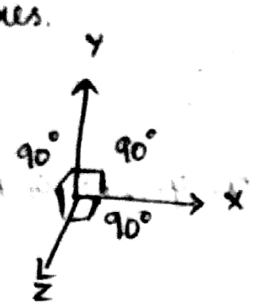
3-D CO-ORDINATE GEOMETRY

→ Every Point P in the space is determined by 3 co-ordinates and in general we denote as $P(x, y, z)$ (or) $P(a, b, c)$.

→ The 3 co-ordinate axes are also known as rectangular co-ordinate axes.



→ The 3 co-ordinate axes intersect at a point known as Origin denoted by $O(0,0,0)$.

→ The 3 co-ordinate axes taken in pairs determine co-ordinate planes known as rectangular co-ordinate planes which are XY plane, YZ plane, XZ plane.

Equations of co-ordinate axes

1) Equation of X-axis is $y=0, z=0$
2) Equation of Y-axis is $x=0, z=0$
3) Equation of Z-axis is $x=0, y=0$

Equations of co-ordinate planes

Equation of XY plane is $z=0$

Equation of YZ plane is $x=0$

Equation of XZ plane is $y=0$.

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Note: The 3 co-ordinate axes divide the space into $2^3 = 8$ regions which are known as octants

Distance b/w 2 points

The distance b/w two points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ is

$$\boxed{AB = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \geq R}$$

→ If P divides the line joining $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ in the ratio $m_1 : m_2$ Then

$$P = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2} \right)$$

Note: let $m_1 : m_2 = K : 1$ $\quad \left[\frac{m_1}{m_2} = K \right]$

$$P = \left[\frac{Kx_2 + x_1}{K+1}, \frac{Ky_2 + y_1}{K+1}, \frac{Kz_2 + z_1}{K+1} \right]$$ applicable only if $K \neq -1$

→ If K is +ve then 'P' divides AB internally

→ If K is -ve then 'P' divides AB externally.

Mid point of line joining 2 points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

→ If G is the centroid of the triangle ABC where $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ & $C(x_3, y_3, z_3)$

Then $$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$

→ Centroid of tetrahedron ABCD is

$$G = \left[\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4} \right]$$

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Direction Cosines Of X-axis

Angles made by x-axis with positive direction of x, y & z axes, are $\alpha = 0°$, $\beta = 90°$, $\gamma = 90°$ respectively.

$\Rightarrow$ direction cosines of x-axis are $(\cos\alpha, \cos\beta, \cos\gamma)$

DC's of x axis - $(1, 0, 0)$

Direction Cosines of Y-axis

DC's of Y-axis are $(\cos 90°, \cos 0°, \cos 90°) = (0, 1, 0)$

Direction Cosines of Z-axis

DC's of Z-axis are $(\cos 90°, \cos 90°, \cos 0°) = (0, 0, 1)$

Direction Ratios of a Line

A set of three numbers of the form (a, b, c) which are proportional to the DC's (l, m, n) of the line L are known as Direction Ratios of line L

$$\boxed{\frac{l}{a} = \frac{m}{b} = \frac{n}{c}}$$

Converting DR's to DC's

Let (a, b, c) be direction ratios, line L.

Let (l, m, n) be direction cosines of line L.

Then $$(l, m, n) = \left(\frac{a}{\pm\sqrt{a^2+b^2+c^2}}, \frac{b}{\pm\sqrt{a^2+b^2+c^2}}, \frac{c}{\pm\sqrt{a^2+b^2+c^2}}\right)$$

* Direction Ratios of the line joining the points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ is equal to $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

(or)

$(x_1 - x_2, y_1 - y_2, z_1 - z_2)$

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Angle b/w two lines L_1 and L_2

Let 'θ' be the angle b/w L_1 and L_2.

→ Let the DR's be (a_1, b_1, c_1) and (a_2, b_2, c_2) of L_1 and L_2 respectively.

Then
$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$$

→ Let the DC's of lines L_1 and L_2 be (l_1, m_1, n_1) & (l_2, m_2, n_2) respectively

Then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$.

Conditions for the lines to be Parallel

If the lines are parallel DR's are proportional

i.e $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

If the lines are parallel DC's are same

i.e $(l_1, m_1, n_1) = (l_2, m_2, n_2)$

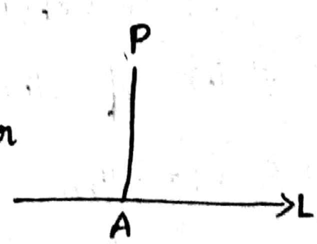
Conditions for the lines to be perpendicular

1) $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (In terms of DR's)

2) $l_1l_2 + m_1m_2 + n_1n_2 = 0$ (In terms of DC's)

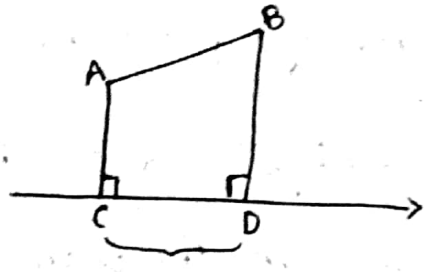
Definition of Projection of a Point P on a line L

Projection of a Point P on a line L is defined as the foot of the perpendicular from the line L.

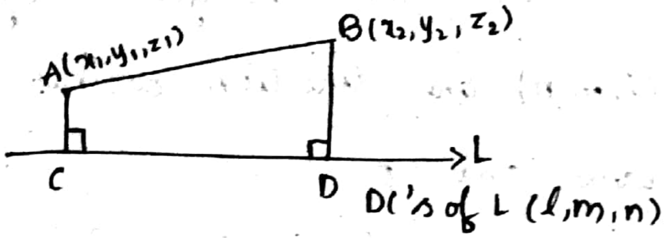


Projection of a line on another line

Projection of line AB on line 'L' is defined as CD where C & D are feet of the perpendiculars drawn from the points A and B on the line L respectively.



Note:- $CD = AB\cos\theta$; 'θ' is angle b/w the lines AB & CD.



$$\boxed{CD = l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)}$$

Plane

A plane is defined as a surface with the property that the line joining any two points on it (surface) lies completely on it.

Normal to a Plane

A normal to a Plane is defined as a line perpendicular to the Plane.

Various Forms of Plane

General Form

Normal Form

Intercept Form

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1) Equation of a plane in the General form

Equation of a plane in the general form is given by $ax+by+cz+d=0$ where (a,b,c) are direction ratios of normal to the plane.

2) Normal Form

Equation of a plane in normal form is given by $lx+my+nz=p$ where (l,m,n) are direction cosines of the normal to the plane and p.

p = length of the perpendicular from the origin to the plane.

3) Intercept Form

Equation of a plane in the intercept form is given by

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$; where (a,b,c) are called intercepts made, by the plane on the co-ordinate axis. Here the plane intersects X-axis in the point $A(a,0,0)$, Y-axis at the point $B(0,b,0)$, Z-axis at the point $C(0,0,c)$.

* Equation of a Plane passing throug the point $A(x_1,y_1,z_1)$ is given by $\boxed{a(x-x_1)+b(y-y_1)+c(z-z_1)=0}$ where a,b,c are direction ratios of the normal to the plane.

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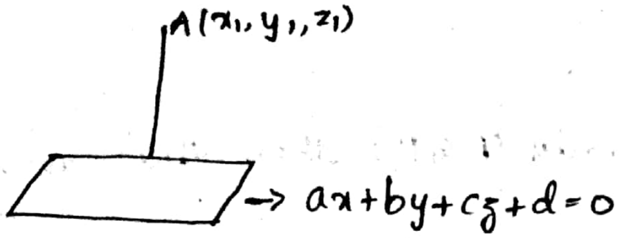
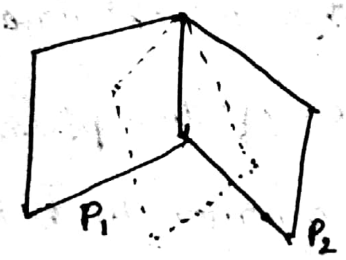


length of the perpendicular from point $A(x_1, y_1, z_1)$ to plane $ax+by+cz+d=0$ is

$$\left| \frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}} \right|$$

* Equation of a plane through the intersection of two planes

Let a plane P_1 & P_2 is



Def of Angle between two planes

Angle b/w two planes is defined as the angle b/w their normals.

Let Eq^ns of 2 planes P_1 & P_2 be

$$a_1x+b_1y+c_1z+d_1=0, \quad a_2x+b_2y+c_2z+d_2=0$$

DR's of normal to $P_1 = (a_1, b_1, c_1)$

DR's of normal to $P_2 = (a_2, b_2, c_2)$

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$\therefore \cos\theta = \dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$

Note: If the planes are parallel (P_1 & P_2) then their normals are also parallel. i.e dr's are proportional

$$\text{i.e} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

- If the planes P_1 & P_2 are perpendicular then their normals are also perpendicular.

$$\text{i.e} \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$$

LINES

- If two planes P_1 & P_2 intersect each other we obtains a line L known as line of intersection of two planes.
- Since every point on the line L is common to both the planes P_1 & P_2 we have ~~all the~~ every point on the line L satisfies the equations of both the planes.

$\therefore$ Eq^n of the L in unsymmetrical form

$$a_1x+b_1y+c_1z+d_1=0, \quad a_2x+b_2y+c_2z+d_2=0$$

(or)

$$a_1x+b_1y+c_1z+d_1 = a_2x+b_2y+c_2z+d_2 = 0$$

- Eq^n of line L in symmetrical form passing through pt (x_1, y_1, z_1) and having DR's as (a, b, c) is given by

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

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* If the DC's of L are (l, m, n)

The equation of L is given by

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

* Equation of a line passing through two points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

* General co-ordinates of a point on a line

Let the equation of 'L' be $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ (say)

then every point on L $= (lr+x_1, mr+y_1, nr+z_1)$

* By giving different values to 'r' we obtain different points on the line L.

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SPHERES

A sphere is a local of a point which remains at a constant distance from a fixed point.

Here the fixed point is known as centre of sphere.

Constant distance is known as radius of the sphere

* Equation of a Sphere whose centre is (a,b,c) and having radius 'r' then it is given by

$$\boxed{(x-a)^2+(y-b)^2+(z-c)^2 = r^2}$$

$$\Rightarrow x^2+y^2+z^2-2ax-2by-2cz+(a^2+b^2+c^2-r^2)=0$$

From the above equation we observe:-

→ A sphere is of 2^{nd} degree - x, y & z

→ The product terms xy, yz & zx -

→ The co-efficients of x^2, y^2 & z^2 is unity i.e 1

Equation of a sphere in the General form

It is given by $\boxed{x^2+y^2+z^2+2ux+2vy+2wz+d=0}$

$$\Rightarrow (x+u)^2+(y+v)^2+(z+w)^2 = u^2+v^2+w^2-d \; [u^2+v^2+w^2=d]$$

$$\Rightarrow [x-(-u)]^2+[y-(-v)]^2+[z-(-w)]^2 = (\sqrt{u^2+v^2+w^2-d})^2$$

∴ Centre of Sphere = $(-u,-v,w)$ and

radius $r = \sqrt{u^2+v^2+w^2-d}$

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→ Equation of a sphere with $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as the end points of a diameter is given by

$$\boxed{(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0}$$

Problems

1) Prove that the equation $ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$ represents a sphere and find its centre and radius.

sol. Consider $ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$

Divide the equation by 'a'.

$$\Rightarrow x^2+y^2+z^2+\frac{2ux}{a}+\frac{2vy}{a}+\frac{2wz}{a}+\frac{d}{a}=0$$

$$\Rightarrow \left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2 = \frac{u^2}{a^2}+\frac{v^2}{a^2}+\frac{w^2}{a^2}-\frac{d}{a}$$

$$\Rightarrow \left[x-\left(-\frac{u}{a}\right)\right]^2+\left[y-\left(-\frac{v}{a}\right)\right]^2+\left[z-\left(-\frac{w}{a}\right)\right]^2 = \frac{u^2+v^2+w^2-ad}{a^2}$$

∴ The given equation represents a sphere whose centre is $\left(\frac{-u}{a}, \frac{-v}{a}, \frac{-w}{a}\right)$ and radius $= \frac{\sqrt{u^2+v^2+w^2-d}}{a}$

2) Find the equations of the sphere passing through $O(0,0,0)$, $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$

sol. Let the general eqⁿ of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

eqⁿ (1) passes through $O(0,0,0)$

$$\Rightarrow \boxed{d=0}$$

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Eq^n ① passes through $A(a,0,0) \Rightarrow a^2 + 2ua = 0$ $[d=0]$

Eq^n ① passes through $B(0,b,0) \Rightarrow b^2 + 2vb = 0$ $[d=0]$

Eq^n ① passes through $C(0,0,c) \Rightarrow c^2 + 2wc = 0$ $[d=0]$

① $\Rightarrow a^2 + 2ua = 0 \Rightarrow a + 2u = 0 \Rightarrow \boxed{u = -\frac{a}{2}}$

② $\Rightarrow b^2 + 2vb = 0 \Rightarrow b + 2v = 0 \Rightarrow \boxed{v = -\frac{b}{2}}$

③ $\Rightarrow c^2 + 2wc = 0 \Rightarrow c + 2w = 0 \Rightarrow \boxed{w = -\frac{c}{2}}$

$\therefore$ The required equation of a sphere is

$$x^2 + y^2 + z^2 + \cancel{2}\left(-\frac{a}{\cancel{2}}\right)x + \cancel{2}\left(-\frac{b}{\cancel{2}}\right)y + \cancel{2}\left(-\frac{c}{\cancel{2}}\right)z = 0$$

$$\therefore \boxed{x^2 + y^2 + z^2 - ax - by - cz = 0}$$

• Find the centres and radii of the following spheres

1) $x^2 + y^2 + z^2 - 6x + 8y - 10z + 1 = 0$

comparing the given eq^n with the standard form

$2u = -6$ $\quad 2v = 8$ $\quad 2w = -10$ $\quad d = 1$

$u = -3$ $\quad v = 4$ $\quad w = -5$

$\therefore$ Centre $= (+3, -4, +5)$

radius $r = \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{9 + 16 + 25 - 1}$

$= \sqrt{49}$

$\therefore r = 7$ units

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2) $2x^2+2y^2+2z^2-2x+4y+2z+3=0$

divide the equation by 2

$$x^2+y^2+z^2-x+2y+z+\frac{3}{2}=0$$

comparing the eqn with standard form

$2u=-1$ $\quad$ $2v=2$ $\quad$ $2w=1$

$u=\frac{-1}{2}$ $\quad$ $v=1$ $\quad$ $w=\frac{1}{2}$

$\therefore$ Centre $=\left(\frac{1}{2},-1,\frac{-1}{2}\right)$

radius $r=\sqrt{\left(-\frac{1}{2}\right)^2+1^2+\left(\frac{1}{2}\right)^2-\frac{3}{2}}$

$=\sqrt{\frac{1}{4}+1+\frac{1}{4}-\frac{3}{2}}$

$=\sqrt{\frac{3}{2}-\frac{3}{2}}$

$\therefore r=0$

Note: A sphere whose radius is zero is known as a point sphere.

• Find the equation of the sphere described on the join of the points $A(2,-3,4)$, $B(-5,6,-7)$

Sol. Eqn of the sphere with $A(x_1,y_1,z_1)$ & $B(x_2,y_2,z_2)$ as the end points of the diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

Required eqn of the sphere is

$$(x-2)(x+5)+(y+3)(y-6)+(z-4)(z+7)=0$$

$$x^2+5x-2x-10+y^2-6y+3y-18+z^2+7z-4z-28=0$$

$$x^2+y^2+z^2+3x-3y+3z-56=0$$

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• Find the equation to the sphere through the points

$O(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$ & $(1,2,3)$

Sol: Let the general eq^n of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

(1) passes through $(0,0,0)$ $\Rightarrow \boxed{d=0}$

(1) passes through $(0,1,-1)$ $\Rightarrow 2+2v-2w=0$

$\Rightarrow 1+v-w=0$ --- (2)

(1) passes through $(-1,2,0)$ $\Rightarrow 5-2u+4v=0$ --- (3)

(1) passes through $(1,2,3)$ $\Rightarrow 14+2u+4v+6w=0$ --- (4)

from (2)

$w=1+v$

from (3)

$2u=5+4v$

$u=\frac{5}{2}+2v$

$14+2\left(\frac{5}{2}+2v\right)+4v+6(1+v)=0$

$14+5+4v+4v+6+6v=0$

$14v+25=0$

$14v=-25$

$v=-\frac{25}{14}$

$w=1-\frac{25}{14}=\frac{-11}{14}$

$u=\frac{5}{2}-\cancel{2}\left(\frac{25}{\cancel{14}_7}\right)=\frac{5}{2}-\frac{25}{7}=\frac{35-50}{14}=\frac{-15}{14}$

$u=\frac{-15}{14}$, $v=\frac{-25}{14}$, $w=\frac{-11}{14}$

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$\therefore$ The general eq^n of the sphere is

$$x^2+y^2+z^2+2\left(\frac{-15}{14}\right)x+2\left(\frac{-25}{14}\right)y+2\left(\frac{-11}{14}\right)z=0$$

$$\therefore\ x^2+y^2+z^2-\frac{15}{7}x-\frac{25}{7}y-\frac{11}{7}z=0$$

PCP-2

Find the equation of the sphere through the four points (0,0,0), (-a,b,c), (a,-b,+c), (a,b,-c)

Sol Let the general eq^n of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

(1) passes through (0,0,0) $\Rightarrow \boxed{d=0}$

(1) passes through $(-a,b,c) \Rightarrow a^2+b^2+c^2-2au+2bv+2cw=0$ --- (2)

(1) passes through $(a,-b,c) \Rightarrow a^2+b^2+c^2+2au-2bv+2cw=0$ --- (3)

(1) passes through $(a,b,-c) \Rightarrow a^2+b^2+c^2+2au+2bv-2cw=0$ --- (4)

(2) - (3)

$$\Rightarrow \cancel{a^2}+\cancel{b^2}+\cancel{c^2}-2au+2bv+\cancel{2cw}-\cancel{a^2}-\cancel{b^2}-\cancel{c^2}-2au+2bv-\cancel{2cw}=0$$

$$-4au+4bv=0$$

$$-au+bv=0$$

$$au-bv=0 \quad \text{--- (5)}$$

(3) - (4)

$$\Rightarrow \cancel{a^2}+\cancel{b^2}+\cancel{c^2}+\cancel{2au}-2bv+2cw-\cancel{a^2}-\cancel{b^2}-\cancel{c^2}-\cancel{2au}-2bv+2cw=0$$

$$-4bv+4cw=0$$

$$bv-cw=0 \quad \text{--- (6)}$$

From (5) $au=bv$ $\qquad$ from (6) $bv=cw$

$u=\frac{bv}{a}$ $\qquad$ $w=\frac{bv}{c}$

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substitute eq^n ② (u, w)

$$a^2+b^2+c^2-2\left(\frac{bv}{a}\right)a+2bv+2\left(\frac{bv}{c}\right)c=0$$

$$\Rightarrow a^2+b^2+c^2-\cancel{2bv}+\cancel{2bv}+2bv=0$$

$$\Rightarrow a^2+b^2+c^2+2bv=0$$

$$\Rightarrow a^2+b^2+c^2=-2bv$$

$$\Rightarrow v=\frac{-(a^2+b^2+c^2)}{2b}.$$

$$\cancel{u=\frac{bv}{a}=\frac{b(-a^2+b^2+c^2)}{}}$$

$$u=\frac{bv}{a}=-\frac{\cancel{b}(a^2+b^2+c^2)}{2a\cancel{b}}. \qquad w=\frac{bv}{c}=\frac{-\cancel{b}(a^2+b^2+c^2)}{2\cancel{b}c}$$

$$u=-\frac{(a^2+b^2+c^2)}{2a} \qquad w=-\frac{(a^2+b^2+c^2)}{2c}.$$

$$v=\frac{-(a^2+b^2+c^2)}{2b}.$$

Substitute (u, v, w) in ①

The general eq^n of the sphere is

$$x^2+y^2+z^2+\cancel{2}\left[\frac{-(a^2+b^2+c^2)}{\cancel{2}a}\right]x+\cancel{2}\left[\frac{-(a^2+b^2+c^2)}{\cancel{2}b}\right]y+\cancel{2}\left[\frac{-(a^2+b^2+c^2)}{\cancel{2}c}\right]z=0$$

$$\therefore x^2+y^2+z^2+\left[\frac{-(a^2+b^2+c^2)}{a}\right]x+\left[\frac{-(a^2+b^2+c^2)}{b}\right]y+\left[\frac{-(a^2+b^2+c^2)}{c}\right]z=0.$$

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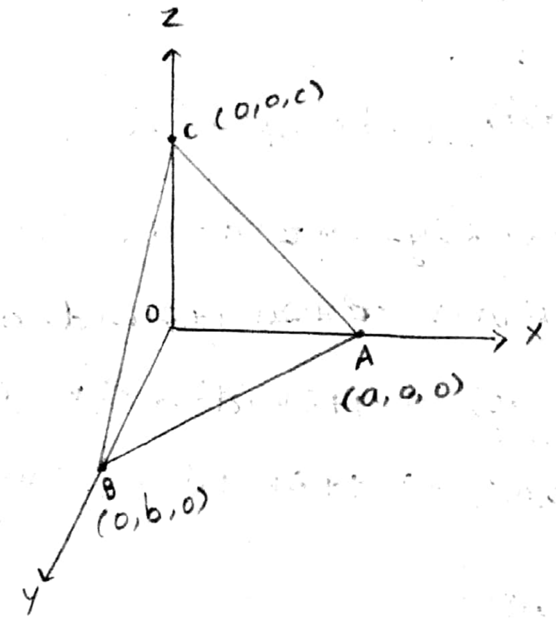
• Find the equation of the sphere circumscribing the tetrahedron whose faces are $x=0, y=0, z=0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$



Sol.

Since, the required sphere circumscribes the tetrahedron we have the sphere passes through the points

$O(0,0,0)$, $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$

Let General Equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{——} (1)$$

(1) passes through $O(0,0,0) \Rightarrow \boxed{d=0}$

(1) passes through $A(a,0,0) \Rightarrow a^2+2ua=0 \rightarrow a+2u=0 \Rightarrow \boxed{a=-u/2}$

(1) passes through $B(0,b,0) \Rightarrow b^2+2vb=0 \Rightarrow b+2v=0 \Rightarrow \boxed{b=-v/2}$

(1) passes through $C(0,0,c) \Rightarrow c^2+2wc \Rightarrow c+2w=0 \Rightarrow \boxed{c=-w/2}$

$u=\frac{-a}{2}, \; v=\frac{-b}{2}, \; w=\frac{-c}{2}$ in (1)

$$(1) \Rightarrow \boxed{x^2+y^2+z^2-ax-by-cz=0}$$

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• Obtain the equation of the sphere which passes through the 3 points (1,0,0), (0,1,0), (0,0,1) and has its radius as small as possible.

Sol. Let the general equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

(1) passes through $(1,0,0) \Rightarrow$ $\cancel{x^2+2u}$ $1+2u+d=0 \Rightarrow u=-\frac{(d+1)}{2}$

(1) passes through $(0,1,0) \Rightarrow 1+2v+d=0 \Rightarrow v=-\frac{(d+1)}{2}$

(1) passes through $(0,0,1) \Rightarrow 1+2w+d=0 \Rightarrow w=-\frac{(d+1)}{2}$

$$u=v=w=-\frac{(d+1)}{2}$$

radius $r=\sqrt{u^2+v^2+w^2-d}$

$$=\sqrt{\frac{(d+1)^2}{4}+\frac{(d+1)^2}{4}+\frac{(d+1)^2}{4}-d}$$

$$=\sqrt{\frac{3(d+1)^2}{4}-d}$$

$$=\sqrt{\frac{3d^2+6d+3-4d}{4}}$$

$$=\sqrt{\frac{3d^2+2d+3}{4}}$$

$$r=\frac{1}{2}\sqrt{3d^2+2d+3}$$

For radius to be as small as possible i.e for minimum radius we equate $\boxed{\frac{dr}{dd}=0}$

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$$\frac{d}{dd}\frac{1}{2}\sqrt{3d^2+2d+3}=0$$

$$\frac{1}{2}\cdot\frac{(6d+2)}{2\sqrt{3d^2+2d+3}}=0$$

$$\frac{1}{4}\cdot\frac{(6d+2)}{\sqrt{3d^2+2d+3}}=0$$

$$\frac{1}{\cancel{4}_2}\cdot\frac{\cancel{2}(3d+1)}{\sqrt{3d^2+2d+3}}=0$$

$$\frac{(3d+1)}{2\sqrt{3d^2+2d+3}}=0$$

$$3d+1=0$$

$$\boxed{d=-\frac{1}{3}}$$

$$u=v=w=-\frac{1}{2}\left(-\frac{1}{3}+1\right)$$

$$=-\frac{1}{2}\left(\frac{-1+3}{3}\right)$$

$$=-\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right)$$

$$=\frac{-1}{3}$$

$$u=v=w=\frac{-1}{3}$$

∴ Req eqn of the sphere is

$$x^2+y^2+z^2+2\left(-\frac{1}{3}\right)x+2\left(-\frac{1}{3}\right)y+2\left(-\frac{1}{3}\right)z-\frac{1}{3}=0$$

$$\therefore\ \boxed{x^2+y^2+z^2-\frac{2}{3}x-\frac{2}{3}y-\frac{2}{3}z-\frac{1}{3}=0}$$

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• Obtain a sphere having its centre on the line $5y+2z = 0 = 2x-3y$ and passing through 2 points $(0,-2,-4)$ & $(2,-1,-1)$.

sol. Let the general eqⁿ of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{—} (1)$$

centre $(-u,-v,-w)$

$\because$ centre lies on the given line

$\Rightarrow -5v-2w=0$

$5v+2w=0 \quad \text{—} (2)$

$\Rightarrow -2u+3v=0$

$-2u+3v=0 \quad \text{—} (3)$

$(2) \Rightarrow 2w=-5v$

$$\boxed{w=\frac{-5v}{2}}$$

$(3) \Rightarrow 2u=+3v$

$$\boxed{u=\frac{+3v}{2}}$$

(1) passes through $(0,-2,-4) \Rightarrow 4+16-4v-8w+d=0 \Rightarrow 20-4v-8w+d=0$ — (4)

(1) passes through $(2,-1,-1) \Rightarrow 4+1+1+4u-2v-2w+d=0 \Rightarrow$

$6+4u-2v-2w+d=0 \quad \text{—} (5)$

substituting u,w in (4)

$$20-4v-\cancel{8}^{4}\left(\frac{-5v}{\cancel{2}}\right)+d=0$$

$$20-4v+20v+d=0$$

$$20+16v+d=0$$

$$\boxed{d=-16v-20}$$

substituting u,w,d in (5)

$$6+\cancel{4}^{2}\left(+\frac{3v}{\cancel{2}}\right)-2v-\cancel{2}\left(\frac{-5v}{\cancel{2}}\right)-16v-20=0$$

$6+6v-2v+5v-16v-20=0 \Rightarrow -7v-14=0 \Rightarrow v=-2.$

$$\boxed{v=-2}$$

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$u = \frac{3v}{2} = \frac{3}{2}(-2) = u = -3$

$w = -\frac{5v}{2} = -\frac{5}{2}(-2) = w = 5$

$w = -16v - 20 \Rightarrow -16(-2) - 20$

$\Rightarrow 32 - 20$

$\boxed{w = 8}$

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$u = \frac{3v}{2} = \frac{3}{2}(-2) = u=-3$

$w = -\frac{5v}{2} = -\frac{5}{2}(-2) = w=5$

$d = -16v-20 \Rightarrow -16(-2)-20$

$\Rightarrow 32-20 \Rightarrow 12.$

• Find the equation of the sphere passing through the origin and the points $(1,0,0)$, $(0,2,0)$ & $(0,0,3)$

sol. Let the general eqⁿ of the sphere be.

