UNIT - 2

TIME RESPONSE ANALYSIS

Syllabus: standard test signals, Time Response of 1^{st} order systems, Characteristic equation of feedback control systems, Transient response of 2^{nd} order systems, Time domain specifications, Steady state response, Steady state errors and error constants, Effects of proportional derivative and integral systems.

LECTURE NOTES

Time Response: Time response of the system is the output of closed loop system as a function of time. It is denoted by $c(t)$. It can be obtained by solving the differential equation governing the system.

The time response of a closed loop control system consists of 2 parts.

i) The transient response ii) steady state response.

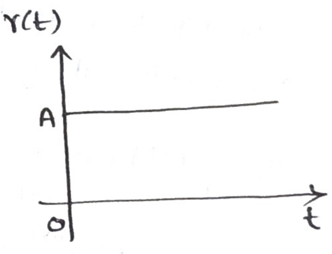
The transient response is the response of the system when the input changes from one state to another.

The steady state response is the response as time approaches ∞

Standard test signals:

To facilitate the time domain analysis, the following deterministic test signals are used.

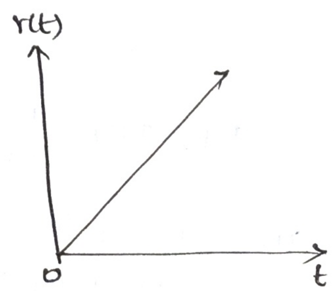
1) Step signal: The step is a signal whose value changes from one level to another level A in zero time. The mathematical representation of the step function is $r(t) = A.u(t)$



where $u(t) = 1$ for $t \geq 0$
$= 0$ for $t < 0$ is called unit step function

$$\boxed{\mathcal{L}[r(t)] = R(s) = \frac{A}{s}}$$

2) Ramp signal: The ramp signal is a signal which starts at a value of zero and increases linearly with time.



Mathematically $r(t) = A.t \; ; \; t > 0$
$= 0 \; ; \; t < 0$

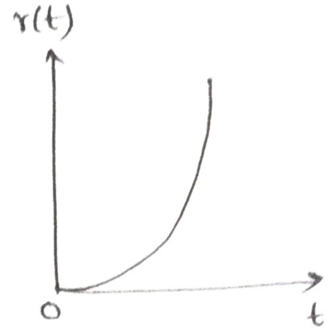
$$\boxed{\mathcal{L}[r(t)] = R(s) = \frac{A}{s^2}}$$

The ramp signal is integral of a step signal.

3) Parabolic Signal:

The mathematical rep^n of a parabolic signal is

$$r(t) = \frac{At^2}{2} \quad \text{for } t>0$$
$$= 0 \quad \text{for } t<0$$

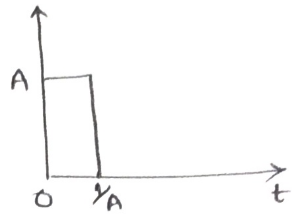


$$\boxed{\mathcal{L}[r(t)] = R(s) = \frac{A}{s^3}}$$

Parabolic signal is an integral of a ramp signal.

4) Impulse signal

A unit impulse signal is defined as a signal which has zero value everywhere except at t=0 where its magnitude is infinity. Generally it is called δ-function and it has the following property



$$\delta(t) = 0 \quad \text{for } t \neq 0$$

$$\text{as } \varepsilon \to 0 \quad \int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1$$

$$\boxed{\mathcal{L}[\delta(t)] = 1}$$

Order of a system:

The input-output relationship of a control system can be expressed by n^{th} order differential equation.

$$a_0 \frac{d^n}{dt^n} p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} p(t) + a_2 \frac{d^{n-2}}{dt^{n-2}} p(t) + \cdots + a_n p(t) = b_0 \frac{d^m}{dt^m} q(t) + b_1 \frac{d^{m-1}}{dt^{m-1}} q(t) + b_2 \frac{d^{m-2}}{dt^{m-2}} q(t) + \cdots + b_m q(t)$$

where $p(t) \to$ response (o/p) & $q(t) \to$ input

The order of the system is given by the order of the differential equation

(or)

$$\text{T.F} \quad T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_{m-1} + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} + a_n}$$

The order of the system is given by the maximum power of s in the denominator polynomial Q(s)

$n \to$ order of the system.

Time Response of 1st order system for a step input

Consider the 1st order unity feedback system shown in figure below

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{Ts}}{1+\frac{1}{Ts}} = \frac{\frac{1}{Ts}}{\frac{Ts+1}{Ts}} = \frac{1}{Ts+1}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{1}{Ts+1} \quad \text{------} \quad (1)$$

here input $r(t) = u(t)$ $\quad \therefore R(s) = \frac{1}{s}$ — (2)

$\therefore$ From eqn (1) & (2)

$$C(s) = \frac{1}{Ts+1} \cdot R(s) = \frac{1}{Ts+1} \cdot \frac{1}{s}$$

C(s) can be expressed as partial fractions as

$$C(s) = \frac{A}{s} + \frac{B}{Ts+1}$$

$$C(s) = \frac{A}{s} + \frac{B/T}{s+\frac{1}{T}}$$

$$A = s \cdot C(s)\Big|_{s=0} = s \frac{1}{(Ts+1)} \cdot \frac{1}{s} = 1$$

$$B = \left(s+\frac{1}{T}\right) C(s)\Big|_{s=-\frac{1}{T}} = \cancel{\left(s+\frac{1}{T}\right)} \frac{\frac{1}{T}}{\cancel{\left(s+\frac{1}{T}\right)}} \cdot \frac{1}{s}\Big|_{s=-\frac{1}{T}} = -1$$

$$\therefore C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Take I.L.T

$$\therefore c(t) = 1 - e^{-t/T} \quad t \geq 0$$

i.e, the o/p rises exponentially from zero value to the final value of unity at $t = T \Rightarrow c(t) = 0.632$ i.e, the response c(t) has reached 63.2% of its final value.

Initial slope of the curve at t=0 is $\frac{dc}{dt}\Big|_{t=0} = +\frac{1}{T}e^{-t/T} = \frac{1}{T}$

T - Time constant

Steady state response: $C_{ss}(t) = \lim_{t\to\infty} c(t)$

$$C_{ss}(t) = \lim_{t\to\infty} (1-e^{-t/T})$$

$$C_{ss}(t) = 1$$

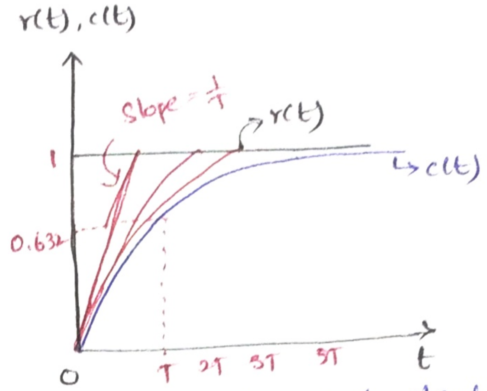
The slope of the response curve decreases monotonically from $\frac{1}{T}$ at t=0 to at $t = \infty$

error signal is $e(t) = r(t) - c(t)$

$$e(t) = r(t) - c(t)$$

$$= 1 - (1 - e^{-t/T})$$

$$e(t) = e^{-t/T}$$



Steady state error is $e_{ss}(t) = \lim_{t\to\infty} e(t)$

$$= \lim_{t\to\infty} e^{-t/T}$$

$$e_{ss}(t) = 0$$

If time constant T is high, it takes more time to reach steady state value. If time constant is less than it takes very less time to reach steady state value.

