UNIT-IV

FIR Digital Filters.

The FIR filters are of non-recursive type, whereby the present output sample depends on the present input sample and previous input samples. In many digital processing applications FIR filters are preferred over their IIR counterparts.

Advantages of FIR filters:-

1. FIR filters are always stable.
2. FIR filters with exactly linear phase can easily be designed
3. FIR filters can be realized in both recursive and non-recursive structures.
4. FIR filters are free of limit cycle oscillations, when implemented on a finite word length digital system.
5. Excellent design methods are available for various kinds of FIR filters.

Disadvantages of FIR filters:-

1. The implementation of narrow transition band FIR filters are very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders and delay elements.
2. memory requirement and execution time are very high.

comparision between FIR and IIR filters:-

FIR filter	IIR filter
1. The impulse response of this filter is restricted to finite number of samples.	1. The impulse response of this filter extends over an infinite duration.
2. FIR filters can have precisely linear phase.	2. These filters do not have linear phase.
3. closed form design equations do not exist.	3. A variety of frequency selective filters can be designed using closed form design formulas.
4. Most of the design methods are iterative procedures, requiring powerful computational facilities for their implementation.	4. These filters can be designed using only a hand calculator and tables of analog filter design parameters.
5. Greater flexibility to control the shape of their magnitude response.	5. less flexibility specially for obtaining non standard frequency responses.
6. In these filters, the poles are fixed at the origin, high selectivity can be achieved by using a relatively high order for the transfer function	6. The poles are placed anywhere inside the unit circle, high selectivity can be achieved with low order transfer functions.
7. Always stable	7. Not always stable.
8. Errors due to round off noise are less severe.	8. IIR filters are more susceptible to errors due to round off noise.

linear phase FIR filters :-

The transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \quad \text{—} (1)$$

where $h(n)$ is the impulse response of filter.

The fourier Transform of $h(n)$ is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

which is periodic in frequency with period 2π.

$$H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j\theta(\omega)}$$

$|H(e^{j\omega})| \rightarrow$ magnitude response

$\theta(\omega) \rightarrow$ phase Response.

phase delay and Group delay of a filter can be defined as

$$T_p = \frac{-\theta(\omega)}{\omega} \quad \& \quad T_g = \frac{-d\theta(\omega)}{d\omega}$$

For FIR filters with linear phase, can define as

$$\theta(\omega) = -\alpha \omega \qquad -\pi \le \omega \le \pi.$$

where $\alpha \rightarrow$ constant phase delay in samples.

$$T_p = \frac{-(-\alpha\omega)}{\omega} = \alpha \quad \& \quad T_g = \frac{-d}{d\omega}(-\alpha\omega) = \alpha$$

$\therefore T_p = T_g = \alpha$, which means α is independent of frequency.

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| e^{j\theta(\omega)}$$

$$\sum_{n=0}^{N-1} h(n) [\cos\omega n - j\sin\omega n] = \pm |H(e^{j\omega})| (\cos\theta(\omega) + j\sin\theta(\omega))$$

$$\sum_{n=0}^{N-1} h(n)\cos\omega n - j \sum_{n=0}^{N-1} h(n)\sin\omega n = \pm |H(e^{j\omega})| \cos\theta(\omega) \pm j |H(e^{j\omega})| \sin\theta(\omega)$$

which gives

$$\sum_{n=0}^{N-1} h(n)\cos\omega n = \pm |H(e^{j\omega})| \cos\theta(\omega) \quad \text{--- (2)}$$

$$-\sum_{n=0}^{N-1} h(n)\sin\omega n = \pm |H(e^{j\omega})| \sin\theta(\omega) \quad \text{--- (3)}$$

By taking ratio of (3) & (2),

$$\frac{-\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n} = \frac{\sin\theta(\omega)}{\cos\theta(\omega)} \qquad \left[\because\ \theta(\omega) = -\alpha\omega\right]$$

$$\frac{-\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n} = \frac{\sin(-\alpha\omega)}{\cos(-\alpha\omega)}$$

$$+\sum_{n=0}^{N-1} h(n)\sin\omega n\cos\alpha\omega = +\sum_{n=0}^{N-1} h(n)\cos\omega n\sin\alpha\omega$$

$$\sum_{n=0}^{N-1} h(n)\sin\omega n\cos\alpha\omega - \sum_{n=0}^{N-1} h(n)\cos\omega n\sin\alpha\omega = 0$$

$$\sum_{n=0}^{N-1} h(n)\left[\sin\omega n\cos\alpha\omega - \cos\omega n\sin\alpha\omega\right] = 0$$

$$\sum_{n=0}^{N-1} h(n)\sin(\alpha - n)\omega = 0 \quad \text{--- (4)}$$

equation (4) will be zero, when

$$\boxed{h(n) = h(N-1-n)} \text{ --- (5)} \quad \& \quad \boxed{\alpha = \frac{N-1}{2}} \text{ --- (6)}$$

Therefore, FIR filters will have constant phase and group delays when the impulse response is symmetrical about $\alpha = \frac{N-1}{2}$.

The impulse response satisfying equation ⑤ & ⑥ for odd and even values of N. when N=7 the centre of symmetry of the sequence occurs at third sample and when N=6, the filter delay is $2\frac{1}{2}$ samples.

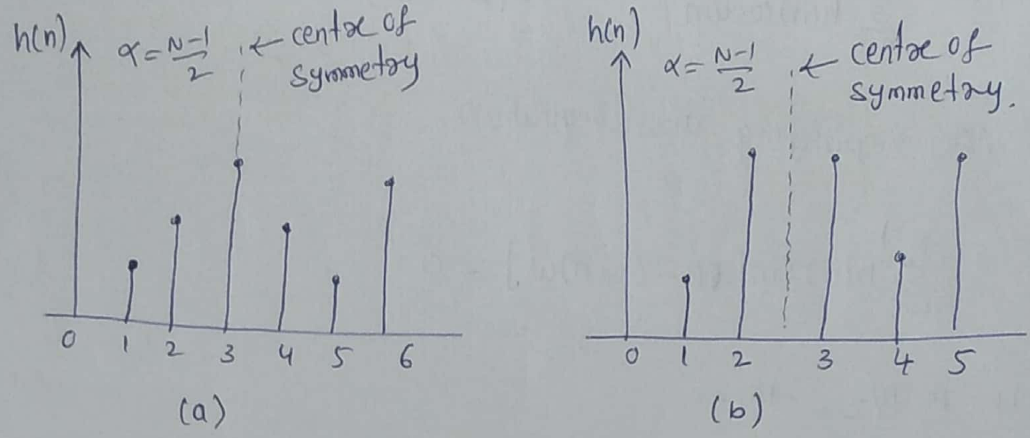


fig: Impulse Response Sequence of symmetric sequences for
(a) N odd (b) N even.

