Unit-2

Convolution, Correlation of Signals
and
Fourier Series

Unit-2

Part-1: (Convolution and Correlation of Signals)

- Convolution and their Properties
- Correlation of Signals
- Cross correlation and Auto Correlation Functions
- Properties of Correlation functions
- Relation between Convolution and Correlation

Part-2: (Fourier Series)

- Fourier Series Properties
- Dirchlet t Conditions
- Trigonometric Fourier Series
- Exponential Fourier Series
- Complex Fourier Spectrum

Part-1:
(Convolution and Correlation of Signals)

Introduction:

Convolution is a mathematical way of combining two signals to form a third signal. It is an important operation because it relates the input signal and impulse response of the system to the output of the system.

Correlation is also a mathematical operation that uses two signals to form a third signal. It compares two signals inorder to determine the degree of similarity between them. It is very widely used in communication engineering

correlation may be cross correlation or auto correlation. when one signal is correlated with another signal to form a third signal, it is called cross correlation. when a signal is correlated with itself to form another signal, it is called auto correlation.

Convolution:

Convolution: Convolution is a mathematical operation which is used to express the input-output relationship of an LTI system. It is an important operation in LTI continuous-time systems. It relates the input and impulse response of the system to the output.

Consider an LTI system which is intially relaxed at $t=0$. If the input to the system is an impulse, then the output of the system is denoted by $h(t)$ and is called the impulse response of the ~~res~~ system.

The impulse response is denoted as

$$h(t) = T[\delta(t)] \quad \text{——} (1)$$

Contd….

We know that any arbitrary signal $x(t)$ can be represented as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \quad \text{—— (2)}$$

The system output is given by

$$y(t) = T[x(t)] \quad \text{—— (3)}$$

$$\therefore \; y(t) = T\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,T[\delta(t-\tau)]\,d\tau \quad \text{—— (4)}$$

If the response of the system due to impulse $\delta(t)$ is $h(t)$, then the response of the system due to delayed input $\delta(t-\tau)$ is

$$h(t,\tau) = T[\delta(t-\tau)] \quad \text{—— (5)}$$

From eqns (4) & (5)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t,\tau)\,d\tau \quad \text{—— (6)}$$

Contd….

For a Time invariant system

$$h(t,\tau) = h(t-\tau) \quad \text{—} (7)$$

From eqn (6) & (7)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

This is called convolution integral or Simply convolution.

The convolution of two ~~sequen~~ signals $x(t)$ and $h(t)$ can be represented as

$$y(t) = x(t) * h(t)$$

Contd…

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

This is called convolution integral or simply convolution.

The convolution of two ~~sequen~~ signals $x(t)$ and $h(t)$ can be represented as

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \quad \text{if both } x(t) \text{ \& } h(t) \text{ are non-causal}$$

$$= \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \quad \text{if } x(t) \text{ is non-causal \& } h(t) \text{ is causal}$$

$$= \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau \quad \text{if } x(t) \text{ is causal \& } h(t) \text{ is non-causal}$$

$$= \int_{0}^{t} x(\tau)h(t-\tau)d\tau \quad \text{if both } x(t) \text{ \& } h(t) \text{ are causal.}$$

Properties of Convolution:

Let us consider two signals $x_1(t)$ and $x_2(t)$. The convolution of two signals $x_1(t)$ and $x_2(t)$ is given by

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau$$

The properties of convolution are as follows:

Commutative property The commutative property of convolution states that

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

Distributive property The distributive property of convolution states that

$$x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$

Associative property The associative property of convolution states that

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

Shift property The shift property of convolution states that if

$$x_1(t) * x_2(t) = z(t)$$

Then $\quad x_1(t) * x_2(t-T) = z(t-T)$

Similarly, $\quad x_1(t-T) * x_2(t) = z(t-T)$

and $\quad x_1(t-T_1) * x_2(t-T_2) = z(t-T_1-T_2)$

Contd…

Convolution with an impulse: Convolution of a signal $x(t)$ with a unit impulse is the signal itself. That is,

$$x(t) * \delta(t) = x(t)$$

Width property Let the duration of $x_1(t)$ and $x_2(t)$ be T_1 and T_2 respectively. Then the duration of the signal obtained by convolving $x_1(t)$ and $x_2(t)$ is $T_1 + T_2$.

Problems:

EXAMPLE 1 Find the convolution of the following signals:

(i) $x_1(t) = e^{-2t} u(t)$; $x_2(t) = e^{-4t} u(t)$

(ii) $x_1(t) = t\,u(t)$; $x_2(t) = t\,u(t)$

(iii) $x_1(t) = \cos t\,u(t)$; $x_2(t) = u(t)$

(iv) $x_1(t) = e^{-3t} u(t)$; $x_2(t) = u(t+3)$

(v) $x_1(t) = r(t)$; $x_2(t) = e^{-2t} u(t)$

Solutions:

Solution:

(i) Given $x_1(t) = e^{-2t}\,u(t);\; x_2(t) = e^{-4t}\,u(t)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} e^{-2\tau}\, u(\tau)\, e^{-4(t-\tau)}\, u(t-\tau)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t-\tau) = 1$ for $(t-\tau) \ge 0$ or for $\tau < t$.
Hence $u(\tau)\, u(t-\tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\, u(t-\tau) = 0$.

$$\therefore \quad x_1(t) * x_2(t) = \int_{0}^{t} e^{-2\tau}\, e^{-4(t-\tau)}\, d\tau$$

$$= e^{-4t} \int_0^t e^{2\tau}\, d\tau = e^{-4t} \left[\frac{e^{2\tau}}{2} \right]_0^t \quad = e^{-4t} \left(\frac{e^{2t} - 1}{2} \right) = \frac{e^{-2t} - e^{-4t}}{2} \quad (\text{for } t \ge 0) \quad = \frac{e^{-2t} - e^{-4t}}{2}\, u(t)$$

Contd…

(ii) Given $x_1(t) = t\,u(t);\ x_2(t) = t\,u(t)$

We know that $$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \tau u(\tau)\,(t-\tau)\, u(t-\tau)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t - \tau) = 1$ for $(t - \tau) \geq 0$ or for $\tau < t$.
Hence $u(\tau)\ u(t - \tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\ u(t - \tau) = 0$.

Contd…

$$\therefore \quad x_1(t) * x_2(t) = \int_0^t \tau(t-\tau)\, d\tau = \int_0^t t\tau\, d\tau - \int_0^t \tau^2 d\tau = t\left[\frac{\tau^2}{2}\right]_0^t - \left[\frac{\tau^3}{3}\right]_0^t$$

$$= t\left(\frac{t^2}{2} - 0\right) - \left(\frac{t^3}{3} - 0\right) = \frac{t^3}{2} - \frac{t^3}{3} = \frac{t^3}{6} \quad (\text{for } t \geq 0) \quad \therefore \quad x_1(t) * x_2(t) = \frac{t^3}{6} u(t)$$

(iii) Given $\quad x_1(t) = \cos t\, u(t);\ x_2(t) = u(t)$

We know that $\quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \;\therefore\; x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \cos\tau\, u(\tau)\, u(t-\tau)\, d\tau$

$u(\tau) = 1$ for $\tau > 0$ and $u(t - \tau) = 1$ for $(t - \tau) \geq 0$ or for $\tau < t$.
Hence $u(\tau)\, u(t - \tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\, u(t - \tau) = 0$.

$$\therefore \quad x_1(t) * x_2(t) = \int_0^t \cos\tau\, d\tau = [\sin\tau]_0^t = \sin t \quad \text{for } t \geq 0$$

$$\therefore \quad x_1(t) * x_2(t) = \sin t\, u(t)$$

Contd…

(iv) Given $x_1(t) = e^{-3t}\, u(t);\ x_2(t) = u(t+3)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} e^{-3\tau}\, u(\tau)\, u(t+3-\tau)$$

In this case, $u(\tau) = 0$ for $\tau < 0$ and $u(t+3-\tau) = 0$ for $\tau > t+3$. $u(\tau)\ u(t+3-\tau) = 1$ only for $0 < \tau < t+3$. For all other values of τ, $u(\tau)\ u(t+3-\tau) = 0$.

$$\therefore \quad x_1(t) * x_2(t) = \int_{0}^{t+3} e^{-3\tau}\, d\tau = \left[\frac{e^{-3\tau}}{-3}\right]_0^{t+3} = \frac{e^{-3(t+3)} - 1}{-3} = \frac{1 - e^{-3(t+3)}}{3}$$

$$\therefore \quad y(t) = 0 \quad (\text{for } t < -3) \qquad = \frac{1 - e^{-3(t+3)}}{3} \quad (\text{for } t > -3)$$

Contd…

(v) Given $x_1(t) = r(t) = tu(t);\ x_2(t) = e^{-2t}\,u(t)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \tau u(\tau)\, e^{-2(t-\tau)}\, u(t-\tau)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t-\tau) = 1$ for $(t-\tau) \geq 0$ or for $\tau < t$.
Hence $u(\tau)\, u(t-\tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\, u(t-\tau) = 0$.

$$\therefore \quad x_1(t) * x_2(t) = \int_0^t \tau e^{-2(t-\tau)}\, d\tau \quad = e^{-2t} \int_0^t \tau e^{2\tau}\, d\tau = e^{-2t} \left\{ \left[\frac{\tau e^{2\tau}}{2} \right]_0^t - \int_0^t \frac{e^{2\tau}}{2}\, d\tau \right\}$$

$$= e^{-2t} \left\{ \left[\frac{te^{2t}}{2} - \left[\frac{e^{2\tau}}{4} \right]_0^t \right] \right\} = e^{-2t} \left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4} \right) \quad = \frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4} \quad (\text{for } t \geq 0)$$

$$\therefore \quad x_1(t) * x_2(t) = \left(\frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4} \right) u(t)$$

Convolution Theorems:

Convolution of signals may be done either in time domain or in frequency domain. So there are following two theorems of convolution associated with Fourier transforms:

1. Time convolution theorem
2. Frequency convolution theorem

Time Convolution Theorem

The time convolution theorem states that convolution in time domain is equivalent to multiplication of their spectra in frequency domain. Mathematically, if

$$x_1(t) \longleftrightarrow X_1(\omega)$$

and $$x_2(t) \longleftrightarrow X_2(\omega)$$

Then $$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)\, X_2(\omega)$$

Proof: $$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\omega t}\, dt$$

We have $$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Contd…

$$\therefore \quad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} [x_1(\tau)\, x_2(t-\tau)\, d\tau] \right\} e^{-j\omega t}\, dt$$

Interchanging the order of integration, we have

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

Letting $t - \tau = p$, in the second integration, we have

$$t = p + \tau \text{ and } dt = dp$$

$$\therefore \quad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega(p+\tau)}\, dp \right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega p}\, dp \right] e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, X_2(\omega)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau\, X_2(\omega)$$

$$= X_1(\omega)\, X_2(\omega)$$

$$\therefore \quad x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)\, X_2(\omega)$$

Contd…

Frequency Convolution Theorem

The frequency convolution theorem states that the multiplication of two functions in time domain is equivalent to convolution of their spectra in frequency domain. Mathematically, if

$$x_1(t) \longleftrightarrow X_1(\omega)$$

and

$$x_2(t) \longleftrightarrow X_2(\omega)$$

Then

$$x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

Proof:

$$F[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} [x_1(t)\, x_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda) e^{j\lambda t}\, d\lambda\right] x_2(t)\, e^{-j\omega t}\, dt$$

Interchanging the order of integration, we get

$$F[x_1(t)\, x_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, e^{j\lambda t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\, e^{-j(\omega-\lambda)t}\, dt\right] d\lambda$$

Contd…

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega - \lambda) \, d\lambda$$

$$= \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)]$$

$\therefore$ $$x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

or $$2\pi x_1(t)\, x_2(t) \longleftrightarrow X_1(\omega) * X_2(\omega)$$

This is frequency convolution theorem in radian frequency. In terms of frequency, we get

$$\mathrm{F}[x_1(t)\, x_2(t)] = X_1(f) * X_2(f)$$

Convolution using Graphical Representation:

Graphical Representation of Convolution :

When two signals $x_1(t)$ and $x_2(t)$ are provided in graphical form, the convolution can be performed by graphical method. It involves the following steps:

1. For the given signals $x_1(t)$ and $x_2(t)$, draw the signals $x_1(\tau)$ and $x_2(\tau)$ as a function of independent time variable τ.
2. Draw the function $x_2(-\tau)$ which is time reversed function of $x_2(\tau)$. Then shifting the function by time t to form $x_2(t-\tau)$.
3. Draw both signals $x_1(t)$ and $x_2(t-\tau)$ on the same time τ-axis with large time shift t along the $-$ve axis.
4. Increase the time shift t along $+$ve time axis. Multiply the signals $x_1(\tau)\, x_2(t-\tau)$ and integrate over the overlapping period of two signals to obtain convolution at t.
5. Increase the time shift t step by step and obtain convolution using step 4.
6. Draw the convolution function $x(t)$ with the values obtained in step 4 and 5 as a function of t.

