UNIT-V

Multirate Digital Signal Processing.

Introduction:

Discrete time systems may be single rate systems or multirate systems. The discrete time systems that use single sampling rate from A/D converter to D/A converter are known as single rate systems. The discrete time systems that process data at more than one sampling rate are known as multirate systems. Different sampling rates can be obtained using up-sampler and down sampler.

The basic operations in multirate processing to achieve this are decimation and Interpolation. Decimation is for reducing the sampling rate and Interpolation is for increasing the sampling rate.

Applications.

1. In high quality data acquisition and storage systems
2. In audio signal processing.
3. In video
4. In speech processing
5. In transmultiplexers
6. For narrow band filtering.

Advantages :-

1. computational requirements are less
2. storage space for filter coefficients is less.
3. Finite arithmetic effects are less.
4. Filter order required in multirate application is low.
5. sensitivity to filter coefficient lengths is less.

→ while designing multirate systems, Effects of aliasing for decimation and pseudo images for interpolators should be avoided.

Sampling :-

A continuous time signal $x(t)$ can be converted into a discrete time signal $x(nT)$ by sampling it at regular intervals of time with sampling period T. The sampled signal $x(nT)$ is given by

$$x(nT) = x(t)\Big|_{t=nT} \qquad -\infty < n < \infty.$$

A sampling process can also be interpreted as a modulation or multiplication process.

Sampling theorem :-

Sampling theorem states that a band limited signal $x(t)$ having finite energy, which has no spectral components higher than f_h Hz can be completely reconstructed from its samples taken at the rate of $2f_h$ or more samples per second.

The sampling rate of $2f_h$ samples per second is the Nyquist rate and its reciprocal $1/2f_h$ is the Nyquist period.

Down Sampling :-

Reducing the sampling rate of a discrete time signal is called down sampling. The sampling rate of the discrete time signal can be reduced by a factor D by taking every D^{th} value of the signal. mathematically, down sampling is represented by

$$y(n) = x(Dn)$$

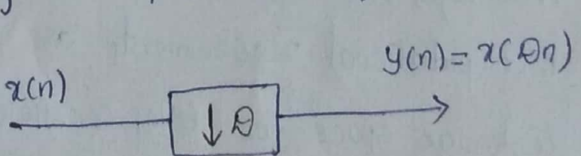


fig: Down Samples.

If $x(n) = \{1, 2, 3, 1, 2, 3, 1, 2, 3 \ldots\}$

Then $x(2n) = \{1, 3, 2, 1, 3 \ldots\}$

$x(3n) = \{1, 1, 1, 1 \ldots\}$.

$x(2n)$ is obtained by keeping every second sample of $x(n)$ & $x(3n)$ is obtained by keeping every 3^{rd} sample of $x(n)$ & removing other samples.

If the input signal $x(n)$ is not band limited, then there will be overlapping of spectra at the output of the down sampler. This overlapping of spectra is called aliasing which is undesirable. This aliasing problem can be eliminated by band limiting the input signal by inserting a low pass filter called anti-aliasing filter before the down sampler. The anti aliasing filter and the down sampler together is called decimator. The decimator is known as sub sampler, down sampler or under sampler.

Decimation is the process of decreasing the sampling rate by an integer factor D by keeping every Dth sample and removing $D-1$ in between samples.

