}$

x=a; ds=dy dz; 0≤y≤a, 0≤z≤a



$$\therefore \overline{F}.\overline{n} = x^3 - yz = a^3 - yz \sin cex = a$$

$$\therefore \iint_{S_1} \overline{F}.\overline{n}dS = \int_{z=0}^{a} \int_{y=0}^{a} (a^3 - yz)dydz$$

$$= \int_{z=0}^{a} \left[a^3 y - \frac{y^2}{2}z\right]_{y=0}^{a} dz$$

$$= \int_{z=0}^{a} \left(a^4 - \frac{a^2}{2}z\right) dz$$

$$= a^5 - \frac{a^4}{4} \text{...(2)}$$

(ii) For S_2 = OCRB; unit outward drawn normal $\bar{n} = -\bar{\imath}$

x=0; ds=dy dz; 0≤y≤a, y≤z≤a

$$\bar{F}.\bar{n} = -(x^3 - yz) = yz \; since \; x = 0$$

$$\int_{S_2}\int \bar{F}.\bar{n} dS = \int_{z=0}^{a}\int_{y=0}^{a} yz\, dy\, dz = \int_{z=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{a} zdz$$

$$= \frac{a^2}{2}\int_{z=0}^{a} zdz = \frac{a^4}{4} \dots (3)$$

(iii) For S_3 = RBQP; Z = a; ds = dxdy; $\bar{n} = \bar{k}$

$0 \le x \le a,\ 0 \le y \le a$

$$\bar{F}.\bar{n} = z = a \;\; since \; z = a$$

$$\therefore \int_{S_3}\int \overline{F}\bar{n} dS = \int_{y=0}^{a}\int_{x=0}^{a} adxdy = a^3 \dots\dots(4)$$

(iv) For S_4 = OASC; z = 0; $\bar{n} = -\bar{k}$, ds = dxdy;

$0 \le x \le a,\ 0 \le y \le a$

$$\bar{F}.\bar{n} = -z = 0 \;\; since \; z = 0$$

$$\int_{S_4}\int \bar{F}.\bar{n} dS = 0 \dots (5)$$

(v) For S_5 = PSCR; y = a; $\bar{n} = \bar{j}$, ds = dzdx;

$0 \le x \le a,\ 0 \le z \le a$

$$\bar{F}.\bar{n} = -2x^2y = -2ax^2 \;\; since \; y = a$$

$$\int_{S_5}\int \bar{F}.\bar{n} dS = \int_{x=0}^{a}\int_{z=0}^{a} (-2ax^2) dzdx$$

$$\int_{x=0}^{a} (-2ax^2 z)_{z=0}^{a} dx$$

$$= -2a^2\left(\frac{x^3}{3}\right)_0^a = \frac{-2a^5}{3} \dots (6)$$

(vi) For S_6 = OBQA; y = 0; $\bar{n} = -\bar{j}$, ds = dzdx;

$0 \le x \le a,\ 0 \le y \le a$

$$\bar{F}.\bar{n} = 2x^2y = 0 \; since \; y = 0$$

$$\int_{S_6}\int \bar{F}.\bar{n} dS = 0$$

$$\int_S \int \bar{F}.\bar{n}dS = \int_{S_1}\int + \int_{S_2}\int + \int_{S_3}\int + \int_{S_4}\int + \int_{S_5}\int + \int_{S_6}\int$$

$$= a^5 - \frac{a^4}{4} - \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$= \frac{a^5}{3} + a^3 = \int\int_V\int \bar{\nabla}.\bar{F}\, dv \, using\ (1)$$

Hence Gauss Divergence theorem is verified

2.Compute $\int(\boldsymbol{ax^2 + by^2 + cz^2})\boldsymbol{dS}$ over the surface of the sphere $x^2+y^2+z^2 = 1$

Sol: By divergence theorem $\int_S \bar{F}.\bar{n}dS = \int_V \bar{\nabla}.\bar{F}\ dv$

$Given\ \bar{F}.\bar{n} = ax^2 + by^2 + cz^2. Let\ \phi = x^2 + y^2 + z^2 - 1$

$\therefore$ *Normal vector* $\bar{n}$ *to the surface* ϕ *is*

$$\bar{\nabla}\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial y}\right)(x^2 + y^2 + z^2 - 1) = 2(x\bar{i} + y\bar{j} + z\bar{k})$$

$\therefore$ Unit normalvector $= \bar{n} = \dfrac{2(x\bar{i} + y\bar{j} + z\bar{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\bar{i} + y\bar{j} + z\bar{k}$ Since $x^2 + y^2 + z^2 = 1$

$\therefore\ \bar{F}.\bar{n} = \bar{F}.(x\bar{i} + y\bar{j} + z\bar{k}) = (ax^2 + by^2 + cz^2) = (a\,x\bar{i} + by\bar{j} + cz\bar{k}).(x\bar{i} + y\bar{j} + z\bar{k})$

i.e., $\bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k}$ $\nabla.\bar{F} = a + b + c$

Hence by Gauss Divergence theorem,

$$\int_S (ax^2 + by^2 + cz^2)dS = \int_V (a + b + c)dv = (a + b + c)V = \frac{4\pi}{3}(a + b + c)$$

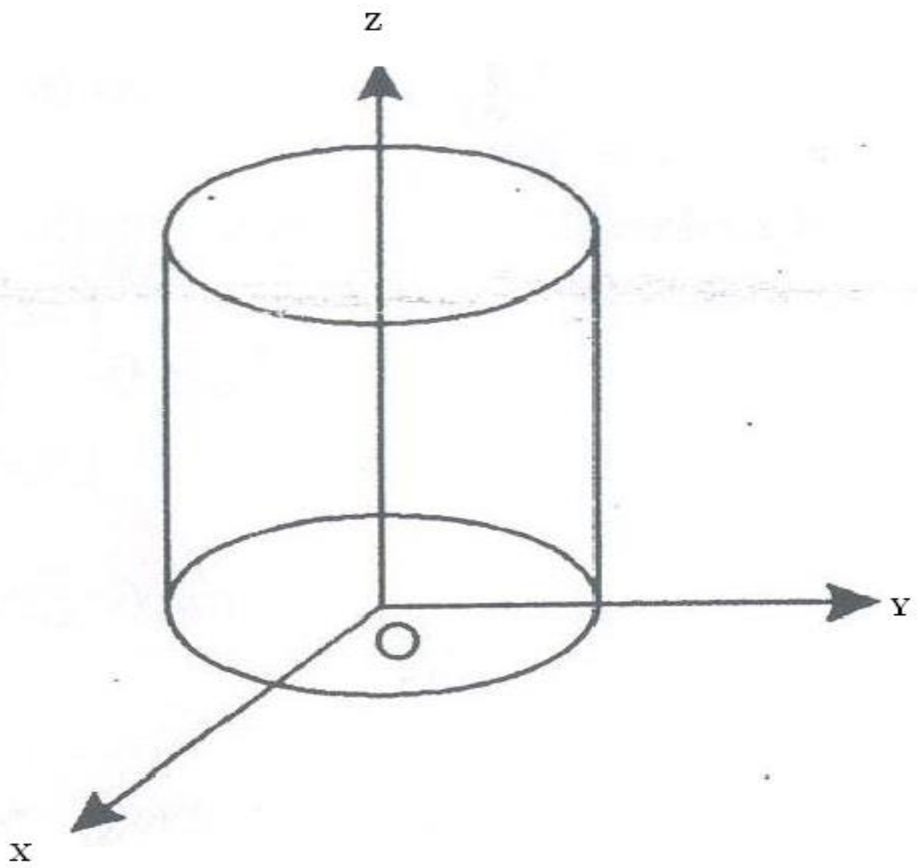
$$\left[SInce\ V = \frac{4\pi}{3}\ is\ the\ volume\ of\ the\ sphere\ of\ unit\ radius\right]$$

3)By transforming into triple integral, evaluate $\int\int \boldsymbol{x^3\ dy\ dz + x^2 y\ dz\ dx + x^2 dx\ dy}$ where S is the closed surface consisting of the cylinder $x^2+y^2 = a^2$ and the circular discs z= 0 , z= b.

Sol: Here $F_1 = x^3, F_2 = x^2y, F_3 = x^2z\ and\ \bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\frac{\partial F_1}{\partial x} = 3x^2, \frac{\partial F_2}{\partial y} = x^2, \frac{\partial F_3}{\partial z} = x^2$$

$$\nabla.\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$



By Gauss Divergence theorem,

$$\int\int F_1 dydz + F_2 dzdx + F_3 dxdy = \int\int\int \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz$$

$$\therefore \int\int_s (x^3 dydz + x^2 ydzdx + x^2 zdxdy = \int\int\int 5x^2 dxdydz$$

$$= 5\int_{-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{b} x^2 dxdydz$$

$$= 20\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \int_{z=0}^{b} x^2 dxdydz \text{ [Integrand is even function]}$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 (z)_0^b dxdy = 20b \int_{x=0}^{a} \int_0^{\sqrt{a^2-x^2}} x^2 dxdy$$

$$= 20b \int_{x=0}^{a} x^2 (y)_0^{\sqrt{a^2-x^2}} dx = 20b \int_0^a x^2 \sqrt{a^2 - x^2}\, dx$$

$$= 20b \int_0^{\frac{\pi}{2}} a^2 \sin^2\theta \sqrt{a^2 - a^2\sin^2\theta}\ (a\cos\theta d\theta)$$

[Put $x = a\sin\theta \Longrightarrow dx = a\cos\theta d\theta$ when $x = a \Rightarrow \theta = \dfrac{\pi}{2}$ and $x = 0 \Rightarrow \theta = 0$]

$$= 20a^4 b \int_0^{\frac{\pi}{2}} \sin^2\theta\ \cos^2\theta d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2\ d\theta\ = 5a^4 b \int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{5a^4 b}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{5a^4 b}{2}\left[\frac{\pi}{2}\right] = \frac{5}{4}\pi a^4 b$$

4: Applying Gauss divergence theorem, Prove that $\int \bar{r}\ .\bar{n}dS = 3V\ or\ \int \bar{r}\ .d\bar{s} = 3V$

Sol: Let $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ we know that div $\bar{r} = 3$

By Gauss divergence theorem, $\int \bar{F}.\bar{n}dS = \int_v div\bar{F}dv$

$$Take\ \bar{F} = \bar{r} \Rightarrow \int_S \bar{r}.\bar{n}dS = \int_V 3\ dV = 3V. Hence\ the\ result$$

5: Show that $\int_S\ (ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}).\bar{n}dS = \frac{4\pi}{3}(a+b+c)$, where S is the surface of the sphere $x^2+y^2+z^2=1$.

Sol: Take $\bar{F} =\ ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}$

$$div\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = a + b + c$$

By Gauss divergence theorem,

$$\int_S\ \bar{F}.\bar{n}dS = \int_V\ \bar{V}.\bar{F}dV = (a+b+c)\int_V\ dV = (a+b+c)V$$

$$We\ have\ V = \frac{4}{3}\pi r^3\ for\ the\ sphere. Here\ r = 1$$

$$\therefore \int_s \bar{F}.\bar{n}dS = (a+b+c)\frac{4\pi}{3}$$

6: Using Divergence theorem, evaluate

$\int\int_S\ (x\ dy\ dz + y\ dz\ dx + z\ dx\ dy), where\ S: x^2+y^2+z^2=a^2$

Sol: We have by Gauss divergence theorem, $\int_s \bar{F}.\bar{n}dS = \int_v div\bar{F}dv$

L.H.S can be written as $\int(F_1 dydz + F_2 dzdx + F_3 dxdy)$ in Cartesian form

Comparing with the given expression, we have $F_1=x$, $F_2=y$, $F_3=z$

Then $div\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3$

$$\therefore \int_v div\overline{F}dv = \int_v 3dv = 3V$$

Here V is the volume of the sphere with radius a.

$$\therefore V = \frac{4}{3}\pi a^3$$

Hence $\int\int (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = 4\pi a^3$

7: Apply divergence theorem to evaluate $\int\int_s (x+z)dydz + (y+z)dzdx + (x+y)dxdy$ S is the surface of the sphere $x^2+y^2+z^2=4$

Sol: Given $\int\int_s (x+z)dydz + (y+z)dzdx + (x+y)dxdy$

Here $F_1 = x+z$, $F_2 = y+z$, $F_3 = x+y$

$\frac{\partial F_1}{\partial x} = 1, \frac{\partial F_2}{\partial y} = 1, \frac{\partial F_3}{\partial z} = 0$ and $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1+1+0 = 2$

By Gauss Divergence theorem,

$$\int\int_s F_1dydz + F_2dzdx + F_3dxdy = \int\int\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz$$

$$= \int\int\int 2dxdydz = 2\int_V dv = 2V$$

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64\pi}{3} \; [for\ the\ sphere, radius = 2]$$

8: Evaluate $\int_S \overline{F}.\overline{n}ds, if\ F = xy\overline{i} + z^2\overline{j} + 2yz\overline{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+y+z=1.

