UNIT-III

Laplace Transforms

The knowledge of Laplace Transform has in recent years become an essential part of mathematical background required of Engineers and scientists. This is because the transform methods provide an easy means for the solution of many problems arising in Engineering.

The method of Laplace tranfdms has the advantage of directly giving the solution of differential Equations with given boundary values. The ready tables of Laplace transforms reduce the problem of Solving differential Equations to mere algebraic manipulation.

Some of applications of Laplace transform are Steady State analysis of electrical Circuits, analysis of impact and mechanical Vibrations, Problems such as deflection of beams etc.

Defination :-

Let $f(t)$ be a given function and defined for all positive values of 't'. Then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ or $\bar{f}(s)$ and is defined by

$$\boxed{L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

Here the parameter 's' is a real or Complex number.

Formulae

1) $L\{K\} = \frac{K}{s}$ where 'K' is a constant
2) $L\{e^{at}\} = \frac{1}{s-a}$ if $s-a>0$.
3) $L\{e^{-at}\} = \frac{1}{s+a}$ if $s+a>0$.
4) $L\{\sin at\} = \frac{a}{s^2+a^2}$
5) $L\{\cos at\} = \frac{s}{s^2+a^2}$
6) $L\{\sinh at\} = \frac{a}{s^2-a^2}$
7) $L\{\cosh at\} = \frac{s}{s^2-a^2}$
8) $L\{t^n\} = \frac{n!}{s^{n+1}}$ if 'n' is a positive integer

1) Find the Laplace transform of $e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$

Sol:- Given $f(t) = e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$

Then $L\{f(t)\} = L\{e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9\}$

By Linearity property

$L\{f(t)\} = L\{e^{3t}\} - 2L\{e^{-2t}\} + L\{\sin 2t\} + L\{\cos 3t\} + L\{\sinh 3t\} - 2L\{\cosh 4t\} + 9.L\{1\}$

$$= \frac{1}{s-3} - 2\left(\frac{1}{s+2}\right) + \frac{2}{s^2+2^2} + \frac{s}{s^2+3^2} + \frac{3}{s^2-3^2} - 2\left\{\frac{s}{s^2-4^2}\right\} + 9\left\{\frac{1}{s}\right\}$$

$$= \frac{1}{s-3} - \frac{2}{s+2} + \frac{2}{s^2+4} + \frac{s}{s^2+9} + \frac{3}{s^2-9} - \frac{2s}{s^2-16} + \frac{9}{s}$$

Note :- While finding the L.T. of elementary functions, it can be noticed that the integral exists under certain conditions, such as $s>0$ or $s>a$ etc. In general, the func $f(t)$ must satisfy the following conditions for existance of L.T

1) The func $f(t)$ must be piece-wise continuous or sectionally continuous in any limited interval $0<a \leq t \leq b$
2) The func $f(t)$ is of exponential order

Linearity Property :-

Statement :- If a, b, c be any constant and f, g, h are any functions of 't' then

$$L\{af(t)+bg(t)+ch(t)\} = a\,L\{f(t)\}+b\,L\{g(t)\}+c\,L\{h(t)\}$$

Proof :- $L\{af(t)+bg(t)+ch(t)\} = \int_0^\infty e^{-st}\{af(t)+bg(t)+ch(t)\}\,dt$ [$\because$ by def of L.T]

$$= \int_0^\infty e^{-st}a\,f(t)\,dt + \int_0^\infty e^{-st}b\,g(t)\,dt + \int_0^\infty e^{-st}c\,h(t)\,dt$$

$$= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt + c\int_0^\infty e^{-st}h(t)\,dt.$$

$$= a\,L\{f(t)\}+b\cdot L\{g(t)\}+c\,L\{h(t)\}$$

Hence proved.