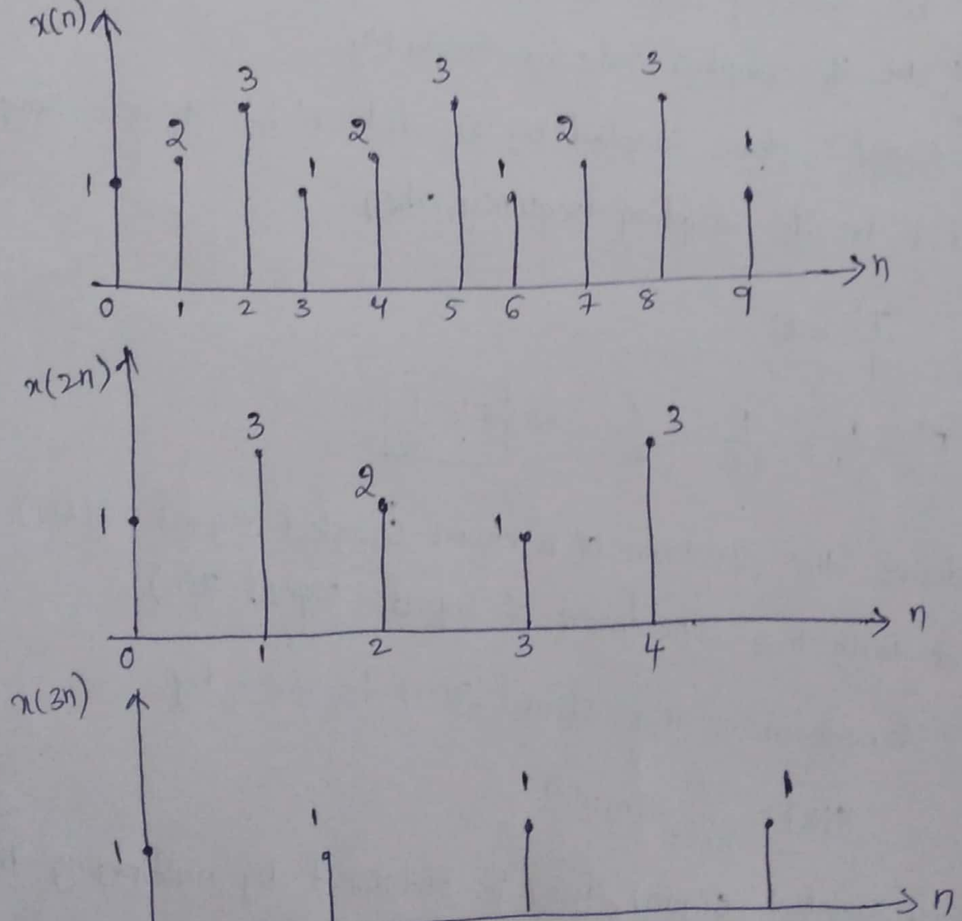


fig: plot of $x(n)$, $x(2n)$ & $x(3n)$

The decimator comprises two blocks such as anti aliasing filter and down sampler. Here the anti-aliasing filter is a low pass filter to band limit the input signal so that aliasing problem is eliminated and the down sampler is used to reduce the sampling rate by keeping every Dth sample and removing D-1 in between samples.

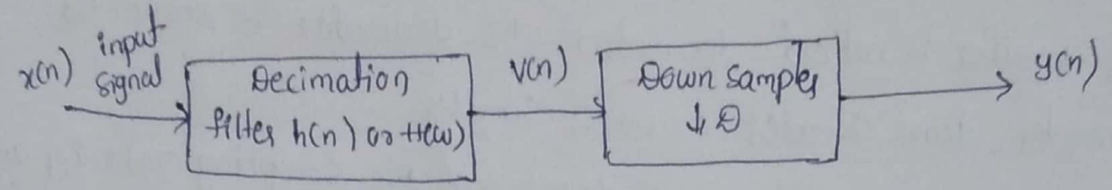


fig: Block diagram of decimator.

Spectrum of down sampled signal :-

let T be sampling period of input signal $x(n)$,

F be its sampling rate or frequency.

when the signal is down sampled by D, let T' be its new sampling period and F' be its sampling frequency, then

$$\frac{T'}{T} = D$$

$$F' = \frac{1}{T'} = \frac{1}{TD} = \frac{F}{D} \Rightarrow \boxed{F' = \frac{F}{D}}$$

let us derive the spectrum of a down sampled signal $x(Dn)$ and compare it with the spectrum of input signal $x(n)$.

The Z-Transform of the signal $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The down sampled signal $y(n)$ is obtained by multiplying the sequence $x(n)$ with a periodic train of impulses $p(n)$ with a period 'D' and then leaving out the D-1 zeros between each pair of samples

The periodic train of impulses is given by

$$p(n) = \begin{cases} 1 & n = 0, \pm D, \pm 2D \\ 0 & \text{otherwise} \end{cases}$$

The discrete Fourier series representation of the signal $p(n)$ is given by

$$p(n) = \frac{1}{D} \sum_{k=0}^{D-1} e^{j2\pi kn/D} \quad -\infty < n < \infty .$$

multiplying the sequence $x(n)$ with $p(n)$ yields

$$x'(n) = x(n)p(n)$$

i.e $$x'(n) = \begin{cases} x(n) & n = 0, \pm D, \pm 2D \\ 0 & \text{otherwise} \end{cases}$$

If we leave $D-1$ zeros between each pair of samples, we get the output of down samples.

$$y(n) = x'(nD) = x(nD)p(nD)$$

$$= x(nD)$$

the Z-Transform of the output sequence is given by

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x'(nD) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x'(n) z^{-n/D}$$

where $x'(n) = 0$ except at multiple of D.

Since $x'(n) = x(n)p(n)$, we get

$$\boxed{Y(z) = \sum_{n=-\infty}^{\infty} x(n) p(n) z^{-n/D}}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{D} \sum_{k=0}^{D-1} e^{j2\pi kn/D} \right] z^{-n/D}$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} \sum_{n=-\infty}^{\infty} x(n) \left(e^{-j2\pi k/D} z^{1/D} \right)^{-n}$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} X\left[e^{-j2\pi k/D} z^{1/D} \right]$$

Substituting $z = e^{j\omega}$, we get the frequency response

$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D} e^{j\omega/D} \right)$$

$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(e^{j(\omega - 2\pi k)/D} \right)$$

$$\boxed{Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left[\frac{(\omega - 2\pi k)}{D} \right]}$$

From the above relation we find that if the fourier transform of the input signal $x(n)$ of a down sampler is $X(\omega)$, then the fourier transform $Y(\omega)$ of the output signal $y(n)$ is a sum of D uniformly shifted and stretched version of $X(\omega)$ scaled by a factor $1/D$.

