P463 Problem Set 3

Assigned: Tuesday, 11 October, 2022
Due: 4:30 pm on Friday, 28 October, 2022

1) Eddington model and Lane-Emden equation

By following the steps below you will perform (most of) the calculation from question 2(d) on the mid-semester exam to find the radius and mass of the star in Eddington's "standard model," which assumes $\beta = P_{\rm gas}/P = \text{const}$ (see also problem 5.5 of Choudhuri). We had

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{2/3}}\frac{d\rho}{dr}\right) = -\frac{3\pi G}{K}\rho. \tag{1}$$

Here $P = K\rho^{4/3}$ and

$$K = \left[\frac{3}{a}\left(\frac{k_{\rm B}}{\mu m_{\rm H}}\right)^4 \frac{1-\beta}{\beta^4}\right]^{1/3}.$$

Recall that $\theta \equiv (\rho/\rho_{\rm c})^{1/n}$ and $\xi \equiv r/\alpha$ with

$$\alpha = \left[\frac{(n+1)K\rho_{\rm c}^{\frac{1-n}{n}}}{4\pi G}\right]^{1/2}.$$

a) Show that equation (1) reduces to the Lane-Emden equation with $n = 3$.

b) The Lane-Emden equation cannot be solved analytically for $n = 3$. To obtain a numerical solution of a second order ordinary differential equation, rewrite the equation as a pair of coupled first order partial differential equations. To do this, you will need to define a new variable (call it φ).

c) What are the boundary conditions at $\xi = 0$?

d) After solving numerically (a bit too much to ask of you for this assignment), one finds $\xi_3 = 6.90$, and $-\xi_3^2(d\theta/d\xi)_{\xi_3} = 2.02$, where $\xi_3 = R/\alpha$ is the dimensionless radius at which $\rho = \rho_{\rm c}\theta^3 = 0$. Obtain an expression for the radius R in terms of μ, β and $\rho_{\rm c}$.

e) Derive the following expression for the mass M:

$$M = \frac{18.1\,M_\odot}{\mu^2}\left(\frac{1-\beta}{\beta^4}\right)^{1/2}. \tag{2}$$

Note that $n = 3$ is the only case for which M is independent of $\rho_{\rm c}$. While the Eddington model is very rough, it does predict, correctly, that radiation pressure becomes more important for more massive stars, as can be seen from equation (2).

f) Calculate μ assuming solar composition, i.e. $X \approx 0.73$, $Z \approx 0.02$, and assuming that the star is fully ionized.

g) What value of β does the model predict for the Sun and what does this tell us about the role played by radiation pressure in determining the Sun's internal structure?

h) Calculate β at the centre of the Sun using values determined from detailed solar models. These give the central temperature and density as $T_c \approx 1.6 \times 10^7\,\mathrm{K}$ and $\rho_c \approx 1.6 \times 10^5\,\mathrm{kg\,m^{-3}}$. (See also problem 3.3 of Choudhuri.)

i) Compare your answers from parts **(g)** and **(h)** and briefly comment.

2) Degeneracy pressure and white dwarfs

(a) Estimate the central temperature T_c for a white dwarf of mass $1\,M_\odot$ and radius $0.01\,R_\odot$. See Carroll & Ostlie §16.2 for a solution if you want.

(b) Based on your answer, and on the rates of nuclear fusion that we had studied, what can you say about the composition of a WD? (Can it be composed of hydrogen? Helium? Carbon? Oxygen? No need for a detailed calculation.)

(c) In our analysis of white dwarfs, we had neglected finite temperature effects, which allowed us to assume that all states with $E < E_\mathrm{F}$ were occupied and all states with $E > E_\mathrm{F}$ were unoccupied. Show that $E_\mathrm{F}/k_\mathrm{B} \gg T$. For a rough estimate you may take $\rho \sim \overline{\rho} = 3M/4\pi R^3$. This does not guarantee that $E - E_\mathrm{F} \gg k_\mathrm{B}T$. Is this a problem?

(d) Estimate the ratio of electron degeneracy pressure to gas pressure as a function of T (again taking $\rho \sim \overline{\rho}$) for the non-relativistic and ultra-relativistic cases. Does your answer convince you that the dominant contribution to the pressure in WDs is electron degeneracy pressure?

3) Write down and explain one important equation related to your reading project

Make sure to (i) note the meaning of the **quantities** in the equation, (ii) explain the role of **each term/part** of the equation, (iii) explain the **physical meaning/implications** of the equation, and (iv) explain the **importance** of this equation in the context of your project.