2) find L.T. of the following

$\rightarrow (t^2+1)^2$

Sol:- $(t^2+1)^2 = t^4+2t^2+1$

$= L\{t^4+2t^2+1\} \Rightarrow L\{t^4\}+2L\{t^2\}+L\{1\}$

$= \frac{4!}{s^5} + 2\frac{2!}{s^3} + \frac{1}{s} \Rightarrow \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}.$

$= \frac{s^4+4s^2+24}{s^5}$

$\rightarrow (\sin t + \cos t)^2 = \sin^2 t + \cos^2 t + 2\sin t\cos t.$

$= 1+2\sin t\cos t.$

$L\{(\sin t+\cos t)^2\} = L\{1+2\sin t\cos t\}$

$= L\{1+\sin 2t\}$

$= L\{1\} + L\{\sin 2t\}.$

$= \frac{1}{s} + \frac{2}{s^2+4} \Rightarrow \frac{s^2+2s+4}{s(s^2+4)}.$

$\rightarrow \cos 3t \sin 5t.$

Sol:- The given func is in the form of ~~sin~~ $\cos A \sin B$.

we know that $2\cos A\sin B = \sin(A+B) + \sin(A-B)$

$\therefore \cos A \sin B = \frac{1}{2}[\sin(A+B) \mp \sin(A-B)]$

$\cos 3t \sin 5t = \frac{1}{2}[\sin(3t+5t) \mp \sin(3t-5t)]$

$= \frac{1}{2}[(\sin 8t) - \sin(-2t)]$

$= \frac{1}{2}[(\sin 8t) + \sin 2t]$ $\quad [\because \sin(-\theta) = -\sin\theta]$

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$$L\{f(t)\} = L\left\{\frac{1}{2}(\sin 2t + \sin 8t)\right\} = \frac{1}{2}\left[L(\sin 2t) + L(\sin 8t)\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^2+2^2} + \frac{8}{s^2+8^2}\right]$$

$$L\{f(t)\} = \frac{1}{s^2+4} + \frac{4}{s^2+64}$$

H/w

$\rightarrow$ $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$. $\left[\text{Ans: } \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^2+9}\right]$

$\rightarrow$ $\sin 2t \cos 3t$ $\left[\text{Ans: } \frac{2(s^2-5)}{(s^2+25)(s^2+1)}\right]$

$\rightarrow$ $L\{3\cos 3t \cdot \cos 4t\}$ $\left[\text{Ans:- } \frac{3s(s^2+25)}{(s^2+1)(s^2+49)}\right]$

First shifting theorem

Theorem :- If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at}f(t)\} = \bar{f}(s-a)$, $s-a>0$

Proof :- By def. of L.T

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= \int_0^\infty e^{-ut} f(t)\, dt \qquad [\text{Let } u = s-a)]$$

$$[\because \text{by def of L.T}]$$

$$= \bar{f}(u)$$

$$= \bar{f}(s-a).$$

→ find the L.T. of

→ $e^{-t}\cos 2t$

Sol!- we know that $L\{\cos 2t\} = \dfrac{s}{s^2+4}$

By first shifting theorem, we get.

$$L\{e^{-t}\cos 2t\}_{s\to s+1} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+1+4} = \frac{s+1}{s^2+2s+5}$$

→ $e^{-3t}(2\cos 5t - 3\sin 5t)$

Sol!- we know that $L\{2\cos 5t - 3\sin 5t\} = 2\left[L\{\cos 5t\}\right] - 3\{L\{\sin 5$

$$= 2\left[\frac{s}{s^2+25}\right] - 3\left[\frac{5}{s^2+25}\right] = \frac{2s-15}{s^2+25}$$

By first shifting thm,

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = \left(\frac{2s-15}{s^2+25}\right)_{s\to s+3} = \frac{2(s+3)-15}{(s+3)^2+25}$$

$$= \frac{2s-9}{s^2+6s+34}$$

$\rightarrow L\{(t+3)^2 e^t\}$

Sol:- $L\{(t+3)^2\} = L\{t^2+6t+9\} = L\{t^2\}+6L\{t\}+9L\{1\}$

$= \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} = F(s)$

By first shifting thm,

$L\{e^t(t+3)^2\} = \left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]_{s\to s-1}$.

$= \frac{2}{(s-1)^3} + \frac{6}{(s-1)^2} + \frac{9}{s-1} = \frac{9s^2-12s+5}{(s-1)^3}$.

$\rightarrow L\{e^{-t}\cos^2 t\}$.

Sol:- $L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2}\left[L\{1\} + L\{\cos 2t\}\right]$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$

$= \frac{s^2+2}{s(s^2+4)}$

By first shifting thm.

