Fourier Transform Introduction

Fourier Series is used to analyse periodic signals. Using Exponential Fourier Series, any CT signal $x(t)$ can be represented as a linear combination of complex exponentials and the Fourier coefficients (spectrum) are discrete. Fourier series can deal only with the periodic signals. This is major drawback of Fourier Series.

However, all the naturally produced signals will be in the form of nonperiodic or aperiodic signals. Therefore the applicability of Fourier series is limited.

∴ Fourier Transform is mostly used to analyze aperiodic signals and can be used to analyse periodic signals also. So it overcomes the limitation of Fourier series.

Fourier Transform is a transformation technique which transforms signals from the continuous-time domain to the corresponding frequency domain and vice versa. Fourier Transform can be developed by finding the Fourier series of a periodic function and then tending T to infinity.

Derivation of the FT of a Non-periodic signal from the FS of a periodic signal

Let $x(t)$ be non-periodic & $x_T(t)$ be periodic with period T

$$x(t) = \lim_{T\to\infty} x_T(t)$$

FS of a periodic signal is

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where $C_n = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t}\, dt$ & $\omega_0 = \frac{2\pi}{T}$

$$TC_n = \int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t}\, dt$$

Let $n\omega_0 \to \omega$ at $T\to\infty$, As $T\to\infty$, $\omega_0 = \frac{2\pi}{T} \to 0$

DFS becomes continuous, Summation → integral & $x_T(t) \to x(t)$

Thus as $T\to\infty$

$$TC_n = \lim_{T\to\infty}\int_{-T/2}^{T/2} x_T(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty}\left(\lim_{T\to\infty} x_T(t)\right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt = X(\omega)$$

$$\therefore X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

FT of $x(t)$

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{TC_n}{T} e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{T} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi} e^{jn\omega_0 t}\, \omega_0$$

$$x(t) = \lim_{T\to\infty} x_T(t) = \lim_{T\to\infty} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(\omega) e^{jn\omega_0 t}\, \omega_0$$

as $T\to\infty$, ω_0 becomes infinitesimally small and may be represented by $d\omega$

Also summation becomes integral

$$\therefore x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

Hence $x(t)$ is called I.F.T of $X(\omega)$

$$\therefore X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$\& \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$X(\omega)$ & $x(t)$ are known as F.T pair

$$x(t) \xleftrightarrow{FT} X(\omega)$$

PART-1 : FOURIER TRANSFORM

Deriving Fourier Transform from Fourier Series:

Consider a periodic signal with period T as shown in Fig ①. If T tends to infinity then $x(t)$ becomes aperiodic signal shown in Fig ②

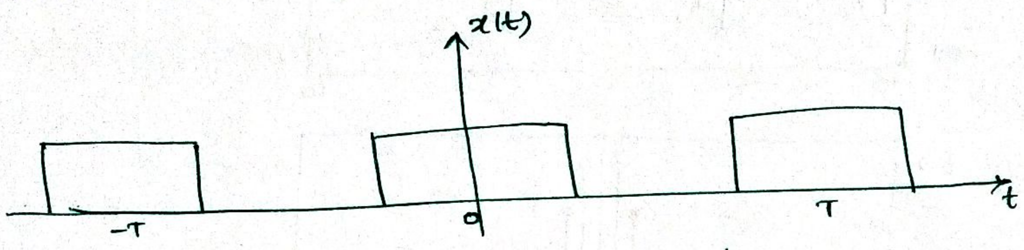


Fig ① periodic signal

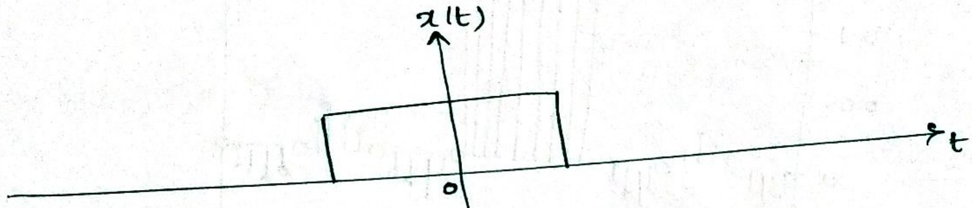


Fig. ② Aperiodic signal

The exponential fourier series representation of the periodic signal signal shown in fig ① is given by

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

by solving, we will get

$$C_0 = \frac{2}{T}$$

$$\& \quad C_n = \frac{2}{T} \frac{\sin n\omega_0}{n\omega_0}$$

$$C_n = \frac{2}{T} \operatorname{sinc}(n\omega_0)$$

The function $\operatorname{sinc}(n\omega_0)$ defines the shape of amplitude spectrum. The spectrum has the following properties