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{—} ①$$

① passes through origin $\Rightarrow \boxed{d=0}$

① passes through $(1,0,0) \Rightarrow 1+2u=0 \Rightarrow u=-\frac{1}{2}$

① passes through $(0,2,0) \Rightarrow 4+4v=0 \Rightarrow v=-1$

① passes through $(0,0,3) \Rightarrow 9+6w=0 \Rightarrow w=-3/2$

$$① \Rightarrow x^2+y^2+z^2+\cancel{2}\left(-\frac{1}{\cancel{2}}\right)x+2(-1)y+\cancel{2}\left(-\frac{3}{\cancel{2}}\right)z=0$$

$$\therefore x^2+y^2+z^2-x-2y-3z=0$$

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PCP-1

Find the equation of the sphere through 4 points $(4,-1,2)$ $(0,-2,3)$, $(1,-5,-1)$, $(2,0,1)$

sol. Let the eqn of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

(1) passes through $(4,-1,2) \Rightarrow 16+1+4+8u-2v+4w+d=0$

(1) passes through $(0,-2,3) \Rightarrow 4+9-4v+6w+d=0$

(1) passes through $(1,-5,-1) \Rightarrow 1+25+1+2u-10v-2w+d=0$

(1) passes through $(2,0,1) \Rightarrow 4+1+4u+2w+d=0$

$$21+8u-2v+4w+d=0 \quad \text{--- (2)}$$

$$13-4v+6w+d=0 \quad \text{--- (3)}$$

$$27+2u-10v-2w+d=0 \quad \text{--- (4)}$$

$$5+4u+2w+d=0 \quad \text{--- (5)}$$

(2) - (3)

$$8+8u+2v-2w=0 \quad \text{--- (6)}$$

(3) - (4)

$$-14-2u+6v+8w=0 \quad \text{--- (7)}$$

(4) - (5)

$$22-2u-10v-4w=0 \quad \text{--- (8)}$$

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Solving (8) & (7)

(7) - (8)

$-36+16v+12w=0$

$\Rightarrow \quad -9+4v+3w=0$ —— (9)

$4 \times (7)$

$-56-8u+24v+32w=0$ — (7)

(7) + (6)

$-48+26v+30w=0$

$-24+13v+15w=0$ —— (10)

(9) $\times 5 \Rightarrow \quad -45+20v+15w=0$

(9) - (10) $\Rightarrow 21-7v=0$

$21=7v$

$\boxed{v=3}$

from (9) $\Rightarrow \quad -9+12+3w=0$

$3+3w=0$

$\boxed{w=-1}$

from (6) $\Rightarrow \quad 8+8u+2(3)-2(-1)=0$

$8+8u+6+2=0$

$8u+16=0$

$\boxed{u=-2}$

from (2)

$\Rightarrow$ ~~21~~ $21-16-6-4+d=0$

$\boxed{d=5}$

$\therefore$ Required eq^n of sphere is

$\therefore \quad x^2+y^2+z^2-4x+6y-2z+5=0$

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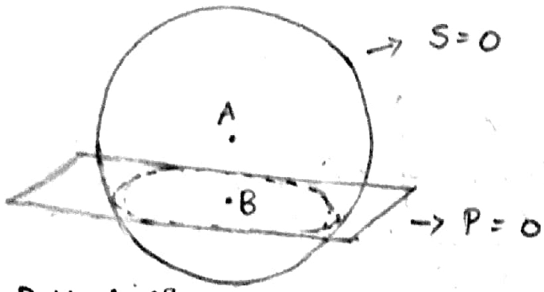
Circle

If a sphere and a plane intersect each other we obtain a circle also known as a plane section of a sphere.



Dotted lines = circle

Note: 1) Normal to a Plane is perpendicular to each and every line lying on the plane.

2) The line joining the centre of the sphere and the centre of the circle is always perpendicular to the plane.

Another def of Circle

The set of all points common to a sphere, $S=0$ and the plane, $P=0$ is defined as a circle.

* Equation of a circle is given by $\boxed{S=0, P=0}$.

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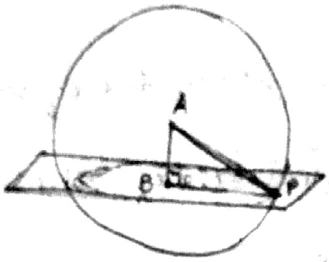
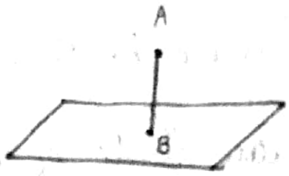
a circle.

Proof: Let the equation of the sphere be $S=0$ and plane $P=0$.





Let A be the centre of the sphere and 'P' is a point on the plane section of the sphere.

Let $\bar{AB}$ be a perpendicular drawn from the point 'A' to the plane.

['B' is the foot of the perpendicular].

Since, the normal to the plane is perpendicular to everyline lying on the plane.

$\Rightarrow AB \perp BP$

$\therefore$ From right angled triangle ΔABP we have,

$AB^2 + BP^2 = AP^2$.

$\Rightarrow \boxed{BP^2 = AP^2 - AB^2}$

Since, A and B are fixed points we get the BP is constants for all the points on the plane section.

$\therefore$ The locus of the point P i.e, the plane section of a sphere is a circle. where in the above diagram B is the centre of circle and the distance BP is radius.

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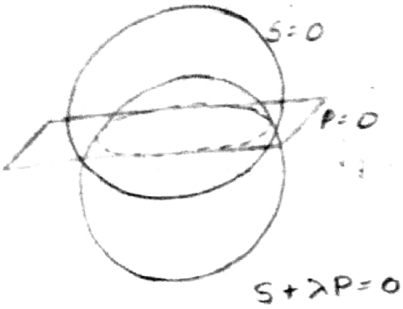
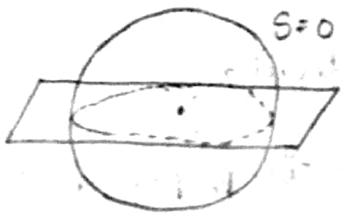


Let the equation of the circle be $S=0, P=0$ then, the equation of the sphere through the circle of intersection of the sphere and the plane is given by $\boxed{S+\lambda P=0}$

Def of great circle

The plane section of a sphere passing through the centre of the sphere is called great circle.



Note:

The centre and radius of a circle are same as the centre and radius of the sphere.

Theorem: Prove that the intersection of two spheres is a circle.

Proof: Let S_1 & S_2 be two spheres, whose equations are given by

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \quad \text{—— (1)}$$

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \quad \text{—— (2)}$$

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let point $P(x_1, y_1, z_1)$ be a locus point i.e, P is common to the spheres S_1 and S_2 substituting the 'P' co-ordinates in ① & ②

$\Rightarrow x_1^2+y_1^2+z_1^2+2u_1x_1+2v_1y_1+2w_1z_1+d_1=0$ —— ③

$\Rightarrow x_1^2+y_1^2+z_1^2+2u_2x_1+2v_2y_1+2w_2z_1+d_2=0$ —— ④

③ - ④

$\Rightarrow 2(u_1-u_2)x_1+2(v_1-v_2)y_1+2(w_1-w_2)z_1+2(d_1-d_2)=0$

$P(x_1, y_1, z_1)$ lies on $2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+2(d_1-d_2)=0$ —— ⑤

Being a first degree eqⁿ in x, y, z & represents a plane.

→ The points of intersection of two spheres are same as anyone of the spheres & the plane.

∴ The intersection of two spheres is a circle.

Note:

1) If the two spheres S_1 & S_2 intersect each other then, we obtain a circle, eqⁿ ⑤ is given by $S_1 - S_2 = 0$.

2) Let the eqⁿ of the circle $S_1 = 0, S_2 = 0$ also represented as $S_1 = 0,\ S_1 - S_2 = 0$

$S_2 = 0,\ S_1 - S_2 = 0$

3) Then, the eqⁿ of the sphere through the circle $S_1=0, S_2=0$ is $\boxed{S_1 + \lambda S_2 = 0}$

Alternatively, The eqⁿ of the sphere through the circle can be written as $\boxed{S_1 + \lambda(S_1 - S_2) = 0}$ (or) $\boxed{S_2 - \lambda(S_1 - S_2) = 0}$

$[\because S + \lambda P = 0]$

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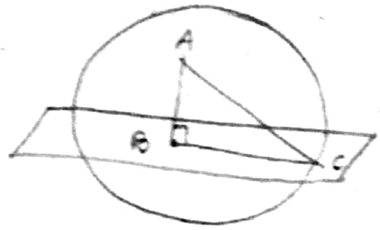
PCP-3

Find the centre and the radius of circle i.e is given by $x+2y+2z=15$, $x^2+y^2+z^2-2y-yz=11$.



A = centre of sphere = $(0,1,2)$

AC = radius of sphere = $\sqrt{0+1+4+11} = \sqrt{16} = 4$

Eq^n of the line AB (normal) passing through A $(0,1,2)$ & having DR's of AB are $(1,2,2)$ is

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{2} = r \text{(say)}$$

$\therefore$ Any point on the line AB, $B = (r, 2r+1, 2r+2)$

let $B = (r, 2r+1, 2r+2)$ for some value of r.

substituting in plane $x+2y+2z=15$

$\Rightarrow r+2(2r+1)+2(2r+2)-15=0$

$r+4r+2+4r+4-15=0$

$9r-9=0$

$r=1$

$\therefore$ B = centre of circle = $(1,3,4)$

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Distance b/w A, B is $AB = \sqrt{(1-0)^2+(3-1)^2+(4-2)^2}$

$= \sqrt{1+4+4}$

$= \sqrt{9}$

$AB = 3$

From ΔABC, Apply pythogaurus theorem

$AB^2 + BC^2 = AC^2$

$3^2 + BC^2 = 4^2$

$9 + BC^2 = 16$

$BC^2 = 16-9 = 7$

$BC = \sqrt{7}$ units

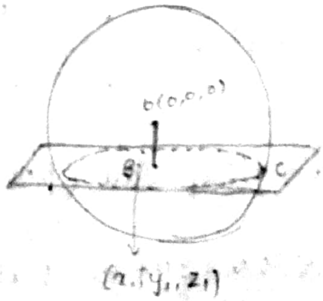
$\therefore$ Radius of the circle $= \sqrt{7}$ units.

• Find the equations of that section of the sphere $x^2+y^2+z^2=a^2$ of which a given internal point (x_1, y_1, z_1) is the centre.

Sol. Centre of the sphere $= O(0,0,0)$

Let B = centre of the circle

$B = (x_1, y_1, z_1)$



•Eq^n of the plane through (x_1, y_1, z_1) is $A(x-x_1) + B(y-y_1) + C(z-z_1) = 0$

where (a,b,c) are direction ratios of the normal to the plane.

Direction Ratios of normal to the plane. [Normal is OB]. $= (x_1-0, y_1-0, z_1-0) = (x_1, y_1, z_1) = (a, b, c)$

① $\Rightarrow x_1(x-x_1) + y_1(y-y_1) + z_1(z-z_1) = 0$

$\Rightarrow \boxed{x_1 x + y_1 y + z_1 z - (x_1^2+y_1^2+z_1^2) = 0}$

$\therefore$ The eq^{n}s of the circle is $\underset{S=0}{x^2+y^2+z^2-a^2=0}$, $\underset{P=0.}{x_1x+y_1y+z_1z-(x_1^2+y_1^2+z_1^2)=0}$

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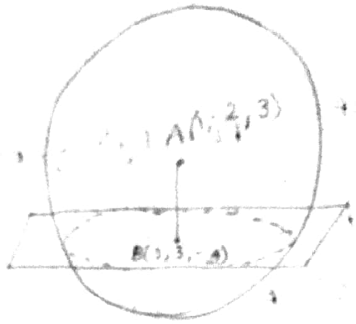
• Obtain the eq^{n}s of the circle lying on the sphere $x^2+y^2+z^2-2x+4y-6z+3=0$ and having its centre at $(2,3,-4)$



Sol: Eq^n of the plane passing through $B(2,3,-4)$ (x_1,y_1,z_1) is

$$a(x-2)+b(y-3)+c(z+4)=0 \quad \text{—(1)}$$

where (a,b,c) are the DR's of the normal to the plane.

DR's of the normal to the plane (line AB) $=(2-1,\ 3+2,\ -4-3)$

$$=(1,5,-7)$$
$$(a,b,c)$$

Substitute DR's in (1)

$$1(x-2)+5(y-3)-7(z+4)=0$$

$$x-2+5y-15-7z-28=0$$

$$\Rightarrow x+5y-7z-45=0$$

$\therefore$ Eq^n of the circle is $x^2+y^2+z^2-2x+4y-6z+3=0,\ x+5y-7z-45=0$

$$[S=0, P=0].$$

Imp (5M)

• Find the eq^n of the sphere through the circle $x^2+y^2+z^2=9,\ 2x+3y+4z=5$ and the point $(1,2,3)$.

Sol: Eq^n of the given circle is

$$x^2+y^2+z^2=9,\ 2x+3y+4z=5$$

$$x^2+y^2+z^2-9=0,\ 2x+3y+4z-5=0$$

$$S=0,\ P=0$$

Eq^n of the sphere through the given circle is

$$\Rightarrow \boxed{S+\lambda P=0}$$

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i.e, $(x^2+y^2+z^2-9)+\lambda(2x+3y+4z-5)=0$ —— ①

① passes through $(1,2,3)$

$\Rightarrow (1+4+9-9)+\lambda(2+6+12-5)=0$

$5+\lambda(15)=0$

$\lambda(15)=-5$

$\boxed{\lambda=-\frac{1}{3}}$

substitute λ in ①

$\Rightarrow (x^2+y^2+z^2-9)-\frac{1}{3}(2x+3y+4z-5)=0$

$\therefore x^2+y^2+z^2-\frac{2}{3}x-y-\frac{4}{3}z-\frac{22}{3}=0$

• Find the eq^n of the sphere through the circle $x^2+y^2+z^2+2x+3y+6=0$, $x-2y+4z-9=0$ and the centre of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$.

Sol: Eq^n of the given circle is

$x^2+y^2+z^2+2x+3y+6=0, x-2y+4z-9=0$

$[S=0, P=0]$

$\therefore$ Eq^n of the sphere through the given circle is

$\boxed{S+\lambda P=0}$

$\Rightarrow (x^2+y^2+z^2+2x+3y+6)+\lambda(x-2y+4z-9)=0$ —— ①

① passes through the centre of the sphere

$x^2+y^2+z^2-2x+4y-6z+5=0$

i.e ① passes through $(1,-2,3)$

$\Rightarrow (1+4+9+2-\cancel{6}+\cancel{6})+\lambda(1+4+12-9)=0$

$16+\lambda(8)=0$

$8\lambda=-16$

$\lambda=-2$

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substitute λ in ①

$$x^2+y^2+z^2+2x+3y+6-2(x-2y+4z-9)=0$$

$$x^2+y^2+z^2+\cancel{2x}+3y+6-\cancel{2x}+4y-8z+18=0$$

$$\therefore \quad x^2+y^2+z^2+7y-8z+24=0$$

• Show that the eqⁿ of the sphere having its co-ordinates on plane $4x-5y-z=3$ and passing through the circles with equations $x^2+y^2+z^2-2x-3y+4z+8=0$, (S_1) $x^2+y^2+z^2+4x+5y-6z+2=0$ (S_2) is $x^2+y^2+z^2+7x+9y-11z-1=0$.

Sol. Eqⁿ of the given circle is of the form $S_1=0, S_2=0$

Eqⁿ of the plane $\Rightarrow \boxed{S_1-S_2=0}$

$$\Rightarrow -6x-8y+10z+6=0 \Rightarrow 3x+4y-5z-3=0$$

$\therefore$ The eqⁿ of the sphere through the given circle is

$$(x^2+y^2+z^2-2x-3y+4z+8)+\lambda(3x+4y-5z-3)=0$$

$$\Rightarrow x^2+y^2+z^2+(3\lambda-2)x+(4\lambda-3)y+(4-5\lambda)z+(8-3\lambda)=0 \quad \text{—— ①}$$

$$\therefore \text{Centre of sphere} = \left(\frac{2-3\lambda}{2}, \frac{3-4\lambda}{2}, \frac{5\lambda-4}{2}\right)$$

Since the centre lies on the plane $\Rightarrow 4x-5y-z-3=0$

$$\Rightarrow \overset{2}{\cancel{4}}\left(\frac{2-3\lambda}{\cancel{2}}\right)-5\left(\frac{3-4\lambda}{2}\right)-\left(\frac{5\lambda-4}{2}\right)-3=0$$

$$\Rightarrow 4-6\lambda-\left(\frac{15-20\lambda}{2}\right)-\left(\frac{5\lambda-4}{2}\right)-3=0$$

$$\Rightarrow 8-12\lambda-15+20\lambda-5\lambda+4\overset{-3}{}=0$$

$$\Rightarrow 3\lambda-\overset{9}{\cancel{8}}=0$$

$$\Rightarrow \boxed{\lambda=3}$$

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substitute λ in 1

$(x^2+y^2+z^2-2x-3y+4z+8)+3(3x+4y-5z-3)=0$

$x^2+y^2+z^2-2x-3y+4z+8+9x+12y-15z-9=0$

$\therefore\ x^2+y^2+z^2+7x+9y-11z-1=0$

$S=0$

• Obtain the eq^n of the sphere having the circle $x^2+y^2+z^2+10y-4z-8=0,\ \underset{P=0}{x+y+z=3}$ as the great circle.

sol. Eq^n of the given circle is of the form $\boxed{S=0, P=0}$

$\therefore$ Eq^n of the sphere having the circle is $\boxed{S+\lambda P=0}$

$\Rightarrow (x^2+y^2+z^2+10y-4z-8)+\lambda(x+y+z-3)=0$ —— ①

$\Rightarrow x^2+y^2+z^2+\lambda x+(10+\lambda)y+(\lambda-4)y+(-8-3\lambda)=0$

Centre of sphere $=\left(\frac{-\lambda}{2}, \frac{-10-\lambda}{2}, \frac{4-\lambda}{2}\right)$

Since the given circle is a great circle for the required sphere, we have the centre of the sphere lies on the plane.

substituting we get, $-\frac{\lambda}{2}+\left(\frac{-10-\lambda}{2}\right)+\left(\frac{4-\lambda}{2}\right)-3=0$

$-\lambda-10-\lambda+4-\lambda-6=0$

$-3\lambda-12=0$

$\boxed{\lambda=-4}$

① $\Rightarrow (x^2+y^2+z^2+10y-4z-8)-4(x+y+z-3)=0$

$x^2+y^2+z^2+10y-4z-8-4x-4y-4z+12=0$

$\therefore\ x^2+y^2+z^2-4x+6y-8z+4=0$

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• A sphere S has the points (0,1,0), (3,−5,2) as the end points of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as the great circle.

Sol Eq^n of the sphere S with the points (0,1,0) & (3,−5,2) as the end points of diameter is

$$\Rightarrow (x-0)(x-3)+(y-1)(y+5)+(z-0)(z-2)=0$$

$$\Rightarrow x^2+y^2+z^2-3x+4y-2z-5=0$$

∴ The eq^n of the sphere passing through the given circle is

$\boxed{S+\lambda P=0}$

$$\Rightarrow (x^2+y^2+z^2-3x+4y-2z-5)+\lambda(5x-2y+4z+7)=0 \quad \text{—(1)}$$

$$\Rightarrow x^2+y^2+z^2+(5\lambda-3)x+(4-2\lambda)y+(4\lambda-2)z+(7\lambda-5)=0$$

Centre of sphere $=\left(\frac{3-5\lambda}{2}, \frac{2\lambda-4}{2}, \frac{2-4\lambda}{2}\right)$

For a great circle, we substitute centre in plane eq^n

$$\Rightarrow 5x-2y+4z+7=0$$

$$5\left(\frac{3-5\lambda}{2}\right)-2\left(\frac{2\lambda-4}{2}\right)+4\left(\frac{2-4\lambda}{2}\right)+7=0$$

$$15-25\lambda-4\lambda+8+8-16\lambda+14=0$$

$$-45\lambda+45=0$$

$$\boxed{\lambda=1}$$

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substitute λ in (1)

$$x^2+y^2+z^2-3x+4y-2z-5+5x-2y+4z+7=0$$

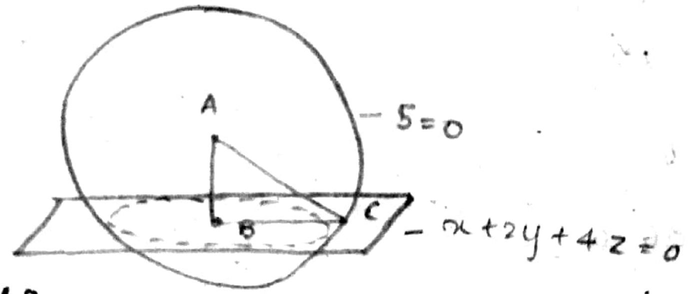
$$\therefore\ x^2+y^2+z^2+2x+2y+2z+2=0$$

* Obtain the eqn of the sphere which passes through the circle $x^2+y^2=4, z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

sol Eqn of given circle is $x^2+y^2=0, z=0$
$[S=0, P=0]$

$\therefore$ The eqn of the sphere passing through the given circle is. $\boxed{S+\lambda P=0}$

$$\Rightarrow (x^2+y^2+z^2-4)+\lambda(z)=0 \quad \text{—— (1)}$$



BC = radius of the circle $=3$ [given]

A = centre of sphere (1) $\Rightarrow (0,0,-\frac{\lambda}{2})$

AC = Radius of the sphere $=\sqrt{0+0+\frac{\lambda^2}{4}+4}$

$$\Rightarrow AC=\sqrt{\frac{\lambda^2}{4}+4}$$

AB = length of the perpendicular from pt to plane $x+2y+2z=0$ $\Rightarrow$ length of perpendicular from $A(0,0,-\frac{\lambda}{2})$ to the plane.

$$= \left| \frac{0+2(0)+2\left(-\frac{\lambda}{2}\right)}{\sqrt{1+4+4}} \right|$$

$$\boxed{AB = \frac{\lambda}{3}}$$

from the right-angled triangle ΔABC we have.,

$$AB^2 + BC^2 = AC^2$$

$$AB = \frac{\lambda}{3}, \; BC = 3, \; \sqrt{\frac{\lambda^2}{4}+4}$$

$$\Rightarrow \left(\frac{\lambda}{3}\right)^2 + 9 = \frac{\lambda^2}{4} + 4$$

$$\Rightarrow \frac{\lambda^2}{9} + 9 = \frac{\lambda^2}{4} + 4$$

$$\Rightarrow \frac{\lambda^2}{9} - \frac{\lambda^2}{4} = -5$$

$$\Rightarrow \frac{\cancel{-5}\lambda^2}{36} = \cancel{-5}$$

$$\Rightarrow \lambda^2 = 36$$

$$\Rightarrow \lambda = \sqrt{36} = 6$$

$$\boxed{\lambda = \pm 6}$$

substitute 'λ' in (1)

$$\therefore \; (x^2+y^2+z^2-4) \pm 6z = 0$$

$$\Rightarrow x^2+y^2+z^2 \pm 6z - 4 = 0$$

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• Show that the two circles $2(x^2+y^2+z^2)+8x-13y+17z-17=0$

~~2x+5~~ $2x+y-3z+1=0$

$x^2+y^2+z^2+3x-4y+3z=0,\ x-y+2z-4=0$

lie on the ~~samp~~ same sphere and find its equation.

Sol Eqⁿ of the fst circle is

$$x^2+y^2+z^2+4x-\frac{13}{2}y+\frac{17}{2}z-\frac{17}{2}=0,\quad 2x+y-3z+1=0$$

$$[S_1=0, P_1=0]$$

Eqⁿ of the 2nd circle is

$$x^2+y^2+z^2+3x-4y+3z=0,\ x-y+2z-4=0$$

$$[S_2=0, P_2=0]$$

Eqⁿ of the sphere passing through the fst circle is

$$\left(x^2+y^2+z^2+4x-\frac{13}{2}y+\frac{17}{2}z-\frac{17}{2}\right)+\lambda_1(2x+y-3z+1)=0$$

$$\Rightarrow x^2+y^2+z^2+(4+2\lambda_1)x+\left(\lambda_1-\frac{13}{2}\right)y+\left(\frac{17}{2}-3\lambda_1\right)z+\left(\lambda_1-\frac{17}{2}\right)=0 \quad \text{--- (1)}$$

Eqⁿ of the sphere passing through 2nd circle is

$$(x^2+y^2+z^2+3x-4y+3z)+\lambda_2(x-y+2z-4)=0$$

$$\Rightarrow x^2+y^2+z^2+(3+\lambda_2)x+(-4-\lambda_2)y+(3+2\lambda_2)z-4\lambda_2=0 \quad \text{--- (2)}$$

If the given circles lie on the same sphere then comparing the co-efficients of x, y, z and constants

$x \Rightarrow\ 4+2\lambda_1 = 3+\lambda_2 \Rightarrow 2\lambda_1-\lambda_2+1=0$ --- (3)

$y \Rightarrow\ \lambda_1-\frac{13}{2} = -4-\lambda_2 \Rightarrow \lambda_1+\lambda_2-\frac{5}{2}=0$ --- (4)

$z \Rightarrow\ \frac{17}{2}-3\lambda_1 = 3+2\lambda_2 \Rightarrow 3\lambda_1+2\lambda_2-\frac{11}{2}=0$ --- (5)

$c \Rightarrow\ \lambda_1-\frac{17}{2} = -4\lambda_2 \Rightarrow \lambda_1+4\lambda_2-\frac{17}{2}=0$ --- (6)

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Solving ③ & ④

③ + ④ $\Rightarrow 3\lambda_1 - \frac{3}{2} = 0 \Rightarrow \boxed{\lambda_1 = \frac{1}{2}}$

substitute λ_1 in ③

$$\Rightarrow \cancel{2}\left(\frac{1}{\cancel{2}}\right) - \cancel{\frac{3}{3}}\lambda_2 + 1 = 0$$

$$\Rightarrow \boxed{\lambda_2 = 2}$$

Substituting λ_1 & λ_2 in ⑤ & ⑥

⑤ $\Rightarrow 3\left(\frac{1}{2}\right) + 2(2) - \frac{11}{2}$

$$\Rightarrow \frac{3}{2} + 4 - \frac{11}{2}$$

$$\Rightarrow -\frac{8}{2} + 4$$

$$\Rightarrow -4 + 4 = 0.$$

⑥ $\Rightarrow \frac{1}{2} + 4(2) - \frac{17}{2}$

$$\Rightarrow \frac{1}{2} + 8 - \frac{17}{2}$$

$$\Rightarrow -8 + 8$$

$$= 0$$

Since, the values of λ_1 & λ_2 satisfy ⑤ & ⑥ we get the two given circles lie on the same sphere.

