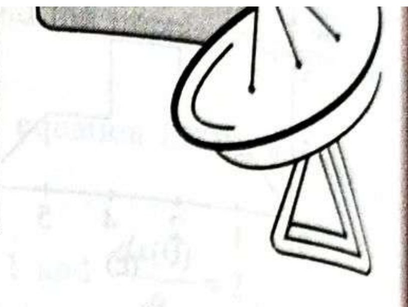
Z-Transforms



10.1 INTRODUCTION

We know that a linear time invariant (LTI) continuous-time system is represented by differential equations and a linear time invariant discrete-time system is represented by difference equations. The direct solution of higher order differential equations as well as higher order difference equations is quite tedious and time consuming. So usually they are solved by indirect methods. To solve the differential equations which are in time domain, they are first converted into algebraic equations in frequency domain using Laplace transforms, the algebraic equations are manipulated in s-domain and the result obtained in frequency domain is converted back into time domain using inverse Laplace transform. The Laplace transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the beginning of the process itself. The Z-transform plays the same role for discrete-time systems as that played by Laplace transform for continuous-time systems. The Z-transform is the discrete-time counterpart of the Laplace transform. It is the Laplace transform of the discretized version of the continuous-time signal $x(t)$. Just as the Laplace transform is a powerful mathematical tool to convert the differential equations into algebraic equations, the Z-transform is a powerful mathematical tool used to convert the difference equations into algebraic equations. To solve the difference equations which are in time domain, they are converted first into algebraic equations in z-domain using Z-transform, the algebraic equations are manipulated in z-domain and the result obtained is converted back into time domain using inverse Z-transform. Like the Laplace transform, the Z-transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and the initial conditions can be introduced in the beginning of the process itself. Also just as the Laplace transform is a very useful tool in the analysis of linear time invariant (LTI) systems, the Z-transform is a very useful tool in the analysis of linear shift invariant (LSI) systems. Like the Laplace transform, the Z-transform may be one-sided (unilateral) or two-sided (bilateral). As in the case of Laplace transform, it is the one-sided or unilateral Z-transform that is more useful because we mostly deal with causal sequences. Further, it is eminently suited for solving difference equations with initial conditions. Just as the range of values of s for which $X(s)$ converges is called the ROC of $X(s)$ the range of values of z for which $X(z)$ converges is called the ROC of $X(z)$.

The *bilateral* or *two-sided* Z-transform of a discrete-time signal or a sequence $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable.

The *one-sided* or *unilateral* Z-transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

Region of convergence (ROC)

For any given sequence, the Z-transform may or may not converge.

The set of values of z or equivalently the set of points in z-plane, for which $X(z)$ converges is called the region of convergence (ROC) of $X(z)$.

If there is no value of z (i.e. no point in the z-plane) for which $X(z)$ converges, then the sequence $x(n)$ is said to be having no Z-transform.

Advantages of Z-transform

1. The Z-transform converts the difference equations of a discrete-time system into linear algebraic equations so that the analysis become easy and simple.
2. Convolution in time domain is converted into multiplication in z-domain.
3. Z-transform exists for most of the signals for which discrete-time Fourier transform (DTFT) does not exist.

Limitation

Frequency domain response cannot be achieved and cannot be plotted.

10.3 Z-TRANSFORMS OF SOME COMMON SEQUENCES

10.3.1 The Unit-sample Sequence (The Unit-impulse Sequence) $[x(n) = \delta(n)]$

We know that

$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{for } n \neq 0$$

$\therefore$

$$X(z) = Z[x(n)] = Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n)\, z^{-n} = 1. \text{ (for all } z)$$

i.e. the ROC is the entire z-plane.

$$\boxed{\delta(n) \xleftrightarrow{\text{ZT}} 1 \quad \text{for all } z}$$

10.3.2 The Unit-step Sequence $[x(n) = u(n)]$

We know that

$$u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

$$\therefore \quad X(z) = Z[x(n)] = Z[u(n)] = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} 1z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

The above series converges if $|z^{-1}| < 1$, i.e. ROC is $|z| > 1$. So, the ROC is the exterior of the unit circle in the z-plane.

$$\boxed{u(n) \xleftrightarrow{\text{ZT}} \frac{z}{z-1} = \frac{1}{1-z^{-1}};\ \text{ROC};\ |z|>1}$$

10.3.3 The Unit-ramp Sequence $[x(n) = r(n) = nu(n)]$

We know that

$$r(n) = n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

$$\therefore \quad X(z) = Z[x(n)] = Z[r(n)] = \sum_{n=0}^{\infty} r(n)\, z^{-n} = \sum_{n=0}^{\infty} nz^{-n}$$

$$= 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + \cdots = z^{-1}(1 + 2z^{-1} + 3z^{-2} + \cdots)$$

$$= z^{-1}(1-z^{-1})^{-2} = z^{-1}\frac{1}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$$

The above series converges if $|z^{-1}| < 1$, i.e ROC is $|z| > 1$. So the ROC is the exterior of the unit circle in the z-plane.

$$\boxed{nu(n) \xleftrightarrow{\text{ZT}} \frac{z^{-1}}{[1-z^{-1}]^2} = \frac{z}{(z-1)^2};\ \text{ROC};\ |z|>1}$$

10.3.4 The Exponential Sequence $[x(n) = e^{-j\omega n}\,u(n)]$

$$x(n) = e^{-j\omega n}\,u(n) = e^{-j\omega n} \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

$$\therefore \quad X(z) = Z[e^{-j\omega n}\,u(n)] = \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} = \sum_{n=0}^{\infty} (z^{-1}e^{-j\omega})^n$$
$$= 1 + (z^{-1}e^{-j\omega}) + (z^{-1}e^{-j\omega})^2 + (z^{-1}e^{-j\omega})^3 + \cdots$$
$$= [1 - (z^{-1}e^{-j\omega})]^{-1} = \frac{1}{1 - z^{-1}e^{-j\omega}} = \frac{z}{z - e^{-j\omega}}$$

The above series converges if $|z^{-1}| < 1$, i.e. ROC $|z| > 1$.

$$\boxed{e^{-j\omega n}\,u(n) \xleftrightarrow{\text{ZT}} \frac{1}{1 - z^{-1}e^{-j\omega}} = \frac{z}{z - e^{-j\omega}};\ \text{ROC; } |z| > 1}$$

On similar lines,

$$\boxed{e^{j\omega n}\,u(n) \xleftrightarrow{\text{ZT}} \frac{1}{1 - z^{-1}e^{j\omega}} = \frac{z}{z - e^{j\omega}};\ \text{ROC; } |z| > |1|}$$

10.3.5 The Sinusoidal Sequence $[x(n) = \sin \omega n\, u(n)]$

$$x(n) = \sin \omega n\, u(n) = \sin \omega n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

$$\therefore \quad X(z) = Z[\sin \omega n\, u(n)] = \sum_{n=0}^{\infty} \sin \omega n\, z^{-n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) z^{-n} = \frac{1}{2j} \sum_{n=0}^{\infty} (e^{j\omega n} z^{-n} - e^{-j\omega n} z^{-n})$$
$$= \sum_{n=0}^{\infty} \frac{(z^{-1}e^{j\omega})^n - (z^{-1}e^{-j\omega})^n}{2j}$$
$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} (z^{-1}e^{j\omega})^n - \sum_{n=0}^{\infty} (z^{-1}e^{-j\omega})^n\right]$$

The above series converges for $|z^{-1}| < 1$, i.e $|z| > 1$.

$$X(z) = \frac{1}{2j}\left(\frac{1}{1-z^{-1}e^{j\omega}} - \frac{1}{1-z^{-1}e^{-j\omega}}\right) = \frac{1}{2j}\left(\frac{z}{z-e^{j\omega}} - \frac{z}{z-e^{-j\omega}}\right)$$

$$= \frac{1}{2j}\left[\frac{z(z-e^{-j\omega}) - z(z-e^{j\omega})}{(z-e^{j\omega})(z-e^{-j\omega})}\right] = \frac{1}{2j}\left[\frac{z(e^{j\omega}-e^{-j\omega})}{z^2 - z(e^{j\omega}+e^{-j\omega})+1}\right]$$

$$= \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}.$$

$\frac{e^{j\omega}+e^{-j\omega}}{2} = \cos\omega$

$e^{j\omega}+e^{-j\omega} = 2\cos\omega$

$$\boxed{\sin\omega n u(n) \xleftrightarrow{ZT} \frac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega + z^{-2}} = \frac{z\sin\omega}{z^2-2z\cos\omega+1};\ \text{ROC};\ |z|>1}$$