$L\{e^{-t}\cos^2 t\} = \left[\frac{s^2+2}{s(s^2+4)}\right]_{s\to s+1} = \frac{(s+1)^2+2}{(s+1)[(s+1)^2+4]}$

$= \frac{s^2+2s+3}{(s+1)(s^2+2s+5)}$

$\rightarrow L\{e^{4t}\sin 2t\cos t\}$

Sol:- $L\{\sin 2t\cos t\} = L\left\{\frac{1}{2}(2\sin 2t\cos t)\right\}$

$= L\left\{\frac{1}{2}(\sin 3t+\sin t)\right\}$. ($\because$ SinA cosB formula

$= \frac{1}{2}\left\{L(\sin 3t) + L(\sin t)\right\}$

$= \frac{1}{2}\left\{\frac{3}{s^2+9} + \frac{1}{s^2+1}\right\}$

By first shifting thm,

$$L\{e^{4t}\sin 2t\cos t\} = \frac{1}{2}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right]_{s\to s-4}$$

$$=\frac{1}{2}\left[\frac{3}{(s-4)^2+9}+\frac{1}{(s-4)^2+1}\right]$$

$$=\frac{1}{2}\left[\frac{3}{s^2-8s+25}+\frac{1}{s^2-8s+17}\right]$$

H/w

1) $e^{-t}(3\sin 2t-5\cosh 2t)$

2) $e^{-at}\sinh bt$

3) $L\{e^{3t}\sin^2 t\}$.

4) $e^{-3t}(\cos 4t+3\sin 4t)$.

Multiplication by 't' :-

Formulae :- If $f(t)$ is sectionally continuous and of Exponential order and if $L\{f(t)\} = \bar{f}(s)$ then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right]$$ where $n=1,2,3,\ldots$

→ Evaluate $L\{t \sin at\}$

Sol:- we know that $L\{\sin at\} = \frac{a}{s^2+a^2} = \bar{f}(s)$

[∵ by def of mult. by 't']

$$L\{t \sin at\} = (-1)\frac{d}{ds}\left[\bar{f}(s)\right]$$

$$= (-1)\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -a\frac{d}{ds}\left[(s^2+a^2)^{-1}\right].$$

$$= (-a)(-1)(s^2+a^2)^{-2} \cdot 2s$$

$$= \frac{2as}{(s^2+a^2)^2}$$

→ Evaluate $L\{t^2 \cos 3t\}$.

Sol:- we know that $L\{\cos 3t\} = \frac{s}{s^2+9} = \bar{f}(s)$

$$L\{t^2 \cos 3t\} = (-1)^2\frac{d^2}{ds^2}\left[\bar{f}(s)\right]$$

$$= (-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+9}\right]$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right] \Rightarrow \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+9}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+9)(1) - s(2s)}{(s^2+9)^2}\right] \qquad \left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+9-2s^2}{(s^2+9)^2}\right] \Longrightarrow \frac{d}{ds}\left[\frac{9-s^2}{(s^2+9)^2}\right]$$

$$= \left[\frac{(s^2+9)^2(-2s) - (9-s^2)\cdot 2(s^2+9)(2s)}{(s^2+9)^4}\right]$$

$$= \left[\frac{-2s(s^2+9)^2 - 4s(s^2+9)(9-s^2)}{(s^2+9)^4}\right]$$

$$= \frac{2s(s^2+9)\left[-(s^2+9) - 2(9-s^2)\right]}{(s^2+9)^4}$$

$$= \frac{2s(s^2+9)\left[-s^2-9-18+2s^2\right]}{(s^2+9)^4}$$

$$= \frac{2s(s^2-27)}{(s^2+9)^3} \Longrightarrow \frac{2s^3-54}{(s^2+9)^3}.$$

$\longrightarrow$ Evaluate $L\{t e^{2t} \sin 3t\}$

Sol:- we know that $L\{\sin 3t\} = \frac{3}{s^2+9}$

also $L\{e^{2t}\sin 3t\} = \left[\frac{3}{s^2+9}\right]_{s\to(s-2)}$

$$= \frac{3}{(s-2)^2+9}$$

$$L\{e^{2t}\sin 3t\} = \frac{3}{s^2-4s+13}$$

$$\Rightarrow L\{te^{2t}\sin 3t\} = (-1)\frac{d}{ds}\left[\frac{3}{s^2-4s+13}\right]$$

$$= (-3)\frac{d}{ds}\left[(s^2-4s+13)^{-1}\right]$$

$$= (-3)\left[(-1)(s^2-4s+13)^{-2}\right].(2s-4)$$

$$= 3\left[\frac{2s-4}{(s^2-4s+13)^2}\right] = \frac{6s-12}{(s^2-4s+13)^2}$$

→ Evaluat $L\{t^3 e^{2t} \sin t\}$

Sol! - $L\{\sin t\} = \frac{1}{s^2+1}$

$$L\{e^{2t}\sin t\} = \left\{\frac{1}{s^2+1}\right\}_{s\to s-2} = \frac{1}{(s-2)^2+1}$$