$\therefore$ The eqn of the sphere containing the two given circles is given by

$$\therefore \boxed{x^2 + y^2 + z^2 + 5x - 6y + 7z - 8 = 0}$$

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Ex: Show that the two circles

$x^2+y^2+z^2-y+2z=0,\ x-y+z-2=0$

$x^2+y^2+z^2+x+3y+z-5=0,\ 2x-y+4z-1=0$

lie on the same sphere and find its equation

Sol. Eqn of the ~~fst~~ sphere containing fst circle.

$(x^2+y^2+z^2-y+2z)+\lambda_1(x-y+z-2)=0$

$x^2+y^2+z^2+\lambda_1 x+(-1-\lambda_1)y+(2+\lambda_1)z-2\lambda_1=0$ — ①

Eqn of the sphere containing the second circle

$(x^2+y^2+z^2+x-3y+z-5)+\lambda_2(2x-y+4z-1)=0$

$x^2+y^2+z^2+(1+2\lambda_2)x+(-3-\lambda_2)y+(1+4\lambda_2)z+(-5-\lambda_2)=0$

If the two circles lie on same sphere then comparing co-eff's of ① & ②, x, y, z, constant — ②

$x \Rightarrow \lambda_1 = 1+2\lambda_2 \Rightarrow \lambda_1-2\lambda_2-1=0$ — ③

$y \Rightarrow -1-\lambda_1=-3-\lambda_2 \Rightarrow \lambda_1-\lambda_2-2=0$ — ④

$z \Rightarrow 2+\lambda_1 = 1+4\lambda_2 \Rightarrow \lambda_1-4\lambda_2+1=0$ — ⑤

$c \Rightarrow -2\lambda_1=-5-\lambda_2 \Rightarrow 2\lambda_1-\lambda_2-5=0$ — ⑥

Solving ③ & ④

③ − ④ $\Rightarrow -\lambda_2+1=0 \Rightarrow \boxed{\lambda_2=1}$

substitute λ_2 in ③

$\Rightarrow \lambda_1-1-2=0$

$\boxed{\lambda_1=3}$

substituting λ_1, λ_2 in ⑤ & ⑥

⑤ $\Rightarrow 3-4+1$

$\Rightarrow -1+1$

$\Rightarrow 0$

⑥ $\Rightarrow 6-1-5$

$\Rightarrow 5-5$

$= 0$

Since the values of λ_1 & λ_2 satisfy ⑤ & ⑥ we get the two given circles lie on the same sphere.

$\therefore$ The eqn of the sphere containing the two given circles is given by.

$$x^2+y^2+z^2-y+2z+3(x-y+z-2)=0$$

$$\Rightarrow \boxed{x^2+y^2+z^2+3x-4y+5z-6=0.}$$

• P/T the circles $x^2+y^2+z^2-2x+3y+4z-5=0, 5y+6z+1=0$

$x^2+y^2+z^2-3x-4y+5z-6=0, x+2y-7z=0$ lies on the same sphere and find its equation.

[Ans $\rightarrow x^2+y^2+z^2-2x-2y-2z-6=0$]

sol) Eqn of the sphere containing fst circle is

$\cancel{(x^2+y^2+z^2-3x-4y+5z-6)+\lambda_1(x+2y}$

$$(x^2+y^2+z^2-2x+3y+4z-5)+\lambda_1(5y+6z+1)=0 \quad \text{—— ①}$$

$$\Rightarrow x^2+y^2+z^2-2x+(3+5\lambda_1)y+(4+6\lambda_1)z+(\lambda_1-5)=0$$

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Eq^n of sphere containing second circle

$$(x^2+y^2+z^2-3x-4y+5z-6)+\lambda_2(x+2y-7z)=0$$

$$x^2+y^2+z^2+(\lambda_2-3)x+(2\lambda_2-4)y+(5-7\lambda_2)z-6=0 \quad \text{— (2)}$$

comparing co-efficients x, y, z in (1) & (2)

$x \Rightarrow -2 = \lambda_2 - 3 \Rightarrow \boxed{\lambda_2 = 1}$

$y \Rightarrow 3+5\lambda_1 = 2\lambda_2 - 4 \Rightarrow 5\lambda_1 - 2\lambda_2 + 7 = 0$ — (3)

$z \Rightarrow 4+6\lambda_1 = 5-7\lambda_2 \Rightarrow 6\lambda_1 + 7\lambda_2 - 1 = 0$ — (4)

$c \Rightarrow \lambda_1 - 5 = -6 \Rightarrow \boxed{\lambda_1 = -1}$

substitue λ_2 in (2)

$$x^2+y^2+z^2-3x-4y+5z-6+x+2y-7z=0$$

$$\Rightarrow \boxed{x^2+y^2+z^2-2x-2y-2z-6=0}$$

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Eq^n of sphere containing second circle

$$(x^2+y^2+z^2-3x-4y+5z-6)+\lambda_2(x+2y-7z)=0$$

$$x^2+y^2+z^2+(\lambda_2-3)x+(2\lambda_2-4)y+(5-7\lambda_2)z-6=0 \quad \text{---} (2)$$

comparing co-efficients x, y, z in (1) & (2)

$x \Rightarrow -2 = \lambda_2 - 3 \Rightarrow \boxed{\lambda_2 = 1}$

$y \Rightarrow 3+5\lambda_1 = 2\lambda_2 - 4 \Rightarrow 5\lambda_1 - 2\lambda_2 + 7 = 0$ --- (3)

$z \Rightarrow 4+6\lambda_1 = 5-7\lambda_2 \Rightarrow 6\lambda_1 + 7\lambda_2 - 1 = 0$ --- (4)

$c \Rightarrow \lambda_1 - 5 = -6 \Rightarrow \boxed{\lambda_1 = -1}$

substitue λ_2 in (2)

$$x^2+y^2+z^2-3x-4y+5z-6+x+2y-7z=0$$

$$\Rightarrow \boxed{x^2+y^2+z^2-2x-2y-2z-6=0}$$

Concyclic Points

A set of given points are said to be concyclic if they lie on the same circle.

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PCP-4

Show that the following points are concyclic

1) (5,0,2), (2,-6,0), (7,-3,8), (4,-9,6)

sol. let A(5,0,2), B(2,-6,0), C(7,-3,8), D(4,-9,6)

Eqn of the plane ABC

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

Eqn of the plane passing through A is

$$a(x-5)+b(y-0)+c(z-2)=0$$

where a,b,c are DR's of normal to the plane

Since, plane passes through B $\Rightarrow a(2-5)+b(-6-0)+c(0-2)=0$

$$\Rightarrow -3a-6b-2c=0$$

$$\Rightarrow 3a+6b+2c=0$$

Since, plane passes through C $\Rightarrow 2a-3b+6c=0$

Solving $\Rightarrow \dfrac{a}{\cancel{42}\ 6} = \dfrac{b}{\cancel{-14}\ -2} = \dfrac{c}{\cancel{-21}\ -3} = K$

$$\Rightarrow \boxed{a=6K}\ \boxed{b=-2K}\ \boxed{c=-3K}$$

substituting in $a(x-5)+b(y-0)+c(z-2)=0$

$$6\cancel{K}(x-5)-2\cancel{K}(y-0)-3\cancel{K}(z-2)=0$$

$$6x-2y-3z-24=0 \quad \text{— (I)}$$

Substituting D(4,-9,6) in (I)

$$6(4)-2(-9)-3(6)-24$$

$\Rightarrow 0 = \text{RHS}$

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$\therefore$ The given points lie on the same plane

Eq^n of the sphere OABC

Eq^n of the sphere be $x^2+y^2+z^2+2ux+2vy+2wz=0$

Sphere passes through $A \Rightarrow$ ~~25~~ $29+10u+4w=0$ —— ①

Sphere passes through $B \Rightarrow 40+4u-12v=0$ —— ②

Sphere passes through $C \Rightarrow 122+14u-6v+16w=0$ —— ③

① $\Rightarrow w = -\left[\frac{29+10u}{4}\right]$

② $\Rightarrow v = \frac{40+4u}{12}$

$\Rightarrow v = \left[\frac{10+u}{3}\right]$

substituting w, v in ③

$$122+14u-\cancel{6}^{2}\left[\frac{10+u}{\cancel{3}}\right]-\cancel{16}^{4}\left[\frac{29+10u}{\cancel{4}}\right]=0$$

$122+14u-20-2u-116-40u=0$

$-28u-14=0$

$-28u=14$

$\boxed{u=-1/2}$

$v = \frac{10-\frac{1}{2}}{3} = \frac{20-1}{6} = \frac{19}{6}$

$\boxed{v=\frac{19}{6}}$

$w = -\left[\frac{29-10(1/2)}{4}\right] = -\left[\frac{29-5}{4}\right] = -\frac{24}{4} = -6$

$\boxed{w=-6}$

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$\therefore$ Eq^n of the sphere OABC is

$$x^2+y^2+z^2+2\left(\frac{-1}{2}\right)x+2\left(\frac{19}{6}\right)y+2(-6)z=0$$

$$x^2+y^2+z^2-x+\frac{19}{3}y-12z=0$$

substituting D(4,-9,6), we get

$$\Rightarrow 16+81+36-4+\frac{19}{3}(-9)-12(6)$$

$$\Rightarrow 0$$

$\therefore$ The given points are lying on same sphere and the plane. Hence, they lie on the same circle. So, they are concyclic.

2) A(-8,5,2), B(-5,2,2), C(-7,6,6), (-4,3,6)

Sol: Eq^n of the plane ABC

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

Eq^n of the plane passing through A is

$$a(x+8)+b(y-5)+c(z-2)=0$$

where a,b,c are DR's of the normal to the plane.

Since, the plane passes through B $\Rightarrow a(-5+8)+b(2-5)+c(2-2)=0$

$$\Rightarrow 3a-3b+0c=0$$

Since, plane passes through C $\Rightarrow a(-7+8)+b(6-5)+c(6-2)=0$

$$\Rightarrow a+b+4c=0$$

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Solving,

$$\frac{a}{-12} = \frac{b}{-12} = \frac{c}{6} = k$$

divide by "-6"

$$\frac{a}{2} = \frac{b}{2} = \frac{c}{-1} = k$$

$\boxed{a = 2k}$, $\boxed{b = 2k}$, $\boxed{c = -k}$

substituting in $a(x+8) + b(y-5) + c(z-2) = 0$

$$2\cancel{k}(x+8) + 2\cancel{k}(y-5) - \cancel{k}(z-2) = 0$$

$$2x + 2y - z + 8 = 0 \quad \text{— I}$$

substituting $D(-4,3,6)$ in I

$\Rightarrow 2(-4) + 2(3) - 6 + 8$

$\Rightarrow -\cancel{8} + \cancel{6} - \cancel{6} + \cancel{8}$

$= 0$

$\therefore$ The given points lie on the same plane

Eq^n of the sphere OABC

Eq^n of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$

Sphere passes through A $\Rightarrow 93 - 16u + 10v + 4w = 0$ — 1

Sphere passes through B $\Rightarrow 33 - 10u + 4v + 4w = 0$ — 2

Sphere passes through C $\Rightarrow 121 - 14u + 12v + 12w = 0$ — 3

1 - 2 $\Rightarrow 60 - 6u + 6v = 0$

$$10 - u + v = 0 \quad \text{— 4}$$

Eliminating w from 2 & 3

2 × 3 $\Rightarrow 99 - 30u + 12\cancel{v} + 12\cancel{w} = 0$

3 $\Rightarrow 121 - 14u + 12\cancel{v} + 12\cancel{w} = 0$

$\quad (-) \quad (+) \quad (-) \quad (-)$

$-22 - 16u = 0$

$16u = -22 \Rightarrow \boxed{u = \frac{-11}{8}}$

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(4) $\Rightarrow 10+\frac{11}{8}+v=0$

$$\frac{80+11}{8}+v=0$$

$$\frac{91}{8}+v=0$$

$$\boxed{v=-\frac{91}{8}}$$

(1) $\Rightarrow 93-\cancel{16}^{2}\left(\frac{-11}{\cancel{8}}\right)+\cancel{10}^{5}\left(\frac{-91}{\cancel{8}_{4}}\right)+4w=0$

$$\Rightarrow 93+22-\frac{455}{4}+4w=0$$

$$\Rightarrow 115-\frac{455}{4}+4w=0$$

$$\Rightarrow \frac{460-455}{4}+4w=0$$

$$\Rightarrow \frac{5}{4}+4w=0$$

$$4w=-\frac{5}{4}$$

$$\boxed{w=-\frac{5}{16}}$$

Eqn of the plane is

$$x^2+y^2+z^2+2\left(\frac{-11}{8}\right)x+2\left(-\frac{91}{8}\right)y+2\left(\frac{-5}{16}\right)z=0$$

$$\Rightarrow x^2+y^2+z^2-\frac{11}{4}x-\frac{91}{4}y-\frac{5}{8}z=0$$

substituting D(-4, 3, 6), we get

$$\Rightarrow 16+9+36-\frac{11}{\cancel{4}}(-\cancel{4})-\frac{91}{4}(3)-\frac{5}{\cancel{8}_4}(\cancel{6}^3)$$

$$\Rightarrow 61+11-\frac{273}{4}-\frac{15}{4}$$

$$\Rightarrow 72-\frac{15}{4}-\frac{273}{4}$$

$$\Rightarrow \frac{288-15}{4}-\frac{273}{4}\Rightarrow \frac{273}{4}-\frac{273}{4}=0$$

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$\therefore$ The given points are lying on same sphere and the plane. Hence, they lie on the same circle. So, they are concyclic.

<u>Intersection of a line with a sphere</u>

Let the equation of the line be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

and equation of the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

let $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$

$\therefore$ Any point on the line $\Rightarrow (lr+\alpha, mr+\beta, nr+\gamma)$

Now substituting the above co-ordinates in the equation of the sphere, we get.

$\Rightarrow (lr+\alpha)^2 + (mr+\beta)^2 + (nr+\gamma)^2 + 2u(lr+\alpha) + 2v(mr+\beta) + 2w(nr+\gamma) + d = 0$

$\Rightarrow l^2r^2 + \alpha^2 + 2rl\alpha + m^2r^2 + \beta^2 + 2rm\beta + n^2r^2 + \gamma^2 + 2rn\gamma + 2rlu$

$+ 2u\alpha + 2rmv + 2v\beta + 2rnw + 2w\gamma + d = 0$

$\Rightarrow r^2(l^2+m^2+n^2) + 2r[l(\alpha+u) + m(\beta+v) + n(\gamma+w)] + (\alpha^2+\beta^2+\gamma^2$

$+ 2u\alpha + 2v\beta + 2w\gamma + d) = 0$ —— (1)

Eq^n (1) is a quadratic equation in r and has two roots, say r_1 & r_2

substituting in the general co-ordinates we get two points of intersection as.

$\Rightarrow (lr_1+\alpha, mr_1+\beta, nr_1+\gamma)$ & $(lr_2+\alpha, mr_2+\beta, nr_2+\gamma)$

- Find the co-ordinates of the points where the line $\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5}$ intersects the sphere $\cancel{x^2+y^2+z^2+2x}$ $x^2+y^2+z^2+2x-10y=23.$

sol. let $\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5} = r$

$\therefore$ Any point on the line $\Rightarrow (4r-3,\ 3r-4,\ -5r+8)$

substituting in sphere equation

$$\Rightarrow x^2+y^2+z^2+2x-10y-23=0$$

$$(4r-3)^2+(3r-4)^2+(-5r+8)^2+2(4r-3) \cancel{+} -10(3r-4)-23=0$$

$$16r^2+9-24r+9r^2+16-24r+25r^2+64-80r+8r-6-30r+40-23=0$$

$$50r^2-150r+100=0$$

$$r^2-3r+2=0$$

$$r^2-r-2r+2=0$$

$$r(r-1)-2(r-1)=0$$

$$(r-1)(r-2)=0$$

$$r=1,2$$

substituting in $\Rightarrow (4r-3,\ 3r-4,\ -5r+8)$

$$\Rightarrow (1,-1,3) \text{ \& } (5,2,-2)$$

The pts of intersection are $(1,-1,3)$ & $(5,2,-2)$

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Tangent line of a sphere

A line touching a sphere is known as a tangent line to the sphere.

Note: If a line touches a sphere at the point 'A' then 'A' is known as the point of contact.

Tangent Plane to a sphere

The locus of all the tangent lines drawn to a sphere at a point (α, β, γ) [(α, β, γ) lies on the sphere]. is known as a tangent plane to the sphere at the point (α, β, γ).

→ Here, the point (α, β, γ) is known as point of contact.

Note:

i) For a tangent line (or) tangent plane to a sphere we have,

Radius of sphere = perpendicular distance from centre of sphere to a tangent line or a tangent plane.

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To find the equation of a tangent plane to a given sphere.

Proof: let the eq^n of sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

let (α, β, γ) be a point on the sphere

Eq^n of the line through (α,β,γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)}$$

Any point on the line $\Rightarrow (lr+\alpha, nr+\beta, mr+\gamma)$

substituting in sphere eq^n, we get

$\cancel{(lr+\alpha}$

$$\Rightarrow r^2(l^2+m^2+n^2) + 2r[l(\alpha+u)+m(\beta+v)+n(\gamma+w)] + (\alpha^2+\beta^2+\gamma^2$$
$$+2u\alpha+2v\beta+2w\gamma+d)=0 \quad \text{—— (1)}$$

The line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ becomes a tangent line to the sphere if discriminant of (1) $=0$.

i.e $\boxed{b^2-4ac=0}$

$$\Rightarrow 4[l(\alpha+u)+m(\beta+v)+n(\gamma+w)]^2 - 4(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2$$
$$+2u\alpha+2v\beta+2w\gamma+d)=0$$

$$\Rightarrow 4[l(\alpha+u)+m(\beta+v)+n(\gamma+w)]^2=0$$

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$\Rightarrow l(\alpha+u)+m(\beta+v)+n(\gamma+w)=0$

$\therefore$ The eqⁿ of the tangent plane is obtained by eliminating (l,m,n) from above condition.

i.e put $l=\frac{x-\alpha}{r}$, $m=\frac{y-\beta}{r}$, $n=\frac{z-\gamma}{r}$

$$\Rightarrow \frac{(x-\alpha)(\alpha+u)}{r}+\frac{(y-\beta)(\beta+v)}{r}+\frac{(z-\gamma)(\gamma+w)}{r}=0$$

$$\Rightarrow (x-\alpha)(\alpha+u)+(y-\beta)(\beta+v)+(z-\gamma)(\gamma+w)=0$$

$$\Rightarrow x\alpha+ux-\alpha^2-u\alpha+y\beta+vy-\beta^2-v\beta+z\gamma+wz-\gamma^2-w\gamma=0$$

$$\Rightarrow \alpha x+\beta y+\gamma z+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d$$
$$=\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d$$

$\Rightarrow \alpha x+\beta y+\gamma z+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d=0$ is the equation of the tangent plane to the ~~plane~~ sphere

Problem

- Find the equation of the tangent plane to the sphere $3(x^2+y^2+z^2)-2x-3y-4z-22=0$ at the point $(1,2,3)$.

Sol. Eqⁿ of the given sphere is

$$x^2+y^2+z^2-\frac{2x}{3}-y-\frac{4z}{3}-\frac{22}{3}=0$$

$\therefore$ Eqⁿ of the tangent plane to sphere at $(1,2,3)$ is

$$\Rightarrow x(1)+y(2)+z(3)-\frac{1}{3}(x+1)-\frac{1}{2}(y+2)-\frac{2}{3}(z+3)-\frac{22}{3}=0$$

$$\Rightarrow x+2y+3z-\frac{1}{3}x-\frac{1}{3}-\frac{y}{2}-1-\frac{2}{3}z-2-\frac{22}{3}=0$$

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$\Rightarrow \frac{2x}{3} + \frac{3y}{2} + \frac{7z}{3} - \frac{32}{3} = 0$

$\Rightarrow 4x + 9y + 14z - 64 = 0$

• Find the value of a for which the plane $x+y+z = a\sqrt{3}$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$

Sol. Centre of the given sphere $= (1,1,1)$

The given plane is a tangent plane to the sphere if radius of sphere = perpendicular distance from the centre of $(\sqrt{u^2+v^2+w^2-d})$ sphere to tangent plane. i.e $x+y+z-a\sqrt{3}=0$

$\Rightarrow \sqrt{1+1+1+6} = \perp^r$ distance from $(1,1,1)$ to $x+y+z-a\sqrt{3}=0$

$\Rightarrow 3 = \left|\frac{1+1+1-a\sqrt{3}}{\sqrt{1+1+1}}\right|$ $\left[\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right]$

$\Rightarrow 3 = \left|\frac{3-a\sqrt{3}}{\sqrt{3}}\right| \Rightarrow 3 = |\sqrt{3}-a|$

$\cancel{3\sqrt{3} = 3-a\sqrt{3}}$ $\Rightarrow 3 = \pm(\sqrt{3}-a)$

$a\cancel{\sqrt{3} = 3 - 3\sqrt{3}}$ $3 = \sqrt{3}-a$, $3 = -\sqrt{3}+a$

$a\cancel{= \frac{3}{\sqrt{3}} - 3}$ $a = \sqrt{3}-3$, $a = \sqrt{3}+3$

$a\cancel{= \sqrt{3}-3}$

$\boxed{a = \sqrt{3} \pm 3}$

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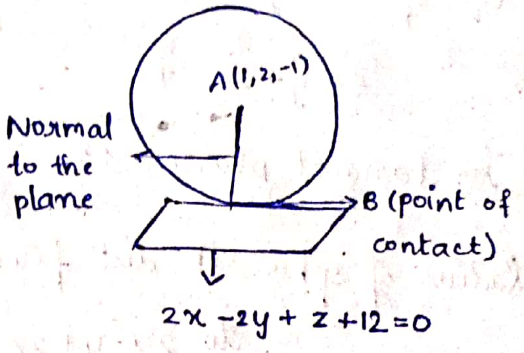
• Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3$ and find the point of contact.

Sol. Centre of given sphere $= (1, 2, -1)$

Radius of the sphere $= \sqrt{1+4+1+3} = 3$ —— (1)

$\perp^r$ distance from centre of the sphere $(1,2,-1)$ to $2x-2y+z+12=0$

$$= \left| \frac{2-4-1+12}{\sqrt{4+4+1}} \right|$$



$$= \frac{\cancel{9}^{3}}{\cancel{3}} = 3 \quad —— (2)$$

(1) & (2) => Radius of sphere = perpendicular distance from centre of sphere to plane

∴ Given plane is a tangent plane to the sphere.

Eq of line AB is

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} = r$$

Any point on line AB $= (2r+1, -2r+2, r-1)$

let $B = (2r+1, -2r+2, r-1)$

substituting in $2x-2y+z+12=0$

$\Rightarrow 2(2r+1) - 2(-2r+2) + r-1 +12 = 0$

$\Rightarrow 9r + 9 = 0 \Rightarrow r = 1$

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Point of contact $B = (-1, 4, -2)$

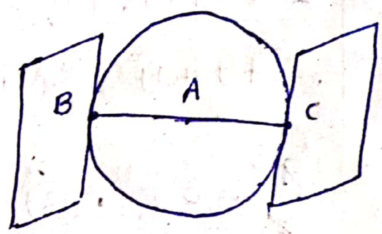
• Find the co-ordinates of the points on the sphere $x^2 + y^2 + z^2 - 4x + 2y = 4$. The tangent plane at which are parallel to the plane $2x - y + 2z = 1$.

Sol: Centre of the sphere $= (2, -1, 0) = A$ (say)

Eq'ns of tangent plane parallel to $2x - y + 2z = 1$ is

$$2x - y + 2z = k$$

For tangent plane



Radius of sphere = ⊥r dist from $(2, -1, 0)$ to $2x - y + 2z = k$

$$\sqrt{4+1+0+4} = \left| \frac{2(2) - (-1) + 2(0) - k}{\sqrt{4+1+4}} \right|$$

$$3 = \left| \frac{5-k}{3} \right| \Rightarrow 3 = \frac{5-k}{\pm 3}$$

$$\Rightarrow 9 = 5 - k \text{ (or) } -9 = 5 - k$$

$$\Rightarrow k = -4 \text{ (or) } k = 14$$

∴ Eq^n's of the tangent plane are

$$2x - y + 2z = -4$$

$$2x - y + 2z = 14$$

Eq^n of line ABC is $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-0}{2} = r$

Any point on line $= (2r+2, -r-1, 2r)$

let $B = (2r+2, -r-1, 2r)$ be some value of r

substituting in $2x + y + 2z = -4$

$\Rightarrow 2(2r+2) - (-r-1) + 2(2r) + 4 = 0$

$\Rightarrow 9r + 9 = 0 \Rightarrow r = -1$

$B = (0, 0, -2)$

let $C = (2r+2, -r-1, 2r)$ be some value of r

substituting in $2x - y + 2z = 14$

$\Rightarrow 2(2r+2) - (-r-1) + 2(2r) = 14$

$\Rightarrow 4r + 4 + r + 1 + 4r = 14$

$\Rightarrow 9r + 5 = 14$

$\Rightarrow 9r = 9$

$r = 1$

$C = (4, -2, 2)$

• Find the two tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+2y=z$.