Problems:

Find the convolution of the following signals by graphical method.

$$x(t) = e^{-3t}u(t) \quad \text{and} \quad h(t) = u(t+3)$$

Soln:

Given $x(t) = e^{-3t}u(t)$

& $h(t) = u(t+3)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

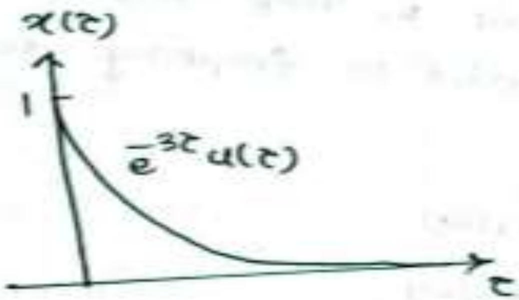
$x(\tau) = e^{-3\tau}u(\tau)$ $\qquad$ $h(\tau) = u(\tau+3)$

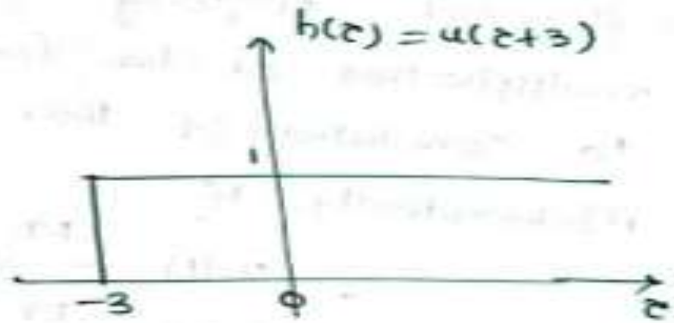
$u(\tau) = 1$ for $\tau \geq 0$ $\qquad$ $u(\tau+3) = 1$ for $\tau \geq -3$

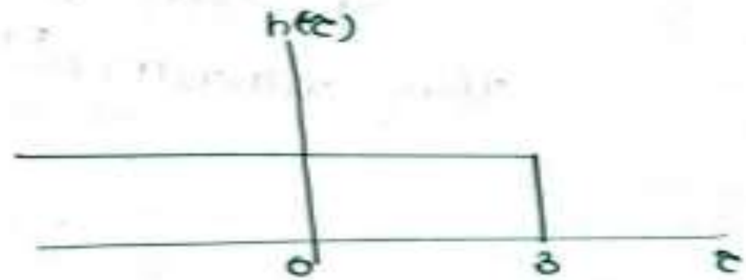
$= 0$ for $\tau < 0$ $\qquad$ $= 0$ for $\tau < -3$

$\therefore x(\tau) = e^{-3\tau}$ $\tau \geq 0$ $\qquad$ $\therefore h(\tau) = u(\tau+3) = 1$ for $\tau \geq -3$

Contd…



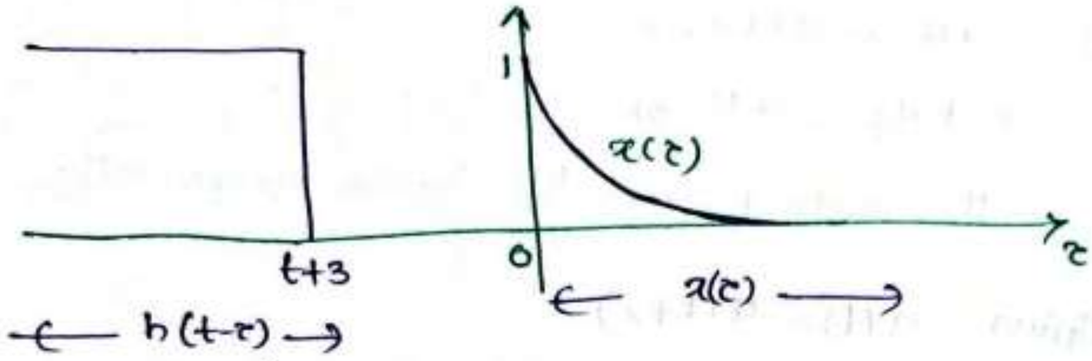




Plot the figures $x(\tau)$ and $h(-\tau)$ together on the same time axis

Contd…

for $t < -3$



for $t+3 < 0$ i.e $t < -3$ $\quad h(t-\tau) = 1$

$h(t-\tau) = 0$ for $\tau > t+3$

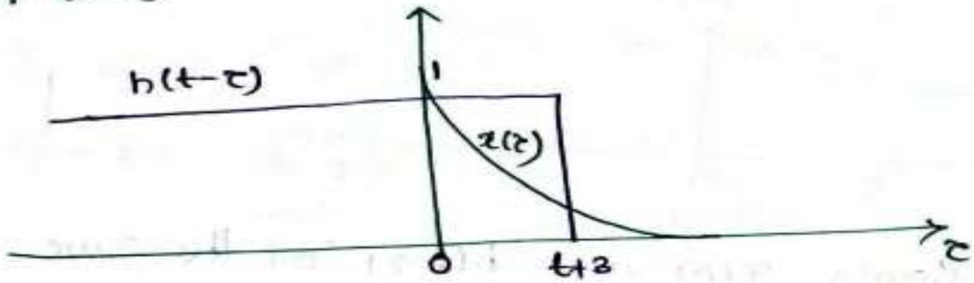
$x(\tau) = 0$ for $\tau < 0$

$x(\tau)$ and $h(t-\tau)$ donot overlap hence $x(\tau)\, h(t-\tau) = 0$

$\therefore \quad y(t) = 0$ for $t < -3$

Contd…

For $t > -3$



$x(\tau)$ and $h(t-\tau)$ are overlapped within the interval

$\tau = [0, t+3]$

$$\therefore\ y(t) = \int_0^{t+3} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^{t+3} e^{-3\tau} d\tau = \left.\frac{e^{-3\tau}}{-3}\right]_0^{t+3}$$

$$= \frac{-1}{3}\left[e^{-3(t+3)} - 1\right] =$$

$$y(t) = \frac{1 - e^{-3(t+3)}}{3} \quad \text{for } t > -3$$

$$\therefore\ y(t) = 0 \quad \text{for } t < -3$$

Contd…

② The input and the impulse response to the system are given by $x(t) = u(t+2)$

& $h(t) = u(t-3)$

Determine the output of the system graphically.

Soln: Given $x(t) = u(t+2)$

& $h(t) = u(t-3)$

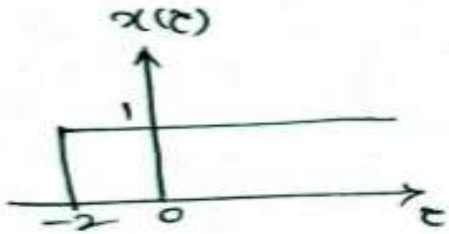
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

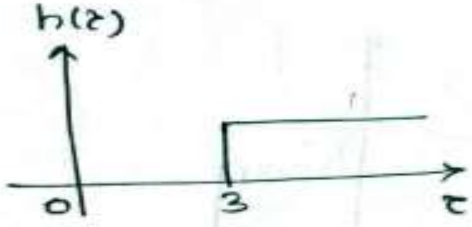
$$x(\tau) = u(\tau+2) = 1 \text{ for } \tau \geq -2$$
$$= 0 \text{ for } \tau < -2$$

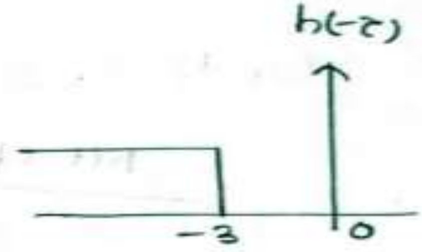
$$h(\tau) = u(\tau-3) = 1 \text{ for } \tau \geq 3$$
$$= 0 \text{ for } \tau < +3$$

$h(-\tau)$

Contd…

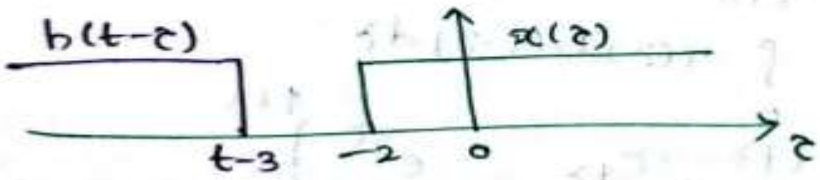






Plot the figures $x(\tau)$ and $h(t-\tau)$ on the same axis

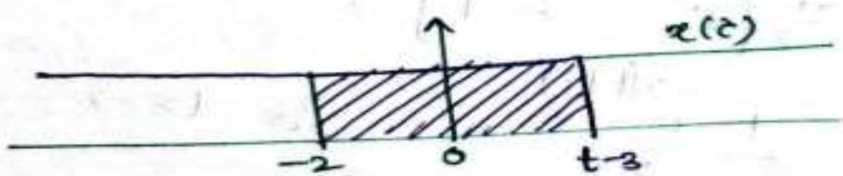
For $t-3 < -2$ i.e $t<1$



no ovelap. hence

$y(t)=0$

For $-2 < \tau < t-3$



Contd…

$$\therefore\ y(t) = \int_{-2}^{t-3} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{-2}^{t-3} (1)(1)\,d\tau$$

$$= \tau\Big]_{-2}^{t-3}$$

$$= t-3+2$$

$$= t-1$$

$$\therefore\ y(t) = t-1 \quad \text{for } t>1$$

or $\quad y(t) = (t-1)\,u(t-1)$

③ The impulse response of the circuit is given as $h(t)=e^{-2t}u(t)$.
This circuit is excited by an input of $x(t)=e^{-4t}\left(u(t)-u(t-2)\right)$
Determine the output of the circuit.

Soln: $h(t)=e^{-2t}u(t)=e^{-2t}$ for $t\geq 0$

& $x(t)=e^{-4t}\left[u(t)-u(t-2)\right]$

$=e^{-4t} \quad 0\leq t\leq 2$

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$x(t)$ and $h(t)$ in terms of τ can be written as

$x(\tau)=e^{-4\tau} \quad 0\leq \tau\leq 2$

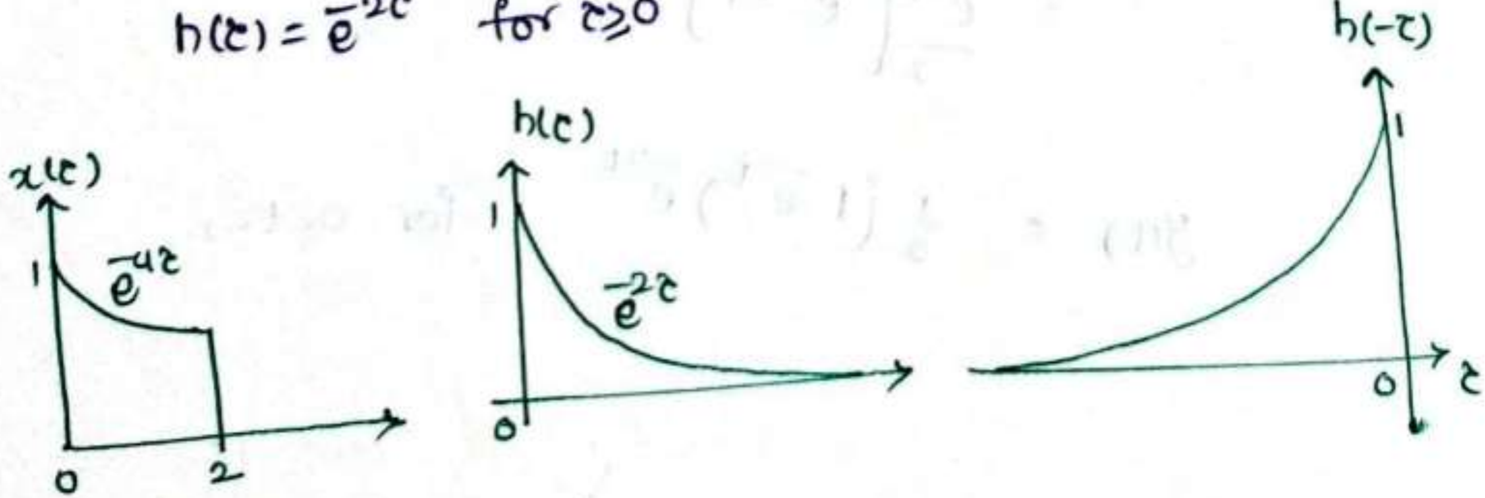
$h(\tau)=e^{-2\tau}$ for $\tau\geq 0$

Contd…

x(t) and h(t) in terms of τ can be written as

$$x(\tau) = e^{-4\tau} \quad 0 \le \tau \le 2$$

$$h(\tau) = e^{-2\tau} \quad \text{for } \tau \ge 0$$

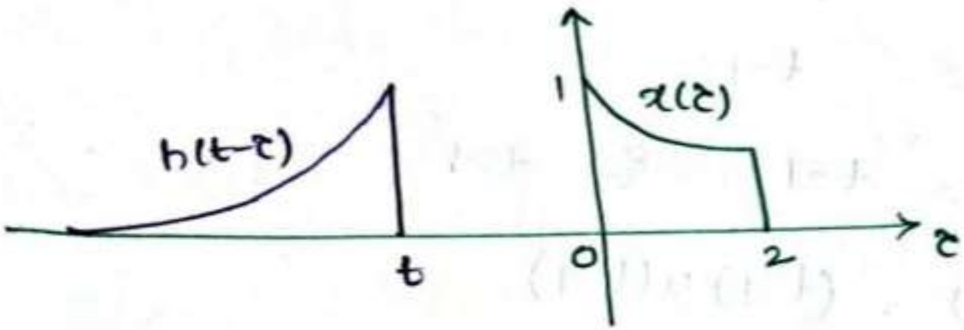


Contd…

$$h(t-\tau) = e^{-2(t-\tau)}u(t-\tau) = e^{-2(t-\tau)} \quad \text{for } t-\tau>0$$
$$\tau<t$$

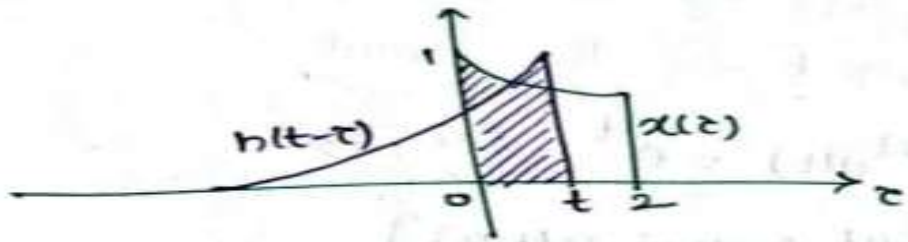
The plots of $x(\tau)$ and $h(t-\tau)$ drawn on the same time axis as shown below.

For $t<0$, the plots do not overlap.