i) amplitude at $\omega = 0$ is $\frac{2}{T}$

ii) fundamental frequency is $\omega_0 = \frac{2\pi}{T}$

iii) Spacing between the freq. components is $\Delta\omega = \frac{2\pi}{T}$

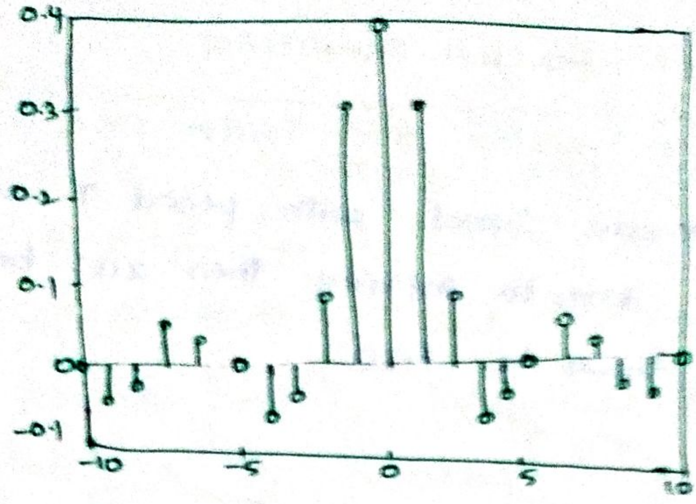


Fig ⓐ

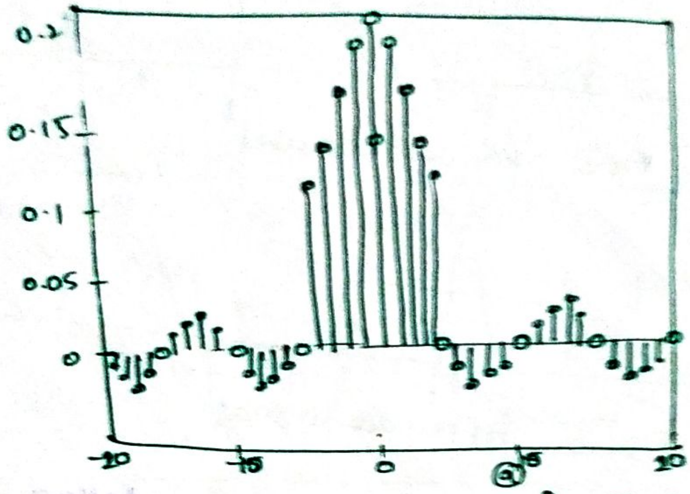


Fig ⓑ

ⓐ Spectrum of the signal $x(t)$ for $T=5$ ⓑ for $T=10$

when $T \to \infty$ the amplitude $\frac{2}{T} \to 0$ and $\Delta\omega = \frac{2\pi}{T} = 0$

That is as T tends ∞, the spectrum consists of an infinite number of zero amplitude components with zero frequency content

$$TC_n = \frac{2 \sin n\omega_0}{n\omega_0}$$

As T increases, the fundamental frequency decreases and occurance of TC_n becomes closer and closer. As a limiting case when $T \to \infty$, TC_n exists for all frequencies and becomes continuous.

The Fourier series of a periodic signal is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \quad \text{—— (1)}$$

and $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t}\, dt \quad \text{—— (2)}$$

Multiply eqn (1) with T on both sides

$$T x(t) = \sum_{n=-\infty}^{\infty} T.C_n e^{jn\omega_0 t}$$

wher $T = \frac{2\pi}{\omega_0}$

let $$X(n\omega_0) = T.C_n = \int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t}\, dt$$

and $$x(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} T C_n e^{jn\omega_0 t}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(n\omega_0) e^{jn\omega_0 t}\, \omega_0$$

As $T \to \infty$, the fundamental frequency ω_0 becomes infinitely small. hence $\omega_0 \to d\omega$. Also the harmonic frequencies get so close together that they become continuous

i.e; $n\omega_0 \to \omega$

$$\therefore\ x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega \quad \text{—— (3)}$$

Similarly $$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \quad \text{—— (4)}$$

These two equations are called the Fourier Transform Pair

Properties of Fourier Transform

① Linearity:

Statement: If $FT[x_1(t)] = X_1(\omega)$

and $FT[x_2(t)] = X_2(\omega)$

Then $FT[a x_1(t) + b x_2(t)] = a X_1(\omega) + b X_2(\omega)$

Proof: $FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$FT[a x_1(t) + b x_2(t)] = \int_{-\infty}^{\infty} (a x_1(t) + b x_2(t)) e^{-j\omega t} dt$$

$$= a\int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + b\int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$$= a X_1(\omega) + b X_2(\omega)$$

$$\therefore FT[a x_1(t) + b x_2(t)] = a X_1(\omega) + b X_2(\omega)$$

2) Time shifting:

Statement: If $FT[x(t)] = X(\omega)$

Then $FT[x(t-t_0)] = e^{-j\omega t_0} X(\omega)$

proof: $FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$FT[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$$

let $t - t_0 = p$ if $t = -\infty \Rightarrow p = -\infty - t_0 = -\infty$

$t = p + t_0$ $t = \infty \Rightarrow p = \infty + t_0 = \infty$

$dt = dp$

$$FT[x(t-t_0)] = \int_{-\infty}^{\infty} x(p) e^{-j\omega(p+t_0)} dp$$

$$= \int_{-\infty}^{\infty} x(p) e^{-j\omega p} \cdot e^{-j\omega t_0} dp$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(p) e^{-j\omega p} dp$$

$$= e^{-j\omega t_0} X(\omega)$$

$$\therefore \quad FT[x(t-t_0)] = e^{-j\omega t_0} X(\omega)$$

3) Time Reversal:

Stmt: If $FT[x(t)] = X(\omega)$

Then $FT[x(-t)] = X(-\omega)$

Proof: $FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$FT[x(-t)] = \int_{-\infty}^{\infty} x(-t) e^{-j\omega t} dt$$

let $-t = p$ limits

$t = -p$ If $t = -\infty \Rightarrow p = \infty$

$dt = -dp$ $t = \infty \Rightarrow p = -\infty$

$$\therefore \quad FT[x(-t)] = -\int_{\infty}^{-\infty} x(p) e^{-j\omega(-p)} dp$$

$$= \int_{-\infty}^{\infty} x(p) e^{-j(-\omega)p} dp$$

$$= X(-\omega)$$

$$\therefore \quad FT[x(-t)] = X(-\omega)$$

4) Frequency shifting property:

Stmt: If $FT[x(t)] = X(\omega)$

Then $FT[e^{j\omega_0 t} x(t)] = X(\omega - \omega_0)$

Proof: $FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$FT[e^{j\omega_0 t} x(t)] = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$$

$$= X(\omega-\omega_0)$$

$$\therefore FT[e^{j\omega_0 t} x(t)] = X(\omega-\omega_0)$$

5) Time scaling:

stmt: If $FT[x(t)] = X(\omega)$

Then $FT[x(at)] = \frac{1}{a} X(\frac{\omega}{a})$

Proof: $FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$FT[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

let $at = p$

$t = \frac{p}{a}$

$dt = \frac{dp}{a}$

limits

If $t = -\infty \Rightarrow p = -\infty$

$t = \infty \Rightarrow p = \infty$

$$FT[x(at)] = \int_{-\infty}^{\infty} x(p) e^{-j\omega(p/a)} \frac{dp}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(p) e^{-j(\frac{\omega}{a})p} dp$$

$$= \frac{1}{a} X(\frac{\omega}{a})$$

$$\therefore FT[x(at)] = \frac{1}{a} X(\frac{\omega}{a})$$

6) Differentiation in Time :

Stmt : If $FT[x(t)] = X(\omega)$

Then $FT\left[\frac{d}{dt}x(t)\right] = j\omega X(\omega)$

Proof :