Time Response of 1st order system to a ramp input

$$\frac{C(S)}{R(S)} = \frac{1}{ST+1}$$

Input $r(t) = t$

$\therefore R(s) = \frac{1}{s^2}$

$$\therefore C(S) = \frac{1}{ST+1} \cdot R(S)$$

$$C(S) = \frac{1}{ST+1} \cdot \frac{1}{S^2} = \frac{1}{S^2(ST+1)}$$

$$C(S) = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{ST+1}$$

$$As(ST+1) + B(ST+1) + CS^2 = 1$$

$$AS^2T + As + BsT + B + Cs^2 = 1$$

$\therefore$ $B = 1$ $\quad AT + C = 0 \quad A + BT = 0$

$A = -BT = -T$

$$AT + C = 0$$

$$-T^2 + C = 0 \Rightarrow C = T^2$$

$$\therefore C(S) = \frac{1}{S^2} - \frac{T}{S} + \frac{T^2}{ST+1}$$

$$= \frac{1}{S^2} - \frac{T}{S} - \frac{T^2}{T(S+\frac{1}{T})}$$

$$C(S) = \frac{1}{S^2} - \frac{T}{S} - \frac{T}{S+\frac{1}{T}}$$

$$\therefore c(t) = t - T - Te^{-t/T} = t - T(1 - e^{-t/T})$$

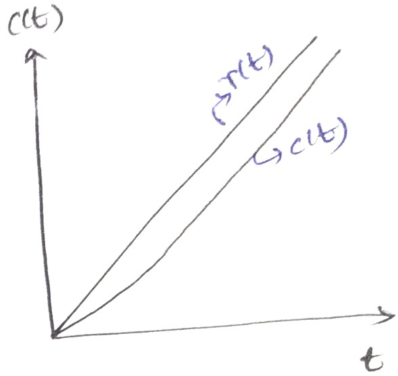
$$c(t) = t - T(1-e^{-t/T}) \quad \text{for } t \geq 0$$

and the error signal is

$$e(t) = r(t) - c(t)$$

$$= t - [t - T(1-e^{-t/T})]$$

$$e(t) = T(1-e^{-t/T})$$



Steady state error is given by

$$e_{ss}(t) = \lim_{t\to\infty} e(t) = \text{Final value}$$

$$\cancel{= \lim_{s\to 0} s\,E(s)}$$

$$\cancel{= \lim_{s\to 0} s\,[R(s) - C(s)]}$$

$$\cancel{= \lim_{s\to 0} s\left[\frac{1}{s^2} - \frac{1}{s^2(Ts+1)}\right]} \quad \cancel{= \lim_{s\to 0} \frac{1}{s}\left[1 - \frac{1}{Ts+1}\right]}$$

$$\cancel{= \lim_{s\to 0} \left[\frac{1}{s} - \frac{1}{s^2\left[s+\frac{1}{T}\right]}\right]}$$

$$e_{ss}(t) = \lim_{t\to\infty} e(t) = T$$

If the time constant is reduced the error between r(t) & c(t) is also reduces and the system becomes fast.

Time Response of 1st order system for the impulse input

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{1}{sT+1}$$

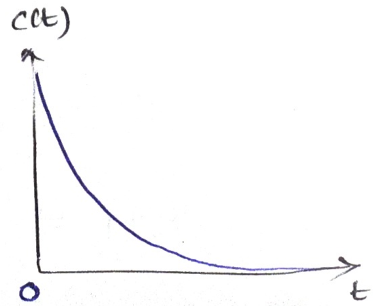
$$r(t) = \delta(t) \Rightarrow R(s) = 1$$

$$C(s) = \frac{1}{sT+1} \cdot R(s)$$

$$C(s) = \frac{1}{sT+1}$$

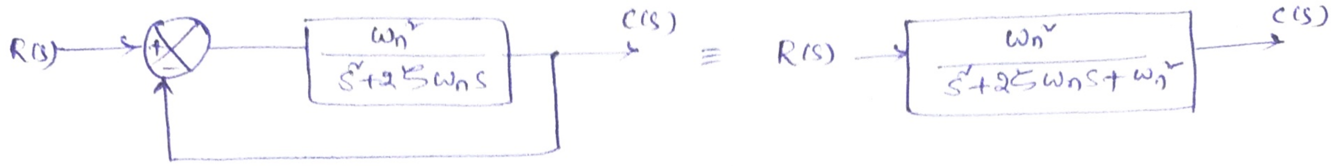
$$C(s) = \frac{\frac{1}{T}}{s+\frac{1}{T}}$$

$$\therefore\ c(t) = \frac{1}{T}e^{-t/T}$$



Second Order System:

The closed loop 2nd order system is shown in fig below



$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

where ω_n = undamped natural frequency, rad/sec

ξ = damping ratio

Depending on the value of ξ, the system can be classified into the following four cases.

case i) $\xi = 0$ → Undamped system

case ii) $0<\xi<1$ → Under damped system

case iii) $\xi = 1$ → Critically damped system

case iv) $\xi > 1$ → Over damped system

The characteristic eqn of the 2nd order system is

$$s^2+2\xi\omega_n s+\omega_n^2=0$$

and the roots are

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2-4\omega_n^2}}{2} = \frac{-2\xi\omega_n \pm \sqrt{4\omega_n^2(\xi^2-1)}}{2}$$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

when $\xi=0 \Rightarrow s_1, s_2 = \pm j\omega_n$: roots are purely imaginary & the system is undamped

when $\xi=1 \Rightarrow s_1 s_2 = -\omega_n$: roots are real & equal and the system is critically damped

when $\xi>1 \Rightarrow s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$: roots are real & unequal & the system is over damped

when $0<\xi<1 \Rightarrow s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

$= -\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$\therefore s_1, s_2 = -\xi\omega_n \pm j\omega_d$ where $\omega_d = \omega_n\sqrt{1-\xi^2}$

↳ damped frequency

Response of undamped 2nd order system for unit step input:

The standard form of closed loop transfer function of 2nd order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

For undamped system $\xi = 0$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+\omega_n^2}$$

Step input means $r(t) = u(t) \Rightarrow R(s) = \frac{1}{s}$

$$\therefore C(s) = \frac{\omega_n^2}{s^2+\omega_n^2} \cdot R(s)$$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2+\omega_n^2} = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+j\omega_n)(s-j\omega_n)}$$

$$C(s) = \frac{A}{s} + \frac{B}{s+j\omega_n} + \frac{C}{s-j\omega_n}$$

$$A = s\,C(s)\Big|_{s=0} \quad \cancel{s} \cdot \frac{\omega_n^2}{\cancel{s}(s+j\omega_n)(s-j\omega_n)}\Big|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+j\omega_n) \cdot C(s)\Big|_{s=-j\omega_n}$$

$$= \cancel{(s+j\omega_n)} \cdot \frac{\omega_n^2}{s\cancel{(s+j\omega_n)}(s-j\omega_n)}\Big|_{s=-j\omega_n} = \frac{\omega_n^2}{(-j\omega_n)(-2j\omega_n)}$$

$$= \frac{\omega_n^2}{-2\omega_n^2} = \frac{-1}{2}$$

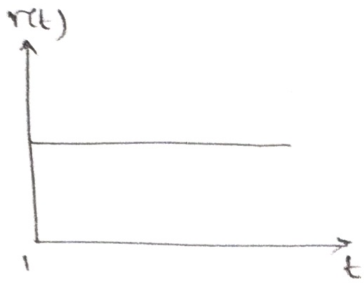
$$C = (s-j\omega_n)\, C(s)\Big|_{s=j\omega_n}$$

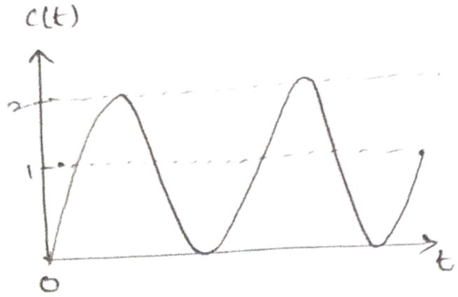
$$= \cancel{(s-j\omega_n)} \frac{\omega_n^2}{s(s+j\omega_n)\cancel{(s-j\omega_n)}}\Big|_{s=j\omega_n} = \frac{\omega_n^2}{(j\omega_n)(2j\omega_n)} = \frac{-1}{2}$$

$$\therefore C(s) = \frac{1}{s} + \frac{(-\frac{1}{2})}{s+j\omega_n} + \frac{(-\frac{1}{2})}{s-j\omega_n}$$

$$c(t) = 1 - \left(\frac{1}{2}\right)\left(e^{j\omega_n t} + e^{-j\omega_n t}\right)$$

$$c(t) = 1 - \cos(\omega_n t)$$





The response is completely oscillating between 0 and 2

Response of underdamped 2nd order system for unit step input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For under damped system $0 < \zeta < 1$

and the roots are complex conjugate

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot R(s)$$

Since $r(t) = 1$

$R(s) = \frac{1}{s}$

$$\therefore C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\therefore C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$A = s \cdot C(s)\Big|_{s=0} = \cancel{s} \cdot \frac{\omega_n^2}{\cancel{s}(s^2 + 2\zeta\omega_n s + \omega_n^2)} = 1$$

$$A(s^2 + 2\zeta\omega_n s + \omega_n^2) + (Bs + C)s = \omega_n^2$$

$\because A = 1$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs = \omega_n^2$$