Contd…

For $0 \leq t \leq 2$



$$\therefore \quad y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

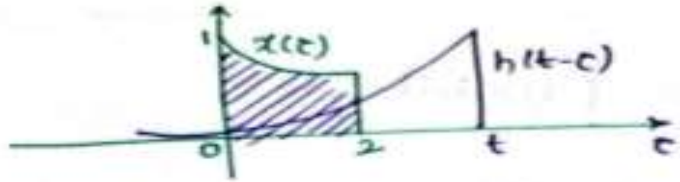
$$= \int_0^t \left(e^{-4\tau}\right)\left(e^{-2(t-\tau)}\right) d\tau$$

$$= e^{-2t} \int_0^t e^{-2\tau}\, d\tau = e^{-2t} \left. \frac{e^{-2\tau}}{-2} \right]_0^t$$

$$= \frac{e^{-2t}}{-2} \left[e^{-2t} - 1 \right]$$

$$y(t) = \frac{1}{2}\left(1 - e^{-2t}\right) e^{-2t} \quad \text{for } 0 \leq t \leq 2$$

For $t>2$



$$\therefore \quad y(t) = \int_0^2 x(\tau)h(t-\tau)\,d\tau$$

$$= \int_0^2 e^{-4\tau} e^{-2(t-\tau)}\,d\tau$$

$$= e^{-2t}\int_0^2 e^{-2\tau}\,d\tau = e^{-2t} \left.\frac{e^{-2\tau}}{-2}\right]_0^2$$

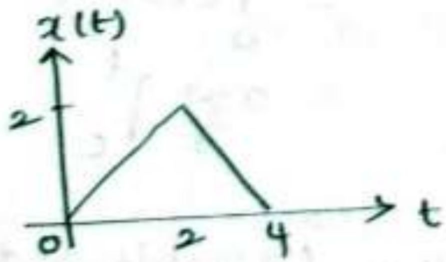
$$= -\frac{1}{2}e^{-2t}\left[e^{-4}-1\right]$$

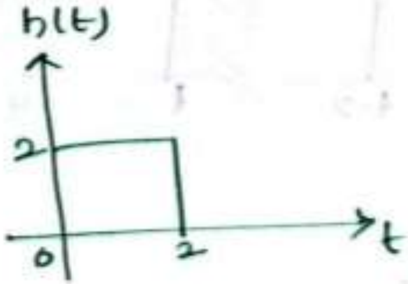
$$y(t) = \frac{1}{2}e^{-2t}(1-e^{-4}) \qquad \text{for } t>2$$

$$\therefore\ y(t) = \begin{cases} 0 & \text{for } t<0 \\ \frac{1}{2}e^{-2t}(1-e^{-2t}) & \text{for } 0<t<2 \\ \frac{1}{2}e^{-2t}(1-e^{-4}) & \text{for } t>2 \end{cases}$$

Contd…

(4) Find the convolution of the signals $x(t)$ and $h(t)$ shown below.





Soln: $x(t) = t \quad \text{for } 0 \le t \le 2$

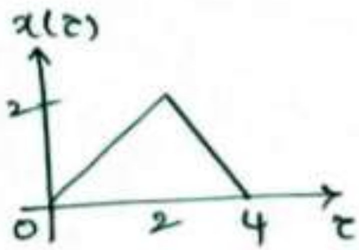
$\qquad\;\; 4-t \quad \text{for } 2 \le t < 4$

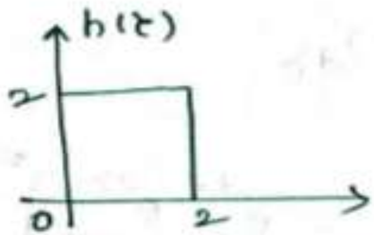
$h(t) = 2 \quad 0 \le t \le 2$

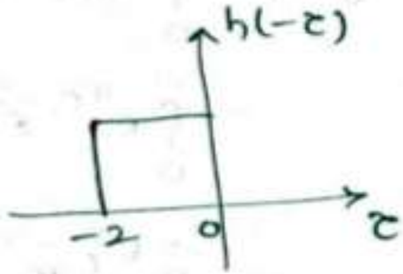
Contd…

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

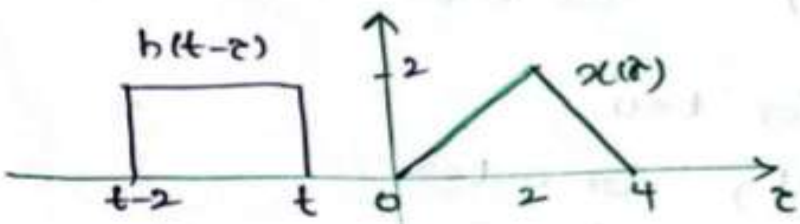
$$x(\tau) = \begin{cases} \tau & 0 \le \tau \le 2 \\ 4-\tau & 2 \le \tau < 4 \end{cases} \qquad h(\tau) = 2 \quad \text{for } 0 \le \tau \le 2$$







For $t<0$

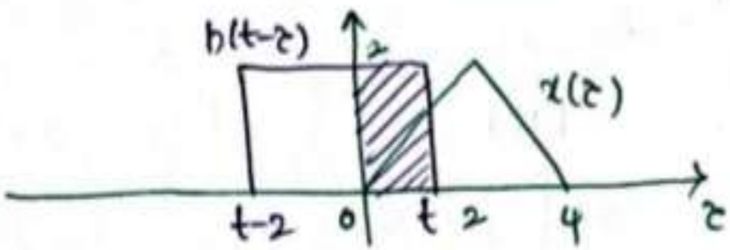


$x(\tau)$ and $h(t-\tau)$ do not overlap

$\therefore\ y(t) = 0$

Contd…

For $0 \le t \le 2$

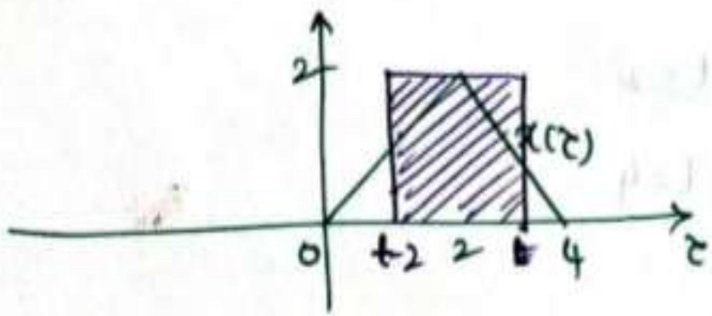


$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t \tau(2)\, d\tau$$

$$= \cancel{2}\frac{\tau^2}{\cancel{2}}\Big]_0^t = t^2$$

For $2 \le t \le 4$



$$y(t) = \int_{t-2}^{2} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-2}^{2} x(\tau)\, h(t-\tau)\, d\tau + \int_{2}^{t} x(\tau)\, h(t-\tau)$$

Contd…

$$y(t) = \int_{t-2}^{2} \tau(2)\, d\tau + \int_{2}^{t} (4-\tau)\,2\, d\tau$$

$$= \cancel{2}\frac{\tau^2}{\cancel{2}}\Big]_{t-2}^{2} + \int_{2}^{t} (8-2\tau)\, d\tau$$

$$= 4-(t-2)^2 + \left(8\tau - \frac{2\tau^2}{2}\right]_{2}^{t}$$

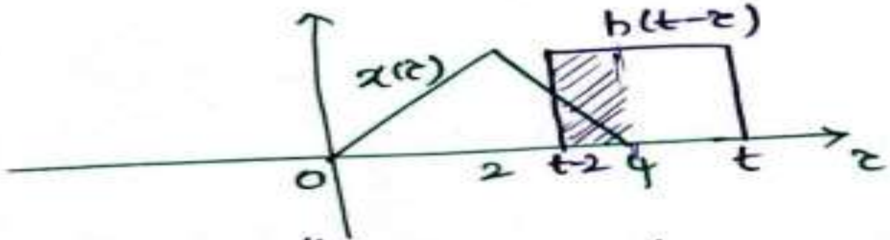
$$= 4-(t^2+4-4t) + \left(8t - \frac{t^2}{\cancel{2}}\right) - (16-4)$$

$$= 4 - t^2 - 4 + 4t + 8t - t^2 - 12$$

$$= -2t^2 + 12t - 12$$

Contd…

for $4 \le t \le 6$



$$y(t) = \int_{t-2}^{4} x(\tau) h(t-\tau) d\tau$$

$$= \int_{t-2}^{4} (4-\tau)(2) d\tau = \int_{t-2}^{4} (8-2\tau) d\tau$$

$$= \left[8\tau - \tau^2 \right]_{t-2}^{4}$$

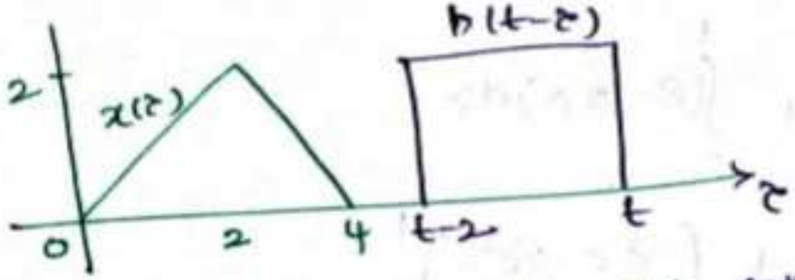
$$= (32-16) - (8t-16-(t-2)^2)$$

$$= 16 - (8t-16-t^2-4+4t)$$

$$y(t) = t^2 - 12t + 36$$

Contd…

For $t \geq 6$



$x(\tau)$ and $h(t-\tau)$ do not overlap

$\therefore \; y(t) = 0 \quad$ for $t \geq 6$

$$\therefore \; y(t) = \begin{cases} 0 & \text{for } t < 0 \\ t^2 & \text{for } 0 \leq t \leq 2 \\ -2t^2 + 12t - 12 & \text{for } 2 \leq t \leq 4 \\ t^2 - 12t + 36 & \text{for } 4 \leq t \leq 6 \\ 0 & \text{for } t \geq 6 \end{cases}$$

Concept of Correlation:

Concept of correlation

The signals may be compared on the basis of similarity of waveforms. Quantitatively, a comparison may be based upon the amount of the component of one waveform contained in the other waveform. If $x_1(t)$ and $x_2(t)$ are two waveforms, then the waveform $x_1(t)$ contains an amount $C_{12}x_2(t)$ of that particular waveform $x_2(t)$ in the interval (t_1, t_2), where

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\,x_2(t)\,dt}{\int_{t_1}^{t_2} x_2^2(t)\,dt}$$

The magnitude of the integral in the numerator might be taken as an indication of similarity. If this integral vanishes, i.e

$$\int_{t_1}^{t_2} x_1(t)\,x_2(t)\,dt = 0$$

then the two signals have no similarity over the interval (t_1, t_2). Such signals are said to be orthogonal over the specified interval.

Contd…

The integral $\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt$ forms the basis of comparison of the two signals $x_1(t)$ and $x_2(t)$ over the interval (t_1, t_2).

In general we are interested in comparing the two signals over the interval $(-\infty, \infty)$. So the test integral becomes

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt$$

However, there is a difficulty with this test integral which can be illustrated with the example of radar pulse. Figure shows a transmitted pulse and a received pulse which is delayed w.r.t. transmitted pulse by T s. Obviously, the two waveforms are identical except that one is delayed w.r.t. the other. Yet the test integral $\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt$ yields zero because the product $x_1(t)\, x_2(t)$ is zero everywhere. This indicates that the two waveforms have no measure of similarity which is obviously a wrong conclusion. Hence in order to search for a

Figure Signal comparison.

Contd…

similarity between the two waveforms, we must shift one waveform w.r.t. the other by various amounts and see whether a similarity exists for some amount of shift of one function w.r.t. the other.

Therefore, the test integral is modified as $\int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$

where τ is the searching or scanning parameter. This integral is a function of τ. This integral is known as the cross correlation function between $x_1(t)$ and $x_2(t)$ and is denoted by $R_{12}(\tau)$.

It is immaterial whether we shift the function $x_1(t)$ by an amount of τ in the negative direction or shift the function $x_2(t)$ by the same amount in the positive direction. Thus

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2(t)\, dt$$

Thus the correlation of two functions or signals or waveforms is a measure of similarity between those signals. The correlation is of two types: cross correlation and autocorrelation. The autocorrelation and cross correlation are defined separately for energy (or aperiodic) signals and power (or periodic) signals.

Cross Correlation:

The cross correlation between two different waveforms or signals is a measure of similarity or match or relatedness or coherence between one signal and the time delayed version of another signal. That means the cross correlation between two signals indicates how much one signal is related to the time delayed version of another signal.

Cross correlation of energy signals

Consider two general complex signals $x_1(t)$ and $x_2(t)$ of finite energy. The cross correlation of these two energy signals denoted by $R_{12}(\tau)$ is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt = \int_{-\infty}^{\infty} x_1(t+\tau) x_2^*(t)\, dt$$

If the two signals $x_1(t)$ and $x_2(t)$ are real, then $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2(t)\, dt$

If $x_1(t)$ and $x_2(t)$ have some similarity, then the cross correlation $R_{12}(\tau)$ will have some finite value over the range of τ. Also if

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = 0 \quad \text{i.e. if} \quad R_{12}(0) = 0$$

then the two signals $x_1(t)$ and $x_2(t)$ are called *orthogonal signals*. That is the cross correlation for orthogonal signals is zero.

Another form of cross correlation between $x_2(t)$ and $x_1(t)$ is defined as: $R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$

Contd…

In the above equations, the cross correlation function $R_{12}(\tau)$ is a function of the variable τ. The variable τ is called the *delay parameter* or the *scanning parameter* or the *searching parameter*. It is time delay or time shift of one of the two signals. The delay parameter τ determines the correlation between two signals. Two signals with no cross correlation at $\tau = 0$ can have significant cross correlation by adjusting the parameter τ. Two signals for which the cross correlation is zero for all values of τ are called *uncorrelated* or *incoherent signals*.

Properties of Cross Correlation:

Properties of cross correlation function for energy signals

Following are the properties of cross correlation for energy signals:

1. The cross correlation functions exhibit conjugate symmetry, i.e.

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

That is unlike convolution, cross correlation is not in general commutative, i.e.

$$R_{12}(\tau) \neq R_{21}(\tau)$$

2. If $\qquad R_{12}(0) = 0$

i.e. if
$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = 0$$

then the two signals are said to be orthogonal over the entire time interval.

