1811061 Dinesh Beniwal PS-03

Q 1 a) $P = k\rho^{4/3}$

$$= \left[\frac{3}{a}\left(\frac{k_B}{\mu m_H}\right)^4 \frac{1-\beta}{\beta^4}\right]^{1/3} \rho^{4/3}$$

$$\frac{d}{dr} = \frac{d\xi}{dr}\frac{d}{d\xi} = \frac{1}{\alpha}\frac{d}{d\xi}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{2/3}}\frac{d\rho}{dr}\right) = \frac{-3\pi G\rho}{k}$$

$$\frac{1}{\left(\frac{r}{\alpha}\right)^2\alpha^2}\frac{1}{\alpha}\frac{d}{d\xi}\left(\frac{r^2}{\alpha^2}\frac{\alpha^2}{\rho^{2/3}}\frac{1}{\alpha}\frac{d\rho}{d\xi}\right) = \frac{-3\pi G\rho}{k}$$

After substituting $\rho = \rho_c\theta^4$, we get the form as,

$$\frac{1}{\xi^2}\frac{1}{\alpha^3}\frac{d}{d\xi}\left(\xi^2\frac{\alpha}{\rho_c^{2/3}\theta^{2/3}}\frac{d\rho_c\theta^n}{d\xi}\right) = \frac{-3\pi\rho_c\theta^4}{k}$$

$$\frac{\rho_c^{1/3}}{a^2\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2}{\theta^{2\cdot4/3}}\frac{d\theta^4}{d\xi}\right) = -\frac{3\pi G}{k}\rho_c\theta^4$$

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(b) $$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \rightarrow \text{Lane-Emden eq}^n$$

For $n=3$, we have

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3$$

To obtain it as a pair of coupled 1st order DE, Lets take

$$\xi^2\frac{d\theta}{d\xi} = -\psi$$

we can write

$$\frac{1}{\xi^2}\frac{d\psi}{d\xi} = \theta^3 \Rightarrow \frac{d\psi}{d\xi} = \theta^3\xi^2$$

Generalising it for any n

$$\frac{d\psi}{d\xi} = \theta^n\xi^2$$

We know that

$$M = 4\pi a^3\rho_c\int_0^{\xi}\xi^2\theta^n\,d\xi$$

Substituting for $\xi^2\theta^n$ we can write

$$m(r) = 4\pi a^3\rho_c\,\psi(\xi)$$

$\therefore$ The Coupled 1st order DE's are:

$$\xi^2\frac{d\theta}{d\xi} = -\psi \quad \& \quad \frac{d\psi}{d\xi} = \theta^n\xi^2 \quad \text{where } \psi(\xi) = \frac{m(r)}{4\pi a^3\rho_c}$$

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(c) Bound^{n} Cond^{n} @ $\xi = 0$.

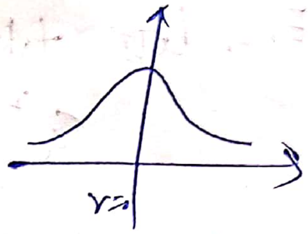
(i) @ $\xi = 0$, $\theta = 1 \rightarrow 1^{st}$ B.C.

as $\xi = \frac{r}{a}$ & at $r=0$, $\xi = 0$

$\Rightarrow \rho = \rho_c$ @ $r = 0$

as $\rho = \rho_c \theta^n$ as $\theta(\xi = 0) = 1$

(ii) $\left(\frac{d\theta}{d\xi}\right)_{\xi=0} = 0 \rightarrow 2^{nd}$ B.C.



(d) $\xi_3 = 6.90$ & $-\xi_3^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_3} = 2.02$ i.e. $\phi(\xi) = 2.02$

$\xi_3 = \frac{R}{\alpha}$ and $\alpha = \left[K\rho^{-2/3} / \pi G \right]^{1/2}$

$\Rightarrow$ $R = \underline{6.90} \times \sqrt{K/\rho^{2/3}\pi G}$

$\Rightarrow$ $R = \frac{6.90}{\sqrt{\pi G \rho_c^{2/3}}} \left[\frac{3}{a} \left(\frac{k_B}{\mu H}\right)^4 \right]^{1/6} \frac{(1-\beta)^{1/6}}{(\mu\beta)^{2/3} \rho_c^{1/3}}$