$1 + B = 0$ $\qquad$ $2\zeta\omega_n + C = 0$

$B = -1$ $\qquad$ $C = -2\zeta\omega_n$

$$\therefore C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + \omega_n^2 - \zeta^2\omega_n^2}$$

let $\omega_d = \omega_n(\sqrt{1 - \zeta^2})$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} + \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} \right]$$

we know $\mathcal{L}[1] = \frac{1}{s}$

$$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$$

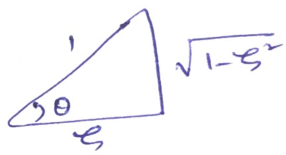
$$\mathcal{L}[e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$

$$\therefore c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]$$

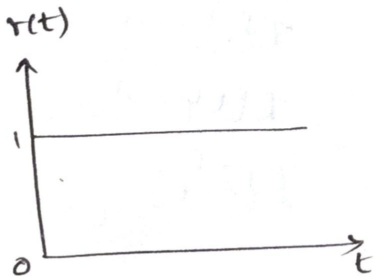
$$\sin\theta = \sqrt{1-\zeta^2}$$
$$\cos\theta = \zeta$$
$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

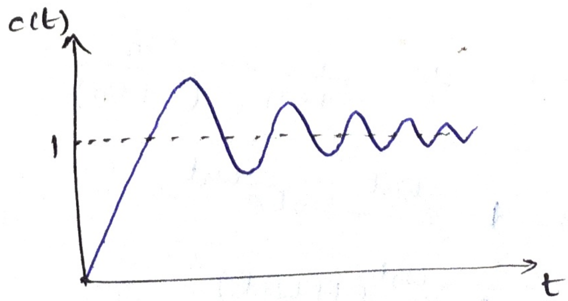


$$\therefore c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\cos\omega_d t\sin\theta + \cos\theta\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_d t + \theta)\right]$$

$$\theta = \tan^{-1}\sqrt{\frac{1-\zeta^2}{\zeta}}$$





The response oscillates before settling to a final value. The oscillations depends on the value of damping ratio

Response of Critically damped 2nd order system for a step input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For critical damping $\zeta = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

For step input $r(t) = 1$

$R(s) = \frac{1}{s}$.

$$\therefore C(s) = R(s) \cdot \frac{\omega_n^2}{(s+\omega_n)^2}$$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\omega_n)^2}$$

$$C(s) = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$A = s \cdot C(s)\Big|_{s=0} = \cancel{s}\frac{\omega_n^2}{\cancel{s}(s+\omega_n)^2} = 1$$

$$B = \frac{d}{ds}\left[(s+\omega_n)^2 C(s)\right]\Big|_{s=-\omega_n} = \frac{d}{ds}\left[\cancel{(s+\omega_n)^2} \cdot \frac{\omega_n^2}{s\cancel{(s+\omega_n)^2}}\right]\Big|_{s=-\omega_n}$$

$$B = \frac{-\omega_n^2}{s^2}\Big|_{s=-\omega_n} = -1$$

$$C = (s+\omega_n)^2 \cdot C(s)\Big|_{s=-\omega_n} = \cancel{(s+\omega_n)^2} \cdot \frac{\omega_n^2}{s\cancel{(s+\omega_n)^2}}\Big|_{s=-\omega_n} = -\omega_n$$

$$\therefore C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

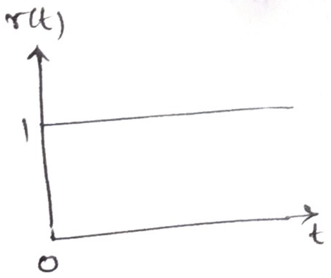
$\mathcal{L}\{1\} = \frac{1}{s}$

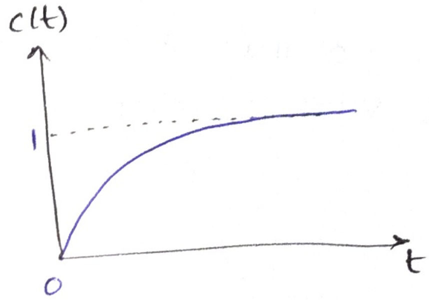
$\mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{e^{-at}t\} = \frac{1}{(s+a)^2}$

$$\therefore c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

$$= 1 - e^{-\omega_n t}(1+\omega_n t)$$





The response does not have any osillations

Response of Over damped second order system for unit step input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For over damped system $\zeta > 1$

The roots of denominator polynomial are

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

$$\therefore \quad s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s-s_1)(s-s_2)}$$

For unit step input $r(t) = u(t) = 1$

$$R(s) = \frac{1}{s}$$

$$\therefore \quad C(s) = R(s) \cdot \frac{\omega_n^2}{(s-s_1)(s-s_2)}$$

$$C(s) = \frac{\omega_n^2}{(s-s_1)(s-s_2)} \cdot \frac{1}{s}$$

Using partial Fraction Expansion

$$C(s) = \frac{A}{s} + \frac{B}{s-s_1} + \frac{C}{s-s_2}$$

$$A = s \cdot C(s)\Big|_{s=0}$$

$$= \cancel{s} \cdot \frac{\omega_n^2}{\cancel{s}\,(s-s_1)(s-s_2)}\Bigg|_{s=0} = \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)}$$

$$= \frac{\omega_n^2}{\zeta^2\omega_n^2 + \cancel{\zeta\omega_n^2\sqrt{\zeta^2-1}} - \cancel{\zeta\omega_n^2\sqrt{\zeta^2-1}} - \omega_n^2(\zeta^2-1)}$$

$$= \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s-s_1)\, C(s) \Big|_{s=s_1}$$

$$= \cancel{(s-s_1)} \frac{\omega_n^2}{s\cancel{(s-s_1)}(s-s_2)} \Bigg|_{s=s_1} = \frac{\omega_n^2}{s_1(s_1-s_2)}$$

$$= \frac{\omega_n^2}{(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(\cancel{-\zeta\omega_n}+\omega_n\sqrt{\zeta^2-1}+\cancel{\zeta\omega_n}+\omega_n\sqrt{\zeta^2-1})}$$

$$= \frac{-\omega_n^2}{(2\omega_n\sqrt{\zeta^2-1})(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})}$$

$$= \frac{-\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_1}$$

$$C = \frac{\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_2}$$

$$\therefore C(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_1} \frac{1}{s-s_1} + \frac{\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_2} \cdot \frac{1}{s-s_2}$$

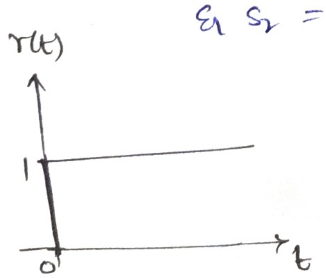
By taking Inverse Laplace Transform.

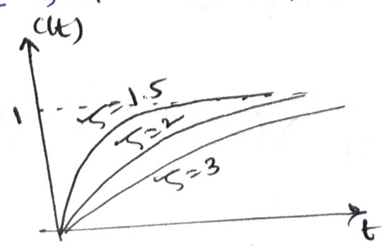
$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}} \frac{1}{s_1} e^{s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_2} e^{s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}} \left(\frac{e^{s_1 t}}{s_1} - \frac{e^{s_2 t}}{s_2} \right)$$

where $s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

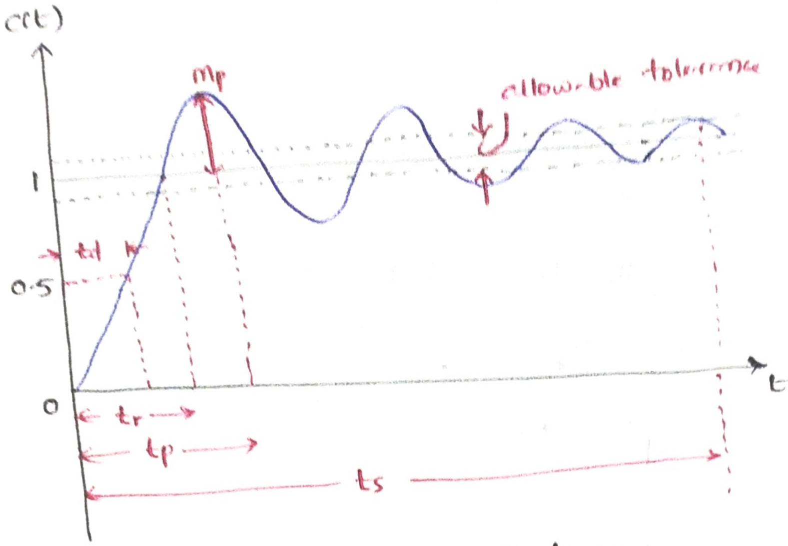
& $s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$





The response has no oscillations but it takes longer time for the response to reach the final steady value.