If only constant group delay is required, and not the phase delay, we can write

$$\theta(\omega) = \beta - \alpha\omega \quad \text{— ⑦}$$

$$H(e^{j\omega}) = \pm |H(e^{j\omega})| \, e^{j(\beta-\alpha\omega)} \quad \text{— ⑧}$$

Above equation can be expressed as

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| \, e^{j(\theta-\alpha\omega)}$$

which gives us,

$$\sum_{n=0}^{N-1} h(n) \cos\omega n = \pm |H(e^{j\omega})| \cos(\beta - \alpha\omega) \quad \text{— ⑨}$$

$$-\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm |H(e^{j\omega})| \sin(\beta - \alpha\omega) \quad \text{— ⑩}$$

By taking the ratio of eq (9) & (10)

$$\frac{-\sum_{n=0}^{N-1} h(n)\sin wn}{\sum_{n=0}^{N-1} h(n)\cos wn} = \frac{\sin(\beta-\alpha w)}{\cos(\beta-\alpha w)}$$

After simplifying above equation,

$$\sum_{n=0}^{N-1} h(n)\sin[\beta-(\alpha-n)w] = 0.$$

If $\beta=\pi/2$, then

$$\sum_{n=0}^{N-1} h(n)\cos(\alpha-n)w = 0$$

Above equation will be satisfied when,

$$h(n) = -h(N-1-n) \quad \text{— (11)}$$

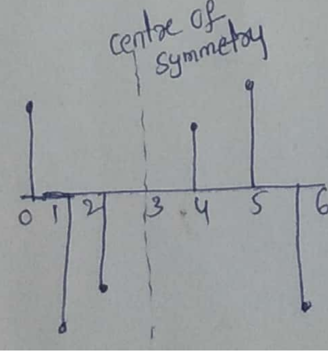
& $$\alpha = \frac{N-1}{2} \quad \text{— (12)}$$

∴ Therefore, FIR filters have constant Group delay T_g, and not constant phase delay when the impulse response is antisymmetrical about $\alpha = \frac{N-1}{2}$.

The impulse response satisfying eq (11) & (12) as shown in fig.

The impulse response satisfying eq (11) & (12) as shown in fig. The centre of antisymmetry occurs at third sample, when $N=7$ the centre of antisymmetry occurs at $2\frac{1}{2}$ sample. and when $N=6$ the centre of antisymmetry occurs at $2\frac{1}{2}$ sample.

we find that $h\left(\frac{N-1}{2}\right)=0$ for antisymmetric odd sequence.



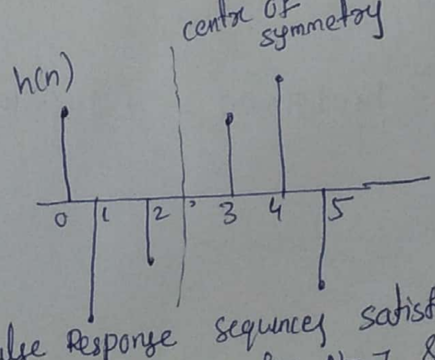


fig: Impulse Response sequences satisfying eq (11) & (12) for $N=7$ & $N=6$.

Frequency Response of linear phase FIR filters :-

The frequency response of impulse response can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{6} h(n) e^{-j\omega n} \quad \text{—(1)}$$

This can be split like

$$H(e^{j\omega}) = \sum_{n=0}^{2} h(n) e^{-j\omega n} + h(3) e^{-j3\omega} + \sum_{n=4}^{6} h(n) e^{-j\omega n}$$

In general for 'N' samples

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega(N-1)/2} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n}$$

let $n = N-1-m$, we have

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega (N-1)/2} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m) e^{-j\omega(N-1-m)}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega (N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$$

for a symmetrical impulse response $h(n) = h(N-1-n)$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega (N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega (N-1-m)}$$

$$= e^{-j\omega(N-1)/2} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) e^{j\omega (N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega (N-1)/2 - n} + h\left(\frac{N-1}{2}\right) \right]$$

$$= e^{-j\omega (N-1)/2} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) . \underbrace{e^{j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega\left(\frac{N-1}{2}\right)}}_{=1} . e^{-j\omega \left(\frac{N-1}{2}\right) - n} + h\left(\frac{N-1}{2}\right) \right]$$

$$H(e^{j\omega}) = e^{-j\omega (N-1)/2} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \left[2\cos\omega \left(\frac{N-1}{2} - n\right)\right] + h\left(\frac{N-1}{2}\right) \right]$$

let $\frac{N-1}{2} - n = p$.

$$= e^{-j\omega(N-1)/2} \left[\sum_{p=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - p\right) \cos\omega p + h\left(\frac{N-1}{2}\right) \right].$$

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}\left[\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n + h\left(\frac{N-1}{2}\right)\right]$$

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}\sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

where

$$a(0) = h\left(\frac{N-1}{2}\right) \qquad a(n) = 2h\left(\frac{N-1}{2}-n\right)$$

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}\,\bar{H}(e^{j\omega}) = \bar{H}(e^{j\omega})e^{j\theta(\omega)}$$

where

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

&

$$\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega.$$

$\bar{H}(e^{j\omega})$ is a real and even function of ω.

The magnitude and phase of $H(e^{j\omega})$ are

$$|H(e^{j\omega})| = |\bar{H}(e^{j\omega})|$$

$$\angle H(e^{j\omega}) = \theta(\omega) = -\alpha\omega \quad \text{when } \bar{H}(e^{j\omega}) \geq 0$$

$$\angle H(e^{j\omega}) = -\alpha\omega + \pi \quad \text{when } \bar{H}(e^{j\omega}) < 0$$

$$\& \ \theta(\omega) = -\alpha\omega \quad \text{when } \bar{H}(e^{j\omega}) < 0$$

$\bar{H}(e^{j\omega})$ is called as zero phase frequency response to distinguish it from the magnitude response.

