10

State-Space Analysis

10.1 MODERN CONTROL THEORY

The modern trend in engineering systems is towards greater complexity, mainly due to the requirements of complex tasks and good accuracy. Complex systems may have multiple-inputs and multiple-outputs and may be time-varying. Because of the necessity of meeting increasingly stringent requirements on the performance of control systems, the increase in system complexity and easy access to large scale computers, modern control theory, which is a new approach to the analysis and design of control systems has been developed. This new approach is based on the concept of state.

10.1.1 Modern Control Theory versus Conventional Control Theory

Basically there are two approaches to the analysis and design of control systems: the transfer function approach and the state variable approach. The transfer function approach is also called the conventional approach or the classical approach and the state variable approach is called the modern approach.

The transfer function approach has certain drawbacks. The transfer function approach is applicable only to linear time-invariant systems and there too, it is generally restricted to single-input-single-output systems. It is cumbersome for multi-input-multi-output systems. It is powerless for nonlinear systems and time-varying systems. The initial conditions are neglected. Eventhough this approach is conceptually simple and provides us with simple and powerful analysis and design techniques, the classical design methods are based on trial and error procedures and do not result in optimal and adaptive systems. They result only in acceptable systems, i.e. systems, which only satisfy the basic requirements, but not the best for the prescribed conditions. Since the transfer function gives only the input-output relationship of the system, in conventional control theory, only the input, output, and error signals are considered important. The input and output variables must be measurable. The design reveals only the system output for a given input and provides no information about the internal state of the system. The internal variables cannot be fed back. It is basically a frequency-domain approach. There may be situations where the output of a system is stable and yet some of the system elements may have a tendency to exceed their specified ratings. It may sometimes be necessary

and advantageous to provide a feedback proportional to some of the internal variables of the system, rather than the output alone, for the purpose of stabilizing and improving the performance of a system.

On the other hand, the state variable approach is applicable to linear as well as nonlinear systems, time-invariant as well as time-varying systems, and single-input-single-output as well as multi-input-multi-output systems. Also all these can be modelled in a unified manner. It is basically a direct time-domain approach which provides a basis for modern control theory and is easily amenable to solution through a digital computer. Initial conditions are taken into account and optimal and adaptive systems can be designed. Also the state variables can be fed back. The state variables need not represent physical variables. They need not even be measurable and observable.

Eventhough the state variable approach has a number of advantages compared to the transfer function approach, the transfer function approach is still very much in use, because it provides the control engineer with a deep physical insight into the system and greatly aids in the preliminary system design. It provides simple and powerful analysis and design techniques like root locus and frequency response methods.

The transfer function formulation requires the Laplace transform for continuous-data control systems and z-transform for discrete-data control systems, but the state variable approach offers us a way to look at both the continuous-data systems and the discrete-data systems with the same formulation.

Although the state model of a system is not unique, the transfer function of a system is unique. All models of a system have the same number of elements in the state vector. This number is referred to as the *order of the system*. State variable approach can be used to solve higher order differential equations. An nth order differential equation can be converted into n first-order differential equations using state variable approach. The solution of n first-order differential equations is simpler compared to the solution of one nth order differential equation.

Table 10.1 summarizes the comparison of the transfer function approach and the state variable approach.

Table 10.1 Transfer function approach versus state variable approach.

Transfer function approach	*State variable approach*
1. The transfer function approach is also called the conventional approach or classical approach.	1. The state variable approach is called the modern approach.
2. It is based on the input-output relationship or transfer function.	2. It is based on the description of the system equations in terms of n first-order differential equations, which may be combined into first-order vector-matrix differential equations.
3. The transfer function approach is applicable only to linear time-invariant systems and there too, it is generally limited to single-input-single-output systems. It is cumbersome for multi-input-multi-output systems.	3. The state variable approach is applicable to linear as well as nonlinear, time-invariant as well as time-varying, single-input-single-output as well as multi-input-multi-output systems.
4. In this initial conditions are neglected.	4. In this initial conditions are considered.

(Contd.)

Table 10.1 Transfer function approach versus state variable approach (contd.)

Transfer function approach	*State variable approach*
5. Classical design methods are based on trial and error procedures and design using this approach yields only acceptable systems.	5. Design is not based on trial and error procedure. Design using this approach yields optimal systems.
6. It is basically a frequency-domain approach.	6. It is basically a time-domain approach.
7. Only input, output and error signals are considered important. The input and output variables must be measurable.	7. The state variables need not represent physical variables. They need not even be measurable and observable.
8. It requires Laplace transform for continuous-data control systems and z-transform for discrete-data control systems.	8. It formulates both the continuous-data control systems and the discrete-data control systems in the same way.
9. The internal variables cannot be fed back.	9. The state variables can be fed back.
10. The transfer function of a system is unique.	10. The state model of a system is not unique.

10.2 CONCEPTS OF STATE, STATE VARIABLES AND STATE MODEL

State: The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the inputs for $t \geq t_0$, completely determine the behaviour of the system for any time $t \geq t_0$.

The concept of state is not limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

State variables: The state variables of a dynamic system are the smallest set of variables that determine the state of the dynamic system, i.e. the state variables are the minimal set of variables such that the knowledge of these variables at any initial time $t = t_0$, together with the knowledge of the inputs for $t \geq t_0$ is sufficient to completely determine the behaviour of the system for any time $t \geq t_0$. If atleast n variables $x_1, x_2, \ldots, x_n$ are needed to completely describe the behaviour of a dynamic system (so that once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future of the system is completely specified), then those n variables are a set of state variables.

The state variables need not be physically measurable or observable quantities. Variables that do not represent physical quantities and those that are neither measurable nor observable can also be chosen as state variables. Such freedom in choosing state variables is an added advantage of the state-space methods.

State vector: If n state variables are needed to completely describe the behaviour of a given system, then these n state variables can be considered as the n components of a vector $\mathbf{x}(t)$. Such a vector is called a *state vector*. A state vector is thus a vector that determines uniquely the system state $\mathbf{x}(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $\mathbf{u}(t)$ for $t \geq t_0$ is specified.

State-space: The n dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ..., x_n axis, where $x_1, x_2, \ldots, x_n$ are state variables is called the *state-space*. Any state can be represented by a point in the state-space.

10.2.1 State-Space Equations

In the state-space analysis, we are concerned with three types of variables that are involved in the modelling of dynamic systems: input variables, output variables, and state variables. The state-space representation for a given system is not unique, except that the number of state variables is the same for any of the different state-space representations of the same system.

The dynamic system must involve elements that memorize the values of the input for $t \geq t_0$. Since integrators in continuous-time control system serve as memory devices, the outputs of such integrators can be considered as the variables that define the internal state of the dynamic system. Thus, the outputs of the integrators serve as state variables. The number of state variables to completely define the dynamics of the system is equal to the number of integrators involved in the system.

In state variable formulation of a system, the state variables are usually represented by $x_1(t)$, $x_2(t), \ldots, x_n(t)$, the inputs by $u_1(t), u_2(t), \ldots, u_m(t)$, and the outputs by $y_1(t), y_2(t), \ldots, y_p(t)$ assuming that the system has m inputs, p outputs and n state variables. For notational economy, the different variables may be represented in vector form by the input vector $\mathbf{u}(t)$ $(m \times 1)$, output vector $\mathbf{y}(t)$ $(p \times 1)$ and state vector $\mathbf{x}(t)$ $(n \times 1)$. The state-space representation in block diagram form is shown in Figure 10.1, where broad arrows are used to represent vector quantities.

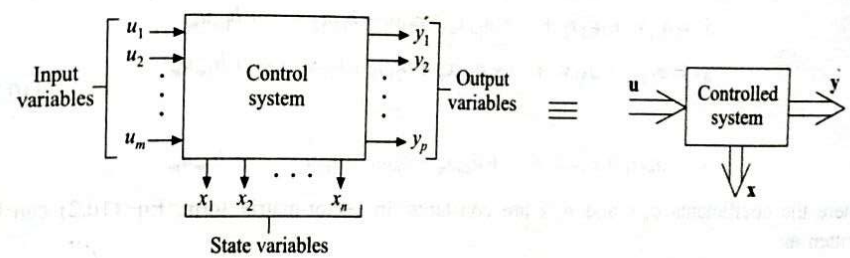


Figure 10.1 State-space representation of a system.

The state variable representation of a system can be arranged in the form of n-first-order differential equations.

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_m) \qquad (10.1)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_m)$$

Integration of Eq. (10.1) gives

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} f_i(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_m)\, dt \qquad i = 1, 2, \ldots, n$$

Thus, the n state variables and hence the state of the system can be determined uniquely at any $t > t_0$, if each state variable is known at $t = t_0$ and all the m control inputs are known throughout the interval t_0 to t.

The n differential equations may be written in vector matrix notation as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$$

for time-invariant systems, where $\mathbf{x}(t)$ is an $n \times 1$ state vector, $\mathbf{u}(t)$ is an $m \times 1$ input vector as already defined earlier and $\mathbf{f}$ is an $n \times 1$ function vector.

For time-varying systems, the function $\mathbf{f}$ is dependent on time as well and the vector equation may be written as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

10.2.2 State Model of Linear Time-invariant Systems

The state equations of a linear time-invariant system are a set of first-order differential equations, where each first derivative of the state variable is a linear combination of system states and inputs, i.e.

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}u_m
\end{aligned} \qquad (10.2)$$

where the coefficients a_{ij}'s and b_{ij}'s are constants. In vector-matrix form, Eq. (10.2) can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

where $\mathbf{x}(t)$ is an $n \times 1$ state vector, $\mathbf{u}(t)$ is a $m \times 1$ input vector, $\mathbf{A}$ is an $n \times n$ system matrix, and $\mathbf{B}$ is an $n \times m$ input matrix defined by

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

Similarly, the output variables at time t are linear combination of the input and state variables at time t, i.e.

$$\begin{aligned}
y_1(t) &= c_{11}x_1(t) + c_{12}x_2(t) + \cdots + c_{1n}x_n(t) + d_{11}u_1(t) + d_{12}u_2(t) + \cdots + d_{1m}u_m(t) \\
&\vdots \\
y_p(t) &= c_{p1}x_1(t) + c_{p2}x_2(t) + \cdots + c_{pn}x_n(t) + d_{p1}u_1(t) + d_{p2}u_2(t) + \ldots + d_{pm}u_m(t)
\end{aligned} \qquad (10.3)$$

where the coefficients c_{ij}'s and d_{ij}'s are constants. This set of equations [Eq. (10.3)] written in vector-matrix form is

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

where $\mathbf{y}(t)$ is a $p \times 1$ output vector, $\mathbf{C}$ is a $p \times n$ output matrix, and $\mathbf{D}$ is a $p \times m$ transmission matrix defined by

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}$$

The state model of linear time-invariant systems is thus given by the following equations

$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$: State equations

$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$: Output equations

The block diagram representation of the state model is shown in Figure 10.2.

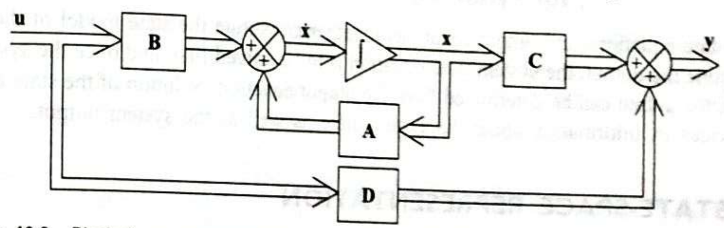


Figure 10.2 Block diagram representation of the state model of a linear multi-input-multi-output system.