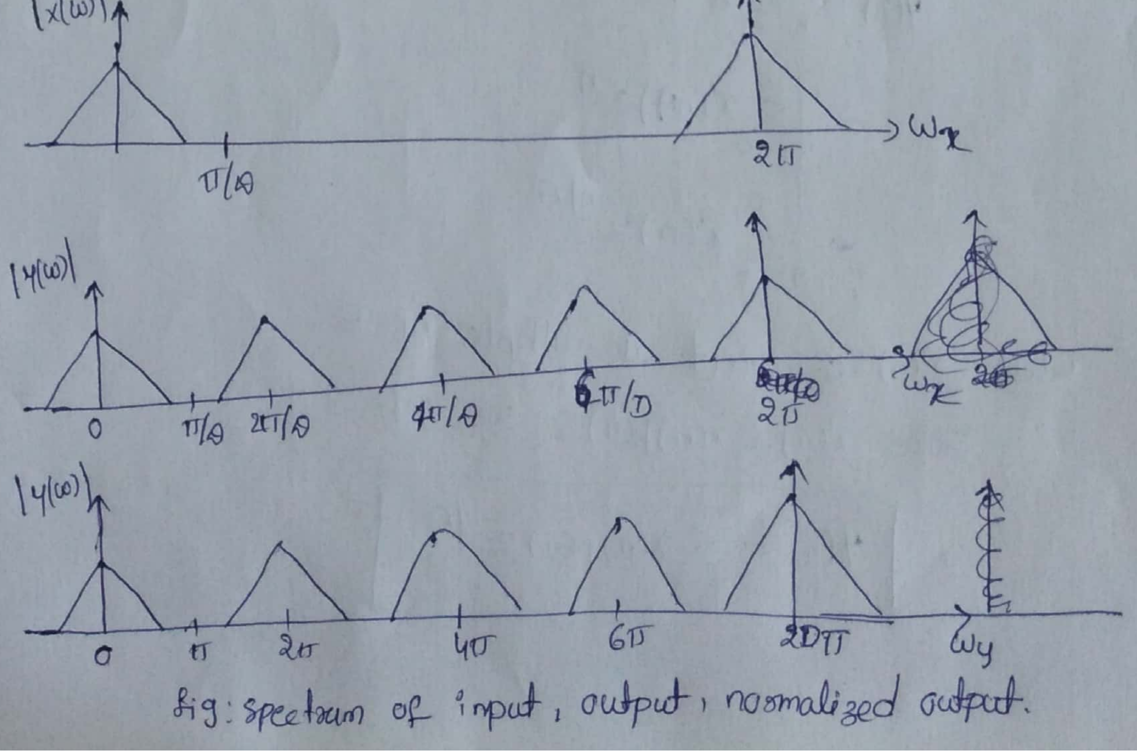


fig: spectrum of input, output, normalized output.

Aliasing Effect and Anti aliasing filter :-

If the original signal spectrum is not band limited to $\omega = \pi/D$, then the spectrum obtained after down sampling will overlap. This overlapping of spectra is called aliasing.

→ Therefore, aliasing due to down sampling a signal by a factor D is absent if and only if the signal $x(n)$ is band limited to $\pm\pi/D$.

If the signal $x(n)$ is not band limited to $\pm\pi/D$, then LPF with cut-off frequency π/D is used prior to down sampling. This lowpass filter which is connected before the down sampler to prevent the effect of aliasing by band limiting the input signal is called the anti aliasing filter.

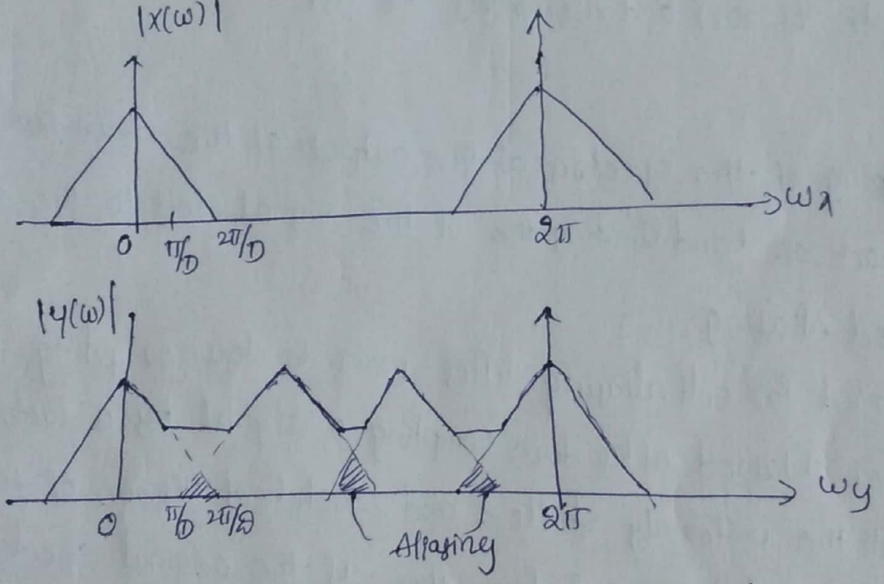


fig: Input spectrum, aliased output spectrum.