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \quad \text{--- (1)}$$

Differentiate w.r.t. time

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega\right]$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,j\omega\,d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega . X(\omega)]e^{j\omega t}\,d\omega \quad \text{--- (2)}$$

Comparing eqn (1) & (2)

$$FT\left[\frac{d}{dt}x(t)\right] = j\omega X(\omega)$$

7) Differentiation in Frequency :

Stmt : If $FT[x(t)] = X(\omega)$

Then $FT[-jt\,x(t)] = \frac{d}{d\omega}X(\omega)$

Proof : $FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ --- (1)

Differentiate w.r.t. ω

$$\frac{d}{d\omega}X(\omega) = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right]$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}(-jt)\,dt$$

$$\frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} (-jt\,x(t))\,e^{-j\omega t}\,dt \quad \text{—— ②}$$

Comparing eqns ① & ②

$$FT[-jt\,x(t)] = \frac{d}{d\omega}X(\omega)$$

(or)

$$FT[t\,x(t)] = j\frac{d}{d\omega}X(\omega)$$

8) Time Integration:

Stmt: If $FT[x(t)] = X(\omega)$

Then $FT\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega}X(\omega)$

Proof: let $x(t)$ can be expressed as

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

Apply Fourier Transform on both sides

$$FT[x(t)] = FT\left[\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]\right]$$

using differentiation property

$$X(\omega) = j\omega\, FT\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$\therefore FT\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega}X(\omega)$$

9) Convolution in Time domain

Stmt: If $FT[x(t)] = X(\omega)$

and $FT[y(t)] = Y(\omega)$

Then $FT[x(t) * y(t)] = X(\omega)Y(\omega)$

ie; Convolution in time-domain is equal to the multiplication in frequency domain.

Proof: $FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ —— ①

$FT[y(t)] = Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$ —— ②

we know

$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau$ —— ③

$FT[x(t) * y(t)] = \int_{-\infty}^{\infty} [x(t) * y(t)] e^{-j\omega t} dt$ —— ④

From eqns ③ & ④

$FT[x(t) * y(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \, e^{-j\omega t} dt$

$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(t-\tau) e^{-j\omega t} dt \, d\tau$

let $t-\tau = p$

$t = p+\tau$

$dt = dp$

limits

$t=-\infty \Rightarrow p=-\infty$

$t=\infty \Rightarrow p=\infty$

$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(p) e^{-j\omega(p+\tau)} dp \, d\tau$

$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(p) e^{-j\omega p} e^{-j\omega\tau} dp \, d\tau$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} y(p) e^{-j\omega p} dp$$

From eqns ① & ②

$$= X(\omega) Y(\omega)$$

$$\therefore FT[x(t) * y(t)] = X(\omega) Y(\omega)$$

10) Multiplication (or) Modulation property :

stmt: If $FT[x(t)] = X(\omega)$

and $FT[y(t)] = Y(\omega)$

Then $FT[x(t) y(t)] = \frac{1}{2\pi} X(\omega) * Y(\omega)$

Proof: $FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ —①

$$FT[x(t) y(t)] = \int_{-\infty}^{\infty} x(t) y(t) e^{-j\omega t} dt \quad \text{—②}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \text{—③}$$

replace ω by λ

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) e^{j\lambda t} d\lambda \quad \text{—④}$$

From eqns ② & ④

$$FT[x(t) y(t)] = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) e^{j\lambda t} d\lambda \, y(t) e^{-j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) \int_{-\infty}^{\infty} y(t) e^{-j(\omega-\lambda)t} dt \, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) Y(\omega-\lambda) d\lambda$$

$$FT[x(t) y(t)] = \frac{1}{2\pi} X(\omega) * Y(\omega)$$

11) Conjugate property:

Stmt: If $FT[x(t)] = X(\omega)$

Then $FT[x^*(t)] = X^*(-\omega)$

Proof: $FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ — (1)

Take conjugate on both sides

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t) e^{j\omega t} dt$$

replace ω by $-\omega$

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt \quad \text{— (2)}$$

Comparing eqns (1) & (2)

$$FT[x^*(t)] = X^*(-\omega)$$

12) Duality:

Stmt: If $FT[x(t)] = X(\omega)$

Then $FT[X(t)] = 2\pi x(\omega)$

Proof: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

replace t by $-t$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$$

Interchanging t and ω

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

$\therefore FT[X(t)] = 2\pi x(-\omega)$

13) Parsevals Theorem :

statement : Parseval's Theorem states that Energy in time domain is equal to the energy in frequency domain.

i.e., $$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

Proof :
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt \quad \text{--- (1)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Take complex conjugate on both sides

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \quad \text{--- (2)}$$

From eqns (1) & (2)

$$E = \int_{-\infty}^{\infty} x(t) \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Problem 1:

Find the Fourier Transform of the following and sketch the magnitude and phase spectrum.

i) $x(t) = \delta(t)$

ii) $x(t) = e^{-at}u(t)$

iii) $x(t) = e^{-|t|}$

iv) $x(t) = e^{2t}u(t)$

Soln: i) Given $x(t) = \delta(t)$

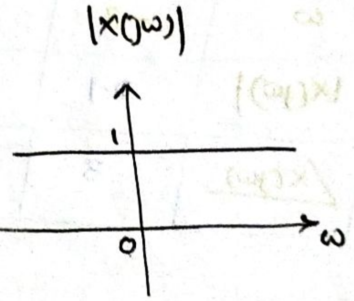
$$\delta(t) = 0 \quad \text{for } t \neq 0$$
$$= 1 \quad \text{for } t = 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$$= \delta(0) e^{-j\omega(0)}$$

$$= 1$$



magnitude spectrum

$\therefore$ $FT[\delta(t)] = 1$

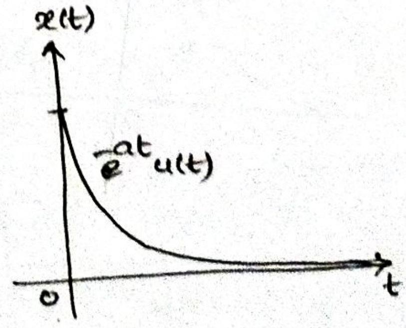
$$\delta(t) \xleftrightarrow{FT} 1$$

$|X(j\omega)| = 1 \quad \forall\ \omega$

$\angle X(j\omega) = 0 \quad \forall\ \omega$

ii) $x(t) = e^{-at}u(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$



$$x(t) = e^{-at}u(t)$$
$$u(t) = 1 \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

$$X(j\omega) = \int_0^\infty e^{-at} \cdot e^{-j\omega t}\, dt = \int_0^\infty e^{-(a+j\omega)t}\, dt$$