(e) $\frac{dm}{dr} = 4\pi r^2 \rho \Rightarrow dm = 4\pi r^2 \rho dr$

$M = \int 4\pi r^2 \rho dr = \int 4\pi \alpha^2 \xi^2 \rho_c \theta^3 \alpha d\xi$

$M = 4\pi\alpha^3 \rho_c \int_0^{\xi_3} \rho_c \theta^3 d\xi$

$= 4\pi\alpha^3 \rho_c \left| -\xi_3^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_3} \right|$

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$$= 4\pi \rho_c \times 2.02 \left[\frac{\rho_c}{\rho}\left[\frac{3}{a}\left(\frac{k_B}{\mu m_H}\right)^4 \frac{1-\beta}{\beta^4}\right]^{1/2}\right] \times \left(\frac{1}{\pi G}\right)^{3/2}$$

$$= \frac{18.32}{\mu^2} M_\odot \left(\frac{1-\beta}{\beta^4}\right)^{1/2}$$

(f) $X = 0.73$, $Z = 0.02$. let's assume the star is fully ionized

$$Y = \cancel{\ } 0.25$$

then $\mu = \left(2X + \frac{3Y}{4} + \frac{Z}{2}\right)^{-1}$

$$= \underline{0.603}$$

(g) $\because M = \frac{M_\odot \times 18.1}{\mu^2}\left(\frac{1-\beta}{\beta^4}\right)^{1/2}$

$$\frac{\mu^2}{18.1} = \left(\frac{1-\beta}{\beta^4}\right)^{1/2} \Rightarrow \frac{1-\beta}{\beta^4} = \left(\frac{0.603^2}{18.1}\right)^2$$

$\Rightarrow$ $\beta = 0.9996$

$\Rightarrow \frac{P_{gas}}{P} = 0.99 \Rightarrow P_{rad} = 4\times10^{-4} P$

$\Rightarrow P_{rad} \ll P_{gas}$

(h) $P_{rad} = aT^4/3$

$P_{gas} = \left(\frac{\rho}{\mu m_H}\right) k_B T$

@ Center of Sun $P_{rad} = \frac{a}{3}(1.6\times10^7)^4 = 1.657\times10^{13} P_4$

$P_{gas} = \left(\frac{1.6\times10^5}{0.603\times1.67\times10^{-27}}\right)\times1.38\times10^{-23}\times1.6\times10^7 P_1$

$= 3.507\times10^{16} P_4$

$P_T = P_{gas} + P_{rad} = 3.509\times10^{16}$

$\beta = \frac{P_{gas}}{P} = \cancel{9.99}\ 0.999$

i) (g) & (h) have very close answer

Q② (a) $\because \frac{dT}{dr} = \frac{-3}{4ac}\frac{k\rho}{T^3}\frac{L}{4\pi r^2}$

$\frac{T_{wd} - T_c}{R_{wd} - 0} = \frac{3}{4ac}\frac{k\rho}{T_c^3}\frac{L_{wd}}{4\pi R_{wd}^2}$

@ Surface Temp $T_{wd} \leq T_c$

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$\Rightarrow$ appr. $\frac{T_c^4}{R_{WD}} = \frac{3}{4ac} \frac{K\rho}{4\pi R_{WD}^2} L_{WD}$

$$T_c = \left[\frac{3}{4ac} \frac{K\rho_c}{4\pi R_{WD}} L_{WD}\right]^{1/4}$$

$M_{WD} = 1 M_\odot$ & $R_{WD} = 0.01 R_\odot$ & $L_{WD} \simeq 0.03 L_\odot$

& $\rho_c \sim \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}$

Now $K\rho = n_e \sigma_{TH}$ $\Big|$ $n_e = \frac{\rho(1+X)}{2m_H}$

$$K = \frac{(1+X)\sigma_{TH}}{2m_H} \simeq 0.02(1+X)$$

But for e^- scattering $X \simeq 0$ $\Rightarrow$ $\underline{K = 0.02}$

$$\Rightarrow T_c = \left[\frac{3^2}{16\sigma} \frac{0.02 \times M}{(4\pi)^2 R_{WD}^4} \times 0.03 L_\odot\right]^{1/4}$$

$$\simeq 5.92 \times 10^7 \, K$$

(b) For He to burn, the required central Temp. ~~lies~~ lies in the range $10^8 - 10^9$ K. ~~B~~ However, ~~we~~ here the core temp. is of order 10^7 K.