Time Response sepecifications



Time Response sepcifications

Rise time. It is time taken for response to raise from 0 to 100% for the very first time.

For underdamped	0 to 100%
Overdamped	0 to 90%
Critically damped	5 to 95%
undamped	10 to 90%

1. Delay time (t_d): It is the time required for the response to reach 50% of the final value the very first time.

2. Rise time (t_r): The rise time is the time required for the response to rise from 0 to 100% of the final value for the undamped systems and from 10% to 90% of the final value for over damped system.

3. Peak time (t_p): The peak time is the time required for the response to reach the first peak of the overshoot.

4. Peak Overshoot (M_p): The peak or maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady state value of the response differs from unity, then it is common to use the max % overshoot

$$\text{Max. percent overshoot} = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100\%$$

5. Settling time (t_s): It is time required for the response curve to reach and stay within a particular tolerance band (usually 2% to 5% of its final value).

6 Steady state error $e_{ss}(t)$: It indicates the error between the actual output and the desired output as t tends to infinity

$$e_{ss} = \lim_{t \to \infty} [r(t) - c(t)]$$

Rise time (tr):

The output of a second order under damped system excited by a unit step input is given by

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \quad \text{—— (1)}$$

Rise time is defined as the time taken by the output to rise from 0 to 100% of the final value.

$\therefore$ At $t = tr$

$$c(tr) = 1$$

From eqn (1)

$$1 = 1 - \frac{e^{-\xi \omega_n tr}}{\sqrt{1-\xi^2}} \sin(\omega_d tr + \theta)$$

$$\frac{e^{-\xi \omega_n tr}}{\sqrt{1-\xi^2}} \sin(\omega_d tr + \theta) = 0$$

$$\frac{e^{-\xi \omega_n tr}}{\sqrt{1-\xi^2}} \neq 0$$

$\therefore$ $\sin(\omega_d tr + \theta)$ must be equal to zero.

$$\sin(\omega_d tr + \theta) = 0 = \sin n\pi$$

$$\omega_d tr + \theta = n\pi$$

$$\omega_d tr = n\pi - \theta$$

$$tr = \frac{n\pi - \theta}{\omega_d}$$

$$tr = \frac{n\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n \sqrt{1-\xi^2}}$$

For smaller value of tr, ω_d must be large.

$\frac{0.7\xi}{\omega_n}$

Delay t.

Peak time (t_p):

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \quad \text{——} (1)$$

Peak time is defined as the time at which the maximum value of magnitude occurs. Therefore at $t = t_p$, the slope of $c(t)$ must be zero

i.e, $\left.\frac{dc(t)}{dt}\right|_{t=t_p} = 0$

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p} = \left. -\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_d t + \theta)\,\omega_d - \frac{\sin(\omega_d t + \theta)}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}(-\xi\omega_n)\right|_{t=t_p} = 0$$

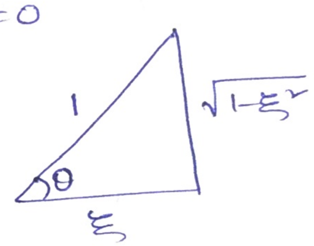
$$\omega_d \cos(\omega_d t_p + \theta) - \xi\omega_n \sin(\omega_d t_p + \theta) = 0$$

$$\omega_n\sqrt{1-\xi^2} \cos(\omega_d t_p + \theta) - \xi\omega_n \sin(\omega_d t_p + \theta) = 0$$

$$\sqrt{1-\xi^2} \cos(\omega_d t_p + \theta) - \xi \sin(\omega_d t_p + \theta) = 0$$

$$\sin\theta \cos(\omega_d t_p + \theta) - \cos\theta \sin(\omega_d t_p + \theta) = 0$$

$$\sin(\omega_d t_p + \theta - \theta) = 0 = \sin\pi$$



Relation b/w θ & ξ

$$\omega_d t_p = n\pi$$

$$t_p = \frac{n\pi}{\omega_d}$$

$$t_p = \frac{n\pi}{\omega_n\sqrt{1-\xi^2}}$$

∴ The first undershoot occurs at $t = \frac{2\pi}{\omega_d}$

The 2nd overshoot occurs at $t = \frac{3\pi}{\omega_d}$ and so on.

Peak Overshoot M_p:

The o/p of a second order underdamped system excited by a unit step input is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

The peak overshoot is the difference between the peak value and the reference input

$$\therefore M_p = c(t_p) - 1$$

$$= 1 - \frac{e^{-\xi \omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta) - 1$$

$$M_p = \frac{-e^{-\xi \omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta)$$

$$M_p = \frac{-e^{-\xi \omega_n \frac{\pi}{\omega_n \sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\omega_d \frac{\pi}{\omega_d} + \theta)$$

$$M_p = \frac{-e^{-\pi\xi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} (-\sin\theta)$$

$$M_p = \frac{e^{-\pi\xi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \sin\theta$$

$$M_p = \frac{e^{-\pi\xi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \sqrt{1-\xi^2} \qquad \because \sin\theta = \sqrt{1-\xi^2}$$

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$\therefore$ Peak percent overshoot $y = 100 \times e^{-\pi\xi/\sqrt{1-\xi^2}} \%$

Steady state error (e_{ss}):

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

$$e_{ss}(t) = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} [r(t) - c(t)]$$

$$= \underset{t\to\infty}{Lt}\left[1 - c(t)\right]$$

$$e_{ss}(t) = \underset{t\to\infty}{Lt}\left[\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right] = 0$$

Steady state error is zero.

Settling time (ts):

The response of 2nd order system has 2 components

1) decaying exponential component $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$

2) Sinusoidal component $\sin(\omega_d t + \theta)$

Decaying exponential term reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component. The settling time can be find out by equating exponential component to % tolerance errors.

For 2% tolerance error band

$$\text{at } t = t_s = \frac{e^{-\xi\omega_n t_s}}{\sqrt{1-\xi^2}} = 0.02$$

For small values of ξ

$$e^{-\xi\omega_n t_s} = 0.02$$

On taking natural logarithm

$$-\xi\omega_n t_s = \ln(0.02)$$

$$-\xi\omega_n t_s = -4$$

$$t_s = \frac{4}{\xi\omega_n}$$

For 5% tolerance

$$e^{-\xi\omega_n t_s} = 0.05$$

On taking natural logarithm

$$-\xi\omega_n t_s = \ln(0.05)$$

$$-\xi\omega_n t_s = -3$$

$$t_s = \frac{3}{\xi\omega_n}$$

Time const for 2nd order system $T = \frac{1}{\xi\omega_n}$

$$\therefore t_s = \frac{4}{\xi\omega_n} = 4T \qquad t_s = 3T$$

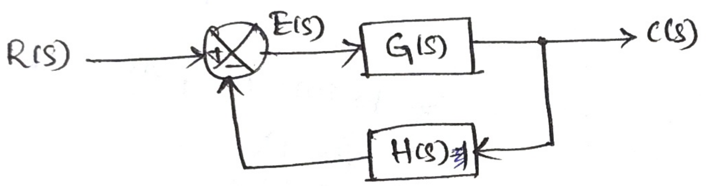
$$\therefore \text{Settling time } t_s = \frac{\ln(\%\ \text{error})}{\xi\omega_n} = \frac{\ln(\%\ \text{error})}{T}$$

Steady state error :

The steady state error is the value of error signal $e(t)$, when t tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non-linearities of system components.

Control systems may be classified according to their ability to follow step inputs, ramp inputs and parabolic inputs and so on as type-0, type-1, type-2 .... The magnitudes of the steady state errors due to these individual inputs are indicative of the goodness of the system.

Consider a system with ~~unity~~ feedback as shown below



The closed loop T.F is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \quad \text{—— (1)}$$

The error signal is

$$E(s) = C(s)/G(s) \quad \text{—— (2)}$$

From eqn (1) & (2)

$$E(s) = \frac{\cancel{G(s)}\, R(s)}{1+G(s)H(s)} \cdot \frac{1}{\cancel{G(s)}}$$

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)}$$

The steady state error e_{ss} can be found by using the final value theorem

$$e_{ss}(t) = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} s\,E(s) = \lim_{s\to 0} \frac{s.\,R(s)}{1+G(s)H(s)}$$

Type number of Control systems.