10.2.3 State Model for Single-Input-Single-Output Linear Systems

The transfer function analysis deals mainly with single-input-single-output linear time-invariant systems. Here we link the transfer function approach with the state variable approach. If we let $m = 1$, and $p = 1$ in the state model of a multi-input-multi-output linear system, we obtain the following state model for a single-input-single-output linear system.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + du(t)$$

where $\mathbf{B}$ and $\mathbf{C}$ are now respectively $(n \times 1)$ and $(1 \times n)$ matrices, d is a constant and u is a scalar control variable. The block diagram representation of this state model is shown in Figure 10.3

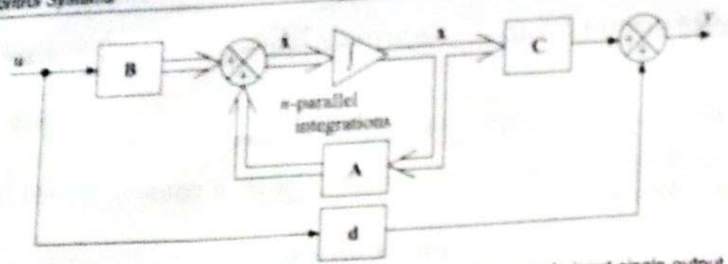


Figure 10.3 Block diagram representation of the state model of a linear single-input-single-output system.

So the state equations are

$$\dot{\mathbf{x}}(t) = f[\mathbf{x}(t), \mathbf{u}(t)]\text{: For time-invariant systems}$$
$$\dot{\mathbf{x}}(t) = f[\mathbf{x}(t), \mathbf{u}(t), t]\text{: For time-varying systems} \qquad (10.4)$$

In the above equation, the state vector $\mathbf{x}(t)$ determines a point (called the state point) in an n dimensional space called the state-space. The curve traced out by the state point from $t = t_0$ to $t = t_1$ in the direction of increasing time is known as the state trajectory. For the two dimensional cases, the state-space reduces to the state plane or phase plane.

To output vector $\mathbf{y}(t)$ can in general be expressed in terms of the state vector $\mathbf{x}(t)$ and the input $\mathbf{u}(t)$ as

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))\text{ : Time-invariant systems}$$
$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t)\text{ : Time-varying systems} \qquad (10.5)$$

The state equations and output equations together constitute the state model of the system. To determine the output, the system state equation is to be solved first, and once the system state is known, the output can be determined from the output equation. Solution of the state equations thus provides us information about the system state as well as the system output.

10.3 STATE-SPACE REPRESENTATION

10.3.1 State-Space Representation Using Physical Variables

Consider the *RLC* circuit shown in Figure 10.4 The input is a voltage source $e(t)$. Let us say the input is applied at $t = t_0$. The desired output information is usually the voltages and currents associated with various elements of the network. This information at any time t can be obtained if the initial voltage across the capacitor $e_c(t_0)$ and the initial current through the inductor $i\ (t_0)$ are known in addition to the values of the input $e(t)$ applied for $t > t_0$. The voltage across the capacitor and the current through the inductor thus constitute a set of characterizing variables of the circuit. The initial state of the circuit is given by $e_c(t_0)$ and $i(t_0)$, and the state of the circuit at any time t is given by $e_c(t)$ and $i(t)$. The values of the characterizing variables at time t describe the state of the network at that time. These variables are therefore called state variables of the circuit.

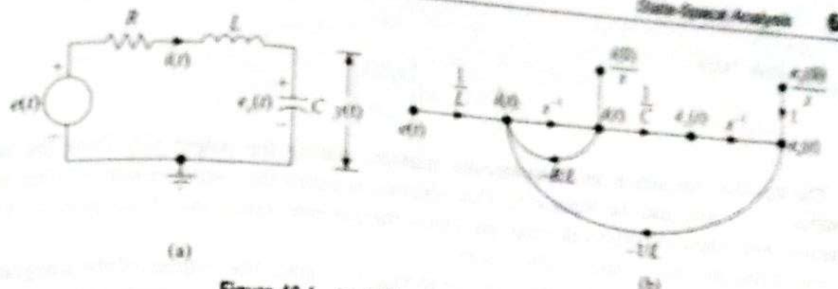


Figure 10.4 (a) RLC network and (b) state diagram.

Circuit analysis usually requires setting up of dynamical equations (using Kirchoff's voltage and current laws) in terms of rates of change of capacitor voltages and inductor currents. The solution of these equations describes the state of the network at time t. Desired output information is then obtained from the state using algebraic relation.

For the circuit shown

$$Ri(t) + \frac{L\,di(t)}{dt} + e_c(t) = e(t) \qquad (10.6)$$

and

$$\frac{C\,de_c(t)}{dt} = i(t) \qquad (10.7)$$

Rearrangement of Eqs. (10.6) and (10.7) gives the rates of change of capacitor voltage and inductor current.

$$\frac{de_c(t)}{dt} = \frac{1}{C}i(t) \qquad (10.8)$$

$$\frac{di(t)}{dt} = \frac{1}{L}e(t) - \frac{R}{L}i(t) - \frac{1}{L}e_c(t) \qquad (10.9)$$

In vector-matrix form, Eqs. (10.8) and (10.9) can be written as

$$\begin{bmatrix} \dfrac{de_c(t)}{dt} \\ \dfrac{di(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} e_c(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} e(t)$$

These equations give the rates of change of state variables (capacitor voltage $e(t)$ and inductor current $i(t)$) in terms of the state variables and the input. These equations are called the *state equations*.

The solution of these equations for given input $e(t)$ applied at $t = 0$ and given initial state $[e_c(0), i(0)]$ yields the state $[e_c(t), i(t)]$ for $t > 0$. If $y(t)$ shown in Figure 10.4(a) is the desired output information, we have the following algebraic relation to obtain $y(t)$.

$$y(t) = e_c(t)$$

In matrix form

$$y(t) = [1 \quad 0]\begin{bmatrix} e_c(t) \\ i(t) \end{bmatrix}$$

The equation for $y(t)$ is an instantaneous relation, reading the output $y(t)$ from the state variables $\{e_c(t), i(t)\}$ and the input $e(t)$. This equation is called the *output equation*. The state equations and output equations together are called the *dynamic equations* of the system. They are also called the *state model of the system*.

The state diagram of the network is shown in Figure 10.4(b). The outputs of the integrators are defined as the state variables. The transfer functions of the system are obtained by applying the signal flow graph gain formula to the state diagram when all the initial states are set to zero.

$$\frac{E_c(s)}{E(s)} = \frac{\frac{1}{L}\cdot s^{-1}\cdot\frac{1}{C}\cdot s^{-1}}{1+\frac{R}{L}\cdot s^{-1}+\frac{1}{LC}s^{-2}} = \frac{1}{LCs^2+RCs+1}$$

$$\frac{I(s)}{E(s)} = \frac{\frac{1}{L}\cdot s^{-1}}{1+\frac{R}{L}\cdot s^{-1}+\frac{1}{LC}s^{-2}} = \frac{Cs}{LCs^2+RCs+1}$$

Example 10.1 Obtain the state model of the network shown in Figure 10.5 assuming $R_1 = R_2 = 1\ \Omega$, $C_1 = C_2 = 1$ F, and $L = 1$ H.

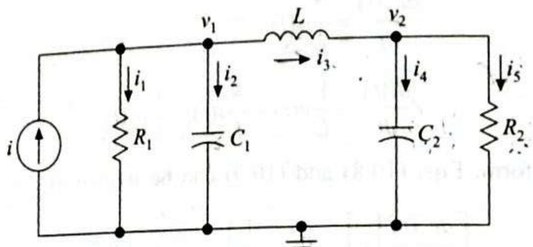


Figure 10.5 Example 10.1: Network.

Solution: The behaviour of the network shown in Figure 10.5 at any time t can be determined if the initial current through the inductor and the initial voltages across the capacitors together with the input i applied for $t > 0$ are known. So select the current through the inductor i_3, and the voltages across the capacitors C_1 and C_2, i.e. $v_1(t)$ and $v_2(t)$ as the state variables. To obtain the state equations, express the first derivatives of the state variables in terms of the input variable and the state variables.

Writing the KCL at node 1

$$i = i_1 + i_2 + i_3$$

i.e.

$$i = \frac{v_1}{R_1} + C_1\frac{dv_1}{dt} + i_3$$

$\therefore$

$$\frac{dv_1}{dt} = \frac{-v_1}{R_1C_1} - \frac{i_3}{C_1} + \frac{i}{C_1}$$

Writing the KCL at node 2

$$i_3 = i_4 + i_5$$

i.e.

$$i_3 = C_2\frac{dv_2}{dt} + \frac{v_2}{R_2}$$

$\therefore$

$$\frac{dv_2}{dt} = \frac{-v_2}{R_2C_2} + \frac{i_3}{C_2}$$

Writing KVL for the loop consisting of L

$$L\frac{di_3}{dt} + v_2 - v_1 = 0$$

$\therefore$

$$\frac{di_3}{dt} = \frac{v_1}{L} - \frac{v_2}{L}$$

If the current through the resister R_2 and the voltage across it are the outputs, the output equations are

$$y_1 = i_5 = \frac{v_2}{R_2} \qquad y_2 = i_5R_2 = v_2$$

In vector-matrix form, the state model is

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di_3}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_1C_1} & 0 & \frac{-1}{C_1} \\ 0 & \frac{-1}{R_2C_2} & \frac{1}{C_2} \\ \frac{1}{L} & \frac{-1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \\ 0 \\ 0 \end{bmatrix} i \qquad \begin{bmatrix} i_5 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{R_2} & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}$$

Substituting the values of components, the state model is

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{i}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} i \qquad \begin{bmatrix} i_5 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}$$

Example 10.2 Obtain the dynamic equations of the network shown in Figure 10.6. The current through R_2 is the output required.

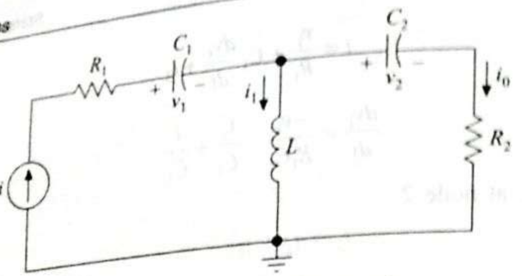


Figure 10.6 Example 10.2: Network.