Sol: Given F = $xy\overline{i} + z^2\overline{j} + 2yz\overline{k}$, then div. F = y+2y = 3y

$$\therefore \int_s \overline{F}.\overline{n}dS = \int_v div\overline{F}dv = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 3ydxdydz$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y[z]_0^{1-x-y}dxdy = 3\int_{x=0}^{1}\int_{y=0}^{1-x} y(1-x-y)dx\,dy$$

$$= 3\int_{x=0}^{1}\left[\frac{y^2}{2}-\frac{xy^2}{2}-\frac{y^3}{3}\right]_0^{1-x} dx = 3\int_0^1\left[\frac{(1-x)^2}{2}-\frac{x(1-x)^2}{2}-\frac{(1-x)^3}{3}\right]dx$$

$$= 3\int_0^1\left[\frac{(1-x)^3}{2}-\frac{(1-x)^3}{3}\right]dx = 3\int_0^1\frac{(1-x)^3}{6}dx = \frac{3}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 = \frac{1}{8}$$

9: Use divergence theorem to evaluate $\iint_S \overline{F}.d\overline{S}$ where $\overline{F}$ =$x^3i+y^3j+z^3k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$

Sol: We have

$$\overline{V}.\overline{F} = \frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(y^3)+\frac{\partial}{\partial z}(z^3) = 3(x^2+y^2+z^2)$$

∴By divergence theorem,

$$\bar{V}.\bar{F}\,dV = \int\int_V\int \bar{V}.\bar{F}\,dV = \iiint_v 3(x^2+y^2+z^2)dxdydz$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2\sin\theta\, dr\, d\theta\, d\phi)$$

$[Changing\ into\ spherical\ polar\ coordinates\ x = r\sin\theta\cos\phi\,, y = r\sin\theta\sin\phi\,, z = r\cos\theta]$

$$\int\int_S \bar{F}.dS = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\left[\int_{\phi=0}^{2\pi} d\phi\right]dr\, d\theta$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta(2\pi-0)drd\theta = 6\pi\int_{r=0}^{a} r^4\left[\int_0^{\pi}\sin\theta d\theta\right]dr$$

$$= 6\pi\int_{r=0}^{a} r^4(-\cos\theta)_0^{\pi}\, dr = -6\pi\int_0^a r^4(\cos\pi-\cos 0)\, dr$$

$$= 12\pi\int_0^a r^4 dr = 12\pi\left[\frac{r^5}{5}\right]_0^a = \frac{12\pi a^5}{5}$$

10: Use divergence theorem to evaluate $\int\int_S\ \bar{F}.dS$ where $\bar{F} = 4xi - 2y^2j + z^2k$ and S is the surface bounded by the region $x^2+y^2=4$, z=0 and z=3.

Sol: We have

$$div\overline{F} = \nabla.\overline{F} = \frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial y}(-2y^2)+\frac{\partial}{\partial z}(z^2) = 4-4y+2z$$

By divergence theorm,

$$\int\int_S \bar{F}.dS = \int\int_V\int \bar{V}.\bar{F}dV$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[(4-4y)z+z^2]_0^3\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[12(1-y)+9]\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)dx\,dy$$

$$= \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}21\,dy - 12\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}ydy\right]dx$$

$$= \int_{-2}^{2}\left[21\times 2\int_{0}^{\sqrt{4-x^2}}dy - 12(0)\right]dx$$

[Since the integrans in forst integral is even and in 2nd integral it is on add function]

$$= 42\int_{-2}^{2}(y)_0^{\sqrt{4-x^2}}\,dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}dx = 42\times 2\int_{0}^{2}\sqrt{4-x^2}dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[0+2.\frac{\pi}{2}-0\right] = 84\pi$$

11: Verify divergence theorem for $\bar{F} = x^2 i + y^2 j + z^2 k$ over the surface S of the solid cut off by the plane x+y+z=a in the first octant.

Sol; By Gauss theorem, $\int_s \bar{F}.\bar{n}dS = \int_v div\bar{F}dv$

Let $\phi = x+y+z-a$ *be the given plane then*

$$\frac{\partial \phi}{\partial x}=1, \frac{\partial \phi}{\partial y}=1, \frac{\partial \phi}{\partial z}=1$$

$$\therefore grad\phi = \sum \bar{i}\frac{\partial \phi}{\partial x} = \bar{i}+\bar{j}+\bar{k}$$

$$Unit\ normal = \frac{grad\ \phi}{|grad\ \phi|} = \frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}$$

Let R be the projection of S on xy-plane

Then the equation of the given plane will be x+y=a $\Rightarrow$ y=a-x

Also when y=0, x=a

$$\therefore \int_s \bar{F}.\bar{n}dS = \iint_R \frac{\bar{F}.\bar{n}dxdy}{|\bar{n}.\bar{k}|}$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x} \frac{\frac{x^2+y^2+z^2}{\sqrt{3}}}{1/\sqrt{3}} dx\ dy \quad = \int_{0}^{a}\int_{y=0}^{a-x} [x^2+y^2+(a-x-y)^2]dx\ dy\ [\,since\ x+y+z=a]$$

$$= \int_0^a \int_0^{a-x} [2x^2+2y^2-2ax+2xy-2ay+a^2]dx\ dy$$

$$= \int_{x=0}^{a} \left[2x^2y+\frac{2y^3}{3}+xy^2-2axy-ay^2+a^2y\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} [2x^2(a-x)+\frac{2}{3}(a-x)^3+x(a-x)^2-2ax(a-x)-a(a-x)^2+a^2(a-x)dx$$

$$\therefore \int_s \bar{F}.\bar{n}dS = \int_0^a \left(-\frac{5}{3}x^3+3ax^2-2a^2x+\frac{2}{3}a^3\right)dx = \frac{a^4}{4}, \text{ on simplification…(1)}$$

Given $\bar{F} = x^2\bar{i}+y^2\bar{j}+z^2\bar{k}$

$$\therefore div\ \bar{F} = \frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial z}(z^2) = 2(x+y+z)$$

$$Now \iiint div\bar{F}.dv = 2\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)dxdydz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left[z(x+y)+\frac{z^2}{2}\right]_0^{a-x-y} dx\ dy$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\left[x+y+\frac{a-x-y}{2}\right]dx\ dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)[a+x+y]dx\ dy$$

$$= \int_0^a \int_0^{a-x} [a^2 - (x+y)^2]\ dy\, dx = \int_0^a \int_0^{a-x} (a^2 - x^2 - y^2 - 2xy) dx\, dy$$

$$= \int_0^a [a^2 y - x^2 y - \frac{y^3}{3} - xy^2]_0^{a-x}\ dx$$

$$= \int_0^a (a-x)(2a^2 - x^2 - ax) dx = \frac{a^4}{4} \ldots\ldots (2)$$

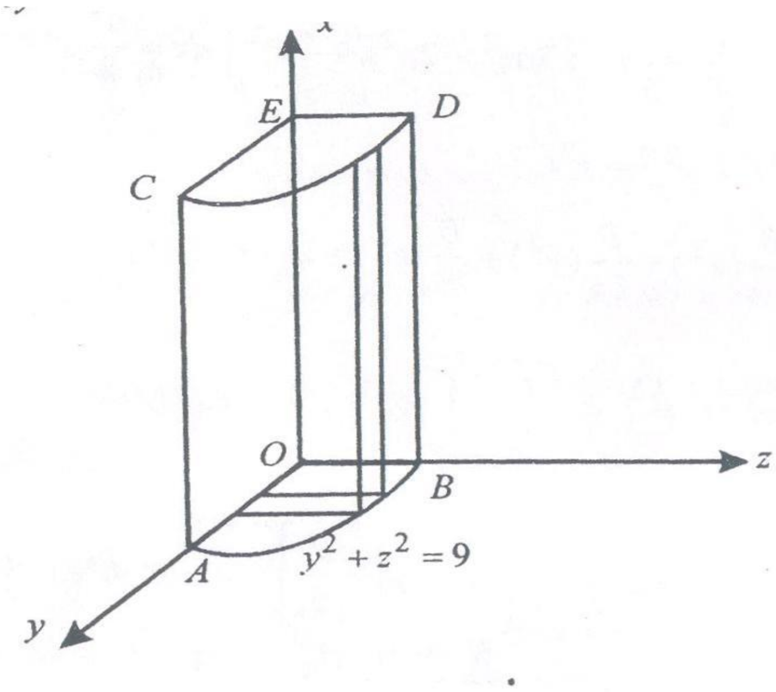
Hence from (1) and (2), the Gauss Divergence theorem is verified.

12: Verify divergence theorem for $2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ taken over the region of first octant of the cylinder $y^2+z^2=9$ and $x=2$.

(or) Evaluate $\iint_S \bar{F}.\bar{n}dS$, where $\bar{F} = 2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2+z^2 = 9$ and the planes $x=0$, $x=2$, $y=0$, $z=0$

Sol: Let $\bar{F} = 2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$

$$\therefore \nabla.\,\bar{F} = \frac{\partial}{\partial x}(2x^2) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2) = 4xy - 2y + 8xz$$



$$\iiint_V \nabla.\bar{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy-2y+8xz)dz\,dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy-2y)z+8x\,\frac{z^2}{2}\right]_{z=0}^{\sqrt{9-y^2}} dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy-2y)\sqrt{9-y^2}+4x(9-y^2)\right]dy\,dx$$

$$= \int_0^2\int_0^3 [(1-2x)(-2y)\sqrt{9-y^2}+4x(9-y^2)]\,dy\,dx$$

$$= \int_0^2 \left\{\left[(1-2x)\frac{(9-y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3 + 4x\left(9y-\frac{y^3}{3}\right)_0^3\right\}dx$$

$$= \int_0^2 \{\frac{2}{3}(1-2x)[0-27]+4x[27-9]\}\,dx = \int_0^2 [-18(1-2x)+72x]dx$$

$$\left[-18(x-x^2)+72\frac{x^2}{2}\right]_0^2 = -18(2-4)+36(4)=36+144=180\ldots(1)$$

$$Now\ we\ sall\ calculate\ \int_S \bar{F}.\bar{n}\ ds\ for\ all\ the\ five\ faces.$$

$$\int_s \bar{F}.\bar{n}dS = \int_{s_1}\bar{F}.\bar{n}dS + \int_{s_2}\bar{F}.\bar{n}dS + \ldots\ldots + \int_{s_5}\bar{F}.\bar{n}dS$$

Where S_1 is the face OAB, S_2 is the face CED, S_3 is the face OBDE, S_4 is the face OACE and S_5 is the curved surface ABDC.

(i) On $S_1 : x=0, \bar{n}=-i \therefore \bar{F}.\bar{n}=0\ Hence \int_{s_1}\bar{F}.\bar{n}dS$

(ii) On $S_2 : x=2, \bar{n}=i \therefore \bar{F}.\bar{n}=8y$

$$\therefore \int_{s_2}\bar{F}.\bar{n}dS = \int_0^3\int_0^{\sqrt{9-z^2}} 8y\,dydz = \int_0^3 8\left(\frac{y^2}{2}\right)_0^{\sqrt{9-z^2}} dz$$

$$= 4\int_0^3 (9-z^2)dz = 4\left(9z - \frac{z^3}{3}\right)_0^3 = 4(27-9) = 72$$

(iii) On $S_3: y = 0, \bar{n} = -j. \therefore \bar{F}.\bar{n} = 0$ *Hence* $\int_{S_3} \bar{F}.\bar{n}dS$

$(iv) On\ S_4: z = 0, \bar{n} = -k. \quad \bar{F}.\bar{n} = 0. \quad Hence \int_{S_4} \bar{F}.\bar{n}ds = 0$

(v) On $S_5: y^2 + z^2 = 9, \bar{n} = \frac{\nabla(y^2+z^2)}{|\nabla(y^2+z^2)|} = \frac{2y\bar{j} + 2z\bar{k}}{\sqrt{4y^2+4z^2}} = \frac{y\bar{j}+z\bar{k}}{\sqrt{4\times 9}} = \frac{y\bar{j}+z\bar{k}}{3}$

$\bar{F}.\bar{n} = \frac{-y^3 + 4xz^3}{3}$ and $\bar{n}.\bar{k} = \frac{z}{3} = \frac{1}{3}\sqrt{9-y^2}$

Hence $\int_{S_5} \bar{F}.\bar{n}ds = \int\int_R \bar{F}.\bar{n}\frac{dx\,dy}{|\bar{n}.\bar{k}|}$ *Where R is the projection of* S_5 *on* $xy-plane.$

$$= \int_R\int \frac{4xz^3 - y^3}{\sqrt{9-y^2}}dx\,dy = \int_{x=0}^{2}\int_{y=0}^{3}\left[4x(9-y^2) - y^3\left(9-y^2\right)^{-\frac{1}{2}}\right]dy\,dx$$

$$= \int_0^2 72x\,dx - 18\int_0^2 dx = 72\left(\frac{x^2}{2}\right)_0^2 - 18(x)_0^2 = 144 - 36 = 108$$

Thus $\int_S \bar{F}.\bar{n}ds = 0 + 72 + 0 + 0 + 108 = 180 \ldots\ldots (2)$

Hence the Divergence theorem is verified from the equality of (1) and (2).