3. The cross correlation of two energy signals corresponds to the multiplication of the Fourier transform of one signal by the complex conjugate of Fourier transform of second signal.

i.e.
$$R_{12}(\tau) \longleftrightarrow X_1(\omega) X_2^*(\omega)$$

This is known as *correlation theorem.*

Contd…

Cross correlation of power (periodic) signals

The cross correlation function $R_{12}(\tau)$ for two periodic signals $x_1(t)$ and $x_2(t)$ may be defined with the help of average form of correlation. If the two periodic signals $x_1(t)$ and $x_2(t)$ have the same time period T, then cross correlation is defined as:

$$R_{12}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_1(t) x_2^*(t-\tau)\, dt$$

then the two signals $x_1(t)$ and $x_2(t)$ are called *orthogonal signals*. That is the cross correlation for orthogonal signals is zero.

Another form of cross correlation between $x_2(t)$ and $x_1(t)$ is defined as:

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$$

Contd…

Properties of cross correlation function for power (periodic) signals

Following are the properties of cross correlation for power signals:

1. The Fourier transform of the cross correlation of two signals is equal to the multiplication of Fourier transform of one signal and complex conjugate of Fourier transform of other signal.

$$R_{12}(\tau) \longleftrightarrow X_1(\omega) X_2^*(\omega)$$

2. If $$R_{12}(0) = 0$$

i.e. if $$\underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t)\, x_2^*(t)\, dt = 0$$

then the signals are said to be orthogonal over the entire time interval.

3. The cross correlation exhibits conjugate symmetry, i.e.

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

4. Unlike convolution, the cross correlation is not commutative, i.e.

$$R_{12}(\tau) \neq R_{21}(\tau)$$

EXAMPLE Prove that $R_{12}(\tau) = R_{21}^*(-\tau)$ i.e. the cross correlation exhibits conjugate symmetry.

Solution: The cross correlation of two signals $x_1(t)$ and $x_2(t)$ is given as:

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

Let $t - \tau = n$ in the above equation for $R_{12}(\tau)$.

$$\therefore \quad R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(n+\tau)\, x_2^*(n)\, dn$$

Also we know that

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$$

Let $t = n$ in the above equation for $R_{21}(\tau)$.

$$\therefore \quad R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(n)\, x_1^*(n-\tau)\, dn$$

$$\therefore \quad R_{21}^*(\tau) = \int_{-\infty}^{\infty} x_2^*(n)\, x_1(n-\tau)\, dn$$

$$\therefore \quad R_{21}^*(-\tau) = \int_{-\infty}^{\infty} x_2^*(n)\, x_1(n+\tau)\, dn$$

Comparing the above two equations for $R_{12}(\tau)$ and $R_{21}^*(-\tau)$, we can write

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

Auto Correlation:

Autocorrelation

The autocorrelation function gives the measure of match or similarity or relatedness or coherence between a signal and its time delayed version. This means that the autocorrelation function is a special form of cross correlation function. It is defined as the correlation of a signal with itself.

The autocorrelation is defined separately for energy signals and power signals.

Autocorrelation for energy signals

The autocorrelation of an energy signal $x(t)$ is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

where τ is called the delay parameter and the signal $x(t)$ is shifted by τ in positive direction.

If $x(t)$ is shifted by τ in negative direction, then

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau)x^*(t)\, dt$$

Properties of Auto Correlation function:

Properties of autocorrelation function of energy signals

Following are the properties of autocorrelation for energy signals:

1. The autocorrelation function exhibits conjugate symmetry, i.e.

$$\boxed{R(\tau) = R^*(-\tau)}$$

Thus, it states that the real part of $R(\tau)$ is an even function of τ and the imaginary part of $R(\tau)$ is an odd function of τ.

Proof: The autocorrelation of an energy signal $x(t)$ is given by

$$R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\,dt$$

Taking the complex conjugate, we have

$$R^*(\tau) = \int_{-\infty}^{\infty} x^*(t)\,x(t-\tau)\,dt$$

$$\therefore \quad R^*(-\tau) = \int_{-\infty}^{\infty} x^*(t)\,x(t+\tau)\,dt = R(\tau)$$

$$\therefore \quad R(\tau) = R^*(-\tau)$$

Contd…

2. The value of autocorrelation function of an energy signal at origin (i.e. at $\tau = 0$) is equal to the total energy of that signal. i.e.

$$R(0) = E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Proof: We have

Proof: We have

$$R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t - \tau) \, dt$$

Putting $\tau = 0$ gives

$$R(0) = \int_{-\infty}^{\infty} x(t)x^*(t) \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = E$$

Contd…

3. If τ is increased in either direction, the autocorrelation $R(\tau)$ reduces. As τ reduces autocorrelation, $R(\tau)$ increases and it is maximum at $\tau = 0$, i.e. at the origin. Therefore,

$$\boxed{|R(\tau)| \leq R(0) \quad (\text{for all } \tau)}$$

Proof: Consider the functions $x(t)$ and $x(t + \tau)$. $[x(t) \pm x(t + \tau)]^2$ is always greater than or equal to zero since it is squared, i.e.

$$x^2(t) + x^2(t+\tau) \pm 2x(t)\, x(t+\tau) \geq 0$$

or
$$x^2(t) + x^2(t+\tau) \geq \pm 2x(t)\, x(t+\tau)$$

Integrating both the sides, we get

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2\, dt \geq 2 \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

$\therefore$ $\quad E + E \geq 2R(\tau) \quad$ [If $x(t)$ is real valued function]

$\therefore$ $$E \geq R(\tau)$$

or $$R(0) \geq |R(\tau)| \quad (\text{Since } R(0) = E)$$

4. The autocorrelation function $R(\tau)$ and energy spectral density function $\psi(\omega)$ of energy signal form a Fourier transform pair.

$$\boxed{R(\tau) \xleftrightarrow{FT} \psi(\omega)}$$

Contd…

Autocorrelation theorem

The autocorrelation theorem states that the Fourier transform of autocorrelation function $R(\tau)$ yields the energy density function of signal $x(t)$, i.e.

$$F[R(\tau)] = |X(\omega)|^2 = \psi(\omega)$$

Proof: The Fourier transform of autocorrelation function $R(\tau)$ is:

$$F[R(\tau)] = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, e^{-j\omega\tau}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \int_{-\infty}^{\infty} x(t-\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$= X(\omega) \int_{-\infty}^{\infty} x(t-\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

Contd…

Letting $t - \tau = n$ in the second integral, we have

$$F[R(\tau)] = X(\omega) \int_{-\infty}^{\infty} x(n)\, e^{j\omega n}\, dn$$

$$= X(\omega)\, X(-\omega) = |X(\omega)|^2$$

$$= \psi(\omega)$$

Autocorrelation function for power (periodic) signals

The autocorrelation function of a periodic signal with any period T is given by

$$R(\tau) = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

Contd…

Properties of autocorrelation function for power signals

Following are the properties of autocorrelation function for power signals:

1. The autocorrelation function exhibits conjugate symmetry, i.e.

$$R(\tau) = R^*(-\tau)$$

Proof: We have

$$R(\tau) = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

$\therefore$
$$R^*(\tau) = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t-\tau)\, dt$$

$\therefore$
$$R^*(-\tau) = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t+\tau)\, dt = R(\tau)$$

$\therefore$
$$R(\tau) = R^*(-\tau)$$

2. The autocorrelation function at origin is equal to the average power of that signal, i.e.

$$R(0) = P = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Contd…

2. The autocorrelation function at origin is equal to the average power of that signal, i.e.

$$R(0) = P = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt$$

3. The autocorrelation function $R(\tau)$ has maximum value at the origin, i.e.

$$|R(\tau)| \leq R(0)$$

The value of autocorrelation reduces as τ increases from origin.

4. The autocorrelation function $R(\tau)$ and power spectral density $S(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \xleftrightarrow{FT} S(\omega)$$

5. The autocorrelation function is periodic with the same period as the periodic signal itself, i.e.

$$R(\tau) = R(\tau \pm nT), \quad n = 1, 2, 3, \ldots$$

Relation between Convolution and Correlation:

RELATION BETWEEN CONVOLUTION AND CORRELATION

There is a striking resemblance between the operation of convolution and correlation. Indeed the two integrals are closely related. To obtain the cross correlation of $x_1(t)$ and $x_2(t)$ according to the equation $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$, we multiply $x_1(t)$ with function $x_2(t)$ displaced by τ sec. The area under the product curve is the cross correlation between $x_1(t)$ and $x_2(t)$ at τ. On the other hand, the convolution of $x_1(t)$ and $x_2(t)$ at $t = \tau$ is obtained by folding $x_2(t)$ backward about the vertical axis at the origin and taking the area under the product curve of $x_1(t)$ and the folded function $x_2(-t)$ displaced by τ. It, therefore, follows that the cross correlation of $x_1(t)$ and $x_2(t)$ is the same as the convolution of $x_1(t)$ and $x_2(-t)$.

The same conclusion can be arrived at analytically as follows:

The convolution of $x_1(t)$ and $x_2(-t)$ is given by

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(\tau - t)\, d\tau$$

Contd…

Replacing the dummy variable τ in the above integral by another variable n, we have

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n-t)\, dn$$

Changing the variable from t to τ, we get

$$x_1(\tau) * x_2(-\tau) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n-\tau)\, dn = R_{12}(\tau)$$

Hence $\qquad R_{12}(\tau) = x_1(t) * x_2(-t)\big|_{t=\tau}$

Similarly, $\qquad R_{21}(\tau) = x_2(t) * x_1(-t)\big|_{t=\tau}$

All of the techniques used to evaluate the convolution of two functions can be directly applied in order to find the correlation of two functions. Similarly, all of the results derived for convolution also apply to correlation.

If one of the function is an even function of t, let us say $x_2(t)$ is an even function of t, i.e.

$$x_2(t) = x_2(-t)$$

then the cross correlation and convolution are equivalent.

Part-2
Fourier Series

Fourier Series:

Fourier Series Representation of Periodic Signals :

A periodic signal is the one which repeat itself periodically over $-\infty < t < \infty$.

For example, $x(t) = A \sin\omega_0 t$ is a periodic signal with period $T_0 = \frac{2\pi}{\omega_0}$.

Let us consider a signal $x(t)$ which is a sum of sine and cosine functions whose frequencies are integer multiple of ω_0 as shown below.

$$x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \cdots + a_k \cos k\omega_0 t + b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \cdots + b_k \sin k\omega_0 t \quad \text{--- (1)}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \quad \text{—— (2)}$$

where $\omega_0 = \frac{2\pi}{T_0}$

$a_0, a_n, b_n \rightarrow$ Constants

$\omega_0 \rightarrow$ fundamental frequency

If the signal $x(t)$ is to be periodic, it has to satisfy the following condition

$$x(t) = x(t+T_0)$$

$$x(t+T_0) = a_0 + \sum_{n=1}^{k} \left[a_n \cos n\omega_0(t+T_0) + b_n \sin n\omega_0(t+T_0) \right]$$

$$= a_0 + \sum_{n=1}^{k} \left[a_n \cos(n\omega_0 t + n\omega_0 T_0) + b_n \sin(n\omega_0 t + n\omega_0 T_0) \right]$$

$$x(t+T_0) = a_0 + \sum_{n=1}^{k} \left[a_n \cos(n\omega_0 t + 2n\pi) + b_n \sin(n\omega_0 t + 2n\pi) \right]$$

$$\because T_0 = \frac{2\pi}{\omega_0}$$

$$x(t+T_0) = a_0 + \sum_{n=1}^{k} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \quad \text{--- (3)}$$

$$\therefore x(t+T_0) = x(t)$$

hence $x(t)$ is a periodic signal with period T_0

As $k \to \infty$ in the above eqn (3) we obtain Fourier series representation of the periodic signal.

Any periodic signal can be represented as an infinite sum of sine and cosine functions. This series of sine and cosine terms are known as Trigonometric Fourier series and can be represented as

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \quad \text{—} (4)$$

where a_0 is dc component

a_0, a_n, b_n are called Fourier Series coefficients.

Evaluation of Fourier Series Coefficients

Evaluation of Fourier Series Coefficients:

The constants a_0, a_n, b_n are called as fourier series coefficients.

To evaluate a_0, we shall integrate both sides of eqn (1) over one period (t_0, t_0+T) of $x(t)$

$\therefore$ eqn (4) becomes

$$\int_{t_0}^{t_0+T} x(t)\,dt = \int_{t_0}^{t_0+T} \left[a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]\right] dt$$

$$= \int_{t_0}^{t_0+T} a_0\, dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t\, dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t\, dt \quad \text{——— (5)}$$

But we know that, the net areas of sine and cosine functions over one complete period is zero.

i.e
$$\int_{t_0}^{t_0+T} \cos n\omega_0 t\, dt = 0 \quad \text{——— (6)}$$

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t\, dt = 0 \quad \text{——— (7)}$$

∴ From eqns (5), (6) & (7)

$$\int_{t_0}^{t_0+T} x(t)\, dt = a_0 \left[t\right]_{t_0}^{t_0+T} + 0 + 0$$

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 \left[t\right]_{t_0}^{t_0+T} + 0 + 0$$

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 . T$$

$$\boxed{\therefore \quad a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt} \quad \text{—— (8)}$$

To evaluate a_n and b_n we have to use the following formulae

$$\int_{t_0}^{t_0+T} \cos n\omega_0 t \cos m\omega_0 t\,dt = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T_0}{2} & \text{for } m = n \end{cases} \quad \text{—— (9)}$$

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t \sin m\omega_0 t\,dt = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T_0}{2} & \text{for } m = n \end{cases} \quad \text{—— (10)}$$

$$\int_{t_0}^{t_0+T} \sin m\omega_0 t \cos n\omega_0 t\,dt = 0 \quad \text{for all } m, n \quad \text{—— (11)}$$

Evaluation of a_n :

To find a_n multiply eqn ④ by $\cos m\omega_0 t$ and integrate over one period

i.e., $\int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt = \int_{t_0}^{t_0+T} a_0 \cos m\omega_0 t \, dt +$

$$\int_{t_0}^{t_0+T} \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \cos m\omega_0 t \, dt$$

$$= \int_{t_0}^{t_0+T} a_0 \cos m\omega_0 t \, dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \cos m\omega_0 t \, dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \cos m\omega_0 t \, dt$$

In the above eqn 1st & 3rd terms are zero using eqn (9) & (11)

$$\therefore \int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt = 0 + a_n \cdot \frac{T}{2} + 0$$

for $m=n$

$$\therefore a_m \cdot \frac{T}{2} = \int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt$$

$$a_m = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt$$

$$\therefore \boxed{a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t \, dt}$$

Evaluation of b_n:

To find b_n multiply the equation (4) by $\sin m\omega_0 t$ and integrate over one period

$$\int_{t_0}^{t_0+T} x(t)\sin m\omega_0 t\,dt = \int_{t_0}^{t_0+T} a_0 \sin m\omega_0 t\,dt +$$

$$\int_{t_0}^{t_0+T} \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]\sin m\omega_0 t\,dt$$

$$= \int_{t_0}^{t_0+T} a_0 \sin m\omega_0 t\,dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \sin m\omega_0 t\,dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \sin m\omega_0 t\,dt$$

From ⑩ & ⑪

$$\int_{t_0}^{t_0+T} x(t) \sin m\omega_0 t \, dt = 0 + 0 + b_m \frac{T}{2} \quad \text{for } m=n$$

$$b_m = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin m\omega_0 t \, dt$$

$$\boxed{\therefore \; b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t \, dt \qquad n = 1, 2, \ldots\ldots}$$

$\therefore$ In Trigonometric Fourier series $x(t)$ can be represented

as $$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

where $$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \, dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T} \int^{t_0+T} x(t) \sin n\omega_0 t \, dt$$

Polar Fourier Series

Polar Fourier series representation (or) Cosine Representation (or) Compact Trigonometric fourier series Representation :

The polar Fourier series is the modified form of the trigonometric fourier series. The polar Fourier series is derived from Trigonometric Fourier series as follows.