→ The zero phase frequency response of the filter may take both positive and negative values, whereas magnitude response

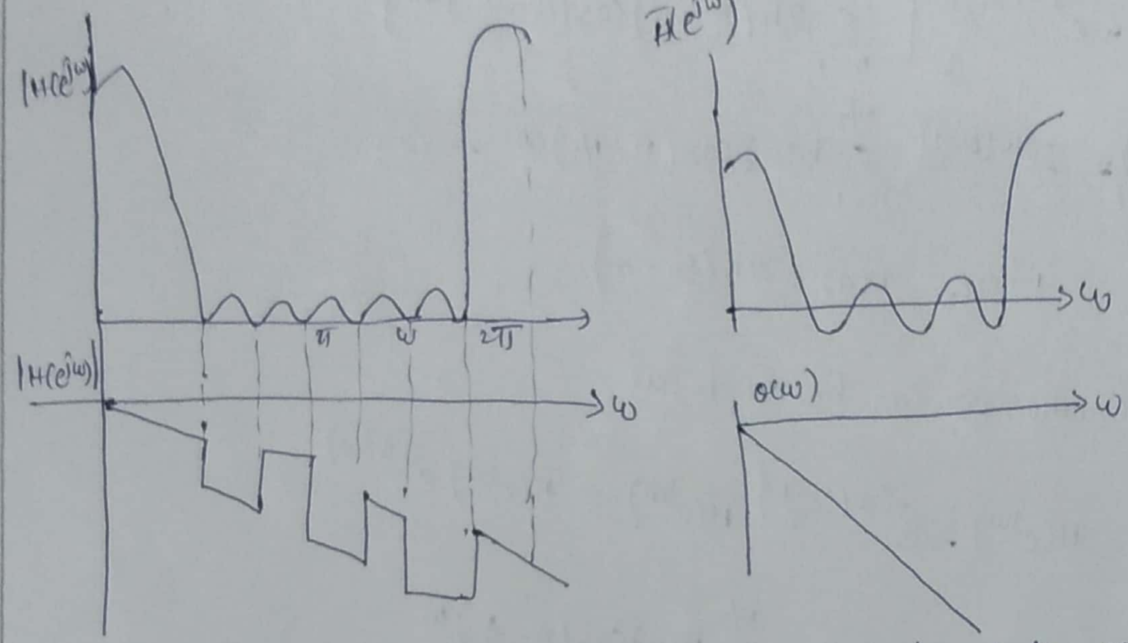


fig: Relation between magnitude Response $|H(e^{j\omega})|$ and the zero phase response $\bar{H}(e^{j\omega})$ & between $\angle H(e^{j\omega})$ & $\theta(\omega)$.

case II: symmetric impulse response for N even.

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \quad \text{--- (1)}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n} \qquad (\because h(n) = h(n-1-N))$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(n-1-n) e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-2}{2}} h(n) e^{j\omega (N-1)/2-n} + \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega\left(\frac{N-1}{2}\right)-n} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-2}{2}} 2h(n) \cos\omega\left(\frac{N-1}{2} - n\right) \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right) \cos\left(n-\frac{1}{2}\right)\omega \right]$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=1}^{N/2} b(n) \cos(n-1/2)\omega \quad \text{--- } (2)$$

where $b(n) = 2h\left(\frac{N}{2}-n\right)$

equation (2) can be write as

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \bar{H}(e^{j\omega}) = \bar{H}(e^{j\omega})\, e^{j\theta(\omega)}.$$

where $\bar{H}(e^{j\omega}) = \sum_{n=1}^{N/2} b(n) \cos\left(n-\frac{1}{2}\right)\omega$

$$\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega$$

The frequency response of linear phase filter with symmetric impulse response for 'N' even.

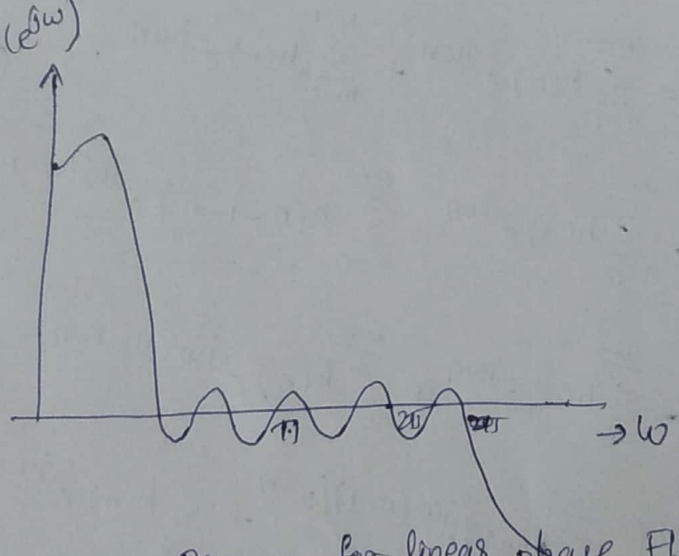


fig: frequency Response for linear phase FIR filter, Symmetrical impulse response 'N' even.

Fourier series method :-

The frequency response $H(e^{j\omega})$ of a system is periodic in 2π. we know that any periodic function can be expressed as a linear combination of complex exponentials.

therefore, the desired frequency response of an FIR filter can be represented by the fourier series.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n} \quad \text{—} (1)$$

where the fourier coefficients $h_d(n)$ are the desired impulse response sequence of the filter.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \quad \text{—} (2)$$

the z-transform of the sequence is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h_d(n) z^{-n} \quad \text{—} (3)$$

Above equation represents a non-causal digital filter of infinite duration. To get an FIR filter transfer function, the series can be truncated by assigning.

$$h(n) = h_d(n) \quad \text{for } |n| \leq \frac{N-1}{2}$$
$$= 0 \quad \text{otherwise.}$$

Then

$$\boxed{H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) z^{-n}}$$

Window type	window sequnces $w(n)$ for $-(N-1)/2 \le n \le (N-1)/2$
1. Rectangular window	1
2. Triangular window	$1 - \frac{2\|n\|}{N-1}$
3. Raised cosine window	$\alpha + (1-\alpha)\cos\frac{2\pi n}{N-1}$ $\alpha = 0.5$ for Hanning window $\alpha = 0.54$ for Hamming window
4. Blackman window	$0.42 + 0.5\cos\frac{2\pi n}{N-1} + 0.08\frac{4\pi n}{N-1}$
5. Kaiser window	$w_k(n) = \frac{I_0\left[\alpha\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right]}{I_0(\alpha)}$

Design an ideal lowpass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad -\frac{\pi}{2} \le \omega \le \pi/2$$
$$= 0 \quad \text{for} \quad \frac{\pi}{2} \le |\omega| \le \pi.$$

find the values of h(n) for, N=11. Find H(z). plot the magnitude response.