The signal obtained after filtering is given by

$$v(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$y(n) = v(nD) = \sum_{k=-\infty}^{\infty} h(k)x(nD-k)$$

Consider a factor of D downsampler, then

$$Y(\omega) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{\omega - 2\pi k}{2}\right)$$

$$Y(\omega) = \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega - 2\pi}{2}\right) \right]$$

$$\boxed{Y(\omega) = \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(-\frac{\omega}{2}\right) \right]}$$

The second term $X(-\frac{\omega}{2})$ is simply obtained by shifting the first term $X(\omega)$ to the right by an amount of 2π.

Decimation:-

Decimation is the process of reducing the sampling rate by an integer factor D, and removes D-1 samples. It is also called down sampling by factor D.

→ Down sampling a sequence $x(n)$ by a factor D is the process of picking every D^{th} sample of $x(n)$ and discarding the rest.

Aliasing:-

The overlapping of the spectra at the output of the down sampler due to the lack of band limiting ~~the~~ of the signal fed to the down sampler is called Aliasing.

what is the need for anti aliasing filter prior to down sampling.

The spectra obtained after down sampling a signal by a factor D is the sum of all the uniformly shifted and stretched version of the original spectrum scaled by a factor $1/D$. If the original spectrum is not band limited to π/D, then down sampling will cause aliasing. In order to avoid aliasing, the signal $x(n)$ [~~with a low pass filter with a cut-off frequency of~~] is to be band limited to $\pm\pi/D$. This can be done by filtering the signal $x(n)$ with a low pass filter with a cut-off frequency of π/m. This is known as Anti aliasing filter.

Up Sampling :-

Increasing the sampling rate of a discrete time signal is called up sampling. The sampling rate of a discrete time signal can be increased by a factor I by placing $(I-1)$ equally spaced zeros between each pair of samples.

mathematically, up sampling is expressed by

$$y(n) = \begin{cases} x\left(\frac{n}{I}\right) & n = 0, \pm I, \pm 2I \ldots \\ 0 & \text{otherwise} \end{cases}$$

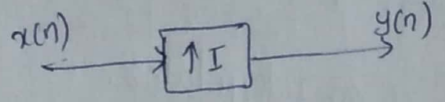


fig: Up samples.

If $x(n) = \{1, 2, 3, 1, 2, 3 \ldots\}$

then $y(n) = x\left(\frac{n}{2}\right) = \{1, 0, 2, 0, 3, 0, 1, 0, 2, 0, 3, 0 \ldots\}$ for an up sampling Sampling factor of $I=2$.

$y(n) = x\left(\frac{n}{3}\right) = \{1, 0, 0, 2, 0, 0, 3, 0, 0, 1, 0, 0 \ldots\}$ for an up sampling factor of $I=3$.

Usually an anti imaging filter is to be kept after the up samples
to remove unwanted images developed due to up sampling. The
anti imaging filter and the up samples together is called interpolator.

Interpolation: Interpolation is the process of increasing the sampling rate by an Integer I and interpolating $(I-1)$ new samples between successive values of the signal.

The block diagram of Interpolator comprises of two blocks such as up sampler and anti imaging filter. Here up sampler is used to increase the sampling rate by introducing zeros between successive input samples and the interpolation filter is also known as anti imaging filter, is used to remove the unwanted images that are yielded by up-sampling.

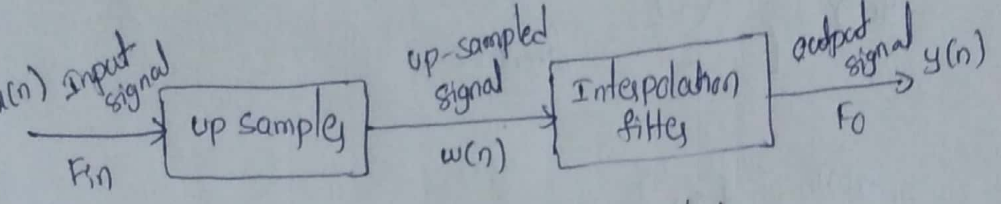


fig: Block diagram of an Interpolator.

Expression for output of Interpolator :-

let I be an integer interpolating factor of the signal. let 'T' be Sampling period and $F = 1/T$ be the sampling frequency of the input signal. After up sampling let T' be the new sampling period and F' be the new sampling frequency, then

$$\frac{T'}{T} = \frac{1}{I}$$

the sampling rate is given by

$$F' = \frac{1}{T'} = \frac{I}{T} = IF \Rightarrow \boxed{F' = IF}$$

let $w(n)$ be the signal obtained by interpolating $(I-1)$ samples between each pair of samples of $x(n)$.

$$w(n) = \begin{cases} x\left(\frac{n}{I}\right), & n = 0, \pm I, \pm 2I \\ 0 & \text{otherwise} \end{cases}$$

The Z-Transform of the signal $w(n)$ is given by

$$w(z) = \sum_{n=-\infty}^{\infty} w(n) z^{-n} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{I}\right) z^{-n}$$

$$w(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-nI}$$

$$w(z) = X(z^I)$$

when considered over the unit circle $z = e^{jw'}$

$$w(e^{jw'}) = X(e^{jw'I}) \quad \text{i.e} \quad w(w') = X.(Iw')$$

where $w' = 2\pi f T'$

the spectra of the signal $w(n)$ contains the images of base band placed at the harmonics of the sampling frequency $\pm 2\pi/I$, $\pm 4\pi/I$.