10.3.6 The Cosinusoidal Sequence $[x(n) = \cos\omega n u(n)]$

$$\cos\omega n u(n) = \cos\omega n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

$$Z[\cos\omega n u(n)] = Z\left[\frac{e^{j\omega n}+e^{-j\omega n}}{2}u(n)\right] = \frac{1}{2}\{Z[e^{j\omega n}u(n)] + Z[e^{-j\omega n}u(n)]\}$$

$$= \frac{1}{2}\left(\frac{z}{z-e^{j\omega}} + \frac{z}{z-e^{-j\omega}}\right) = \frac{1}{2}\left[\frac{z(z-e^{-j\omega})+z(z-e^{j\omega})}{(z-e^{j\omega})(z-e^{-j\omega})}\right];\ |z|>1$$

$$= \frac{1}{2}\left\{\frac{z[2z-(e^{j\omega}+e^{-j\omega})]}{z^2-z(e^{j\omega}+e^{-j\omega})+1}\right\} = \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1};\ \text{ROC};\ |z|>1$$

$$\boxed{\cos\omega n u(n) \xleftrightarrow{ZT} \frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}} = \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1};\ \text{ROC};\ |z|>1}$$

Laplace transform	*Z-transform*
1. It is used to analyse LTI continuous-time systems.	1. It is used to analyse LTI discrete-time systems.
2. It converts differential equations which are in time domain into algebraic equations in s-domain.	2. It converts difference equations which are in time domain into algebraic equations in z-domain.
3. It is a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the beginning of the process itself.	3. It is also a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the beginning of the process itself.
4. Laplace transform may be one-sided (unilateral) or two-sided (bilateral).	4. Z-transform also may be one-sided (unilateral) or two-sided (bilateral).
5. The range of values of s for which $X(s)$ converges is called ROC of $X(s)$.	5. The range of values of z for which $X(z)$ converges is called ROC of $X(z)$
6. The ROC of $X(s)$ consists of strips parallel to $j\omega$ axis in s-plane.	6. The ROC of $X(z)$ consists of a ring or disc in z-plane centred at the origin.
7. If the real part of s, i.e. $\sigma = 0$, then the Laplace transform becomes the continuous-time Fourier transform.	7. If the magnitude of z, i.e. $\|z\| = 1$, then the Z-transform becomes DTFT.
8. Convolution in time domain is equal to multiplication in s-domain.	8. Convolution in time domain is equal to multiplication in z-domain.

EXAMPLE 10.3 Prove that, for causal sequences, the ROC is the exterior of a circle of radius r.

Solution: Causal sequences are the sequences defined for only positive integer values of n and do not exist for negative times, i.e.,

$$x(n) = 0 \qquad \text{for } n < 0$$

Consider a causal sequence,

$$x(n) = r^n u(n)$$

From the definition of Z-transform of $x(n)$, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} r^n\, u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} r^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (rz^{-1})^n$$

The above summation converges for

$$|rz^{-1}| < 1, \text{ i.e. for } |z| > r$$

Hence, for the causal sequences, the ROC is the exterior of a circle of radius r.

EXAMPLE 10.6 Prove that the sequences

(a) $x(n) = a^n u(n)$ and (b) $x(n) = -a^n u(-n-1)$

have the same $X(z)$ and differ only in ROC. Also plot their ROCs.

Solution:

(a) The given sequence $a^n u(n)$ is a causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n < 0 \end{cases} \quad \text{because} \quad u(n) = \begin{cases} 1 & \text{for } n \ge 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$\therefore \quad Z[x(n)] = Z[a^n u(n)] = \sum_{n=0}^{\infty} a^n u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} [az^{-1}]^n = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

This is a geometric series of infinite length, and converges if $|az^{-1}| < 1$, i.e. if $|z| > |a|$.

$$\therefore \quad X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}; \text{ ROC; } |z| > |a|$$

which implies that the ROC is exterior to the circle of radius a as shown in Figure 10.1(a)

$$\boxed{a^n u(n) \xleftrightarrow{\text{ZT}} \frac{1}{1-az^{-1}} = \frac{z}{z-a}; \text{ ROC; } |z| > a}$$

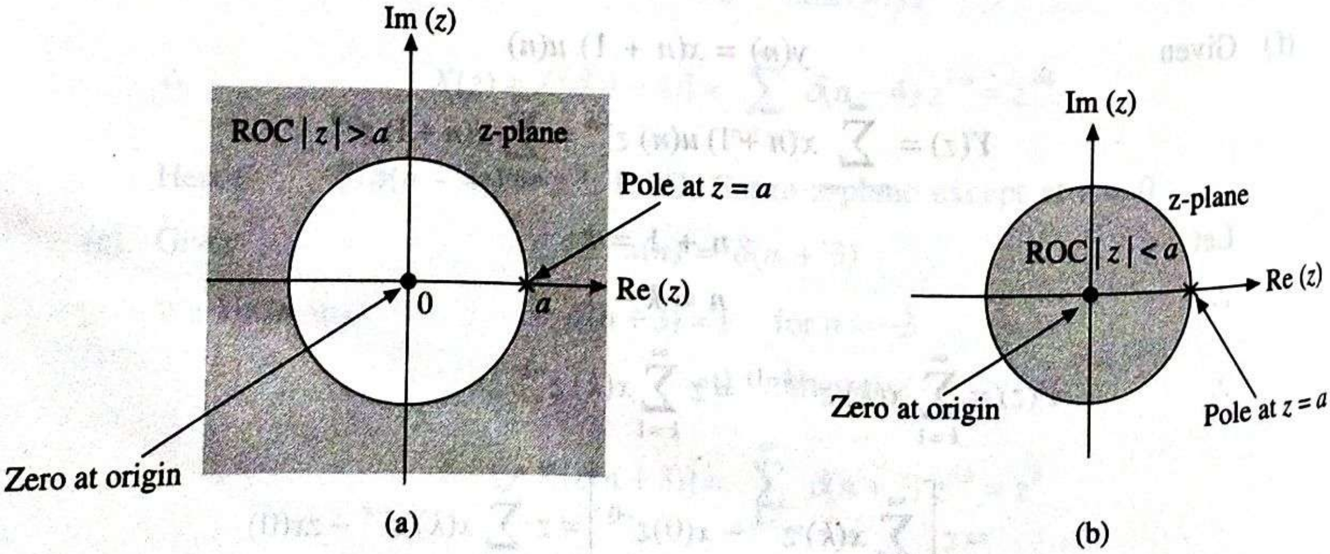


Figure 10.1 (a) ROC of $a^n u(n)$, (b) ROC of $-a^n u(-n-1)$.

(b) The given signal $x(n) = -a^n u(-n-1)$ is a non-causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} -a^n, & n \leq -1 \\ 0, & n \geq 0 \end{cases} \quad \text{because } u(-n-1) = \begin{cases} 1 & \text{for } n \leq -1 \\ 0 & \text{for } n \geq 0 \end{cases}$$

$$\therefore \quad X(z) = Z[-a^n u(-n-1)] = \sum_{n=-\infty}^{\infty} -a^n u(-n-1)\, z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= \sum_{n=1}^{\infty} -a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1} z)^n$$

This is a geometric series of infinite length and converges if $|a^{-1}z| < 1$ or $|z| < |a|$.

Hence

$$X(z) = -\left[\sum_{n=0}^{\infty} (a^{-1}z)^n - 1\right] = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1 - a^{-1}z} = -\frac{a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}; \text{ ROC; } |z| < |a|$$

That is, the ROC is the interior of the circle of radius a as shown in Figure 10.1(b).

From this example, we can observe that the Z-transform of the sequences $a^n u(n)$ and $-a^n u(-n-1)$ are same, even though the sequences are different. Only ROC differentiates them. Therefore, to find the correct inverse Z-transform, it is essential to know the ROC. The ROCs are shown in Figure 10.1[(a) and (b)].

In general, the ROC of a causal signal is $|z| > a$ and the ROC of a non-causal signal is $|z| < a$, where a is some constant.

EXAMPLE 10.8 Find the Z-transform and ROC of $x(n) = a^n u(-n)$.

Solution: The given sequence is a non-causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} a^n, & n \le 0 \\ 0, & n > 0 \end{cases}$$

$$\therefore \quad Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} a^n u(-n) z^{-n} = \sum_{n=-\infty}^{0} a^n z^{-n} = \sum_{n=0}^{\infty} (a^{-1} z)^n$$

The above series converges if $|a^{-1}z| < 1$ or $|z| < a$.

$$\therefore \quad X(z) = \frac{1}{1 - a^{-1}z} = \frac{a}{a - z}; \text{ ROC; } |z| < a$$

EXAMPLE 10.9 Determine the Z-transform and ROC of

$$x(n) = a^n u(n) - b^n u(-n-1)$$

Solution: The given sequence is a two-sided infinite duration sequence.

$$\therefore \quad Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} [a^n u(n) - b^n u(-n-1)]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} [az^{-1}]^n - \sum_{n=1}^{\infty} [b^{-1} z]^n$$

The first series converges if $|az^{-1}| < 1$ or $|z| > |a|$ and the second series converges if $|b^{-1}z| < 1$ or $|z| < |b|$. If $|b| < |a|$, the two ROCs do not overlap as shown in Figure 10.2(a) and the conditions $|z| > |a|$ and $|z| < |b|$ cannot be satisfied simultaneously, so the Z-transform $X(z)$ does not exist.