Sol. The frequency response of LPF with $\omega_c = \pi/2$

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad -\frac{\pi}{2} \le \omega \le \pi/2$$
$$0 \quad \text{for} \quad \frac{\pi}{2} \le |\omega| \le \pi$$

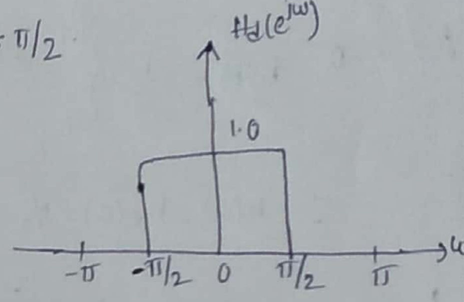


fig: Ideal frequency response.

From the frequency response we can find that $\alpha = 0$.

$\therefore$ we get a non causal filter coefficients symmetrical about $n=0$.

i.e $h_d(n) = h_d(-n)$.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi . jn} \left[e^{j\pi n/2} - e^{-j\pi n/2} \right]$$

$$= \frac{1}{\pi n} \left[\frac{e^{j\pi n/2} - e^{-j\pi n/2}}{2j} \right]$$

$$\boxed{h_d(n) = \frac{\sin(\frac{\pi}{2} n)}{\pi n} \qquad -\infty \le n \le \infty}$$

Truncating $h_d(n)$ to 11 samples,

$$h(n) = \frac{\sin(\frac{\pi}{2}n)}{\pi n} \quad \text{for } |n| \leq 5$$

$$= 0 \quad \text{otherwise.}$$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

for $n=0$, $h(0) = \lim_{n \to 0} \frac{\sin(\frac{\pi}{2}n)}{\pi n} = \left(\frac{1}{2}\right) \lim_{n \to 0} \frac{\sin(\frac{\pi}{2}n)}{(\frac{\pi}{2}n)} = \underline{\frac{1}{2}}$

for $n=0$, $h_d(0) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega(0)} \, d\omega = \frac{1}{2\pi} \Big[\omega \Big]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi} \left[\frac{\pi}{2} - (-\pi/2) \right] = \underline{\underline{1/2}}$

$\therefore \; h(0) = h_d(0) = 1/2$

for $n=1$, $h(1) = h(-1) = \frac{\sin \pi/2}{\pi} = \frac{1}{\pi} = 0.3183$.

$n=2 \quad h(2) = h(-2) = \frac{\sin \frac{2\pi}{2}}{\pi(2)} = \frac{\sin \pi}{2\pi} = \underline{0}$

$h(3) = h(-3) = \frac{\sin(\frac{3\pi}{2})}{3\pi} = \frac{-1}{3\pi} = -0.106$

$h(4) = h(-4) = \frac{\sin(\frac{4\pi}{2})}{4\pi} = 0$

$h(5) = h(-5) = \frac{\sin(\frac{5\pi}{2})}{5\pi} = \frac{1}{5\pi} = 0.06366$.

The transfer function of filter is given by

$$H(z) = \sum_{n=-5}^{5} h(n) z^{-n} \qquad \text{i.e } \frac{N-1}{2} = \frac{11-1}{2} = \underline{5}$$

$$H(z) = h(-5)z^5 + h(-4)z^4 + h(-3)z^3 + h(-2)z^2 + h(-1)z^{1} + h(0)$$
$$h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5}$$

$$H(z) = h(+5)(z^5 + z^{-5}) + h(4)(z^4 + z^{-4}) + h(3)(z^3 + z^{-3}) +$$
$$h(2)(z^2 + z^{-2}) + h(1)(z^1 + z^{-1}) + h(0).$$

$$= 0.06366(z^5 + z^{-5}) + 0 + (-0.106)(z^3 + z^{-3}) + 0 + 0.3183(z^1 + z^{-1})$$

$$H(z) = 0.06366(z^5 + z^{-5}) - 0.106(z^3 + z^{-3}) + 0.3183(z^1 + z^{-1}) + 0.5$$

$$\therefore\ H(z) = 0.5 + 0.3183(z^1 + z^{-1}) - 0.106(z^3 + z^{-3}) + 0.06366(z^5 + z^{-5})$$

The frequency response of LPF is given by

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}.$$

$$H(e^{j\omega}) = 0.5 + 0.3183(e^{j\omega} + e^{-j\omega}) - 0.106(e^{j3\omega} - e^{-j3\omega}) + 0.06366(e^{j5\omega} - e^{-j5\omega})$$

$$H(e^{j\omega}) = 0.5 + 0.3183 \times 2\cos\omega - 0.106 \times 2\cos 3\omega + 0.06366(2\cos 5\omega)$$

$$\boxed{H(e^{j\omega}) = 0.5 + 0.6366\cos\omega - 0.212\cos 3\omega + 0.127\cos 5\omega}$$

The magnitude in dB is calculated by varying ω from 0 to π.