Sol: Eq^n of the tangent plane parallel to $2x+2y-z=0$ is

$2x+2y-z=k$

Centre of sphere $= (2,-1,3)$, radius of sphere $= 3$

For tangent plane

radius of sphere = ⊥^{r} dist from $(2,-1,3)$ to $2x+2y-z=k$

$$3 = \left| \frac{2(2)+2(-1)-3-k}{\sqrt{4+4+1}} \right|$$

$$3 = \left| \frac{-1-k}{3} \right| \Rightarrow \frac{1+k}{\pm 3} = 3$$

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$\Rightarrow 9 = 1+k$ (or) $-9 = 1+k$

$k = 8$ (or) $k = -10$

$\therefore$ The eq^{n}s of tangent plane are

$2x+2y-z = 8$ (and) $2x+2y-z = -10$

• Find the eq^n of the sphere to the circle $x^2+y^2+z^2=1, 2x+4y+5z=6$ and touching the plane $z=0$.

Sol. The eq^n of the given circle is $x^2+y^2+z^2-1=0$, $2x+4y+5z-6=0$
[$S=0, P=0$]

Eq^n of the sphere to the given circle is

$$S+\lambda P=0$$

$$(x^2+y^2+z^2-1)+\lambda(2x+4y+5z-6)=0 \quad \text{①}$$

$$\Rightarrow x^2+y^2+z^2+2\lambda x+4\lambda y+5\lambda z+(-1-6\lambda)=0$$

Centre of the sphere ① $= \left(-\lambda, -2\lambda, -\frac{5\lambda}{2}\right)$

Radius of sphere $= \sqrt{\lambda^2+4\lambda^2+\frac{25\lambda^2}{4}+1+6\lambda}$

For tangent plane,

radius of sphere = $\perp^r$ dist from $\left(-\lambda, -2\lambda, -\frac{5\lambda}{2}\right)$ to plane

$$0x+0y+z+0=0$$

$$\sqrt{\lambda^2+4\lambda^2+\frac{25\lambda^2}{4}+1+6\lambda} = \left|\frac{0(-\lambda)+0(-2\lambda)+1\left(\frac{-5\lambda}{2}\right)+0}{\sqrt{0+0+1}}\right|$$

$$= 5\lambda^2+6\lambda+1$$

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$\lambda = -1, -\frac{1}{5}$ sub in ①

$x^2+y^2+z^2-1-2x-4y-5z+6=0$

$x^2+y^2+z^2-2x-4y-5z+5=0$

$(x^2+y^2+z^2-1)-\frac{1}{5}(2x+4y+5z-6)=0$

$5x^2+5y^2+5z^2-5-2x-4y-5z+6=0$

$5x^2+5y^2+5z^2-2x-4y-5z+1=0$

• Find the equations of two tangent planes to the sphere $x^2+y^2+z^2=9$, passes through the line $x+y=6$, $x-2z=3$.

Sol Eq^n of the line is $x+y-6=0, x-2z-3=0$
$[P_1=0, P_2=0]$

Eq^n of the tangent plane to the given line is

$P_1+\lambda P_2=0$

$\Rightarrow (x+y-6)+\lambda(x-2z-3)=0$

$\Rightarrow (1+\lambda)x+y-2\lambda z+(-6-3\lambda)=0$ — ①

① is a tangent plane to the plane if, radius of the sphere = ⊥^{r} distance from centre of sphere $(0,0,0)$ to ①

radius = 3

$$\Rightarrow 3 = \left|\frac{(1+\lambda)0+0-2\lambda(0)+(-6-3\lambda)}{\sqrt{(1+\lambda)^2+1+4\lambda^2}}\right|$$

$$\Rightarrow 3 = \frac{6+3\lambda}{\sqrt{1+\lambda^2+2\lambda+4\lambda^2+1}}$$

$$\Rightarrow \cancel{3} = \frac{\cancel{3}(2+\lambda)}{\sqrt{5\lambda^2+2\lambda+2}}$$

$\Rightarrow 5\lambda^2+2\lambda+2 = 4+\lambda^2+4\lambda$.

$\Rightarrow 4\lambda^2-2\lambda-2=0 \Rightarrow 2\lambda^2-\lambda-1=0$

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$2\lambda^2 - \lambda - 1 = 0$

$2\lambda^2 - 2\lambda + \lambda - 1 = 0$

$2\lambda(\lambda - 1) + (\lambda - 1) = 0$

$(\lambda - 1)(2\lambda + 1) = 0$

$\lambda = 1, \lambda = \frac{-1}{2}$

put $\lambda = 1$ in eq^n ①

$(x + y - 6) + (x - 2z - 3) = 0$

$2x + y - 2z - 9 = 0$

put $\lambda = \frac{-1}{2}$ in eq^n ①

$(x + y - 6) - \frac{1}{2}(x - 2z - 3) = 0$

$2x + 2y - 12 - x + 2z + 3 = 0$

$x + 2y + \cancel{2z}\,2z - 9 = 0$

• Obtain the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$ which passes through the line $3(16 - x) = 3z = 2y + 30$

sol. Writing the equation in unsymmetrical form we get,

$\cancel{3}(16 - x) = \cancel{3}z \quad ; \quad 2y - 3z + 30 = 0$

i.e $\quad x + z - 16 = 0, \; 2y - 3z + 30 = 0$

$[P_1 = 0 ; P_2 = 0]$

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$\therefore$ Eq^n of the tgt ~~given~~ plane to the ~~tangent plane~~ given line is

$$(x+z-16)+\lambda(2y-3z+30)=0$$

$$x+2\lambda y+(1-3\lambda)z+(30\lambda-16)=0 \quad \text{--- (1)}$$

Centre of sphere $=(-3,0,1)$

radius of sphere $=\sqrt{9+0+1-1}=3.$

for tangent plane,

radius of sphere $=\perp^r$ distance from $(-3,0,1)$ to (1)

$$\Rightarrow 3=\left|\frac{-3+2\lambda(0)+(1-3\lambda)+30\lambda-16}{\sqrt{1+4\lambda^2+(1-3\lambda)^2}}\right|$$

$$\Rightarrow 3=\left|\frac{-3+1-3\lambda+30\lambda-16}{\sqrt{1+4\lambda^2+1+9\lambda^2-6\lambda}}\right|$$

$$\Rightarrow 3=\left|\frac{-18+27\lambda}{\sqrt{13\lambda^2-6\lambda+2}}\right|$$

$$\Rightarrow \cancel{3}=\frac{\cancel{3}(9\lambda-6)}{\sqrt{13\lambda^2-6\lambda+2}}$$

S.O.B.S.

$$\Rightarrow 13\lambda^2-6\lambda+2=(9\lambda-6)^2$$

$$\Rightarrow 13\lambda^2-6\lambda+2=81\lambda^2+36-108\lambda.$$

$$\Rightarrow 68\lambda^2 \cancel{+6\lambda}-102\lambda+34=0$$

$$\Rightarrow 2\lambda^2-3\lambda+1=0 \quad \Rightarrow 2\lambda^2-2\lambda-\lambda+1=0$$

$$2\lambda(\lambda-1)-(\lambda-1)=0$$

$$(2\lambda-1)(\lambda-1)=0$$

$$\lambda=\frac{1}{2},\ \lambda=1$$

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substitute $\lambda=\frac{1}{2}$ in ①

$$(x+z-16)+\frac{1}{2}(2y-3z+30)=0$$

$$2x+2z-32+2y-3z+30=0$$

$$2x+2y-z-2=0$$

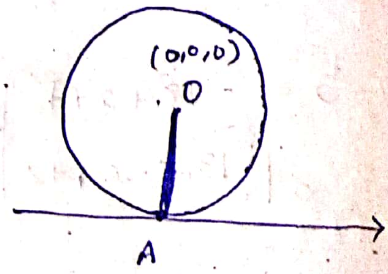
sub $\lambda=1$ in ①

$$(x+z-16)+(2y-3z+30)=0$$

$$x+2y-2z+14=0.$$

• Find the eqn of the sphere which has centre at origin and which touches the line $2(x+1)=2-y=z+3$

sol. Since, the given line touches the sphere it is a tangent line.



its' DR's are $\left(\frac{1}{2},-1,1\right)$

let the line eqn be

$$\frac{x+1}{1/2}=\frac{y-2}{-1}=\frac{z+3}{1}=r$$

Any point on the line $=\left(\frac{r}{2}-1,-r+2,r-3\right)$

let $A=\left(\frac{r}{2}-1,-r+2,r-3\right)$

DR's of OA $=\left(\frac{r}{2}-1,-r+2,r-3\right)$

DR's of tangent line $\left(\frac{1}{2},-1,1\right)$

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Since OA is $\perp^r$ to tangent line

$$\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow \frac{1}{2}\left(\frac{r}{2}-1\right) - 1(-r+2) + 1(r-3) = 0$$

$$\Rightarrow \frac{r}{4} - \frac{1}{2} + r - 2 + r - 3 = 0$$

$$\Rightarrow 2r + \frac{r}{4} - 5 - \frac{1}{2} = 0$$

$$\Rightarrow \frac{9r}{4} - \frac{11}{2} = 0$$

$$\Rightarrow \frac{9r}{\cancel{4}_{2}} = \frac{11}{\cancel{2}}$$

$$\Rightarrow r = \frac{22}{9}$$

$$A = \left(\frac{22}{18} - 1, -\frac{22}{9} + 2, \frac{22}{9} - 3\right)$$

$$A = \left(\frac{4}{18}, \frac{-4}{9}, \frac{-5}{9}\right)$$

$$A = \left(\frac{2}{9}, \frac{-4}{9}, \frac{-5}{9}\right).$$

radius $= OA$

$$= \sqrt{\frac{4}{81} + \frac{16}{81} + \frac{25}{81}} = \sqrt{\frac{45}{81}} = \sqrt{\frac{5}{9}}$$

eqn of the sphere is $(x-0)^2 + (y-0)^2 + (z-0)^2 = \frac{5}{9}$

$$\Rightarrow \boxed{9(x^2+y^2+z^2) = 5}$$

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Imp

• Obtain the equations of the sphere which pass through the circle $x^2+y^2+z^2-2x+2y+4z-3=0$, $2x+y+z=4$ and touches the plane $3x+4y=14$

Sol Eqⁿ of the sphere through the circle $[S=0, P=0]$ is

$$S+\lambda P=0$$

$$(x^2+y^2+z^2-2x+2y+4z-3)+\lambda(2x+y+z-4)=0$$

$$x^2+y^2+z^2+(2\lambda-2)x+(2+\lambda)y+(4+\lambda)z+(-3-4\lambda)=0 \quad \text{—} ①$$

Eqⁿ ① touches tangent plane $3x+4y=14$

Centre of sphere $= \left(1-\lambda, \dfrac{-2-\lambda}{2}, \dfrac{-4-\lambda}{2}\right)$

$$r=\sqrt{(1-\lambda)^2+\left(\frac{-2-\lambda}{2}\right)^2+\left(\frac{-4-\lambda}{2}\right)^2+3+4\lambda}$$

For a tangent plane,

radius of the sphere = ⊥ʳ dist from centre of sphere to $3x+4y=14$

$$\sqrt{1+\lambda^2-2\lambda+\frac{4+\lambda^2+2\lambda}{4}+\frac{16+\lambda^2+8\lambda}{4}+3+4\lambda}=\left|\frac{3(1-\lambda)+4\left(\frac{-2-\lambda}{2}\right)-14}{\sqrt{9+16}}\right|$$

$$\sqrt{\frac{6\lambda^2+20\lambda+36}{4}}=\left|\frac{-5\lambda-15}{5}\right|$$

$$\sqrt{\frac{3\lambda^2+10\lambda+18}{2}}=\lambda+3$$

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$$\Rightarrow \frac{3\lambda^2+10\lambda+18}{2} = \lambda^2+6\lambda+9$$

$$\Rightarrow 3\lambda^2+10\lambda+18 = 2\lambda^2+12\lambda+18$$

$$\Rightarrow \lambda^2-2\lambda = 0$$

$$\lambda(\lambda-2) = 0$$

$$\lambda = 0, \lambda = 2$$

substituting in ① we get,

$\lambda = 0$

$$x^2+y^2+z^2-2x+2y+4z-3=0$$

$\lambda = 2$

$$x^2+y^2+z^2-2x+2y+4z-3+2(2x+y+z-4)=0$$

$$x^2+y^2+z^2-2x+2y+4z-3+4x+2y+2z-8=0$$

$$x^2+y^2+z^2+2x+4y+6z-11=0.$$

Note:

i) If the two spheres touch externally at a point c then the distance b/w the centres = Sum of the radii.

'c' [point of contact] divides the line joining the centres internally in the ratio $r_1 : r_2$.

2) If the two spheres touch internally at a point 'c' then distance b/w the centres = difference of the radii.

and 'c' divides the line joining the centres externally in the ratio $r_1 : r_2$.

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PCP-6

Show that the spheres $x^2+y^2+z^2=25$, $x^2+y^2+z^2-24x-40y-18z$ $+225=0$ touch externally and find point of contact.

Sol. Let $S_1 \Rightarrow x^2+y^2+z^2-25=0$

$S_2 = x^2+y^2+z^2-24x-40y-18z+225=0.$

A = centre of $S_1 = (0,0,0)$, B = centre of $S_2 = (12,20,9)$

r_1 = radius of $S_1 = 5$, r_2 = radius of $S_2 = \sqrt{144+400+81-225}$

$= \sqrt{400}$

$r_2 = 20$

distance b/w centres $= AB$

$= \sqrt{(12-0)^2+(20-0)^2+(9-0)^2}$

$= \sqrt{144+400+81}$

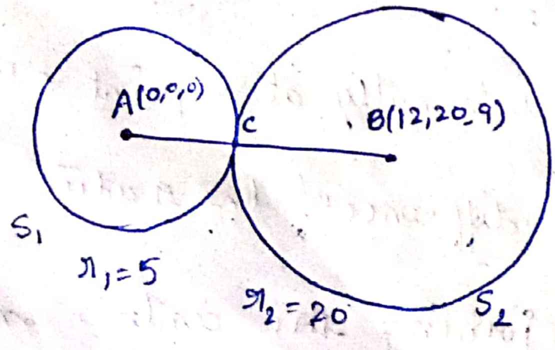
$= \sqrt{625} = 25$

$= r_1 + r_2$

$= 20+5 = 25$

$\therefore$ distance b/w centres = sum of radii

$\Rightarrow S_1$ & S_2 touch externally at 'c'.



$r_1 : r_2 = 1:4$

$\therefore C = \left(\frac{1\times12 + 4\times0}{1+4}, \frac{1\times20 + 4\times0}{1+4}, \frac{1\times9 + 4\times0}{1+4}\right)$

$C = \left(\frac{12}{5}, \frac{20}{5}, \frac{9}{5}\right)$

$C = \left(\frac{12}{5}, \frac{20}{5}, \frac{9}{5}\right)$ is point of contact.

• Show that the spheres $x^2+y^2+z^2=64$ and $x^2+y^2+z^2-12x+4y-6z+48=0$ touch internally and find the point of contact.

Sol. $S_1 = x^2+y^2+z^2-64=0$

A = centre of $S_1 = (0,0,0)$

r_1 = radius of $S_1 = 8$

$S_2 = x^2+y^2+z^2-12x+4y-6z+48=0$

B = centre of $S_2 = (6,-2,3)$

r_2 = radius of $S_2 = \sqrt{36+4+9-48}$

$= 1$

distance b/w centres $= AB$

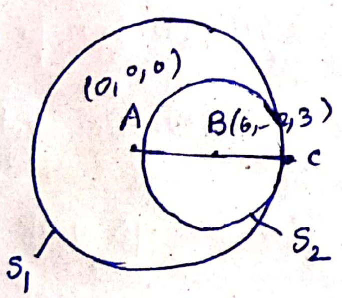
$= \sqrt{(6-0)^2+(-2-0)^2+(3-0)^2}$

$= \sqrt{36+4+9}$

$= \sqrt{49} = 7$

difference b/w radii $= r_1 - r_2 = 8-1 = 7$

$\therefore S_1$ & S_2 touch internally at 'c'.



$r_1 : r_2 = 8:1 = m_1 : m_2$

C divides AB ~~internally~~ externally in ratio $8:1$

$C = \left[\frac{(8\times6)-(1\times0)}{8-1}, \frac{(8\times-2)-(1\times0)}{8-1}, \frac{(8\times3)-(1\times0)}{8-1}\right] = \left(\frac{48}{7}, \frac{-16}{7}, \frac{24}{7}\right)$

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Angle Of Intersection of Two Spheres

If Two spheres intersect each other then angle b/w the two spheres is defined as the angle b/w the tangent planes drawn to the spheres.

(or)

The angle b/w the two spheres is also defined as the angle b/w the radii of the two spheres drawn at the common point.

Orthogonal Spheres

If the angle b/w the two spheres is right angle then the spheres are known as Orthogonal Spheres.

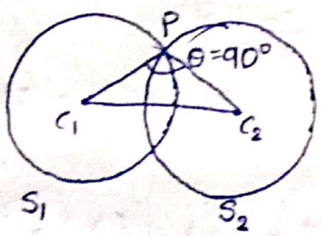
Condition for Orthogonality Of Two Spheres

Let $S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

Let $S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$

C_1 = Centre of $S_1 = (-u_1, -v_1, -w_1)$; radius of S_1 (C_1P) $= \sqrt{u_1^2+v_1^2+w_1^2-d_1}$

C_2 = Centre of $S_2 = (-u_2, -v_2, -w_2)$; radius of S_2 (C_2P) $= \sqrt{u_2^2+v_2^2+w_2^2-d_2}$



Let 'P' be a common point on S_1 & S_2

From right-angled ΔC_1PC_2,

$$C_1P^2 + C_2P^2 = C_1C_2^2$$

$\Rightarrow (u_1^2+v_1^2+w_1^2-d_1)+(u_2^2+v_2^2+w_2^2-d_2) = (u_1-u_2)^2+(v_1-v_2)^2 + (w_1-w_2)^2$

$\Rightarrow (\cancel{u_1^2}+\cancel{v_1^2}+\cancel{w_1^2}-d_1)+(\cancel{u_2^2}+\cancel{v_2^2}+\cancel{w_2^2}-d_2) = \cancel{u_1^2}+\cancel{u_2^2}-2u_1u_2+\cancel{v_1^2}+\cancel{v_2^2}-2v_1v_2 + \cancel{w_1^2}+\cancel{w_2^2}-2w_1w_2$

$\Rightarrow \quad -d_1-d_2 = -2u_1u_2-2v_1v_2-2w_1w_2$

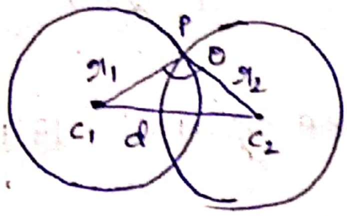
$\Rightarrow \quad d_1+d_2 = 2u_1u_2+2v_1v_2+2w_1w_2$

$$\boxed{2u_1u_2+2v_1v_2+2w_1w_2 = d_1+d_2}$$

- If 'd' is the distance b/w the centres of two spheres of radii r_1 and r_2 then Prove that angle b/w them is $\cos^{-1}\left(\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)$.

Sol. Let $\angle C_1PC_2 = \theta$

From cosine formula we have,



$d^2 = r_1^2+r_2^2-2r_1r_2\cos\theta$

~~cosθ~~ $\Rightarrow 2r_1r_2\cos\theta = r_1^2+r_2^2-d^2$

$\Rightarrow \quad \cos\theta = \frac{r_1^2+r_2^2-d^2}{2r_1r_2}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)$

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1) Find the equation of the sphere that passes through the circle $x^2+y^2+z^2-2x+3y-4z+6=0$, $3x-4y+5z-15=0$ and cuts the sphere $x^2+y^2+z^2+2x+4y-6z+11=0$ Orthogonally.

Sol: Eqn of the sphere to the given circle $[S=0, P=0]$ is

$$\boxed{S+\lambda P=0}$$

$$\Rightarrow (x^2+y^2+z^2-2x+3y-4z+6)+\lambda(3x-4y+5z-15)=0 \quad \text{--- (1)}$$

$$\Rightarrow x^2+y^2+z^2+(3\lambda-2)x+(3-4\lambda)y+(5\lambda-4)z+(6-15\lambda)=0$$

(1) cuts $x^2+y^2+z^2+2x+4y-6z+11=0$ Orthogonally

Using the condition for Orthogonality

$$\Rightarrow [2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2]$$

$$\Rightarrow \cancel{2}\left(\frac{3\lambda-2}{\cancel{2}}\right)(1)+\cancel{2}\left(\frac{3-4\lambda}{\cancel{2}}\right)(2)+\cancel{2}\left(\frac{5\lambda-4}{\cancel{2}}\right)(-3)=6-15\lambda+11$$

$$\Rightarrow 3\lambda-2+6-8\lambda-\cancel{15\lambda}+12=17-\cancel{15\lambda}$$

$$\Rightarrow -5\lambda+16=17$$

$$\Rightarrow -5\lambda=1$$

$$\lambda=\frac{-1}{5}$$

substitute 'λ' in (1)

$$(x^2+y^2+z^2-2x+3y-4z+6)-\frac{1}{5}(3x-4y+5z-15)=0$$

$$\Rightarrow 5x^2+5y^2+5z^2-10x+15y-20z+30-3x+4y-5z+15=0$$

$$\Rightarrow 5x^2+5y^2+5z^2-13x+19y-25z+45=0$$

$$\therefore x^2+y^2+z^2-\frac{13}{5}x+\frac{19}{5}y-5z+9=0$$

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Show that every sphere through the circle $x^2+y^2-2ax+r^2=0$, $z=0$ cuts orthogonally every sphere through the circle $x^2+z^2=r^2$, $y=0$.

Sol. The sphere through the 1st circle is

$$(x^2+y^2+z^2-2ax+r^2)+\lambda_1 z=0 \quad — (1)$$

Sphere through the second circle is

$$(x^2+y^2+z^2-r^2)+\lambda_2 y=0 \quad — (2)$$

To P/T (1) cuts (2) Orthogonally. $[2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2]$

$\Rightarrow 2(-a)(0)+2(0)(\frac{\lambda_2}{2})+2(\frac{\lambda_1}{\lambda_2})(0)=0$

$d_1+d_2 \Rightarrow r^2-r^2=0$

$\Rightarrow 2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

$\therefore$ (1) cuts (2) Orthogonally.

PCP-7

Find the equation of the sphere that passes through 2 points $(0,3,0)$ $(-2,-1,-4)$ and cuts orthogonally two spheres $x^2+y^2+z^2+x-3z-2=0$, $2(x^2+y^2+z^2)+x+3y+4=0$.

Sol. Let the general eq^n of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad — (1)$$

(1) passes through $(0,3,0) \Rightarrow 9+6v+d=0 \quad — (2)$

(1) passes through $(-2,-1,-4) \Rightarrow 21-4u-2v-8w+d=0 \quad — (3)$

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① cuts $x^2+y^2+z^2+x-3z-2=0$ orthogonally

$\Rightarrow \cancel{2}\cdot u\cdot\left(\frac{1}{\cancel{2}}\right)+2v(0)+\cancel{2}(w)\left(-\frac{3}{\cancel{2}}\right)=d-2$

$\Rightarrow \quad u-3w=d-2$

$\Rightarrow \quad u-3w-d+2=0$ — ④

① cuts $x^2+y^2+z^2+\frac{x}{2}+\frac{3}{2}y+2=0$ orthogonally

$\Rightarrow \quad \cancel{2}\cdot u\left(\frac{1}{\cancel{4}_2}\right)+\cancel{2}v\left(\frac{3}{\cancel{4}_2}\right)+2w(0)=d+2$

$\Rightarrow \quad \frac{u}{2}+\frac{3v}{2}=d+2$

$\Rightarrow \quad u+3v=2(d+2)$

$\Rightarrow \quad u+3v-2d-4=0$ — ⑤

② $\Rightarrow \boxed{v=\frac{-d-9}{6}}$

substitute 'v' in ⑤

$\Rightarrow \quad u+\cancel{3}\left(\frac{-d-9}{\cancel{6}_2}\right)-2d-4=0$

$\Rightarrow \quad 2u-d-9-4d-8=0$

$\Rightarrow \quad 2u-5d-17=0$

$\Rightarrow \quad \boxed{u=\frac{5d+17}{2}}$

substitute 'u' in ④

$\frac{5d+17}{2}-3w-d+2=0$

$5d+17-6w-2d+4=0$

$-6w+3d+21=0$

$-2w+d+7=0 \Rightarrow \boxed{w=\frac{d+7}{2}}$

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substitute u, v, w in (3).

$$\Rightarrow 21 - \cancel{7}^{2}\left(\frac{5d+17}{\cancel{2}}\right) - \cancel{2}\left(-\frac{d-9}{\cancel{6}_{3}}\right) - \cancel{8}^{4}\left(\frac{d+7}{\cancel{2}}\right) + d = 0$$

$$\Rightarrow 21 - 10d - 34 + \left(\frac{d+9}{3}\right) - 4d - 28 + d = 0$$

$$\Rightarrow 63 - 30d - 102 + d + 9 - 12d - 84 + 3d = 0$$

$$\Rightarrow -114 - 38d = 0$$

$$\Rightarrow 38d = -114$$

$$\Rightarrow \boxed{d = -3}$$

$$u \Rightarrow \frac{5(-3)+17}{2} = \frac{-15+17}{2} = \frac{2}{2} = 1 \Rightarrow \boxed{u=1}$$

$$v \Rightarrow \frac{-(-3)-9}{6} = \frac{-6}{6} = -1 \Rightarrow \boxed{v=-1}$$

$$w \Rightarrow \frac{-3+7}{2} = \frac{4}{2} = 2 \Rightarrow \boxed{w=2}$$

substitute u, v, w, d in (1)

$$\therefore x^2+y^2+z^2+2x-2y+4z-3=0.$$

• Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4=0$.