$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^\infty$$

$$= -\frac{1}{a+j\omega}\left[e^{-\infty} - e^{-0}\right] = \frac{1}{a+j\omega}$$

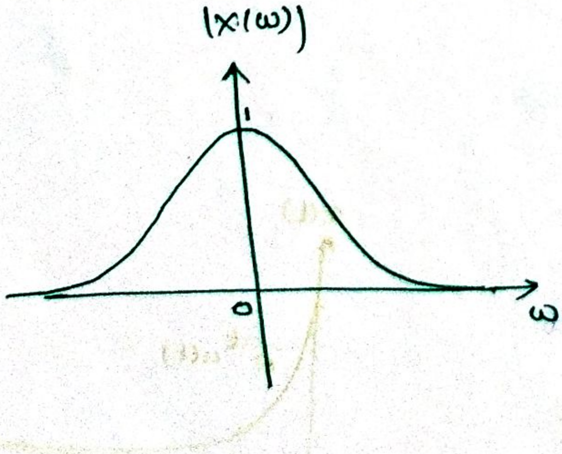
$$\therefore \; FT\left[e^{-at}u(t)\right] = \frac{1}{a+j\omega}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

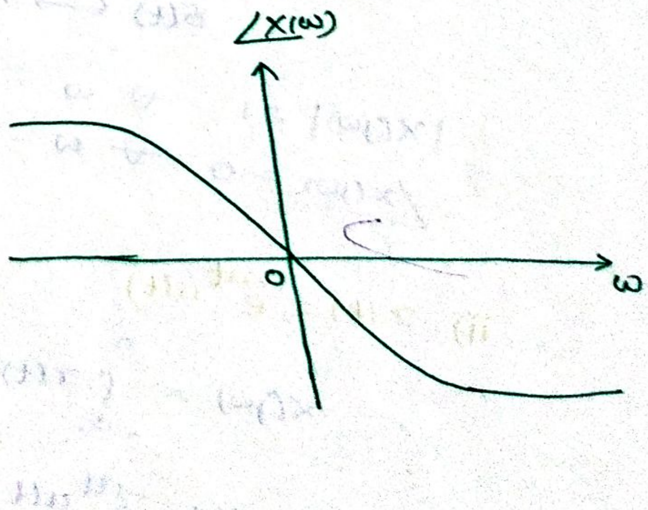
$$\angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right) \qquad \text{let } a=1$$

ω	-3	-2	-1	0	1	2	3
$\lvert X(j\omega)\rvert$	0.1	0.2	0.5	1	0.5	0.2	0.1
$\angle X(j\omega)$	3	2	1	0	-1	-2	-3

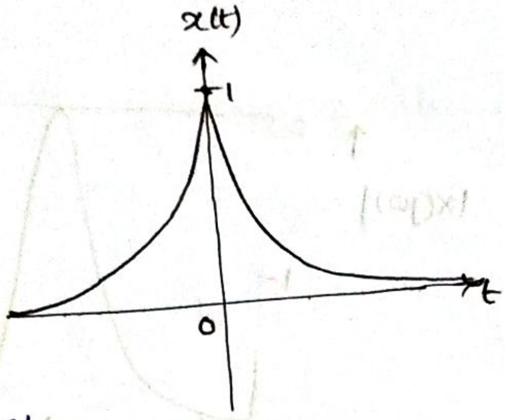
Magnitude spectrum



phase spectrum



iii) $x(t) = e^{-|t|}$



$$x(t) = e^{-t} \quad \text{for } t \geq 0$$
$$\quad = e^{t} \quad \text{for } t < 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} x(t) e^{-j\omega t}\, dt + \int_{0}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{t} . e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-t} . e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(1-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(1+j\omega)t}\, dt$$

$$= \left.\frac{e^{(1-j\omega)t}}{1-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{(1-j\omega)}\left(e^{0} - e^{-\infty}\right) - \frac{1}{(1+j\omega)}\left[e^{-\infty} - e^{0}\right]$$

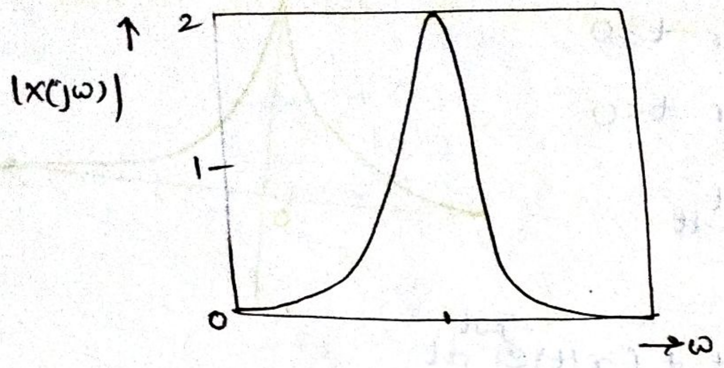
$$= \frac{1}{1-j\omega} + \frac{1}{1+j\omega}$$

$$= \frac{1+j\omega + 1-j\omega}{1+\omega^2}$$

$$= \frac{2}{1+\omega^2}$$

$$|X(j\omega)| = \frac{2}{1+\omega^2} \quad \text{and} \quad \underline{/X(j\omega)} = 0 \quad \forall\, \omega$$

ω	$-\infty$	-10	-5	-3	-2	-1	0	1	2	3	5	10	∞
$\lvert X(j\omega)\rvert$	0	0.019	0.0769	0.2	0.4	1	2	1	0.4	0.2	0.076	0.019	0



Magnitude spectrum.

iv) $x(t) = e^{2t} u(t)$

The given signal is not absolutely integrable.

i.e $\int_0^{\infty} e^{2t} dt = \infty$

$\therefore$ Fourier Transform does not exist.