The type number is specified for the loop transfer function. The no. of Poles of the loop transfer function lying at the origin decides the type number of the system. In general, if N is the number of poles at the origin then the type number is N.

loop transfer Function can be expressed as

$$G(s)H(s) = K \cdot \frac{(s+z_1)(s+z_2)(s+z_3)\cdots}{s^N (s+P_1)(s+P_2)(s+P_3)\cdots} \quad \text{(Pole-zero form)}$$

(or)

$$G(s)H(s) = \frac{K(1+T_{z1}s)(1+T_{z2}s)\cdots}{s^n (1+T_{P1}s)(1+T_{P2}s)\cdots}$$

where $z_1, z_2, \ldots$ zeros of T.F

$P_1, P_2, \ldots$ Poles of T.F

N = no. of Poles at the origin

if N=0 → type-0 System

if N=1 → type-1 system

if N=2 → type-2 System

⋮

Static error constants

Position error constant $K_p = \lim_{s\to 0} G(s)H(s)$

velocity error constant $K_v = \lim_{s\to 0} s\,G(s)H(s)$

Acceleration error constant $K_a = \lim_{s\to 0} s^2 G(s)H(s)$

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number & the input signal.

Type-0 system will have a constant steady state error when the I/p is step

Type-1 ⇒ when the I/p is a ramp signal

Type-2 ⇒ when the I/p is a parabolic signal.

Static Position error constant (K_p):

The steady state error of the system for a step input is

$r(t) = 1$

$R(s) = \frac{1}{s}$

$$e_{ss} = \underset{s\to 0}{Lt}\, sE(s)$$

$$= \underset{s\to 0}{Lt}\, s.\frac{R(s)}{1+G(s)H(s)} = \underset{t\to 0}{Lt}\, \frac{s.\frac{1}{s}}{1+G(s)H(s)}$$

$$e_{ss} = \frac{1}{1+\underset{s\to 0}{Lt}\, G(s)H(s)} = \frac{1}{1+G(0)H(0)} = \frac{1}{1+K_p}$$

where $K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = G(0)H(0)$ is defined as position error constant

Static Velocity Error constant (K_v):

Steady state error for a unit-ramp input is

$r(t) = t$

$R(s) = \frac{1}{s^2}$

$$e_{ss} = \underset{s\to 0}{Lt}\, s\,E(s) = \underset{s\to 0}{Lt}\, s.\frac{R(s)}{1+G(s)H(s)}$$

$$= \underset{s\to 0}{Lt}\, s.\frac{\frac{1}{s^2}}{1+G(s)H(s)} = \underset{s\to 0}{Lt}\, \frac{1}{s(1+G(s)H(s))}$$

$$= \underset{s\to 0}{Lt}\, \frac{1}{s+sG(s)H(s)} = \cancel{Lt}\, \frac{1}{\underset{s\to 0}{Lt}\, sG(s)H(s)} = \frac{1}{K_v}$$

where $K_v = \underset{s\to 0}{Lt}\, sG(s)H(s) \Rightarrow$ velocity error constant

Static Acceleration error constant

Steady state error for a unit parabolic input is

$r(t) = \frac{t^2}{2}$

$R(s) = \frac{1}{s^3}$

$$e_{ss}(\cancel{t}) = \underset{s\to 0}{Lt}\, sE(s)$$

$$= \underset{s\to 0}{Lt}\, s.\frac{R(s)}{1+G(s)H(s)} = \underset{s\to 0}{Lt}\, \frac{s.\frac{1}{s^3}}{1+G(s)H(s)}$$

$$= \underset{s\to 0}{Lt}\, \frac{1}{s^2+s^2G(s)H(s)} = \frac{1}{0+\underset{s\to 0}{Lt}\, s^2G(s)H(s)}$$

$$e_{ss} = \frac{1}{\cancel{1+}K_a}$$

where $K_a = \underset{s\to 0}{Lt}\, s^2G(s)H(s) \Rightarrow$ acceleration error constant

Steady state error : Type - 0 System.

For a type-zero system

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots}{(s+P_1)(s+P_2)\cdots}$$

$$\therefore K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ \frac{K(s+z_1)(s+z_2)\cdots}{(s+P_1)(s+P_2)\cdots} = K$$

$$\therefore e_{ss}\,(\text{Position}) = \frac{1}{1+K_p} = \frac{1}{1+K} = \text{finite value}$$

$$K_v = \underset{s\to 0}{Lt}\ s\,G(s)H(s) = \underset{s\to 0}{Lt}\ s.\frac{K(s+z_1)(s+z_2)\cdots}{(s+P_1)(s+P_2)\cdots} = 0$$

$$e_{ss}\,(\text{velocity}) = \frac{1}{K_v} = \frac{1}{0} = \infty$$

$$K_a = \underset{s\to 0}{Lt}\ s^2 G(s)H(s) = \underset{s\to 0}{Lt}\ s^2\,\frac{K(s+z_1)(s+z_2)\cdots}{(s+P_1)(s+P_2)\cdots} = 0$$

$$e_{ss}\,(\text{acceleration}) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Steady state Error For a type-1 system.

For a type-1 system

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots}{s(s+P_1)(s+P_2)\cdots}$$

$$K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ \frac{K(s+z_1)(s+z_2)\cdots}{s(s+P_1)(s+P_2)\cdots} = \frac{1}{0} = \infty$$

$$e_{ss}\,(\text{Position}) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

$$K_v = \underset{s\to 0}{Lt}\ s\,G(s)H(s) = \underset{s\to 0}{Lt}\ \cancel{s}\,\frac{K(s+z_1)(s+z_2)\cdots}{\cancel{s}(s+P_1)(s+P_2)\cdots} = K$$

$$e_{ss}\,(\text{velocity}) = \frac{1}{K_v} = \frac{1}{K} = \text{Finite value}$$

$$K_a = \underset{s\to 0}{Lt}\ s^2 G(s)H(s) = \underset{s\to 0}{Lt}\ \cancel{s^2}\,s.\frac{K(s+z_1)(s+z_2)\cdots}{\cancel{s}(s+P_1)(s+P_2)\cdots} = 0$$

$$e_{ss}\,(\text{acceleration}) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Steady state error for a type-2 system

For a type-2 System

$$G(S)H(S) = \frac{K(S+Z_1)(S+Z_2)\cdots}{S^2(S+P_1)(S+P_2)\cdots}$$

$$K_p = \lim_{S\to 0} G(S)H(S) = \lim_{S\to 0} \frac{K(S+Z_1)(S+Z_2)\cdots}{S^2(S+P_1)(S+P_2)\cdots} = \frac{1}{0} = \infty$$

$$e_{ss}(\text{Position}) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

$$K_v = \lim_{S\to 0} S\,G(S)H(S) = \lim_{S\to 0} \cancel{S}\cdot\frac{K(S+Z_1)(S+Z_2)\cdots}{S^{\cancel{2}}(S+P_1)(S+P_2)\cdots} = \frac{1}{0} = \infty$$

$$e_{ss}(\text{velocity}) = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

$$K_a = \lim_{S\to 0} S^2 G(S)H(S) = \lim_{S\to 0} \frac{\cancel{S^2}\cdot K(S+Z_1)(S+Z_2)\cdots}{\cancel{S^2}(S+P_1)(S+P_2)\cdots} = K$$

$$e_{ss}(\text{acceleration}) = \frac{1}{K_a} = \frac{1}{K} = \text{Finite value}$$

Static error constants for various type number of system

Error constant	Type-0	Type-1	Type-2
K_p	Const	∞	∞
K_v	0	Const	∞
K_a	0	0	Const

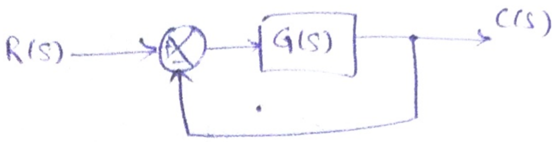
Steady state error for various types of inputs

Input signal	Type-0	Type-1	Type-2
Unit step	$\frac{1}{1+K_p}$	0	0
Unit ramp	∞	$\frac{1}{K_v}$	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$

Problems:

① Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step?