13: Use Divergence theorem to evaluate $\int\int\left(x\bar{i} + y\bar{j} + z^2\bar{k}\right).\bar{n}ds.$ Where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane z = 4.

Sol: Given $\int\int(x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}).\bar{n}.ds$ *Where* S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane z = 4.

Let $\bar{F} = x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}$

By Gauss Divergence theorem,we have

$$\int\int(x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}).\bar{n}.ds = \int\int_V\int \bar{\nabla}.\bar{F}\,dv$$

$$Now\ \nabla.\bar{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2) = 1+1+2z = 2(1+z)$$

On the cone, $x^2+y^2=z^2$ and z=4 $\Rightarrow x^2+y^2=16$

The limits are $z=0$ *to* $4, y=o$ *to* $\sqrt{16-x^2}, x=0$ *to* $4.$

$$\iiint_V \bar{\nabla}.\bar{F}\,dv = \int_0^4\int_0^{\sqrt{16-x^2}}\int_0^4 2(1+z)dx\,dy\,dz$$

$$=2\int_0^4\int_0^{\sqrt{16-x^2}}\left\{[z]_0^4+\left[\frac{z^2}{2}\right]_0^4\right\}dx\,dy$$

$$=2\int_0^4\int_0^{\sqrt{16-x^2}}[4+8]dxdy=2\times 12\int_0^4[y]_0^{\sqrt{16-x^2}}dx$$

$$=24\int_0^4\sqrt{16-x^2}\,dx=24\int_0^{\frac{\pi}{2}}\sqrt{16-16\sin^2\theta}\,.4\cos\theta d\theta$$

$[put x=4\sin\theta\Rightarrow dx=4\cos\theta d\theta.\ Also\ x=0\Rightarrow\theta=0\ and\ \ x=4\Rightarrow\theta=\frac{\pi}{2}]$

$$\therefore\iiint_V\nabla.\bar{F}dv=96\times 4\int_0^{\frac{\pi}{2}}4\sqrt{1-\sin^2\theta}\cos\theta d\theta=96\times 4\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta$$

$$\iiint_V\bar{\nabla}.\bar{F}\,dv=96\ X\ 4\int_0^{\frac{\pi}{2}}4\sqrt{1-\sin^2\theta}\ \cos\theta d\theta=96\ X\ 4\int_0^{\frac{\pi}{2}}\cos^2\theta\ d\theta$$

$$=96\ X\ 4\int_0^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}d\theta=96X4\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}+\frac{\cos 2\theta}{2}\right]d\theta$$

$$=384\left[\frac{1}{2}\theta+\frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=96\pi$$

14: Use Gauss Divergence theorem to evaluate $\int\int_S\left(yz^2\bar{\imath}+zx^2\bar{\jmath}+2z^2\bar{k}\right).ds, where$ S is the closed surface bounded by the xy-plane and the upper half of the sphere $x^2+y^2+z^2=a^2$

above this plane.

Sol: Divergence theorem states that

$$\iint_S\bar{F}.ds=\iiint_V\bar{\nabla}.\bar{F}\,dv$$

Here $\nabla.\overline{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(2z^2) = 4z$

$$\therefore \int\int_s \overline{F}.ds = \int\int\int_V 4z dxdydz$$

Introducing spherical polar coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi,$

$z = r\cos\theta$ then $dxdydz = r^2 drd\theta d\phi$

$$\therefore \int\int_s \overline{F}.ds = 4\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\cos\theta)(r^2\sin\theta drd\theta d\phi)$$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3 \sin\theta\cos\theta \left[\int_{\phi=0}^{2\pi} d\phi\right] dr\, d\theta$$

$$= 4.\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3 \sin\theta\cos\theta\ (2\pi - 0) dr\, d\theta$$

$$= 4\pi\int_{r=0}^{a} r^3 \left[\int_0^{\pi} \sin 2\theta\, d\theta\right] dr = 4\pi\int_{r=0}^{a} r^3 \left(-\frac{\cos 2\theta}{2}\right)_0^{\pi} dr$$

$$= (-2\pi)\int_0^a r^3(1-1)dr = 0$$

15: Verify Gauss divergence theorem for $\overline{\boldsymbol{F}} = \boldsymbol{x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}}$ taken over the cube bounded by

x = 0, x = a, y= 0, y = a, z = 0, z = a.

Sol: We have $\bar{F} = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$

$$\nabla.\overline{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

$$\int\int_V\int \bar{\nabla}.\bar{F}\, dv = \int\int_V\int (3x^2 + 3y^2 + 3z^2) dx\, dy\, dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a}(x^2 + y^2 + z^2) dx\, dy\, dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{x^3}{3} + xy^2 + z^2x\right)_0^a dy\, dz$$

$$= 3 \int_{z=0}^{a} \int_{y=0}^{a} \left(\frac{a^3}{a} + ay^2 + az^2 \right) dy\, dz$$

$$= 3 \int_{z=0}^{a} \left(\frac{a^3}{3} y + a\frac{y^3}{3} + az^2 y \right)_0^a dz$$

$$= 3 \int_{0}^{a} \left(\frac{a^4}{3} + \frac{a^4}{3} + a^2 z^2 \right) dz = 3 \int_{0}^{a} \left(\frac{2}{3} a^4 + a^2 z^2 \right) dz$$

$$= 3 \left(\frac{2}{3} a^4 z + a^2 . \frac{z^3}{3} \right)_0^a = 3 \left(\frac{2}{3} a^5 + \frac{1}{3} a^5 \right)$$

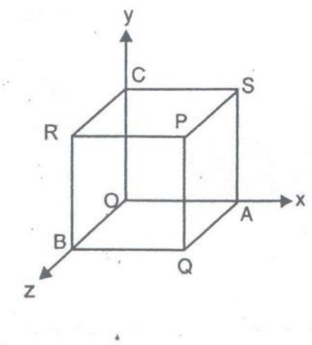
$$= 3a^5$$

To evaluate the surface integral divide the closed surface S of the cube into 6 parts.

i.e., S_1 : The face DEFA ; S_4 : The face OBDC

S_2 : The face AGCO ; S_5 : The face GCDE

S_3 : The face AGEF ; S_6: The face AFBO



$$\int_S \int \bar{F}.\bar{n} ds = \int_{S_1} \int \bar{F}.\bar{n} ds + \int_{S_2} \int \bar{F}.\bar{n} ds + \cdots + \int_{S_6} \int \bar{F}.\bar{n} ds$$

$On\ S_1, we\ have\ \bar{n} = \bar{\imath}, x = a$

$$\therefore \iint_{S_1} \bar{F}.\bar{n} ds = \int_{z=0}^{a} \int_{y=0}^{a} \left(a^3 \bar{i} + y^3 \bar{j} + z^3 \bar{k} \right) . \bar{i} dydz$$

$$\int_{S_1} \int \bar{F}.\bar{n} ds = \int_{z=0}^{a} \int_{y=0}^{a} \left(a^3 \bar{\imath} + y^3 \bar{\jmath} + z^3 \bar{k} \right) . \bar{\imath}\, dy\, dz$$

$$= \int_{z=0}^{a} \int_{y=0}^{a} a^3 dy\, dz = a^3 \int_{0}^{a} (y)_0^a\, dz$$

$$= a^4 (z)_0^a = a^5$$

$On\ S_2, we\ have\ \bar{n} = -\bar{\imath}, x = 0$

$$\iint_{S_2} \bar{F}.\bar{n} ds = \int_{z=0}^{a} \int_{y=0}^{a} \left(y^3 \bar{j} + z^3 \bar{k} \right) . \left(-\bar{i} \right) dydz = 0$$

$On\ S_3, we\ have\ \bar{n} = \bar{\jmath}, y = a$

$$\iint_{S_3} \bar{F}.\bar{n} ds = \int_{z=0}^{a} \int_{x=0}^{a} \left(x^3 \bar{i} + a^3 \bar{j} + z^3 \bar{k} \right) . \bar{j} dxdz = a^3 \int_{z=0}^{a} \int_{x=0}^{a} dxdz = a^3 \int_0^a adz = a^4 \left(z \right)_0^a$$

$$= a^5$$

$On\ S_4, we\ have\ \bar{n} = -\bar{j}, y = 0$

$$\int_{S_4}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a}(x^3\bar{i} + z^3\bar{k}).(-\bar{j})dx\,dz = 0$$

$On\ S_5, we\ have\ \bar{n} = \bar{k}, z = a$

$$\int_{S_5}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a}(x^3\bar{i} + y^3\bar{j} + a^3\bar{k}).\bar{k}\,dx\,dy$$

$$= \int_{y=0}^{a}\int_{x=0}^{a} a^3 dx\,dy = a^3\int_0^a (x)_0^a dy = a^4(y)_0^a = a^5$$

$On\ S_6, we\ have\ \bar{n} = -\bar{k}, z = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a}(x^3\bar{i} + y^3\bar{j}).(-\bar{k})dx\,dy = 0$$

$$Thus \int_S\int \bar{F}.\bar{n}ds = a^5 + 0 + a^5 + 0 + a^5 + 0 = 3a^5$$

$$Hence \int_S\int \bar{F}.\bar{n}ds = \int_V\int \bar{\nabla}.\bar{F}\,dv$$

$\therefore$ *The* Gauss divergence theorem is verified.

Assignment

1. Evaluate $\iint_s xdy\,dz + y\,dz\,dx + zdxdy$ over $x^2 + y^2 + z^2$=1

2. Compute $\iint (a^2x^2 + b^2y^2 + c^2z^2)^{\frac{1}{2}}\,dS$ over the ellipsoid $ax^2 + by^2 + cz^2 = 1$

(Hint: Volume of the ellipsoid , V= $\frac{4\pi}{3\sqrt{abc}}$)

3. Find $\int_s \bar{F}.\bar{n}dS$ where $\bar{F}$ =2 $x^2\bar{i} - y^2\bar{j}$+ 4xz$\bar{k}$ and S is the region in the first octant bounded by $y^2 + z^2$=9 and x=0,x=2.

4. Find $\int_s (4.x\bar{i} - 2y^2\bar{j} + z^2k).\bar{n}\,dS$ Where S Is the region bounded by $x^2 + y^2$ =4, z=0 and z=3.

5. Verify divergence theorem for F=6z$\bar{i}$ + (2x+y) $\bar{j}$ -x$\bar{k}$, taken over the region bounded by the surface of the cylinder $x^2 + y^2$ =9 included in z=0, z=8, x=0 and y=0. [JNTU 2007 S(Set No.2)]

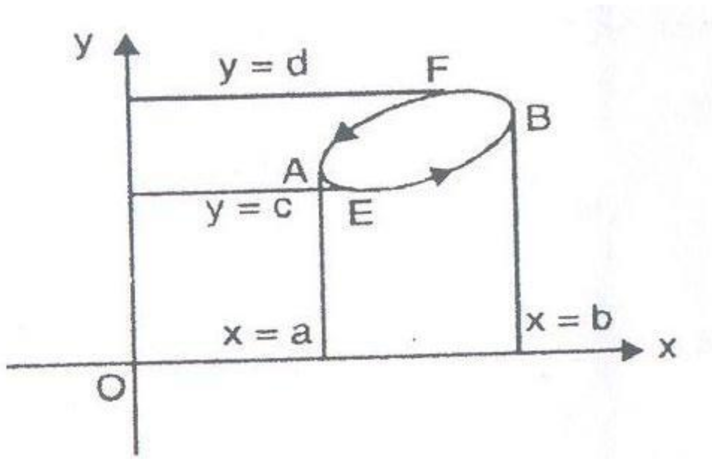
II. GREEN'S THEOREM IN A PLANE

(Transformation Between Line Integral and Surface Integral) [JNTU 2001S].