Solution: In the network shown in Figure 10.6, the current though C_1 is the same as the source current. Therefore,

$$C_1\frac{dv_1}{dt} = i$$

i.e.

$$\frac{dv_1}{dt} = \frac{1}{C_1}i$$

The current through C_2 is the difference between the source current and the current through the inductor, i.e.

$$C_2\frac{dv_2}{dt} = i - i_1$$

$\therefore$

$$\frac{dv_2}{dt} = -\frac{1}{C_2}i_1 + \frac{1}{C_2}i$$

The voltage across the inductor is the same as the voltage across the combination of R_2 and C_2 i.e.

$$L\frac{di_1}{dt} = v_2 + R_2(i - i_1)$$

$\therefore$

$$\frac{di_1}{dt} = \frac{1}{L}v_2 - \frac{R_2}{L}i_1 + \frac{R_2}{L}i$$

The output variable, that is the current i_0 through the resistor R_2 is

$$i_0 = i - i_1$$

So, the dynamic equations in matrix form are as follows:

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di_1}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{C_2} \\ 0 & \frac{1}{L} & \frac{-R_2}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_1 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \\ \frac{1}{C_2} \\ \frac{R_2}{L} \end{bmatrix} i$$

$$i_0 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_1 \end{bmatrix} + (1)i$$

The above matrix form gives the state model.

Example 10.3 Write the state variable formulation of the parallel *RLC* network shown in Figure 10.7. The current through the inductor and the voltage across the capacitor are the output variables.

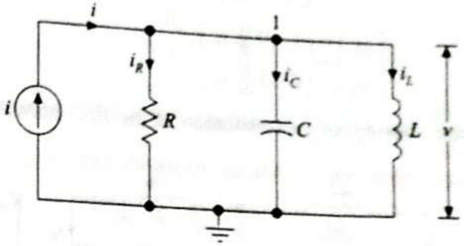


Figure 10.7 Example 10.3: Network.

Solution: In the network shown in Figure 10.7, choose the current through the inductor i_L and the voltage across the capacitor v as the state variables, and the current i_0 through L and the voltage v_0 across L as the output variable.

Writing the KCL at node 1

$$i = i_R + i_C + i_L = \frac{v}{R} + C\frac{dv}{dt} + i_L$$

i.e.

$$\frac{dv}{dt} = \frac{-1}{C}i_L - \frac{v}{RC} + \frac{1}{C}i$$

The voltage across the inductor is

$$v = L\frac{di_L}{dt}$$

i.e.

$$\frac{di_L}{dt} = \frac{v}{L}$$

Also,

$$i_o = i_L$$

and

$$v_o = v$$

Based on the above equations, the state model is

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{-1}{C} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} i_L \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} i$$

$$\begin{bmatrix} i_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ v \end{bmatrix}$$

If the state variables are selected as i_L is x_1 and v is x_2, and the output i_0 is y_1 and v_0 is y_2 and the input current i is u, then the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{-1}{C} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 10.4 Obtain the state-space representation of the *RLC* network shown in Figure 10.8.

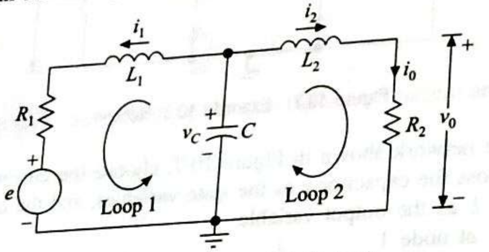


Figure 10.8 Example 10.4: Network.

Solution: The network shown in Figure 10.8 has three energy storage elements, a capacitor C and two inductors L_1 and L_2. History of the network is completely specified by the voltage across the capacitor and the currents through the inductors at $t = 0$. If we have a knowledge of initial conditions $i_1(0)$, $i_2(0)$, $v_c(0)$ and the input signal $e(t)$ for $t \geq 0$, then the behaviour of the network is completely specified for $t \geq 0$. However, if one (or more) of the initial conditions is not known, we are unable to determine the complete response of the network to a given input. Therefore, the initial conditions $i_1(0)$, $i_2(0)$, $v_c(0)$ together with the input signal $e(t)$ for $t \geq 0$ constitute the minimum information needed. Hence select the currents through the inductors $i_1(t)$ and $i_2(t)$ and the voltage across the capacitor $v_c(t)$ as the state variables. The output variables are the current through R_2, i.e. $i_0(t)$ and the voltage across R_2, i.e. $v_0(t)$ and the input variable is $e(t)$. Hence let

$$x_1(t) = i_1(t) \qquad y_1(t) = i_0(t)$$
$$x_2(t) = i_2(t) \qquad y_2(t) = v_0(t)$$
$$x_3(t) = v_c(t) \qquad u(t) = e(t)$$

The differential equations governing the behaviour of the *RLC* network are obtained by writing the KCL equation at the node and the KVL equations around the two loops.

Writing the KCL equation at the node

$$i_1 + i_2 + C\frac{dv_c}{dt} = 0$$

Writing KVL equation around loop 1

$$L_1\frac{di_1}{dt} + R_1 i_1 + e - v_c = 0$$

Writing KVL equation around loop 2

$$L_2\frac{di_2}{dt} + R_2 i_2 - v_c = 0$$

Expressing the first derivatives of the state variables $\frac{di_1}{dt}, \frac{di_2}{dt}$ and $\frac{dv_c}{dt}$ as linear combinations of the state variables i_1, i_2, v_c, and the input variable e, the state equations are

$$\frac{di_1}{dt} = \frac{-R_1}{L_1}i_1 + \frac{1}{L_1}v_c - \frac{1}{L_1}e$$

$$\frac{di_2}{dt} = \frac{-R_2}{L_2}i_2 + \frac{1}{L_2}v_c$$

$$\frac{dv_c}{dt} = \frac{-1}{C}i_1 - \frac{1}{C}i_2$$

and the output equations are

$$i_0(t) = i_2(t)$$

and

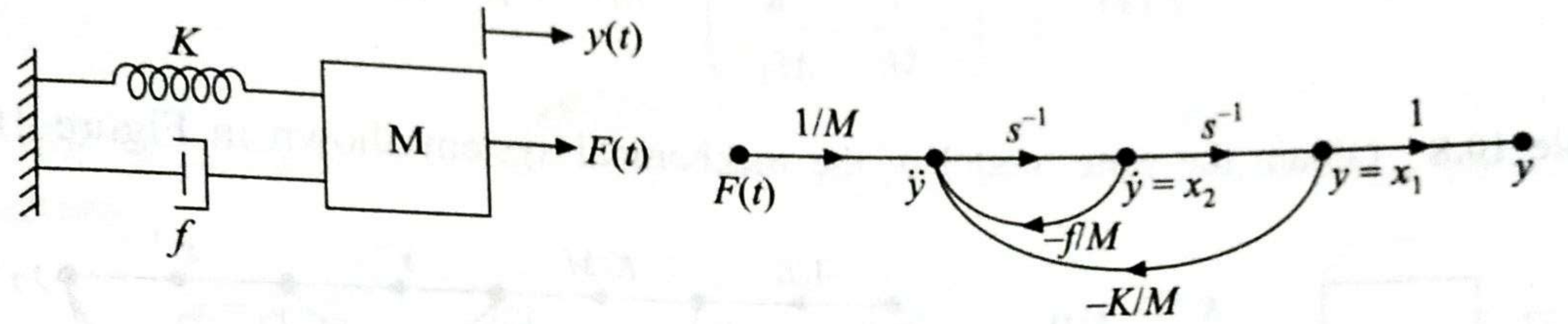
$$v_0(t) = i_2(t)\,R_2$$

In terms of the state variables and the outputs defined earlier, the state variable formulation in matrix form is

$$\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_c \end{bmatrix} + \begin{bmatrix} -\frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} e, \text{ i.e. } \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & \frac{-R_2}{L_2} & \frac{1}{L_2} \\ -\frac{1}{C} & \frac{-1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$\begin{bmatrix} i_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_c \end{bmatrix} \text{ i.e. } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 10.7 Obtain a state model for the mechanical system shown in Figure 10.12(a).



(a) (b)

Figure 10.12 Example 10.7: (a) mechanical system and (b) state diagram.

Solution: The differential equation governing the behaviour of the given mechanical system is

$$F(t) = M\frac{d^2y}{dt^2} + f\frac{dy}{dt} + Ky$$

$$\therefore \quad \frac{d^2y}{dt^2} = -\frac{f}{M}\frac{dy}{dt} - \frac{K}{M}y + \frac{1}{M}F(t)$$

The state diagram of the system is constructed as shown in Figure 10.12(b). By defining the outputs of the integrators on the state diagram as state variables x_1 and x_2, the state equations are as follows:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\frac{K}{M}x_1 - \frac{f}{M}x_2 + \frac{1}{M}F(t)$$

and the output equation is

$$y(t) = x_1(t)$$

In vector-matrix form, the state model is

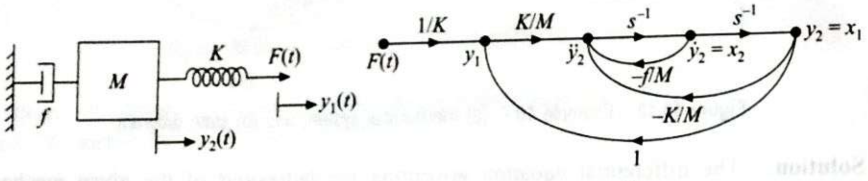
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{f}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From the state diagram, the transfer function of the system is

$$\frac{Y(s)}{F(s)} = \frac{\frac{1}{Ms^2}}{1 - \left(-\frac{f}{Ms} - \frac{K}{Ms^2}\right)} = \frac{1}{Ms^2 + fs + K}$$

Example 10.8 Obtain the state model of the mechanical system shown in Figure 10.13(a).



(a) (b)

Figure 10.13 Example 10.8: (a) mechanical system and (b) state diagram.

Solution: The differential equations governing the behaviour of the given mechanical system are

$$F(t) = K[y_1(t) - y_2(t)]$$

$$M\frac{d^2y_2(t)}{dt^2} + K(y_2(t) - y_1(t)) + f\frac{dy_2(t)}{dt} = 0$$

Let the output be $y_1(t)$. The equations are arranged as

$$\frac{d^2y_2(t)}{dt^2} = \frac{K}{M}y_1(t) - \frac{K}{M}y_2(t) - \frac{f}{M}\frac{dy_2(t)}{dt}$$

and

$$y_1(t) = \frac{1}{K}F(t) + y_2(t)$$

Using the last two equations, the state diagram of the system is drawn as shown in Figure 10.13(b). The outputs of the integrators are taken as the state variables. So defining the state variables as $x_1(t) = y_2(t)$ and $x_2(t) = \dot{y}_2(t)$, the state equations and the output equation written directly from the state diagram are

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = -\frac{f}{M}x_2(t) + \frac{1}{M}F(t)$$

and

$$y_1(t) = y_2(t) + \frac{1}{K}F(t)$$

So in vector-matrix form, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{f}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{K}F(t)$$

The transfer functions of the system are obtained by applying the Mason's gain formula to the state diagram.

$$\frac{Y_2(s)}{F(s)} = \frac{\frac{1}{K}\cdot\frac{K}{M}s^{-1}s^{-1}}{1 - \left(-\frac{f}{M}s^{-1} - \frac{K}{M}s^{-2} + \frac{K}{M}s^{-2}\right)} = \frac{\frac{1}{Ms^2}}{1 + \frac{f}{Ms}} = \frac{1}{Ms^2 + fs}$$

$$\frac{Y_1(s)}{F(s)} = \frac{\frac{1}{K}\left(1 + \frac{f}{M}s^{-1} + \frac{K}{M}s^{-2}\right)}{1 - \left(-\frac{f}{M}s^{-1} - \frac{K}{M}s^{-2} + \frac{K}{M}s^{-2}\right)} = \frac{Ms^2 + fs + K}{Ks(Ms + f)}$$

Example 10.9 Consider the mechanical system shown in Figure 10.14. Choosing suitable state variables, construct a state model of the system.