$$= \frac{1}{s^2-4s+4+1} = \frac{1}{s^2-4s+5}$$

$$L\{t^3 e^{2t}\sin t\} = (-1)^3\frac{d^3}{ds^3}\left\{\frac{1}{s^2-4s+5}\right\}$$

$$= (-1)\frac{d^2}{ds^2}\left[\frac{d}{ds}\left(\frac{1}{s^2-4s+5}\right)\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{d}{ds}(s^2-4s+5)^{-1}\right]$$

$$= -\frac{d^2}{ds^2}\left[(-1)(s^2-4s+5)^{-2}.(2s-4)\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{2s-4}{(s^2-4s+5)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{2s-4}{(s^2-4s+5)^2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2-4s+5)^2(2)-(2s-4)(2(s^2-4s+5)(2s-4))}{(s^2-4s+5)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{-3s^2+12s-11}{(s^2-4s+5)^3}\right]$$

$$= \frac{241(s-2)(s^2-4s+3)}{(s^2-4s+5)^4}$$

10 :- 1) $L\{t^2 \sin 2t\}$

2) $L\{t^2 e^{2t} \cos t\}$.

3) $L\{t \sin 3t \cos 2t\}$.

Division by 't'

Statement :- If $L\{f(t)\} = \bar{f}(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\,ds$

Provided the integral Exists.

1) find the L.T. of

→ $\frac{\sin t}{t}$

Sol:- we know that $L\{\sin t\} = \frac{1}{s^2+1} = \bar{f}(s)$

By division by 't' we have.

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(s)\,ds$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \frac{1}{s^2+1}\,ds = \left(\tan^{-1}s\right)_s^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}s.$$

$$= \frac{\pi}{2} - \tan^{-1}s. \qquad [\because \text{by formula}]$$

$$= \cot^{-1}s$$

→ $\frac{\sin 3t \cos t}{t}$

Sol:- $\sin 3t \cos t = \frac{1}{2}(2\sin 3t \cos t) = \frac{1}{2}[\sin(4t) + \sin 2t]$

$$L\{\sin 3t \cos t\} = \frac{1}{2}[L(\sin 4t) + L(\sin 2t)]$$

$$= \frac{1}{2}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right] = \bar{f}(s)$$

$$L\left\{\frac{\sin 3t \cos t}{t}\right\} = \int_s^{\infty} \bar{f}(s)\,ds = \frac{1}{2}\int_s^{\infty}\left(\frac{4}{s^2+16} + \frac{2}{s^2+4}\right) ds.$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+4}{s^2+9}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right) = \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)^{-1}$$

$$= \log\left(\frac{s^2+9}{s^2+4}\right)^{1/2}.$$

$\longrightarrow$ find $L\left\{\int_0^t \frac{1-e^{-t}}{t}\,dt\right\}$.

Sol!- $L(1-e^{-t}) = L(1) - L(e^{-t})$

$$= \frac{1}{s} - \frac{1}{s+1} = \bar{f}(s)$$

$$L\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right) ds.$$

$$= \left[\log s - \log(s+1)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s+1}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1}{1+1/s}\right)\right]_s^\infty$$

$$= \log\left(\frac{1}{1+0}\right) - \log\left(\frac{1}{1+1/s}\right)$$

$$= 0 - \log\left(\frac{s}{s+1}\right)$$

$$= \log\left(\frac{s}{s+1}\right)^{-1}$$

$$= \log\left(\frac{s+1}{s}\right)$$

$\rightarrow \dfrac{\cos 4t \sin 2t}{t}$

Sol:- $L(\cos 4t \sin 2t) = \frac{1}{2} L\{2\cos 4t \sin 2t\}$

$= \frac{1}{2} L\{\sin 6t - \sin 2t\}$

$[\because 2\cos A \sin B = \sin(A+B) - \sin(A-B)]$

$= \frac{1}{2}\left[\frac{6}{s^2+6^2} - \frac{2}{s^2+2^2}\right]$

$= \frac{3}{s^2+6^2} - \frac{1}{s^2+2^2} = \bar{f}(s)$

$L\left\{\frac{\cos 4t \sin 2t}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left[\frac{3}{s^2+6^2} - \frac{1}{s^2+2^2}\right] ds.$