Sol. Let the general eq^n of sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{—— (1)}$$

(1) passes through $(1,-2,1)$

$$\Rightarrow 6+2u-4v+2w+d=0 \quad \text{—— (2)}$$

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① cuts $x^2+y^2+z^2-4x+6y+4=0$ Orthogonally

$\Rightarrow 2\cdot u\cdot(-2)+2\cdot v\cdot(3)+2w(0)=d+4$

$\Rightarrow -4u+6v=d+4$

$\Rightarrow -4u+6v-d-4=0$ — ③ $\Rightarrow 4u-6v+d+4=0$

Eqn of the tangent plane to eq ① at $(1,-2,1)$ is

$x(1)+y(-2)+z(1)+u(x+1)+v(y-2)+w(z+1)+d=0$

$\Rightarrow x(u+1)+(v-2)y+(w+1)z+(x-2y+z+d)=0$

But eqn of tangent plane is given as $3x+2y-z+2=0$

Comparing the co-efficients of similar terms we get,

$$\frac{1+u}{3}=\frac{v-2}{2}=\frac{w+1}{-1}=\frac{u-2v+w+d}{2}$$

$\Rightarrow 2+2u=3v-6 \Rightarrow 2u-3v+8=0$ — ④

$\Rightarrow -v+2=2w+2 \Rightarrow v=-2w$ — ⑤

④ $\Rightarrow 2u=3v-8 \Rightarrow 2u=6w-8$

$\Rightarrow u=-3w-\cancel{8}\,4$

③ $\Rightarrow 4(-3w-\overset{4}{\cancel{8}})-6(-2w)+d+4=0$

$\Rightarrow -\cancel{12w}-16+\cancel{12w}+d+4=0$

$\Rightarrow \boxed{d=12}$

② $\rightarrow$ $6+2(-3w-4)-4(-2w)+2w+12=0$

$\Rightarrow$ $6-6w-8+8w+2w+12=0$

$\Rightarrow$ $4w+10=0$

$\Rightarrow$ $\cancel{4w=10}$ $4w=-10$

$\boxed{w=\frac{-5}{2}}$

$v = +\cancel{2}\left(+\frac{5}{\cancel{2}}\right) = 5 \Rightarrow \boxed{v=5}$

$u = -3\left(-\frac{5}{2}\right)-4 \Rightarrow \frac{15}{2}-4 \Rightarrow \frac{15-8}{2} \Rightarrow \frac{7}{2}.$

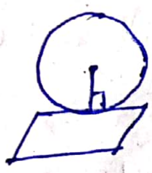
$\boxed{u=7/2}$

Substitute u, v, w, d in ①

$\therefore \quad x^2+y^2+z^2+7x+10y-5z+12=0.$

PCP-5

Find the centres of two spheres which touch the plane $4x+3y=47$ at the point $(8,5,4)$ and which touches the sphere $x^2+y^2+z^2=1$.



Sol. Eq^n of line C_1P is $\frac{x-8}{4} = \frac{y-5}{3} = \frac{z-4}{0} = r$

Any point on line C_1P is $(4r+8, 3r+5, 4)$

let $C_1=(4r+8, 3r+5, 4)$ for some value of 'r'

let r_1 = radius of the required sphere

$\therefore r_1 = C_1P.$

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$$r_1 = \sqrt{(4r+8-8)^2+(3r+5-5)^2+(4-4)^2}$$

$$r_1 = \sqrt{16r^2+9r^2}$$

$$r_1 = \sqrt{25r^2}$$

$$\boxed{r_1 = 5r}$$

The required sphere touches externally or internally the sphere $x^2+y^2+z^2=1$. [centre - $(0,0,0)$, radius - 1]

Sum or diff of the radii = distance b/w the ~~sphere~~ centres

$$5r \pm 1 = \sqrt{(4r+8-0)^2+(3r+5-0)^2+(4-0)^2}$$

$$5r \pm 1 = \sqrt{16r^2 + \cancel{16r}\,64r + 64 + 9r^2 + 25 + 30r + 16}$$

$$5r \pm 1 = \sqrt{25r^2 + \cancel{100r}\,94r + 105}$$

Consider $5r \pm 1 = 25r^2 + 94r + 105$

(i) $$(5r+1)^2 = 25r^2+94r+105$$

$$\cancel{25r^2}+1+10r = \cancel{25r^2}+94r+105$$

$$94r - 10r + 105 - 1 = 0$$

$$84r + 104 = 0$$

$$r = -\frac{26}{21}$$

$$C_1 = \left[4\left(\frac{-26}{21}\right)+8,\ \cancel{3}\left(\frac{-26}{\cancel{21}_{7}}\right)+5, 4\right]$$

$$C_1 = \left(\frac{-104+168}{21}, \frac{35-26}{7}, 4\right) = \left(\frac{64}{21}, \frac{9}{7}, 4\right)$$

$$C_1 = \left(\frac{64}{21}, \frac{9}{7}, 4\right)$$

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(ii) $(5r-1)^2 = 25r^2+94r+105$

$\cancel{25r^2}+1-10r = \cancel{25r^2}+94r+105$

$94r+10r+105-1=0$

$104r+104=0$

$\boxed{r=-1}$

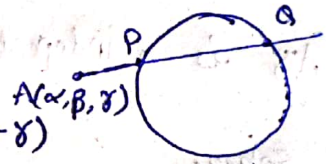
$C_2 = (4(-1)+8,\ 3(-1)+5,\ 4)$

$C_2 = (4, 2, 4)$

$\therefore$ The centres of the two spheres are $\left(\frac{64}{21}, \frac{9}{7}, 4\right)$, $(4,2,4)$

<u>Power of a point w.r.t a Sphere</u>

Eq^n of a line through A is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ [where (l,m,n) are d c's of line].



Any point on the line $= (lr+\alpha,\ mr+\beta,\ nr+\gamma)$

Substituting the above co-ordinates in sphere eq^n

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get

$$r^2(l^2+m^2+n^2) + 2r\left[l(\alpha+u)+m(\beta+v)+n(\gamma+w)\right]$$
$$+\left[\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d\right]=0$$

The above eq^n is a quadratic eq^n in r & has two roots r_1 & r_2.

$P = (lr_1+\alpha,\ mr_1+\beta,\ nr_1+\gamma)$

$Q = (lr_2+\alpha,\ mr_2+\beta,\ nr_2+\gamma)$

$$AP = \sqrt{(lr_1+\cancel{\alpha}-\cancel{\alpha})^2+(mr_1+\cancel{\beta}-\cancel{\beta})^2+(nr_1+\cancel{\gamma}-\cancel{\gamma})^2}$$

$$= \sqrt{r_1^2(l^2+m^2+n^2)}$$

$$= \sqrt{r_1^2}\ [l^2+m^2+n^2=1] \rightarrow \boxed{AP = r_1}$$

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Illy $\boxed{AQ = r_2}$

$AP \times AQ = r_1 \cdot r_2$

= Product of roots of the quadratic Equation in 'r'

$$= \frac{\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d}{l^2+m^2+n^2}$$

[$\because$ Product of roots of $ax^2+bx+c=0$ is $\frac{c}{a}$]

$$\boxed{AP \cdot AQ = \alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d} \quad [\because l^2+m^2+n^2=1]$$

Def of Power of a Point w.r.t Sphere

If from a given point $A(\alpha, \beta, \gamma)$ lines are drawn in any direction to meet the sphere in the points, P & Q, then, AP. AQ is a independent of l, m, n which is known as power of point w.r.t Sphere.

Def of a radical plane

The locus of a point whose power w.r.t two spheres are equal is a plane known as radical plane of two spheres.

To find the eq^n of radical plane of two spheres

Proof: let two spheres be S_1 & S_2

$$S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1y+d_1=0$$

$$S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2y+d_2=0$$

Let $P(x_1, y_1, z_1)$ be a point on the locus

By def. equal power of P w.r.t S_1 = Power of P w.r.t S_2

$$\cancel{x_1^2}+\cancel{y_1^2}+\cancel{z_1^2}+2u_1x_1+2v_1y_1+2w_1z_1+d_1 = \cancel{x_1^2}+\cancel{y_1^2}+\cancel{z_1^2}+2u_2x_1+2v_2y_1+2w_2z_1 +d_2$$

$$\Rightarrow 2(u_1-u_2)x_1+2(v_1-v_2)y_1+2(w_1-w_2)z_1+(d_1-d_2)=0$$

$\therefore$ Eq^n of the Radical Plane is

$$\boxed{2(u_1-u_2)x_1+2(v_1-v_2)y_1+2(w_1-w_2)z_1+(d_1-d_2)=0}$$

Note:

1) Radical Plane -

$$2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+(d_1-d_2)=0$$

$\Rightarrow$ DR's of normal to Radical Plane are

$$[2(u_1-u_2), 2(v_1-v_2), 2(w_1-w_2)]$$

$$= (u_1-u_2, v_1-v_2, w_1-w_2)$$

Also DR's of line joining the centres of two spheres

$$= (u_1-u_2, v_1-v_2, w_1-w_2)$$

$\therefore$ line joining the centres of two spheres is normal to radical plane

Def of Radical Line

The three radical planes of the three spheres intersect at a line known as Radical Line.

Let the three spheres be $S_1=0, S_2=0, S_3=0$ and radical planes are $S_1-S_2=0$, $S_2-S_3=0$, $S_1-S_3=0$.

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The Eqn of Radical Line is $\boxed{S_1 = S_2 = S_3}$

Radical Centre

The four radical lines of the four spheres intersect at a point known as Radical Centre of four spheres.

Theorem: If $S_1 = 0$, $S_2 = 0$ are two spheres then the equation $S_1 + \lambda S_2 = 0$ [λ-parameter] represents a system of spheres such that any two members of the system have the same radical plane.

Proof: The eqn of given system of spheres is $\boxed{S_1 + \lambda S_2 = 0}$

$\therefore$ The 2 members of given system are

$$S_1 + \lambda_1 S_2 = 0, \quad S_1 + \lambda_2 S_2 = 0 \; (\lambda_1 \neq \lambda_2).$$

Reducing above eqns to Standard Form.

$$(x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1) + \lambda_1 (x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2) = 0$$

dividing by ~~λ~~ $1+\lambda_1$, $1+\lambda_2$

$$\Rightarrow \frac{S_1 + \lambda_1 S_2}{1+\lambda_1} = 0, \quad \frac{S_1 + \lambda_2 S_2}{1+\lambda_2} = 0$$

$\therefore$ The eqn of Radical plane is

$$\frac{S_1 + \lambda_1 S_2}{1+\lambda_1} - \frac{S_1 + \lambda_2 S_2}{1+\lambda_2} = 0$$

$$(1+\lambda_2)(S_1 + \lambda_1 S_2) - (1+\lambda_1)(S_1 + \lambda_2 S_2) = 0$$

$$\cancel{S_1} + \lambda_1 S_2 + S_1 \lambda_2 + \cancel{\lambda_1 \lambda_2 S_2} - \cancel{S_1} - \lambda_2 S_2 - S_1 \lambda_1 - \cancel{\lambda_1 \lambda_2 S_2} = 0$$

$$\lambda_1 S_2 - \lambda_2 S_2 + \lambda_2 S_1 - \lambda_1 S_1 = 0$$

$$(\lambda_1 - \lambda_2) S_2 - (\lambda_1 - \lambda_2) S_1 = 0$$

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$(\lambda_1 - \lambda_2)(S_2 - S_1) = 0$

$\Rightarrow \lambda_1 - \lambda_2 = 0$ $\left[\begin{array}{l}\lambda_1 \neq \lambda_2 \\ \lambda_1 - \lambda_2 \neq 0\end{array}\right]$

$\Rightarrow \boxed{S_2 - S_1 = 0}$

which is independent of λ.

Def - Coaxial System of spheres

Note: 1) Eq^n of Co-axial Spheres is $S_1 + \lambda S_2 = 0$.

2) ~~Aslo~~ Also the eq^n of Coaxial System of Spheres is

$S_1 + \lambda(S_1 - S_2) = 0$ (or) $S_2 + \lambda(S_1 - S_2) = 0$

Def of Limiting Points

For limiting Points of the co-axial System we have radius = 0.

***

Ex: Find the Limiting Points of the co-axial System of spheres

$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$, $x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$.

Sol. The Eq^n of Radical Plane is $3x + 3y + 6z = 0$ $[\because S_1 - S_2 = 0]$

$x + y + 2z = 0$

$\therefore$ The eq^n of Coaxial Spheres is $S_1 + \lambda(S_1 - S_2) = 0$

$\Rightarrow x^2 + y^2 + z^2 + 3x - 3y + 6 + \lambda(x + y + 2z) = 0$

$\Rightarrow x^2 + y^2 + z^2 + (3 + \lambda)x + (\lambda - 3)y + (2\lambda)z + 6 = 0$

Centre $= \left(\frac{-\lambda - 3}{2}, \frac{3 - \lambda}{2}, -\lambda\right)$

For limiting points $\lambda = 0$.

$$\Rightarrow \sqrt{\left(\frac{-\lambda-3}{2}\right)^2+\left(\frac{3-\lambda}{2}\right)^2+(\bar{2}\lambda)^2-6} = 0.$$

$$\Rightarrow \sqrt{\frac{\lambda^2+9+6\lambda}{4}+\frac{9+\lambda^2-6\lambda}{4}+\cancel{4}\lambda^2-6} = 0$$

$$\Rightarrow \sqrt{\frac{\lambda^2+9+\cancel{6\lambda}+9+\lambda^2-\cancel{6\lambda}+\cancel{16}\,4\lambda^2-24}{2}} = 0$$

S.O.B.S.

$$\Rightarrow \cancel{\sqrt{18\lambda^2-18}=0}$$

$$\Rightarrow 6\lambda^2-6=0$$

$$\Rightarrow 6\lambda^2=6$$

$$\Rightarrow \lambda^2=1$$

$$\Rightarrow \lambda=\pm 1$$

Centre $= (-2, 1, -1)$, $(-1, 2, 1)$

Limiting Points are $(-2,1,-1)$ & $(-1,2,1)$

PCP-8

Find the limiting points of co-axial system of spheres

$x^2+y^2+z^2-20x+30y-40z+29+\lambda(2x-3y+4z)=0$

Sol. $x^2+y^2+z^2+(2\lambda-20)x+(30-3\lambda)y+(4\lambda-40)z+29=0$

Centre of sphere $\Rightarrow \left(10-\lambda,\ \frac{3\lambda-30}{2},\ 20-2\lambda\right)$

For limiting points, radius $=0$.

$$\Rightarrow \sqrt{(10-\lambda)^2+\left(\frac{3\lambda-30}{2}\right)^2+(20-2\lambda)^2-29} = 0$$

S.O.B.S

$$\Rightarrow 4(10-\lambda)^2+(3\lambda-30)^2+4(20-2\lambda)^2-116=0$$

$$\Rightarrow 4(100+\lambda^2-20\lambda)+9\lambda^2+900-180\lambda+4(400+4\lambda^2-80\lambda)-116=0$$

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$\Rightarrow 400 + 4\lambda^2 - 80\lambda + 9\lambda^2 + 900 - 180\lambda + 1600 + 16\lambda^2 - 320\lambda - 116 = 0$

$\Rightarrow 29\lambda^2 - 580\lambda + 2784 = 0$

$\Rightarrow \lambda^2 - 20\lambda + 96 = 0$

$\Rightarrow \lambda^2 - 12\lambda - 8\lambda + 96 = 0$

$\Rightarrow \lambda(\lambda - 12) - 8(\lambda - 12) = 0$

$\Rightarrow (\lambda - 12)(\lambda - 8) = 0$

$\Rightarrow \lambda = 12, 8$

~~Centre~~ Substituting 'λ' values in centre we get,

Limiting points are $(-2, 3, -4)$, $(2, -3, +4)$

PCP-10

Show that the Radical Planes of sphere of a co-axial system and of any given sphere pass through a line.

Sol. let the eqn of co-axial system of spheres be

$$S_1 + \lambda S_2 = 0$$

Reducing to Standard form $\Rightarrow \dfrac{S_1 + \lambda S_2}{1+\lambda} = 0$ — (1)

let $S_3 = 0$ be any given Sphere — (2)

$\therefore$ Radical Plane of (1) & (2) is

$$\frac{S_1 + \lambda S_2}{1+\lambda} - S_3 = 0$$ [$\because$ Eqn of Radical Plane]

$\Rightarrow S_1 + \lambda S_2 - S_3 - \lambda S_3 = 0$

$\Rightarrow (S_1 - S_3) + \lambda(S_2 - S_3) = 0$

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It is in the form of $\boxed{P_1 + \lambda P_2 = 0}$ $\begin{bmatrix} S_1 - S_3 = P_1 \\ S_2 - S_3 = P_2 \end{bmatrix}$

which represents a plane passing through the line whose

eqn is $P_1 = 0, P_2 = 0$

$\Rightarrow S_1 - S_3 = 0, \ S_2 - S_3 = 0$

PCP-9

Find the equation ~~of~~ to two spheres of the co-axial system $x^2+y^2+z^2-5+\lambda(2x+y+3z-3)=0$ which touch the plane $3x+4y=15$.

Sol. Eqn of the co-axial system of spheres is

$$x^2+y^2+z^2-5+\lambda(2x+y+3z-3)=0 \quad \text{—— (1)}$$

Centre of (1) $\Rightarrow \left(-\lambda, -\frac{\lambda}{2}, -\frac{3\lambda}{2}\right)$

(1) touches the plane $3x+4y-15=0$ (Tangent Plane)

For a tangent Plane,

radius of sphere $=$ $\perp^r$ distance from $\left(-\lambda, -\frac{\lambda}{2}, -\frac{3\lambda}{2}\right)$ to $3x+4y-15=0$

$$\Rightarrow \sqrt{\lambda^2+\frac{\lambda^2}{4}+\frac{9\lambda^2}{4}+5+3\lambda} = \left|\frac{3(-\lambda)+\cancel{4}^{2}\left(-\frac{\lambda}{\cancel{2}}\right)-15}{\sqrt{9+16}}\right|$$

$$\Rightarrow \sqrt{\frac{4\lambda^2+\lambda^2+9\lambda^2+20+12\lambda}{4}} = \left|\frac{-3\lambda-2\lambda-15}{5}\right|$$

$$\Rightarrow \sqrt{\lambda^2+\frac{5\lambda^2}{2}+5+3\lambda} = \left|\frac{-5\lambda-15}{5}\right| = \lambda+3$$

S.O.B.S

$$\lambda^2+\frac{5\lambda^2}{2}+5+3\lambda=(\lambda+3)^2$$

$$\cancel{\lambda^2}+\frac{5\lambda^2}{2}+5+3\lambda=\cancel{\lambda^2}+9+6\lambda$$

$$\frac{5\lambda^2}{2}-3\lambda-4=0$$

$$5\lambda^2-6\lambda-8=0$$

$$5\lambda^2-10\lambda+4\lambda-8=0$$

$$5\lambda(\lambda-2)+4(\lambda-2)=0$$

$$(\lambda-2)(5\lambda+4)=0$$

$$\lambda=+2,-\frac{4}{5}$$

Substituting 'λ' value in (1)

$$x^2+y^2+z^2-5+2(2x+y+3z-3)=0$$

$$\Rightarrow x^2+y^2+z^2+4x+2y+6z-11=0$$

$$\lambda=-\frac{4}{5}$$

$$x^2+y^2+z^2-5-\frac{4}{5}(2x+y+3z-3)=0$$

$$\Rightarrow x^2+y^2+z^2-\frac{8}{5}x-\frac{4}{5}y-\frac{12}{5}z-\frac{13}{5}=0$$

$$\Rightarrow 5(x^2+y^2+z^2)-8x-4y-12z-13=0$$

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• Show that the sphere $x^2+y^2+z^2+2vy+2wz-d=0$ passes through the limiting points of the co-axial system

$x^2+y^2+z^2+2kx+d=0$.

sol. Eq^n of the co-axial system is $x^2+y^2+z^2+2kx+d=0$

centre = $(-k,0,0)$

For limiting points, radius = 0

$\Rightarrow \sqrt{k^2+0+0-d}=0$

$\Rightarrow k^2-d=0$

$\Rightarrow k=\pm\sqrt{d}$

$\therefore$ Limiting points are $(\sqrt{d},0,0)$, $(-\sqrt{d},0,0)$

Substituting $(-\sqrt{d},0,0)$ in $x^2+y^2+z^2+2vy+2wz-d=0$

$\Rightarrow$ LHS = $\cancel{d}+0+0+0+0-\cancel{d}$

$=0$

LHS = RHS

Illy

Substituting $(\sqrt{d},0,0)$ in $x^2+y^2+z^2+2vy+2wz-d=0$

$\Rightarrow$ LHS = $\cancel{d}+0+0+0+0-\cancel{d}$

$=0$

LHS = RHS

$\therefore$ The given sphere passes through the limiting points of the co-axial system.

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Def. of Plane Of Contact

The locus of the points of contact of the tangent planes which pass through a given point (α, β, γ) is known as plane of contact of the given sphere w.r.t the point (α, β, γ)

To Find the Eqn of the Plane Of Contact

Proof: Let the eqn of the given sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

Let the fixed point be (α, β, γ)

Let $P(x_1, y_1, z_1)$ be the point of contact

$\therefore$ Eqn of the tangent plane at the 'P' w.r.t given sphere

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0 \quad \text{--- (1)}$$

(1) passes through (α, β, γ)

$$\Rightarrow \alpha x_1+\beta y_1+\gamma z_1+u(\alpha+x_1)+v(\beta+y_1)+w(\gamma+z_1)+d=0$$

Required Locus (Plane of Contact) the is obtained by generalizing (x_1, y_1, z_1) we get,

Eqn of Plane of Contact is

$$\alpha x+\beta y+\gamma z+u(\alpha+x)+v(\beta+y)+w(\gamma+z)+d=0$$

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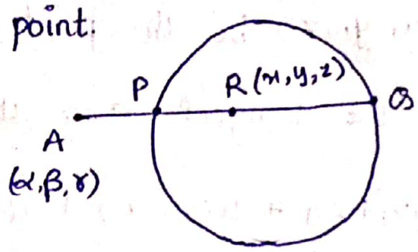
Polar Plane: If a line drawn through a fixed Point $A(\alpha,\beta,\gamma)$ intersects the sphere in two points say P & Q and a Point 'R' is taken on the line so that the line segment AR is divided internally & externally by the points P & Q in the same radius then the locus of R is known as Polar Plane.

• Here the point A is known as the pole of the plane.

Theorem:

To Find the Equation of the Polar Plane of the Sphere $x^2+y^2+z^2=a^2$

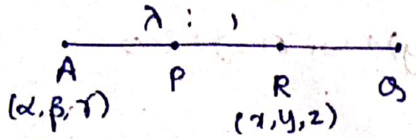
Proof: Let $A(\alpha,\beta,\gamma)$ be a fixed point.



Through point A, line is drawn that intersects the sphere in the point P & Q.

Let $R(x,y,z)$ be a point on the line such that line AR cuts ~~is~~ sphere externally & internally by its points P & Q in the same ratio.

Let the ~~radius~~ ratio be $\boxed{\lambda:1}$



$$R=\left(\frac{\lambda x+\alpha}{\lambda+\cancel{\alpha}1},\frac{\lambda y+\beta}{\lambda+\cancel{\beta}1},\frac{\lambda z+\gamma}{\lambda+\cancel{\gamma}1}\right)$$

substituting in the eq of sphere, we get

$$\Rightarrow \cancel{\lambda x+\lambda\alpha}\left(\frac{\lambda x+\alpha}{\lambda+\cancel{\alpha}1}\right)^2+\left(\frac{\lambda y+\beta}{\lambda+\cancel{\beta}1}\right)^2+\left(\frac{\lambda z+\gamma}{\lambda+\cancel{\gamma}1}\right)^2=a^2$$

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$\Rightarrow (\lambda x+\alpha)^2+(\lambda y+\beta)^2+(\lambda z+\gamma)^2 = a^2(\lambda+1)^2$

$\Rightarrow \lambda^2x^2+\alpha^2+2\lambda\alpha x+\lambda^2y^2+\beta^2+2\lambda\beta y+\lambda^2z^2+\gamma^2+2\lambda\gamma z$

$= a^2(\lambda^2+2\lambda+1)$

$\Rightarrow \lambda^2(x^2+y^2+z^2)+2\lambda(\alpha x+\beta y+\gamma z)+(\alpha^2+\beta^2+\gamma^2) = a^2\lambda^2+2a^2\lambda+a^2$

$\Rightarrow \lambda^2(x^2+y^2+z^2-a^2)+2\lambda(\alpha x+\beta y+\gamma z-a^2)+(\alpha^2+\beta^2+\gamma^2-a^2)=0$

let the roots of the quadratic Equation be λ_1 & λ_2 which corresponds to two ratios (by internal & external division)

By def, $\Rightarrow \lambda_1=-\lambda_2 \Rightarrow \boxed{\lambda_1+\lambda_2=0}$

$\Rightarrow$ Sum of the roots of Quadratic Equation.

$$\Rightarrow \frac{-2(\alpha x+\beta y+\gamma z-a^2)}{x^2+y^2+z^2-a^2}=0$$

[$\because$ Sum of the roots of $ax^2+bx+c=0$ is $-\frac{b}{a}$]

$$\Rightarrow \boxed{\alpha x+\beta y+\gamma z-a^2=0}$$

Polar Plane of the point $A(\alpha,\beta,\gamma)$ w.r.t the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is

$$\alpha x+\beta y+\gamma z+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d=0$$

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Theorem: To find the pole of the plane $lx+my+nz=p$ w.r.t the sphere $x^2+y^2+z^2=a^2$.

Proof: Given that, $lx+my+nz=p$ is a polar plane ——(1)

Let $A(\alpha, \beta, \gamma)$ be the required pole

By previous theorem, Polar plane of A w.r.t given sphere is $\alpha x+\beta y+\gamma z-a^2=0$ ——(2)

$$\alpha x+\beta y+\gamma z=a^2$$

(1) & (2) represent a polar plane

$$\frac{l}{\alpha}=\frac{m}{\beta}=\frac{n}{\gamma}=\frac{p}{a^2}$$

consider $\frac{l}{\alpha}=\frac{p}{a^2} \Rightarrow \alpha p=a^2 l$

$$\boxed{\alpha=\frac{a^2 l}{p}}$$

lly $\boxed{\beta=\frac{a^2 m}{p}}$

$$\boxed{\gamma=\frac{a^2 n}{p}}$$

$\therefore$ Required pole $=\left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$

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~~Theorem: If the Pole~~

Note: If the Pole of a Plane π_1 lie on Plane π_2, then show that the pole of π_2 lies on π_1.