Excercise problems:

Find the Fourier Transform of the following

i) $x(t) = e^{at} u(-t)$

ii) $x(t) = e^{-a|t|}$

iii) $x(t) = e^{-a|t|} \operatorname{sgn}(t)$

Answers:

i) $\frac{1}{a-j\omega}$

ii) $\frac{2a}{a^2+\omega^2}$

iii) $\frac{-j2\omega}{a^2+\omega^2}$

Problem 2:

Find the Fourier Transform of the following signals.

i) $x(t) = 1$

ii) $x(t) = \text{sgn}(t)$

iii) $x(t) = u(t)$

Soln: $x(t) = 1$

$$FT[\delta(t)] = 1$$

here $x(t) = \delta(t)$ and $X(j\omega) = 1$

$$\therefore \quad FT[\delta(t)] = X(j\omega)$$

Using duality property

$$FT[X(t)] = 2\pi x(-j\omega)$$

here $X(t) = 1$ then $x(-j\omega) = \delta(-j\omega)$

Then $FT[1] = 2\pi\delta(-\omega)$

$\delta(\omega)$ is even function. Since it is impulse function

hence $\delta(-\omega) = \delta(\omega)$

$$\therefore \quad FT[1] = 2\pi\delta(\omega)$$

ii) $x(t) = \text{sgn}(t)$

Given $x(t) = \text{sgn}(t)$

The relation between signum and unit step function is

de'

$$\text{sgn}(t) = 2u(t) - 1$$

$$x(t) = 2u(t) - 1$$

Differentiate both sides w.r.t time

$$\frac{d}{dt}x(t) = 2\frac{d}{dt}u(t) - 0$$

$$\frac{d}{dt}x(t) = 2\,\delta(t) \qquad \because \frac{d}{dt}u(t) = \delta(t)$$

Apply F.T on both sides

$$FT\left[\frac{d}{dt}x(t)\right] = 2FT[\delta(t)]$$

$$j\omega . X(\omega) = 2$$

$$X(\omega) = \frac{2}{j\omega}$$

$$\therefore FT[sgn(t)] = \frac{2}{j\omega}$$

iii) $x(t) = u(t)$

Relation between $u(t)$ and signum function is

$$sgn(t) = 2u(t) - 1$$

$$2u(t) = sgn(t) + 1$$

Apply FT on both sides

$$2FT[u(t)] = FT[sgn(t)] + FT[1]$$

$$2FT[u(t)] = \frac{2}{j\omega} + 2\pi\delta(\omega)$$

$$\therefore FT[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)$$

Problem 3:

Find the Fourier Transform of the following signals

i) $x(t) = \cos n\omega_0 t$

ii) $x(t) = \sin n\omega_0 t$

Soln:

i) Given $x(t) = \cos n\omega_0 t$

we know $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

here $\theta = n\omega_0 t$

$$\therefore x(t) = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

$$x(t) = \frac{1}{2} e^{jn\omega_0 t} + \frac{1}{2} e^{-jn\omega_0 t}$$

Apply F.T on both sides.

$$FT[x(t)] = FT\left[e^{jn\omega_0 t} \cdot \tfrac{1}{2}\right] + FT\left[e^{-jn\omega_0 t} \cdot \tfrac{1}{2}\right]$$

we know $FT[1] = 2\pi\delta(\omega)$

$FT\left[\frac{1}{2}\right] = \pi\delta(\omega)$

Using Frequency shift property

$$FT[e^{jn\omega_0 t} x(t)] = X(\omega - n\omega_0)$$

$$\therefore FT\left[e^{jn\omega_0 t} \cdot \tfrac{1}{2}\right] = \pi\,\delta(\omega - n\omega_0)$$

$$\& \quad FT\left[e^{-jn\omega_0 t} \cdot \tfrac{1}{2}\right] = \pi\,\delta(\omega + n\omega_0)$$

$$\therefore FT[\cos n\omega_0 t] = \pi\left(\delta(\omega - n\omega_0) + \delta(\omega + n\omega_0)\right)$$

ii) $x(t) = \sin n\omega_0 t$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

here $\theta = n\omega_0 t$

$$\therefore x(t) = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

$$x(t) = \frac{1}{2j} e^{jn\omega_0 t} - \frac{1}{2j} e^{-jn\omega_0 t}$$

Apply F.T on both sides

$$FT[x(t)] = \frac{1}{j} FT\left[e^{jn\omega_0 t} \cdot \frac{1}{2}\right] - \frac{1}{j}\left[FT\left(e^{-jn\omega_0 t} \cdot \frac{1}{2}\right)\right]$$

we know $FT[1] = 2\pi\delta(\omega)$

$$FT\left[\tfrac{1}{2}\right] = \pi\delta(\omega)$$

$$FT\left[e^{jn\omega_0 t} \cdot \tfrac{1}{2}\right] = \pi\delta(\omega - n\omega_0)$$

$$FT\left[e^{-jn\omega_0 t} \cdot \tfrac{1}{2}\right] = \pi\delta(\omega + n\omega_0)$$

$$\therefore FT[x(t)] = \frac{1}{j}\left[\pi\delta(\omega - n\omega_0)\right] - \frac{1}{j}\pi\delta(\omega + n\omega_0)$$

$$FT[x(t)] = \frac{\pi}{j}\left[\delta(\omega - n\omega_0) - \delta(\omega + n\omega_0)\right]$$

$$FT[\sin n\omega_0 t] = j\pi\left[\delta(\omega + n\omega_0) - \delta(\omega - n\omega_0)\right]$$

Find the Fourier Transform of the following

1) $x(t)\cos\omega_0 t$

2) $x(t)\sin\omega_0 t$

Soln:

1) Given $x(t)\cos\omega_0 t$

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

we know $\cos\omega_0 t = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

$$\therefore FT[x(t)\cos\omega_0 t] = FT\left[x(t)\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)\right]$$

$$= \frac{1}{2} FT[e^{j\omega_0 t} x(t)] + \frac{1}{2} FT[e^{-j\omega_0 t} x(t)]$$

Using Freq. shifting property

$$FT[e^{j\omega_0 t} x(t)] = X(\omega - \omega_0)$$

$$\& \quad FT[e^{-j\omega_0 t} x(t)] = X(\omega + \omega_0)$$

$$\therefore FT[x(t)\cos\omega_0 t] = \frac{1}{2} X(\omega-\omega_0) + \frac{1}{2} X(\omega+\omega_0)$$

$$FT[x(t)\cos\omega_0 t] = \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)]$$