Soln. The closed loop system is shown in fig



Closed loop T.F is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

and $G(s) = \frac{4}{s(s+5)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{4}{s(s+5)}}{1+\frac{4}{s(s+5)}} = \frac{\frac{4}{\cancel{s(s+5)}}}{\frac{s(s+5)+4}{\cancel{s(s+5)}}} = \frac{4}{s^2+5s+4}$$

$$\frac{C(s)}{R(s)} = \frac{4}{(s+1)(s+4)}$$

$$C(s) = R(s) \times \frac{4}{(s+1)(s+4)}$$

Since the input is unit step $R(s) = \frac{1}{s}$

$$\therefore C(s) = \frac{4}{s(s+1)(s+4)}$$

By Partial Fraction Expansion

$$C(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}.$$

$$A = s \cdot C(s)\Big|_{s=0} = s \cdot \frac{4}{s(s+1)(s+4)}\Big|_{s=0} = 1$$

$$B = (s+1)C(s)\Big|_{s=-1} = \cancel{(s+1)} \frac{4}{s\cancel{(s+1)}(s+4)}\Big|_{s=-1} = \frac{-4}{3}$$

$$C = (s+4)C(s)\Big|_{s=-4} = \cancel{(s+4)} \frac{4}{s(s+1)\cancel{(s+4)}}\Big|_{s=-4} = \frac{1}{3}$$

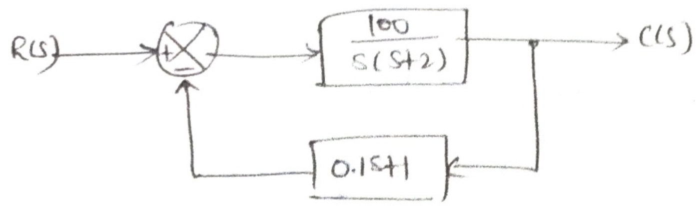
$$\therefore C(s) = \frac{1}{s} - \frac{4}{3} \cdot \frac{1}{s+1} + \frac{1}{3} \cdot \frac{1}{s+4}$$

By taking I.L.T

$$c(t) = u(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{-4t} u(t)$$

$$\therefore c(t) = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right] \quad \text{for } t \geq 0$$

(2) A positional control system with velocity feedback is shown in Fig. what is the response of the system for unit step input



Ans: $c(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right]$

(3) The response of a servomechanism is $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ when subject to a unit step input. obtain an expression for the closed loop transfer function. Determine the underdamped natural frequency and damping ratio.

Soln: Given $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$

$$\therefore C(s) = \frac{1}{s} + 0.2 \frac{1}{s+60} - 1.2\frac{1}{s+10}$$

$$C(s) = \frac{(s+60)(s+10) + 0.2s(s+10) - 1.2s(s+60)}{s(s+10)(s+60)}$$

$$C(s) = \frac{600}{s(s+10)(s+60)} = \frac{1}{s}\cdot\frac{600}{(s+10)(s+60)}$$

$$C(s) = \frac{1}{s}\cdot\frac{600}{s^2+70s+600}$$

Since the unit step is the input $R(s) = \frac{1}{s}$.

$$\therefore C(s) = R(s)\cdot\frac{600}{s^2+70s+600}$$

$\therefore$ Closed loop T.F is $\frac{C(s)}{R(s)} = \frac{600}{s^2+70s+600}$

Comparing the above eqn with standard 2nd order T.F

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

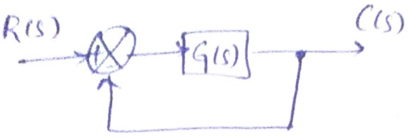
$\therefore \omega_n^2 = 600 \Rightarrow \omega_n = 24.49$ rad/sec

$2\xi\omega_n = 70$

$\xi = \frac{35}{\omega_n} = \frac{35}{24.49} = 1.43$

(4) The unity feedback system is characterized by an open loop T.F $G(s) = \frac{K}{s(s+10)}$. Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine settling time, peak overshoot and time at peak overshoot for a unit step input.

Soln: The unity feedback system is shown fig



$\therefore$ Closed loop T.F is $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given $G(s) = \frac{K}{s(s+10)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{\frac{K}{s(s+10)}}{\frac{s(s+10)+K}{s(s+10)}} = \frac{K}{s^2+10s+K}$$

$$\frac{C(s)}{R(s)} = \frac{K}{s^2+10s+K}$$

Comparing with the standard 2nd order T.F

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$\omega_n^2 = K$ $\qquad$ $2\xi\omega_n = 10$

$\omega_n = \sqrt{K}$ $\qquad$ $\xi = 0.5$

$\therefore$ $2 \times 0.5 \times \omega_n = 10 \Rightarrow \omega_n = 10$

$\therefore$ $\sqrt{K} = 10$

$K = 100$

Settling time for 2% tolerance band $t_s = \frac{4}{\xi\omega_n}$

$t_s = \frac{4}{0.5 \times 10} = \frac{4}{5} = 0.8$ sec. (2% tolerance.)

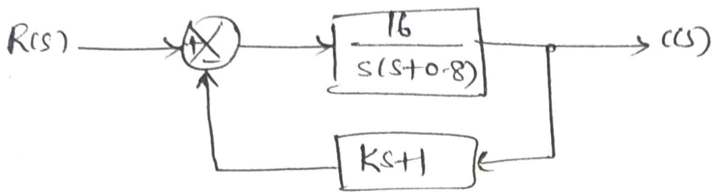
$t_s = \frac{3}{\xi\omega_n} = \frac{3}{0.5 \times 10} = \frac{3}{5} = 0.6$ sec (5% tolerance)

Peak overshoot $\%M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$

$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 16.3\%$

Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363$ sec

(5) A Positional control system with velocity feedback is shown in Figure. What is the response c(t) to the unit step input. Given that $\xi = 0.5$. Also calculate rise time, peak time, maximum overshoot & settling time



Ans

$$c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$$

$t_r = 0.6046$ sec

$t_p = 0.907$ sec

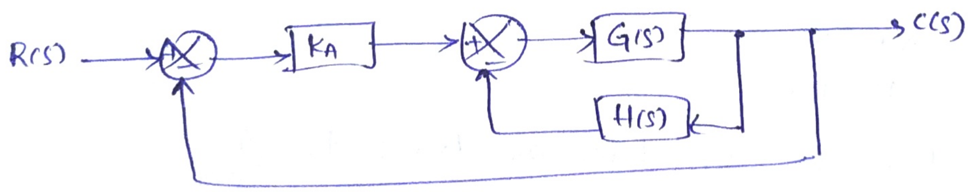
$\% M_p = 16.3\%$

$t_s = 1.5$ sec for 5% error

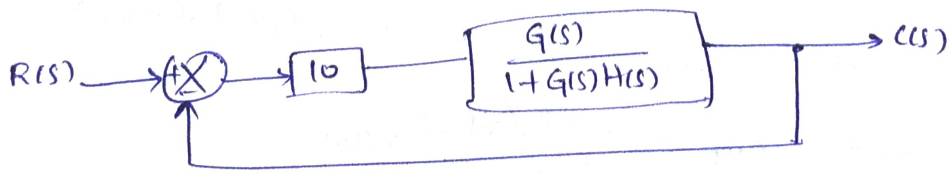
$= 2$ sec for 2% error

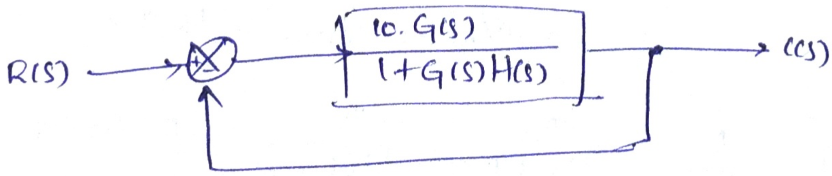
(6) A unity feedback control system has an ampr with the gain $K_A = 10$ and gain Ratio $G(s) = \frac{1}{s(s+2)}$ in the feed forward path. A derivative feedback $H(s) = sK_o$ is introduced as a minor loop around G(s). Determine the const K_o, so that the system damping ratio is 0.6

Soln: The given system can be represented by the block diagram as shown below



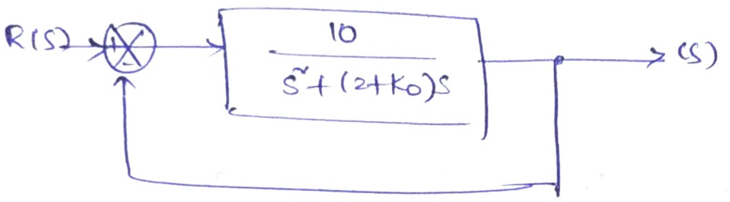
here $K_A = 10$ & $G(s) = \frac{1}{s(s+2)}$ $H(s) = sK_o$





$$\frac{G(S)}{1+G(S)H(S)} = \frac{\frac{1}{s(s+2)}}{1+\frac{1}{s(s+2)}\cdot sK_o} = \frac{\frac{1}{s(s+2)}}{\frac{s(s+2)+sK_o}{s(s+2)}}$$

$$\frac{G(S)}{1+G(S)H(S)} = \frac{1}{s^2+2s+sK_o} = \frac{1}{s^2+(2+K_o)s}$$



$$\frac{C(S)}{R(S)} = \frac{\frac{10}{s^2+(2+K_o)s}}{1+\frac{10}{s^2+(2+K_o)s}} = \frac{\frac{10}{s^2+(2+K_o)s}}{\frac{s^2+(2+K_o)s+10}{s^2+(2+K_o)s}}$$

$$\frac{C(S)}{R(S)} = \frac{10}{s^2+(2+K_o)s+10}$$

Comparing with the standard 2nd order T.F $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 10$

$\omega_n = \sqrt{10} = 3.162$ rad/sec

$2+K_o = 2\xi\omega_n$

$\xi = 0.6$

$2+K_o = 2(0.6)(3.162)$

$K_o = 2\times0.6\times3.162-2$

$K_o = 1.7944$

(7) A unity feedback control system has an open loop T.F $G(s) = \frac{10}{s(s+2)}$ Find the rise time, % overshoot, peak time and settling time for a step input of 12 units.