Proof: Let the equation of plane π_1 be $l_1x + m_1y + n_1z = p_1$

equation of plane π_2 be $l_2x + m_2y + n_2z = p_2$

Let the eq^n of the sphere be $x^2 + y^2 + z^2 = a^2$

$\therefore$ Pole of π_1 w.r.t given sphere $= \left(\frac{a^2 l_1}{p_1}, \frac{a^2 m_1}{p_1}, \frac{a^2 n_1}{p_1}\right)$

[~~Pole of π_2 w.r.t given sphere $= \left(\frac{a^2 l_2}{p_2}, \frac{a^2 m_2}{p_2}, \frac{a^2 n_2}{p_2}\right)$~~] $\times$

Since, Pole of π_1 lies on π_2 we get,

$$l_2\left(\frac{a^2 l_1}{p_1}\right) + m_2\left(\frac{a^2 m_1}{p_1}\right) + n_2\left(\frac{a^2 n_1}{p_1}\right) = p_2$$

$$\Rightarrow \quad a^2 l_1 l_2 + a^2 m_1 m_2 + a^2 n_1 n_2 = p_1 p_2$$

$$\Rightarrow \quad l_1\left(\frac{a^2 l_2}{p_2}\right) + m_1\left(\frac{a^2 m_2}{p_2}\right) + n_1\left(\frac{a^2 n_2}{p_2}\right) = p_1$$

$\Rightarrow$ Pole of π_2 $\left(\frac{a^2 l_2}{p_2}, \frac{a^2 m_2}{p_2}, \frac{a^2 n_2}{p_2}\right)$ lies on $l_1x + m_1y + n_1z = p_1$

Polar line

Two lines such that the polar plane of every point on either passes through the other are known as polar lines.

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1) Find the polar line of $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ w.r.t. the sphere $x^2+y^2+z^2=16$.

Sol. let $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r$

Any point on the same line $= (2r+1, 3r+2, 4r+3)$

$\therefore$ The polar plane of $(2r+1, 3r+2, 4r+3)$ w.r.t given sphere is $(2r+1)x + (3r+2)y + (4r+3)z - 16 = 0$

$\Rightarrow (x+2y+3z-16) + r(2x+3y+4z) = 0$ — ①

① which represents a polar plane passing through the line (polar line). whose eqn is given by $x+2y+3z-16=0$,

$2x+3y+4z=0$.

2) Find the Polar line of $\frac{x+1}{2} = \frac{y-2}{3} = \frac{z+3}{1}$ w.r.t the sphere $x^2+y^2+z^2=1$.

sol. let $\frac{x+1}{2} = \frac{y-2}{3} = \frac{z+3}{1} = r$

Any point on the same line $= (2r-1, 3r+2, r-3)$

$\therefore$ the plane of $(2r-1, 3r+2, r-3)$ w.r.t given sphere is

$(2r-1)x + (3r+2)y + (r-3)z - 1 = 0$

$\Rightarrow (-x+2y-3z-1) + r(2x+3y+z) = 0$ — ①

① represents a polar plane passing through the polar line whose eqns are given by,

$x-2y+3z+1=0$

$2x+3y+z=0$

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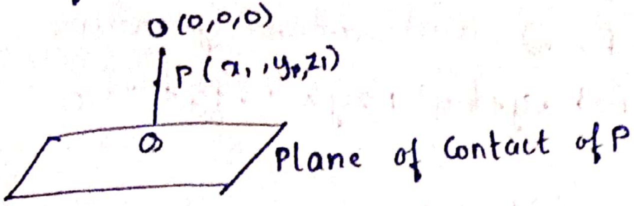
3) Show that the line joining any point P & to the ~~centre~~ given of sphere is perpendicular to the plane of contact of P and if OP meets it in Q then $OP \times OQ = a^2$.

sol.



DR's of OP = (x_1, y_1, z_1) — ①

Plane of Contact of P w.r.t $x^2+y^2+z^2=a^2$ is

$xx_1+yy_1+zz_1-a^2=0$

DR's of normal to the plane of contact = (x_1, y_1, z_1) — ②

① & ② represent that OP is $\perp^r$ to (normal) to the plane of contact.

$$OP = \sqrt{(x_1-0)^2+(y_1-0)^2+(z_1-0)^2} = \sqrt{x_1^2+y_1^2+z_1^2}$$

OQ = $\perp^r$ distance from O(0,0,0) to $xx_1+yy_1+zz_1-a^2=0$

$$= \left|\frac{0(x_1)+0(y_1)+0(z_1)-a^2}{\sqrt{x_1^2+y_1^2+z_1^2}}\right|$$

$$= \frac{a^2}{\sqrt{x_1^2+y_1^2+z_1^2}}$$

$$\therefore OP \times OQ = \cancel{\sqrt{x_1^2+y_1^2+z_1^2}} \cdot \frac{a^2}{\cancel{\sqrt{x_1^2+y_1^2+z_1^2}}} = a^2$$

$= (\text{radius})^2$

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Cone:

Theorem: To find the equation of a cone whose vertex is the point (α, β, γ) and whose generators intersect in $ax^2+2hxy+by^2+2gx+2fy+c=0$, $z=0$.

Proof: Let the eqn of line passing through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{--- (1)}$$

(1) becomes a generator if it intersects $z=0$

put $z=0$ in (1)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$\therefore$ point of intersection $= \left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

Substituting in $ax^2+2hxy+by^2+2gx+2fy+c=0$

we get,

$$\Rightarrow a\left(\alpha-\frac{l\gamma}{n}\right)^2 + 2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right) + b\left(\beta-\frac{m\gamma}{n}\right)^2 + 2g\left(\alpha-\frac{l\gamma}{n}\right) + 2f\left(\beta-\frac{m\gamma}{n}\right) + c = 0$$

Eliminating (l, m, n) using (1) we get required locus.

$$\Rightarrow a\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2 + 2h\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right] + b\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2$$

$$+ 2g\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right] + 2f\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right] + c = 0$$

$$\Rightarrow a\left[\frac{\alpha z-\alpha\gamma-x\gamma+\alpha\gamma}{z-\gamma}\right]^2 + 2h\left[\frac{z\alpha-\gamma\alpha-x\gamma+\alpha\gamma}{z-\gamma}\right]\left[\frac{\beta z-\beta\gamma-y\gamma+\beta\gamma}{z-\gamma}\right]$$

$$+ b\left[\frac{\beta z-\beta\gamma-y\gamma+\beta\gamma}{z-\gamma}\right]^2 + 2g\left[\frac{\alpha z-\alpha\gamma-x\gamma+\alpha\gamma}{z-\gamma}\right] + 2f\left[\frac{\beta z-\beta\gamma-y\gamma+\beta\gamma}{z-\gamma}\right] + c = 0$$

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$$\Rightarrow a(\alpha z - x\gamma)^2 + 2h(\alpha z - x\gamma)(\beta z - y\gamma) + b(\beta z - y\gamma)^2 + 2g(\alpha z - x\alpha)$$

$$(z-\gamma) + 2f(\beta z - y\gamma)(z-\gamma) + c(z-\gamma)^2 = 0$$

$\rightarrow$ Eq^n of the required cone.

Problems :

1) Find the equation of the cone whose vertex is (α, β, γ) and base is ~~ax²+2hxy+by²~~ $ax^2+by^2=1, z=0$.

Sol. Eq^n of line through vertex (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{— ①}$$

① is a generator if it intersects $z=0$

put $z=0$ in ① $\Rightarrow \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$

$\Rightarrow$ point of intersection $= \left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

substituting point of intersection in $ax^2+by^2=1$, we get

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + b\left(\beta - \frac{m\gamma}{n}\right)^2 = 1$$

eliminating (l, m, n) using ① we get eq^n of cone as

$$a\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2 + b\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2 = 1$$

$$\Rightarrow a\left[\frac{\alpha z - \cancel{\alpha\gamma} - x\gamma + \cancel{\alpha\gamma}}{z-\gamma}\right]^2 + b\left[\frac{\beta z - \cancel{\beta\gamma} - y\gamma + \cancel{\beta\gamma}}{z-\gamma}\right]^2 = 1$$

$$\Rightarrow \boxed{a(\alpha z - x\gamma)^2 + b(\beta z - y\gamma)^2 = (z-\gamma)^2}$$

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4) Find the equation of the cone whose vertex is (α, β, γ) and base is $y^2 = 4ax, z = 0$.

Sol: Eqⁿ of the line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{--- (1)}$$

put $z = 0$ in (1)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

point of intersection $= \left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

substituting in $y^2 = 4ax$ we get

$$\left(\beta - \frac{m\gamma}{n}\right)^2 = 4a\left(\alpha - \frac{l\gamma}{n}\right)$$

Eliminating (l, m, n) using (1) we get, eqⁿ of cone as

$$\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2 = 4a\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]$$

$$\Rightarrow \left[\frac{\beta z - \cancel{\beta\gamma} - y\gamma + \cancel{\beta\gamma}}{z-\gamma}\right]^2 = 4a\left[\frac{\alpha z - \cancel{\alpha\gamma} - x\gamma + \cancel{\alpha\gamma}}{z-\gamma}\right]$$

$$\Rightarrow \frac{(\beta z - y\gamma)^2}{(z-\gamma)^{\cancel{2}}} = \frac{4a(\alpha z - x\gamma)}{\cancel{(z-\gamma)}}$$

$$\Rightarrow \boxed{(\beta z - y\gamma)^2 = 4a(\alpha z - x\gamma)(z-\gamma)}$$

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3) Obtain the locus of the lines which pass through the point (α, β, γ) and through the points of the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$. (ans –

Sol. Eqⁿ of the line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{--- (1)}$$

put $z=0$ in (1)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{0-\gamma}{n}$$

Point of intersection $= \left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

substituting P.I in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we get

$$\frac{\left(\alpha - \frac{l\gamma}{n}\right)^2}{a^2} + \frac{\left(\beta - \frac{m\gamma}{n}\right)^2}{b^2} = 1$$

Eliminating (l, m, n) using (1), we get eqⁿ. of cone as

$$\Rightarrow \frac{\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2}{a^2} + \frac{\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2}{b^2} = 1$$

$$\Rightarrow \frac{\left[\frac{\alpha z - \cancel{\alpha\gamma} - x\gamma + \cancel{\alpha\gamma}}{z-\gamma}\right]^2}{a^2} + \frac{\left[\frac{\beta z - \cancel{\beta\gamma} - y\gamma + \cancel{\beta\gamma}}{z-\gamma}\right]^2}{b^2} = 1$$

$$\Rightarrow \frac{(\alpha z - x\gamma)^2}{a^2(z-\gamma)^2} + \frac{(\beta z - y\gamma)^2}{b^2(z-\gamma)^2} = 1$$

$$\Rightarrow \boxed{\frac{(\alpha z - x\gamma)^2}{a^2} + \frac{(\beta z - y\gamma)^2}{b^2} = (z-\gamma)^2}$$

PCP-11 :

Find the equation of the cone whose vertex is (1, 1, 0) and whose guiding curve is $y=0,\ x^2+z^2=4$.

sol. Equation of the line through (1, 1, 0)

$$\frac{x-1}{l}=\frac{y-1}{m}=\frac{z-0}{n} \quad \text{—} \ (1)$$

put $y=0$ in (1)

$$\frac{x-1}{l}=\frac{-1}{m}=\frac{z}{n}$$

point of intersection = $\left(1-\frac{l}{m}, 0, -\frac{n}{m}\right)$

substituting P.I in $x^2+z^2=4$, we get,

$$\Rightarrow \left(1-\frac{l}{m}\right)^2+\left(\frac{-n}{m}\right)^2=4$$

Eliminating (l, m, n) ~~from~~ using (1), we get eq^n of cone as

$$\Rightarrow \left[1-\left(\frac{x-1}{y-1}\right)\right]^2+\left[-\left(\frac{z}{y-1}\right)\right]^2=4$$

$$\Rightarrow \left[\frac{y\cancel{-1}-x\cancel{+1}}{y-1}\right]^2+\left[\frac{z}{y-1}\right]^2=4$$

$$\Rightarrow \left[\frac{y-x}{y-1}\right]^2+\left[\frac{z}{y-1}\right]^2=4$$

$$\Rightarrow (y-x)^2+z^2=4(y-1)^2$$

$$\Rightarrow y^2+x^2-2yx+z^2=4(y^2+1-2y)$$

$$\Rightarrow y^2+x^2-2xy+z^2-4y^2-4+8y=0$$

$$\Rightarrow \boxed{x^2-3y^2+z^2-2xy+8y-4=0}$$

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• Find the equation to the cone whose vertex is at the origin and which pass through the curve by the eqn $z=2$, $x^2+y^2=4$.

sol Equation of line through (0,0,0)

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n} \quad \text{——} \quad (1)$$

put $z=2$ in (1)

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{2}{n}$$

$$\frac{x}{l}=\frac{y}{m}=\frac{2}{n}$$

point of intersection $=\left(\frac{2l}{n}, \frac{2m}{n}, 2\right)$

substituting P.I in $x^2+y^2=4$

$$\Rightarrow \left(\frac{2l}{n}\right)^2+\left(\frac{2m}{n}\right)^2=4 \Rightarrow \frac{4l^2}{n^2}+\frac{4m^2}{n^2}=4 \Rightarrow \frac{l^2}{n^2}+\frac{m^2}{n^2}=1$$

Eliminating (l,m,n) using (1)

$$\Rightarrow \left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2=1$$

$$\Rightarrow x^2+y^2=z^2$$

$$\Rightarrow \boxed{x^2+y^2-z^2=0}$$

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• Find the equation of the cone whose vertex is $(1,2,3)$ and guiding curve is the circle $x^2+y^2+z^2=4,\ x+y+z=1$

Sol. Equation of line through $(1,2,3)$

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = r \quad (1)$$

Any point on the line $= (lr+1,\ mr+2,\ nr+3)$

Substituting in $x+y+z=1$, we get,

$$\Rightarrow lr + \cancel{1} + mr + 2 + nr + 3 = \cancel{1}$$

$$\Rightarrow lr + mr + nr + 5 = 0$$

$$\Rightarrow r(l+m+n) = -5$$

$$\Rightarrow r = \frac{-5}{l+m+n}$$

[Substitute 'r' in the point $\Rightarrow \left(\frac{-5l}{l+m+n}+1,\ \frac{-5m}{l+m+n}+2,\ \frac{-5n}{l+m+n}+3\right)$]

Again substituting $(lr+1,\ mr+2,\ nr+3)$

$$\Rightarrow (lr+1)^2 + (mr+2)^2 + (nr+3)^2 = 4$$

$$\Rightarrow \left[\frac{-5l}{l+m+n}+1\right]^2 + \left[\frac{-5m}{l+m+n}+2\right]^2 + \left[\frac{-5n}{l+m+n}+3\right]^2 = 4$$

$$\Rightarrow \left[\frac{-5l+l+m+n}{l+m+n}\right]^2 + \left[\frac{-5m+2l+2m+2n}{l+m+n}\right]^2 + \left[\frac{-5n+3l+3m+3n}{l+m+n}\right]^2 = 4$$

$$\Rightarrow \left[\frac{-4l+m+n}{l+m+n}\right]^2 + \left[\frac{2l-3m+2n}{l+m+n}\right]^2 + \left[\frac{3l+3m-2n}{l+m+n}\right]^2 = 4$$

$$\Rightarrow (-4l+m+n)^2 + (2l-3m+2n)^2 + (3l+3m-2n)^2 = 4(l+m+n)^2$$

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$\Rightarrow (16l^2+m^2+n^2-8lm-8ln+2mn)+(4l^2+9m^2+4n^2-12lm+8ln$
$-12mn)$

$+(9l^2+9m^2+4n^2-12ln+18lm-12mn)$

$=4l^2+4m^2+4n^2+8lm+8mn+8ln$

$\Rightarrow 29l^2+19m^2+\overset{9m^2}{9m^2}-2lm-28ln-22mn=4l^2+4m^2+4n^2$
$+8lm+8mn+8ln$

$\Rightarrow 25l^2+15m^2+5n^2+6lm-36ln-30mn=0$

$\Rightarrow 25m^2+15m^2+5n^2-10lm-30mn-20ln=0$

$\Rightarrow 5l^2+3m^2+n^2-2lm-6mn-4ln=0$

Eliminating (l,m,n) using ① we get,

$$\Rightarrow 5\left(\frac{x-1}{r}\right)^2+3\left(\frac{y-2}{r}\right)^2+\left(\frac{z-3}{r}\right)^2-2\left(\frac{x-1}{r}\right)\left(\frac{y-2}{r}\right)-6\left(\frac{y-2}{r}\right)\left(\frac{z-3}{r}\right)$$
$$-4\left(\frac{x-1}{r}\right)\left(\frac{z-3}{r}\right)=0$$

$\Rightarrow 5(x^2-2x+1)+3(y^2-4y+4)+(z^2-6z+9)-2(-2x-y+xy+2)$
$-6(-3y-2z+yz+6)-4(-3x-z+xz+3)=0$

$\Rightarrow 5x^2-10x+5+3y^2-12y+12+z^2-6z+9+4x+2y+2xy-4+18y$
$+12z-6yz-36+12x+4z-4xz-12=0$

$\Rightarrow 5x^2+3y^2+z^2+2xy-6yz-4zx+6x+8y+10z-36=0$

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• Find the eqn of the cone whose vertex is (1,2,3) and base is $y^2 = 4ax, z=0$ [ans- $(2z-3y)^2 = 4a(z-3x)(z-3)$]

sol Eqn of the line through (1,2,3) is

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} \quad \text{--- (1)}$$

put $z=0$ in (1)

$$\frac{x-1}{l} = \frac{y-2}{m} = -\frac{3}{n}$$

point of intersection $= \left(-\frac{3l}{n}+1, -\frac{3m}{n}+2, 0\right)$

substituting P.I in $y^2 = 4ax$

$$\left(2-\frac{3m}{n}\right)^2 = 4a\left(1-\frac{3l}{n}\right)$$

Eliminating (l, m, n) using (1) we get,

$$\left[2-3\left(\frac{y-2}{z-3}\right)\right]^2 = 4a\left[1-3\left(\frac{x-1}{z-3}\right)\right]$$

$$\left[\frac{2z-\cancel{6}-3y+\cancel{6}}{z-3}\right]^2 = 4a\left[\frac{z-\cancel{3}-3x+\cancel{3}}{z-3}\right]$$

$$\left[\frac{2z-3y}{z-3}\right]^2 = 4a\left[\frac{z-3x}{z-3}\right]$$

$$\frac{(2z-3y)^2}{(z-3)^{\cancel{2}}} = \frac{4a(z-3x)}{(z-\cancel{3})}$$

$$(2z-3y^2) = 4a(z-3x)(z-3)$$

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PCP-15

Find the equation of the cone whose vertex is at the origin and DC's of the generating curve satisfy the relation

$3l^2 4m^2+5n^2=0.$

Sol. Eqn of generator through origin is

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n}\quad \left[\text{where } (l,m,n) \text{ are DC's of the generator}\right]$$

$$\Rightarrow \frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{— } (1)$$

Given relation is

$$3l^2-4m^2+5n^2=0$$

$$\Rightarrow 3\left(\frac{l}{n}\right)^2-4\left(\frac{m}{n}\right)^2+5=0 \quad \text{eliminating } (l,m,n) \text{ using } (1)$$

$$\Rightarrow 3\left(\frac{x}{z}\right)^2-4\left(\frac{y}{z}\right)^2+5=0$$

$$\Rightarrow \boxed{3x^2-4y^2+5z^2=0}$$

• Find the eqn of the cone whose generators pass through (α,β,γ) and having DC's satisfying the relation $al^2+bm^2+cn^2=0$.

Sol. Eqn of generator through (α,β,γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{— } (1) \quad [(l,m,n) \text{ are DC's of generator}]$$

Given relation $\Rightarrow al^2+bm^2+cn^2=0$

$$\Rightarrow a\left(\frac{l}{n}\right)^2+b\left(\frac{m}{n}\right)^2+c=0$$

Eliminating (l,m,n) using (1)

$$\Rightarrow a\left(\frac{x-\alpha}{z-\gamma}\right)^2+b\left(\frac{y-\beta}{z-\gamma}\right)^2+c=0$$

$$\Rightarrow \boxed{a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0}$$

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Note: The cones whose equations are in second degree are called Quadratic cones (or) Quadric Cones.

PCP-14

Find the equation of the quadric cone whose vertex is at the origin and passes through the curves given by the equation $ax^2+by^2+cz^2=1$, $lx+my+nz=p$.

sol. Eq^n of line through origin is

$$\frac{x-0}{l_1} = \frac{y-0}{m_1} = \frac{z-0}{n_1} = r$$

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} = r \text{ (say)} \quad \text{—} \ (1)$$

Any point on the line $= (l_1 r, m_1 r, n_1 r)$.

Substituting in $lx+my+nz=p$, we get,

$\Rightarrow l(l_1 r) + m(m_1 r) + n(n_1 r) = p$

$\Rightarrow (ll_1)r + (mm_1)r + (nn_1)r = p$

$\Rightarrow r(ll_1 + mm_1 + nn_1) = p$

$$\Rightarrow r = \frac{p}{ll_1 + mm_1 + nn_1}$$

Again substituting $(l_1 r, m_1 r, n_1 r)$ in $ax^2+by^2+cz^2=1$,

$a(l_1 r)^2 + b(m_1 r)^2 + c(n_1 r)^2 = 1$

$\Rightarrow a l_1^2 r^2 + b m_1^2 r^2 + c n_1^2 r^2 = 1$

$\Rightarrow r^2 (a l_1^2 + b m_1^2 + c n_1^2) = 1$

substituting 'r' in the above equation, we get,

$$\Rightarrow \frac{p^2}{(ll_1+mm_1+nn_1)^2}(a l_1^2 + b m_1^2 + c n_1^2) = 1$$

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Eliminating (l_1, m_1, n_1) using eq^n ① we get, eq^n of cone as.

$$\Rightarrow \frac{p^2}{\left[l\left(\frac{x}{r}\right)+m\left(\frac{y}{r}\right)+n\left(\frac{z}{r}\right)\right]^2}\left[a\left(\frac{x}{r}\right)^2+b\left(\frac{y}{r}\right)^2+c\left(\frac{z}{r}\right)^2\right]=1$$

$$\Rightarrow \frac{p^2}{\left[l\left(\frac{x}{r}\right)+m\left(\frac{y}{r}\right)+n\left(\frac{z}{r}\right)\right]^2}\left[a\left(\frac{x^2}{r^2}\right)+b\left(\frac{y^2}{r^2}\right)+c\left(\frac{z^2}{r^2}\right)\right]=1$$

$$\Rightarrow \frac{p^2 \cancel{r^2}}{(lx+my+nz)^2}\left[\frac{ax^2+by^2+cz^2}{\cancel{r^2}}\right]=1$$

$$\left[\Rightarrow \frac{\cancel{p^2}}{\cancel{p^2}}\ \cancel{(ax^2+by^2+cz^2)=1}\right.$$

$$\left.\Rightarrow \cancel{ax^2+by^2+cz=1}\right]^{\times}$$

$$\Rightarrow \boxed{p^2(ax^2+by^2+cz^2)=(lx+my+nz)^2}$$

P(P-12

~~The~~ Find the section of a cone whose vertex is P and guiding curve is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ by the plane $x=0$.

Sol. Let P be (α, β, γ) the vertex

Eq^n of the line through vertex $=(\alpha, \beta, \gamma)$ is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{—— ①}$$

Put $z=0$

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{-\gamma}{n}$$

Point of intersection $=\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$

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Substituting in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we get,

$$\frac{\left(\alpha-\frac{l\gamma}{n}\right)^2}{a^2}+\frac{\left(\beta-\frac{m\gamma}{n}\right)^2}{b^2}=1 \qquad \text{Eliminating } (l,m,n)$$

$$\Rightarrow \frac{1}{a^2}\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2+\frac{1}{b^2}\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2=1$$

$$\Rightarrow \frac{1}{a^2}\left[\frac{\alpha z-\cancel{\alpha\gamma}-x\gamma+\cancel{\alpha\gamma}}{z-\gamma}\right]^2+\frac{1}{b^2}\left[\frac{\beta z-\cancel{\beta\gamma}-y\gamma+\cancel{\beta\gamma}}{z-\gamma}\right]^2=1$$

$$\Rightarrow \frac{1}{a^2}\left[\frac{\alpha z-x\gamma}{z-\gamma}\right]^2+\frac{1}{b^2}\left[\frac{\beta z-y\gamma}{z-\gamma}\right]^2=1$$

$$\Rightarrow \frac{1}{a^2}(\alpha z-x\gamma)^2+\frac{1}{b^2}(\beta z-y\gamma)^2=(z-\gamma)^2$$

— eq of cone (2)

(2) meets $\boxed{x=0}$

$$\Rightarrow \frac{1}{a^2}\alpha^2 z^2+\frac{1}{b^2}(\beta z-y\gamma)^2=(z-\gamma)^2$$

For rectangular Hyperbola, $\boxed{\text{co-eff of } y^2+\text{co-eff of } z^2=0}$

~~$\Rightarrow \left(\frac{\gamma^2}{b^2}\right)+\left(\frac{\alpha^2}{a^2}\right)+$~~

$$\Rightarrow \left(\frac{\gamma^2}{b^2}\right)+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)=0$$

$\therefore$ locus of $P(\alpha,\beta,\gamma)$ is $\frac{z^2}{b^2}+\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

$$\Rightarrow \boxed{\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1}$$

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Enveloping Cone: The locus of all the tangent lines drawn to a surface from a fixed point is known as Enveloping Curve.