Trigonometric Fourier series is given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t] \quad \text{--- (1)}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2+b_n^2}} \cos n\omega_0 t + \frac{b_n}{\sqrt{a_n^2+b_n^2}} \sin n\omega_0 t \right] \quad \text{--- (2)}$$

let $\cos\phi = \frac{a_n}{\sqrt{a_n^2+b_n^2}}$ ——— ③

& $\sin\phi = \frac{b_n}{\sqrt{a_n^2+b_n^2}}$ ——— ④

$\therefore$ From eqn ③ & ④

$$\tan\phi = \frac{b_n}{a_n}$$

$$\phi = \tan^{-1}\left(\frac{b_n}{a_n}\right) \text{ ——— ⑤}$$

From eqns ②, ③ & ④

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2}\left[\cos\phi \cos n\omega_0 t + \sin\phi \sin n\omega_0 t\right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2} \cos(n\omega_0 t - \phi)$$

let $D_0 = a_0$

& $D_n = \sqrt{a_n^2 + b_n^2}$

$$\therefore x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t - \phi)$$

where $\phi = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

Exponential Fourier Series

Exponential Fourier series is simpler and more compact hence it is mostly used. We will derive an exponential Fourier series in the following way.

Exponential Fourier series:

Trigonometric Fourier series is given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \quad \text{—} (1)$$

Using Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \Rightarrow \cos n\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \quad \text{—} (2)$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \Rightarrow \sin n\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \quad \text{—} (3)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \quad \text{--- (1)}$$

Using Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \Rightarrow \cos n\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \quad \text{--- (2)}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \Rightarrow \sin n\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \quad \text{--- (3)}$$

From eqns (1), (2) & (3)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \right) + b_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right) \right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} e^{jn\omega_0 t}\left[\frac{a_n}{2} + \frac{b_n}{2j}\right] +$$

$$\sum_{n=1}^{\infty} e^{-jn\omega_0 t}\left[\frac{a_n}{2} - \frac{b_n}{2j}\right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} e^{jn\omega_0 t}\left[\frac{a_n}{2} + \frac{j\, b_n}{2j^2}\right] +$$

$$\sum_{n=1}^{\infty} e^{-jn\omega_0 t}\left[\frac{a_n}{2} - j\frac{b_n}{2j^2}\right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} e^{jn\omega_0 t}\left[\frac{a_n - jb_n}{2}\right] +$$

$$\sum_{n=1}^{\infty} e^{-jn\omega_0 t}\left[\frac{a_n + jb_n}{2}\right]$$

let $c_0 = a_0$

$$c_n = \frac{1}{2}[a_n - jb_n]$$

$$c_{-n} = \frac{1}{2}[a_n + jb_n]$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega_0 t}$$

Change the index $-n$ to n in the 2^{nd} summation

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega_0 t}$$

$$\boxed{\therefore \; x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}}$$

Evaluation of C_n :

$$C_n = \frac{1}{2}[a_n - jb_n] \quad \text{——} ①$$

but $$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t \, dt \quad \text{——} ②$$

& $$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin n\omega_0 t \, dt \quad \text{——} ③$$

From eqns ①, ② & ③

$$C_n = \frac{1}{2}\left[\frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t \, dt - j\frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin n\omega_0 t \, dt\right]$$

$$C_n = \frac{1}{T}\left[\int_{t_0}^{t_0+T} x(t)\left[\cos n\omega_0 t - j\sin n\omega_0 t\right] dt\right.$$

$$\boxed{C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t} \, dt}$$

Relation Between Trigonometric and Exponential FS

Derive the relation between Trigonometric Fourier series and exponential Fourier series:

In Trigonometric Fourier series $x(t)$ is given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t] \quad \text{---} \quad (1)$$

In Exponential Fourier series $x(t)$ is given by

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{+jn\omega_0 t} \quad \text{---} \quad (2)$$

$$x(t) = C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + C_3 e^{j3\omega_0 t} + \cdots\cdots +$$

$$C_{-1} e^{-j\omega_0 t} + C_{-2} e^{-j2\omega_0 t} + C_{-3} e^{-j3\omega_0 t} + \cdots\cdots$$

$$x(t) = C_0 + C_1 [\cos\omega_0 t + j\sin\omega_0 t] + C_2[\cos 2\omega_0 t + j\sin 2\omega_0 t] +$$

$$\cdots\cdots + C_{-1}[\cos\omega_0 t - j\sin\omega_0 t] +$$

$$C_{-2}[\cos 2\omega_0 t - j\sin 2\omega_0 t] + \cdots\cdots$$

$$x(t) = C_0 + (C_1 + C_{-1})\cos\omega_0 t + (C_2 + C_{-2})\cos 2\omega_0 t + \cdots\cdots$$

$$\cdots\cdots + j(C_1 - C_{-1})\sin\omega_0 t + j(C_2 - C_{-2})\sin 2\omega_0 t + \cdots\cdots$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} (C_n + C_{-n}) \cos n\omega_0 t +$$

$$j\sum_{n=1}^{\infty} (C_n - C_{-n}) \sin n\omega_0 t$$

$$x(t) = C_0 + \sum_{n=1}^{\infty}\left[(C_n + C_{-n})\cos n\omega_0 t + j(C_n - C_{-n})\sin n\omega_0 t\right] \quad \text{--- (3)}$$

Compare eqns (1) & (3)

$$C_0 = a_0 \quad \text{--- (4)}$$

$$a_n = C_n + C_{-n} \quad \text{--- (5)}$$

$$b_n = j(C_n - C_{-n}) \quad \text{--- (6)}$$

eqn (6) $\times j$

$$j b_n = (-1)(C_n - C_{-n}) \quad \text{--- (7)}$$

add eqn (5) & (7)

$$a_n + j b_n = 2C_{-n}$$

$$C_{-n} = \frac{1}{2}\left[a_n + j b_n\right]$$

eqns (5) - (7)

$$a_n = c_n + \cancel{c_{-n}}$$

$$jb_n = -c_n + \cancel{c_{-n}}$$

$$- \qquad + \qquad -$$

$$a_n - jb_n = 2c_n$$

$$c_n = \frac{1}{2}\left[a_n - jb_n\right]$$

$\therefore$ The relation b/w T.F.S & E.F.S is

$$\boxed{\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{1}{2}\left[a_n - jb_n\right] \\ c_{-n} &= \frac{1}{2}\left[a_n + jb_n\right] \end{aligned}}$$

Derive the polar Fourier series from Exponential Fourier series & hence prove that $D_n = 2|C_n|$

In Exponential Fourier Series $x(t)$ is given by

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \quad \text{—— (1)}$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} C_n e^{jn\omega_0 t}$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_{-n} e^{-jn\omega_0 t} \quad \text{—— (2)}$$

But we have

$$C_0 = a_0 \quad \text{—— (3)}$$

$$C_n = \frac{1}{2}[a_n - jb_n] \quad \text{—— (4)}$$

$$C_{-n} = \frac{1}{2}[a_n + jb_n] \quad \text{—— (5)}$$

From eqns ②, ③, ④ & ⑤

$$x(t) = a_0 + \sum_{n=1}^{\infty} \frac{1}{2}[a_n - jb_n] e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{1}{2}[a_n + jb_n] e^{-jn\omega_0 t}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right] + \sum_{n=1}^{\infty} b_n \left(\frac{e^{-jn\omega_0 t} - j e^{jn\omega_0 t}}{2}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right] \quad ⑥$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2+b_n^2}} \cos n\omega_0 t + \frac{b_n}{\sqrt{a_n^2+b_n^2}} \sin n\omega_0 t\right] \quad ⑦$$

let $\cos\phi = \dfrac{a_n}{\sqrt{a_n^2+b_n^2}}$ ⑧ & $\sin\phi = \dfrac{b_n}{\sqrt{a_n^2+b_n^2}}$ ⑨

$$\therefore \tan\phi = \frac{b_n}{a_n}$$

$$\phi = \tan^{-1}\left(\frac{b_n}{a_n}\right) \quad ⑨$$

From eqn (7), (8), (9)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2} \left[\cos\phi \cos n\omega_0 t + \sin\phi \sin n\omega_0 t\right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2} \cos(n\omega_0 t - \phi) \quad \text{—— (10)}$$

Let $a_0 = D_0$ & $\sqrt{a_n^2+b_n^2} = D_n$ —— (11)

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t - \phi) \quad \text{—— (12)}$$

$$C_n = \frac{1}{2}[a_n - j b_n]$$

$$|C_n| = \frac{1}{2}|a_n - j b_n| = \frac{1}{2}\sqrt{a_n^2+b_n^2}$$

From eqn. (11)

$$|C_n| = \frac{1}{2} D_n$$

$$\boxed{\therefore \quad D_n = 2|C_n|}$$

Dirchlet Conditions

Convergence of Fourier Series: (Dirchlet Conditions):

The Fourier series is convergent if the signal $x(t)$ satisfies the following conditions. Those conditions are called Dirchlet conditions.

i) Single valued property:

$x(t)$ must have only one value at any time instant within the given interval.

ii) Finite peaks:

$x(t)$ should have finite number of maxima and minima within the given interval

iii) Finite Discontinuities:

$x(t)$ should have at the most finite number of discontinuities within the given interval.

iv) Absolute integrability:

$x(t)$ should be absolute integrable

i.e. $\int_{\langle T \rangle} |x(t)|\,dt < \infty$

Symmetry Conditions:

Any signal can be represented as a sum of even and odd functions.

i.e., $x(t) = x_e(t) + x_o(t)$ —— ①

where $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$ —— ②

and $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$ —— ③

These equations are used to derive the symmetry conditions.

i) Odd functions have only sine terms:

In Trigonometric Fourier series

$$a_n = \frac{2}{T}\int_{<T>} x(t)\cos n\omega_0 t \, dt \quad \text{---} (4)$$

& $$b_n = \frac{2}{T}\int_{<T>} x(t)\sin n\omega_0 t \, dt \quad \text{---} (5)$$

For odd functions $x_e(t) = 0$

$\therefore$ from eqn (1) $x(t) = x_o(t)$ --- (6)

From eqns (4) & (6)

$$a_n = \frac{2}{T}\int_{<T>} x_o(t)\cos n\omega_0 t \, dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{x_o(t)}_{\text{odd}} * \underbrace{\cos n\omega_0 t}_{\text{Even}} \, dt \Rightarrow \text{odd}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x_o(t)\cos n\omega_0 t\, dt$$

$$\Downarrow \qquad \Downarrow$$

odd * Even ⇒ odd

$$\boxed{\therefore a_n = 0} \quad \text{--- (7)}$$

$\because$ Since for odd functions $\int_{-a}^{a} (\text{odd fn})\,dt = 0$

From eqn (5) & (6)

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x_o(t)\sin n\omega_0 t\, dt$$

$$\Downarrow \qquad \Downarrow$$

odd * odd ⇒ even

$$\therefore b_n = \frac{2}{T}\times 2\int_{0}^{T/2} x(t)\sin n\omega_0 t\, dt$$

$$\boxed{b_n = \frac{4}{T}\int_{0}^{T/2} x(t)\sin n\omega_0 t} \quad \text{--- (8)}$$

∴ In Trigonometric F.S.

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \quad \text{—— (9)}$$

From the above eqn if x(t) is odd, it contain only sine terms

ii) Even functions have only cosine terms:

For even functions $x_o(t) = 0$

$$\therefore \quad x(t) = x_e(t)$$

$$a_n = \frac{2}{T}\int_{<T>} x(t)\cos n\omega_0 t\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x_e(t)\cos n\omega_0 t\,dt$$

even * even ⇒ even

$$a_n = \frac{2}{T}\times 2\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$\boxed{a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t} \quad (10)$$

$$b_n = \frac{2}{T}\int_{<T>} x(t)\sin n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{<T>} x(t) \sin n\omega_0 t \, dt$$

$$= \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{x_e(t)}_{\text{even}} \underbrace{\sin n\omega_0 t}_{\text{odd}} \, dt$$

even * odd $\Rightarrow$ odd

$$\int_{-T/2}^{T/2} (\text{even function})\,dt = 2\int_{0}^{T/2} (\text{even fn})\,dt$$

$$\int_{-T/2}^{T/2} (\text{odd function})\,dt = 0$$

$$\boxed{b_n = 0} \rightarrow (11)$$

$$\therefore \quad x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right]$$

From eqns (10) & (11)

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$\therefore$ $\boxed{\text{Even function contain only cosine terms}}$

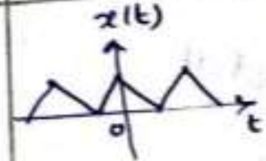
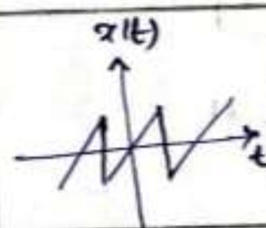
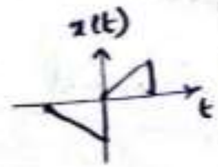
Half wave Symmetry:

A periodic signal is said to have a halfwave symmetry if it satisfies the following condition

$$\boxed{x(t) = -x\left(t \pm \frac{T}{2}\right)}$$

The Fourier series expansion of such type of periodic signals contain odd harmonics only.