If S is Closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

Where C is traversed in the positive(anti clock-wise) direction

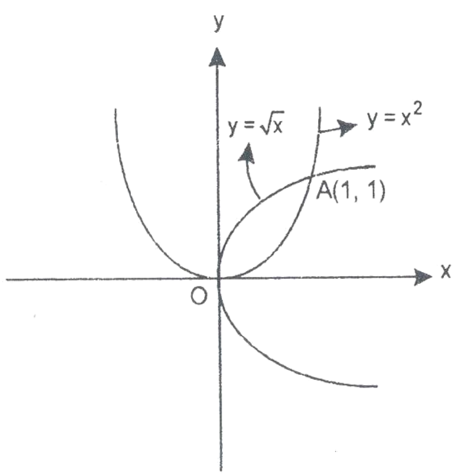


SOLVED PROBLEMS

1: Verify Green's theorem in plane for $\oint (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by y=$\sqrt{x}$ and y=x^2 .

Solution: Let M=3x^2-8y^2 and N=4y-6xy. Then

$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$



We have by Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R (16y - 6y) dxdy$

$$=10 \iint_R ydxdy = 10 \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} ydydx = 10 \int_{x=0}^{1} \left(\frac{y^2}{2} \right)_{x^2}^{\sqrt{x}} dx$$

$$=5 \int_0^1 (x - x^4)dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$$

….(1)

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2=x(from A to O)]

=I_1+I_2(say)

Now $I_1 = \int_{x=0}^{1} \{[3x^2 - 8(x^2)^2]dx + [4x^2 - 6x(x^2)]2xdx\}$ $\left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$

$= \int_0^1 (3x^3 + 8x^3 - 20x^4)dx = -1$

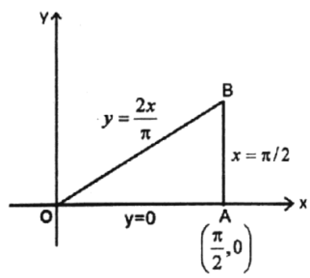
And $I_2 = \int_1^0 \left[(3x^2 - 8x)dx + \left(4\sqrt{x} - 6x^{3/2}\right) \frac{1}{2\sqrt{x}} dx \right] = \int_1^0 (3x^2 - 11x + 2)dx = \frac{5}{2}$

$\therefore I_1 + I_2 = -1 + 5/2 = 3/2.$

From(1) and (2), we have $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$

Hence the verification of the Green's theorem.

2: Evaluate by Green's theorem $\int_c (y - \sin x)dx + \cos x \, dy$ where C is the triangle enclosed by the lines y=0, x=$\frac{\pi}{2}$, $\pi y = 2x$.



Solution : Let M=y-$\sin x$ $and\ N = \cos x$ Then

$\frac{\partial M}{\partial y}$=1 and $\frac{\partial N}{\partial x}$=-$\sin x$

$\therefore$ By Green's theorem $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$

$$\Rightarrow \int_c (y-\sin x)dx+\cos x dy = \iint_R (-1-\sin x)dxdy$$

$$=-\int_{x=0}^{\pi/2}\int_{y=0}^{\frac{2x}{\pi}}(1+\sin x)\,dxdy$$

$$= - \int_{x=0}^{\pi/2}(\sin x+1)\,[y]_0^{2x/\pi}\,dx$$

$$=\frac{-2}{\pi}\int_{x=0}^{\pi/2} x(\sin x+1)dx$$

$$=\frac{-2}{\pi}\left[x(-\cos x+x)\right]_0^{\pi}-\int_0^{\pi/2}1(-\cos x+x)dx$$

$$=\frac{-2}{\pi}\left[x(-\cos x+x)+\sin x-\frac{x^2}{2}\right]_0^{\pi/2}$$

$$=$$

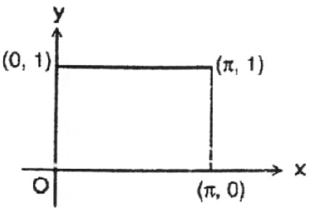
$$\frac{-2}{\pi}\left[-x\cos x+\frac{x^2}{2}+\sin x\right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8}+1\right] = -\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$$

3: Evaluate by Green's theorem for $\oint_c (x^2-\cosh y)dx+(y+\sin x)dy$ where C is the rectangle with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.

Solution: Let M=$x^2-\cosh y$, $N = y+\sin x$

$$\therefore \frac{\partial M}{\partial y} = -\sinh y \;\; and \;\; \frac{\partial N}{\partial x} = \cos x$$

By Green's theorem, $\oint_C Mdx+Ndy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy.$



$$\Rightarrow \oint_c (x^2-\cosh y)dx+(y+\sin x)dy = \int\int_R (\cos x+\sinh y)dxdy$$

$$\Rightarrow \oint_c (x^2-\cosh y)dx+(y+\sin x)dy = \int_s\int(\cos x+\sinh$$

$$=\int_{x=0}^{\pi}\int_{y=0}^{1}(\cos x+\sinh y)dydx = \int_{x=0}^{\pi}(y\cos x+\cosh y)_0^1\,dx$$

$$=\int_{x=0}^{\pi}(\cos x+\cosh 1-1)dx$$

$$=\pi(\cosh 1-1)$$

4: A Vector field is given by $\overline{F} = (\sin y)\overline{i}+x(1+\cos y)\overline{j}$

Evaluate the line integral over the circular path $x^2+y^2 = a^2$, z=0

(i) Directly (ii) By using Green's theorem

Solution : (i) Using the line integral

$$\oint_c \overline{F}.d\overline{r} = \oint_c F_1 dx + F_2 dy = \oint_c \sin y dx + x(1+\cos y)dy$$

$$= \oint_c \sin y dx + x\cos y dy + x dy = \oint_c d(x\sin y) + x dy$$

Given Circle is $x^2 + y^2 = a^2$. Take x=a $\cos\theta$ and y=a $\sin\theta$ so that dx=-a $\sin\theta\ d\theta$ and dy=a $\cos\theta$ dθ and $\theta = 0 \to 2\pi$

$$\therefore\ \oint \overline{F}.d\overline{r} = \int_0^{2\pi} d[a\ \cos\theta \sin(a\ \sin\theta)] + \int_0^{2\pi} a(\cos\theta)a\ \cos\theta\, d\theta$$

$$=[a\ \cos\theta \sin(a\ \sin\theta)]_0^{2x} + 4a^2 \int_0^{\pi/2} cos^2\theta\, d\theta$$

$$=0 + 4a^2.\frac{1}{2}.\frac{\pi}{2} = \pi a^2$$

(ii)Using Green's theorem

Let M=$\sin y$ and N=x$(1+\cos y)$. Then

$\frac{\partial M}{\partial y}$=$\cos y$ and $\frac{\partial N}{\partial x}$=$(1+\cos y)$

By Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$\therefore \oint_c \sin y dx + x(1+\cos y)dy = \iint_R (-\cos y + 1 + \cos y)dxdy == \iint dxdy$$

$$= \iint_R dA = A = \pi a^2 (\because area\ \text{of circle} = \pi a^2)$$

We observe that the values obtained in (i) and (ii) are same to that Green's theorem is verified.

5: Show that area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$ and hence find the area of

(i)The ellipse x=$a\cos\theta, y = bsin\theta$ $(i.e)\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii)The Circle x=a$\cos\theta$, $y = a\sin\theta$ $(i.e) x^2 + y^2 = a^2$

<u>**Solution:**</u> We have by Green's theorem $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here M=-y and N=x so that $\frac{\partial M}{\partial y} = -1\ and \frac{\partial N}{\partial x} = 1$

$\oint_c xdy - ydx = 2\int_R dxdy = 2A$ where A is the area of the surface.

$\therefore \frac{1}{2}\int xdy - ydx = A$

(i)For the ellipse x=acosθ and y=bsinθ and $\theta = 0 \to 2\pi$

$\therefore Area, A = \frac{1}{2}\oint xdy - ydx = \frac{1}{2}\int_0^{2\pi}[(a\ \cos\theta)(b\cos\theta) - (b\ \sin\theta\ (-a\ \sin\theta))]d\ \theta$

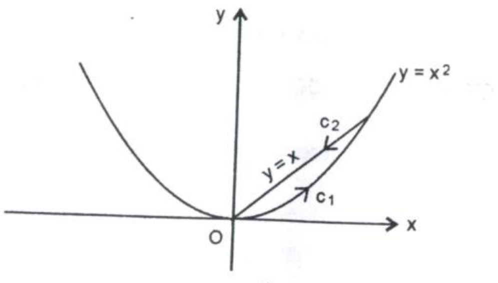
$$= \frac{1}{2}ab\int_0^{2\pi}(\cos^2\theta + \sin^2\theta)\, d\theta = \frac{1}{2}ab(\theta)_0^{2\pi} = \frac{ab}{2}(2\pi - 0) = \pi ab$$

(ii)Put a=b to get area of the circle A=πa^2

6: Verify Green's theorem for $\int_c[(xy + y^2)dx + x^2dy]$, where C is bounded by y=x and y=x^2

Solution: By Green's theorem, we have $\oint_C Mdx + Ndy = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here M=xy +y^2 and N=x^2



The line y=x and the parabola y=x^2 intersect at O$(0,0)$ and A$(1,1)$

Now $\oint_c Mdx + Ndy = \int_{c_1} Mdx + Ndy + \int_{c_2} Mdx + Ndy$......(1) .....(1)

Along C_1 $(i.e. y = x^2)$, the line integral is

$$\int_{c_1} Mdx + Ndy = \int_{c_1}[x(x^2) + x^4]dx + x^2d(x^2)\int_c(x^3 + x^4 + 2x^3)dx = \int_0^1(3x^3 + x^4)dx$$

$$= \left(3.\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \qquad ....(2)$$

Along C_2 $(i.e. y = x)$ from $(1,1)$ to $(0,0)$, the line integral is

$$\int_{c_2} Mdx + Ndy = \int_{c_2}(x.x + x^2)dx + x^2dx\ [\because dy = dx]$$

$$= \int_{c_2} 3x^2dx = 3\int_1^0 x^2dx = 3\left(\frac{x^3}{3}\right)_1^0 = (x^3)_1^0 = 0-1=-1 \qquad ....(3)$$

From (1), (2) and (3), we have

$\int_c Mdx + Ndy = \frac{19}{20} - 1 = \frac{-1}{20}$

…(4)

Now

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (2x - x - 2y)dxdy$$

$$= \int_0^1 [(x^2 - x^2) - (x^3 - x^4)]dx = \int_0^1 (x^4 - x^3)dx$$

$$= \left(\frac{x^5}{5} + \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

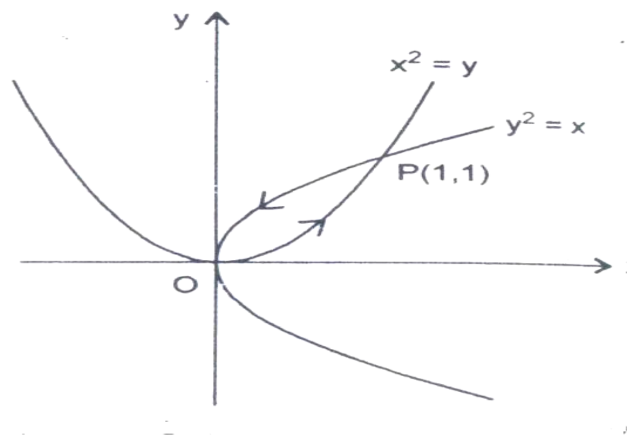
….(5)

From(4) and (5), $We\ have$ $\oint_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$dxdy

Hence the verification of the Green's theorem.

7: Using Green's theorem evaluate $\int_c (2xy - x^2)dx + (x^2 + y^2)dy$, Where "C" is the closed curve of the region bounded by $y=x^2$ and $y^2 = x$

Solution:



The two parabolas $y^2 = x$ and $y = x^2$ are intersecting at O$(0,0)$, and P(1,1)

Here M=2xy-x^2 and N=x^2 +y^2

$\therefore \frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$

Hence $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2x = 0$

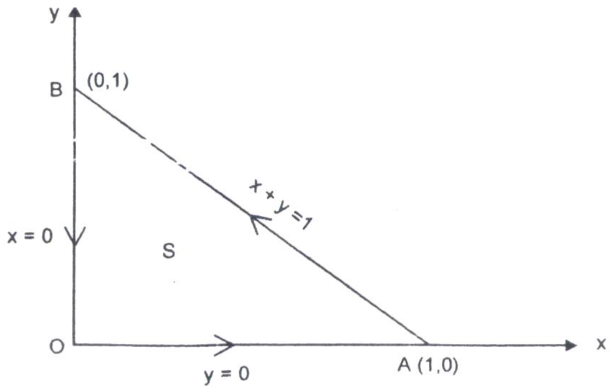
By Green's theorem $\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$dxdy i.e., $\int_c (2xy - x^2)dx + (x^2 + y^2)dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (0)dxdy = 0$

8: Verify Green's theorem for $\int_c [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where c is the region bounded by x=0, y=0 and x+y=1.