$\Rightarrow$ PP-1 Chain occurs due to this Temp.

& CNO cycle can't occurs, hence Limiting the presence of havier elements

(c) $$E_F = \frac{h^2}{2m}(3\pi^2 n)^{2/3}$$

$$(E_F)^{e^-} = \frac{h^2}{2m_e}\left(3\pi^2 \frac{\rho}{4 m_H}\right)^{2/3}$$

Avg thermal energy of e^- is $\frac{3}{2}k_B T$. If $\frac{3}{2}k_B T < E_F$ then the probability of an e^- to make a transition to an unoccupied state is less & the e^- gas will not be generated. For degenerate gas

$$\frac{3}{2}k_B T < E_F$$

$$E_F > k_B T \Rightarrow \frac{E_F}{k_B} \geq T$$

(d) $$P_{gas} = \frac{\rho}{\mu m_H} k_B T$$

for e^-, $\mu_e \approx 2$

$$\Rightarrow P_{gas} = \frac{\rho}{2 m_H} k_B T$$

For non-radiative: $$P_e = \frac{8\pi p_F^5}{15 h^3 m_e}$$ $$p_F = \left(\frac{3\rho}{8\pi \mu_e m_H}\right)^{1/3} h$$

$$= K_{NR}\rho^{5/3} = \frac{1\times 10^7 \rho^{5/3}}{\mu_e}$$

$$\frac{P_e}{P_{gas}} = \frac{10^7 m_H}{k_B}\left(\frac{\rho}{4}\right)^{2/3} T^{-1}$$

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Upon taking $\rho = \bar{\rho} = \frac{3M}{4\pi R^3}$

we get $\frac{P_e}{P} = 1.9924 \times 10^{7}\, T^{-1}$

For radiative Cor

$$P_e = \frac{2\pi c\, P_F^4}{3h^3}$$

$$= K_{ug}\, \rho^{4/3} = \frac{1.24 \times 10^{10} \rho^{4/3}}{\mu_e^{4/3}}$$

then $\frac{P_e}{P_{gas}} = \frac{1.24 \times 10^{10}}{\frac{\rho k_B T}{\mu m_H}} \left(\frac{\rho}{\mu_e}\right)^{4/3} = 1.33 \times 10^{7}\, T^{-1}$

It can be observed that the dominant distribution of pressure in WDs is due to degenracy pressure than gas pressure.

Q(3) Project : Spiral str in Galaxies

Formula : $r(\phi) = \frac{A}{\log\left(B \tan \frac{\phi}{2N}\right)}$

(i) & (ii) Meaning & role of each quantities

A : Scale parameter for entire St^n

B : Determine the spiral pitch

N : winding number (Need not to be an integer)

(iii) Assumption in formula : All galaxies have "bars"

N ↑ tighter winding

B ↑ greater arm sweep & smaller bar/bulge

B ↓ Larger bar/bulge with sharper bar/arm junction

⇒ B : Controls "bulge-to-arm" size

N : Controls "tightness"

(iv) It explains/Fit the spiral galaxies arms.
The examples for different N & B are shown below:

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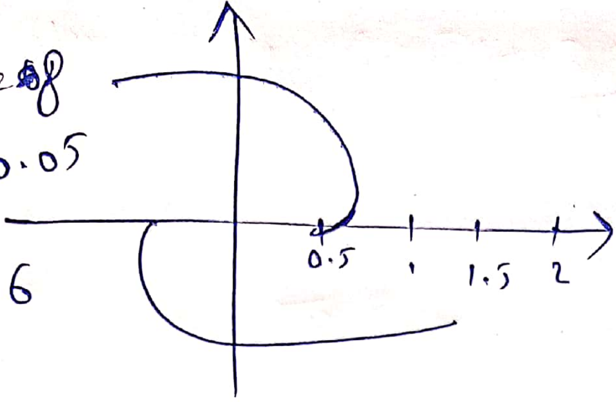
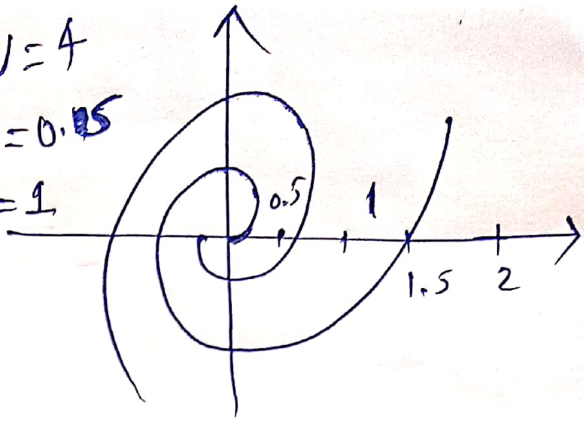
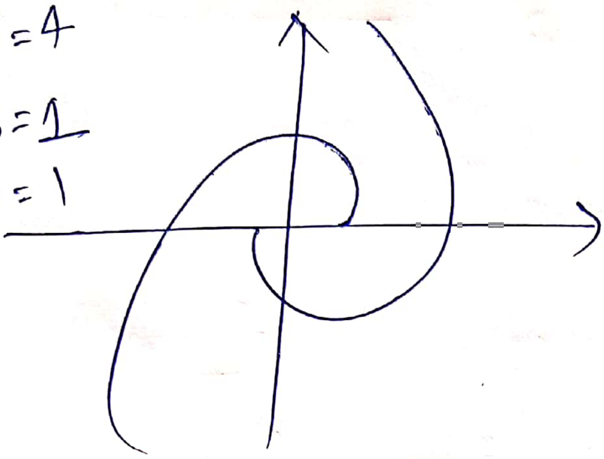
N=8
B=0.05
A=6



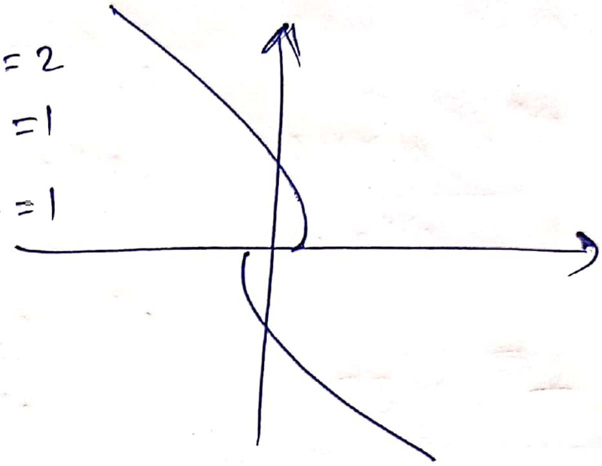
N=4
B=0.5
A=1



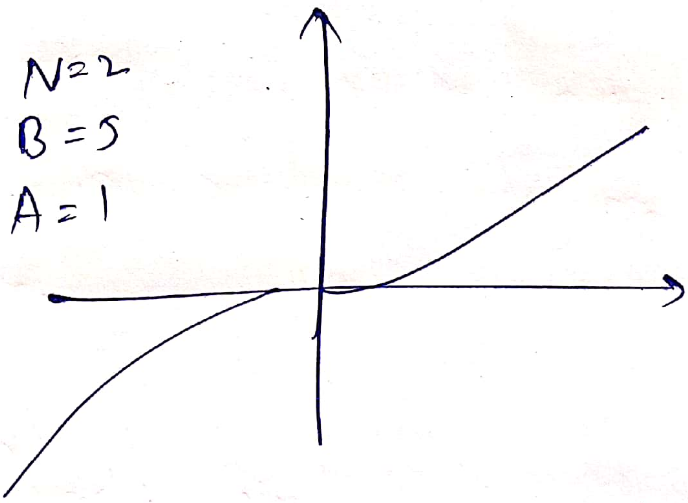
N=4
B=1
A=1



N=2
B=1
A=1



N=2
B=5
A=1



Ex <u>Comparision with Data</u> : It perfectly fits for NGC 1365, M51, NGC 1097

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