Theorem: To Find the equation of the cone whose vertex is the point (α, β, γ) and whose generators touch the sphere $x^2+y^2+z^2=a^2$

Proof: Let $A(\alpha, \beta, \gamma)$ be the fixed point (vertex)

Eq^n of the line through A is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ —— ①

Any point on the line = $(lr+\alpha, mr+\beta, nr+\gamma)$

Substituting in the eq^n of sphere we get

$$(lr+\alpha)^2+(mr+\beta)^2+(nr+\gamma)^2=a^2$$

$$l^2r^2+2rl\alpha+\alpha^2+m^2r^2+2rm\beta+\beta^2+n^2r^2+2rn\gamma+\gamma^2=a^2$$

$$r^2(l^2+m^2+n^2)+2r(l\alpha+m\beta+n\gamma)+(\alpha^2+\beta^2+\gamma^2-a^2)=0$$

The line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ becomes a tangent line to the sphere if discriminant $=0$ $[b^2-4ac=0]$

$$\Rightarrow 4(l\alpha+m\beta+n\gamma)^2-4(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)=0$$

Eliminating (l,m,n) using ① we get,

$$\Rightarrow \left[\left(\frac{x-\alpha}{r}\right)\alpha+\left(\frac{y-\beta}{r}\right)\beta+\left(\frac{z-\gamma}{r}\right)\gamma\right]^2-\left[\left(\frac{x-\alpha}{r}\right)^2+\left(\frac{y-\beta}{r}\right)^2+\left(\frac{z-\gamma}{r}\right)^2\right](\alpha^2+\beta^2+\gamma^2-a^2)$$

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$$\Rightarrow [(x-\alpha)\alpha+(y-\beta)\beta+(z-\gamma)\gamma]^2-[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2](\alpha^2+\beta^2+\gamma^2-a^2)=0$$

$$\Rightarrow [(\alpha x+\beta y+\gamma z)-(\alpha^2+\beta^2+\gamma^2)]^2-[(x^2+y^2+z^2)+(\alpha^2+\beta^2+\gamma^2)-2(\alpha x+\beta y+\gamma z)](\alpha^2+\beta^2+\gamma^2-a^2)=0 \quad (2)$$

let $S=x^2+y^2+z^2-a^2$, $S_1=\alpha^2+\beta^2+\gamma^2-a^2$ and $T=\alpha x+\beta y+\gamma z-a^2$

(2) $\Rightarrow$

$$[(\alpha x+\beta y+\gamma z-a^2)-(\alpha^2+\beta^2+\gamma^2-a^2)]^2-[(x^2+y^2+z^2-a^2)+(\alpha^2+\beta^2+\gamma^2-a^2)-2(\alpha x+\beta y+\gamma z-a^2)](\alpha^2+\beta^2+\gamma^2-a^2)=0$$

$$\Rightarrow [T-S_1]^2-[S+S_1-2T](S_1)=0$$

$$\Rightarrow T^2+\cancel{S_1^2}-\cancel{2TS_1}-SS_1-\cancel{S_1^2}+\cancel{2TS_1}=0$$

$$\Rightarrow T^2-SS_1=0$$

$$\Rightarrow \boxed{T^2=SS_1}$$

PCP-13 : Find the enveloping cone of the sphere $x^2+y^2+z^2-2x+4z=1$, with vertex at $(1,1,1)$.

Sol. let $S=x^2+y^2+z^2-2x+4z-1=0$

vertex $=(1,1,1)=(\alpha,\beta,\gamma)$

$\cancel{S_1=\alpha^2+\beta^2+\gamma^2-a^2\Rightarrow 1^2+1^2+1^2}$ $S_1=1^2+1^2+1^2-2(1)+4(1)-1$

$\boxed{S_1=4}$

$T=x(1)+y(1)+z(1)-1(x+1)+2(z+1)-1$

$T=x+y+z-x-1+2z+2-1$

$T=y+3z$

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Enveloping cone $\Rightarrow T^2 = SS_1$

$\Rightarrow (y+3z)^2 = (x^2+y^2+z^2-2x+4z-1)(4)$

$\Rightarrow y^2+9z^2+6zy = 4x^2+4y^2+4z^2-8x+16z-4$

$\Rightarrow y^2+9z^2+6yz-4x^2-4y^2-4z^2+8x-16z+4=0$

$\Rightarrow -4x^2-3y^2+5z^2+6yz+8x-16z+4=0$

$\Rightarrow \boxed{4x^2+3y^2-5z^2-6yz-8x+16z-4=0}$

Theorem: Prove that the equation of a cone whose vertex is origin is homogeneous and converse also holds

Proof: Let the general equation of a cone be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \quad \text{---}(1)$$

(1) represents a cone whose vertex is origin

$\Leftrightarrow u=v=w=d=0$

Suppose that (1) represents a cone whose vertex is origin

let $P(x',y',z')$ be any point on the cone

$\Rightarrow$ eq^n of line OP (generator) is

$$\frac{x-0}{x'-0} = \frac{y-0}{y'-0} = \frac{z-0}{z'-0} \quad \text{i.e} \Rightarrow \frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = r$$

$\therefore$ Any point on the generator $=(rx', ry', rz')$

Substituting in the eq^n of cone we get,

$\Rightarrow ar^2x'^2+br^2y'^2+cr^2z'^2+2fr^2y'z'+2gr^2z'x'+2hr^2x'y'$

$+2urx'+2vry'+2wrz'+d=0 \;\forall$ values of r

$\Rightarrow r^2(ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y')+2r(ux'+vy'+wz')$

$+d=0$ — ②

$\forall$ values of r

$\Rightarrow ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'=0$

$ux'+vy'+wz'=0$ — ③

$d=0$ — ④

if u, v & w are not all zero

③ $\Rightarrow P(x', y', z')$ lies on $ux+vy+wz=0$

$\Rightarrow$ 'p' lies on a plane which is a contradiction

$u=v=w=0$ & ④ $\Rightarrow d=0$

Eqn of cone is homogenious

Converting that eqn. of a cone in homogenous given by

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ — ⊛

let $P(x', y', z')$ which satisfies ⊛

$\Rightarrow ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'=0.$

The point $=(rx', ry', rz')$

$\Rightarrow ar^2x'^2+br^2y'^2+cr^2z'^2+2fr^2y'z'+2gr^2z'x'+2hr^2x'y'=0$

The point (rx', ry', rz') which satisfies ⊛ $\forall$ values of r.

$\Rightarrow$ If P. lies on a surface whose eqn is given by ⊛

$\Rightarrow$ All the points on the line OP satisfies ⊛

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Note: Let the eq^n of the cone be from

$\Rightarrow$ eq^n of cone $= ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Eq^n of generator through vertex $=(0,0,0)$ is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r\text{(say)}$$

Any point on the generator $=(lr, mr, nr)$

substituting in the eq^n of cone

$$al^2r^2+bm^2r^2+cn^2r^2+2fr^2mn+2gr^2ln+2hr^2lm=0$$

$\forall$ value of r.

$\Rightarrow r^2(al^2+bm^2+cn^2+2fmn+2gln+2hlm)=0 \ \forall$ value of r.

$\Rightarrow al^2+bm^2+cn^2+2fmn+2gln+2hlm=0$

* The DR's or DC's of the generator satisfy the eq^n of cone.

• Show that the general eq^n of a cone which passes through the three axes is $fyz+gzx+hxy=0$.

Sol. G/T x, y & z axes are generators of the cone $\Rightarrow$ vertex of the cone is origin.

$\Rightarrow$ Eq^n of cone is homogenous

$\therefore \Rightarrow$ Eq^n of the cone is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ — ①

DC's of x-axis $=(1,0,0)$ satisfies ①

~~DC's of Y-axis = (0,1,0)~~ $\Rightarrow \boxed{a=0}$

DC's of Y-axis $=(0,1,0)$ satisfies ① $\Rightarrow \boxed{b=0}$

DC's of Z-axis $=(0,0,1)$ satisfies ① $\Rightarrow \boxed{c=0}$

① $\Rightarrow 2fyz + 2gzx + 2hxy = 0$

$\Rightarrow \boxed{fyz + gzx + hxy = 0.}$

• Find the eqn of the cone which passes through the co-ordinate axes as well as the lines. $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$

Sol. Eqn of the cone through co-ordinate axes as its generators is $fyz + gzx + hxy = 0$ — ①

DR's of generators = $(1, -2, 3)$ satisfy ①

$\Rightarrow -6f + 3g - 2h = 0$

Ily DR's of generators = $(3, -1, 1)$ satisfy ①

$\Rightarrow -f + 3g - 3h = 0$

$$\left.\begin{array}{l} -6f + 3g - 2h = 0 \\ -f + 3g - 3h = 0 \end{array}\right\} \Rightarrow \frac{f}{-9+6} = \frac{g}{2-18} = \frac{h}{-18+3} = k \text{ (say)}$$

$$\Rightarrow \frac{f}{-3} = \frac{g}{-16} = \frac{h}{-15} = k$$

$\Rightarrow f = -3k,\ g = -16k,\ h = -15k$

substituting in ① we get,

$\Rightarrow (-3k)yz + (-16k)zx + (-15k)xy = 0$

$\Rightarrow -k(3yz + 16zx + 15xy) = 0$

$\Rightarrow \boxed{3yz + 16zx + 15xy = 0}$

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• Find the eq^n of cone passes through co-ordinate axes as well as 3 mutually $\perp^r$ lines $\frac{x}{2} = y = -z$;

$x = \frac{y}{3} = \frac{z}{5}$, $\frac{x}{8} = \frac{y}{-11} = \frac{z}{5}$?

Sol. Eq^n of cone through co-ordinate axes as its generators is $fyz + gzx + hxy = 0$ — ①

DR's of generators $= (1, 3, 5)$ satisfy ①

$$\Rightarrow 15f + 5g + 3h = 0$$

DR's of generator $= (2, 1, -1)$ satisfy ①

$$\cancel{\Rightarrow -55f + 40g - 88h = 0} \Rightarrow -f - 2g + 2h = 0$$

Solving,

$$\left.\begin{array}{l} -f - 2g + 2h = 0 \\ 15f + 5g + 3h = 0 \end{array}\right\} \Rightarrow \frac{f}{-6-10} = \frac{g}{30+3} = \frac{h}{-5+30} = k$$

$$\Rightarrow \frac{f}{-16} = \frac{g}{33} = \frac{h}{25} = k$$

$$\Rightarrow f = -16k, g = 33k, h = 25k$$

$\therefore$ eq^n of cone is $\Rightarrow (-16k)yz + (33k)zx + (25k)xy = 0$

$$\Rightarrow 16yz - 33zx - 25xy = 0.$$

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• Find the eqn of the cone which passes through 3 co-ordinates axes and two lines through origin with DC's (l_1, m_1, n_1) & (l_2, m_2, n_2).

sol. Eqn of the cone through co-ordinate axes as its generators

$\rightarrow fyz + gzx + hxy = 0$ — ①

DC's of generators (l_1, m_1, n_1) satisfy ①

$\cancel{\Rightarrow m_1 n_1 f + m_1 n_1 \, l_1 m_1 g + l_1 n}$

$\Rightarrow m_1 n_1 f + l_1 n_1 g + l_1 m_1 h = 0$

Ily. $\Rightarrow m_2 n_2 f + l_2 n_2 g + l_2 m_2 h = 0$

$$\rightarrow \frac{f}{l_1 l_2 m_2 n_1 - l_1 l_2 m_1 n_2} = \frac{g}{m_1 m_2 l_2 n_1 - m_1 m_2 n_2 l_1} = \frac{h}{n_1 n_2 m_1 l_2 - n_1 n_2 m_2 l_1} = k$$

//

To Find the co-ordinates of a vertex of a cone (Non-homogeneous equation).

1) Let $f(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$

2) Homogenizing the above eq^n using new variable 't' we get

$$f(x,y,z,t) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2uxt+2vyt+2wzt+dt^2=0$$

3) Calculate partial derivatives

$$f_x, f_y, f_z, f_t$$

4) Put $t=1$ & equate to zero

Problems:

1) Show that the equation $4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$ represents a cone with vertex $(-1,-2,-3)$

Sol let $f(x,y,z) = 4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$

Homogenizing the above equation

$$f(x,y,z,t) = 4x^2-y^2+2z^2+2xy-3yz+12xt-11yt+6zt+4t^2=0$$

$\Rightarrow$ $f_x = 8x+2y+12t$

$f_y = -2y+2x-3z-11t$

$f_z = 4z-3y+6t$

$f_t = 12x-11y+6z+8t$

put $t=1$ in above eq^{n}s and equate to zero.

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$\Rightarrow 8x + 2y + 12 = 0$ —— ①

$\Rightarrow$ ~~4xy~~ $2x - 2y - 3z - 11 = 0$ —— ②

$\Rightarrow -3y + 4z + 6 = 0$ —— ③

$\Rightarrow 12x - 11y + 6z + 8 = 0$ —— ④

Solving ①, ②, ③

~~from~~ Eliminating 'x' from ① & ②

① $\Rightarrow \cancel{8x} + 2y + 12 = 0$

② × 4 $\Rightarrow \underset{(-)}{\cancel{8x}} \underset{(+)}{- 8y} \underset{(+)}{- 12z} \underset{(+)}{- 44} = 0$

$10y + 12z + 56 = 0$

$\Rightarrow 5y + 6z + 28 = 0$ —— ⑤

Solving ③ & ⑤

⑤ × 3 $\Rightarrow \cancel{15y} + 18z + 84 = 0$

③ × 5 $\Rightarrow -\cancel{15y} + 20z + 30 = 0$

$38z + 114 = 0$

$\boxed{z = -3}$

substitute 'z' in ⑤

$\Rightarrow 5y + 6(-3) + 28 = 0$

$5y - 18 + 28 = 0 \Rightarrow 5y + 10 = 0$

$\boxed{y = -2}$

substitute 'y' in ①

$\Rightarrow 8x + 2(-2) + 12 = 0$

$\Rightarrow 8x - 4 + 12 \Rightarrow 0 \quad 8x + 8 = 0$

Substitute x, y, z in ④ $\boxed{x = -1}$

$\Rightarrow 12(-1) - 11(-2) + 6(-3) + 8 \Rightarrow -12 + 22 - 18 + 8 \Rightarrow 0$

$\therefore$ Given eqⁿ is a cone with vertex $(-1, -2, -3)$.

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1) Prove that $2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0$

represents a cone with the vertex $(2,2,1)$.

Sol. Let $f(x,y,z)=0$

$\Rightarrow 2x^2+2y^2+7x^2-10yz-10zx+2x+2y+26z-17=0$

Homogenizing the above equation

$f(x,y,z,t) \Rightarrow 2x^2+2y^2+7x^2-10yz-10zx+2xt+2yt+26zt-17t^2=0$

$\Rightarrow f_x = 4x-10z+2t$

$f_y = 4y-10z+2t$

$f_z = 14z-10x+26t-10y$

$f_t = 2x+2y+26z-34t$

put $t=1$ in above eq's and equate to zero

$\Rightarrow 4x-10z+2=0$ — (1)

$\Rightarrow 4y-10z+2=0$ — (2)

$\Rightarrow -10x+14z+26=0$ — (3)

$\Rightarrow 2x+2y+26z-34=0$ — (4)

Solving (1),(2),(3)

(1)-(2) $\Rightarrow 4x-4y=0$

$\Rightarrow \boxed{x=y}$

Substitute $x=y$ in (3)

$\Rightarrow -10y-10y+14z+26=0$

$\Rightarrow -20y+14z+26=0$

$\Rightarrow -10y+7z+13=0$ — (5)

Eliminating 'y' from eqⁿs (2) & (5)

(2) $\Rightarrow 40y-100z+20=0$

(5) $\Rightarrow -40y+28z+52=0$

$-72z+72=0 \Rightarrow \boxed{z=1}$

$-72z=-72$

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put $z=1$ in ②

$\Rightarrow 4y-10+2=0$

$\Rightarrow 4y-8=0$

$\boxed{y=2}$

Since $x=y \Rightarrow \boxed{x=2}$

Substitute x,y,z in ④

$\Rightarrow 2(2)+2(2)+26(1)-34$

$\Rightarrow 4+4+26-34$

$\Rightarrow 34-34$

$\Rightarrow 0$

$\therefore$ Given eq^n is a cone with vertex $(2,2,1)$

3) Show that the eq^n $2y^2-8yz-4zx-8xy+6x-4y-2z+5=0$ represents a cone with vertex $\left(-\frac{1}{6},\frac{1}{3},\frac{5}{6}\right)$.

Sol. Let $f(x,y,z) \Rightarrow 2y^2-8yz-4zx-8xy+6x-4y-2z+5=0$

Homogenizing the above equation

$f(x,y,z,t) \Rightarrow 2y^2-8yz-4zx-8xy+6xt-4yt-2zt+5t^2=0$

$\Rightarrow f_x = -4z-8y+6t$

$f_y = 4y-8z-8x-\cancel{4y}\,4t$

$f_z = -8y-4x-2t$

$f_t = 6x-4y-2z+10t$

put $t=1$ in above eq^{n}s and equate to zero

$\Rightarrow \cancel{-4z-8y} \quad -8y-4z+6=0$ — ①

$\Rightarrow -8x+4y-8z-4=0$ — ②

$\Rightarrow -4x-8y-2=0$ — ③

$\Rightarrow 6x-4y-2z+10=0$ — ④

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Solving ①, ②, ③

Eliminating 'z' from ① & ②

$① \times 2 \Rightarrow -16y - \cancel{8z} + 12 = 0$

$② \Rightarrow \underset{(+)}{-8x} \underset{(-)}{+4y} \underset{(+)}{-\cancel{8z}} \underset{(+)}{-4} = 0$

$8x - 20y + 16 = 0$

$4x - 10y + 8 = 0$

$2x - 5y + 4 = 0$ — ⑤

Solving ⑤ & ③

$⑤ \times 2 \Rightarrow \cancel{4x} - 10y + 8 = 0$

$③ \Rightarrow -\cancel{4x} - 8y - 2 = 0$

$-18y + 6 = 0$

$-18y = -6$

$y = \frac{6}{18} = \frac{1}{3} \Rightarrow \boxed{y = \frac{1}{3}}$

Substitute 'y' in ⑤

$\Rightarrow 2x - \frac{5}{3} + 4 = 0$

$\Rightarrow 2x + \frac{7}{3} = 0$

$\Rightarrow 2x = -\frac{7}{3} \Rightarrow \boxed{x = \frac{-7}{6}}$

Substitute 'y' in ①

$\Rightarrow -\frac{8}{3} - 4z + 6 = 0$

$\Rightarrow -4z + 6 - \frac{8}{3} = 0$

$\Rightarrow -4z + \frac{10}{3} = 0 \Rightarrow -4z = -\frac{10}{3}$

$\Rightarrow \cancel{4}z = \frac{10}{12} = \frac{5}{6} \Rightarrow \boxed{z = \frac{5}{6}}$

Substitute (x, y, z) in ④

$\Rightarrow \cancel{6}\left(\frac{-7}{\cancel{6}}\right) - 4\left(\frac{1}{3}\right) - \cancel{2}\left(\frac{5}{\cancel{6}_3}\right) + 10 \Rightarrow -7 - \frac{4}{3} - \frac{5}{3} + 10 \Rightarrow -7 - \frac{\cancel{9}^3}{\cancel{3}} + 10 \Rightarrow \cancel{-10+10=0}$

$-10 + 10 = 0$

Given eqn is a cone with vertex $\left(-\frac{7}{6}, \frac{1}{3}, \frac{5}{6}\right)$.

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Ex:

Prove that the equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$

Sol: Let $f(x,y,z)=ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$

Homogenizing,

$$f(x,y,z,t)=ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt^2=0$$

$\Rightarrow f_x = 2ax+2ut$

$f_y = 2by+2vt$

$f_z = 2cz+2wt$

$f_t = 2ux+2vy+2wz+2dt$

put $t=1$ & equate to zero

$\Rightarrow ax+u=0 \Rightarrow x=-\frac{u}{a}$

$\Rightarrow by+v=0 \Rightarrow y=-\frac{v}{b}$

$\Rightarrow cz+w \Rightarrow z=-\frac{w}{c}$

$\Rightarrow ux+vy+wz+d=0$

Given eq^n represents a cone if $u\left(-\frac{u}{a}\right)+v\left(-\frac{v}{b}\right)+w\left(-\frac{w}{c}\right)+d=0$

$$\Rightarrow \boxed{\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d}$$

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Equation of a line of intersection of a plane with cone.

. Find the eq^ns to the lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$.

Sol. Let the eq^n of line of intersection of given plane with given be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Since, the line is common to both plane & cone, we get

$2l+m-n=0$ — ① $\Rightarrow n=2l+m$

$4l^2-m^2+3n^2=0$ — ②

Substituting in ② $\Rightarrow 4l^2-m^2+3(2l+m)^2=0$

$\Rightarrow 4l^2-m^2+3(4l^2+m^2+4lm)=0$

$\Rightarrow 4l^2-m^2+12l^2+3m^2+12lm=0$

$\Rightarrow 16l^2+12lm+2m^2=0$

$\Rightarrow 8l^2+6lm+m^2=0$

$\Rightarrow 8\left(\frac{l}{m}\right)^2+6\left(\frac{l}{m}\right)+1=0$

$$\frac{l}{m}=\frac{-6\pm\sqrt{36-4(8)(1)}}{2(8)}$$

$\frac{l}{m} \Rightarrow \frac{-6\pm 2}{16} = \frac{-6+2}{16} = \frac{-4}{16}$

$\Rightarrow \frac{-6-2}{16} = -\frac{8}{16}$

$\frac{l}{m} = \frac{-4}{16}$ (or) $\frac{-8}{16}$ $= \frac{-1}{4}$ (or) $\frac{-1}{2}$

① $\Rightarrow 2l+m-n=0$

$\Rightarrow \frac{2l}{m}+1-\frac{n}{m}=0$ — ③

Case 1: let $\frac{l}{m} = \frac{-1}{4}$

③ $\Rightarrow \cancel{2}\left(\frac{-1}{\cancel{4}_{2}}\right) + 1 - \frac{n}{m} = 0$

$\Rightarrow \boxed{\frac{n}{m} = \frac{1}{2}}$

$\frac{l}{m} = \frac{-1}{4}, \frac{n}{m} = \frac{1}{2}$

$\frac{l}{m} = \frac{-1}{4}, \frac{m}{n} = 2$

$\frac{l}{m} = \frac{-1}{4}; \frac{m}{n} = \frac{4}{2}$

$\Rightarrow \boxed{\frac{l}{-1} = \frac{m}{4} = \frac{n}{2}} = k$

Eqn of the lines is $\frac{x}{-k} = \frac{y}{4k} = \frac{z}{2k} \Rightarrow \boxed{\frac{x}{-1} = \frac{y}{4} = \frac{z}{2}}$

Case 2:

let $\frac{l}{m} = \frac{-1}{2}$

③ $\Rightarrow \frac{2l}{m} + 1 - \frac{n}{m} = 0$

substituting $\frac{l}{m}$ in ③

$\Rightarrow \cancel{2}\left(\frac{-1}{\cancel{2}}\right) + 1 - \frac{n}{m} = 0$

$\Rightarrow \frac{n}{m} = 0$

$\frac{l}{m} = -\frac{1}{2}, \frac{n}{m} = 0$

$\frac{l}{m} = \frac{-1}{2}, \frac{n}{m} = \frac{0}{2}$

$\frac{l}{m} = \frac{-1}{2}, \frac{m}{n} = \frac{2}{0}$

$\Rightarrow \boxed{\frac{l}{-1} = \frac{m}{2} = \frac{n}{0} = k}$

Eqn of line is $\boxed{\frac{x}{-1} = \frac{y}{2} = \frac{z}{0}}$

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• Find the equations of the line of intersection of the plane & cone

$3x+4y+z=0, 15x^2-32y^2-7z^2=0.$

sol: Let the eqⁿ of line be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

Since the line is common to plane and a cone we get

$3l+4m+n=0$ — (1) , $15l^2-32m^2-7n^2=0$ — (2)

(1) $\Rightarrow$ $n=-3l-4m.$

(2) $\Rightarrow 15l^2-32m^2-7(-3l-4m)^2=0$

$\Rightarrow 15l^2-32m^2-7(9l^2+16m^2+24lm)=0$

$\Rightarrow 15l^2-32m^2-63l^2-112m^2-168lm=0$

$\Rightarrow -48l^2-168lm-144m^2=0$

$\Rightarrow 48l^2+168lm+144m^2=0$

$\Rightarrow 4l^2+14lm+12m^2=0$

$\Rightarrow 2l^2+7lm+6m^2=0$

$\Rightarrow 2l^2+3lm+4lm+6m^2=0$

$\Rightarrow l(2l+3m)+2m(2l+3m)=0$

$\Rightarrow (2l+3m)(l+2m)=0$

$2l+3m=0$; $l+2m=0$

$2l=-3m$; $l=-2m$

$\frac{l}{m}=-\frac{3}{2}$; $\frac{l}{m}=\frac{-2}{1}$

case I: $\frac{l}{m}=\frac{-3}{2}$

(1) $\Rightarrow 3l+4m+n=0$

$\Rightarrow \frac{3l}{m}+4+\frac{n}{m}=0$

Case 1: let $\frac{l}{m} = \frac{-3}{2}$

(3) $\Rightarrow \frac{3l}{m} + 4 + \frac{n}{m} = 0$

$\Rightarrow 3\left(\frac{-3}{2}\right) + 4 + \frac{n}{m} = 0$

$\Rightarrow -\frac{9}{2} + 4 + \frac{n}{m} = 0$

$\Rightarrow \frac{n}{m} = \frac{9}{2} - 4$

$\Rightarrow \frac{n}{m} = \frac{+1}{2}$

$\frac{l}{m} = \frac{-3}{2}, \frac{n}{m} = \frac{+1}{2}$

$$\boxed{\frac{l}{-3} = \frac{m}{2} = \frac{n}{+1} = k}$$

Eq^n of the line is $\boxed{\frac{x}{-3} = \frac{y}{2} = \frac{z}{+1}}$

Case 2: let $\frac{l}{m} = \frac{-2}{1}$

(3) $\Rightarrow \frac{3l}{m} + 4 + \frac{n}{m} = 0$

$\Rightarrow 3\left(\frac{-2}{1}\right) + 4 + \frac{n}{m} = 0$

$\Rightarrow -6 + 4 + \frac{n}{m} = 0$

$\Rightarrow \frac{n}{m} = \frac{2}{1}$

$\frac{l}{m} = \frac{-2}{1}, \frac{n}{m} = \frac{2}{1}$

$\frac{l}{m} = \frac{-2}{1}, \frac{m}{n} = \frac{1}{2}$

$$\boxed{\frac{l}{-2} = \frac{m}{1} = \frac{n}{2} = k}$$

Eq^n of line is $\boxed{\frac{x}{-2} = \frac{y}{1} = \frac{z}{2}}$

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Find the equations of line of intersection of $x+7y-5z=0$, $3yz+14zx-30xy=0$.