2) $x(t)\sin\omega_0 t$

We know that

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\therefore FT[x(t)\sin\omega_0 t] = FT\left[x(t)\left(\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right)\right]$$

$$= \frac{1}{2j}\left\{FT[x(t)e^{j\omega_0 t}] - FT[x(t)e^{-j\omega_0 t}]\right\}$$

$$= \frac{1}{2j}[X(\omega-\omega_0) - X(\omega+\omega_0)]$$

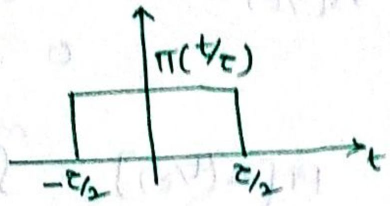
$$\therefore FT[x(t)\sin\omega_0 t] = \frac{1}{2j}[X(\omega-\omega_0) - X(\omega+\omega_0)]$$

Fourier Transform of the standard signal

i) Rectangular pulse (Unit gate function):

Rectangular pulse is defined as

$$\Pi\left(\frac{t}{\tau}\right) = 1 \quad \text{for } |t| \le \tau/2$$
$$= 0 \quad \text{for } |t| > \tau/2$$



let $x(t) = \Pi(t/\tau)$

Then $X(\omega) = FT[x(t)] = FT[\Pi(t/\tau)] = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$

$$X(\omega) = \int_{-\tau/2}^{\tau/2} (1)\, e^{-j\omega t}\, dt$$

$$= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-\tau/2}^{\tau/2}$$

$$= \frac{-1}{j\omega}\left[e^{-j\omega\tau/2} - e^{j\omega\tau/2} \right]$$

$$= \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega}$$

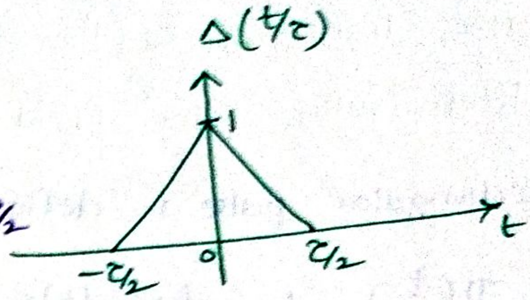
$$= \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j} \cdot \frac{2}{\omega}$$

$$= \frac{2}{\omega} \sin \omega\tau/2 = \frac{\sin \omega\tau/2}{\omega\tau/2} \cdot \tau/2 \times 2$$

$$\boxed{FT[\Pi(t/\tau)] = \tau\, \mathrm{sinc}(\omega\tau/2)} \qquad \because \mathrm{sinc}(\theta) = \frac{\sin\theta}{\theta}$$

ii) Triangular Pulse:

Triangular pulse is defined as



$$\Delta(t/\tau) = \begin{cases} 1-2\frac{|t|}{\tau} & \text{for } |t|<\tau/2 \\ 0 & \text{otherwise} \end{cases}$$

$$FT[\Delta(t/\tau)] = \int_{-\infty}^{\infty} \Delta(t/\tau) e^{-j\omega t}\, dt$$

$$\Delta(t/\tau) = 1+\frac{2t}{\tau} \quad \text{for } -\tau/2<t<0$$

$$= 1-\frac{2t}{\tau} \quad \text{for } 0<t<\tau/2$$

$$\therefore FT[\Delta(t/\tau)] = \int_{-\tau/2}^{0}\left(1+\frac{2t}{\tau}\right)e^{-j\omega t}\,dt + \int_{0}^{\tau/2}\left(1-\frac{2t}{\tau}\right)e^{-j\omega t}\,dt$$

$$= \frac{e^{-j\omega t}}{-j\omega}\Big|_{-\tau/2}^{0} + \int_{-\tau/2}^{0} 2\frac{t}{\tau}e^{-j\omega t}\,dt + \frac{e^{-j\omega t}}{-j\omega}\Big|_{0}^{\tau/2}$$

$$- \int_{0}^{\tau/2} 2t/\tau\, e^{-j\omega t}\,dt$$

$$= \cancel{\frac{-1}{j\omega}} + \frac{e^{j\omega\tau/2}}{j\omega} + \frac{2}{\tau}\left[\frac{te^{-j\omega t}}{-j\omega}\Big|_{-\tau/2}^{0} - \int_{-\tau/2}^{0}\frac{e^{-j\omega t}}{-j\omega}\,dt\right]$$

$$-\frac{e^{-j\omega\tau/2}}{j\omega} + \cancel{\frac{1}{j\omega}} - \frac{2}{\tau}\left[\frac{te^{-j\omega t}}{-j\omega}\Big|_{0}^{\tau/2} - \int_{0}^{\tau/2}\frac{e^{-j\omega t}}{-j\omega}\,dt\right]$$

$$= \frac{e^{j\omega\tau/2}}{j\omega} - \frac{e^{-j\omega\tau/2}}{j\omega} + \frac{2}{\tau}\left(\frac{te^{-j\omega t}}{-j\omega} + \frac{e^{-j\omega t}}{(j\omega)(-j\omega)}\right]_{-\tau/2}^{0}$$

$$-\frac{2}{\tau}\left[\frac{te^{-j\omega t}}{-j\omega} + \frac{e^{-j\omega t}}{(j\omega)(-j\omega)}\right]_{0}^{\tau/2}$$

$$= \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega} + \frac{2}{\tau}\left[\frac{-(-\tau/2)\, e^{-j\omega(-\tau/2)}}{-j\omega} - \frac{1}{(j\omega)^2} + \frac{e^{j\omega\tau/2}}{(j\omega)^2} \right.$$

$$\left. + \frac{\tau/2\, e^{-j\omega\tau/2}}{j\omega} + \frac{1}{(j\omega)^2} e^{-j\omega\tau/2} - \frac{1}{(j\omega)^2} \right]$$

$$= \cancel{\frac{e^{j\omega\tau/2}}{j\omega}} - \cancel{\frac{e^{-j\omega\tau/2}}{j\omega}} - \cancel{\frac{e^{j\omega\tau/2}}{j\omega}} - \frac{2}{\tau}\frac{1}{(j\omega)^2} + \frac{2}{\tau}\cdot\frac{e^{j\omega\tau/2}}{(j\omega)^2}$$