$= \left[3 \cdot \frac{1}{6}\tan^{-1}\frac{s}{6} - \frac{1}{2}\tan^{-1}\frac{s}{2}\right]_s^\infty$

$\left[\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{2}\right) - \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{6}\right) - \tan^{-1}\left(\frac{s}{2}\right)\right]$

$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{s}{6}\right)\right]$

$$= \frac{1}{2}\left(\tan^{-1}\frac{s}{4} + \tan^{-1}\frac{s}{2}\right)_s$$

$$= \frac{1}{2}\left[\left(\tan^{-1}\infty + \tan^{-1}\infty\right) - \left(\tan^{-1}\frac{s}{4} + \tan^{-1}\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(\tan^{-1}\frac{s}{4} + \tan^{-1}\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}(\pi) - \frac{1}{2}\left[\tan^{-1}\frac{s}{4} + \tan^{-1}\frac{s}{2}\right)$$

$$= \frac{\pi}{2} - \frac{1}{2}\left(\tan^{-1} s/4 + \tan^{-1} s/2\right)$$

$\rightarrow$ $\dfrac{\cos 2t - \cos 3t}{t}$

Sol:- $L\{\cos 2t - \cos 3t\} = \dfrac{s}{s^2+4} - \dfrac{s}{s^2+9} = \bar{f}(s)$

$$L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds.$$

$$= \frac{1}{2}\left[\int_s^\infty \left(\frac{2s}{s^2+4} - \frac{2s}{s^2+9}\right)\right] ds.$$

$$= \frac{1}{2}\left[\log(s^2+4) - \log(s^2+9)\right] \qquad \left[\because \int \frac{f'(x)}{f(x)} = \log(f(x))\right.$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1+4/s^2}{1+9/s^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{1+4/s^2}{1+9/s^2}\right)\right]$$

Evaluation of Integrals by L.T

Evaluate $\int_0^\infty t e^{-3t} dt$.

Sol!- Given $\int_0^\infty t e^{-3t} dt$ ——(1)

By def of L.T $\int_0^\infty e^{-st} f(t) dt$ ——(2)

∴ from (1) & (2) $s=3$. $f(t)=t$.

Now $L\{f(t)\} = L\{t\} = \frac{1}{s^2}$ —— (3)

Now Sub. $s=3$ in (3)

$$\therefore \int_0^\infty t e^{-3t} dt = \frac{1}{s^2} = \frac{1}{3^2} = \frac{1}{9}$$

Also

Evaluate $\int_0^\infty e^{-4t} \sin 3t \, dt$.

→ using L.T Evaluate $\int_0^\infty t e^{-t} \sin t \, dt$.

Sol!- Here $s=1$.

$$\int_0^\infty t e^{-st} \sin t \, dt = \int_0^\infty e^{-st} (t \sin t) \, dt.$$

Here func is mul. by 't' So apply multiplication by 't' method

$$L\{\sin t\} = \frac{1}{s^2+1} = F(s)$$

By mul. by t.

$$L\{t \sin t\} = (-1) \frac{d}{ds} F(s) \Rightarrow (-1) \frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -1 \left[\frac{d}{ds} (s^2+1)^{-1}\right]$$

$\frac{d}{ds} - [(-1)(s^2+1)^{-2} \cdot 2s]$

$$L\{t\sin t\} = \frac{2s}{(s^2+1)^2}$$

$$\therefore \int_0^\infty t e^{-st} \sin t \, dt = \frac{2s}{(s^2+1)^2} \quad \text{Put } s=1.$$

$$= \frac{2(1)}{(1^2+1)^2} = \frac{2}{4} = \frac{1}{2}.$$

→ Show that $\int_0^\infty t^2 e^{-4t} \sin 2t \, dt = \frac{11}{500}$

sol!- we have $s=4$.