Summary of symmetry Conditions

Type of Symmetry	Condition	Example	a_0	a_n	b_n	Property
Even	$x(t)=x(-t)$		$a_0=\frac{2}{T}\int_0^{T/2}x(t)dt$ $\omega_0=\frac{2\pi}{T}$ T - period	$a_n=\frac{4}{T}\int_0^{T/2}x(t)\cos n\omega_0 t\,dt$	0	Cosine terms only
odd	$x(t)=-x(-t)$		0	0	$b_n=\frac{4}{T}\int_0^{T/2}x(t)\sin n\omega_0 t\,dt$	Sine terms only
Halfwave	$x(t)=-x(t\pm\frac{T}{2})$		0	$a_n=\frac{4}{T}\int_0^{T/2}x(t)\cos n\omega_0 t\,dt$	$b_n=\frac{4}{T}\int_0^{T/2}x(t)\sin n\omega_0 t\,dt$	odd n only

Properties of Fourier Series

Properties of Fourier Series:

1) Linearity:

Let us consider two signals $x_1(t)$ and $x_2(t)$ with period T.

If $FS[x_1(t)] = C_n$

& $FS[x_2(t)] = D_n$

Then $FS[Ax_1(t) + Bx_2(t)] = AC_n + BD_n$

Proof: $FS[x(t)] = C_n = \frac{1}{T}\int_{<T>} x(t)e^{-jn\omega_0 t}\,dt$

$$FS[Ax_1(t)+Bx_2(t)] = \frac{1}{T}\int_{<T>} (Ax_1(t)+Bx_2(t))e^{-jn\omega_0 t}\,dt$$

$$= A\cdot\frac{1}{T}\int_{<T>} x_1(t)e^{-jn\omega_0 t}\,dt + B\cdot\frac{1}{T}\int_{<T>} x_2(t)e^{-jn\omega_0 t}\,dt$$

$$= AC_n + B\cdot D_n$$

$$\boxed{\therefore\ FS[Ax_1(t)+Bx_2(t)] = AC_n + BD_n}$$

2) Time shifting:

stmt: If $FS[x(t)] = C_n$

Then $FS[x(t-t_0)] = e^{-jn\omega_0 t_0} \cdot C_n$

proof: $FS[x(t)] = C_n = \frac{1}{T}\int_{<T>} x(t) e^{-jn\omega_0 t} dt$

$$FS[x(t-t_0)] = \frac{1}{T}\int_{<T>} x(t-t_0) e^{-jn\omega_0 t} dt$$

let $t - t_0 = p \Rightarrow t = p + t_0$

$dt = dp$

$$FS[x(t-t_0)] = \frac{1}{T}\int_{<T>} x(p)\, e^{-jn\omega_0 (p+t_0)} dp$$

$$= \frac{1}{T}\int_{<T>} x(p)\, e^{-jn\omega_0 p}\, e^{-jn\omega_0 t_0} dp$$

$$= e^{-jn\omega_0 t_0} \cdot \frac{1}{T}\int_{<T>} x(p)\, e^{-jn\omega_0 p} dp$$

$$\boxed{FS[x(t-t_0)] = e^{-jn\omega_0 t_0} C_n}$$

3) Time reversal:

stmt: If $FS[x(t)] = C_n$

Then $FS[x(-t)] = C_{-n}$

proof: $FS[x(t)] = C_n = \frac{1}{T}\int_{<T>} x(t)e^{-jn\omega_0 t}\,dt$

$$FS[x(-t)] = \frac{1}{T}\int_{<T>} x(-t)e^{-jn\omega_0 t}\,dt$$

let $-t = p$

$dt = -dp$

$$FS[x(-t)] = \frac{1}{T}\int_{<T>} x(p)e^{-jn\omega_0(-p)}(-dp)$$

$$= \frac{1}{T}\int_{<T>} x(p)e^{-j(-n)\omega_0 p}\,dp$$

$$= C_{-n}$$

$$\boxed{FS[x(-t)] = C_{-n}}$$

4) Time scaling :

The time scaled signal of $x(t)$ is denoted as $x(at)$. If $a<1$, the resulting time scaled signal is expanded version of $x(t)$. If $a>1$, then $x(at)$ is compressed version of $x(t)$.

If the fundamental ~~frequen~~ period of $x(t)$ is T, then the fundamental period of $x(at)$ is $\frac{T}{a}$ and the fundamental frequency is $a\omega_0$. The Fourier series coefficients of $x(at)$ is same as $x(t)$ but the harmonics are now at the frequencies $\pm a\omega_0, \pm 2a\omega_0, \pm 3a\omega_0, \pm 4a\omega_0, \cdots$

5) Frequency shifting:

Stmt: If $FS[x(t)] = C_n$

Then $FS[e^{jn_o\omega_o t} x(t)] = C_{n-n_o}$

Proof: $FS[x(t)] = C_n = \frac{1}{T}\int_{\langle T\rangle} x(t) e^{-jn\omega_o t} dt$

$$FS[e^{jn_o\omega_o t} x(t)] = \frac{1}{T}\int_{\langle T\rangle} e^{jn_o\omega_o t} x(t) e^{-jn\omega_o t} dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t) e^{-j(n-n_o)\omega_o t} dt$$

$$= C_{n-n_o}$$

$$\boxed{FS[e^{jn_o\omega_o t} x(t)] = C_{n-n_o}}$$

6) Time differentian :

Stmt : If $F_S[x(t)] = C_n$

Then $F_S\left[\frac{d}{dt}x(t)\right] = jn\omega_0 C_n$

Proof : $F_S[x(t)] = C_n = \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jn\omega_0 t}\,dt$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\frac{d}{dt}x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}(jn\omega_0)$$

$$\frac{d}{dt}x(t) = \sum_{n=-\infty}^{\infty} [C_n\, jn\omega_0]\, e^{jn\omega_0 t}$$

$$\boxed{\therefore F_S\left[\frac{d}{dt}x(t)\right] = jn\omega_0 C_n}$$

7) Convolution in Time domain:

stmt: If $Fs[x(t)] = C_n$ & $Fs[y(t)] = D_n$

Then $Fs[x(t) * y(t)] = T C_n D_n$

Proof: $Fs[x(t)] = C_n = \frac{1}{T}\int_{<T>} x(t) e^{-jn\omega_0 t} dt$ —— ①

$Fs[y(t)] = D_n = \frac{1}{T}\int_{<T>} y(t) e^{-jn\omega_0 t} dt$ —— ②

$x(t) * y(t) = \int_{<T>} x(p) y(t-p) dp$ —— ③

$Fs[x(t) * y(t)] = \frac{1}{T}\int_{<T>} [x(t) * y(t)] e^{-jn\omega_0 t} dt$ —— ④

From eqns ③ & ④

$Fs[x(t) * y(t)] = \frac{1}{T}\int_{<T>}\int_{<T>} x(p) y(t-p) dp\, e^{-jn\omega_0 t} dt$

$= \frac{1}{T}\int_{<T>} x(p) \int_{<T>} y(t-p) e^{-jn\omega_0 t} dt\, dp$

let $t - p = m$

$t = m + p$

$dt = dm$

$$\therefore \; Fs[x(t) * y(t)] = \frac{1}{T} \int_{\langle T \rangle} x(p) \int_{\langle T \rangle} y(m) e^{-jn\omega_0 (m+p)} \, dm \, dp$$

$$= \frac{1}{T} \int_{\langle T \rangle} x(p) \int_{\langle T \rangle} y(m) e^{-jn\omega_0 m} \cdot e^{-jn\omega_0 p} \, dm \, dp$$

$$= \frac{1}{T} \int_{\langle T \rangle} x(p) e^{-jn\omega_0 p} \, dp \, \frac{T}{T} \int_{\langle T \rangle} y(m) e^{-jn\omega_0 m} \, dm$$

$$= T C_n D_n$$

$$\boxed{\therefore \; Fs[x(t) * y(t)] = T C_n D_n}$$

8) Multiplication (or) Modulation Theorem:

stmt: If $FS[x(t)] = C_n$ & $FS[y(t)] = D_n$

Then $FS[x(t)y(t)] = C_n * D_n = \sum_{l=-\infty}^{\infty} C_l D_{n-l}$

Proof: $FS[x(t)] = C_n = \frac{1}{T}\int_{<T>} x(t) e^{-jn\omega_0 t} dt$

$FS[y(t)] = D_n = \frac{1}{T}\int_{<T>} y(t) e^{-jn\omega_0 t} dt$

$$FS[x(t)y(t)] = \frac{1}{T}\int_{<T>} x(t)y(t) e^{-jn\omega_0 t} dt$$

but $x(t) = \sum_{l=-\infty}^{\infty} C_l e^{+jl\omega_0 t}$

$$\therefore FS[x(t)y(t)] = \frac{1}{T}\int_{<T>} \sum_{l=-\infty}^{\infty} C_l e^{+jl\omega_0 t} y(t) e^{-jn\omega_0 t} dt$$

$$= \sum_{l=-\infty}^{\infty} C_l \frac{1}{T}\int_{<T>} y(t) e^{-j(n-l)\omega_0 t} dt$$

$$FS[x(t)y(t)] = \sum_{l=-\infty}^{\infty} C_l D_{n-l}$$

$$= C_n * D_n$$

9) Parsevals Theorem:

Statement:- If $x(t)$ is the periodic signal with Fourier coefficients C_n, then the average power of the periodic signal is given by $\sum_{n=-\infty}^{\infty} |C_n|^2$

Proof: $P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt \qquad \because x(t).x^*(t) = |x(t)|^2$$

but $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

Take complex conjugate on both sides

$$x^*(t) = \sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t}$$

$$\therefore P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t} dt$$

Take complex Conjugate on both sides

$$x^*(t) = \sum_{n=-\infty}^{\infty} c_n^* e^{-jn\omega_0 t}$$

$$\therefore \quad P = \underset{T\to\infty}{lt} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sum_{n=-\infty}^{\infty} c_n^* e^{-jn\omega_0 t} \, dt$$

$$= \underset{T\to\infty}{lt} \sum_{n=-\infty}^{\infty} c_n^* \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_0 t} \, dt$$

$$= \underset{T\to\infty}{lt} \sum_{n=-\infty}^{\infty} c_n^* \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_0 t} \, dt$$

$$= \sum_{n=-\infty}^{\infty} c_n^* c_n$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2$$

10) Conjugation:

Stmt: i) If $FS[x(t)] = C_n$

Then $FS[x^*(t)] = C_{-n}^*$

Proof: $$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$x^*(t) = \sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t}$$

$$x^*(t) = \sum_{n=-\infty}^{\infty} C_{-n}^* e^{jn\omega_0 t}$$

$\therefore FS[x^*(t)] = C_{-n}^*$

Problems

Problems:

① obtain the trigonometric fourier series for the periodic signal shown below

Soln: Period $T = 0.5$

and $x(t) = e^{-t}$

Trigonometric fourier series representation of $x(t)$ is

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right]$$

where $a_0 = \frac{1}{T} \int_{<T>} x(t)\, dt$

$$a_n = \frac{2}{T} \int_{<T>} x(t) \cos n\omega_0 t\, dt$$

& $b_n = \frac{2}{T} \int_{<T>} x(t) \sin n\omega_0 t\, dt$

Evaluation of a_0:

$$a_0 = \frac{1}{T}\int_{<T>} x(t)\,dt$$

$$= \frac{1}{0.5}\int_0^{0.5} e^{-t}\,dt$$

$$= 2\left[-e^{-t}\right]_0^{0.5}$$

$$= -2\left[e^{-0.5} - e^{0}\right]$$

$$= 0.7869$$

Evaluation of a_n:

$$a_n = \frac{2}{T}\int_{<T>} x(t)\cos n\omega_0 t\,dt$$

$n\omega_0 t = n.\frac{2\pi}{T}t$

$= n.\frac{2\pi}{0.5}t = 4n\pi t$

$$a_n = \frac{2}{0.5}\int_0^{0.5} e^{-t}\cos n\omega_0 t\,dt$$

$$\boxed{\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]}$$

$$\therefore\ a_n = 4\left.\frac{e^{-t}}{1+(4n\pi)^2}\left(-\cos 4n\pi t + 4n\pi\sin 4n\pi t\right)\right|_0^{0.5}$$

$$a_n = \frac{4e^{-0.5}}{1+(4n\pi)^2}\left[-\cos 4n\pi(0.5) + 4n\pi\sin 4n\pi(0.5)\right]$$

$$- \frac{4.e^{0}}{1+(4n\pi)^2}\left[-1+0\right]$$

$$= \frac{4(0.606)}{1+(4n\pi)^2}\left[-\cos 2n\pi + 4n\pi\sin 2n\pi\right] + \frac{4}{1+(4n\pi)^2}$$

$$= \frac{4}{1+(4n\pi)^2}\left[-0.606+0+1\right]$$

$$a_n = \frac{4\, e^{-0.5}}{1+(4n\pi)^2}\left[-\cos 4n\pi(0.5) + 4n\pi \sin 4n\pi(0.5)\right]$$

$$- \frac{4 \cdot e^{0}}{1+(4n\pi)^2}\left[-1+0\right]$$

$$= \frac{4\,(0.606)}{1+(4n\pi)^2}\left[-\cos 2n\pi + 4n\pi \sin 2n\pi\right] + \frac{4}{1+(4n\pi)^2}$$