To remove the images an anti imaging filter is used.

The ideal characteristics of low pass filter is given by

$$H(e^{jw'}) = \begin{cases} G & |w'| \le 2\pi f T'/2 = \pi/I \\ 0 & \text{otherwise} \end{cases}$$

where G is the gain of the filter & it should be I in the pass band. The frequency response of the output signal is given by

$$Y(e^{jw'}) = H(e^{jw'}) X(e^{jw'I})$$

$$= \begin{cases} G X(e^{jw'I}) & |w'| \le \pi/I \\ 0 & \text{otherwise} \end{cases}$$

The output signal $y(n)$ is given by

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k) w(k)$$

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} h(n-k)\, x(k/I)}$$

k/I is an integer

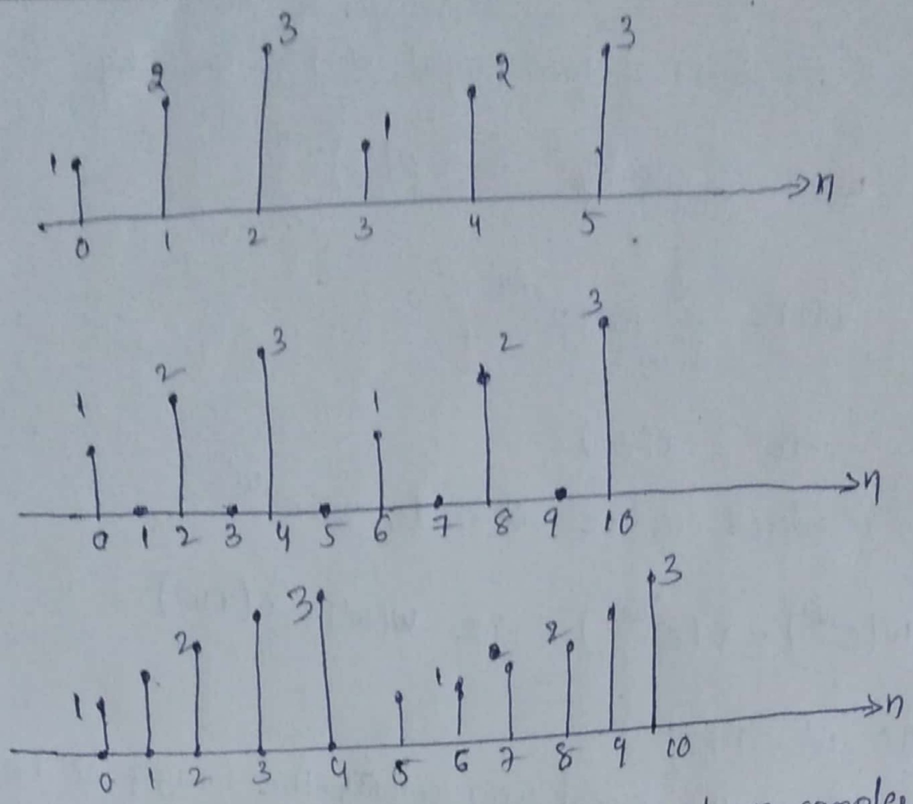


fig (a) Input signal $x(n)$, (b) output of 2 fold up-sampler $y_1(n) = x(n/2)$
(c) output of interpolator $y_2(n) = x(n/2)$.

→ The frequency spectrum $X(3\omega)$ is three-fold repetition of $X(\omega)$.
i.e inserting $I-1$ zeros between successive values of $x(n)$ results in a signal whose spectrum $X(I\omega)$ is an I fold periodic repetition of the input spectrum $X(\omega)$. These additional spectra created are called image spectra and the phenomenon is known as imaging.

Anti Imaging filter :-
The low pass filter placed after the up sampler to remove the images created due to upsampling is called the anti-imaging filter.