Solution : By Green's theorem, we have

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Here M=$3x^2 - 8y^2$ and N=4y-6xy



$$\therefore \frac{\partial M}{\partial y} = -16y \text{ and } \frac{\partial N}{\partial x} = -6y$$

Now $\int_c Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy$...(1)

Along OA, y=0 $\quad \therefore dy = 0$

$$\int_{OA} Mdx + Ndy = \int_0^1 3x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = 1$$

Along AB, x+y=1 $\therefore dy = -dx$ and x=1-y and y varies from 0 to 1.

$$\int_{AB} Mdx + Ndy = \int_0^1 [3(y-1)^2 - 8y^2](-dy) + [4y + 6y(y-1)]dy$$

$$= \int_0^1 (-5y^2 - 6y + 3)(-dy) + (6y^2 - 2y)dy$$

$$= \int_0^1 (11y^2 + 4y - 3)dy = \left(11\frac{y^3}{3} + 4\frac{y^2}{2} - 3y\right)_0^1$$

$$= \frac{11}{3} + 2 - 3 = \frac{8}{3}$$

Along BO, x=0 $\therefore dx = 0$ and limits of y are from 1 to 0

. $\int_{BO} Mdx + Ndy = \int_1^0 4ydy = \left(4\frac{y^2}{2}\right)_1^0 = (2y^2)_0^1 = -2$

from (1), we have $\int_c Mdx + Ndy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{x=0}^{1}\int_{y=0}^{1-x} (-6y+16y)dxdy$

$$=10\int_{x=0}^{1} \left[\int_{y=0}^{1-x} ydy\right] dx = 10\int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$=5\int_0^1 (1-x)^2\, dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$=-\frac{5}{3}[(1-1)^3 - (1-0)^3] = \frac{5}{3}$$

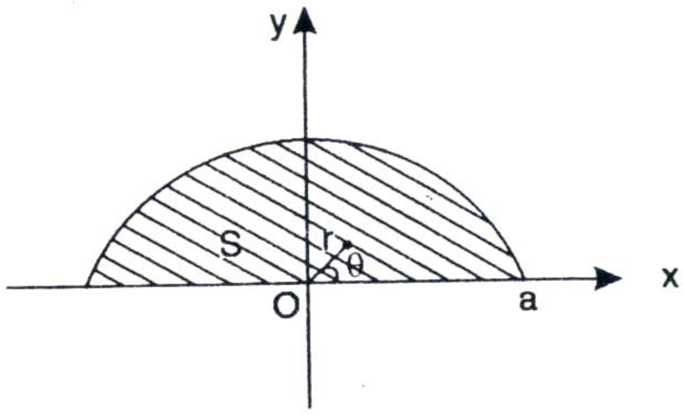
From (2) and (3), we have $\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the verification of the Green's Theorem.

9: Apply Green's theorem to evaluate $\oint_c (2x^2 - y^2)dx + (x^2 + y^2)\, dy, where\ c\ is$ the boundary of the area enclosed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$

Solution : Let M=$2x^2 - y^2$ and N=$x^2 + y^2$ Then

$\frac{\partial M}{\partial y} = -2y\ and\ \frac{\partial N}{\partial x} = 2x$



Figure

$\therefore$ *By Green's Theorem,* $\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$\oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = \iint_R (2x + 2y)dxdy$

$=2\iint_R (x+y)dy$

$$=2\int_0^a \int_0^\pi r(\cos\theta+\sin\theta).rd\theta dr$$

[Changing to polar coordinates (r,θ), r varies from 0 to a and θ varies from 0 to π]

$$\therefore \oint_c [(2x^2-y^2)dx+(x^2+y^2)dy]=2\int_0^a r^2dr\int_0^\pi(\cos\theta+\sin\theta)d\theta$$

$$=2.\frac{a^3}{3}(1+1)=\frac{4a^3}{3}$$

10: Find the area of the Folium of Descartes $x^3+y^3=3axy(a>0) using\ Green's$ Theorem.

Solution: from Green's theorem, we have

$$\int Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

By Green's theorem, Area = $\frac{1}{2}\oint(xdy-ydx)$

Considering the loop of folium Descartes(a>0)

Let x= $\frac{3at}{1+t^3}, y=\frac{3at^2}{1+t^3}, Then\ dx=\left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right]dt$ and $dy=\left[\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right]dt$

The point of intersection of the loop is $\left(\frac{3a}{2},\frac{3a}{2}\right)\Rightarrow t=1$

Along OA, t varies from 0 to1.

$$\therefore \frac{1}{2}\oint(xdy-ydx)=\frac{1}{2}\int_0^1\left(\frac{3at}{1+t^3}\right)\left[\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right]dt-\left(\frac{3at^2}{1+t^3}\right)\left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right]dt$$

$$=\frac{1}{2}\int_0^1\left\{\frac{3at}{1+t^3}\left[\frac{3at(2-t^3)}{(1+t^3)^2}\right]-\frac{3at^2}{1+t^3}\left[\frac{3a(1-2t^3)}{(1+t^3)^2}\right]\right\}dt$$

$$=\frac{9a^2}{2}\int_0^1\left[\frac{t^2(2-t^3)}{(1+t^3)^3}-\frac{t^2(1-2t^3)}{(1+t^3)^3}\right]dt=\frac{9a^2}{2}\int_0^1\frac{2t^2-t^5-t^2+2t^5}{(1+t^3)^3}dt$$

$$=\frac{9a^2}{2}\int_0^1\frac{t^2+t^5}{(1+t^3)^3}dt=\frac{9a^2}{2}\int_0^1\frac{t^2(1+t^3)}{(1+t^3)^3}dt$$

$=\frac{9a^2}{2}\int_0^1\frac{t^2}{(1+t^3)^2}dt$ [Put $1+t^3=x\Rightarrow 3t^2\,dt=dx$

L.L. : x=1, U.L.:x=2]

$$=\frac{9a^2}{2}\int_1^2\frac{t^2}{x^2}.\frac{dx}{3t^2}=\frac{9a^2}{6}\int_1^2\frac{1}{x^2}dx=\frac{3a^2}{4}sq.\text{units(a>0).}$$

11: Verify Green's theorem in the plane for $\int_c(x^2-xy^3)\,dx+(y^2-2xy)dy$

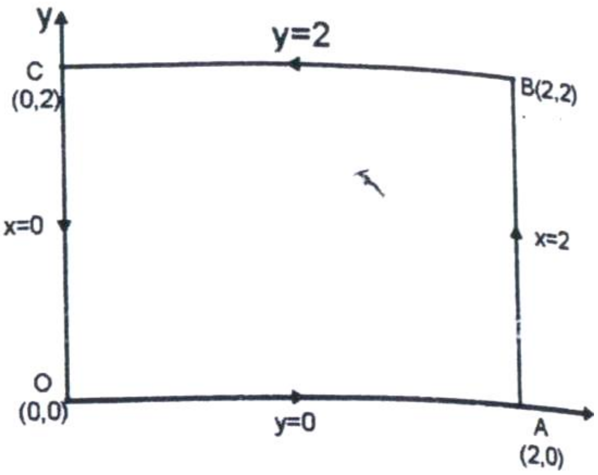
Where C is square with vertices (0,0), (2,0), (2,2), (0,2).

Solution: The Cartesian form of Green's theorem in the plane is

$$\int_c M\,dx + N\,dy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Here M=$x^2 - xy^3$ and N=$y^2 - 2xy$

$\therefore\ \frac{\partial M}{\partial y}$ = - 3 xy^2 and $\frac{\partial N}{\partial x} = -2y$



Evaluation of $\int_C (\boldsymbol{Mdx + Ndy})$

To Evaluate $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$, we shall take C in four different segments viz (i) along OA(y=0) (ii) along AB(x=2) (iii) along BC(y=2) (iv) along CO(x=0).

(i)Along OA(y=0)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

…..(1)

(ii)Along AB(x=2)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 (y^2 - 4y)\,dy \quad [\because x = 2, dx = 0]$$

$$= \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \left(\frac{8}{3} - 8\right) = 8\left(-\frac{2}{3}\right) = -\frac{16}{3}$$

….(2)

(iii)Along BC(y=2)

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 (x^2 - 8x)\,dx \quad [\because y = 2, dy = 0]$$

$$= \left(\frac{x^3}{3} - 4x^2\right)_0^2 = -\left(\frac{8}{3} - 16\right) = \frac{40}{3} \text{......(3)}$$

(iv)Along CO(x=0)

$\int_c (x^2 - xy^3)\, dx + (y^2 - 2xy) dy = \int_2^0 y^2\, dx \quad [\because x = 0, dx = 0] = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3}$

…..(4)

Adding(1),(2),(3) and (4), we get

$$\int_c \left(x^2 - xy^3\right)dx + \left(y^2 - 2xy\right)dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8 \qquad \ldots(5)$$

Evaluation of $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here x ranges from 0 to 2 and y ranges from 0 to 2.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^2 \int_0^2 (-2y + 3xy^2) dxdy$$

$$= \int_0^2 \left(-2xy + \frac{3x^2}{2} y^2\right)_0^2 dy$$

$$= \int_0^2 (-4y + 6y^2) dy = \left(-2y^2 + 2y^3\right)_0^2$$

$$= -8 + 16 = 8 \qquad \ldots(6)$$

From (5) and (6), we have

$$\int_c M\,dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Hence the Green's theorem is verified.

Assignments

(1) Evaluate $\oint_c (3x + 4y) dx + (2x - 3y)\, dy$ where c is the circle $x^2 + y^2 = 4$

(2) Verify Green's theorem in the plane for $\oint (x^2 - xy^3) dx + (y^2 - 2xy) dy$ where c is the square with vertices (0,0), (2,0), (2,2) and (0,2).

(3) Use Green's theorem to evaluate $\oint_c x^2(1 + y) dx + (y^3 + x^3)\, dy$ where c is the square bounded by y=± 1 and $x = \pm 1$.

(4) Find the area bounded by one arc of the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta), a > 0$ and the $x - axis$.

(5) Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}, a > 0$.

(6) Find $\oint (x^2 + y^2) dx + 3xy^2 dy$ where c is the circle $x^2 + y^2 = 4$ in xy plane.

Answers

(1)-8π (3)$\frac{8}{3}$ (4)3πa^2 (5)$\frac{3\pi a^2}{8}$ (6)12π

III. STOKE'S THEOREM

(Transformation between Line Integral and Surface Integral) [JNTU 2000]

Let S be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentieable vector point function then $\oint_C \bar{F}.d\,\bar{r}=$ $\int_S curl\ \ \bar{F}.\bar{n}\ ds\ where\ c\ is\ traversed\ in\ the\ positive$ direction and $\bar{n}\ is\ unit\ outward\ drawn\ normal\ at\ any\ point\ of\ the\ surface.$

PROBLEMS:

1: Prove by Stokes theorem, Curl grad $\phi=\bar{0}$

Solution: Let S be the surface enclosed by a simple closed curve C.