If $|b| > |a|$, the two ROCs overlap as shown in Figure 10.2(b) and the conditions $|z| > |a|$ and $|z| < |b|$ can be satisfied simultaneously, so $X(z)$ exists. Therefore, the ROC of $X(z)$ is $|a| < |z| < |b|$. This implies that *for an infinite duration two-sided signal, the ROC is a ring in the z-plane.*

$$\therefore \quad X(z) = \frac{z}{z-a} + \frac{z}{z-b}; \text{ ROC; } |a|<|z|<|b|$$

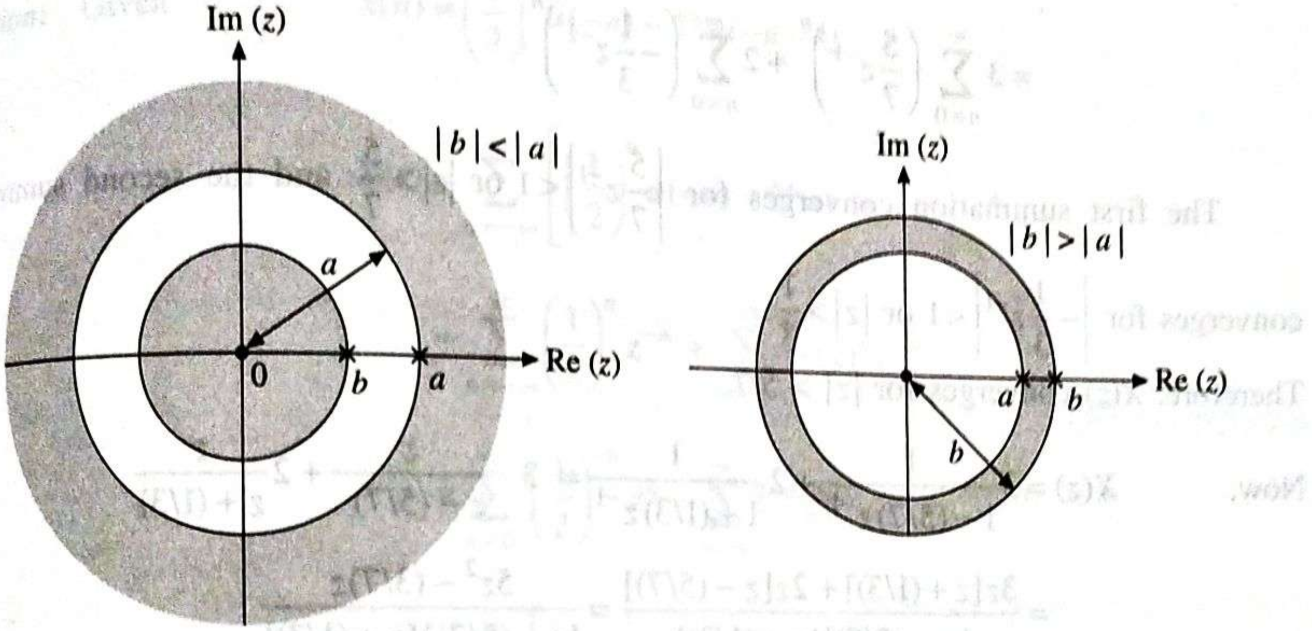


Figure 10.2 ROC of two-sided sequence for (a) $|b| < |a|$, and (b) $|b| > |a|$.

From the above discussion, the following conclusions can be drawn:

1. The ROC of a causal sequence is $|z| > |a|$, i.e. it is the exterior of a circle of radius a, where $z = a$ is the largest pole in $X(z)$.
2. The ROC of a non-causal sequence is $|z| < |a|$, i.e. it is the interior of a circle of radius a, where $z = a$ is the smallest pole in $X(z)$.
3. The ROC of a two-sided sequence is a ring in z-plane or the Z-transform does not exist at all.
4. The ROC of the sum of two or more sequences is equal to the intersection of the ROCs of those sequences.

EXAMPLE 10.10 Find the Z-transform and ROC of $X(z)$ for

$$x(n) = 3\left(\frac{5}{7}\right)^n u(n) + 2\left(-\frac{1}{3}\right)^n u(n)$$

Also find the pole-zero location.

Solution: Given $\quad x(n) = 3\left(\frac{5}{7}\right)^n u(n) + 2\left(-\frac{1}{3}\right)^n u(n)$

We have $$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty}\left[3\left(\frac{5}{7}\right)^n u(n) + 2\left(-\frac{1}{3}\right)^n u(n)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty}\left[3\left(\frac{5}{7}\right)^n z^{-n} + 2\left(-\frac{1}{3}\right)^n z^{-n}\right] = 3\sum_{n=0}^{\infty}\left(\frac{5}{7}\right)^n z^{-n} + 2\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^{-n}$$

$$= 3\sum_{n=0}^{\infty}\left(\frac{5}{7}z^{-1}\right)^n + 2\sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n$$

The first summation converges for $\left|\frac{5}{7}z^{-1}\right| < 1$ or $|z| > \frac{5}{7}$ and the second summation converges for $\left|-\frac{1}{3}z^{-1}\right| < 1$ or $|z| > \frac{1}{3}$.

Therefore, $X(z)$ converges for $|z| > 5/7$.

Now, $$X(z) = 3\frac{1}{1-(5/7)z^{-1}} + 2\frac{1}{1+(1/3)z^{-1}} = 3\frac{z}{z-(5/7)} + 2\frac{z}{z+(1/3)}$$

$$= \frac{3z[z+(1/3)] + 2z[z-(5/7)]}{[z-(5/7)][z+(1/3)]} = \frac{5z^2-(3/7)z}{[z-(5/7)][z+(1/3)]}$$

$$= \frac{5z^2-(3/7)z}{[z-(5/7)][z+(1/3)])} = \frac{5z[z-(3/35)]}{[z-(5/7)][z+(1/3)]}$$

The poles of $X(z)$ are at $z = (5/7)$ and $z = -(1/3)$. So the ROC is $|z| > (5/7)$, i.e. the ROC is the exterior of the circle with radius 5/7. The zeros are at $z = 0$ and $z = (3/35)$. The pole-zero location is shown in Figure 10.3.

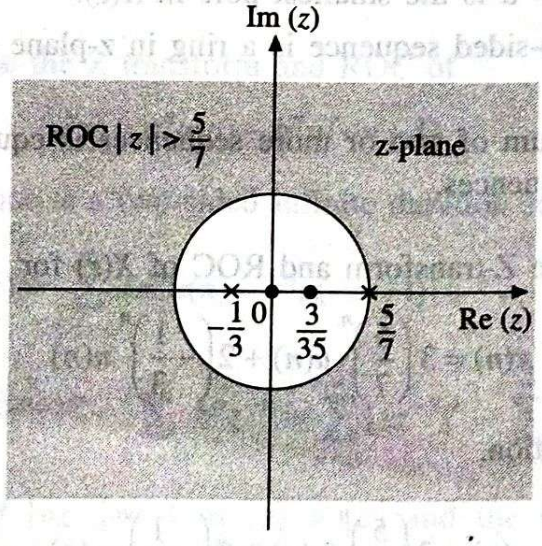


Figure 10.3 Pole-zero location and ROC for Example 10.10.

EXAMPLE 10.13 Find the Z-transform and ROC of

$$x(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{2}\right)^{2n} u(n)$$

Sketch the ROC and pole-zero location.

Solution: Given $x(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{2}\right)^{2n} u(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{4}\right)^n u(n)$

We have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} \left[2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{4}\right)^n u(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 2\left(\frac{5}{6}\right)^n u(-n-1)\, z^{-n} + \sum_{n=-\infty}^{\infty} 3\left(\frac{1}{4}\right)^n u(n)\, z^{-n} = \sum_{n=-\infty}^{-1} 2\left(\frac{5}{6}\right)^n z^{-n} + \sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n z^{-n}$$

$$= \sum_{n=1}^{\infty} 2\left[\left(\frac{5}{6}\right)^{-1} z\right]^n + \sum_{n=0}^{\infty} 3\left(\frac{1}{4} z^{-1}\right)^n$$

The first series converges if $(5/6)^{-1}\ z < 1$ or $z < (5/6)$ and the second series converges if $(1/4)\ z^{-1} < 1$ or $z > (1/4)$.