$$+ \cancel{\frac{e^{-j\omega\tau/2}}{j\omega}} + \frac{2}{\tau}\cdot\frac{e^{-j\omega\tau/2}}{(j\omega)^2} - \frac{2}{\tau}\cdot\frac{1}{(j\omega)^2}$$

$$= \frac{2}{\tau}\cdot\frac{1}{(j\omega)^2}\left[e^{j\omega\tau/2} + e^{-j\omega\tau/2} - 2 \right]$$

$$= \frac{2}{\tau}\cdot\frac{1}{(j\omega)^2}\left[\left(e^{j\omega\tau/4}\right)^2 + \left(e^{-j\omega\tau/4}\right)^2 - 2\left(e^{j\omega\tau/4}\right)\left(e^{-j\omega\tau/4}\right) \right]$$

$$= \frac{2}{\tau}\cdot\frac{1}{(j\omega)^2}\left(e^{j\omega\tau/4} - e^{-j\omega\tau/4}\right)^2$$

$$= \frac{2}{\tau}\cdot\left(\frac{e^{j\omega\tau/4} - e^{-j\omega\tau/4}}{j\omega}\right)^2$$

$$= \frac{2}{\tau}\cdot\left[\frac{e^{j\omega\tau/4} - e^{-j\omega\tau/4}}{2j} \times \frac{2}{\omega}\right]^2$$

$$= \frac{2}{\tau}\cdot\left[\frac{\sin\omega\tau/4}{\omega}\right]^2 \times 4$$

$$= \frac{2}{\tau}\left(\frac{\sin\omega\tau/4}{\omega\cdot\tau/4}\cdot\tau/4\right)^2 \times 4 = \frac{8}{\tau}\,\mathrm{sinc}^2\left(\frac{\omega\tau}{4}\right)\left(\frac{\tau^2}{16}\right)$$

$$\boxed{FT[\Delta(t/\tau)] = \tau/2\ \mathrm{sinc}^2(\omega\tau/4)}$$

Find the Fourier Transform of the following

i) $x(t) = e^{-2t} u(t-1)$ ii) $x(t) = t \cdot e^{-3t} u(t)$

Soln: i) $x(t) = e^{-2t} u(t-1)$

We know $FT[e^{-at} u(t)] = \frac{1}{a+j\omega}$

$\therefore FT[e^{-2t} u(t)] = \frac{1}{2+j\omega}$

Using Time shifting property

$FT[x(t-t_0)] = e^{-j\omega t_0} \cdot X(\omega).$

$\therefore FT[e^{-2(t-1)} u(t-1)] = e^{-j\omega} \cdot \frac{1}{2+j\omega}$

$FT[e^{-2t} \cdot e^{2} u(t-1)] = \frac{e^{-j\omega}}{2+j\omega}$

$e^{2} FT[e^{-2t} u(t-1)] = \frac{e^{-j\omega}}{2+j\omega}$

$FT[e^{-2t} u(t-1)] = \frac{e^{-j\omega} \cdot e^{-2}}{2+j\omega} = \frac{e^{-(2+j\omega)}}{2+j\omega}$

ii) $x(t) = t \cdot e^{-3t} u(t)$

We know $FT[e^{-at} u(t)] = \frac{1}{a+j\omega}$

$\therefore FT[e^{-3t} u(t)] = \frac{1}{3+j\omega}$

From the differentiation property

$FT[-jt x(t)] = \frac{d}{d\omega} X(\omega)$

$FT[t x(t)] = j \frac{d}{d\omega} X(\omega)$

$\therefore FT[t \cdot e^{-3t} u(t)] = j \frac{d}{d\omega}\left[\frac{1}{3+j\omega}\right] = j\left[\frac{-1(j)}{(3+j\omega)^2}\right]$

$= \frac{1}{(3+j\omega)^2}$

Find the inverse Fourier Transform of $X(\omega) = \frac{j\omega}{(3+j\omega)^2}$

We know that $FT[t e^{-at} u(t)] = \frac{1}{(a+j\omega)^2}$

$\therefore FT[t e^{-3t} u(t)] = \frac{1}{(3+j\omega)^2}$

let $t e^{-3t} u(t) = x_1(t)$

Then $X_1(\omega) = \frac{1}{(3+j\omega)^2}$

From Time differentiation property

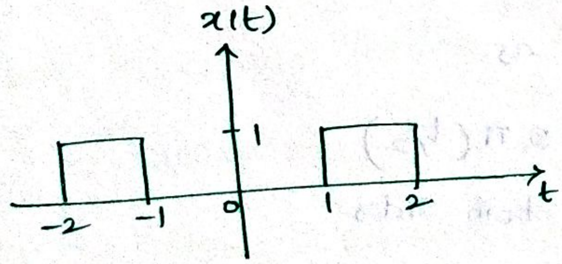
$$FT\left[\frac{d}{dt} x_1(t)\right] = j\omega . X_1(\omega)$$

$$\therefore FT\left[\frac{d}{dt} t e^{-3t} u(t)\right] = \frac{j\omega}{(3+j\omega)^2}$$

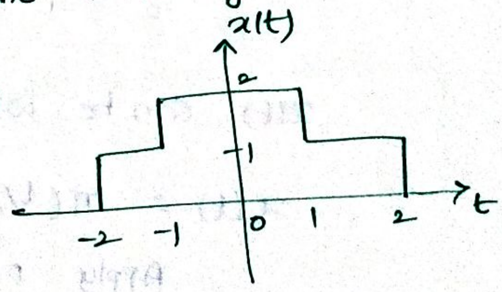
$$\therefore IFT\left[\frac{j\omega}{(3+j\omega)^2}\right] = \frac{d}{dt}\left[t e^{-3t} u(t)\right]$$

Find the Fourier Transform of the following

1)



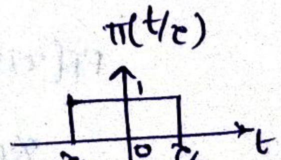
2)



Soln: Rectangular pulse is defined as

$\pi(t/\tau) = 1$ for $|t| \leq \tau/2$

$= 0$ for $|t| > \tau/2$



i) Signal x(t) can be written as

$$x(t) = \pi(t + 3/2) + \pi(t - 3/2)$$

Apply F.T on both sides.