Ans: $t_r = 0.63$ sec

$t_p = 1.047$ sec

$\% M_p = 35.12\%$

Peak overshoot $= 4.2144$ units

$t_s = 3$ sec for 5% error

4 sec for 2% error

(8) For a unity feedback control system with the open loop T.F

$G(s) = \dfrac{10(s+2)}{s^2(s+1)}$ Find

i) the position, velocity & acceleration error constants

ii) the steady state error when the i/p is R(s)

where $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$

Soln i) Static error constants

For a unity feedback system $H(s) = 1$

PEC $K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \dfrac{10(s+2)}{s^2(s+1)}(1) = \infty$

VEC $K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} \dfrac{s\,10(s+2)}{s^2(s+1)}(1) = \infty$

AEC $K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \dfrac{s^2\,10(s+2)}{s^2(s+1)}(1) = \dfrac{10\times 2}{1} = 20$

ii) Steady state error ess

Error signal in s-domain $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

Given $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$, $G(s) = \dfrac{10(s+2)}{s^2(s+1)}$ & $H(s) = 1$

$\therefore E(s) = \dfrac{\dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}}{1 + \dfrac{10(s+2)}{s^2(s+1)}(1)} = \dfrac{\dfrac{9s^2 - 6s + 1}{3s^3}}{\dfrac{s^3 + s^2 + 10s + 20}{s^2(s+1)}}$

$E(s) = \dfrac{9s^2 - 6s + 1}{3s} \times \dfrac{s+1}{s^3 + s^2 + 10s + 20}$

Steady state error $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$

$e_{ss} = \lim_{s\to 0} \cancel{s} \dfrac{9s^2 - 6s + 1}{3\cancel{s}} \times \dfrac{s+1}{s^3 + s^2 + 10s + 20}$

$= \dfrac{1}{3\times 20} = \dfrac{1}{60}$

$e_{ss} = \dfrac{1}{60}$

(Q) For servomechanisms with open loop T.F given below explain what type of input signal gives rise to a constant steady state error and calculate their values.

i) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$ ii) $G(s) = \frac{10}{(s+2)(s+3)}$

iii) $G(s) = \frac{10}{s^2(s+1)(s+2)}$

Soln.

i) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$ Assume $H(s)=1$

The open loop system has a pole at origin. Hence it is a type-1 system. In type-1 systems velocity (ramp) input will give a const ess

$$e_{ss}\,(\text{velocity}) = \frac{1}{K_v}$$

$$K_v = \lim_{s\to 0} s\,G(s)\,H(s) = \lim_{s\to 0} s\,G(s)$$

$$K_v = \lim_{s\to 0} \cancel{s}\,\frac{20(s+2)}{\cancel{s}(s+1)(s+3)} = \frac{20\times 2}{3} = \frac{40}{3}$$

$$e_{ss}\,(\text{velocity}) = \frac{1}{K_v} = \frac{3}{40} = 0.075$$

ii) $G(s) = \frac{10}{(s+2)(s+3)}$

No pole is lying at origin. Hence it is type-0 system

For a type-0 system the step input will give a const ess

$$e_{ss}\,(\text{step}) = \frac{1}{1+K_p}$$

$$K_p = \lim_{s\to 0} G(s)\,H(s) = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{10}{(s+2)(s+3)} = \frac{10}{6} = \frac{5}{3}$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{5}{3}} = \frac{1}{\frac{3+5}{3}} = \frac{3}{8} = 0.375$$

iii) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$ Assume $H(s)=1$

Two poles are lying at origin. Hence it is type-2 system. In type-2 system, parabolic input will give a const ess

$$e_{ss}\,(\text{Parabolic}) = \frac{1}{K_a}$$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^2 G(s)$$

$$K_a = \lim_{s\to 0} s^2 \frac{10}{s^2(s+1)(s+2)} = \frac{10}{1\times 2} = \frac{10}{2} = 5$$

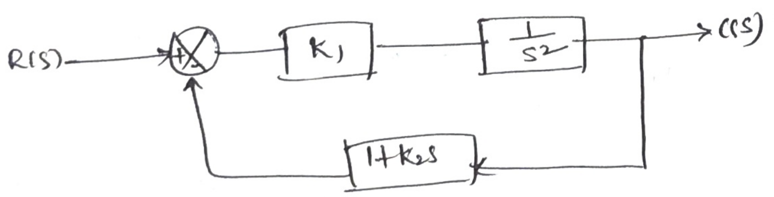
$$e_{ss} = \frac{1}{K_a} = \frac{1}{5} = 0.2$$

(10) Find the error constants and steady state error for a velocity input $r(t) = 2t$ and a step input of 2units. The system is described by $G(s)H(s) = \dfrac{10}{s(s+5)}$ (No/Dec 2012 3.6)

(11) Determine the step, ramp and parabolic error constants of unity feedback control system. whose open loop T.F is

$$G(s) = \frac{1000}{(1+0.1s)(1+10s)}$$

(12) For the control system shown in Fig below, Find the values of K_1 and K_2 so that $M_p = 25\%$ and $T_p = 4s$. Assume unit step input



Ans: $K_1 = 0.7639$
$K_2 = 0.9406$

(13) The open loop T.F of a unity feedback system is given by $G(s) = \dfrac{K}{s(1+Ts)}$. By what factor should the amplifier gain K be reduced so that the peak overshoot of unit step response of the system is reduced from 75% to 25%.

Ans: 19.6 times

Methods to improve Time Response:

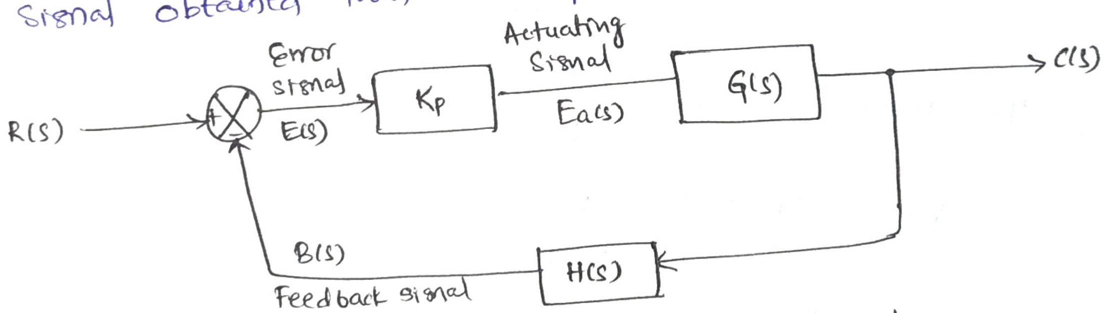
The following linear control methods are used to improve the system performance.

1. proportional control
2. proportional Derivative (PD) Control
3. proportional plus Integral (PI) control
4. Proportional plus Integral plus Derivative (PID) Control

A controller is device which introduced in feedback or forward path of a system that controls the steady state and transient response according to the requirement.

1. proportional controller

The proportional controller is a device that produces a control signal which is proportional to the input error signal e(t). The error signal is the difference between the reference input signal and the feedback signal obtained from the output



Block diagram of proportional control system.

The ~~act~~ actuating signal is proportional to the error signal, hence the name proportional control

From the Figure

$$e_a(t) = K_p e(t)$$

$$E_a(s) = K_p E(s)$$

$$K_p = \frac{E_a(s)}{E(s)}$$

The proportional controller amplifies the error signal by an amount K_p. It also increases the forward path gain by an amount K_p. If forward path gain is increased, the peak overshoot increases while the steady state error is reduced. In actual systems both peak overshoot and 'ess' are desired to be small. Hence a compromised value of the forward path gain K_p is selected for which peak overshoot & 'ess' are within specified values.