So the region of convergence for $X(z)$ is $(1/4) < z < (5/6)$, i.e. it is a ring with $(1/4) < z < (5/6)$.

$$\therefore \qquad X(z) = 2\left\{-1 + \sum_{n=0}^{\infty}\left[\left(\frac{5}{6}\right)^{-1} z\right]^n\right\} + \sum_{n=0}^{\infty} 3\left(\frac{1}{4}z^{-1}\right)^n$$

$$= \left\{2\left[-1 + \frac{1}{1-(5/6)^{-1}z}\right] + 3\frac{1}{1-(1/4)z^{-1}}\right\} = -2\frac{z}{z-(5/6)} + 3\frac{z}{z-(1/4)}$$

$$= \frac{-2z^2 + (z/2) + 3z^2 - (5z/2)}{[z-(5/6)][z-(1/4)]} = \frac{z(z-2)}{[z-(5/6)][z-(1/4)]}$$

The ROC and the pole-zero plot are shown in Figure 10.6.

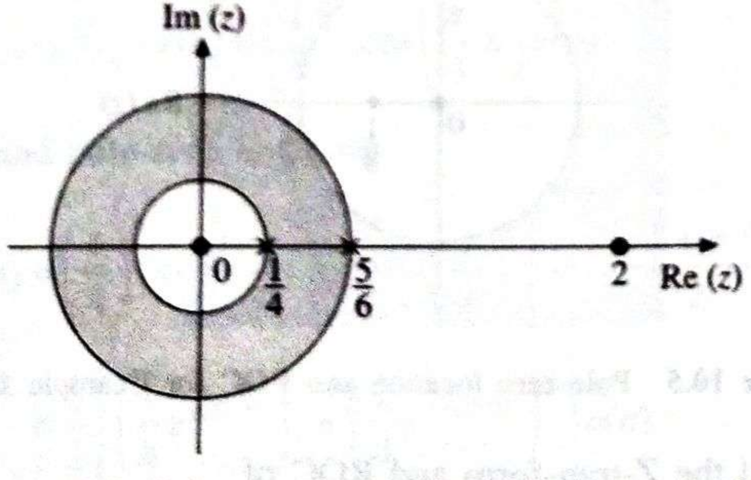


Figure 10.6 Pole-zero plot and ROC for Example 10.13.

10.4 Z-TRANSFORM AND ROC OF FINITE DURATION SEQUENCES

Finite duration sequences are sequences having a finite number of samples. Finite duration sequences may be right hand sequences or left hand sequences or two-sided sequences.

Right hand sequence

A right hand sequence is one for which $x(n) = 0$ for $n < n_0$, where n_0 is positive or negative but finite. If $n_0 \geq 0$, the resulting sequence is a causal or a positive time sequence. For a causal or a positive time sequence, the ROC is entire z-plane except at $z = 0$.

EXAMPLE 10.22 Find the ROC and Z-transform of the causal sequence

$$x(n) = \{\underset{\uparrow}{1}, 0, -2, 3, 5, 4\}$$

Solution: The given sequence values are:

$$x(0) = 1,\ x(1) = 0,\ x(2) = -2,\ x(3) = 3,\ x(4) = 5 \text{ and } x(5) = 4.$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(0) + x(1)z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + x(4)\, z^{-4} + x(5)\, z^{-5}$$

$$\therefore \qquad Z[x(n)] = X(z) = 1 - 2z^{-2} + 3z^{-3} + 5z^{-4} + 4z^{-5}$$

The $X(z)$ converges for all values of z except at $z = 0$.

EXAMPLE 10.23 A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, -2, 0, 4, -3\}$. Find $X(z)$ and its ROC.

Solution: Given $\qquad x(n) = \{5, 3, -2, 0, 4, -3\}$

$$\therefore \qquad x(n) = 5\delta(n) + 3\delta(n-1) - 2\delta(n-2) + 4\delta(n-4) - 3\delta(n-5)$$

The given sequence is a right-sided sequence. So the ROC is entire z-plane except at $z = 0$. Taking Z-transform on both sides of the above equation, we have

$$\therefore \qquad X(z) = 5 + 3z^{-1} - 2z^{-2} + 4z^{-4} - 3z^{-5}$$

ROC: Entire z-plane except at $z = 0$.

Left hand sequence

A left hand sequence is one for which $x(n) = 0$ for $n \geq n_0$ where n_0 is positive or negative but finite. If $n_0 \leq 0$, the resulting sequence is anticausal sequence. For such type of sequence, the ROC is entire z-plane except at $z = \infty$.

EXAMPLE 10.24 Find the Z-transform and ROC of the anticausal sequence.

$$x(n) = \{4, 2, 3, -1, -2, \underset{\uparrow}{1}\}$$

Solution: The given sequence values are:

$$x(-5) = 4,\ x(-4) = 2,\ x(-3) = 3,\ x(-2) = -1,\ x(-1) = -2,\ x(0) = 1$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values, $X(z)$ is:

$$X(z) = x(-5)\, z^5 + x(-4)\, z^4 + x(-3)\, z^3 + x(-2) z^2 + x(-1) z + x(0)$$

$$\therefore \quad Z[x(n)] = X(z) = 4z^5 + 2z^4 + 3z^3 - z^2 - 2z + 1$$

The $X(z)$ converges for all values of z except at $z = \infty$.

Two-sided sequence

A sequence that exists on both the left and right sides is known as a two-sided sequence. For a two-sided sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.

EXAMPLE 10.25 Find the Z-transform and ROC of the sequence

$$x(n) = \{2, 1, -3, 0, \underset{\uparrow}{4}, 3, 2, 1, 5\}$$

Solution: The given sequence values are:

$x(-4) = 2,\ x(-3) = 1,\ x(-2) = -3,\ x(-1) = 0,\ x(0) = 4,\ x(1) = 3,\ x(2) = 2,\ x(3) = 1,\ x(4) = 5$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 2z^4 + z^3 - 3z^2 + 4 + 3z^{-1} + 2z^{-2} + z^{-3} + 5z^{-4}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

10.5 PROPERTIES OF ROC

1. The ROC is a ring or disk in the z-plane centred at the origin.
2. The ROC cannot contain any poles.
3. If $x(n)$ is an infinite duration causal sequence, the ROC is $|z| > \alpha$, i.e. it is the exterior of a circle of radius α.

   If $x(n)$ is a finite duration causal sequence (right-sided sequence), the ROC is entire z-plane except at $z = 0$.
4. If $x(n)$ is an infinite duration anticausal sequence, the ROC is $|z| < \beta$, i.e. it is the interior of a circle of radius β.

   If $x(n)$ is a finite duration anticausal sequence (left-sided sequence), the ROC is entire z-plane except at $z = \infty$.

5. If $x(n)$ is a finite duration two-sided sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.
6. If $x(n)$ is an infinite duration, two-sided sequence, the ROC consists of a ring in the z-plane (ROC; $\alpha < |z| < \beta$) bounded on the interior and exterior by a pole, not containing any poles.
7. The ROC of an LTI stable system contains the unit circle.
8. The ROC must be a connected region. If $X(z)$ is rational, then its ROC is bounded by poles or extends up to infinity.
9. $x(n) = \delta(n)$ is the only signal whose ROC is entire z-plane.

10.6 PROPERTIES OF Z-TRANSFORM

The Z-transform has several properties that can be used in the study of discrete-time signals and systems. They can be used to find the closed form expression for the Z-transform of a given sequence. Many of the properties are analogous to those of the DTFT. They make the Z-transform a powerful tool for the analysis and design of discrete-time LTI systems. In general, both one-sided and two-sided Z-transforms have almost same properties.

10.6.1 Linearity Property

The linearity property of Z-transform states that, the Z-transform of a weighted sum of two signals is equal to the weighted sum of individual Z-transforms. That is, the linearity property states that

If
$$x_1(n) \xleftrightarrow{ZT} X_1(z), \text{ with ROC} = R_1$$
and
$$x_2(n) \xleftrightarrow{ZT} X_2(z), \text{ with ROC} = R_2$$
Then
$$ax_1(n) + bx_2(n) \xleftrightarrow{ZT} aX_1(z) + bX_2(z); \text{ with ROC} = R_1 \cap R_2$$

Proof: We know that
$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$\therefore$
$$Z[ax_1(n) + bx_2(n)] = \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)] z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} ax_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} bx_2(n) z^{-n}$$
$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$
$$= aX_1(z) + bX_2(z); \text{ ROC; } R_1 \cap R_2$$

$$\boxed{ax_1(n) + bx_2(n) \xleftrightarrow{ZT} aX_1(z) + bX_2(z)}$$

The ROC for the Z-transform of a sum of sequences is equal to the intersection of the ROCs of the individual transforms.

10.6.2 Time Shifting Property

The time shifting property of Z-transform states that

If $x(n) \xleftrightarrow{ZT} X(z)$, with zero initial conditions with ROC = R

Then $$x(n-m) \xleftrightarrow{ZT} z^{-m} X(z)$$

with ROC = R except for the possible addition or deletion of the origin or infinity.