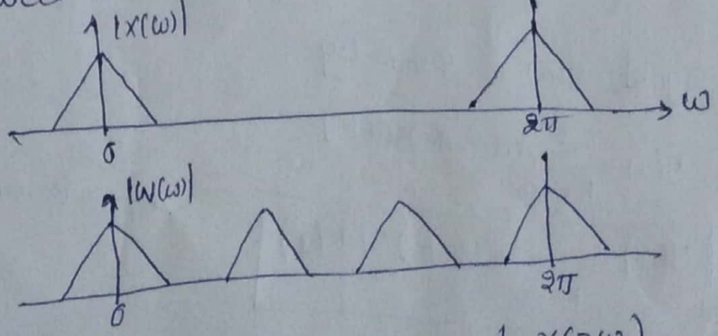


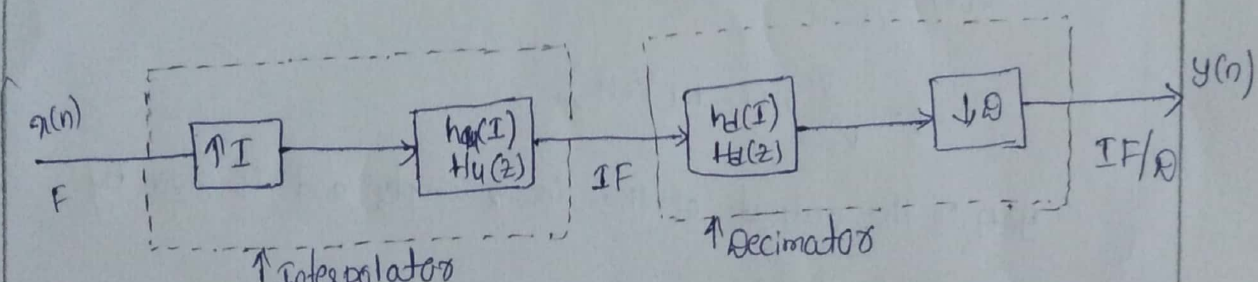
fig: spectrum of $X(\omega)$ and $X(3\omega)$.

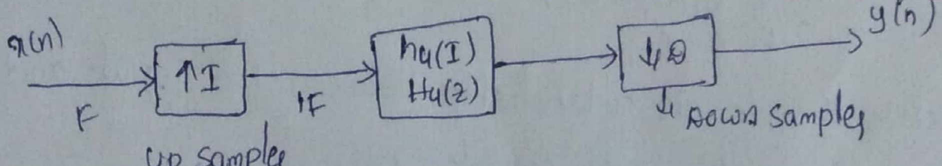
Sampling Rate Conversion :-

A Sampling rate conversion by a factor I/D can be achieved by first performing interpolation by factor I and then performing decimation by factor D.

→ In some applications sampling rate conversion by a non-integer factor may be required. for example transfering data from a compact disc at a rate of 44.1 KHZ to a digital audio tape at 48 KHZ. Here the sampling rate conversion factor is 48/44.1 KHZ which is a non-integer.

→ fig 1(a) shows the cascade connection configuration of interpolator and decimator. The anti imaging filter $H_u(z)$ and the anti aliasing filter $H_d(z)$ are operated at the Sampling rate, hence can be replaced by a simple low pass filter with transfer function $H(z)$ as shown in fig 1(b). where the low pass filter has a cut-off frequency of $\omega_c = \min\left[\frac{\pi}{I}, \frac{\pi}{D}\right]$.





fig(1) cascading of sampling rate converters.

Time domain and frequency domain relations of Sampling rate converters:-

$h_u(l)$ is Anti Imaging filter and $h_d(l)$ is Anti aliasing filter. The overall cut-off frequency of the two cascaded low pass filters will be the minimum of the two cut-off frequencies.

The frequency response of $h_u(l)$ is given as:

$$H_u(\omega) = \begin{cases} I & -\frac{\pi}{I} \le \omega \le \frac{\pi}{I} \\ 0 & \text{elsewhere} \end{cases}$$

The frequency response of $h_d(l)$ (anti aliasing filter) is given as

$$H_d(\omega) = \begin{cases} I & -\frac{\pi}{D} \le \omega \le \frac{\pi}{D} \\ 0 & \text{elsewhere.} \end{cases}$$

Time domain relationship:-

~~the output of the low pass filter is given as~~

The output of the low pass filter is given as

$$w(l) = \sum_{k=-\infty}^{\infty} h(l-k)v(k)$$

$$= \sum_{k=-\infty}^{\infty} h(l-kI)x(k)$$

$y(d)$ is the output of the down sampler and is given by

$$y(d) = w(dD)$$

$$= \sum_{k=-\infty}^{\infty} h(dD-kI)x(k)$$

Therefore time domain relationship between the input and output of a sampling rate converter is

$$y(n) = \sum_{k=-\infty}^{\infty} h(nD-kI)x(k)$$

Frequency domain relationship:-

from fig(1), $v(k)$ = output of up sampler with frequency w_y.

Therefore output of samples with frequency w_v is expressed as

$$V(w_v) = X(w_v I)$$

The output of the up sampler is passed through a LPF and hence we obtain $w(l)$ with frequency w_v. Therefore output of ~~frequency~~ the low pass filter with frequency w_v is given as

$$W(w_v) = H(w_v)\,X(w_v)$$

$$W(w_v) = \begin{cases} I\,X(w_v I) & |w| \le \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 & \text{elsewhere} \end{cases}$$

The spectrum of the output sequence is given by

$$Y(w_y) = \frac{1}{D} X\left(\frac{w_y}{D}\right)$$

$$Y(w_y) = \frac{1}{D} W\left(\frac{w_y}{D}\right)$$

we know that $w_v = \frac{w_y}{D}$

$$Y(w_y) = \frac{1}{D} W(w_y)$$

Substituting $W(w_v) = I\,X(w_v I)$

$$Y(w_y) = \begin{cases} \frac{I}{D} X(w_v I) & |w_v| \le \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 & \text{elsewhere} \end{cases}$$

$$Y(w_y) = \begin{cases} \frac{I}{D} X\left(\frac{I w_y}{D}\right) & \left|\frac{w_y}{D}\right| \le \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 & \text{elsewhere} \end{cases}$$

So the frequency domain relationship between input and output of a sampling rate converter is

$$Y(w_y) = \begin{cases} \frac{I}{D} X\left(\frac{I w_y}{D}\right) & |w_y| \le \min\left(\pi, \frac{\pi D}{I}\right) \\ 0 & \text{elsewhere} \end{cases}$$

Fig(2) shows the sampling rate conversion by a factor of 5/3.

fig(2)(a) shows the actual signal $x(n)$

fig 2(b) shows the signal, i.e the sampling rate is increased by 5, by inserting 4 zero valued samples between successive values of $x(n)$. The output of anti imaging filter is shown in fig 2(c).