$\therefore\ By\ Stokes\ theorem$

$$\int_s (curl\ grand\phi).\bar{n}\ ds = \int_s (\nabla x \nabla\phi).\bar{n}\ dS = \oint_C \nabla\phi.d\bar{r} = \oint_C \nabla\phi.d\bar{r}$$

$$= \oint_c \left(\frac{\bar{i}\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} \right).\left(\bar{i}dx + \bar{j}dy + \bar{k}dz \right)$$

$$= \oint_c \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \right) = \int d\phi = [\phi]_p$$ where P is any point on C.

$$\therefore \int curl\ grad\,\phi.\bar{n}\ ds = \bar{0} \Rightarrow curl\ grad\phi = \bar{0}$$

2: prove that $\int_s \phi curl\ \ \bar{f}.dS = \int_c \phi\bar{f}.d\bar{r} - \int_s curl\, grad\,\phi \times \bar{f}dS$

Solution: Applying Stokes theorem to the function $\phi\ \bar{f}$

$$\int_c \phi\bar{f}.d\bar{r} = \int curl\left(\phi\bar{f}\right).\bar{n}ds = \int_s \left(grad\phi \times \bar{f} + \phi curl\,\bar{f} \right)ds$$

$$\therefore \int_c \phi curl\,\bar{f}.ds = \int_c \phi\bar{f}.d\bar{r} - \int \nabla\phi \times \bar{f}.ds$$

3: Prove that $\oint_c f\nabla f.d\bar{r} = 0.$

Solution: By Stokes Theorem,

$$\oint_c (f\nabla f).d\bar{r} = \int_s curl f\nabla \bar{f}.\bar{n}\ ds = \int_s [f curl\nabla f + \nabla f \times \nabla f].\bar{n}\ ds$$

$$= \int \bar{0}.\bar{n}ds = 0[\because curl\nabla f = \bar{0} \text{ and } \nabla f \times \nabla f = \bar{0}]$$

4: Prove that $\oint_c f\nabla g.d\bar{r} = \int (\nabla f \times \nabla g).\bar{n}ds$

Solution: By Stokes Theorem,

$$\oint_c (f\nabla g.d\bar{r}) = \int_s [\nabla \times (f\nabla g)]\bar{n}ds = \int_s [\nabla f \times \nabla g + f curl\, grad g].\bar{n}ds$$

$$= \int [\nabla f \times \nabla g].\bar{n}ds \left[\because curl(grad g) = \bar{0}\right]$$

5: Verify Stokes theorem for $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$, Where S is the circular disc $x^2 + y^2 \le 1, z = 0.$

Solution: Given that $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$. The boundary of C of S is a circle in xy plane. $x^2 + y^2 \le 1, z = 0.$ We use the parametric co-ordinates x=cos $\theta, y = sin\theta, z = 0, 0 \le \theta \le 2\pi;$

dx=-sin$\theta\ d\theta$ and dy =cos$\theta\ d\theta$

$$\therefore \oint_c \bar{F}.dr = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c -y^3dx + x^3dy$$

$$= \int_0^{2\pi} [-sin^3\theta(-sin\theta) + cos^3\theta cos\theta]d\theta = \int_0^{2\pi}(cos^4\theta + sin^4\theta)d\theta$$

$$= \int_0^{2\pi}(1 - 2sin^2\theta\ cos^2\theta)d\theta = \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi}(2sin\theta\ cos\theta)^2 d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} sin^2 2d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi}(1 - cos4\theta)d\theta$$

$$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}sin4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

$$\text{Now} \nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$$

$$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$$

We have $(\bar{k}.\bar{n})ds = dxdy$ and R is the region on xy-plane

$$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2)\ dx\ dy$$

Put x=r cos$\emptyset, y = r\ sin\emptyset \therefore dxdy = rdr\ d\emptyset$

r is varying from 0 to 1 and 0$\le \emptyset \le 2\pi.$

$$\therefore \int (\nabla \times \bar{F}).\bar{n}ds = 3\int_{\emptyset=0}^{2\pi}\int_{r=0}^{1} r^2.rdr\ d\emptyset = \frac{3\pi}{2}$$

L.H.S=R.H.S.Hence the theorem is verified.

6: If $\overline{F} = y\bar{i} + (x-2xz)\bar{j} - xy\bar{k}$, evaluate $\int_s (\nabla \times F).\bar{n}ds$. Where S is the surface of sphere $x^2+y^2+z^2=a^2, above\ the\ xy-plane.$

Solution: Given $\bar{F} = y\bar{\imath} + (x-2xz)\bar{\jmath} - xy\bar{k}.$

By Stokes Theorem,

$\int_s (\nabla \times \bar{F}).\bar{n}ds = \int_c \overline{F}.d\bar{r} = \int_c F_1\,dx + F_2\,dy + F_3\,dz = \int_c y dx + (x-2xz)dy - xydz$

Above the xy plane the sphere is $x^2+y^2+= a^2, z=0$

$\therefore \int_c \bar{F}.d\bar{r} = \int_c ydx + xdy.$

Put x=a cos θ,y=asinθ $so\ that\ dx = -a\ sin\theta d\theta, dy = acos\theta d\theta\ and \theta = 0 \to 2\pi$

$$\int_c \bar{F}.d\bar{r} = \int_0^{2\pi} (a\ sin\theta)(-a\ sin\theta)\,d\theta + (acos\theta)(acos\theta)d\theta$$

$$=a^2\int_0^{2\pi} cos2\theta\ d\theta = a^2\left[\frac{sin2\theta}{2}\right]_0^{2\pi} = \frac{a^2}{2}(0) = 0$$

7: Verify Stokes theorem for $\bar{F} = (2x-y)\bar{\imath} - \dot{y}z^2\bar{\jmath} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$bounded by the projection of the xy-plane.

Solution: The boundary C of S is a circle in xy plane i.e x^2+y^2=1, z=0

The parametric equations are x=$cos\theta, y = sin\theta, \theta = 0 \to 2\pi$

$\therefore\ dx = -sin\theta\ d\theta, dy = cos\theta\ d\theta$

$$\int_c \overline{F}.d\bar{r} = \int_c \overline{F_1}dx + \overline{F_2}dy + \overline{F_3}dz = \int_c (2x-y)dx - yz^2dy - y^2zdz$$

$$=\int_c (2x-y)dx(since\ z = 0\ and\ dz = 0)$$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta d\theta = \int_0^{2\pi} \sin^2\theta d\theta - \int_0^{2\pi} \sin 2\theta d\theta$$

$$=\int_{\theta=0}^{2\pi} \frac{1-cos2\theta}{2}d\theta - \int_0^{2\pi} sin2\theta\ d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}sin2\theta + \frac{1}{2}.cos2\theta\right]_0^{2\pi}$$

$$=\frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}.(cos4\pi - cos0) = \pi$$

Again $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \bar{\imath}(-2yz+2yz) - \bar{\jmath}(0-0) + \bar{k}(0+1) = \bar{k}$

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_S \bar{k}.\bar{n}ds = \int_R \int dxdy$

Where R is the projection of S on xy plane and $\bar{k}.\bar{n}ds = dxdy$

Now $\int\int_R dxdy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dydx = 4\int_{x=0}^{1}\sqrt{1-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$

$$=4\left[\frac{1}{2}\sin^{-1}1\right]=2\frac{\pi}{2}=\pi$$

$\therefore$ *The* Stokes theorem is verified.

8: Verify Stokes theorem for the function $\bar{F}=x^2\,\bar{\imath}+xy\,\bar{\jmath}$ integrated round the square in the plan z=0 whose sides are along the lines x=0, y=0, x=a, y=a.

Solution: Given $\bar{F}=x^2\,\bar{\imath}+xy\,\bar{\jmath}$

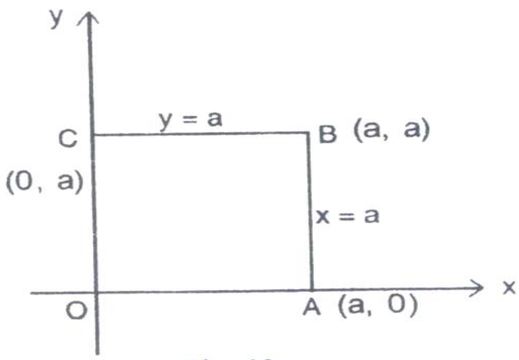


Fig. 13

By Stokes Theorem, $\int_S(\nabla\times\bar{F}).\bar{n}ds=\int_c \bar{F}.d\bar{r}$

Now $\nabla\times\bar{F}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 & xy & 0\end{vmatrix}=\bar{k}$y

L.H.S.$=\int_S(\nabla\times\bar{F}).\bar{n}ds=\int_s y(\bar{n}.\bar{k})ds=\int_s ydxdy$

$\therefore \bar{n}.\bar{k}.ds=dxdy$ and R is the region bounded for the square.

$\therefore \int_S(\nabla\times\bar{F}).\bar{n}ds=\int_0^a\int_0^a ydydx=\frac{a^3}{2}$

R.H.S. $=\int_C \bar{F}.d\bar{r}=\int_C (x^2dx+xydy)$

But $\int \bar{F}.d\bar{r}=\int_{OA}\bar{F}.d\bar{r}+\int_{AB}\bar{F}.d\bar{r}+\int_{BC}\bar{F}.d\bar{r}+\int_{CO}\bar{F}.d\bar{r}$

(i)Along OA: y=0, z=0, dy=0, dz=0

$\therefore \int_{OA}\bar{F}.d\bar{r}=\int_0^a x^2dx=\frac{a^3}{3}$

(ii)Along AB:x=a, z=0,dx=0,dz=0

$$\int_{AB}\bar{F}.d\bar{r}=\int_0^a aydy=\frac{1}{2}a^3$$

(iii)Along BC: y=a,z=0,dy=0,dz=0

$\therefore \int_{BC} \bar{F} \cdot d\bar{r} = \int_{a}^{0} 0dx = \frac{1}{3}a^3$

(iv)Along CO: x=0, z=0, dx=0, dz=0

$\therefore \int_{CO} \bar{F} \cdot d\bar{r} = \int_{a}^{0} 0dy = 0$

Adding $\int_{c} \bar{F} \cdot d\bar{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 + \frac{1}{3}a^3 + 0 = \frac{1}{2}a^3$

Hence the verification.

9: Apply Stokes theorem, to evaluate $\int_{c}(ydx + zdy + xdz)$ where c is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and x+z=a.

Solution : The intersection of the sphere $x^2 + y^2 + z^2 = a^2$ *and* the plane x+z=a. is a circle in the plane x+z=a. with AB as diameter.

Equation of the plane is x+z=a$\Longrightarrow \frac{x}{a} + \frac{z}{a} = 1$

$\therefore OA = OB = a$ i.e., $A = (a, 0, 0)$ and B=(0,0,a)

$\therefore$ Length of the diameter AB $= \sqrt{a^2 + a^2 + 0}$ =a$\sqrt{2}$

Radius of the circle, r=$\frac{a}{\sqrt{2}}$

Let $\bar{F}.d\bar{r} = ydx + zdy + xdz \Longrightarrow \bar{F}.d\bar{r} = \bar{F}.(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz) = ydx + zdy + xdz$

$\Longrightarrow \bar{F} = y\,\bar{\imath} + z\,\bar{\jmath} + x\,\bar{k}$

$$\therefore curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(\bar{\imath} + \bar{\jmath} + \bar{k})$$

Let $\bar{n}$ be the unit normal to this surface. $\bar{n} = \frac{\nabla S}{|\nabla S|}$

Then s=x+z-a, $\nabla S = \bar{i} + \bar{k}$ $\therefore$ $\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{\bar{\imath} + \bar{k}}{\sqrt{2}}$

Hence $\oint_{c} \bar{F}.d\bar{r} = \int curl\ \bar{F}.\bar{n}\, ds\ (by\ Stokes\ Theorem)$

$$= -\int(\bar{\imath} + \bar{\jmath} + \bar{k}).\left(\frac{\bar{\imath}+\bar{k}}{\sqrt{2}}\right)ds = -\int\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)ds$$

$$= -\sqrt{2}\int_{S} ds = -\sqrt{2}S = -\sqrt{2}\left(\frac{\pi a^2}{2}\right) = \frac{\pi a^2}{\sqrt{2}}$$

10: Apply the Stoke's theorem and show that $\int_{S}\int curl\ \bar{F}.\bar{n}d\bar{s} = 0\ where\ \bar{F}$ is any vector and S =$x^2 + y^2 + z^2 = 1$

Solution: Cut the surface if the Sphere $x^2 + y^2 + z^2 = 1\ by\ any\ plane,$ Let $S_1\ and\ S_2$ denotes its upper and lower portions a C, be the common curve bounding both these portions.

$$\therefore \int_{S} curl\bar{F}.d\bar{s} = \int_{S_1} \bar{F}.d\bar{s} + \int_{S_2} \bar{F}.d\bar{s}$$

Applying Stoke's theorem,

$$\int_s curl\,\overline{F}.d\overline{s} = \int_{s_1} \overline{F}.d\overline{R} + \int_{s_2} \overline{F}.d\overline{R} = 0$$

The 2nd integral curl $\bar{F}.d\bar{s}$ is negative because it is traversed in opposite direction to first integral.