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore \quad Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

Let $n - m = p$ in the summation, then $n = m + p$.

$$Z[x(n-m)] = \sum_{p=-\infty}^{\infty} x(p) z^{-(m+p)}$$

$$\therefore \quad = z^{-m} \sum_{p=-\infty}^{\infty} x(p) z^{-p}$$

$$= z^{-m} X(z)$$

$$\boxed{x(n-m) \xleftrightarrow{ZT} z^{-m} X(z)}$$

Similarly

$$\boxed{x(n+m) \xleftrightarrow{ZT} z^{m} X(z)}$$

If the initial conditions are not neglected, we have

(a) Time delay $$Z[x(n-m)] = z^{-m} X(z) + z^{-m} \sum_{k=1}^{m} x(-k) z^{k}$$

(b) Time advance $$Z[x(n+m)] = z^{m} X(z) - z^{m} \sum_{k=0}^{m-1} x(k) z^{-k}$$

Proof:

(a)

$$Z[x(n-m)] = \sum_{n=0}^{\infty} x(n-m) z^{-n} = \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)} z^{-m}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$

Let $n - m = p$ in the summation, then

$$Z[x(n-m)] = z^{-m} \sum_{p=-m}^{\infty} x(p)\, z^{-p}$$

$$= z^{-m}\left[\sum_{p=0}^{\infty} x(p)\, z^{-p} + \sum_{p=-m}^{-1} x(p)\, z^{-p}\right]$$

Substituting $p = -k$ in the second summation, we obtain

$$Z[x(n-m)] = z^{-m} X(z) + z^{-m} \sum_{k=1}^{m} x(-k)\, z^{k}$$

$$\therefore \quad Z[x(n-m)] = z^{-m} X(z) + z^{-(m-1)} x(-1) + z^{-(m-2)} x(-2) + z^{-(m-3)} x(-3) + \ldots$$

(b) $$Z[x(n+m)] = \sum_{n=0}^{\infty} x(n+m)\, z^{-n} = \sum_{n=0}^{\infty} x(n+m)\, z^{-(n+m)}\, z^{m}$$

$$= z^{m} \sum_{n=0}^{\infty} x(n+m)\, z^{-(n+m)}$$

Let $p = n + m$, then

$$Z[x(n+m)] = z^{m} \sum_{p=m}^{\infty} x(p)\, z^{-p}$$

$$= \left[z^{m} \sum_{p=0}^{\infty} x(p)\, z^{-p} - \sum_{p=0}^{m-1} x(p)\, z^{-p} \right]$$

$$= z^{m} X(z) - z^{m} \sum_{p=0}^{m-1} x(p)\, z^{-p}$$

i.e. $$Z[x(n+m)] = z^{m} X(z) - z^{m} x(0) - z^{m-1} x(1) - z^{m-2} x(2) - \ldots$$

This time shifting property is very useful in finding the output $y(n)$ of a system described in difference equation for an input $x(n)$.

$$Z[x(n+1)] = zX(z) - zx(0)$$

$$Z[x(n+2)] = z^{2} X(z) - z^{2} x(0) - zx(1)$$

$$Z[x(n+3)] = z^{3} X(z) - z^{3} x(0) - z^{2} x(1) - zx(2)$$

$$Z[x(n-1)] = z^{-1} X(z) + x(-1)$$

$$Z[x(n-2)] = z^{-2} X(z) + z^{-1} x(-1) + x(-2)$$

$$Z[x(n-3)] = z^{-3} X(z) + z^{-2} x(-1) + z^{-1} x(-2) + x(-3)$$

10.6.3 Multiplication by an Exponential Sequence Property

The multiplication by an exponential sequence property of Z-transform states that

If
$$x(n) \xleftrightarrow{\text{ZT}} X(z) \text{ with ROC} = R$$

Then
$$a^n x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{a}\right) \text{ with ROC} = |a|\, R$$

where a is a complex number.

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \quad Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

$$\boxed{a^n x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{a}\right)}$$

Note:
$$e^{j\omega n} x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{e^{j\omega}}\right) = X(e^{-j\omega} z)$$

$$e^{-j\omega n} x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{e^{-j\omega}}\right) = X(e^{j\omega} z)$$

10.6.4 Time Reversal Property

The time reversal property of Z-transform states that

If
$$x(n) \xleftrightarrow{\text{ZT}} X(z), \text{ with ROC} = R$$

Then
$$x(-n) \xleftrightarrow{\text{ZT}} X\left(\frac{1}{z}\right), \text{ with ROC} = \frac{1}{R}$$

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$\therefore$
$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

Let $p = -n$ in the above summation, then

$$Z[x(-n)] = \sum_{p=-\infty}^{\infty} x(p)\, z^{p}$$

$$Z[x(-n)] = \sum_{p=-\infty}^{\infty} x(p)\, (z^{-1})^{-p}$$

$$= X(z^{-1}) = X\left(\frac{1}{z}\right)$$

$$\boxed{x(-n) \xleftrightarrow{\text{ZT}} X(z^{-1})}$$

$$\boxed{x\left(\frac{n}{k}\right) \xleftrightarrow{ZT} X(z^k)}$$

10.6.6 Multiplication by *n* or Differentiation in *z*-domain Property

The multiplication by n or differentiation in z-domain property of Z-transform states that

If
$$x(n) \xleftrightarrow{ZT} X(z), \text{ with ROC} = R$$

Then
$$nx(n) \xleftrightarrow{ZT} -z\frac{d}{dz}X(z), \text{ with ROC} = R$$

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Differentiating both sides with respect to z, we get

$$\frac{d}{dz}X(z) = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\right] = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}(z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-n)\, z^{-n-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty} [nx(n)]z^{-n}$$

$$= -z^{-1}Z[nx(n)]$$

$\therefore$
$$Z[nx(n)] = -z\frac{d}{dz}X(z)$$

$$\boxed{nx(n) \xleftrightarrow{ZT} -z\frac{d}{dz}X(z)}$$

In the same way,
$$Z[n^k x(n)] = (-1)^k z^k \frac{d^k X(z)}{dz^k}$$

10.6.7 Conjugation Property

The conjugation property of Z-transform states that

If
$$x(n) \xleftrightarrow{ZT} X(z), \text{ with ROC} = R$$

Then
$$x^*(n) \xleftrightarrow{ZT} X^*(z^*), \text{ with ROC} = R$$

Proof: We have

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \quad Z[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n}\right]^*$$

$$= [X(z^*)]^* = X^*(z^*)$$

$$\boxed{x^*(n) \xleftrightarrow{ZT} X^*(z^*)}$$

10.6.8 Convolution Property

The convolution property of Z-transform states that the Z-transform of the convolution of two signals is equal to the multiplication of their Z-transforms, i.e.

If
$$x_1(n) \xleftrightarrow{ZT} X_1(z), \text{ with ROC} = R_1$$
and
$$x_2(n) \xleftrightarrow{ZT} X_2(z) \text{ with ROC} = R_2$$
Then
$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z)\, X_2(z), \text{ with ROC} = R_1 \cap R_2$$

Proof: We know that

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Let

$$x(n) = x_1(n) * x_2(n)$$

We have

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \quad Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\, z^{-(n-k)}\, z^{-k}$$

Interchanging the order of summations,

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-(n-k)}$$

Replacing $(n - k)$ by p in the second summation, we get

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{p=-\infty}^{\infty} x_2(p)\, z^{-p}$$

$$= X_1(z)\, X_2(z)$$

$$\boxed{x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z)\, X_2(z);\ \text{ROC};\ R_1 \cap R_2}$$

10.6.13 Initial Value Theorem

The initial value theorem of Z-transform states that, for a causal signal $x(n)$

If $$x(n) \xleftrightarrow{ZT} X(z)$$

Then $$\underset{n \to 0}{\text{Lt}}\ x(n) = x(0) = \underset{z \to \infty}{\text{Lt}}\ X(z)$$

Proof: We know that for a causal signal

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots$$

Taking the limit $z \to \infty$ on both sides, we have

$$\underset{z \to \infty}{\text{Lt}}\ X(z) = \underset{z \to \infty}{\text{Lt}} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \cdots \right] = x(0) + 0 + 0 + \cdots = x(0)$$

$$\therefore \quad \underset{n \to 0}{\text{Lt}}\ x(n) = x(0) = \underset{z \to \infty}{\text{Lt}}\ X(z)$$

$$\boxed{x(0) = \underset{z \to \infty}{\text{Lt}}\ X(z)}$$

This theorem helps us to find the initial value of $x(n)$ from $X(z)$ without taking its inverse Z-transform.