The filtered data is then reduced for every three samples as shown in fig 2(d).

A cascade of a factor of D down sampler and a factor I upsampler is interchangeable with no change in the input and output relation if and only if I and D are co-prime.

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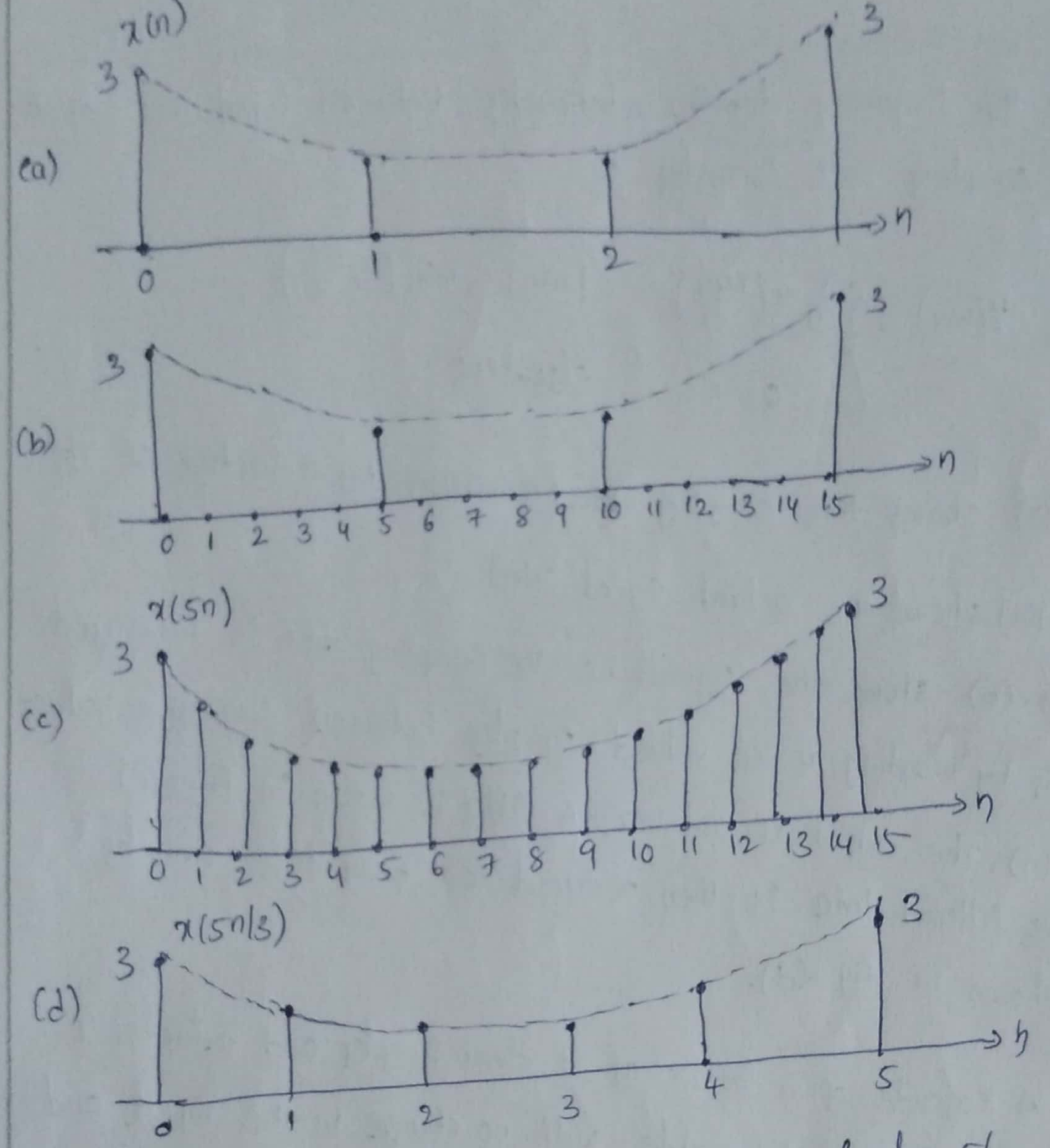

fig (2) Sampling rate conversion by a factor 5/3.

Multistage Decimators and Interpolators :-

In practical applications, mostly sampling rate conversion by a rational factor I/D is required.

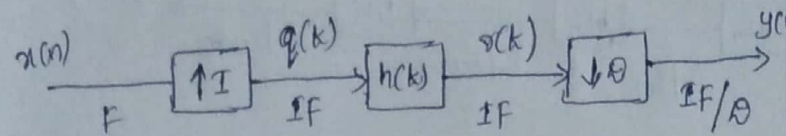

fig: sampling rate conversion by a rational factor I/D.