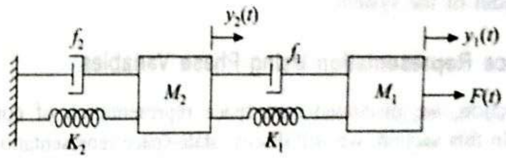


Figure 10.14 Example 10.9: Mechanical system.

Solution: Select the displacements y_1 and y_2 and their derivatives, i.e. velocities $\dot{y}_1$ and $\dot{y}_2$ as the state variables.

Let
$$x_1 = y_1$$
$$x_2 = \dot{y}_1$$
$$x_3 = y_2$$
and
$$x_4 = \dot{y}_2$$
Let
$$F(t) = u(t)$$

Therefore, $\dot{x}_1 = x_2$ and $\dot{x}_3 = x_4$. To obtain the first derivatives of x_2 and x_4, i.e. $\dot{x}_2$ and $\dot{x}_4$, write the differential equations describing the behaviour of the system. The equations governing the behaviour of the mechanical system are as follows:

$$F(t) = M_1\ddot{y}_1 + f_1(\dot{y}_1 - \dot{y}_2) + K_1(y_1 - y_2)$$
$$M_2\ddot{y}_2 + f_2\dot{y}_2 + K_2y_2 + f_1(\dot{y}_2 - \dot{y}_1) + K_1(y_2 - y_1) = 0$$
$$\dot{x}_2 = \ddot{y}_1 = \frac{F(t)}{M_1} - \frac{f_1x_2}{M_1} + \frac{f_1x_4}{M_1} - \frac{K_1x_1}{M_1} + \frac{K_1x_3}{M_1}$$
$$\dot{x}_4 = \ddot{y}_2 = -\frac{(f_1+f_2)x_4}{M_2} - \frac{(K_1+K_2)x_3}{M_2} + \frac{f_1x_2}{M_2} + \frac{K_1x_1}{M_2}$$

Therefore, the state equations and the output equations in matrix form are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_1}{M_1} & \frac{-f_1}{M_1} & \frac{K_1}{M_1} & \frac{f_1}{M_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_1}{M_2} & \frac{f_1}{M_2} & \frac{-(K_1+K_2)}{M_2} & \frac{-(f_1+f_2)}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

This is the state model of the system.

Example 10.10 Obtain a state model for the system described by

$$T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 6s^2 + 10s + 5}$$

Solution: The differential equation corresponding to the given transfer function is obtained by cross-multiplying and taking the inverse Laplace transform. So, we have

$$\dddot{y} + 6\ddot{y} + 10\dot{y} + 5y = u$$

Since the derivatives of the input are not present in the differential equation, phase variables can be selected as the state variables. Therefore,

$$\begin{aligned} x_1 &= y & \text{i.e.} \quad y &= x_1 \\ x_2 &= \dot{y} = \dot{x}_1 & \dot{x}_1 &= x_2 \\ x_3 &= \ddot{y} = \dot{x}_2 & \dot{x}_2 &= x_3 \\ \dddot{y} &= -6\ddot{y} - 10\dot{y} - 5y + u & \dot{x}_3 &= -5x_1 - 10x_2 - 6x_3 + u \end{aligned}$$

Therefore, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 10.11 Obtain a state model of the system described by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{5}{s^3 + 6s + 7}$$

Solution: The transfer function does not have any zeros. So the matrix **A** will be in bush (companion) form with the elements in the last row as −7, −6, 0. The matrix **B** has the last element as 5 and all other elements as zeros.

Expressing the system in terms of the differential equation by cross-multiplying the terms of the transfer function and taking the inverse Laplace transform, we have

$$\dddot{y} + 6\dot{y} + 7y = 5u$$

Define the state variables as

$$x_1 = y$$
$$x_2 = \dot{y} = \dot{x}_1$$
$$x_3 = \ddot{y} = \ddot{x}_1 = \dot{x}_2$$

Equating the highest-order term $\dddot{y}$ to all other terms in the differential equation, we have

$$\dddot{y} = -6\dot{y} - 7y + 5u$$

i.e.

$$\dot{x}_3 = -6x_2 - 7x_1 + 5u$$

So, the first-order differential equations constituting the state equations are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -7x_1 - 6x_2 + 5u$$

The output equation is

$$y = x_1$$

The state model based on the above equations is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State-space representation of an *n*th-order system in which the forcing function involves derivative terms: Consider an *n*th-order system represented by the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y^{(1)} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \dots + b_{n-1} u^{(1)} + b_n u$$

Its transfer function is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

Since the derivatives of the input are present in the differential equation, the output and its derivatives cannot be chosen as the state variables, i.e. phase variables cannot be selected as the state variables. The state model for the above equation may be obtained as follows.

Define the following n variables as a set of state variables:

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{x}_1 - \beta_1 u$$
$$x_3 = \dot{x}_2 - \beta_2 u$$
$$\vdots$$
$$x_{n-1} = \dot{x}_{n-2} - \beta_{n-2} u$$
$$x_n = \dot{x}_{n-1} - \beta_{n-1} u$$

Therefore, the output equation is

$$y = x_1 + \beta_0 u$$

and the state equations are

$$\dot{x}_1 = x_2 + \beta_1 u$$
$$\dot{x}_2 = x_3 + \beta_2 u$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n + \beta_{n-1} u$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n + \beta_n u$$

where

$$\beta_0 = b_0$$
$$\beta_1 = b_1 - a_1\beta_0$$
$$\beta_2 = b_2 - a_2\beta_0 - a_1\beta_1$$
$$\beta_3 = b_3 - a_3\beta_0 - a_2\beta_1 - a_1\beta_2$$
$$\vdots$$
$$\beta_n = b_n - a_n\beta_0 - a_{n-1}\beta_1 - \dots - a_2\beta_{n-2} - a_1\beta_{n-1}$$

In vector-matrix form, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$y(t) = [1 \ 0 \ \cdots \ 0] \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 u$$

Example 10.12 Construct a state model for the system described by

$$\ddot{y} + 7\dot{y} + 5y + 6\int_0^t y.dt = \ddot{u} + 3u + 2\int_0^t u\,dt$$

Solution: Differentiating the given integro-differential equation, we have

$$\dddot{y} + 7\ddot{y} + 5\dot{y} + 6y = \ddot{u} + 3\dot{u} + 2u$$

Comparing this with the standard third-order differential equation, we have

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

$$a_1 = 7,\ a_2 = 5,\ a_3 = 6 \quad \text{and} \quad b_0 = 0,\ b_1 = 1,\ b_2 = 3,\ b_3 = 2$$

Therefore,

$$\beta_0 = b_0 = 0$$
$$\beta_1 = b_1 - a_1\beta_0 = 1 - 7 \times 0 = 1$$
$$\beta_2 = b_2 - a_2\beta_0 - a_1\beta_1 = 3 - 5 \times 0 - 7 \times 1 = -4$$
$$\beta_3 = b_3 - a_3\beta_0 - a_2\beta_1 - a_1\beta_2 = 2 - 6 \times 0 - 5 \times 1 - 7 \times (-4) = 25$$

The state variables are defined as

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{x}_1 - \beta_1 u$$
$$x_3 = \dot{x}_2 - \beta_2 u$$

Also $\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u$

The state and output equations are therefore

$$y = x_1 + \beta_0 u$$
$$\dot{x}_1 = x_2 + \beta_1 u$$
$$\dot{x}_2 = x_3 + \beta_2 u$$
$$\dot{x}_3 = -6x_1 - 5x_2 - 7x_1 + 25u$$

The state model in vector-matrix form is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \\ 25 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 10.13 The transfer function of a control system is given by

$$\frac{Y(s)}{U(s)} = \frac{s^2 + 3s + 4}{s^3 + 2s^2 + 3s + 2}$$

Obtain a state model.

Solution: The corresponding differential equation obtained by cross-multiplying and taking the inverse Laplace transform is

$$\dddot{y} + 2\ddot{y} + 3\dot{y} + 2y = \ddot{u} + 3\dot{u} + 4u$$

Comparing this differential equation with the standard differential equation of a third-order system, we have

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

i.e. $a_1 = 2,\ a_2 = 3,\ a_3 = 2$ and $b_0 = 0,\ b_1 = 1,\ b_2 = 3,\ b_3 = 4$

Therefore,

$$\beta_0 = b_0 = 0$$
$$\beta_1 = b_1 - a_1\beta_0 = 1 - 2 \times 0 = 1$$
$$\beta_2 = b_2 - a_2\beta_0 - a_1\beta_1 = 3 - 3 \times 0 - 2 \times 1 = 1$$
$$\beta_3 = b_3 - a_3\beta_0 - a_2\beta_1 - a_1\beta_2 = 4 - 2 \times 0 - 3 \times 1 - 2 \times 1 = -1$$

The state variables are as follows:

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{x}_1 - \beta_1 u$$
$$x_3 = \dot{x}_2 - \beta_2 u$$

Also $\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u$

The state and output equations are as follows:

$$y = x_1 + \beta_0 u = x_1$$
$$\dot{x}_1 = x_2 + \beta_1 u = x_2 + u$$
$$\dot{x}_2 = x_3 + \beta_2 u = x_3 + u$$
$$\dot{x}_3 = -2x_1 - 3x_2 - 2x_3 - u$$

Hence the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

An alternative method using signal flow graphs is presented as follows:

Alternative way of obtaining the state model using signal flow graph when the transfer functions have poles and zeros: Let us consider an nth-order transfer function

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n}$$

Let the state variables be $x_1, x_2, \dots, x_n$.

The signal flow graph must have n integrators. By dividing the numerator and denominator of $T(s)$ by s^n, the above transfer function may be rearranged as

$$T(s) = \frac{b_0 + b_1 s^{-1} + b_2 s^{-2} + \dots + b_{n-1}s^{-n+1} + b_n s^{-n}}{1 + a_1 s^{-1} + a_2 s^{-2} + \dots + a_{n-1}s^{-n+1} + a_n s^{-n}}$$

$$= \frac{b_0 + b_1 s^{-1} + b_2 s^{-2} + \dots + b_{n-1}s^{-n+1} + b_n s^{-n}}{1 - (-a_1 s^{-1} - a_2 s^{-2} - \dots - a_{n-1}s^{-n+1} - a_n s^{-n})}$$

Earlier we have seen that the transfer function and signal flow graph are related by Mason's gain formula:

$$T(s) = \sum_k \frac{M_k \Delta_k}{\Delta}$$

where M_k = the path gain of the kth forward path

Δ (Determinant of the signal flow graph)

= 1 − (sum of loop gains of all individual loops)

+ (sum of gain products of all possible combinations of two non touching loops)

− (sum of gain products of all possible combinations of three non touching loops) +...

Δ_k = the value of Δ for that part of the graph not touching the kth forward path.