The magnitude $|H(e^{j\omega})|_{dB} = 20\log|H(e^{j\omega})|$

ω (in degree)	0	10	20	30	40	50	60	70	80	90
$\lvert H(e^{j\omega})\rvert$	0.4	0.21	-0.26	-0.517	0.27	0.42	0.77	0.21	-1.79	-6

100	110	120	130	140	150	160	170	180
-14.56	-31.89	-20.6	-26	~~-24.07~~ -32	-30.55	-30.55	-32	-26

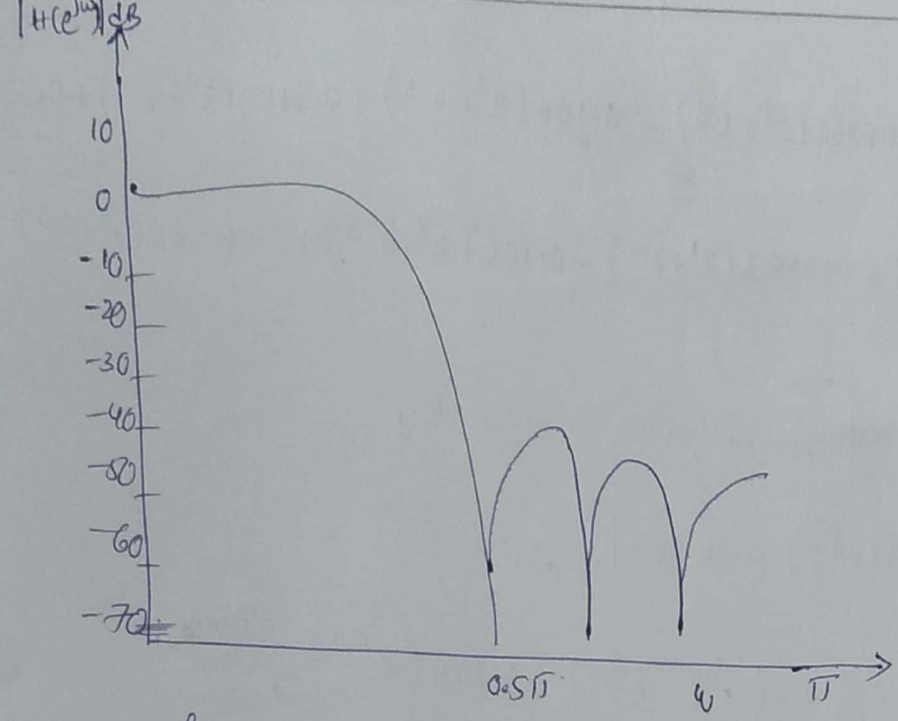


fig: frequency response of LPF.

Design an ideal highpass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \le |\omega| \le \pi$$
$$= 0 \quad \text{for } |\omega| \le \pi/4.$$

find the values of $h(n)$ for $N=11$. Find $H(z)$. plot the magnitude response.

sol

$$H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \le |\omega| \le \pi$$
$$= 0 \quad \text{for } |\omega| \le \pi/4$$

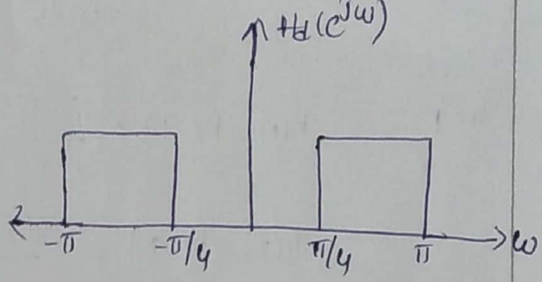


fig: ideal frequency response of HPF.

The desired frequency response is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/4} e^{j\omega n}\, d\omega + \int_{\pi/4}^{\pi} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)_{-\pi}^{-\pi/4} + \left(\frac{e^{j\omega n}}{jn}\right)_{\pi/4}^{\pi}\right]$$

$$= \frac{1}{2\pi(jn)}\left[e^{-jn\pi/4} - e^{-jn\pi} + e^{jn\pi} - e^{+jn\pi/4}\right]$$

$$= \frac{1}{\pi n}\cdot\frac{1}{2j}\left[e^{j\pi n} - e^{-j\pi n} + (e^{j\pi n/4} - e^{-j\pi n/4})\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} - \frac{e^{j\pi n/4} - e^{-j\pi n/4}}{2j}\right]$$

$$= \frac{1}{\pi n}\left[\sin \pi n - \sin\left(\frac{\pi}{4}n\right)\right]$$

$$\therefore\ h_d(n) = \frac{1}{\pi n}\left[\sin \pi n - \sin\left(\frac{\pi}{4}n\right)\right]$$

Truncating $h_d(n)$ to 11 samples, we have

$$h(n) = h_d(n) \quad \text{for } |n| \le 5$$
$$= 0 \quad \text{otherwise.}$$

for $n=0$, $h(0) = \lim_{n\to 0} \frac{\sin \pi n}{\pi n} - \lim_{n\to 0} \frac{\sin(\frac{\pi}{4} n)}{\pi n}$

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$= \left(1 - \frac{1}{4}\right)$

$\lim_{\theta\to 0} \frac{\sin n\theta}{\theta} = n.$

$= 3/4$

$= 0.75$

$n=1$ $h(1) = h(-1) = \frac{\sin\pi - \sin\pi/4}{\pi} = -0.225$

$h(2) = h(-2) = \frac{\sin 2\pi - \sin\pi/2}{2\pi} = -0.159.$

$h(3) = h(-3) = \frac{\sin 3\pi - \sin\frac{3\pi}{4}}{3\pi} = -0.075$

$h(4) = h(-4) = \frac{\sin 4\pi - \sin\pi}{4\pi} = 0$

$h(5) = h(-5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} = 0.045$

The transfer function of the filter is given by

$H(z) = \sum_{n=-5}^{5} h(n) z^{-n}$

$= h(-5)z^5 + h(-4)z^4 + h(-3)z^3 + h(-2)z^2 + h(-1)z^1 + h(0) +$
$h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5}$

$= h(0) + h(1)(z^1+z^{-1}) + h(2)(z^2+z^{-2}) + h(3)(z^3+z^{-3}) +$
$h(4)(z^4+z^{-4}) + h(5)(z^5+z^{-5})$

$\therefore H(z) = 0.75 - 0.225(z+z^{-1}) - 0.159(z^2+z^{-2}) - 0.075(z^3+z^{-3})$
$+ 0.045(z^5+z^{-5})$

The frequency response is given by

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

$$H(e^{j\omega}) = 0.75 - 0.225\left(e^{j\omega}+e^{-j\omega}\right) - 0.159\left(e^{j2\omega}+e^{-j2\omega}\right) - 0.075\left(e^{j3\omega}+e^{-j3\omega}\right) + 0.045\left(e^{j5\omega}+e^{-j5\omega}\right).$$

$$H(e^{j\omega}) = 0.75 - 0.45\cos\omega - 0.318\cos 2\omega - 0.15\cos 3\omega + 0.09\cos 5\omega$$