10.6.14 Final Value Theorem

The final value theorem of Z-transform states that, for a causal signal

If $$x(n) \xleftrightarrow{ZT} X(z)$$

and if $X(z)$ has no poles outside the unit circle, and it has no double or higher order poles on the unit circle centred at the origin of the z-plane, then

$$\underset{n\to\infty}{\text{Lt}}\; x(n) = x(\infty) = \underset{z\to 1}{\text{Lt}}\; (z-1)\, X(z)$$

Proof: We know that for a causal signal

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$Z[x(n+1)] = zX(z) - zx(0) = \sum_{n=0}^{\infty} x(n+1)\, z^{-n}$$

$\therefore$ $$Z[x(n+1)] - Z[x(n)] = zX(z) - zx(0) - X(z) = \sum_{n=0}^{\infty} x(n+1)\, z^{-n} - \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

i.e. $$(z-1)\, X(z) - zx(0) = \sum_{n=0}^{\infty} [x(n+1) - x(n)]\, z^{-n}$$

i.e. $$(z-1)\, X(z) - zx(0) = [x(1) - x(0)]\, z^{-0} + [x(2) - x(1)]\, z^{-1} + [x(3) - x(2)]\, z^{-2} + \cdots$$

Taking limit $z \to 1$ on both sides, we have

$$\underset{z\to 1}{\text{Lt}}\; (z-1)\, X(z) - x(0) = [x(1) - x(0) + x(2) - x(1) + x(3) - x(2) + \cdots + x(\infty) - x(\infty - 1)$$

$$= x(\infty) - x(0)$$

$\therefore$ $$x(\infty) = \underset{z\to 1}{\text{Lt}}\; (z-1)\, X(z)$$

or $$x(\infty) = \underset{z\to 1}{\text{Lt}}\; (1-z^{-1})\, X(z)$$

$$\boxed{x(\infty) = \underset{z\to 1}{\text{Lt}}\; (z-1)\, X(z)}$$

There are four methods often used to find the inverse Z-transform.

1. Power series method or long division method
2. Partial fraction method
3. Complex inversion integral method or residue method
4. Convolution integral method

The long division method is simple and the advantage of this method is: it is more general and can be applied to any problem, but the disadvantage is: it is difficult to get the solution in closed form. Further it can be used only if the ROC of the given $X(z)$ is either of the form $|z| > \alpha$ or of the form $|z| < \alpha$, i.e. it is useful only if the sequence $x(n)$ is either purely right-sided or purely left-sided. It cannot be used for bidirectional sequences.

If $X(z)$ is a ratio of two polynomials, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

We can generate a series in z by dividing the numerator by the denominator.

If ROC is $|z| > a$, it yields a causal sequence.

$$X(z) = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

So $X(z)$ is to be expressed in negative powers of z.

If ROC is $|z| < a$, it yields an anticausal sequence.

$$X(z) = x(0) + x(-1)\, z^{1} + x(-2)\, z^{2} + \cdots$$

So $X(z)$ is to be expressed in positive powers of z.

In long division method to realise a causal sequence (i.e. if ROC is $|z| > a$) both numerator and denominator are expressed either in descending powers of z or in ascending powers of z^{-1}, and the numerator is divided by the denominator continuously. To realise an anticausal sequence (i.e. ROC is $|z| < a$) both numerator and denominator are expressed either in ascending powers of z or in descending power of z^{-1} and the numerator is divided by the denominator continuously.

For partial fraction expansion method, $X(z)/z$ must be proper and the denominator should be in factored form. If it is not proper, it is to be written as the sum of a polynomial and a proper transfer function. The proper function $X(z)/z$ is written in terms of partial fractions and inverse Z-transform of each partial fraction is found by using the table of standard Z-transform pairs and all of them are added.

In the residue method, the inverse Z-transform of $X(z)$ can be obtained using the equation:

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

where c is a circle in the z-plane in the ROC of $X(z)$. The above equation can also be written as

$$x(n) = \sum \text{Residues of } X(z)\, z^{n-1} \text{ at the poles inside } c$$

$$= \sum_i (z - z_i)\, X(z)\, z^{n-1}\Big|_{z=z_i}$$

EXAMPLE 10.57 Using long division, determine the inverse Z-transform of

$$X(z) = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1}; \text{ ROC; } |z| > 1$$

Solution: Since ROC is $|z| > 1$, $x(n)$ must be a causal sequence. For getting a causal sequence, the $N(z)$ and $D(z)$ of $X(z)$ must be put either in descending powers of z or in ascending powers of z^{-1} before performing long division.

In the given $X(z)$ both $N(z)$ and $D(z)$ are already in descending powers of z.

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\
\hline
z^3 - 3z^2 + 4z + 1 & z^2 + 2z \\
 & \underline{z^2 - 3z + 4 + z^{-1}} \\
 & \quad 5z - 4 - z^{-1} \\
 & \quad \underline{5z - 15 + 20z^{-1} + 5z^{-2}} \\
 & \qquad 11 - 21z^{-1} - 5z^{-2} \\
 & \qquad \underline{11 - 33z^{-1} + 44z^{-2} + 11z^{-3}} \\
 & \qquad\quad 12z^{-1} - 49z^{-2} - 11z^{-3} \\
 & \qquad\quad \underline{12z^{-1} - 36z^{-2} + 48z^{-3} + 12z^{-4}} \\
 & \qquad\qquad -13z^{-2} - 59z^{-3} - 12z^{-4}
\end{array}$$

$\therefore$ $$X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \ldots$$

$\therefore$ $$x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Writing $N(z)$ and $D(z)$ of $X(z)$ in ascending powers of z^{-1}, we have

$$X(z) = \frac{N(z)}{D(z)} = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1} = \frac{z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 4z^{-2} + z^{-3}}$$

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\
\hline
1 - 3z^{-1} + 4z^{-2} + z^{-3} & z^{-1} + 2z^{-2} \\
 & \underline{z^{-1} - 3z^{-2} + 4z^{-3} + z^{-4}} \\
 & \quad 5z^{-2} - 4^{-3} - z^{-4} \\
 & \quad \underline{5z^{-2} - 15z^{-3} + 20z^{-4} + 5z^{-5}} \\
 & \qquad 11z^{-3} - 21z^{-4} - 5z^{-5} \\
 & \qquad \underline{11z^{-3} - 33z^{-4} + 44z^{-5} + 11z^{-6}} \\
 & \qquad\quad 12z^{-4} - 49z^{-5} - 11z^{-6} \\
 & \qquad\quad \underline{12z^{-4} - 36z^{-5} + 48z^{-6} + 12z^{-7}} \\
 & \qquad\qquad -13z^{-5} - 59z^{-6} - 12z^{-7}
\end{array}$$

$\therefore$ $$X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \ldots$$

$\therefore$ $$x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Observe that both the methods give the same sequence $x(n)$.

EXAMPLE 10.58 Using long division, determine the inverse Z-transform of

$$X(z)=\frac{z^2+z+2}{z^3-2z^2+3z+4}; \text{ ROC; } |z|<1$$

Solution: Since ROC is $|z|<1$, $x(n)$ must be a non-causal sequence. For getting a non-causal sequence, the $N(z)$ and $D(z)$ must be put either in ascending powers of z or in descending powers of z^{-1} before performing long division.

$$X(z)=\frac{z^2+z+2}{z^3-2z^2+3z+4}=\frac{2+z+z^2}{4+3z-2z^2+z^3}$$

$$
\begin{array}{r|l}
 & \frac{1}{2}-\frac{1}{8}z+\frac{19}{32}z^2-\frac{81}{128}z^3+\frac{411}{512}z^4 \\
\hline
4+3z-2z^2+z^3 & 2+z+z^2 \\
 & 2+\frac{3}{2}z-z^2+\frac{1}{2}z^3 \\
\cline{2-2}
 & -\frac{1}{2}z+2z^2-\frac{1}{2}z^3 \\
 & -\frac{1}{2}z-\frac{3}{8}z^2+\frac{1}{4}z^3-\frac{1}{8}z^4 \\
\cline{2-2}
 & \frac{19}{8}z^2-\frac{3}{4}z^3+\frac{1}{8}z^4 \\
 & \frac{19}{8}z^2+\frac{57}{32}z^3-\frac{19}{16}z^4+\frac{19}{32}z^5 \\
\cline{2-2}
 & -\frac{81}{32}z^3+\frac{21}{16}z^4-\frac{19}{32}z^5 \\
 & -\frac{81}{32}z^3-\frac{243}{128}z^4+\frac{81}{64}z^5-\frac{81}{128}z^6 \\
\cline{2-2}
 & \frac{411}{128}z^4-\frac{119}{64}z^5+\frac{81}{128}z^6
\end{array}
$$

$$\therefore \quad X(z)=\frac{1}{2}-\frac{1}{8}z+\frac{19}{32}z^2-\frac{81}{128}z^3+\frac{411}{512}z^4\ \dots$$

$$\therefore \quad x(n)=\left\{\dots,\frac{411}{512},-\frac{81}{128},\frac{19}{32},-\frac{1}{8},\underset{\uparrow}{\frac{1}{2}}\right\}$$

Also
$$X(z)=\frac{2+z+z^2}{4+3z-2z^2+z^3}=\frac{2z^{-3}+z^{-2}+z^{-1}}{4z^{-3}+3z^{-2}-2z^{-1}+1}$$