$$= \frac{4}{1+(4n\pi)^2}\left[-0.606+0+1\right]$$

$$= \frac{4\,(0.394)}{1+(4n\pi)^2}$$

$$a_n = \frac{1.576}{1+(4n\pi)^2}$$

Evaluation of bn :

$$b_n = \frac{2}{T}\int_{<T>} x(t)\, \sin n\omega_0 t\, dt$$

$$= \frac{2}{0.5}\int_0^{0.5} e^{-t} \sin n\frac{2\pi}{T} t\, dt$$

$$= 4\int_0^{0.5} e^{-t} \sin \frac{2n\pi}{0.5} t\, dt = 4\int_0^{0.5} e^{-t} \sin 4n\pi t\, dt$$

$$\boxed{\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]}$$

$$b_n = 4\left(\frac{e^{-t}}{1+(4n\pi)^2}\left[-\sin 4n\pi t - 4n\pi \cos 4n\pi t\right]\right)\Bigg|_0^{0.5}$$

$$b_n = 4\left(\frac{e^{-t}}{1+(4n\pi)^2}\left[-\sin 4n\pi t - 4n\pi\cos 4n\pi t\right]\right)\Big|_0^{0.5}$$

$$= 4\cdot\left[\frac{e^{-0.5}}{1+(4n\pi)^2}\left(-\sin 4n\pi(0.5) - 4n\pi\cos 4n\pi(0.5)\right) - \frac{e^{0}}{1+(4n\pi)^2}\left[-0 - 4n\pi(1)\right]\right]$$

$$= \frac{4}{1+(4n\pi)^2}\left[0.606\left((-\sin 2n\pi) - 4n\pi\cos(2n\pi)\right)\right] - \frac{4}{1+(4n\pi)^2}\left[-4n\pi\right]$$

$$= \frac{4}{1+(4n\pi)^2}\left[0 - 4n\pi(0.606) + 4n\pi\right]$$

$$= \frac{4\times 4n\pi}{1+(4n\pi)^2}\left[1-0.606\right]$$

$$= \frac{6.32n\pi}{1+(4n\pi)^2}$$

$$x(t) = 0.7869 + \sum_{n=1}^{\infty} \frac{1.576}{1+(4n\pi)^2}\cos n\omega_0 t + \frac{6.32n\pi}{1+(4n\pi)^2}\sin n\omega_0 t$$

(2) Obtain the exponential Fourier series for the figure shown in problem (1)

Exponential Fourier series representation of $x(t)$ is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{-jn\omega_0 t}$$

where $C_n = \frac{1}{T}\int_{\langle T \rangle} x(t) e^{-jn\omega_0 t} dt$

$$C_n = \frac{1}{0.5}\int_0^{0.5} e^{-t} e^{-jn\omega_0 t} dt$$

$$= 2\int_0^{0.5} e^{-t} \cdot e^{-jn\frac{2\pi}{T}t} dt$$

$$= 2\int_0^{0.5} e^{-t} \cdot e^{-jn\frac{2\pi}{0.5}t} dt$$

$$= 2\int_0^{0.5} e^{-t} \cdot e^{-j4n\pi t} dt$$

$$C_n = \frac{1}{0.5}\int_0^{0.5} e^{-t}\, e^{-jn\omega_0 t}\, dt$$

$$= 2\int_0^{0.5} e^{-t} \cdot e^{-jn\frac{2\pi}{T}t}\, dt$$

$$= 2\int_0^{0.5} e^{-t} \cdot e^{-jn\frac{2\pi}{0.5}t}\, dt$$

$$= 2\int_0^{0.5} e^{-t} \cdot e^{-j4n\pi t}\, dt$$

$$= 2\int_0^{0.5} e^{-(1+j4n\pi)t}\, dt$$

$$= 2 \cdot \left. \frac{e^{-(1+j4n\pi)t}}{-(1+j4n\pi)} \right|_0^{0.5}$$

$$= \frac{-2}{1+j4n\pi}\left[e^{-(1+j4n\pi)0.5} - 1 \right]$$

$$= \frac{-2}{1+j4n\pi}\left[e^{-0.5} e^{-j4n\pi(0.5)} - 1\right]$$

$$= \frac{-2}{1+j4n\pi}\left[e^{-0.5} \cdot e^{-j2n\pi} - 1\right] \qquad e^{-j2n\pi} = \cos 2n\pi - j\sin 2n\pi = 1$$

$$= \frac{-2}{1+j4n\pi}\left[e^{-0.5} - 1\right]$$

$$= \frac{-2}{1+j4n\pi}\left[0.606 - 1\right]$$

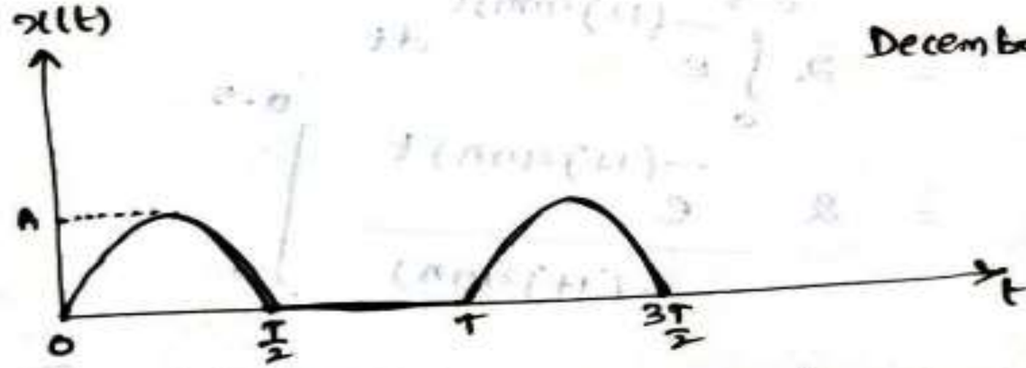
$$C_n = \frac{0.7869}{1+j4n\pi}$$

$\therefore$ Exponential Fourier series representation is

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{0.7869}{1+j4n\pi} e^{jn\omega_0 t}$$

(3) Obtain the Trigonometric Fourier series for the halfwave rectified sine wave shown below

December 2011



Soln: Functional representation of the given waveform y

$$x(t) = \begin{cases} A \sin \omega_0 t & \text{for } 0 \le t \le \frac{T}{2} \\ 0 & \text{for } \frac{T}{2} \le t \le T \end{cases}$$

$$\omega_0 = \frac{2\pi}{T}$$

here $T = 2\pi$

$$\therefore \quad \omega_0 = \frac{2\pi}{2\pi} = 1$$

Trigonometric Fourier Series representation of $x(t)$ is

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right]$$

Where
$$a_0 = \frac{1}{T} \int_{\langle T \rangle} x(t)\,dt$$

$$a_n = \frac{2}{T} \int_{\langle T \rangle} x(t) \cos n\omega_0 t\,dt$$

$$\& \quad b_n = \frac{2}{T} \int_{\langle T \rangle} x(t) \sin n\omega_0 t\,dt$$

Evaluation of a_0:

$$a_0 = \frac{1}{T}\int_{<T>} x(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} x(t)\,dt + \int_{\pi}^{2\pi} x(t)\,dt\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} A \sin\omega_0 t\,dt + 0$$

$$= \frac{A}{2\pi}\int_0^{\pi} \sin t\,dt$$

$$= \frac{-A}{2\pi} \cos t\Big]_0^{\pi} = \frac{-A}{2\pi}\left[\cos\pi - \cos 0\right]$$

$$= \frac{-A}{2\pi}\left[-1-1\right] = \frac{2A}{2\pi} = \frac{A}{\pi}$$

$$\boxed{a_0 = \frac{A}{\pi}}$$

Evaluation of a_n:

$$a_n = \frac{2}{T} \int_{<T>} x(t) \cos n\omega_0 t \, dt$$

$$= \frac{2}{2\pi} \int_0^{\pi} A \sin \omega_0 t \cos n\omega_0 t \, dt$$

$$= \frac{A}{\pi} \int_0^{\pi} \sin t \cos nt \, dt \qquad \because \omega_0 = 1$$

$$\boxed{\sin(A+B) + \sin(A-B) = 2 \sin A \cos B}$$

$$a_n = \frac{A}{2\pi} \int_0^{\pi} 2 \sin t \cos t \, dt$$

$$= \frac{A}{2\pi} \int_0^{\pi} \left(\sin(t+nt) + \sin(t-nt) \right) dt$$

$$= \frac{A}{2\pi} \int_0^{\pi} \left(\sin(1+n)t + \sin(1-n)t \right) dt$$

$$= \frac{-A}{2\pi}\left[\frac{\cos(1+n)t}{1+n} + \frac{\cos(1-n)t}{1-n}\right]_0^{\pi}$$

$$= \frac{-A}{2\pi}\frac{1}{(1+n)}\left[\cos\pi(n+1) - 1\right] -$$

$$\frac{A}{2\pi}\frac{1}{(1-n)}\left[\cos\pi(1-n) - 1\right]$$

For $n = \pm 1, \pm 3, \pm 5, \ldots\ldots$

$$\cos(1+n)\pi = 1 \quad \& \quad \cos(1-n)\pi = 1$$

$$\therefore\ a_n = \frac{-A}{2\pi}\frac{1}{(1+n)}[1-1] - \frac{A}{2\pi}\frac{1}{(1-n)}[1-1]$$

$a_n = 0$ for $n = \pm 1, \pm 3, \pm 5, \ldots.$

For $n = \pm 2, \pm 4, \pm 6, \ldots$

$$\cos(1+n)\pi = -1 \quad \& \quad \cos(1-n)\pi = -1$$

$$\therefore \quad a_n = -\frac{A}{2\pi} \cdot \frac{1}{1+n}\left[-1-1\right] - \frac{A}{2\pi}\left(\frac{1}{1-n}\right)\left[-1-1\right]$$

$$= \frac{A}{\pi}\left(\frac{1}{1+n}\right) + \frac{A}{\pi}\left(\frac{1}{1-n}\right)$$

$$= \frac{A}{\pi}\left(\frac{1}{1+n} + \frac{1}{1-n}\right)$$

$$= \frac{A}{\pi}\left[\frac{1-\cancel{n}+1+\cancel{n}}{1-n^2}\right]$$

$$a_n = \frac{2A}{\pi(1-n^2)}$$

Evaluation of b_n:

$$b_n = \frac{2}{T}\int_{\langle T\rangle} x(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} A\sin\omega_0 t\sin n\omega_0 t\,dt$$

$$= \frac{A}{\pi}\int_0^{\pi}\sin t\sin nt\,dt \qquad \because \omega_0 = 1$$

$$= \frac{A}{2\pi}\int_0^{\pi} 2\sin t\sin nt\,dt$$

$$= \frac{A}{2\pi}\int_0^{\pi}\left(\cos(1-n)t - \cos(1+n)t\right)dt$$

$$= \frac{A}{2\pi}\left[\frac{\sin(1-n)t}{1-n} - \frac{\sin(1+n)t}{1-n}\right]_0^{\pi}$$

$$= \frac{A}{2\pi}\left[0-0+0-0\right]$$

$$= 0$$

$$a_0 = \frac{A}{\pi}$$

$$a_n = \frac{2A}{\pi(1-n^2)} \quad \text{for } n = 2,4,6,8,\ldots$$

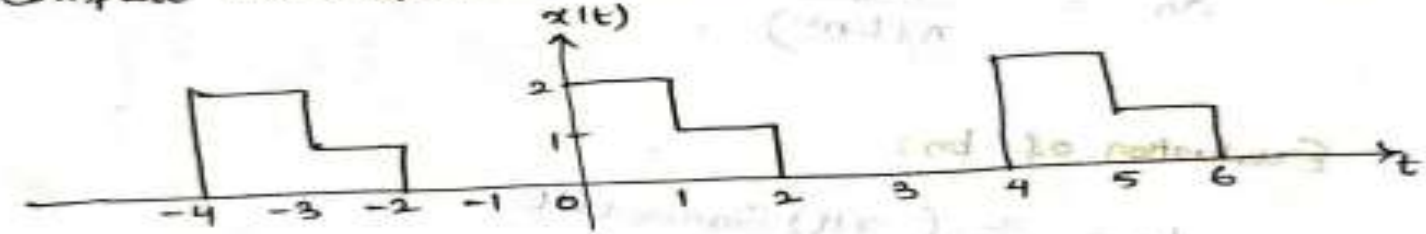
$$b_n = 0$$

$$\therefore x(t) = \frac{A}{\pi} + \sum_n a_n \cos n\omega_0 t \quad \text{for } n = 2,4,6,8,\ldots$$

$$x(t) = \frac{A}{\pi} + \frac{2A}{\pi(1-2^2)} \cos 2\omega_0 t + \frac{2A}{\pi(1-4^2)} \cos 4\omega_0 t + \cdots$$

$$x(t) = \frac{A}{\pi} + \frac{2A}{3\pi} \cos 2\omega_0 t - \frac{2A}{15\pi} \cos 4\omega_0 t + \cdots$$

4) Compute the exponential Fourier series of the following signal



Soln: $T = 4-0 = 4$ and $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

In E.F.S $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

$$C_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{4}\int_0^4 x(t)dt = \frac{1}{4}\left[\int_0^1 2dt + \int_1^2 dt\right]$$

$$= \frac{1}{2}(t)_0^1 + \frac{1}{4}[t]_1^2 = \frac{1}{2}(1-0) + \frac{1}{4}(2-1) = \frac{1}{2} + \frac{1}{4} = 3/4$$

$$C_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t} dt = \frac{1}{4}\int_0^4 x(t) e^{-jn\frac{\pi}{2}t} dt$$

$$= \frac{2}{4}\int_0^1 e^{-jn\frac{\pi}{2}t} dt + \frac{1}{4}\int_1^2 e^{-jn\pi/2 t} dt$$

$$= \frac{1}{2} \left.\frac{e^{-jn\frac{\pi}{2}t}}{-jn\pi/2}\right|_0^1 + \frac{1}{4} \left.\frac{e^{-jn\pi/2 t}}{-jn\pi/2}\right|_1^2$$

$$= \frac{-1}{jn\pi}\left[e^{-jn\pi/2} - 1\right] - \frac{1}{2jn\pi}\left[e^{-jn\pi} - e^{-jn\pi/2}\right]$$

$$= \frac{1}{jn\pi}\left[1 - e^{-jn\pi/2} - \frac{1}{2}e^{-jn\pi} + \frac{1}{2}e^{-jn\pi/2}\right] = \frac{1}{jn\pi}\left(1 - \frac{1}{2}e^{-jn\frac{\pi}{2}} - \frac{1}{2}e^{-jn\pi}\right)$$

$$= \frac{1}{jn\pi}\left(1 - \frac{1}{2}e^{-jn\pi/2} - \frac{1}{2}(-1)^n\right)$$

Fourier Spectrum

Fourier Spectrum:

Fourier spectrum of a periodic signal $x(t)$ can be obtained by plotting the Fourier coefficients versus ω. The plot of amplitude of fourier coefficients versus ω is known as amplitude spectra. The plot of phase of fourier coefficients versus ω is known as phase spectra. The two plots together are known as Fourier frequency spectra of $x(t)$. That is in Fourier spectra the amplitude and phase of the Fourier coefficients are plotted as a function of frequency. So this type of representation is also known as frequency domain representation of $x(t)$.