ω (degrees)	0	10	20	30	40	50	60
$H(e^{j\omega})$	-0.08	-0.066	-0.0086	0.122	0.34	0.61	0.88
$\lvert H(e^{j\omega})\rvert$	-22	-23.62	-41.3	-18.2	-9.36	-4.2	-1.1

70	80	90	100	110	120	130	140	150	160
1.05	1.11	1.07	0.98	0.93	0.94	0.995	1.26	1.05	1.01
0.504	0.95	0.587	-0.132	-0.625	-0.537	-0.037	2	0.48	0.16

170	180
0.96	0.94
-0.31	-0.537

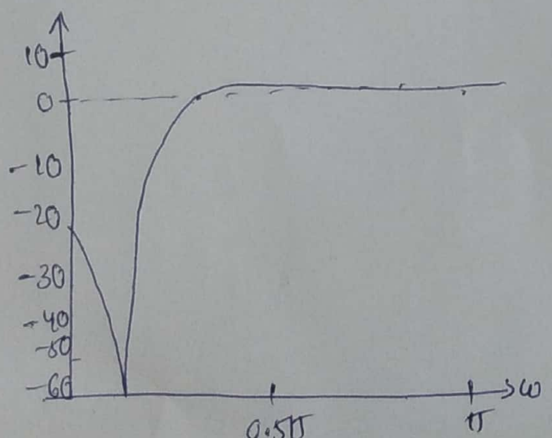


fig: Frequency Response of highpass filter.

Design an ideal HPF with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \le |\omega| \le \pi$$
$$= 0 \quad \text{for } |\omega| \le \pi/4.$$

by using (a) Hanning window (b) Hamming window.

Sol

$$H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \le |\omega| \le \pi$$
$$= 0 \quad \text{for } |\omega| \le \pi/4.$$

the desired frequency response is

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega.$$

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)_{-\pi}^{-\pi/4} + \left(\frac{e^{j\omega n}}{jn}\right)_{\pi/4}^{\pi} \right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-jn\pi/4} - e^{-jn\pi} + e^{jn\pi} - e^{jn\pi/4}}{jn} \right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{jn\pi} - e^{-jn\pi}}{2j} - \frac{e^{jn\pi/4} - e^{-jn\pi/4}}{2j} \right]$$

$$= \frac{1}{\pi n}\left[\sin \pi n - \sin \frac{\pi}{4} n \right]$$

$$\therefore \ h_d(n) = \frac{\sin \pi n - \sin\left(\frac{\pi}{4} n\right)}{\pi n}$$

$n \Rightarrow 0 \qquad h_d(0) = \lim_{n \to 0} \frac{\sin \pi n - \sin\left(\frac{\pi}{4}\right) n}{\pi n} = 1 - \frac{1}{4} = \frac{3}{4} = \underline{\underline{0.75}}$

$$h_d(-1) = h_d(1) = \frac{\sin \pi - \sin \frac{\pi}{4}}{\pi} = -0.225$$

$$h_d(-2) = h_d(2) = \frac{\sin 2\pi - \sin(\pi/2)}{2\pi} = -0.159$$

$$h_d(-3) = h_d(3) = \frac{\sin 3\pi - \sin 3\frac{\pi}{4}}{3\pi} = -0.075$$

$$h_d(-4) = h_d(4) = \frac{\sin 4\pi - \sin \pi}{4\pi} = 0.$$

$$h_d(-5) = h_d(5) = \frac{\sin 5\pi - \sin \pi}{5\pi} = 0.045$$

Hanning window $(N=11)$

$$w_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$

$$= 0 \quad \text{otherwise.}$$

for $N=11$.

$$w_{Hn}(n) = 0.5 + 0.5\cos\frac{\pi n}{5} \quad \text{for } -5 \le n \le 5$$

$$= 0 \quad \text{otherwise}$$

$$w_{Hn}(0) = 0.5 + 0.5\cos 0 = 1$$

$$w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5\cos\frac{\pi}{5} = 0.9045$$

$$w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5\cos\frac{2\pi}{5} = 0.655$$

$$w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5\cos\frac{3\pi}{5} = 0.345$$

$$w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5\cos\frac{4\pi}{5} = 0.0945$$

$$w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5\cos \pi = 0.5 - 0.5 = 0.$$

The filter coefficients using Hanning window are

$$h(n) = h_d(n)\, w_{Hn}(n) \quad \text{for } -5 \le n \le 5$$
$$= 0 \quad \text{otherwise.}$$

$h(0) = h_d(0) w_{Hn}(0) = 0.75(1) = 0.75$

$h(-1) = h(1) = h_d(1) w_{Hn}(1) = (-0.225)(0.905) = -0.204$

$h(2) = h(-2) = h_d(2) w_{Hn}(2) = (-0.159)(0.655) = -0.104$

$h(3) = h(-3) = h_d(3) w_{Hn}(3) = (-0.075)(0.345) = -0.026$

$h(-4) = h(4) = h_d(4) w_{Hn}(4) = (0).(0.8145) = 0$

$h(-5) = h(5) = h_d(5) w_{Hn}(5) = (0.045)(0) = 0.$

The transfer function of the filter is given by

$$H(z) = \sum_{n=-5}^{5} h(n) z^{-n}.$$

$= h(-5) z^{+5} + h(-4) z^4 + h(3) z^3 + h(-2) z^2 + h(-1) z^{-1} + h(0) +$
$h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5}$

$= h(0) + h(+1)[z^1 + z^{-1}] + h(2)[z^2 + z^{-2}] + h(3)[z^3 + z^{-3}]$
$h(4)[z^4 + z^{-4}] + h(5)[z^5 + z^{-5}]$

$H(z) = 0.705 - 0.204(z + z^{-1}) - 0.104(z^2 + z^{-2}) - 0.026(z^3 + z^{-3}) + 0 + 0.$