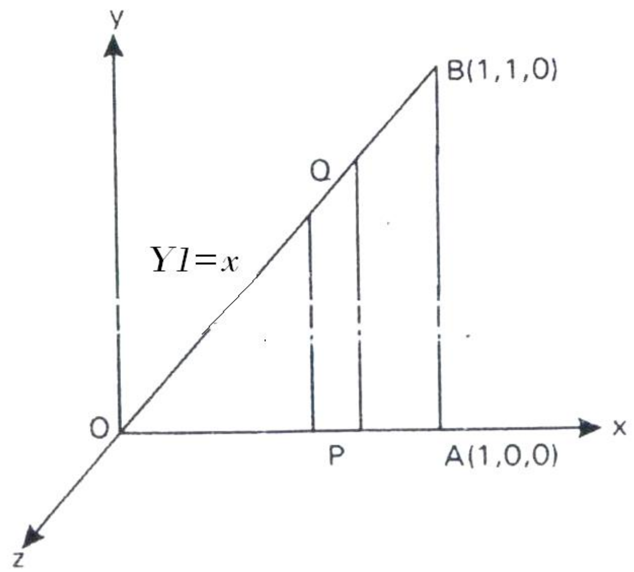
The above result is true for any closed surface S.

11: Evaluate by Stokes theorem $\oint_c (x+y)dx + (2x-z)dy + (y+z)dz$ where C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

Solution: Let $\bar{F}.d\bar{r} = \bar{F}.(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz) = (x+y)dx + (2x-z)dy + (y+z)dz$

Then $\bar{F} = (x+y)\bar{\imath} + (2x-z)\bar{\jmath} + (y+z)\bar{k}$

By Stokes theorem, $\oint_C \bar{F}.d\bar{r} = \int\int_S curl\,\bar{F}.\bar{n}\,ds$



Where S is the surface of the triangle OAB which lies in the xy plane. Since the z Co-ordinates of O,A and B Are zero. Therefore$\bar{n} = \bar{k}$. Equation of OA is y=0 and that of OB, y=x in the xy plane.

$$\therefore\ curl\,\bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\bar{\imath} + \bar{k}$$

$\therefore\ curl\,\bar{F}.\bar{n}ds = curl\,\bar{F}.\bar{K}\,dx\,dy = dx\,dy$

$\therefore \oint_c \bar{F}.d\bar{r} = \int\int_s dx\,dy = \int\int_s dA = A = area\ of$ the $\Delta\ OAB$

$$=\frac{1}{2}OA \times AB=\frac{1}{2}\times 1\times 1=\frac{1}{2}$$

12: Use Stoke's theorem to evaluate $\int\int_S curl\ \bar{F}.\bar{n}\ dS$ over the surface of the paraboloid $z+x^2+y^2=1, z\geq 0$ where $\bar{F}=y\,\bar{\imath}+z\,\bar{\jmath}+x\,\bar{k}.$

Solution : By Stoke's theorem

$$\int_s curl\bar{F}.d\bar{s}=\oint_c \bar{F}.d\bar{r}=\int_c (y\bar{i}+z\bar{j}+x\bar{k}).(\bar{i}dx+\bar{j}dy+\bar{k}dz)$$

$$=\int_c ydx \text{ (Since z=0,dz=0) ......(1)}$$

Where C is the circle $x^2+y^2=1$

The parametric equations of the circle are x=$cos\theta, y=sin\theta$

$\therefore dx=-\,sin\theta\ d\theta$

Hence (1) becomes

$$\int_s curl\bar{F}.ds=\int_{\theta=0}^{2\pi}\sin\theta(-\sin\theta)d\theta=-\int_{\theta=0}^{2\pi}\sin^2\theta d\theta=-4\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta=-4\times\frac{1}{2}\times\frac{\pi}{2}=-\pi$$

13: Verify Stoke's theorem for $\bar{F}=(x^2+y^2)\bar{\imath}-2xy\bar{\jmath}$ taken round the rectangle bounded by the lines x=$\pm a, y=0, y=b.$

Solution: Let ABCD be the rectangle whose vertices are (a,0), (a,b), (-a,b) and (-a,0).

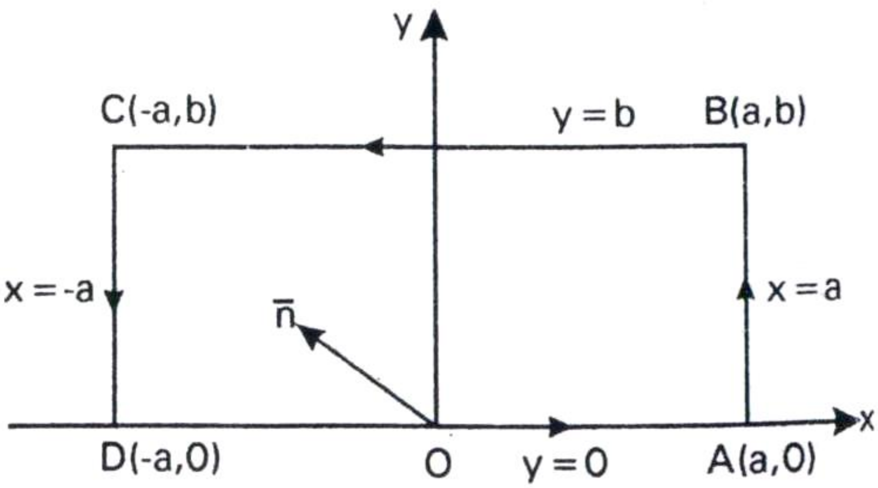
Equations of AB, BC, CD and DA are x=a, y=b, x=-a and y=0.

We have to prove that $\oint_c \bar{F}.d\bar{r}=\int_S curl\ \bar{F}.\bar{n}ds$

$$\oint_c \bar{F}.d\bar{r}=\oint_c\{(x^2+y^2)\bar{\imath}-2xy\bar{\jmath}\}.\{\bar{\imath}dx+\bar{\jmath}dy\}$$

$$=\oint_C (x^2+y^2)\,dx-2xydy$$

$$=\int_{AB}+\int_{BC}+\int_{CD}+\int_{DA} \qquad \text{.....(1)}$$



(i) Along AB, x=a, dx=0

from (1), $\int_{AB} = \int_{y=0}^{b} -2ay\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$

(ii)Along BC, y=b, dy=0

from (1), $\int_{BC} = \int_{x=a}^{x=-a} (x^2+b^2)dx = \left[\frac{x^3}{3}+b^2x\right]_{x=a}^{-a} = \frac{-2a^3}{3} - 2ab^2$

(iii) Along CD, x=-a, dx=0

from (1), $\int_{CD} = \int_{y=b}^{0} 2aydy = \cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_{y=b}^{0} = -ab^2$

(iv)Along DA, y=0, dy=0

from (1), $\int_{DA} = \int_{x=-a}^{x=a} x^2dx = \left[\frac{x^3}{3}\right]_{x=-a}^{a} = \frac{2a^3}{3}$

(i)+(ii)+(iii)+(iv) gives

$\therefore \oint_c \bar{F}.d\bar{r} = -ab^2 - -\frac{-2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2$ ….(2)

Consider $\int_S curl\ \bar{F}.\bar{n}\,dS$

Vector Perpendicular to the xy-plane is $\bar{n} = k$

$$\therefore\ curl\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = 4y\bar{k}$$

Since the rectangle lies in the xy plane,

$\bar{n} = \bar{k}$ and ds =dx dy

$\int_S curl\ \bar{F}.\bar{n}dS = \int_S -4y\bar{k}.\bar{k}dx\ dy = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\ dx\ dy$

$$=\int_{y=0}^{b}\int_{x=-a}^{a} -4y\ dx\ dy = 4\int_{y=0}^{b} y[x]_{-a}^{a}\ dy = -4\int_{y=0}^{b} 2aydy$$

$$=-4a[y^2]_{y=0}^{b} = -4ab^2 \qquad \text{.....(3)}$$

Hence from (2) and (3), the Stoke's theorem is verified.

14: Verify Stoke's theorem for $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the cube x =0, y=0, z=0, x=2, y=2,z=2 above the xy plane.

Solution: Given $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the cube. x=0, y=0, z=0, x=2, y=2, z=2 above the xy plane.

By Stoke's theorem, we have $\int curl\ \bar{F}.\bar{n}ds = \int \bar{F}.d\bar{r}$

$$\nabla\times\bar{F}=\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & y+4 & -xz \end{vmatrix} = \bar{\imath}(0+y) - \bar{\jmath}(-z+1) + \bar{k}(0-1) = y\bar{\imath} - (1-z)\bar{\jmath} - \bar{k}$$

$\therefore\ \nabla\times\bar{F}.\bar{n} = \nabla\times\bar{F}.k = (y\bar{\imath} - (1-z)\bar{\jmath} - \bar{k}).k = -1$

$\therefore \int \nabla\times\bar{F}.\bar{n}.ds = \int_0^2\int_0^2 -1\ dx\ dy\ \ (\because z = 0, dz = 0) = -4$

.....(1)

To find $\int \bar{F}.d\bar{r}$

$\int \bar{F}.d\bar{r} = \int\left((y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}\right).(dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k})$

$\qquad = \int[(y - z + 2)dx + (yz + 4)dy - (xz)dz]$

Sis the surface of the cube above the xy-plane

$\therefore z = 0 \ \Rightarrow dz = 0$

$\therefore \int \bar{F}.d\bar{r} = \int(y + 2)dx + \int 4dy$

Along $\overline{OA}, y = 0, z = 0, dy = 0, dz = 0, x\ change\ from\ 0\ to\ 2.$

$\int_0^2 2dx = 2[x]_0^2 = 4 \qquad$ ....(2)

Along $\overline{BC}, y = 2, z = 0, dy = 0, dz = 0, x\ change\ from\ 2\ to\ 0.$

$\int_2^0 4dx = 4[x]_2^0 = -8 \qquad$ ....(3)

Along $\overline{AB}, x = 2, z = 0, dx = 0, dz = 0, y\ change\ from\ 0\ to\ 2.$

$\int \bar{F}.d\bar{r} = \int_0^2 4dy = [4y]_0^2 = 8 \qquad$ ......(4)

Along $\overline{CO}, x = 0, z = 0, dx = 0, dz = 0, y\ change\ from\ 2\ to\ 0.$

.$\int_2^0 4dy = -8 \qquad$ .....(5)

Above the surface When z=2

Along $0'A'$, $\int_0^2 \bar{F}\,\overline{dr} = 0$ ....(6)

Along $A'B'$, $x = 2, z = 2, dx = 0, dz = 0,$ y changes from 0 to 2

$$\int_0^2 \bar{F}.d\bar{r} = \int_0^2 (2y+4)dy = 2\left[\frac{y^2}{2}\right]_0^2 + 4[y]_0^2 = 4+8 = 12 \quad ....(7)$$

Along $B'C'$, $y = 2, z = 2, dy = 0, dz = 0,$ x changes from 2 to 0

$\int_0^2 \bar{F}.\overline{dr} = 0$ ....(8)

Along $C'D'$, $x = 0, z = 2, dx = 0, dz = 0,$ y changes from 2 to 0.

$$\int_2^0 (2y+4) = 2\left[\frac{y^2}{2}\right]_2^0 + 4[y]_2^0 = -12 \quad .....(9)$$

(2)+(3)+(4)+(5)+(6)+(7)+(8)+(9) gives

$\int_C \bar{F}.d\bar{r} = 4 - 8 + 8 - 8 + 0 + 12 + 0 - 12 = -4$ .....(10)

By Stokes theorem, We have

$\int \bar{F}.d\bar{r}$=$\int curl\ \bar{F}.\bar{n}$ds=-4

Hence Stoke's theorem is verified.