Comparing the above expressions for $T(s)$, we observe that the signal flow graph for $T(s)$ may consist of

1. n feedback loops (touching each other) with gains $-a_1 s^{-1}, -a_2 s^{-2}, \dots, -a_{n-1}s^{-n+1}, -a_n s^{-n}$
2. $n + 1$ forward paths which touch the loops and have gains $b_0, b_1 s^{-1}, b_2 s^{-2}, \dots, b_{n-1}s^{-n+1}, b_n s^{-n}$

The signal flow graph configuration which satisfies the above requirements is shown in Figure 10.16.

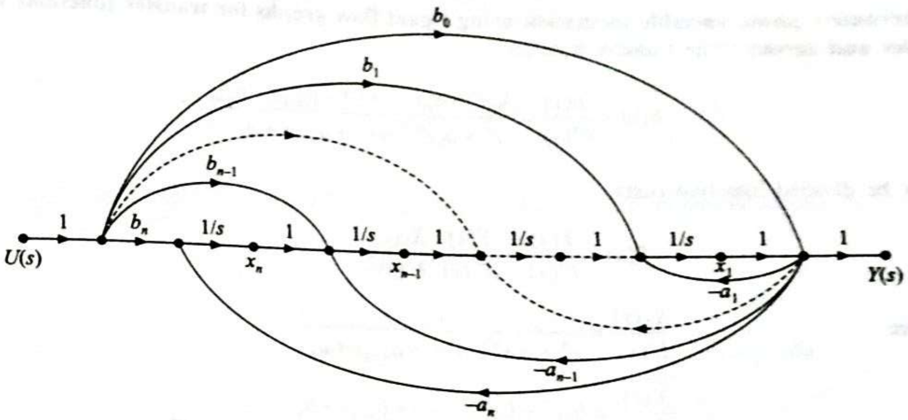


Figure 10.16 Signal flow graph for nth-order system with zeros.

Choosing the outputs of the integrators as the state variables and ordering the state variables x_1 to x_n from right to left, we have

$$y = x_1 + b_0 u$$
$$\dot{x}_1 = -a_1 x_1 + x_2 + (b_1 - a_1 b_0)u$$
$$\dot{x}_2 = -a_2 x_1 + x_3 + (b_2 - a_2 b_0)u$$
$$\vdots$$
$$\dot{x}_{n-1} = -a_{n-1}x_1 + x_n + (b_{n-1} - a_{n-1}b_0)u$$
$$\dot{x}_n = -a_n x_1 + (b_n - a_n b_0)u$$

In matrix form, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_{n-1} - a_{n-1}b_0 \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad \cdots \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Diagonalization:

The diagonal matrix plays an important role in matrix algebra. The eigen values and inverse of a diagonal matrix can be very easily obtained just by an inspection.

When a matrix is diagonalized, then the elements along its principle diagonal are the eigen values. The eigen values are the closed loop poles of the system, from which stability of the system can be analyzed.

Consider n^{th} order state model in which matrix A is not diagonal

$$\dot{X}(t) = AX(t) + BU(t) \quad \text{—(1)}$$
$$Y(t) = CX(t) + DU(t) \quad \text{—(2)}$$

let $Z(t)$ is new state vector such that

$$X(t) = MZ(t) \quad \text{—(3)}$$

here M is modal matrix of A

$$\therefore \dot{X}(t) = M\dot{Z}(t) \quad \text{—(4)}$$

From (1), (2), (3) & (4)

$$M\dot{Z}(t) = AMZ(t) + BU(t) \quad \text{—(5)}$$
$$Y(t) = CMZ(t) + DU(t) \quad \text{—(6)}$$

Multiply (5) both sides by M^{-1}

$$M^{-1}M\dot{Z}(t) = M^{-1}AMZ(t) + M^{-1}BU(t)$$
$$\dot{Z}(t) = M^{-1}AMZ(t) + M^{-1}BU(t) \quad \text{—(7)}$$
$$\& \quad Y(t) = CMZ(t) + DU(t) \quad \text{—(8)}$$

eqn (7) & (8) gives canonical state model in which $M^{-1}AM$ is a diagonal matrix denoted as Λ

$\therefore$ New canonical state model is represented as

$$\dot{Z}(t) = \Lambda Z(t) + \tilde{B}U(t)$$
$$\& \quad Y(t) = \tilde{C}Z(t) + DU(t)$$

where
$$\Lambda = M^{-1}AM$$
$$\tilde{B} = M^{-1}B$$
$$\tilde{C} = CM$$

Problem:

Consider a state model with matrix A as (1)

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{bmatrix}$$

Determine a) characteristic eqn b) Eigen values c) Eigen vectors and d) Modal matrix

Soln: a) char. eqn $|\lambda I - A| = 0$

$$\lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda & -2 & 0 \\ -4 & \lambda & -1 \\ 48 & 34 & \lambda+9 \end{vmatrix} = 0$$

$$\lambda^2(\lambda+9) + 2 \times 48 + 0 + 0 - 8(\lambda+9) + 34\lambda = 0$$

$$\lambda^3 + 9\lambda^2 + 26\lambda + 24 = 0$$

b) To find the eigen values, test $\lambda = -2$ for its root

$$\begin{array}{r|cccc} -2 & 1 & 9 & 26 & 24 \\ & & -2 & -14 & -24 \\ \hline & 1 & 7 & 12 & 0 \end{array}$$

$$(\lambda+2)(\lambda^2+7\lambda+12)=0$$

$$(\lambda+2)(\lambda+3)(\lambda+4)=0$$

$$\therefore \lambda_1=-2, \; \lambda_2=-3 \; \& \; \lambda_3=-4.$$

These are the eigen values of matrix A

c) Eigen vector $\lambda I - A$

for $\lambda_1 = -2$ $\quad [\lambda_1 I - A] = \begin{bmatrix} -2 & -2 & 0 \\ -4 & -2 & -1 \\ 48 & 34 & 7 \end{bmatrix}$

$m_1 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix}$ $\quad c_{11}, c_{12}, c_{13}$ are co-factors of row$_1$

$\therefore m_1 = \begin{bmatrix} 20 \\ -20 \\ 40 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ $\quad \because$ Common factor can be taken out

For $\lambda_2 = -3$ $\quad (\lambda_2 I - A) = \begin{bmatrix} -3 & -2 & 0 \\ -4 & -3 & -1 \\ 48 & 34 & 6 \end{bmatrix}$

$\therefore m_2 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} 16 \\ -24 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

For $\lambda_3 = -4$ $\quad (\lambda_3 I - A) = \begin{bmatrix} -4 & -2 & 0 \\ -4 & -4 & 1 \\ 48 & 34 & 5 \end{bmatrix}$

$m_3 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} 14 \\ -28 \\ 56 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

m_1, m_2, m_3 are eigen vectors corresponding to eigen values $\lambda_1, \lambda_2 \& \lambda_3$.

d) Modal matrix is

$$M = [m_1 : m_2 : m_3] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -3 & -2 \\ -2 & 1 & 4 \end{bmatrix}$$

Reduce the given state model into its canonical form by diagonalizing matrix A

$$\dot{X} = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(t)$$

& $Y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X(t)$

Soln: From the given state model

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \& \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

let us find eigen values, eigen vectors & modal matrix of A

$$|\lambda I - A| = \left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix} \right| = 0$$

$$\therefore \begin{vmatrix} \lambda & -1 & 1 \\ 6 & \lambda+11 & -6 \\ 6 & 11 & \lambda-5 \end{vmatrix} = 0$$

$$\lambda(\lambda+11)(\lambda-5) + 66 + 6(\lambda-5) + 36 + 66 - 6(\lambda+11)$$

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$$(\lambda+1)(\lambda+2)(\lambda+3) = 0$$

$\therefore$ Eigen values are $\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3$

To find eigen vectors

For $\lambda_1 = -1 \Rightarrow [\lambda_1 I - A] = \begin{bmatrix} -1 & -1 & 1 \\ 6 & 10 & -6 \\ 6 & 11 & -6 \end{bmatrix}$

$$m_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -2 \Rightarrow [\lambda_2 I - A] = \begin{bmatrix} -2 & -1 & 1 \\ 6 & 9 & -6 \\ 6 & 11 & -7 \end{bmatrix}$

$$m_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

For $\lambda_3 = -3$

$$[\lambda_3 I - A] = \begin{bmatrix} -3 & -1 & 1 \\ 6 & 8 & -6 \\ 6 & 11 & -8 \end{bmatrix}$$

$$m_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$$

4

$$\therefore m = [m_1 : m_2 : m_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 1 & 4 & 9 \end{bmatrix} = \text{Modal matrix}$$

$$m^{-1} A m = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

while $\tilde{B} = m^{-1} B$

$$m^{-1} = \frac{\text{adj } m}{|m|} = \frac{\begin{bmatrix} -6 & 6 & -2 \\ -5 & 8 & -3 \\ 4 & -6 & 2 \end{bmatrix}^T}{-14} = \frac{\begin{bmatrix} -6 & -5 & 4 \\ 6 & 8 & -6 \\ -2 & -3 & 2 \end{bmatrix}}{-14}$$

$$= \begin{bmatrix} 0.43 & 0.36 & -0.28 \\ -0.43 & -0.57 & 0.43 \\ 0.14 & 0.21 & -0.14 \end{bmatrix}$$

$$\therefore \tilde{B} = m^{-1} B = \begin{bmatrix} -0.28 \\ 0.43 \\ -0.14 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

and $\tilde{C} = Cm = [1 \; 0 \; 0] \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} = [1 \; 1 \; 1]$

$$= [1 \; 0 \; 0] \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 1 & 4 & 9 \end{bmatrix} = [1 \; 1 \; 1]$$

Canonical state model is

$$\dot{z}(t) = \Lambda z(t) + \tilde{B} u(t)$$

$$y(t) = \tilde{C} z(t) +$$

To solve the homogeneous (unforced) state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t),\ \mathbf{x}(0) = \mathbf{x}_0$$

Consider a scalar case,

$$\frac{dx}{dt} = ax(t), x(0) = x_0$$

This equation has the solution

$$x(t) = e^{at}x_0$$
$$= \left(1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots\right)x_0$$

By analogy with the scalar case, the vector state equation has the solution

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots\right)\mathbf{x}_0$$

Each of the terms inside the brackets is an $n \times n$ matrix and the entire term is called a matrix exponential, which may be written as

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots$$

The solution $\mathbf{x}(t)$ can therefore be written as $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$. This solution of homogeneous state equation shows that the initial state $\mathbf{x}_0$ at $t = 0$, is driven to a state $\mathbf{x}(t)$ at time t. Since this transition in state is carried out by the matrix exponential e^{At}, e^{At} is known as the state transition matrix (STM) and is denoted by $\phi(t)$, i.e. $\phi(t) = e^{At}$. Since the STM depends only on the system matrix **A**, it is also called the STM of **A**. In general, for linear time-invariant systems, if the initial time is $t = t_0$, the state transition matrix becomes

$$\phi(t - t_0) = e^{A(t-t_0)}$$

Since the STM depends only on the length of the time $(t - t_0)$ and not on the initial time t_0, the initial time is conveniently regarded as zero. Given the system matrix **A**, the STM can be computed by expanding e^{At} into a power series in t and then adding the corresponding elements in the matrix terms of the infinite series. This is practical only for simple cases.