$$\int_0^\infty e^{-4t} t^2 \sin 2t \, dt = \quad \text{Here } f(t) = t^2 \sin 2t \, dt.$$

$$L\{\sin 2t\} = \frac{2}{s^2+4} = \bar{f}(s)$$

$$L\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right) \quad [\because \text{by mul. by } t]$$

$$= 1 \frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right]$$

$$= 2 \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s^2+4}\right)\right]$$

$$= 2 \frac{d}{ds}\left[\frac{d}{ds}(s^2+4)^{-1}\right] \Rightarrow \frac{2d}{ds}\left[-1(s^2+4)^{-2} 2s\right]$$

$$= -2 \frac{d}{ds}\left[\frac{2s}{(s^2+4)^2}\right]$$

$$= -4 \frac{d}{ds}\left(\frac{s}{(s^2+4)^2}\right) \quad \left(\text{apply } \frac{u}{v} \text{ formula}\right)$$

$$= -4\left[\frac{(s^2+4)^2(1) - s^2(s^2+4)2s}{(s^2+4)^4}\right]$$

$$= -4\left(\frac{(s^2+4)^2 - 4s^2(s^2+4)}{(s^2+4)^4}\right]$$

$$= -4\left[\cancel{(s^2+4)\{1-4s^2}\right.$$

$$= -4\left[\frac{\cancel{(s^2+4)}\{(s^2+4) - 4s^2\}}{(s^2+4)^{\cancel{4}3}}\right]$$

$$= -4\left(\frac{s^2+4-4s^2}{(s^2+4)^3}\right) \Rightarrow -4\left[\frac{4-3s^2}{(s^2+4)^3}\right]$$

$$= \left[\frac{12s^2-16}{(s^2+4)^3}\right].$$

Now sub $s=4$ we get.

$$\int_0^\infty t^2 e^{-4t}\sin 2t \, dt = \frac{12(16)-16}{(16+4)^3} = \frac{11}{500}$$

A/w

ST $\rightarrow \int_0^\infty te^{-3t}\sin t \, dt = \frac{3}{50}$.

$\rightarrow \int_0^\infty t^3 e^{-t}\sin t \, dt = 0$.

Evaluate $\int_0^{\infty} \frac{e^{-t} - e^{-2t}}{t} \, dt$.

Sol:- Here $s=0$ by def of L.T.

The given transform is in the form of division by 't'

So:- $L\{e^{-t} - e^{-2t}\} = \frac{1}{s+1} - \frac{1}{s+2} = \bar{f}(s)$

$$L\left\{\frac{e^{-t} - e^{-2t}}{t}\right\} = \int_s^{\infty} \bar{f}(s)\, ds.$$

$$= \int_s^{\infty} \left(\frac{1}{s+1} - \frac{1}{s+2}\right) ds.$$

$$= \left[\log(s+1) - \log(s+2)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{s+1}{s+2}\right)\right]_s^{\infty} = \log\left[\frac{1+1/s}{1+2/s}\right]_s^{\infty}.$$

$$= \log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{1+1/s}{1+2/s}\right).$$

$$= \log 1 - \log\left(\frac{s+1}{s+2}\right)$$

$$= -\log\left(\frac{s+1}{s+2}\right) \Rightarrow \log\left(\frac{s+1}{s+2}\right)^{-1} \Rightarrow \log\left(\frac{s+2}{s+1}\right)$$

$\therefore$ Now put $s=0$ we get

$$\int_0^{\infty} \frac{e^{-t} - e^{-2t}}{t}\, dt = \log\left(\frac{0+2}{0+1}\right) = \log 2$$

H/w Evaluate $\int_0^{\infty} \frac{\cos at - \cos bt}{t}\, dt$.

Laplace Transform of Periodic functions

Theorem :- If $f(t)$ is a periodic function with period 'T'

then $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$.

→ find the L.T. of Square-wave function of Period '2a' defined as

$f(t) = k$ when $0<t<a$.

$= -k$ when $a<t<2a$

Sol:- Since $f(t)$ is a periodic function with Period $T = 2a$.

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt \qquad [\because \text{by def}]$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt.$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t)\, dt + \int_a^{2a} e^{-st} f(t)\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a k e^{-st}\, dt + \int_a^{2a} (-k) e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[k \int_0^a e^{-st}\, dt - k \int_a^{2a} e^{-st}\, dt \right].$$

$$= \frac{1}{1-e^{-2as}} \left[k \left[\frac{e^{-st}}{-s} \right]_0^a - k \left[\frac{e^{-st}}{-s} \right]_a^{2a} \right]$$

$$= \frac{k}{T(1-e^{-sT})} \int_0^T \overset{u}{t} . \overset{v}{e^{-st}} \, dt.$$