Inverse z-Transform using partial Fraction Expansion

① $$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

for i) ROC: $|z|>1$ (ii) ROC: $|z|<0.5$ (iii) ROC: $0.5<|z|<1$

Soln:
$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$X(z) = \frac{z^2}{z^2-1.5z+0.5}$$

$$\frac{X(z)}{z} = \frac{z}{z^2-1.5z+0.5} = \frac{z}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{A}{z-1} + \frac{B}{z-0.5}$$

$$A = (z-1)\left.\frac{X(z)}{z}\right|_{z=1}$$

$$= \cancel{(z-1)}\left.\frac{z}{\cancel{(z-1)}(z-0.5)}\right|_{z=1} = \frac{1}{0.5} = 2$$

$$B = (z-0.5)\frac{X(z)}{z}\Bigg|_{z=0.5}$$

$$= \cancel{(z-0.5)}\frac{z}{(z-1)\cancel{(z-0.5)}}\Bigg|_{z=0.5} = -1$$

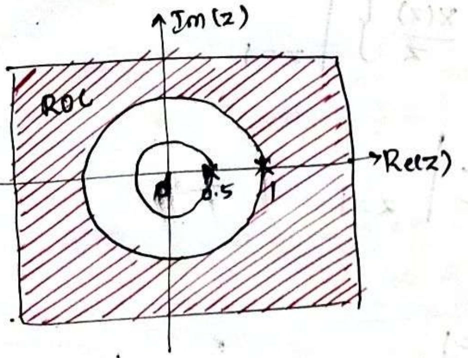
$$\therefore \quad \frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$X(z) = 2.\frac{z}{z-1} - \frac{z}{z-0.5}$$

$$x(n) = 2\, z^{-1}\left\{\frac{z}{z-1}\right\} - z^{-1}\left\{\frac{z}{z-0.5}\right\}$$

i) ROC: $|z| > 1$

There are two poles, at $z=1$ and $z=0.5$

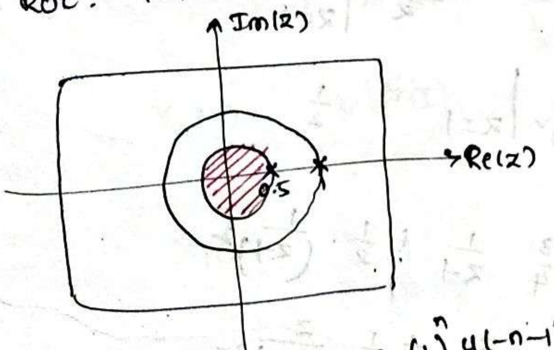


From the figure, it is clear that for both the poles ROC is the outside of the circle

$\therefore$ For Both the poles $x(n)$ must be causal

$$\therefore \quad x(n) = 2(1)^n u(n) - (0.5)^n u(n)$$
$$= [2-(0.5)^n]\,u(n)$$

ii) ROC: $|z| < 0.5$

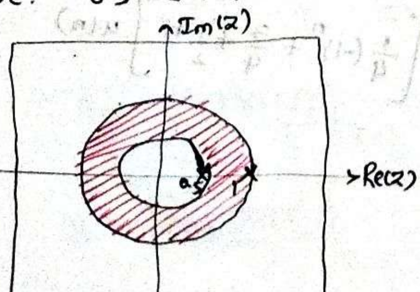


For both the poles ROC is inside of the circle

$\therefore$ $x(n)$ is non causal at both the poles $z=1$ and $z=0.5$

$$\therefore \quad x(n) = -2(1)^n u(-n-1) + (0.5)^n u(-n-1)$$
$$= [-2+(0.5)^n]\, u(-n-1)]$$

iii) ROC: $0.5 < |z| < 1$



For the poles $z=0.5$, ROC is outside of the circle $\therefore$ $x(n)$ is causal for the pole $z=0.5$

ROC is inside of the circle for $z=1$

For the $z=1$ $x(n)$ is non-causal

$$\therefore \quad x(n) = -2\,u(n) - (0.5)^n u(n)$$

$\therefore$

(2) $X(z) = \dfrac{1}{(1+z^{-1})(1-z^{-1})^2}$, ROC : $|z| > 1$

$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$A = (z+1)\frac{X(z)}{z}\bigg|_{z=-1}$$

$$A = \frac{\cancel{(z+1)}\, z^2}{\cancel{(z+1)}(z-1)^2}\bigg|_{z=-1} = \frac{1}{4}.$$

$$B = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{(z-1)^2\frac{X(z)}{z}\right\}\bigg|_{z=1}$$

$$= \frac{d}{dz}\,\cancel{(z-1)^2}\cdot\frac{z^2}{(z+1)\cancel{(z-1)^2}}\bigg|_{z=1}$$

$$= \frac{(z+1)(2z) - z^2}{(z+1)^2}\bigg|_{z=1}$$

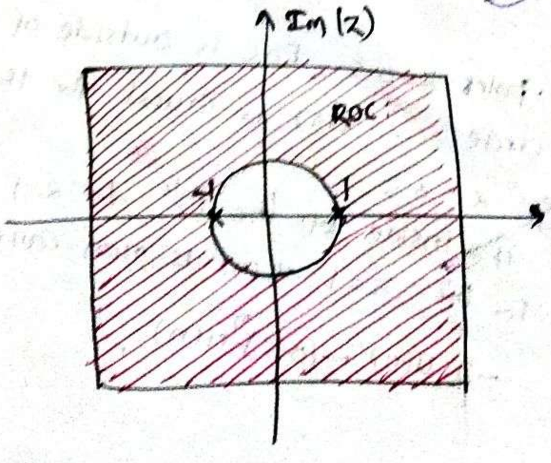
$$= \frac{2z^2+2z-z^2}{(z+1)^2}\bigg|_{z=1} = \frac{z^2+2z}{(z+1)^2}\bigg|_{z=1} = \frac{1+2}{(1+1)^2} =$$

$$C = \frac{1}{(2-2)!}\frac{d^{2-2}}{dz^{2-2}}\,(z-1)^2\cdot\frac{X(z)}{z}\bigg|_{z=1}$$

$$= \frac{\cancel{(z-1)^2}\cdot z^2}{(z+1)\cancel{(z-1)^2}}\bigg|_{z=1} = \frac{1}{2}$$

$$\therefore\ \frac{X(z)}{z} = \frac{1}{4}\cdot\frac{1}{z+1} + \frac{3}{4}\cdot\frac{1}{z-1} + \frac{1}{2}\cdot\frac{1}{(z-1)^2}$$

$$X(z) = \frac{1}{4}\cdot\frac{z}{z+1} + \frac{3}{4}\cdot\frac{z}{z-1} + \frac{1}{2}\cdot\frac{z}{(z-1)^2}.$$



$$\therefore\ x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}(1)^n u(n) + \frac{1}{2}\, n\, u(n)$$

$$= \left[\frac{1}{4}(-1)^n + \frac{3}{4} + \frac{1}{2}n\right]u(n)$$

Inverse z-transform using Contour integration (Residue method)

→ Define $X_0(z)$ as

$$X_0(z) = X(z)\, z^{n-1}$$

→ For causal signal

$$x(n) = \text{Sum of residues of } X_0(z)\Big|_{\text{For the poles inside the circle}}$$

For non-causal signal

$$x(n) = -\text{Sum of residues of } X_0(z)\Big|_{\text{For the poles outside the circle}}$$

Problems:

① Find the inverse z-transform of

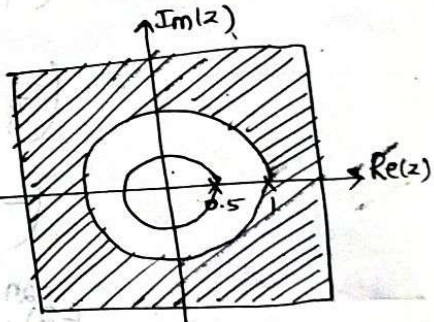
$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

For the ROC

i) $|z|>1$ (ii) $|z|<0.5$ (iii) $0.5<|z|<1$

Soln:
$$X(z) = \frac{z^2}{z^2-1.5z+0.5} = \frac{z^2}{(z-1)(z-0.5)}$$

$$X_0(z) = X(z).z^{n-1}$$



i) $|z|>1$

From the ROC, all the poles lie inside the circle. hence $x(n)$ should be causal

$$\therefore x(n) = \text{Residues of } X_0(z)\Big|_{z=1 \text{ and } z=0.5}$$

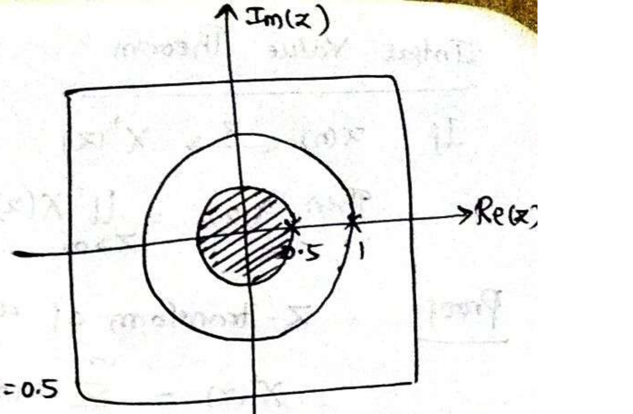
$$= \text{Residue of } X_0(z)\Big|_{z=1} + \text{Residue of } X_0(z)\Big|_{z=0.5}$$

$$\cancel{(z-1)}\frac{z^2 . z^{n-1}}{\cancel{(z-1)}(z-0.5)}\Bigg|_{z=1} + \frac{\cancel{(z-0.5)}\, z^2 . z^{n-1}}{(z-1)\cancel{(z-0.5)}}\Bigg|_{z=0.5}$$

$$\frac{1}{0.5} + \frac{(0.5)^2(0.5)^{n-1}}{-0.5}$$

$$[2-(0.5)^n]\,u(n)$$

ii) $|z| < 0.5$



From the figure, poles lie outside the circle.