If the decimation factor D and/or interpolation factor I are much larger than unity, the implementation of sampling rate conversion in a single stage is computational inefficient. Therefore, for performing sampling rate conversion for either $D>>1$ and/or $I>>1$, we go in for multistage implementation.

Consider a system for decimating a signal by an integer factor D. let the input signal sampling frequency be f_x, then the then the decimated signal frequency will be $f_y = f_x/D$.

If $D>>1$, the we express D as a product of positive integers as

$$D = \prod_{i=1}^{N} D_i$$

Each decimator D_i is decimated and cascaded to get N stages of filtering and decimators as shown in fig (1)

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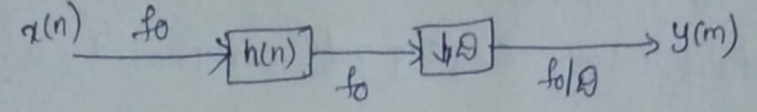
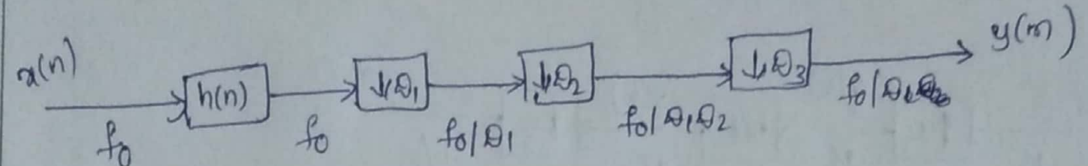
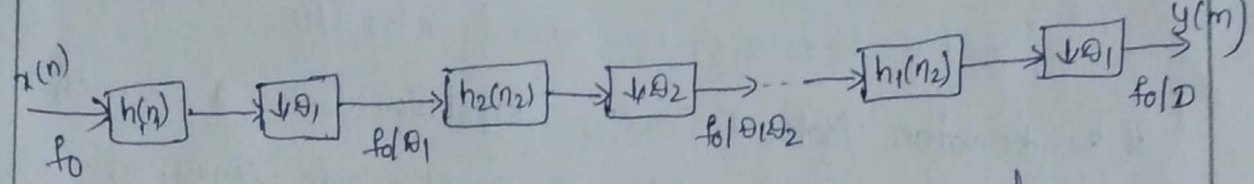






fig(1) multistage Decimator.

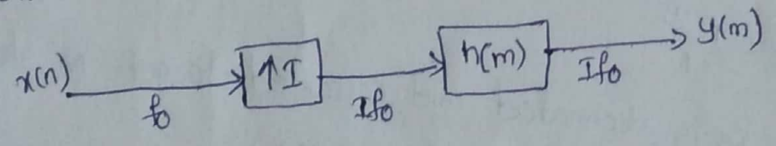
$$D = D_1 D_2 \cdots D_I$$

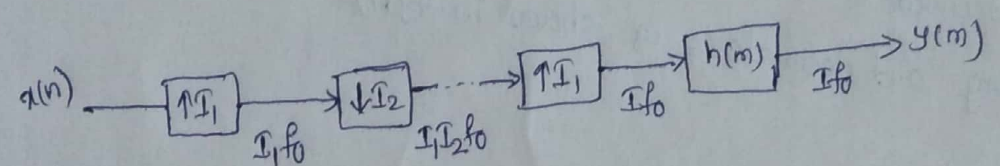
$$\boxed{D = \prod_{i=1}^{I} D_i}$$

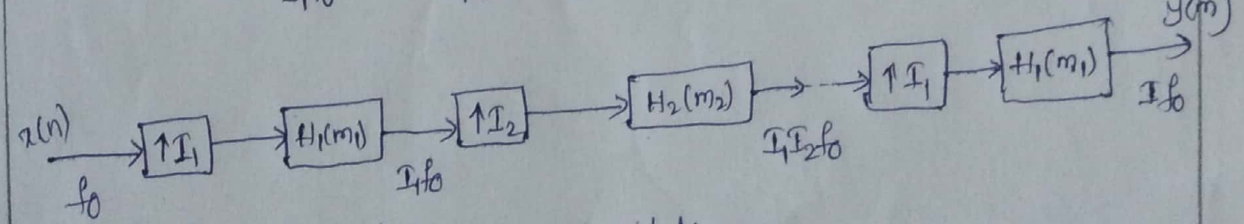
Similarly, if the interpolation factor $I >> 1$, then express I as a product of positive integers as

$$\boxed{I = \prod_{i=1}^{N} I_i}$$

then each interpolator I_i is implemented and cascaded to get N stages of implementation and filtering as shown in fig.







fig(2): multistage Interpolation

$$I = I_1 I_2 \cdots I_I = \prod_{i=1}^{I} I_i$$

If the Sampling rate alteration system is designed as a cascade system, the computational ~~effective~~ efficiency is improved significantly. The reasons for using multistage structures are as follows:

1. multistage system requires less computation
2. storage space required is less.
3. Filter design problem is simple.
4. Finite word length Effects are less.

The demerits of the system are that proper control structure is required in implementing the system and proper values of I should be chosen.