$\therefore \boxed{H(z) = 0.75 - 0.204(z + z^{-1}) - 0.104(z^2 + z^{-2}) - 0.026(z^3 + z^{-3})}$

frequency response of filter,

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.2052(e^{-j\omega}+e^{j\omega}) - 0.1084(e^{-j2\omega}+e^{j2\omega}) - 0.1084(e^{-j2\omega}+e^{j2\omega})$$
$$- 0.03(e^{-j3\omega}+e^{j3\omega}) + 0.0036(e^{-j5\omega}+e^{j5\omega})$$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.204(e^{j\omega}+e^{-j\omega}) + 0.104(e^{j2\omega}+e^{-j2\omega}) - 0.026(e^{j3\omega}+e^{-j3\omega})$$

$$\boxed{\bar{H}(e^{j\omega}) = 0.75 - 0.408\cos\omega - 0.208\cos 2\omega - 0.052\cos 3\omega}$$

ω (in degrees)	0	15	30	45	60	75	90	105
$\bar{H}(e^{j\omega})$	0.082	0.139	0.292	0.498	0.702	0.86	0.96	0.999
$\lvert H(e^{j\omega})\rvert$	-21.72	-17.14	-10.67	-6.05	-3.07	-1.297	-0.3726	-0.0087

120	135	150	165	180
1.006	1.0017	1	1	1.002
0.052	0.015	0	0	0.017

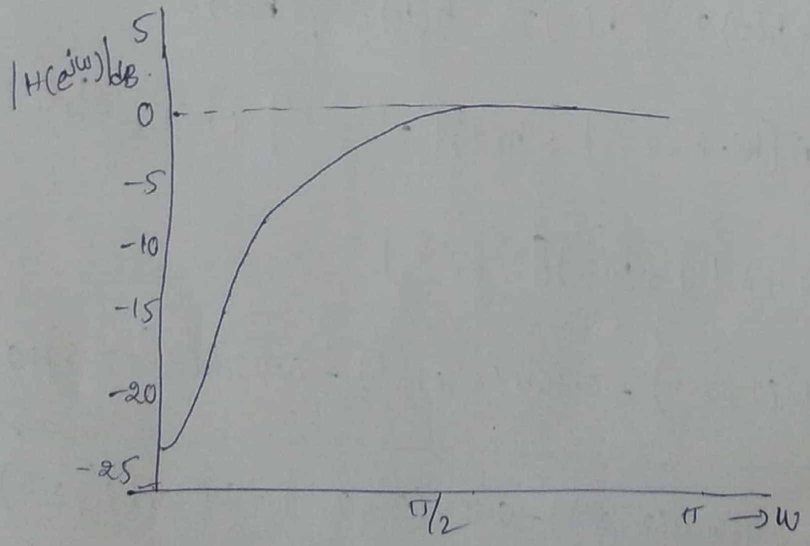


fig: magnitude response

(b) Hamming window

Hamming window sequence is given by

$$w_H(n) = 0.54 + 0.46\cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{otherwise.}$$

The window sequence for $N=11$ is given by

$$w_H(n) = 0.54 + 0.46\cos\left(\frac{\pi n}{5}\right) \quad \text{for } -5 \le n \le 5$$
$$= 0 \quad \text{otherwise.}$$

$w_H(0) = 0.54 + 0.46\cos 0 = 1$

$w_H(1) = w_H(-1) = 0.54 + 0.46\cos\left(\frac{\pi}{5}\right) = 0.912$

$w_H(2) = w_H(-2) = 0.54 + 0.46\cos\left(\frac{2\pi}{5}\right) = 0.682$

$w_H(3) = w_H(-3) = 0.54 + 0.46\cos\left(\frac{3\pi}{5}\right) = 0.398$

$w_H(4) = w_H(-4) = 0.54 + 0.46\cos\left(\frac{4\pi}{5}\right) = 0.1678$.

$w_H(5) = w_H(-5) = 0.54 + 0.46\cos\pi = \underline{0.08}$

The filter coefficients using Hamming window sequences are

$$h(n) = h_d(n)\, w_H(n) \quad \text{for } -5 \le n \le 5$$
$$= 0 \quad \text{otherwise.}$$

$h(0) = h_d(0) w_{Hn}(0) = 1(0.75) = 0.75$

$h(1) = h(-1) = h_d(1) w_{Hn}(1) = (-0.225)(0.912) = -0.2052$

$h(2) = h(-2) = h_d(2) w_{Hn}(2) = (-0.159)(0.682) = -0.1084$

$h(3) = h(-3) = h_d(3) w_{Hn}(3) = (-0.075)(0.398) = -0.03$.

$h(4) = h(-4) = h_d(4) w_{Hn}(4) = 0(0.1678) = 0$

$h(5) = h(-5) = h_d(5) w_{Hn}(5) = (-0.045)(0.08) = 0.0036$

The transfer function of filter.

$$H(z) = \sum_{n=-5}^{5} h(n) z^{-n}$$

$$H(z) = h(-5)z^{5} + h(-4)z^{4} + h(-3)z^{3} + h(-2)z^{2} + h(-1)z^{1} + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5}.$$

$$H(z) = h(0) + h(1)(z^{1}+z^{-1}) + h(2)(z^{2}+z^{-2}) + h(3)(z^{3}+z^{-3}) + h(4)(z^{4}+z^{-4})$$
$$h(5)(z^{5}+z^{-5})$$

$$H(z) = 0.75 - 0.2052(z^{-1}+z) - 0.1084(z^{-2}+z^{2}) + 0.03(z^{-3}+z^{3}) +$$
$$0.0036(z^{-5}+z^{5})$$

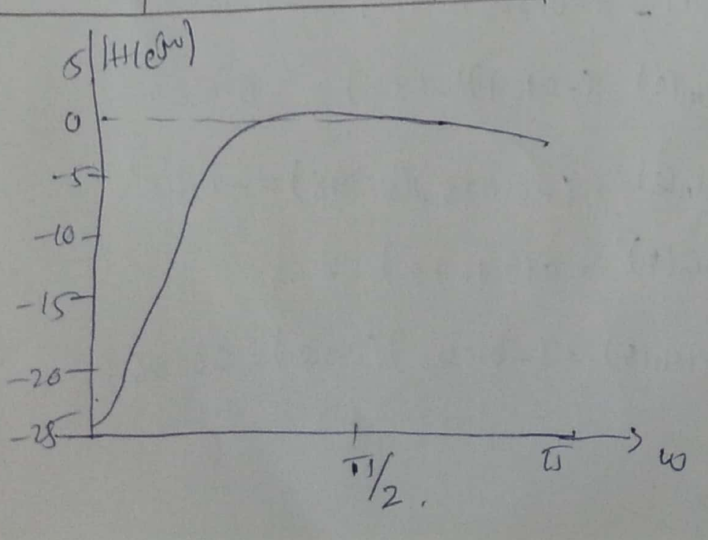
Frequency Response of filter.