$\therefore$ $x(n)$ is non causal

$x(n) = -[$~~sum of~~ residues of $X_0(z)]$ $z=1$ & $z=0.5$

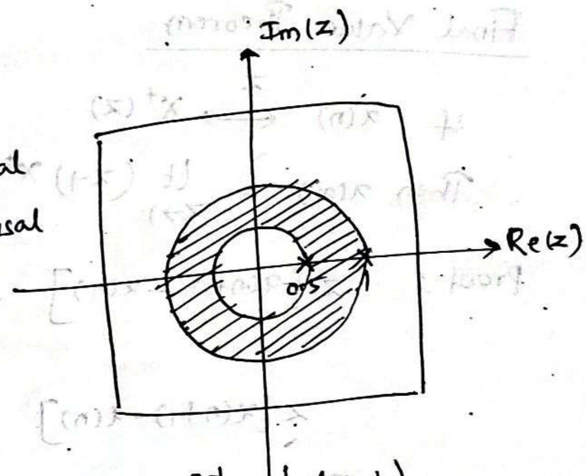
$$-\left[\text{Residue of } X_0(z)\Big|_{z=1} + \text{Residue of } X_0(z)\Big|_{z=0.5}\right]$$

$$= -\left[\frac{\cancel{(z-1)}\, z^2\, z^{n-1}}{\cancel{(z-1)}(z-0.5)}\Big|_{z=1} + \frac{\cancel{(z-0.5)}\, z^2 z^{n-1}}{(z-1)\cancel{(z-0.5)}}\Big|_{z=0.5}\right] u(-n-1)$$

$$\left[\frac{1}{0.5} + \frac{(0.5)^2 (0.5)^{n-1}}{-0.5}\right] u(-n-1)$$

$$= \left[-2 + (0.5)^n\right] u(-n-1)$$

(iii) $0.5 < |z| < 1$



at the pole $z=0.5$ $x(n)$ is causal

For the pole $z=1$ $x(n)$ is non causal

$\therefore$ $x(n) =$ Residue of $X_0(z)\Big|_{z=0.5}$

$-$ Residue of $X_0(z)\Big|_{z=1}$

$$= \frac{\cancel{(z-0.5)}\, z^2\, z^{n-1}}{(z-1)\cancel{(z-0.5)}}\Bigg|_{z=0.5} u(n) - \frac{\cancel{(z-1)}\, z^2\, z^{n-1}}{\cancel{(z-1)}\,(z-0.5)}\Bigg|_{z=1} u(-n-1)$$

$$= \frac{(0.5)^2 (0.5)^{n-1}}{-0.5} u(n) - \frac{1}{0.5} u(-n-1)$$

$$x(n) = -(0.5)^n u(n) - 2u(-n-1)$$

Transfer function of the LTI System

Let us consider the linear constant coefficient difference eqn is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Take z- transform

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k} Y(z)\right] = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Problem:

A difference equation of the system given below is

$y(n) = 0.5\,y(n-1) + x(n)$

Determine

① system function ② pole-zero plot ③ unit sample response impulse response

Soln: D.E is $y(n) = 0.5\,y(n-1) + x(n)$

Apply z- transform

$$Y(z) = 0.5\, z^{-1} Y(z) + X(z)$$

$$Y(z)\left[1 - 0.5 z^{-1}\right] = X(z)$$

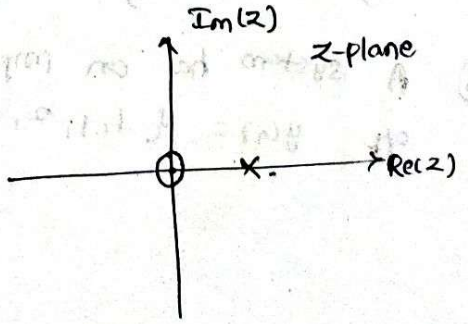
$$\frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-1}}$$

$$H(z) = \frac{z}{z-0.5}$$

Pole-zero plot

Zeros : $z - 0 = 0 \Rightarrow z = 0$

Poles : $z - 0.5 = 0 \Rightarrow z = 0.5$



unit sample response $h(n)$

$$H(z) = \frac{z}{z-0.5}$$

$$\therefore h(n) = (0.5)^n u(n)$$

Causality and stability interms of z-transform

The condn for LTI system to be causal is given as

$$h(n) = 0 \quad \text{for } n<0$$

The necessary and sufficient condn for the system to be BIBO stable is given as

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

An LTI system is BIBO stable if and only if the ROC of system function includes the unit circle.

Find the impulse and step response of the following DE

~~$y(n) - y(n-1) + 0.9y(n-2) = x(n)$~~

~~impulse response~~

~~i/p $x(n) = \delta(n)$~~

~~$X(z) = 1$~~

~~$y(n) - y(n-1) + 0.9y(n-2)$~~

$y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$

impulse response

$x(n) = \delta(n)$

$X(z) = 1$

Given DE is $y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$

Apply z-transform

$Y(z) = 0.6z^{-1}Y(z) - 0.08z^{-2}Y(z) + X(z)$ — (1)

$Y(z)[1 - 0.6z^{-1} + 0.08z^{-2}] = 1$

$Y(z) = \frac{1}{1-0.6z^{-1}+0.08z^{-2}} = \frac{z^2}{z^2-0.6z+0.08}$

$\frac{Y(z)}{z} = \frac{z}{(z-0.4)(z-0.2)}$

$\frac{Y(z)}{z} = \frac{A}{z-0.4} + \frac{B}{z-2}$

$A = (z-0.4)\frac{Y(z)}{z}\Big|_{z=0.4} = \cancel{(z-0.4)}\frac{z}{\cancel{(z-0.4)}(z-0.2)}\Big|_{z=0.4} = \frac{0.4}{0.2} = 2$

$B = \cancel{(z-0.2)}\frac{z}{(z-0.4)\cancel{(z-0.2)}}\Big|_{z=0.2} = \frac{0.2}{0.2-0.4} = -1$

$\frac{Y(z)}{z} = \frac{2}{z-0.4} - \frac{1}{z-2}$

$Y(z) = 2\frac{z}{z-0.4} - \frac{z}{z-2}$

$\therefore y(n) = 2(0.4)^n u(n) - (2)^n u(n)$

step response

i/p $x(n) = u(n)$

$X(z) = \frac{z}{z-1}$

From (1) $Y(z)[1 - 0.6z^{-1} + 0.08z^{-2}] = X(z)$

$Y(z) = \frac{z^2}{(z-0.4)(z-0.2)}\frac{z}{(z-1)}$

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)(z-0.2)(z-0.4)}$$

$$\frac{Y(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.2)} + \frac{C}{(z-0.4)}$$

$$A = \cancel{(z-1)}\frac{z^2}{\cancel{(z-1)}(z-0.2)(z-0.4)}\Bigg|_{z=1} = \frac{1}{(0.8)(0.6)} = 2$$

$$B = \cancel{(z-0.2)}\frac{z^2}{(z-1)\cancel{(z-0.2)}(z-0.4)}\Bigg|_{z=0.2} = \frac{0.04}{(-0.8)(-0.2)} = 1/4$$

$$C = \cancel{(z-0.4)}\frac{z^2}{(z-1)(z-0.2)\cancel{(z-0.4)}}\Bigg|_{z=0.4} = \frac{0.16}{(-0.6)(0.2)} = \frac{4}{3}$$

$$\frac{Y(z)}{z} = \frac{2}{z-1} + \frac{1/4}{z-0.2} + \frac{4/3}{z-0.4}$$

$$Y(z) = 2\left[\frac{z}{z-1}\right] + 1/4\left[\frac{z}{z-0.2}\right] + 4/3\left(\frac{z}{z-0.4}\right)$$

$$y(n) = 2(1)^n u(n) + \frac{1}{4}(0.2)^n u(n) + \frac{4}{3}(0.4)^n u(n)$$