15: Verify the Stoke's theorem for $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$ and surface is the part of the sphere $x^2 + y^2 + z^2 = 1$ *above the xy plane.*

Solution: Given $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$ over the surface $x^2 + y^2 + z^2 = 1$ *is xy plane.*

We have to prove $\int_C \bar{F}.d\bar{r}$=$\int\int_s Curl\ \bar{F}.\bar{n}ds$

$\bar{F}.d\bar{r}$ =. $(y\bar{i} + z\bar{j} + x\bar{k}).(dx\bar{i} + dy\bar{j} + dz\bar{k})$=ydx + zdy + xdz

$\int_C(ydx + zdy + xdz) = \int ydx \quad (in\ xy\ plane\ z = 0, dz = 0)$

Let x= $\cos\theta, y = \sin\theta \Rightarrow dx = -\sin\theta\, d\theta, dy = \cos\theta\, d\theta$

$\therefore \int_C \bar{F}.d\bar{r} = \int_C y.dx = \int_0^{2\pi} ydx \qquad [\because x^2 + y^2 = 1, z = 0]$

$=\int_0^{2\pi} \sin\theta\,(-\sin\theta)d\theta = -4\int_0^{\pi/2} \sin^2\theta\, d\theta$

$=-4\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta = -4\left[\left(\frac{1}{2}.\frac{\pi}{2}\right) - \frac{1}{4}(\sin\pi)\right]$

$=-4\left[\left(\frac{1}{2}.\frac{\pi}{2}\right) - 0\right] = -4\left[\frac{\pi}{4}\right] = -\pi$ .....(1)

$$\text{Curl}\bar{F}=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = -(\bar{i} + \bar{j} + \bar{k})$$

Unit normal vector $\bar{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\bar{\imath}+2y\bar{\jmath}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$

Substituting the spherical polar coordinates, we get

$\bar{n} = \sin\theta\cos\phi\,\bar{\imath} + \sin\theta\sin\phi\,\bar{\jmath} + \cos\theta\,\bar{k}$

$\therefore Curl\,\bar{F}.\bar{n} = -(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)$

$$\iint curl\bar{F}.\bar{n}ds = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\sin\theta d\theta d\phi$$

$$= -\int_0^{\pi/2}[\sin\theta\sin\phi - \sin\theta\cos\phi + \phi\cos\theta]_0^{2\pi}\ \sin\theta d\theta$$

$$= -2\pi\int_0^{\pi/2}\cos\theta\sin\theta d\theta = -\pi\int_0^{\pi/2}\sin 2\theta d\theta = (-\pi)\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}(-1-1) = -\pi \qquad \ldots..(2)$$

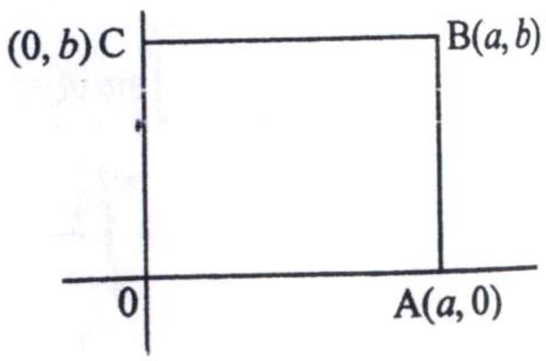
From (1) and (2), we have

$\int_C \bar{F}.d\bar{r} = \int\int_s Curl\,\bar{F}.\bar{n}ds = -\pi$

$\therefore$ Stoke's theorem is verified.

16: Verify Stoke's theorem for $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes x=0,x=a,y=0,y=b.

Solution :



Stoke"s theorem states that $\int_c \bar{F}.d\bar{r} = \int_s Curl\bar{F}.\bar{n}ds$

Given $\bar{F} = (x^2 - y^2)\bar{\imath} + 2xy\bar{\jmath}$

$$\text{Curl}\bar{F}=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix} = \bar{i}(0,0)-\bar{j}(0,0)+\bar{k}(2y+2y)=4y\bar{k}$$

R.H.S= $\int_s Curl\,\overline{F}.\overline{n}ds = \int_s 4y\left(\overline{k}.\overline{n}\right)ds$

Let R be the region bounded by the rectangle

$(\bar{k}.\bar{n})ds = dx\ dy$

$$\int_s Curl\,\overline{F}.\overline{n}ds = \int_{x=0}^{a}\int_{y=0}^{b} 4y dx dy = \int_{x=0}^{a}\left[4\frac{y^2}{2}\right]_0^b dx = 2b^2\int_{x=0}^{a} 1dx$$

$=2b^2(x)_0^a$ =2ab^2

To Calculate L.H.S

$\bar{F}.d\bar{r} = (x^2-y^2)dx + 2xy\ dy$

Let O=$(0,0), A = (a,0), B = (a,b)$ and

C=(0,b) are the vertices of the rectangle.

(i)Along the line OA

y=0; dy=0, x ranges from 0 to a.

$\int_{OA} \bar{F}.d\bar{r} = \int_{x=0}^{a} x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$

(ii)Along the line AB

x=a; dx=0, y ranges from 0 to b.

$\int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^{b}(2xy)\,dy = \left[2a\frac{y^2}{2}\right]_0^b$=a$b^2$

(iii)Along the line BC

y=b; dy=0, x ranges from a to 0

$$\int_{BC} \overline{F}.d\overline{r} = \int_{x=a}^{0}(x^2-y^2)dx = \left[\frac{x^3}{3}-b^2x\right]_a^0 = 0-\left(\frac{a^3}{3}-b^2a\right)$$

=a$b^2 - \frac{a^3}{3}$

(iv) Along the line CO

x=0,dx=0,y changes from b to 0

$\int_C \bar{F}\,.\,d\bar{r} = \int_{y=b}^{0} 2xydy = 0$

Adding these four values

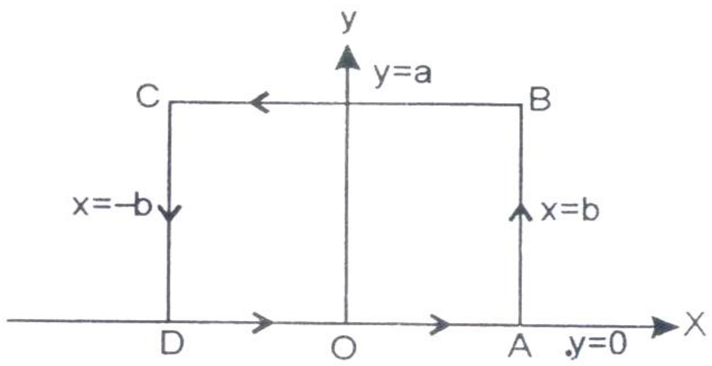
$\int_{CO} \bar{F}\,.\,d\bar{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2ab^2$

L.H.S = R.H.S

Hence the verification of the stoke's theorem.

17: Verify Stoke's theorem for $\bar{F}=y^2\,\bar{\imath} - 2xy\bar{\jmath}$ taken round the rectangle bounded by $x=\pm b$, $y=0, y=a$.

Solution:



$$\text{Curl } \bar{A} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -2xy & 0 \end{vmatrix} = -4y\bar{k}$$

For the given surface S, $\bar{n} = \bar{k}$

$\therefore (Curl\ \bar{F}).\bar{n} = -4y$

Now $\iint_s (Curl\ \bar{F}).\bar{n}\,dS = \iint_s -4y\,dx\,dy$

$$= \int_{y=0}^{a} \left[\int_{x=-b}^{b} -4y\,dx \right] dy$$

$$= \int_0^a [-4xy]_{-b}^{b}\,dy$$

$$= \int_0^a -8by\,dy = \left[-4by^2\right]_0^a = -4a^2b \text{........(1)}$$

$\int_c \bar{F}\,.\,d\bar{r} = \int_{DA} + \int_{AB} + \int_{BC} + \int_{CD}$

$\int \bar{F}\,.\,d\bar{r} = y^2\,dx - 2xy\,dy$

Along DA , y=0,dy=0 $\Rightarrow \int_{DA} \bar{F}\,.\,d\bar{r} = 0$ ($\because \overline{F}.dr = 0$)

Along AB, x=b,dx=0

$\int_{AB} \bar{F} \cdot d\bar{r} = \int_{y=0}^{a} -2bydy = \left[-by^2\right]_0^a = -a^2b$

Along BC,y=a,dy=0

$\int_{BC} \bar{F} \cdot d\bar{r} = \int_{b}^{-b} a^2\, dx = -2a^2b$

Along CD, x=-b,dx=0

$\int_{CD} \bar{F} \cdot d\bar{r} = \int_{a}^{0} 2bydy = \left[-by^2\right]_a^0 = -a^2b$.

$\int_{C} \bar{F} \cdot d\bar{r} = 0 - a^2b - 2a^2b - a^2b = -4a^2b$ -------(2)

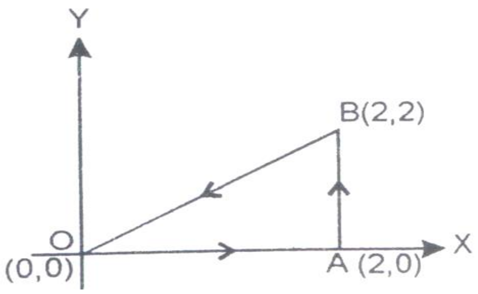
From (1),(2) $\int_{C} \bar{F} \cdot d\bar{r} = \iint_{s} (Curl\ \bar{F}).\bar{n}\, dS$

Hence the theorem is verified.

19: Using Stroke's theorem evaluate the integral $\int_{C} \bar{F} \cdot d\bar{r}$ where $\bar{F} = 2y^2\,\bar{\imath} + 3x^2\,\bar{\jmath} - (2x+z)\bar{k}$ and C is the boundary of the triangle whose vertices are (0,0,0),(2,0,0),(2,2,0).

<u>**Solution:**</u>

$$\text{Curl } \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^2 & 3x^2 & -2x-z \end{vmatrix} = 2\bar{\jmath} + (6x-4y)\,\bar{k}$$



Since the z-coordinate of each vertex of the triangle is zero , the triangle lies in the xy-plane .

$\therefore \bar{n} = k$

$\therefore$ (Curl $\bar{F}$).$\bar{n}$= 6x-4y

Consider the triangle in xy-plane .

Equation of the straight line OB is y=x.

By Stroke's theorem

$$\int_{c} \bar{F}.d\bar{r} = \int\int_{s} (curl \bar{F}).\bar{n} ds$$

$$=\int_{x=0}^{2}\int_{y=0}^{y=x}(6x-4y)\,dx\,dy \qquad = \int_{x=0}^{2}\left[\int_{y=0}^{x}(6x-4y)\,dy\right]dx$$

$$= \int_{x=0}^{2}\left[6xy-2y^2\right]_0^x dx = \int_0^2(6x^2-2x^2)\,dx$$

$$= 4\left[\frac{x^3}{3}\right]_0^2 \quad = \frac{32}{3}$$

OBJECTIVE TYPE QUESTIONS

(1) For any closed surface S, $\iint_s (curl\,\overline{F}).\overline{n}ds =$

(a) 0 (b) $2\ \overline{F}$ (c) $\overline{n}$ (d) $\oint \overline{F}.d\overline{r}$

(2) If S is any closed surface enclosing a volume V and $\overline{F} = x\overline{i} + 2y\,\overline{j} + 3z\overline{k}$ then

$\iint_s \overline{F}.\overline{n}ds =$

(a) V (b)3V (c)6V (d)None

(3) If $\overline{r} = x\overline{i} + y\,\overline{j} + z\overline{k}$ then $=\oint \overline{r}.d\overline{r}$

(a) 0 (b) $\overline{r}$ (c) x (d) None

(4) $\int \overline{r}$ x $\overline{n}\,dS$ =

(a) 0 (b) r (c) 1 (d) None

(5) $\int_s \overline{r}\ .\overline{n}\,dS$ =

(a) V (b) 3V (c) 4V (d) None

(6) If $\overline{n}$ is the unit outward drawn normal to any closed surface then $\int_s$ div $\overline{n}\,dV$ =

(a) S (b)2S (c) 3S (d) None

(7) $\oint f\nabla f.d\overline{r} =$

(a) f (b)2f (c) 0 (d) None

(8) The value of the line integral $\int grad(x+y-z)d\overline{r}$ from (0, 1,-1) to (1 , 2, 0) is

(a) -1 (b) 0 (c) 2 (d) 3

(9) A necessary and sufficient condition that the line integral $\int_c A\,.\,dr$ =0 for every closed curve c is that

(a) div A=0 (b)div A$\neq$0 (c) curl A=0 (d) curl A$\neq$0

(10) If $\overline{F}$ =axi + byj+ czk where a, b, c are constants then $\iint \overline{F}\ .\overline{n}dS$ where S is the surface of the unit sphere is

(a) 0 (b) $\frac{4}{3}\pi(a+b+c)$ (c) $\frac{4}{3}\pi(a+b+c)^2$ (d) none

(11) $\int_V D \times \bar{F} dv$ =_________________

(a) $\int_S \bar{n} \times \bar{F} ds$ (b) 0 (c) V (d) S

(12) $\int_V \phi \times dv$ = __________

(a) $\int \bar{n}\ \phi ds$ (b) 0 (c) V (d) ϕ

(13) $\int fog . d\bar{r}$= _____________

(a) 0 (b) $\int_S (\nabla f \times \bar{F} Dg)$ (c) $\bar{r}$ (d) S

(14) $\iint_S xdydx + ydzdx + zdxdy$ where S: $x^2 + y^2 + z^2 = a^2$ as

(a) 4p (b) $\frac{4}{3}\pi a^3$ (c) $4\pi a^3$ (d) 4π

ANSWERS

(1) d (2) c (3) a (4) a (5) b (6) a (7) c (8) d (9) c

(10) b (11) a (12) a (13) b (14) c