Let us now consider the solution of the non-homogeneous state equation.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t);\ \mathbf{x}(0) = \mathbf{x}_0$$

By writing this equation as

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

and pre-multiplying both sides of this equation by e^{-At}, we obtain

$$e^{-\mathbf{A}t}[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)] = \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Integrating this with respect to t between the limits 0 and t, gives

$$\int_0^t \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{x}(t)\Big|_0^t = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

i.e.
$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Now pre-multiplying both sides by e^{At}

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left[\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau\right]$$

In terms of STM,

$$\mathbf{x}(t) = \phi(t)\mathbf{x}(0) + \int_0^t \phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

If the initial time is t_0 instead of $t = 0$, the solution of the non-homogeneous state equation, also called the state transition equation becomes

$$\mathbf{x}(t) = \phi(t - t_0)\mathbf{x}(t_0) + \int_0^t \phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Significance of the STM: Since the STM satisfies the homogeneous state equations, it represents the free response of the system. In other words, it governs the response that is excited by the initial conditions only. The STM is dependent only on the system matrix **A**, and therefore it is sometimes referred to as the STM of **A**. As the name implies, the STM describes the change of state from the initial time $t = 0$, to any time t, when the inputs are zero.

Properties of STM

1. $\phi(0) = \mathbf{I}$, *Proof:* $\phi(0) = e^{A \times 0} = \mathbf{I}$
2. $\phi^{-1}(t) = \phi(-t)$ *Proof:* $\phi^{-1}(t) = \frac{1}{\phi(t)} = \frac{1}{e^{At}} = e^{-At} = \phi(-t)$
3. $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0)$ for any t_2, t_1, t_0

   Proof: $[\phi(t_2 - t_1)\phi(t_1 - t_0) = e^{A(t_2-t_1)} \cdot e^{A(t_1-t_0)} = e^{A(t_2-t_1+t_1-t_0)} = e^{A(t_2-t_0)} = \phi(t_2 - t_0)]$
4. $[\phi(t)]^k = \phi(kt)$ *Proof:* $[\phi(t)]^k = \phi(t) \cdot \phi(t) \cdots k \text{ times} = e^{At} \cdot e^{At} \ldots k \text{ times} = e^{Akt} = \phi(kt)$
5. $\phi(t_1 + t_2) = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

   Proof: $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2} = e^{At_2} \cdot e^{At_1} = \phi(t_1) \cdot \phi(t_2) = \phi(t_2) \cdot \phi(t_1)$

10.7 COMPUTATION OF THE STATE TRANSITION MATRIX

10.7.1 Computation of the STM by Infinite Series Method

Example 10.22 Compute the STM by infinite series method.

(a) $\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$ (b) $\mathbf{A}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

Solution: (a) For the given system matrix **A**, the state transition matrix (STM) is

$$\phi(t)=e^{\mathbf{A}t}=\mathbf{I}+\mathbf{A}t+\frac{\mathbf{A}^2t^2}{2!}+\frac{\mathbf{A}^3t^3}{3!}+\cdots$$

$$\mathbf{A}^2=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}=\begin{bmatrix}-1 & -2\\ 2 & 3\end{bmatrix},\ \mathbf{A}^3=\mathbf{A}^2.\mathbf{A}=\begin{bmatrix}-1 & -2\\ 2 & 3\end{bmatrix}\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}=\begin{bmatrix}2 & 3\\ -3 & -4\end{bmatrix}$$

$$\therefore\quad \phi(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}t+\begin{bmatrix}-1 & -2\\ 2 & 3\end{bmatrix}\frac{t^2}{2!}+\begin{bmatrix}2 & 3\\ -3 & -4\end{bmatrix}\frac{t^3}{3!}+\cdots$$

$$=\begin{bmatrix}1-\frac{t^2}{2}+\frac{t^3}{3}+\cdots & t-t^2+\frac{t^3}{2}+\cdots\\ -t+t^2-\frac{t^3}{2}+\cdots & 1-2t+\frac{3t^2}{2}-\frac{2t^3}{3}+\cdots\end{bmatrix}$$

$$=\begin{bmatrix}e^{-t}+te^{-t} & te^{-t}\\ -te^{-t} & e^{-t}-te^{-t}\end{bmatrix}$$

(b) For the given system matrix **A**, the state transition matrix (STM) is

$$\phi(t)=e^{\mathbf{A}t}=\mathbf{I}+\mathbf{A}t+\frac{\mathbf{A}^2t^2}{2!}+\frac{\mathbf{A}^3t^3}{3!}+\cdots$$

$$\mathbf{A}^2=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix},\ \mathbf{A}^3=\mathbf{A}^2\cdot\mathbf{A}=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$$

$$\therefore\quad \phi(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}t+\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\frac{t^2}{2!}+\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\frac{t^3}{3!}+\cdots$$

$$=\begin{bmatrix}1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots & t+t^2+\frac{t^3}{2}+\cdots\\ 0 & 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\end{bmatrix}$$

$$=\begin{bmatrix}1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots & t\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)\\ 0 & 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\end{bmatrix}=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}$$

10.7.2 Computation of the STM by Laplace Transformation

Consider an unforced system. Its homogeneous state equation is

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)$$

where **A** is a constant matrix. Taking the Laplace transform of this equation,

$$s\mathbf{X}(s)-\mathbf{x}(0)=\mathbf{A}\mathbf{X}(s)$$

where $\mathbf{X}(s)$ is the Laplace transform of the free response, and $\mathbf{x}(0)$ is the initial condition vector. The above equation may be arranged as

$$[s\mathbf{I}-\mathbf{A}]\,\mathbf{X}(s)=\mathbf{x}(0)$$

Pre-multiplying both sides by $[s\mathbf{I}-\mathbf{A}]^{-1}$

$$\mathbf{X}(s)=[s\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{x}(0)$$
$$=\phi(s)\mathbf{x}(0)$$

where $\phi(s)=[s\mathbf{I}-\mathbf{A}]^{-1}$ is called the resolvent matrix.

Taking the inverse Laplace transform, we get

$$\mathbf{x}(t)=L^{-1}[s\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{x}(0)$$

where $\mathbf{x}(t)$ is the free response of the system. This solution must be identical with the one obtained earlier. Comparing them

$$\phi(t)=e^{\mathbf{A}t}=L^{-1}\,[s\mathbf{I}-\mathbf{A}]^{-1}=L^{-1}\,[\phi(s)]$$

we can say, the state transition matrix STM is the inverse Laplace transform of the resolvent matrix.

Let us now consider the forced response of the system.

The non homogeneous state equation of the system is

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)$$

Taking the Laplace transform on both sides,

$$s\mathbf{X}(s)-\mathbf{x}(0)=\mathbf{A}\mathbf{X}(s)+\mathbf{B}\mathbf{U}(s)$$

or

$$[s\mathbf{I}-\mathbf{A}\,]\,\mathbf{X}(s)=\mathbf{x}(0)+\mathbf{B}\mathbf{U}(s)$$

Pre-multiplying both sides by $[s\mathbf{I}-\mathbf{A}]^{-1}$,

$$\mathbf{X}(s)=[s\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{x}(0)+[s\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{B}\mathbf{U}(s)$$

$$= \phi(s)\,\mathbf{x}(0) + \phi(s)\,\mathbf{BU}(s)$$
$$= \phi(s)\,[\mathbf{x}(0) + \mathbf{BU}(s)]$$

Taking the inverse Laplace transform on both sides,

$$\mathbf{x}(t) = \mathbf{L}^{-1}[\phi(s)\,[\mathbf{x}(0) + \mathbf{BU}(s)]]$$
$$= \phi(t)\,[\mathbf{x}(0)] + \mathbf{L}^{-1}\,[\phi(s)\,\mathbf{BU}(s)]$$

Applying convolution theorem,

$$\mathbf{x}(t) = \phi(t)\mathbf{x}(0) + \int_0^t \phi(t-\tau)\mathbf{Bu}(\tau)\,d\tau$$
$$= \phi(t)\left[\mathbf{x}(0) + \int_0^t \phi(-\tau)\mathbf{Bu}(\tau)\,d\tau\right]$$

This solution of linear non-homogeneous state equation is called the state transition equation.

Example 10.23 Obtain the STM for the state model whose **A** matrix is given by

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution: (a) For the given system matrix **A**

$$[s\mathbf{I}-\mathbf{A}] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}$$

$$\therefore \quad \phi(s) = [s\mathbf{I}-\mathbf{A}]^{-1} = \frac{\text{adj}\,[s\mathbf{I}-\mathbf{A}]}{|s\mathbf{I}-\mathbf{A}|} = \frac{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}^T}{(s-1)^2} = \frac{\begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}}{(s-1)^2}$$

$$\therefore \quad \text{STM} = \phi(t) = L^{-1}[\phi(s)] = L^{-1}[s\mathbf{I}-\mathbf{A}]^{-1} = L^{-1}\begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

(b) For the given system matrix **A**

$$[s\mathbf{I}-\mathbf{A}] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}$$

$$\phi(s) = [s\mathbf{I}-\mathbf{A}]^{-1} = \frac{\text{Adj}\,[s\mathbf{I}-\mathbf{A}]}{|s\mathbf{I}-\mathbf{A}|} = \frac{\begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}^T}{|s\mathbf{I}-\mathbf{A}|} = \frac{\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}}{(s+1)^2}$$

$$\therefore \quad \text{STM} = \phi(t) = L^{-1}\phi(s) = L^{-1}[s\mathbf{I}-\mathbf{A}]^{-1} = L^{-1}\begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

(c) For the given system matrix **A**

$$[s\mathbf{I}-\mathbf{A}] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\phi(s) = [s\mathbf{I}-\mathbf{A}]^{-1} = \frac{\text{Adj}\,[s\mathbf{I}-\mathbf{A}]}{|s\mathbf{I}-\mathbf{A}|} = \frac{\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^T}{|s\mathbf{I}-\mathbf{A}|} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2+3s+2}$$

$$\therefore \quad \text{STM} = \phi(t) = L^{-1}\phi(s) = L^{-1}[s\mathbf{I}-\mathbf{A}]^{-1} = L^{-1}\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} L^{-1}\left(\frac{s+3}{(s+1)(s+2)}\right) & L^{-1}\left(\frac{1}{(s+1)(s+2)}\right) \\ L^{-1}\left(\frac{-2}{(s+1)(s+2)}\right) & L^{-1}\left(\frac{s}{(s+1)(s+2)}\right) \end{bmatrix}$$

$$= \begin{bmatrix} L^{-1}\left(\frac{2}{s+1} - \frac{1}{s+2}\right) & L^{-1}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) \\ L^{-1}\left(\frac{-2}{s+1} + \frac{2}{s+2}\right) & L^{-1}\left(\frac{-1}{s+1} + \frac{2}{s+2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

10.7.3 Computation of the STM using Cayley–Hamilton Theorem

The STM may be computed using the technique based on the Cayley–Hamilton theorem. For large systems this method is far more convenient computationally compared to other methods.

The Cayley–Hamilton theorem states that every square matrix **A** satisfies its own characteristic equation. This theorem provides a simple procedure for evaluating the function of a matrix.