$$FT[x(t)] = FT[\Pi(t+3/2)] + FT[\Pi(t-3/2)]$$

we know $\quad FT[\Pi(t/\tau)] = \tau\, \text{sinc}(\omega\tau/2)$

$$\therefore \quad FT[\Pi(t)] = \text{sinc}(\omega/2)$$

& $\quad FT[\Pi(t-3/2)] = e^{-j\omega 3/2}\, \text{sinc}(\omega/2)$

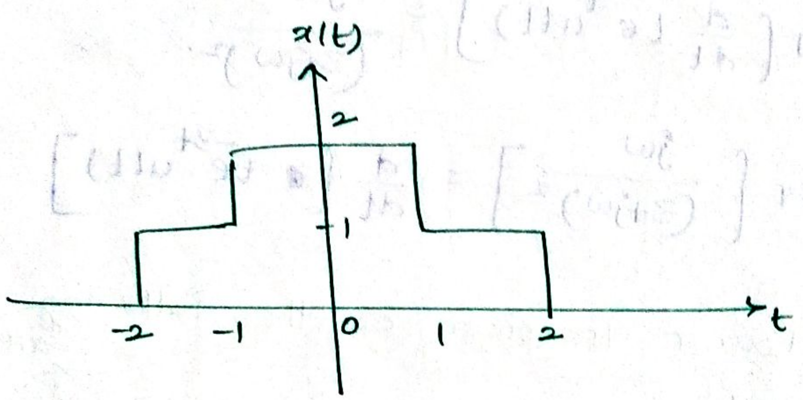
$$FT[\Pi(t+3/2)] = e^{j\omega 3/2}\, \text{sinc}(\omega/2)$$

$$\therefore \quad FT[x(t)] = e^{j\omega 3/2}\, \text{sinc}(\omega/2) + e^{-j\omega 3/2}\, \text{sinc}(\omega/2)$$

$$X(\omega) = \left[e^{j\omega 3/2} + e^{-j\omega 3/2}\right] \text{sinc}(\omega/2)$$

$$X(\omega) = 2\cos(3\omega/2)\, \text{sinc}(\omega/2)$$

ii)



$x(t)$ can be written as

$$x(t) = \Pi(t/4) + 2\Pi(t/2)$$

Apply FT on both sides

$$FT[x(t)] = FT[\Pi(t/4)] + 2\, FT[\Pi(t/2)]$$

$$X(\omega) = 4\, \text{sinc}(\omega 4/2) + 2\, \text{sinc}(2\omega/2)$$

$$X(\omega) = 4\, \text{sinc}(\omega/2) + 2\, \text{sinc}(\omega)$$

Fourier Transform of a periodic signal:

Any signal can be represented as a sum of complex exponentials. Therefore, we can represent a periodic signal using the Fourier integral. Let us consider a signal $x(t)$ with a period T. Then we can express $x(t)$ interms of fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

The Fourier Transform of $x(t)$ is

$$X(\omega) = FT\left[\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right]$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} C_n \; FT[e^{jn\omega_0 t} \cdot 1]$$

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \; \delta(\omega - n\omega_0)$$

where C_n represent Fourier Coefficient

and $C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt$

Using Parseval's Theorem for power signals, evaluate

$$\int_{-\infty}^{\infty} e^{-2t} u(t)\, dt$$

Soln: Using Parseval's Theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega,$$

let $x(t) = e^{-t}u(t)$

Then $X(\omega) = \frac{1}{1+j\omega}$

$$\therefore \int_{-\infty}^{\infty} |e^{-t}u(t)|^2 dt = \int_{-\infty}^{\infty} e^{-2t}u(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{1+j\omega}\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{1+\omega^2}}\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\omega^2}\, d\omega$$

$$= \frac{1}{2\pi}\tan^{-1}(\omega)\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi}\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{1}{2}$$

Find the Fourier Transform of

1) Gaussian signal $x(t) = e^{-at^2}$

2) Gaussian modulated signal $x(t) = e^{-at^2}\cos\omega_0 t$

Soln: 1) Given that $x(t) = e^{-at^2}$

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$FT[e^{-at^2}] = \int_{-\infty}^{\infty} e^{-at^2} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-(at^2+j\omega t)}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-\left((\sqrt{a}t)^2+\left(\frac{j\omega}{2\sqrt{a}}\right)^2-\left(\frac{j\omega}{2\sqrt{a}}\right)^2+2\sqrt{a}\cdot\frac{j\omega}{2\sqrt{a}}t\right)} dt$$

$$= e^{-\omega^2/4a}\int_{-\infty}^{\infty} e^{-\left(t\sqrt{a}+\frac{j\omega}{2\sqrt{a}}\right)^2} dt$$

let $t\sqrt{a}+\frac{j\omega}{2\sqrt{a}} = p$

$\sqrt{a}\,dt = dp$

$dt = \frac{dp}{\sqrt{a}}$

$$= e^{-\omega^2/4a} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{a}}$$

$$= \frac{e^{-\omega^2/4a}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-p^2} dp$$

$$= \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a} \qquad \because \int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$$

2) $x(t) = e^{-at^2} \cos\omega_0 t$

$$FT[e^{-at^2}] = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

$$FT[e^{-at^2} \cos\omega_0 t] = ?$$

let $x_1(t) = e^{-at^2} \Rightarrow X_1(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$

$$\therefore FT[x_1(t) \cos\omega_0 t] = \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)$$

$$\therefore FT[e^{-at^2} \cos\omega_0 t] = \frac{1}{2}\left(\sqrt{\frac{\pi}{a}}\, e^{-(\omega-\omega_0)^2/4a} + \sqrt{\frac{\pi}{a}}\, e^{-(\omega+\omega_0)^2/4a}\right]$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{a}}\left[e^{-(\omega-\omega_0)^2/4a} + e^{-(\omega+\omega_0)^2/4a}\right]$$

Excercise Problems:

Find the Fourier Transform of the following signals.

1) $x(t) = 3\cos 10t + 4\sin 10t$

2) $x(t) = e^{-3t}[u(t+2) - u(t-3)]$

3) $x(t) = 5\sin^2(3t)$

4) $x(t) = t\cos 2t$

* Find the Fourier Transform of the following functions

i) A single symmetrical Triangular pulse with period $T = 8$ sec and amplitude $A = 10V$.

ii) A single symmetrical gate pulse

iii) A single cosine wave at $t=0$

* Find the Fourier Transform of the waveform shown below