2. Proportional Plus Derivative (PD) controller

In PD controllers, the actuating signal $e_a(t)$ is proportional to the error signal $e(t)$ and also proportional to derivative of the error signal.

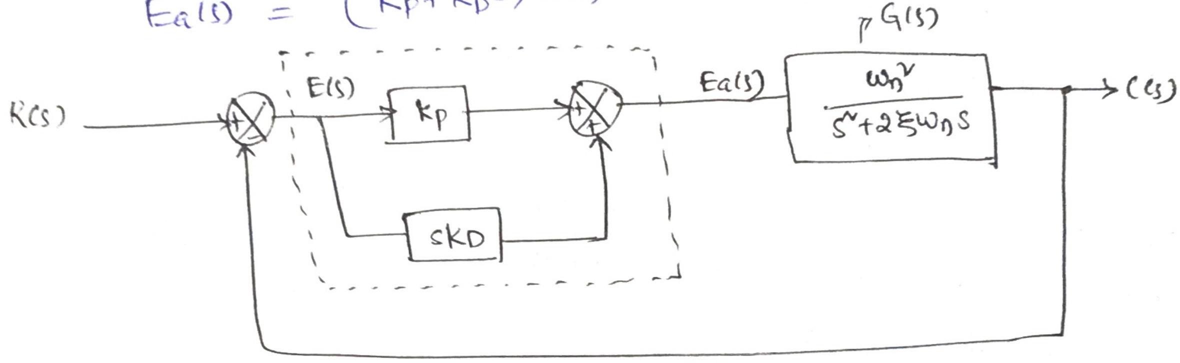
Thus

$$e_a(t) = K_P e(t) + K_D \frac{d}{dt} e(t)$$

Apply L.T

$$E_a(s) = K_P E(s) + K_D sE(s)$$

$$E_a(s) = (K_P + K_D s) E(s)$$



Block diagram of PD control system

From the figure $G(s) = \frac{C(s)}{E(s)}$

$$G(s) = (K_P + sK_D)\frac{\omega_n^2}{s^2+2\xi\omega_n s}$$

and $H(s) = 1$

$\therefore$ Closed loop T.F is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{(K_P+sK_D)\frac{\omega_n^2}{s^2+2\xi\omega_n s}}{1+(K_P+sK_D)\frac{\omega_n^2}{s^2+2\xi\omega_n s}}$$

$$\frac{C(s)}{R(s)} = \frac{(K_P+sK_D)\omega_n^2}{s^2+(2\xi\omega_n+K_D\omega_n^2)s+\omega_n^2 K_P}$$

The characteristic equation of the system given by the denominator

$$s^2 + (2\xi\omega_n + K_D\omega_n^2)s + K_p\omega_n^2 = 0 \quad \text{——(1)}$$

The standard eqn of a 2nd order system is

$$s^2 + 2\xi'\omega_n s + \omega_n^2 = 0 \quad \text{——(2)}$$

Comparing eqn (1) & (2)

$$2\xi\omega_n + K_D\omega_n^2 = 2\xi'\omega_n$$

$$2\xi'\omega_n = 2\left[\xi + \frac{1}{2}K_D\omega_n\right]\omega_n$$

$$\therefore \xi' = \xi + \frac{1}{2}K_D\omega_n$$

$\therefore$ damping ratio is increased using PD control. This makes the system response slower with lesser overshoots increasing delay time. PD controller will not affect the steady state error of the system.

3. Proportional plus Integral (PI) control

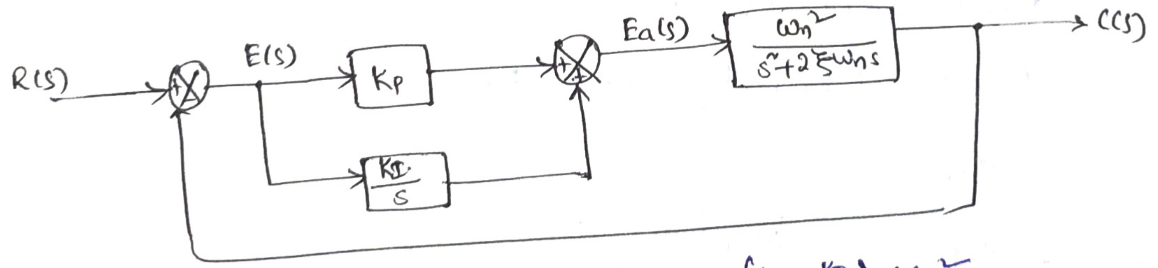
In PI control, the actuating signal is proportional to the error signal $e(t)$ and also proportional to the integral of the error signal

Thus $\quad e_a(t) = K_p e(t) + K_I \int e(t)\,dt$

Apply L.T

$$E_a(s) = K_p E(s) + K_I \frac{E(s)}{s}$$

$$E_a(s) = \left(K_p + \frac{K_I}{s}\right)E(s)$$



Open loop T.F $\quad G(s) = \dfrac{C(s)}{E(s)} = \left(K_p + \dfrac{K_I}{s}\right)\dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s}$

$$G(s) = \frac{(K_I + K_p s)\,\omega_n^2}{s^2(s + 2\xi\omega_n)} \quad \& \quad H(s) = 1$$

$\therefore$ Closed loop T.F is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{(K_P s + K_I)\omega_n^2}{s^2(s+2\xi\omega_n)}}{1+\dfrac{(K_P s+K_I)\omega_n^2}{s^2(s+2\xi\omega_n)}}$$

$$\frac{C(s)}{R(s)} = \frac{(K_P s+K_I)\omega_n^2}{s^3+2\xi\omega_n s^2+K_P\omega_n^2 s+K_I\omega_n}$$

From the closed loop T.F it is observed that the PI controller introduces a zero in the system and increases the order by one. The increase in the order of a system results in a less stable system than the original one because higher order systems are less stable than the lower order systems.

The characteristic eqn is

$$s^3+2\xi\omega_n s^2+K_P\omega_n^2 s+K_I\omega_n = 0 \rightarrow \text{3rd order equation}$$

Thus, a 2nd order system has changed to a 3rd order system by adding an integral control in the system.

$\therefore$ The effect of PI controller on the system performance is that it increases the order of the system by one, which results in the reduction of the steady state error.

4. Proportional - Integral - Derivative (PID Control):

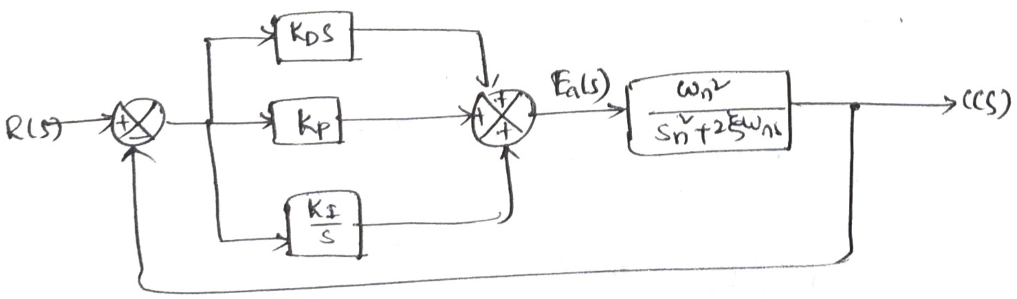
A PID Controller produces an output signal consisting of three terms - one proportional to the error signal, second one proportional to integral of error signal and the third one proportional to the derivative of error signal.

Thus $e_a(t) = K_P e(t) + K_I \int e(t)dt + K_D \frac{d}{dt} e(t)$

Apply L.T

$$E_a(s) = K_P E(s) + K_I \frac{E(s)}{s} + K_D s E(s)$$

$$E_a(s) = \left(K_P + \frac{K_I}{s} + K_D s\right) E(s)$$



* The proportional controller stabilizes the gain but produces a steady state error.
* The integral controller reduces or eliminates steady state error.
* The derivative controller reduces the rate of change of error.

∴ The main advantages of PID controllers are higher stability, no offset and reduced overshoot.

Closed loop T.F of PID controller is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2 (K_D s^2 + K_P s + K_I)}{s^3 + (2\xi\omega_n + K_D\omega_n^2)s^2 + K_P\omega_n^2 s + K_I\omega_n^2}$$