If **A** is a nonsingular $n \times n$ matrix, then to determine the matrix polynomial

$$f(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2 + \cdots + \alpha_{n-1}\mathbf{A}^{n-1}$$

first determine $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ by considering the scalar case

$$f(\lambda_i) = \alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \cdots + \alpha_{n-1}\lambda_i^{n-1}$$

and write simultaneous equations for $i = 1, 2, \ldots, n-1$ and solve them.

The formal procedure to determine a matrix polynomial is

Step 1. Find the eigenvalues of system matrix **A**.

Step 2. If all the eigenvalues are distinct, solve n simultaneous equations given by the equation

$$f(\lambda_i) = \alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \cdots + \alpha_{n-1}\lambda_i^{n-1}$$

for the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. If any of the eigenvalues are repeated, then obtain one independent equation by substituting that eigenvalue in the above equation.

Step 3. Substitute the coefficients α_i obtained in step 2 in equation

$$f(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2 + \cdots + \alpha_{n-1}\mathbf{A}^{n-1}$$ to obtain the matrix polynomial.

Example 10.24 Find $f(\mathbf{A}) = \mathbf{A}^{10}$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Solution: The characteristic equation is

$$|\lambda\mathbf{I} - \mathbf{A}| = \left|\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\right| = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{bmatrix} = (\lambda+1)^2 = 0$$

Matrix **A** has repeated eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1$. So we can get only one independent equation. Since **A** is of second-order,

$$f(\lambda) = \lambda^{10} = \alpha_0 + \alpha_1\lambda$$

Putting $\lambda = \lambda_1 = -1$ in the above equation, we get

$$f(\lambda_1) = \lambda_1^{10} = \alpha_0 + \alpha_1\lambda_1$$

i.e. $$(-1)^{10} = \alpha_0 - \alpha_1$$

i.e. $$\alpha_0 - \alpha_1 = 1$$

To obtain the second equation, differentiating the expression for $f(\lambda)$ on both sides,

$$\frac{d}{d\lambda}[\lambda^{10}]\bigg|_{\lambda=-1} = \alpha_1$$

i.e. $$10\lambda^9\Big|_{\lambda=-1} = \alpha_1$$

i.e. $$\alpha_1 = -10$$

$$\alpha_0 = 1 + \alpha_1 = 1 - 10 = -9$$

$\therefore$ $$f(\mathbf{A}) = \mathbf{A}^{10} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A}$$

$$= \alpha_0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_0 & \alpha_1 \\ -\alpha_1 & \alpha_0 - 2\alpha_1 \end{bmatrix} = \begin{bmatrix} -9 & -10 \\ 10 & 11 \end{bmatrix}$$

Example 10.25 Obtain the STM for the state model whose **A** matrix is given below using Cayley–Hamilton theorem.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution: (a) The eigenvalues of the system matrix **A** are the roots of the characteristic equation

$$|\lambda\mathbf{I} - \mathbf{A}| = \left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right| = \begin{vmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda-1)(\lambda-1) = 0$$

Therefore, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$.

We know that

$$e^{\lambda t} = \alpha_0 + \alpha_1\lambda$$

Substituting $\lambda = 1$ in the above equation, we have

$$e^t = \alpha_0 + \alpha_1$$

Differentiating the equation $e^{\lambda t} = \alpha_0 + \alpha_1\lambda$, with respect to λ

$$te^{\lambda t}\Big|_{\lambda=1} = te^t = \alpha_1$$

Substituting this value of α_1 in the expression for $e^t = \alpha_0 + \alpha_1$, we have

$$\alpha_0 = -\alpha_1 + e^t = -te^t + e^t$$

Therefore, the STM is given by

$$\boldsymbol{\phi}(t) = e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A}$$

$$= \alpha_0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \alpha_1 \\ 0 & \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 + \alpha_1 & \alpha_1 \\ 0 & \alpha_0 + \alpha_1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

(b) For the given system matrix **A**, the characteristic equation is

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

i.e. $$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{bmatrix} = \lambda^2 + 2\lambda + 1 = 0$$

The roots of this equation are the eigenvalues. Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -1$, i.e. the eigenvalues are repeated.

For the second-order system

$$f(\lambda) = e^{\lambda t} = \alpha_0 + \alpha_1 \lambda$$

∴ $$f(\lambda_1) = e^{\lambda_1 t} = e^{-t} = \alpha_0 - \alpha_1$$

Differentiating $f(\lambda)$ with respect to λ,

$$te^{\lambda t} = \alpha_1$$

For $\lambda = \lambda_1 = -1$, $$\alpha_1 = te^{-t}$$

∴ $$\alpha_0 = \alpha_1 + e^{-t} = te^{-t} + e^{-t} = e^{-t}(1 + t)$$

The state transition matrix is

∴ $$f(\mathbf{A}) = e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}$$

$$= \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_1 \\ -\alpha_1 & -2\alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_1 \\ -\alpha_1 & \alpha_0 - 2\alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

(c) For the given system matrix **A**, the eigenvalues are the roots of the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

i.e. $$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = 0$$

i.e. $$\lambda^2 + 3\lambda + 2 = 0$$

i.e. $$(\lambda + 1)(\lambda + 2) = 0$$

∴ $\lambda_1 = -1$, $\lambda_2 = -2$ are the eigenvalues.

For a second-order system,

$$f(\lambda) = e^{\lambda t} = \alpha_0 + \alpha_1 \lambda$$

∴ $$f(\lambda_1) = e^{\lambda_1 t} = \alpha_0 + \alpha_1 \lambda_1$$

i.e. $$e^{-t} = \alpha_0 - \alpha_1$$

$$f(\lambda_2) = e^{\lambda_2 t} = \alpha_0 + \alpha_1 \lambda_2$$

i.e. $$e^{-2t} = \alpha_0 - 2\alpha_1$$

Solving the equations for $f(\lambda_1)$ and $f(\lambda_2)$,

$$\alpha_0 = 2e^{-t} - e^{-2t}$$
$$\alpha_1 = e^{-t} - e^{-2t}$$

∴ $$f(\mathbf{A}) = e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A} = \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_1 \\ -2\alpha_1 & -3\alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_1 \\ -2\alpha_1 & \alpha_0 - 3\alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

10.7.4 Computation of STM by Canonical Transformation

The STM can be computed by using the modal matrix. Consider the homogeneous state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

with initial condition vector $\mathbf{x}(0) = \mathbf{x}_0$.

The solution vector is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

Suppose the matrix **A** is non-diagonal and has distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the above state equation can be transformed into another state equation using the linear transformation

$$\mathbf{x}(t) = \mathbf{M}\mathbf{z}(t)$$

where **M** is the diagonalizing or modal matrix. Substitution of this value of $\mathbf{x}(t)$ in the original state equation yields

$$\mathbf{M}\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{M}\mathbf{z}(t)$$

Pre-multiplying both sides by $\mathbf{M}^{-1}$, the transformed homogeneous state equation is

$$\dot{\mathbf{z}}(t) = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{z}(t) = \bar{\mathbf{A}}\mathbf{z}(t)$$

where $\bar{\mathbf{A}}$ is a diagonal matrix with the eigenvalues of **A** as its main diagonal elements.

$$\bar{\mathbf{A}} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

The new solution vector is

$$\mathbf{z}(t) = e^{\bar{\mathbf{A}}t}\mathbf{z}(0)$$

where $\mathbf{z}(0)$ is the transformed initial condition vector.

$$\mathbf{x}(t) = \mathbf{M}\mathbf{z}(t)$$

Since

$$\mathbf{z}(t) = \mathbf{M}^{-1}\mathbf{x}(t)$$

Substituting this value of $\mathbf{z}(t)$ in the new solution vector,

$$\mathbf{M}^{-1}\mathbf{x}(t) = e^{\bar{\mathbf{A}}t}\mathbf{M}^{-1}\mathbf{x}(0)$$

Pre-multiplying both sides by **M**

$$\mathbf{x}(t) = \mathbf{M}e^{\bar{\mathbf{A}}t}\mathbf{M}^{-1}\mathbf{x}(0)$$

Comparison of this with the original solution vector yields

$$e^{\mathbf{A}t} = \mathbf{M}e^{\bar{\mathbf{A}}t}\mathbf{M}^{-1}$$

Since

$$\bar{\mathbf{A}} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$e^{\bar{\mathbf{A}}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

If the system matrix **A** involves multiple eigenvalues, then $\mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ will not yield a diagonal matrix. The resulting state equation is in Jordan canonical form. Suppose the matrix **A** has eigenvalues as $\lambda_1, \lambda_1, \lambda_1, \lambda_4, \ldots, \lambda_n$ then

$$\dot{\mathbf{z}}(t) = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{z}(t) = \mathbf{J}\mathbf{z}(t)$$

where

$$\mathbf{J} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_4 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

and the STM is

$$e^{\mathbf{A}t} = \mathbf{M}e^{\mathbf{J}t}\mathbf{M}^{-1}$$

where

$$e^{\mathbf{J}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

Example 10.26 Obtain the STM for the state model whose **A** matrix is given below using canonical transformation method.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution: (a) Since the matrix **A** is not in companion form, the modal matrix can not be written directly. It can be determined by obtaining eigenvectors either by obtaining the co-factors along the rows or by the solution of simultaneous equations.

The eigenvalues of **A** are nothing but the roots of the characteristic equation

$$|\lambda\mathbf{I} - \mathbf{A}| = \left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)(\lambda - 1) = 0$$

Therefore, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$.

$$[\lambda\mathbf{I} - \mathbf{A}] = \begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{bmatrix}$$

10.8 TRANSFER FUNCTION FROM THE STATE MODEL

In the previous sections, we discussed how to obtain the state model of a system from its transfer function. Here we discuss how to obtain the transfer function of the system from its state model. If the state model is using phase variables, the transfer function can be obtained directly by inspection of dynamic equations. Another way is to draw a signal flow graph for the given phase variable state model and then obtain the transfer function by using Mason's gain formula. In general, given a state model, the transfer function can be obtained algebraically as shown below.

The state model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{10.19}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{10.20}$$

Taking the Laplace transform of Eqs. (10.19) and (10.20), we get

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

i.e.
$$[s\mathbf{I} - \mathbf{A}]\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

i.e.
$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}\mathbf{U}(s) \tag{10.21}$$

Substituting the value of $\mathbf{X}(s)$ from Eq. (10.21) in the expression $\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$, we get

$$\mathbf{Y}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{x}(0) + \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \tag{10.22}$$

Since transfer function is defined under zero initial conditions, neglecting $\mathbf{x}(0)$, Eq. (10.22) becomes

$$\mathbf{Y}(s) = [\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}]\ \mathbf{U}(s)$$

Therefore, the transfer function

$$\frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{T}(s) = \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}$$

Eventhough the state model of a system is not unique, the transfer function of a system is unique that means all the state models of a system result in the same transfer function.