Note: The spectrum exists only at discrete frequencies $n\omega_0$ where $n = 0, 1, 2, \ldots$ Thus the spectrum is not continuous but exists only at some discrete values of ω and is known as discrete spectrum or line spectrum.

The trigonometric representation of periodic signal $x(t)$ contains both sine and cosine terms with +ve & -ve amplitude coefficients but with no phase angles. In cosine representation all the Fourier coefficients are +ve with a phase angle θ_n. ∴ we can plot amplitude spectra (C_n vs ω) and phase spectra (θ_n vs ω). In this, Fourier coefficients present only for +ve frequencies and may be called as single sided spectra.

In Exponential Fourier series the periodic signal is expressed as sum of exponential function of complex frequencies: $0, \pm j\omega_0, \pm 2j\omega_0, \ldots\ldots$ The amplitudes C_n are complex and can be represented by magnitude and phase. Therefore we can plot two spectra. The spectra can be plotted for both +ve and -ve frequencies. Hence the name two sided spectra.

Complex Fourier Spectrum:

The complex Fourier series representation of a function $x(t)$ is equivalent to resolving the function in terms of harmonically related components of the fundamental frequency ω_0 (or f_0). A complex weighting factor F_n (or C_n), called the *spectral amplitude* is assigned to each harmonic component. Graphical representation of a spectral amplitude along with a spectral phase is called the *complex frequency spectrum.*

An amplitude spectrum without phase information does not specify the waveform; because, in general, F_n (or C_n) is a complex quantity. Therefore, such a spectrum is called the *complex frequency spectrum*.

However, if F_n (or C_n) is purely real or purely imaginary, the phase spectrum can be disregarded.

Figure 4.5 below shows a typical amplitude spectrum, where a vertical line has been drawn at each harmonic frequency and the height of the line represents the amplitude at the corresponding harmonic frequency. This spectrum is known as the *discrete spectrum* or *line spectrum* and exists only at discrete frequencies that are harmonically related. Figure 4.5(a) represents the spectrum of a trigonometric Fourier series extending from 0 to ∞, producing a *one-sided spectrum* as no negative frequencies exist here. Figure 4.5(b) represents the spectrum of a complex exponential Fourier series extending from $-\infty$ to ∞, producing a *two-sided spectrum*.

The amplitude spectrum of the exponential series is symmetrical about the vertical axis. This is true for all real periodic functions.

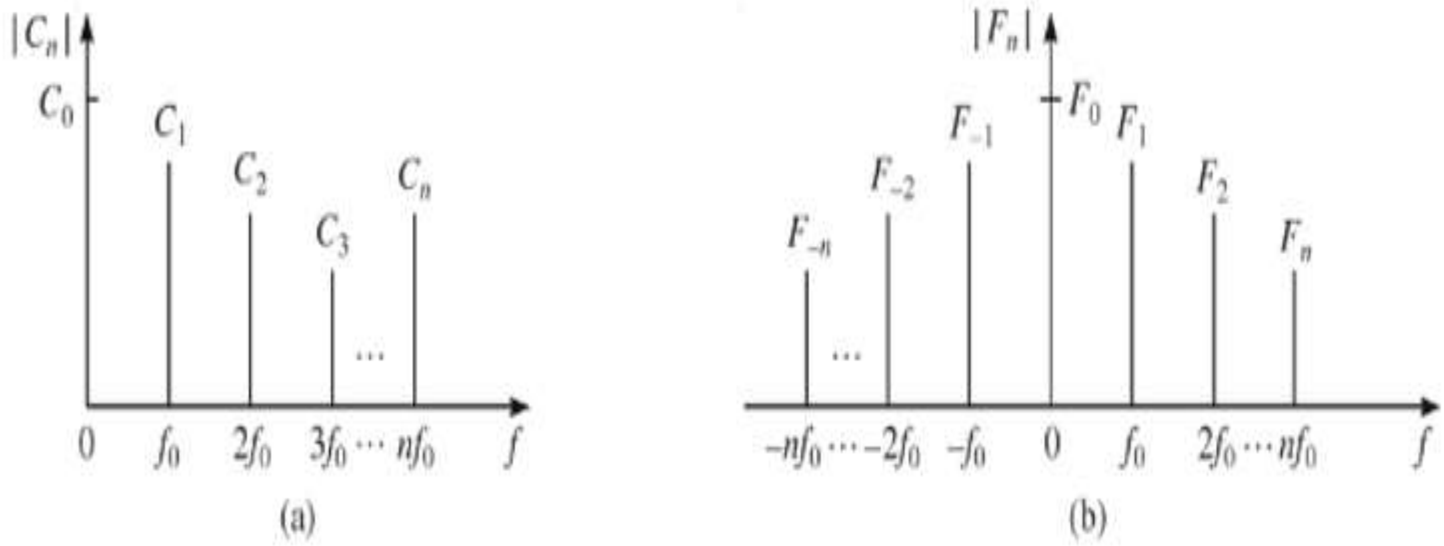


Figure 4.5 Complex frequency spectrum for (a) trigonometric Fourier series and (b) complex exponential Fourier series.

If F_n is a general complex number, then

$$F_n = |F_n| e^{j\theta_n}$$

$$F_{-n} = |F_n| e^{-j\theta_n}$$

$$\therefore \qquad |F_n| = |F_{-n}|$$

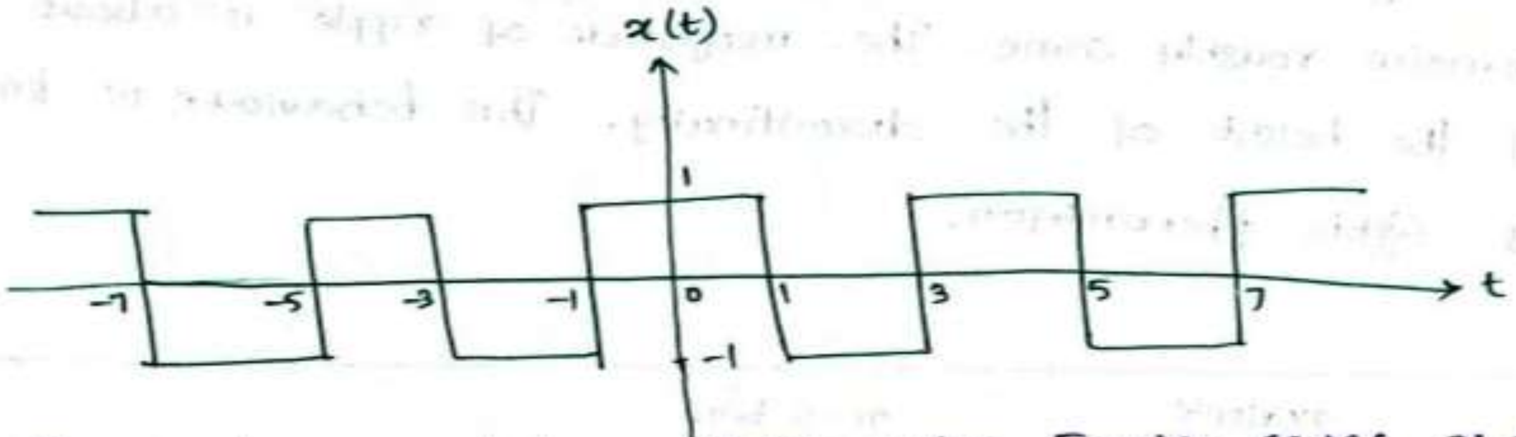
Hence, the magnitude spectrum is symmetrical about the vertical axis passing through the origin and, thus, it is an even function of ω_n. It is called the *even symmetry* of the magnitude spectrum. Also, θ_n is the phase of F_n and $-\theta_n$ is the phase of F_{-n}. So, the phase spectrum is anti-symmetrical about the vertical axis. It is called the *odd symmetry* of phase spectrum.

Accordingly, for a real-valued periodic signal, $x(t)$, the magnitude spectrum is symmetrical and the phase spectrum is anti-symmetrical about the vertical axis passing through the origin.

When $x(t)$ is real, then $F_{-n} = F_n^*$, the complex conjugate.

Gibbs phenominon:

Consider the square wave with time period T as shown in figure below



It can be represented in Trigonometric Fourier series as follows

$$x(t) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi}{2} t\right) \quad \text{——} (1)$$

$$x(t) = \frac{4}{\pi}\left[\cos\left(\frac{\pi}{2} t\right) - \frac{1}{3}\cos\left(\frac{3\pi}{2} t\right) + \frac{1}{5}\cos\left(\frac{5\pi}{2} t\right) - \frac{1}{7}\cos\left(\frac{7\pi}{2} t\right) + \cdots\cdots\right] \quad \text{——} (2)$$

The above eqn is in an infinite series. Now let us approximate $x(t)$ by a finite value n

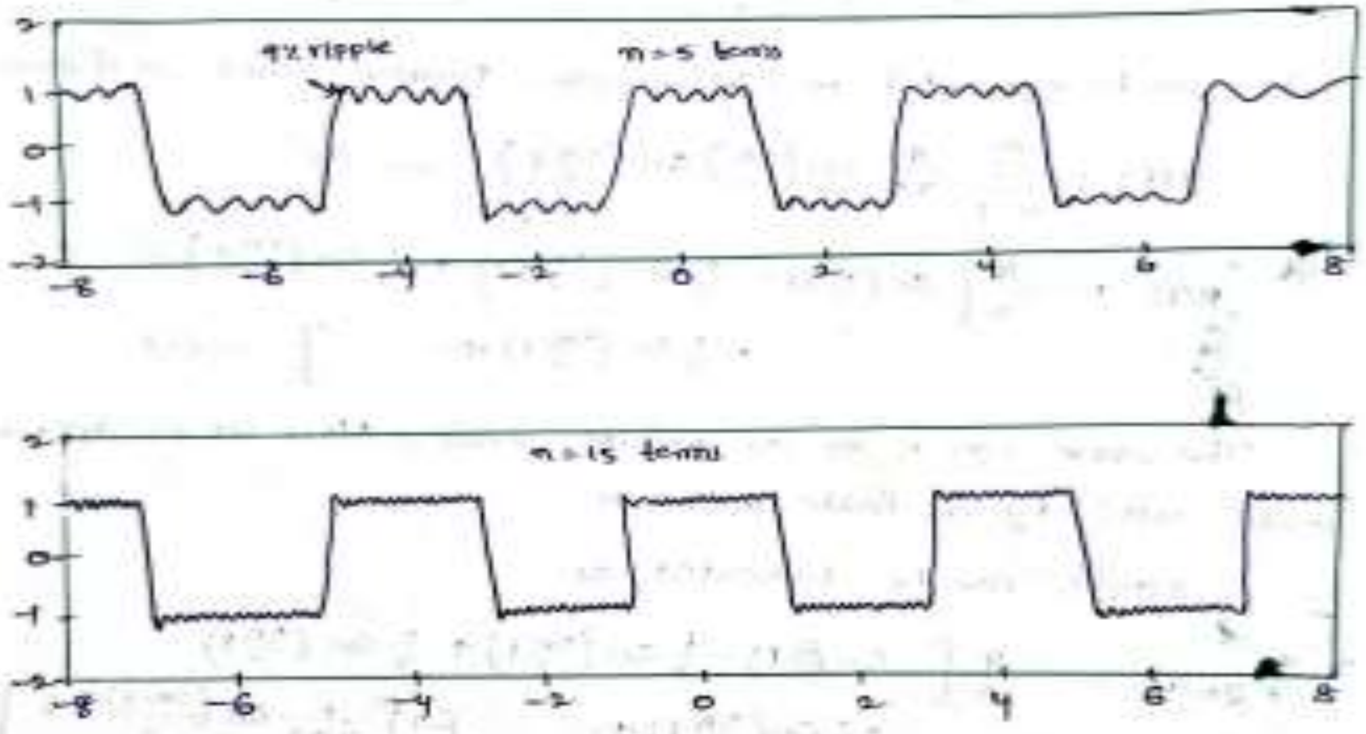
$\therefore$ eqn (2) can be truncated as

$$x_n(t) = \frac{4}{\pi}\Big[\cos\left(\frac{\pi}{2}t\right) - \frac{1}{3}\cos\left(\frac{3\pi}{2}t\right) + \frac{1}{5}\cos\left(\frac{5\pi}{2}t\right) - \frac{1}{7}\cos\left(\frac{7\pi}{2}t\right) + \cdots\cdots (-1)^n \frac{1}{2n-1}\cos\left(\frac{(2n-1)}{2}\pi t\right)\Big]$$

The below figure shows the plot of $x_n(t)$ for $n=5$ and $n=15$

We can observe that the truncated Fourier series approaches $x(t)$ as n increases. That is an error between $x(t)$ and $x_n(t)$ decreases as n increases.

However at the discontinuity, $x_n(t)$ exhibits an oscillatory behaviour and has ripples on both sides. As n increases the frequency of the ripple increases, but the amplitude of the ripples remains roughly same. The magnitude of ripple is about 9% of the height of the discontinuity. This behaviour is known as Gibbs phenomenon.



Gibbs phenominon