$\because \int uv\,dx = u\int v\,dx - \int(u'\int v\,dx)\,dx$

$$= \frac{k}{T(1-e^{-sT})} \left[t.\left(\frac{e^{-st}}{-s}\right)_0^T - \int_0^T 1.\left[\frac{e^{-st}}{-s}\right] dt \right]$$

$$= \frac{k}{T(1-e^{-sT})} \left[t.\left(\frac{e^{-st}}{-s}\right)_0^T + \left(\frac{e^{-st}}{-s^2}\right)_0^T \right]$$

$$= \frac{k}{T(1-e^{-sT})} \left[\frac{e^{-st}\,t}{-s} - \frac{e^{-st}}{s^2} \right]_0^T$$

$$= \frac{k}{T(1-e^{-sT})} \left[\left(\frac{e^{-sT}T}{-s} - \frac{e^{-sT}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right]$$

$$= \frac{k}{T(1-e^{-sT})} \left[\frac{e^{-st}T}{-s} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{ke^{-st}\cancel{T}}{\cancel{T}s(1-e^{-st})} - \frac{ke^{-st}}{Ts^2(1-e^{-st})} + \frac{k}{Ts^2(1-e^{-st})}.$$

$$= \frac{ke^{-st}}{s(1-e^{-st})} - \frac{k}{Ts^2}\left[\frac{e^{-st}+1}{(1-e^{-st})}\right].$$

→ find $L\{f(t)\}$ where $f(t)$ is a periodic function of period 2π and it is given by

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi. \end{cases}$$

Sol:- since $f(t)$ is a periodic fun^c with period 2π

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} f(t)\,dt + \int_{\pi}^{2\pi} e^{-st} f(t)\,dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt + \int_{\pi}^{2\pi} e^{-st} \cdot 0\,dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt \right]$$

$$\left[\because \int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1} (-s \sin t - \cos t) \right]_0^{\pi} \quad \left[\text{Here } \begin{array}{l} a=-s \\ b=1 \end{array} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-\pi s}}{s^2+1} (-\pi \sin \pi - \cos \pi) - \frac{e^{-0}}{s^2+1} (-s \sin 0 - \cos 0) \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-\pi s}}{s^2+1} (-\pi(0) - (-1)) - \frac{1}{s^2+1} (-s(0) - 1) \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-\pi s}}{s^2+1} (1) + \frac{1}{s^2+1} \right]$$

$$= \frac{e^{-\pi s}+1}{(s^2+1)(1-e^{-2\pi s})} = \frac{\cancel{1+e^{-\pi s}}}{(s^2+1)(1-e^{-\pi s})\cancel{(1+e^{-\pi s})}}$$

$$= \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

→ If $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \end{cases}$ is a Periodic func with

Period '2'

Sol:- $f(t)$ is a periodic func with period '2'

$$L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} f(t)\,dt + \int_1^2 e^{-st} f(t)\,dt \right].$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} (1)\,dt + \int_1^2 e^{-st} (-1)\,dt \right].$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st}\,dt - \int_1^2 e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\left(\frac{e^{-st}}{-s}\right)_0^1 - \left(\frac{e^{-st}}{-s}\right)_1^2 \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\left(\frac{e^{-s}-1}{-s}\right) - \left(\frac{e^{-2s}-e^{-s}}{-s}\right) \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{1}{s}(e^{-s}-1) + \frac{1}{s}(e^{-2s}-e^{-s}) \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{s}\left[-e^{-s}+1+e^{-2s}-e^{-s}\right) \right]$$

$$= \frac{1}{s(1-e^{-2s})} \left[1-2e^{-s}+e^{-2s}\right]$$

$$= \frac{1}{s(1-e^{-2s})} (1-e^{-s})^2 \Rightarrow \cancel{\frac{(1-e^{-s})(1+e^{-s})}{s(1-e^{-}}}$$

$$= \frac{(1-e^{-s})^{\cancel{2}}}{s\cancel{(1-e^{-s})}(1+e^{-s})} \Rightarrow \frac{(1-e^{-s})}{s(1+e^{-s})}$$

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Inverse Laplace Transform

Inverse L.T is useful in solving differential Equations without finding the general solution and arbitrary constants.

Defination :- If $\bar{f}(s)$ is the Laplace transform of a func $f(t)$ then $f(t)$ is called the inverse L.T. of $\bar{f}(s)$ and is denoted by $L^{-1}\{\bar{f}(s)\}$. L^{-1} is called the Inverse Laplace Transform Operator.