Example 10.32 Obtain the transfer function of a system described by the state model

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Solution: Given the system matrix **A**

$$[s\mathbf{I}-\mathbf{A}] = s\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix} = \begin{bmatrix}s & -1 & 0\\ 0 & s & -1\\ 1 & 2 & s+3\end{bmatrix}$$

$$\therefore \quad [s\mathbf{I}-\mathbf{A}]^{-1} = \frac{\begin{bmatrix}(s+2)(s+1) & -1 & -s\\ (s+3) & s(s+3) & -(2s+1)\\ 1 & s & s^2\end{bmatrix}^T}{s(s^2+3s+2)+1(1)} = \frac{\begin{bmatrix}(s+2)(s+1) & (s+3) & 1\\ -1 & s(s+3) & s\\ -s & -(2s+1) & s^2\end{bmatrix}}{s^3+3s^2+2s+1}$$

The transfer function of the system is given by

$$\frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{T}(s) = \mathbf{C}[s\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}$$

$$\therefore \quad \mathbf{T}(s) = \frac{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}(s+2)(s+1) & (s+3) & 1\\ -1 & s(s+3) & s\\ -s & -(2s+1) & s^2\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}}{s^3+3s^2+2s+1}$$

$$= \frac{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}s+3 & 1\\ s(s+3) & s\\ -(2s+1) & s^2\end{bmatrix}}{s(s+1)(s+2)} = \frac{\begin{bmatrix}s+3 & 1\\ -(2s+1) & s^2\end{bmatrix}}{s^3+3s^2+2s+1}$$

Example 10.33 Obtain the transfer function of the system described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u; \quad x(0) = \begin{bmatrix}1\\ 1\end{bmatrix}, \quad y = [1 \;\; 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Solution: The transfer function of the system is given by

$$\frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{T}(s) = \mathbf{C}[s\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

$$\mathbf{T}(s) = [1 \;\; 0]\begin{bmatrix}s & -1\\ 2 & s+3\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{[1 \;\; 0]\begin{bmatrix}s+3 & -2\\ 1 & s\end{bmatrix}^T\begin{bmatrix}0\\ 1\end{bmatrix}}{s^2+3s+2}$$

$$= [1 \;\; 0]\frac{\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}}{(s+1)(s+2)}$$

$$= \frac{[1 \;\; 0]\begin{bmatrix}1\\ s\end{bmatrix}}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

10.9 STABILITY

Stability of a system represented by its state model: Since the eigenvalues of the system matrix **A** are the same as the roots of the characteristic equation which are nothing but the poles of the closed-loop transfer function, the stability of a system can be determined by determining the location of the eigenvalues.

Example 10.34 Determine the stability of the system whose **A** matrix is

(a) $\mathbf{A} = \begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix}0 & 0\\ -1 & -1\end{bmatrix}$ (d) $\mathbf{A} = \begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$

(e) $\mathbf{A} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & -5 & -4\end{bmatrix}$ (f) $\mathbf{A} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}$ (g) $\mathbf{A} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -3 & 3\end{bmatrix}$

Solution: (a) The characteristic equation of the system is

$$|\lambda\mathbf{I}-\mathbf{A}| = \left|\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}\right| = \begin{vmatrix}\lambda & 2\\ -1 & \lambda+3\end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$$

The roots of this characteristic equation, i.e. the eigenvalues of **A** are $\lambda_1 = -1$ and $\lambda_2 = -2$. Since all the eigenvalues have only negative real parts, the system represented by the state model is stable.

(b) For the given system matrix **A**, the eigenvalues are given by the roots of the characteristic equation

$$|\lambda\mathbf{I}-\mathbf{A}| = \left|\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right| = \begin{vmatrix}\lambda-1 & -1\\ 0 & \lambda-1\end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda-1)(\lambda-1) = 0$$

Therefore, the eigenvalues of **A** are $\lambda_1 = 1$ and $\lambda_2 = 1$. Since the eigenvalues have positive real parts, the system represented by that state model is unstable.

$$\therefore \quad e^{At} = \begin{bmatrix} 2e^{-t}-e^{-2t} & -e^{-t}+e^{-2t} \\ 2e^{-t}-2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

- This is the required state transition matrix.

14.25 Controllability and Observability

May-10, 8 Marks

In a control system analysis, it is necessary to find an optimal control solution for a given control problem. The existence of such a solution depends on the answers of two basic questions which are,

1. For a given system, is it possible to transfer any initial state to any other desired state in a finite time under the effect of suitable control input force ?
2. If the output is measured for finite time then with the knowledge of the input, is it possible to determine initial state of the system ?

The answer to the first question gives the concept of the controllability while the answer to the second question gives the concept of observatility of the system.

14.25.1 Controllability

The answer to the first question means the concept of controllability of a system which is related to the transfer of any initial state of the system to any other desired state, in a finite length of time by application of proper inputs. Hence controllability can be defined as,

A system is said to be **completely state controllable** if it is possible to transfer the system state from any initial state $X(t_0)$ to any other desired state $X(t_f)$ in a specified finite time interval (t_f) by a control vector U(t).

The concept of controllability and observability were originally introduced by Kalman hence Kalman's tests are used to find out whether the system is controllable and observable or not.

14.25.1.1 Kalman's Test for Controllability

May-08, 10, 4 Marks

Consider n^{th} order multiple input linear time invariant system represented by its state equation as,

$$\dot{X} = AX(t) + BU(t)$$

where A has order $n \times n$ matrix

and U(t) is $m \times 1$vector i.e. there are m inputs.

X(t) is n dimensional state vector.

The system is completely state controllable if and only if the rank of the composite matrix Q_c is 'n'.

The composite matrix Q_c is given by,

$$\boxed{Q_c = [B : AB : A^2B : \ldots\ldots A^{n-1}B]}$$

In the matrix Q_c, B, AB, A^2B ..... are the various columns.

To find the rank of a matrix means to search for a highest order determinant which is nonsingular i.e. whose value is nonzero. This order of determinant is the rank of the given matrix. Thus if $r \times r$ determinant has a nonzero value in a given matrix then the rank of a matrix is r and any determinant having order r + 1 or more than that has a zero value.

If such a rank r of Q_c is n which is order of the system then the system is completely state controllable. So for the complete controllability of the system,

$$\boxed{\text{rank of } Q_c = n}$$

The matrix Q_c is called test matrix for controllability.

Example 14.16 : *Evaluate the controllability of the system*

with, $\dot{X} = AX + BU$

and $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}_{2\times 2}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution : $n = 2$

$\therefore \quad Q_c = [B \quad AB]$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Now $\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$ = determinant of $2\times 2 = 0$

Hence rank of Q_c $r = 1$ and

rank of $Q_c \neq n$

$\therefore$ The system is **not state controllable**

Example 14.17 : Find the controllability of the system,

$$\dot{X} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t)$$

$$n = 2$$

Solution :

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore$ $$Q_c = [B \;\; AB] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$\therefore$

Now $\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$ which is nonzero

$\therefore$ Rank of $Q_c = 2 = n$

Hence the given system is **completely controllable.**

Example 14.18 : *A linear dynamic time invariant system is represented by*

AB

$$\dot{X} = A\,X(t) + B\,U(t)$$

where $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

+ (3×2

3×3 3×2

Find if the system is completely controllable.

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Solution :

For the system, $n = 3$

$\therefore$ $$Q_c = [B : AB : A^2B]$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -3 & 0 \end{bmatrix}$$

$$A^2B = A\,[A\,B] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 7 & 0 \end{bmatrix}$$

$\therefore$ $$Q_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 1 & 0 & -3 & 0 & 7 & 0 \end{bmatrix}$$

Consider the determinant,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -3 \end{bmatrix} = 1 = \text{nonzero}$$

Hence rank of $Q_c = 3 = n$

Thus the given system is **completely controllable**

Example 14.19 : *Consider the system with state equation*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(t)$$

Evaluate its state controllability.

Solution : The system has $n = 3$

$\therefore$ $$Q_c = [B : AB : A^2B]$$

$\therefore$ $$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$$

$\therefore$ $$A^2B = A[A\,B] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix}$$

$\therefore$ $$Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & -25 \end{bmatrix}$$

Consider the determinant,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & -25 \end{bmatrix} = -1 = \text{nonzero}$$

$\therefore$ Rank of $Q_c = 3 = n$

Thus the given system is **completely controllable.**

14.25.2 Observability

The observability is related to the problem of determining the system state by measuring the output for finite length of time. Hence observability can be defined as,

A system is said to be **completely observable**, if every state $X(t_0)$ can be completely identified by measurements of the outputs $Y(t)$ over a finite time interval. If the system is not completely observable means that few of its state variables are not practically measurable and are shielded from the observation.

Similar to the controllability, the observability of the system can be obtained by using Kalman's test.

14.25.2.1 Kalman's Test for Observability

May-08, 10, 4 Marks

Consider n^{th} order multiple input multiple output linear time invariant system, represented by its state equation as,

$$\dot{X} = A\,X(t) + B\,U(t)$$

and $$Y(t) = C\,X(t)$$

where $$Y(t) = p\times 1 \text{ output vector}$$

and $$C = 1\times n \text{ matrix}$$

The system is completely observable if and only if the rank of the composite matrix Q_0 is 'n'.

The composite matrix Q_0 is given by,

$$\boxed{Q_o = [C^T : A^T C^T : \ldots\ldots\ldots (A^T)^{n-1} C^T]}$$

where C^T = Transpose of matrix C

and A^T = Transpose of matrix A

Thus if, rank of $Q_0 = n$

then system is completely observable.

The rank of Q_0 can be obtained by the same method as discussed earlier to obtain the rank of Q_c.

The matrix Q_0 is called test matrix for the observability.

Example 14.20 : *Evaluate the observability of the system*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t)$$

and $$Y(t) = [1 \;\; 0]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Solution : The order of the system is, $n = 2$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \text{ and } C = [1 \;\; 0]$$

$$\therefore \quad A^T = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \text{ and } C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Q_0 = [C^T \;\; A^T C^T]$$

$$A^T C^T = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad Q_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Consider the determinant

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 = \text{nonzero}$$

$\therefore$ Rank of $Q_0 = 2 = n$

Hence the system is **completely observable.**

Example 14.21 : *Evaluate the observability of the system with*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix},\; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } C = [3 \;\; 4 \;\; 1]$$

Solution : The order of the system is $n = 3$

$$\therefore \quad Q_0 = [C^T \;\; A^T C^T \;\; (A^T)^2 C^T]$$

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$(A^T)^2C^T = A^T[A^TC^T] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$$

$$\therefore \quad Q_0 = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$

Consider the determinant,

$$\begin{vmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{vmatrix} = -6 + 0 + 0 - 0 + 0 - 6 = 0$$

Hence a nonzero determinant existing in Q_0 is having order less than 3.

$\therefore$Rank of $Q_0 \neq 3 \neq n$

Hence the system is **not completely observable.**

Example 14.22 : *Consider the system represented by,*

$$\dot{X} = \begin{bmatrix} -0.2 & 0.4 \\ 0.1 & -0.1 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(t)$$

$$Y(t) = [1 \quad 0] \, X(t)$$

Find the complete observability of the system.

Solution : For the system, n = 2

$$A = \begin{bmatrix} -0.2 & 0.4 \\ 0.1 & -0.1 \end{bmatrix} \quad C = [1 \quad 0]$$

$$\therefore \quad A^T = \begin{bmatrix} -0.2 & 0.1 \\ 0.4 & -0.1 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Q_0 = [C^T \quad A^TC^T]$$

$$\therefore \quad A^TC^T = \begin{bmatrix} -0.2 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.4 \end{bmatrix}$$

$$\therefore \quad Q_0 = \begin{bmatrix} 1 & -0.2 \\ 0 & 0.4 \end{bmatrix}$$

Consider the determinant,