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

$$H(e^{j\omega}) = 0.75 - 0.2052(e^{-j\omega}+e^{j\omega}) - 0.1084(e^{j2\omega}+e^{-j2\omega}) - 0.03(e^{-j3\omega}+e^{j3\omega})$$
$$+ 0.0036(e^{-j5\omega}+e^{j5\omega}).$$

ω (in degrees)	0	15	30	45	60	75	90	105	120	135
$H(e^{j\omega})$	0.07	0.125	0.28	0.497	0.7168	0.88	0.9668	0.9945	1	1.0026
$\|H(e^{j\omega})\|$	-23.1	-18	-11	-6.07	-2.89	-1.1	-0.29	-0.0478	0	0.0299

150	165	180
1.003	1	1.0108
0.028	0	0.093



Frequency Sampling method of designing FIR filters:-

let $h(n)$ is the filter coefficients of an FIR filter and $H(K)$ is the DFT of $h(n)$. then

$$h(n) = \frac{1}{N}\sum_{K=0}^{N-1} H(K)\, e^{j2\pi Kn/N} \quad , n = 0,1,2 \ldots N-1$$

$$H(K) = \sum_{K=0}^{N-1} h(n)\, e^{-j2\pi Kn/N} \qquad K = 0,1,2 \ldots N-1 \quad \text{—(1)}$$

The DFT samples $H(K)$ for an FIR sequence can be regarded as samples of the filter z-transform evaluated at N-points equally spaced around the unit circle. i.e

$$H(K) = H(z)\Big|_{z = e^{j2\pi K/N}}$$

The transfer function $H(z)$ of an FIR filter with impulse response is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{K=0}^{N-1} H(K) e^{j2\pi Kn/N}\right] z^{-n}$$

$$= \sum_{K=0}^{N-1}\frac{H(K)}{N}\sum_{n=0}^{N-1} H(K)\left(e^{j2\pi K/N} z^{-1}\right)^{n}$$

$$= \sum_{K=0}^{N-1}\frac{H(K)}{N}\;\frac{1-\left(e^{j2\pi K/N} z^{-1}\right)^{N}}{1-e^{j2\pi K/N} z^{-1}}$$

$$H(z) = \frac{1-z^{-1}}{N}\sum_{K=0}^{N-1}\frac{H(K)}{1-e^{j2\pi K/N} z^{-1}} \quad \text{—(2)}$$

we know,

$$H(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = H(e^{j2\pi kn/N}) = H(K)$$

i.e $H(k)$ is the K^{th} DFT component obtained by sampling the frequency response $H(e^{j\omega})$. As such this approach for designing FIR filter is called the frequency sampling method.

Frequency Sampling Realisation:-

equation ② can be written as

$$H(z) = \frac{1-z^{-N}}{N} \sum_{K=0}^{N-1} G_K(z) \quad — ③$$

where

$$G_K(z) = \frac{H(K)}{1-e^{j2\pi K/N} z^{-1}} \qquad 0 \le K \le N-1. \quad — ④$$

is the transfer function of first order FIR filters, where poles lie on the unit circle at equidistant points.

$G_K(z)$ in equation ③ are sometimes called resonant filters, resonant filters, because they are resonant at the sample values of K^{th} frequency.

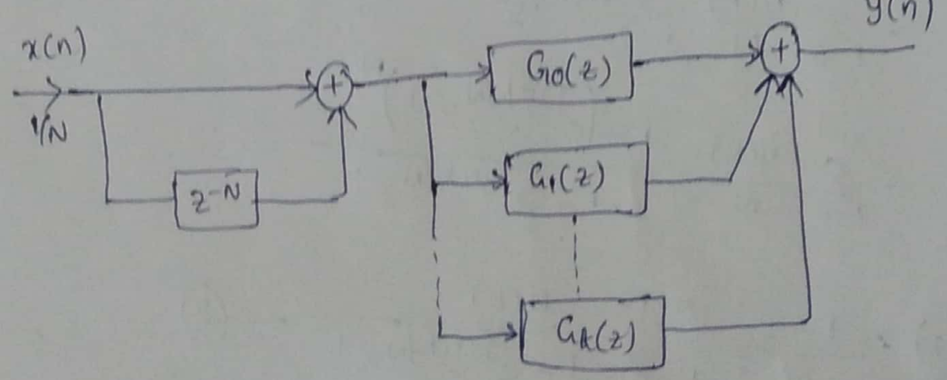


fig: Frequency Sampling realization

Frequency Response :-

The frequency of the FIR filter can be obtained by setting $z = e^{j\omega}$ in eq (3)

$$H(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j2\pi k/N} e^{-j\omega}}$$

$$= \frac{e^{-j\omega N/2}\left(e^{j\omega N/2} - e^{-j\omega N/2}\right)}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{-j(\omega - 2\pi k/N)}}$$

$$= \frac{e^{-j\omega N/2}}{N} \sum_{k=0}^{N-1} \frac{H(k)\left(e^{j\omega N/2} - e^{-j\omega N/2}\right)}{e^{-j\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)}\left[e^{j\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)} - e^{-j\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)}\right]}$$

$$= \frac{e^{-j\omega\left(\frac{N-1}{2}\right)}}{N} \sum_{k=0}^{N-1} \frac{H(k)\, e^{-j\pi k/N} \sin(\omega N/2)}{\sin\left(\frac{\omega}{2} - \pi k/N\right)}$$

$$H(e^{j\omega}) = \frac{e^{-j\omega\left(\frac{N-1}{2}\right)}}{N} \sum_{k=0}^{N-1} \frac{H(k)(-1)^k e^{-j\pi k/N} \sin N\left(\frac{\omega}{2} - \frac{k\pi}{N}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)}$$

$$\left[\because \sin\left(\frac{\omega N}{2} - k\pi\right) = (-1)^k \sin\left(\frac{\omega N}{2}\right)\right].$$