Formulae

	$\bar{f}(s)$	$L^{-1}\{\bar{f}(s)\} = f(t)$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
3)	$\frac{1}{s-a}$	e^{at}
4)	$\frac{1}{s+a}$	e^{-at}
5)	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
6)	$\frac{s}{s^2+a^2}$	$\cos at$
7)	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
8)	$\frac{s}{s^2-a^2}$	$\cosh at$

Note :- Inverse L.T of a given function $F(s)$ can be obtained either by use of the above standard results & by splitting the given func into partial fractions and then applying above results.

→ find the Inverse L.T. of

$$\frac{2s-5}{s^2-4}$$

Sol:- $$L^{-1}\left\{\frac{2s-5}{s^2-4}\right\} = L^{-1}\left\{\frac{2s}{s^2-4}\right\} - L^{-1}\left\{\frac{5}{s^2-4}\right\}$$

$$= 2L^{-1}\left\{\frac{s}{s^2-4}\right\} - 5L^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$= 2\cosh 2t - 5\cdot\frac{1}{2}\sinh 2t$$

→ $$\frac{2s+1}{s(s+1)}$$

$$L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\} = L^{-1}\left\{\frac{s+(s+1)}{s(s+1)}\right\}$$

$$= L^{-1}\left\{\frac{s}{s(s+1)} + \frac{s+1}{s(s+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+1} + \frac{1}{s}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s}\right\}$$

$$= e^{-t} + 1.$$

$\rightarrow L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$

Sol:- $\rightarrow L^{-1}\left\{\frac{3s}{4s^2+25}\right\} - 8L^{-1}\left\{\frac{1}{4s^2+25}\right\}$

$\rightarrow 3\, L^{-1}\left\{\frac{s}{4s^2+25}\right\} - 8\, L^{-1}\left\{\frac{1}{4s^2+25}\right\}$

$\rightarrow 3.\, L^{-1}\left\{\frac{s}{s^2+\left(\frac{5}{2}\right)^2}\right\} - 8L^{-1}\left\{\frac{1}{s^2+\left(\frac{5}{2}\right)^2}\right\}$

$\Rightarrow 3\cos\frac{5t}{2} - 2\left(\frac{2}{5}\right)\sin\frac{5t}{2}$

$\rightarrow L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}$

Sol:- $L^{-1}\left\{\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right\}$

$= L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$

$= 1 - 3t + 4\frac{t^2}{2!}$

$\rightarrow L^{-1}\left\{\frac{s^2+9s-9}{s^3-9s}\right\}$

Sol:- $L^{-1}\left\{\frac{s^2+9s-9}{s^3-9s}\right\} = L^{-1}\left\{\frac{(s^2-9)+9s}{s^3-9s}\right\}$

$= L^{-1}\left\{\frac{(s^2-9)+9s}{s(s^2-9)}\right\}$

$= L^{-1}\left\{\frac{s^2-9}{s(s^2-9)} + \frac{9s}{s(s^2-9)}\right\} \Rightarrow L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{9}{s^2-}\right.$

$= 1 + 3\sinh 3t.$

$$= \frac{1}{1-e^{-2as}} \left[\frac{k}{s} \left[-e^{-st} \right]_0^a + \frac{k}{s} \left(e^{-st} \right)_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{k}{s} \left[-e^{-as} + e^0 \right] + \frac{k}{s} \left[e^{-2as} - e^{-as} \right] \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{k}{s} \left[-e^{-as} + 1 + e^{-2as} - e^{-as} \right] \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{k}{s} \left[1 - 2e^{-as} + e^{-2as} \right] \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{k}{s} \left(1 - e^{-as} \right)^2 \right]$$

$$= \frac{1}{1-(e^{-as})^2} \left[\frac{k}{s} \left(1 - e^{-as} \right)^2 \right] \qquad [\because (a^2-b^2) = (a+b)(a-b)]$$

$$= \frac{1}{(1+e^{-as})\cancel{(1-e^{-as})}} \frac{k}{s} (1-e^{-as})^{\cancel{2}}$$

$$= \frac{k(1-e^{-as})}{s(1+e^{-as})}$$

→ find the L.T. of saw-toothed wave of period 'T'

given $f(t) = \frac{k}{T} t$, when $0 < t < T$.

Sol:- Since $f(t)$ is a periodic func with period T.

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot \frac{kt}{T}\, dt.$$