Sol: let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be the line of intersection

$l+7m-5n=0$ — (1)

$3mn+14ln-30lm=0$ — (2)

(1) $\Rightarrow$ $\boxed{l=5n-7m}$

Substituting in (2)

$\Rightarrow$ $3mn+14(5n-7m)n-30(5n-7m)m=0$

$\Rightarrow$ $3mn+70n^2-98mn-150mn+210m^2=0$

$\Rightarrow$ $210m^2-245mn+70n^2=0$

$\Rightarrow$ $6m^2-7mn+2n^2=0$

$\Rightarrow$ $6\left(\frac{m}{n}\right)^2-7\left(\frac{m}{n}\right)+2=0$

$\Rightarrow$ $\frac{m}{n}=\frac{7\pm\sqrt{49-48}}{12}$

$\Rightarrow$ $\frac{m}{n}=\frac{7\pm1}{12}$ $\Rightarrow$ $\frac{m}{n}=\frac{2}{3},\frac{1}{2}$

$\frac{m}{n}=\frac{2}{3}$ (or) $\frac{m}{n}=\frac{1}{2}$

(1) $\Rightarrow$ $l+7m-5n=0$ $\Rightarrow$ $\frac{l}{n}+7\left(\frac{m}{n}\right)-5=0$ — (3)

Case 1: let $\frac{m}{n}=\frac{2}{3}$

(3) $\Rightarrow$ $\frac{l}{n}+7\left(\frac{2}{3}\right)-5=0$

$\Rightarrow$ $\frac{l}{n}+\frac{14-15}{3}=0$

$\Rightarrow$ $\frac{l}{n}-\frac{1}{3}=0$ $\Rightarrow$ $\frac{l}{n}=\frac{1}{3}$, $\frac{m}{n}=\frac{2}{3}$

$\frac{l}{1}=\frac{m}{2}=\frac{n}{3}$

Eq^n of line is $\boxed{\frac{x}{1}=\frac{y}{2}=\frac{z}{3}}$

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Case 2: let $\frac{m}{n} = \frac{1}{2}$

③ $\Rightarrow \frac{l}{n} + 7\left(\frac{1}{2}\right) - 5 = 0$

$\Rightarrow \frac{l}{n} + \frac{7-10}{2} = 0$

$\Rightarrow \frac{l}{n} - \frac{3}{2} = 0$

$\Rightarrow \frac{l}{n} = \frac{3}{2}, \frac{m}{n} = \frac{1}{2}$

$\frac{l}{3} = \frac{m}{1} = \frac{n}{2}$

Eqⁿ of the line is $\boxed{\frac{x}{3} = \frac{y}{1} = \frac{z}{2}}$

• Find the angle between the lines of intersection $x - 3y + z = 0$ & $x^2 - 5y^2 + z^2 = 0$.

Sol: let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be the line of intersection

$\Rightarrow l - 3m + n = 0$ — ①

$\Rightarrow l^2 - 5m^2 + n^2 = 0$ — ②

① $\Rightarrow l = 3m - n$

② $\Rightarrow (3m-n)^2 - 5m^2 + n^2 = 0$

$\Rightarrow 9m^2 + n^2 - 6mn - 5m^2 + n^2 = 0$

$\Rightarrow 4m^2 - 6mn + 2n^2 = 0 \Rightarrow 2m^2 - 3mn + n^2 = 0$

$\Rightarrow 2m^2 - 2mn - mn + n^2 = 0$

$\Rightarrow 2m(m-n) - n(m-n) = 0$

$\Rightarrow (m-n)(2m-n) = 0$

$\Rightarrow m = n$ (or) $2m - n = 0$

$\frac{m}{n} = 1$ $\qquad 2m = n$

$\frac{m}{n} = \frac{1}{2}$

① $\Rightarrow l - 3m + n = 0 \Rightarrow \frac{l}{n} - 3\frac{m}{n} + 1 = 0$

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case 1: $\frac{m}{n} = 1$

③ $\Rightarrow \frac{l}{n} - 3 + 1 = 0$

$$\frac{l}{n} = \frac{2}{1}, \quad \frac{m}{n} = \frac{1}{1}$$

$$\frac{l}{2} = \frac{m}{1} = \frac{n}{1}$$

DR's of the line are $\boxed{\frac{x}{2} = \frac{y}{1} = \frac{z}{1}}$

case 2: $\frac{m}{n} = \frac{1}{2}$

③ $\Rightarrow \frac{l}{n} - 3\left(\frac{1}{2}\right) + 1 = 0$

$$\Rightarrow \frac{l}{n} + 1 - \frac{3}{2} = 0$$

$$\Rightarrow \frac{l}{n} - \frac{1}{2} = 0$$

$$\Rightarrow \frac{l}{n} = \frac{1}{2}, \quad \frac{m}{n} = \frac{1}{2} \Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{2}$$

DR's of the line are $\boxed{\frac{x}{1} = \frac{y}{1} = \frac{z}{2}}$

DR's of 1st line = (2,1,1), DR's of 2nd line = (1,1,2)

if θ is the angle b/w the line then,

$$\Rightarrow \cos\theta = \frac{2(1) + 1(1) + 1(2)}{\sqrt{4+1+1} \cdot \sqrt{1+1+4}}$$

$$\cos\theta = \frac{5}{6}$$

$$\theta = \cos^{-1}\left(\frac{5}{6}\right)$$

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Intersection of a line with a Cone

Let the eq^n of the line be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ and eq^n of the cone be $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Any point on the line $= (lr+\alpha, mr+\beta, nr+\gamma)$

Substituting in eq^n of cone, we get

$$\Rightarrow a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2+2f(mr+\beta)(nr+\gamma)+2g(nr+\gamma)(lr+\alpha)$$
$$+2h(lr+\alpha)(mr+\beta)=0$$

$$\Rightarrow a(l^2r^2+\alpha^2+2rl\alpha)+b(m^2r^2+\beta^2+2rm\beta)+c(n^2r^2+\gamma^2+2rn\gamma)$$
$$+2f(mnr^2+rm\gamma+rn\beta+\beta\gamma)+2g(lnr^2+rn\alpha+rl\gamma+\alpha\gamma)$$
$$+2h(lmr^2+rl\beta+rm\alpha+\alpha\beta)=0$$

$$\Rightarrow r^2(al^2+bm^2+cn^2+2fmn+2gln+2hlm)+2r[l(a\alpha+h\beta+g\gamma)$$
$$+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)]$$
$$+(a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\alpha\gamma+2h\alpha\beta)=0$$

[$\because$ which is a quadratic eq^n in 'r' which has two roots]

$\therefore$ We get two intersection lines with a cone. (or) There are two points of intersection.

Tangent plane to a Cone.

A Tangent plane to a cone is defined as the locus of all tangent lines drawn to a cone at a point (α, β, γ) on the cone.

To find eqn of tangent plane to a cone

Consider the homogenous eqn of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \text{--- (1)}$$

let (α, β, γ) on the cone.

$$\Rightarrow a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\alpha\gamma+2h\alpha\beta=0 \quad \text{--- (1)}$$

Let the eqn of line through (α, β, γ) be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)}$$

The point $\Rightarrow (lr+\alpha, mr+\beta, nr+\gamma)$

Substituting in the eqn of cone we get,

$$r^2(al^2+bm^2+cn^2+2fmn+2gln+2hlm)+2r[l(a\alpha+h\beta+g\gamma)$$
$$+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)]+(a\alpha^2+b\beta^2+c\gamma^2+2f\gamma\beta$$
$$+2g\alpha\gamma+2h\alpha\beta)=0 \quad \text{--- (2)}$$

The line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ becomes a tangent line if discriminant of eq (2) $=0$ $[b^2-4ac=0]$

$$\Rightarrow 4[l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)]^2-4(al^2+bm^2+cn^2+2fmn$$
$$+2gln+2hlm)(a\alpha^2+b\beta^2+c\gamma^2+2f\gamma\beta+2g\alpha\gamma+$$
$$2h\alpha\beta)=0$$

$$\Rightarrow 4[l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)]^2=0$$

$$\Rightarrow l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)=0 \quad \text{--- (3)}$$

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To get the eq^n of tangent plane we need to eliminate (l, m, n)

$$① \Rightarrow \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$$

③ $\Rightarrow$

$$\Rightarrow \left(\frac{x-\alpha}{r}\right)(a\alpha+h\beta+g\gamma)+\left(\frac{y-\beta}{r}\right)(h\alpha+b\beta+f\gamma)+\left(\frac{z-\gamma}{r}\right)(g\alpha+f\beta+c\gamma)=0$$

$$\Rightarrow (x-\alpha)(a\alpha+h\beta+g\gamma)+(y-\beta)(h\alpha+b\beta+f\gamma)+(z-\gamma)(g\alpha+f\beta+c\gamma)=0$$

$$\Rightarrow x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=\alpha(a\alpha+h\beta+g\gamma)$$
$$+\beta(h\alpha+b\beta+f\gamma)+\gamma(g\alpha+f\beta+c\gamma).$$

$$\Rightarrow x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=a\alpha^2+h\alpha\beta+g\alpha\gamma$$
$$+h\alpha\beta+b\beta^2+f\beta\gamma+g\alpha\gamma+f\beta\gamma+c\gamma^2.$$

$$\Rightarrow x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=a\alpha^2+b\beta^2+c\gamma^2$$
$$+2f\beta\gamma+2g\alpha\gamma+2h\alpha\beta.$$

[from ①]

- Eq^n of the tangent plane is

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0.$$

[It is in the form of $\boxed{Ax+By+Cz=0}$ passes through origin].

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Condition of Tangency

To find the condition for the plane $lx+my+nz=0$ to be a tangent plane to the give cone $\rightarrow ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

let (α,β,γ) be the point of contact.

Eq^n of a tangent plane to a cone at (α,β,γ) is

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0 \quad \text{—} (1)$$

$$lx+my+nz=0 \quad \text{—} (2)$$

Comparing the co-eff's of x,y,z in (1) & (2)

$$\Rightarrow \frac{a\alpha+h\beta+g\gamma}{l}=\frac{h\alpha+b\beta+f\gamma}{m}=\frac{g\alpha+f\beta+c\gamma}{n}=k \text{ (say)}$$

$\Rightarrow a\alpha+h\beta+g\gamma-lk=0$

$\Rightarrow h\alpha+b\beta+f\gamma-mk=0$

$\Rightarrow g\alpha+f\beta+c\gamma-nk=0$

$\Rightarrow l\alpha+m\beta+n\gamma+0k=0$

$$\begin{vmatrix} a & h & g & -l \\ h & b & f & -m \\ g & f & c & -n \\ l & m & n & 0 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}=0 \rightarrow \text{condition for tangency.}$$

$- \quad + \quad - \quad +$

$$\Rightarrow -l\begin{vmatrix} h & g & l \\ b & f & m \\ f & c & n \end{vmatrix}+m\begin{vmatrix} a & g & l \\ h & f & m \\ g & c & n \end{vmatrix}-n\begin{vmatrix} a & h & l \\ h & b & m \\ g & f & n \end{vmatrix}+0=0$$

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$$\Rightarrow l \begin{vmatrix} h & g & l \\ b & f & m \\ f & c & n \end{vmatrix} - m \begin{vmatrix} a & g & l \\ h & f & m \\ g & c & n \end{vmatrix} + n \begin{vmatrix} a & h & l \\ h & b & m \\ g & f & n \end{vmatrix} = 0$$

$$\Rightarrow l[l(bc-f^2)-m(ch-fg)+n(hf-bg)]-m[l(ch-fg)-m(ac-g^2)+n(af-$$

$$+n[l(hf-bg)-m(af-hg)+n(ab-h^2)]$$

$$\Rightarrow (bc-f^2)l^2+(ac-g^2)m^2+(ab-h^2)n^2-2(af-gh)mn+2(hf-bg)ln$$
$$-2(ch-fg)lm$$

$$\Rightarrow Al^2+Bm^2+Cn^2+2Fmn+2Gln+2Hlm=0.$$

<u>Reciprocal</u> <u>Cone</u> : The locus of all the lines which are drawn through the vertex of the cone.

<u>To find</u> <u>Eqn</u> <u>of</u> <u>Reciprocal</u> <u>Cone</u>

Given cone is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$, vertex $=(0,0,0)$

let the eqn of the tangent plane be $lx+my+nz=0$

Draw line i.e. $\perp^r$ to the tangent plane passing through vertex of given cone.

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} = r \text{ (say)}$$

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r \quad \text{— (1)}$$

consider,

$$Al^2+Bm^2+Cn^2+2Fmn+2Gln+2Hlm=0.$$

Eliminating (l,m,n) using (1)

$$\Rightarrow \frac{Ax^2}{r^2}+\frac{By^2}{r^2}+\frac{Cz^2}{r^2}+2F\frac{yz}{r^2}+2G\frac{zx}{r^2}+2H\frac{xy}{r^2}=0$$

$$\Rightarrow Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$$

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1. Prove that the perpendiculars drawn from the origin to the tangent planes of the cone $ax^2+by^2+cz^2=0$ lies on the cone

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0.$$

Sol Eq^n of given cone is $ax^2+by^2+cz^2=0$ — (1)

Standard form $\rightarrow$ $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

comparing above eq^n with (1)

$a=a, b=b, c=c, f=0, g=0, h=0$

Reciprocal cone is given by

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0.$$

$A=bc-f^2$ $\quad F=gh-af$
$B=ac-g^2$ $\quad G=hf-bg$
$C=ab-h^2$ $\quad H=fg-ch$

$A=bc, B=ac, C=ab, F=0, G=0, H=0$

$\therefore$ Eq^n of the Reciprocal Cone is

$$bcx^2+acy^2+abz^2=0$$

divide by abc on both sides

$$\Rightarrow \frac{\cancel{bc}x^2}{a\cancel{bc}}+\frac{\cancel{ac}y^2}{\cancel{a}b\cancel{c}}+\frac{\cancel{ab}z^2}{\cancel{ab}c}=0$$

$$\Rightarrow \frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0.$$

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2. Prove that the tangents planes to the cone $x^2-y^2+2z^2-3yz+4zx-5xy=0$ are perpendicular to the generators of the cone $17x^2+8y^2+29z^2+28yz-46zx-16xy=0$.

Sol. Eqn of the given cone is

$$x^2-y^2+2z^2-3yz+4zx-5xy=0.$$

Standard form. $\rightarrow ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

$$a=1, b=-1, c=2, f=-\frac{3}{2}, g=2, h=-\frac{5}{2}$$

$$A=bc-f^2=(-1)(2)-\left(-\frac{3}{2}\right)^2=-2-\frac{9}{4}=-\frac{8+9}{4}=\frac{-17}{4}$$

$$B=ac-g^2=(1)(2)-(2)^2=2-4=-2$$

$$C=ab-h^2=(1)(-1)-\left(-\frac{5}{2}\right)^2=-1-\frac{25}{4}=\frac{-29}{4}$$

$$F=gh-af=(2)\left(-\frac{5}{2}\right)-(1)\left(-\frac{3}{2}\right)=-5+\frac{3}{2}=\frac{-7}{2}$$

$$G=hf-bg=\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)-(-1)(2)=\frac{15}{4}+2=\frac{23}{4}$$

$$H=fg-ch=\left(-\frac{3}{2}\right)(2)-(2)\left(-\frac{5}{2}\right)=-3+5=2$$

Eqn of Reciprocal cone

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$$

$$-\frac{17}{4}x^2-2y^2-\frac{29}{4}z^2+2\left(-\frac{7}{2}\right)yz+2\left(\frac{23}{4}\right)zx+2(2)xy=0$$

$$\Rightarrow -17x^2-8y^2-29z^2-28yz+46zx+16xy=0$$

$$\Rightarrow 17x^2+8y^2+29z^2+28yz-46zx-16xy=0.$$

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3. Prove that the cone $ayz+bzx+cxy=0$ and $(ax)^{1/2}+(by)^{1/2}+(cz)^{1/2}=0$ are reciprocal cones.

Sol: Consider $ayz+bzx+cxy=0$.

comparing with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

we get $a=$ co-eff of $x^2=0$ $\quad f=a/2$

$b=$ co-eff of $y^2=0$ $\quad g=b/2$

$c=$ co-eff of $z^2=0$ $\quad h=c/2$

$A=bc-f^2=(0)(0)-\left(\frac{a}{2}\right)^2=-\frac{a^2}{4}$

$B=ac-g^2=(0)(0)-\left(\frac{b}{2}\right)^2=-\frac{b^2}{4}$

$C=ab-h^2=(0)(0)-\left(\frac{c}{2}\right)^2=-\frac{c^2}{4}$

$F=gh-af=\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)-(0)\left(\frac{a}{2}\right)=\frac{bc}{4}$

$G=hf-bg=\left(\frac{c}{2}\right)\left(\frac{a}{2}\right)-(0)\left(\frac{b}{2}\right)=\frac{ac}{4}$

$H=fg-ch=\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)-(0)\left(\frac{c}{2}\right)=\frac{ab}{4}$

Eqn of reciprocal cone

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

$-\frac{a^2}{4}x^2-\frac{b^2}{4}y^2-\frac{c^2}{4}z^2+2\left(\frac{bc}{4}\right)yz+2\left(\frac{ac}{4}\right)zx+2\left(\frac{ab}{4}\right)xy=0$

$\Rightarrow -a^2x^2-b^2y^2-c^2z^2+2bcyz+2aczx+2abxy=0.$

$\Rightarrow a^2x^2+b^2y^2+c^2z^2-2bcyz-2aczx-2abxy=0$

$\Rightarrow$ ~~ax+by+cz~~ $(ax+by-cz)^2=4abxy$

$\Rightarrow (ax+by-cz)=\pm 2\sqrt{abxy}$

$\Rightarrow ax+by\pm 2\sqrt{abxy}=cz$

$\Rightarrow [\sqrt{ax}\pm\sqrt{by}]^2=[\sqrt{cz}]^2$

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$\Rightarrow \sqrt{ax} \pm \sqrt{by} = \pm\sqrt{cz}$

$\Rightarrow \sqrt{ax} \pm \sqrt{by} \pm \sqrt{cz} = 0$

$\Rightarrow \cancel{\sqrt{ax}\pm}\ \sqrt{ax} + \sqrt{by} + \sqrt{cz} = 0$

$\Rightarrow (ax)^{1/2} + (by)^{1/2} + (cz)^{1/2} = 0$

4. Prove that $fyz + gzx + hxy = 0$. and $\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ are reciprocal cones.

Sol. Consider $fyz + gzx + hxy = 0$

standard eq^n $\longrightarrow$ $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

comparing

$a = 0$ $\quad$ $f = $ co-eff of $yz = \frac{f}{2}$

$b = 0$ $\quad$ $g = $ co-eff of $zx = \frac{g}{2}$

$c = 0$ $\quad$ $h = $ co-eff of $xy = \frac{h}{2}$

$A = bc - f^2 = (0)(0) - \left(\frac{f}{2}\right)^2 = -\frac{f^2}{4}$

$B = ac - g^2 = (0)(0) - \left(\frac{g}{2}\right)^2 = -\frac{g^2}{4}$

$C = ab - h^2 = (0)(0) - \left(\frac{h}{2}\right)^2 = -\frac{h^2}{4}$

$F = gh - af = \left(\frac{g}{2}\right)\left(\frac{h}{2}\right) - (0)\frac{f}{2} = \frac{gh}{4}$

$G = hf - bg = \left(\frac{h}{2}\right)\left(\frac{f}{2}\right) - (0)\cdot\frac{g}{2} = \frac{hf}{4}$

$H = fg - ch = \left(\frac{f}{2}\right)\left(\frac{g}{2}\right) - (0)\left(\frac{h}{2}\right) = \frac{fg}{4}$

Eq^n of reciprocal cone

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$$

$$-\frac{f^2}{4}x^2 - \frac{g^2}{4}y^2 - \frac{h^2}{4}z^2 + 2\left(\frac{gh}{4}\right)yz + 2\left(\frac{hf}{4}\right)zx + 2\left(\frac{fg}{4}\right)xy = 0$$

$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0.$

$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfx - 2fgxy = 0.$

$\Rightarrow (fx + gy - hz)^2 = 4fgxy$

$\Rightarrow (fx + gy - hz) = \pm 2\sqrt{fgxy}$

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$\Rightarrow fx+gy \pm 2\sqrt{fxgy} = hz$

$\Rightarrow [\sqrt{fx} \pm \sqrt{gy}]^2 = [\sqrt{hz}]^2$

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$\Rightarrow$

$\Rightarrow \sqrt{fx} \pm \sqrt{gy} = \pm\sqrt{hz}$

$\Rightarrow \sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

$\Rightarrow \sqrt{fx} + \sqrt{gy} \pm \sqrt{hz} = 0$ //.

Note:

1) The cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ has three mutually perpendicular generators iff $a+b+c=0$ i.e co-eff x^2 + co-eff y^2 + co-eff $z^2 = 0$.

2) Given cone is

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$. —— ①

Recipocal cone $\Rightarrow Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

If ① has 3 mutually perpendicular tangent planes then the reciprocal cone has 3 mutually perpendicular generators.

$\Rightarrow A+B+C=0$.

$\Rightarrow bc-f^2+ac-g^2+ab-h^2=0$

$\Rightarrow \boxed{ab+bc+ca = f^2+g^2+h^2}$

5. Prove that the $lx+my+nz=0$ cuts the cone $(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$ (or) perpendicular lines of $(b-c)l^2+(c-a)m^2+(a-b)n^2+2fmn+2gln+2hlm=0$.

Sol: coeff of x^2 + co-eff y^2 + co-eff of z^2

$\Rightarrow (b-c)+(c-a)+(a-b)$.

$\Rightarrow 0$.

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Eq^n of the generators through vector $(0,0,0)$ is perpendicular to $\frac{x}{l} = \frac{y}{m} = \frac{z}{m}$

DR's of generators satisfy eq^n of the cone

$\Rightarrow (b-c)l^2 + (c-a)m^2 + (a-b)n^2 + 2f\,mn + 2gln + 2hlm = 0$

Right Circular Cone - ~~to the locus~~

To find eq^n of a right circular cone whose vertex is at the point (α, β, γ) and whose axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and semi vertical angle θ.

Proof: let $A(\alpha, \beta, \gamma)$ be the vertex & AB be the axis & let $P(x, y, z)$ be any point on the cone.

$\rightarrow$ AP is the generator of the cone

eq^n of the line AB (α, β, γ) is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

DR's of axis $\Rightarrow (l, m, n) = (a_1, b_1, c_1)$

DR's of AP $\rightarrow (x-\alpha, y-\beta, z-\gamma) = (a_2, b_2, c_2)$.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\cos\theta = \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\cdot\sqrt{(x-\alpha)^2 + (y-\beta)^2 + (n-\gamma)^2}}$$

Eq^n of RC cone

$$[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2 = \cos^2\theta\,(l^2+m^2+n^2)\left[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\right]$$

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Properties.

1) $(lx+my+nz)^2 = \cos^2\theta\,(l^2+m^2+n^2)(x^2+y^2+z^2)$ —— (2)

If $(\alpha,\beta,\gamma)=(0,0,0) \rightarrow$

2) Vertex $=(0,0,0)$ & Axis of Right Circular ~~Reciprocal~~ cone is Z axis.

DR's of Z-axis - $(0,0,1)$

$$\Rightarrow (0x+0y+1z)^2 = \cos^2\theta\,(0+0+1)(x^2+y^2+z^2)$$

$$\Rightarrow z^2 = \cos^2\theta\,(x^2+y^2+z^2)$$

$$\Rightarrow z^2\sec^2\theta = x^2+y^2+z^2$$

$$\Rightarrow z^2(\sec^2\theta-1) = x^2+y^2$$

$$\Rightarrow z^2\tan^2\theta = x^2+y^2$$

$$\Rightarrow \boxed{x^2+y^2-z^2\tan^2=0}$$

3) Three mutually $\perp^r$ generators then;

co-eff of x^2 + co-eff of y^2 + co-eff of $z^2=0$

$$1+1-\tan^2\theta=0$$

$$\tan^2\theta=2 \Rightarrow \tan\theta=\sqrt{2}$$

$$\therefore\ \theta=\tan^{-1}\sqrt{2}$$

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1) Find the equation of the right circular cone which passes through point $(1,1,2)$ & has its vertex at origin and axis is the line

$$\frac{x}{2} = \frac{-y}{4} = \frac{z}{3}.$$

Sol. Eqn of right circular cone

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = \cos^2\theta\,(l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]$$

$$\Rightarrow (2x-4y+3z)^2 = \cos^2\theta\,(4+16+9)(x^2+y^2+z^2)$$

$$\Rightarrow (2x-4y+3z)^2 = 29\cos^2\theta\,(x^2+y^2+z^2) \quad \text{--- (1)}$$

(1) passes through $(1,1,2)$

$$\Rightarrow (2-4+6)^2 = 29\cos^2\theta\,(1+1+4)$$

$$\Rightarrow 16 = 29\cos^2\theta \cdot 6.$$

$$\Rightarrow \cos^2\theta = \frac{16}{29\times 6}.$$

$$\Rightarrow (2x-4y+3z)^2 = \cancel{29}\cdot\frac{\cancel{16}^{8}}{\cancel{29}\times\cancel{6}_{3}}(x^2+y^2+z^2)$$

$$\Rightarrow 3(2x-4y+3z)^2 = 8x^2+8y^2+8z^2$$

$$\Rightarrow 3(4x^2+16y^2+9z^2-16xy-24yz+12xz) = 8x^2+8y^2+8z^2$$

$$\Rightarrow 12x^2+48y^2+27z^2-48xy-72yz+36xz = 8x^2+8y^2+8z^2$$

$$\Rightarrow 4x^2+40y^2+19z^2-48xy-72yz+36xz.$$

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2) Prove that $x^2-y^2+z^2-4x+2y+6z+12=0$ represents a right circular cone whose vertex is $(2,1,-3)$, axis is parallel to OY(Y-axis) and whose semi vertical angle is $45°$

Sol. $(\alpha,\beta,\gamma)=(2,1,-3)$

$(l,m,n)=(0,1,0)$ - DR's of Y-axis

Eqn of right circular cone

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2=\cos^2\theta(l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]$$

$$\Rightarrow [0(x-2)+1(y-1)+0(z+3)]^2 = \cos^2\theta(0+1+0)[(x-2)^2+(y-1)^2+(z+3)^2]$$

$$\Rightarrow (y-1)^2=\frac{1}{2}[(x-2)^2+(y-1)^2+(z+3)^2]$$

$$\Rightarrow 2(y-1)^2=\{(x-2)^2+(y-1)^2+(z+3)^2$$

$$\Rightarrow (x-2)^2-(y-1)^2+(z+3)^2=0$$

$$\Rightarrow x^2+4-4x-y^2-1+2y+z^2+9+6z=0$$

$$\Rightarrow x^2-y^2+z^2-4x+2y+6z+12=0$$

3) Find the equation of Right Circular cone whose vertex is $(3,2,1)$ & axis is the line $\frac{x-3}{4}=\frac{y-2}{1}=\frac{z-1}{3}$ & semivertical angle is $30°$.

Sol. $(\alpha,\beta,\gamma)=(3,2,1)$

$(l,m,n)=(4,1,3)$

Eqn of Right Circular Cone is

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2=\cos^2\theta(l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]$$

$$\Rightarrow [4(x-3)+1(y-2)+3(z-1)]^2=\frac{3}{4}(16+1+9)[(x-3)^2+(y-2)^2+(z-1)^2]$$

$$\Rightarrow 2[4x+y+3z-17]^2=3(13)[x^2+9-6x+y^2+4-4y+z^2+1-2z]$$

$\Rightarrow 2\left[(4x+y)^2+(3z-17)^2+2(4x+y)(3z-17)\right]$

$$= 39x^2-234x+351+39y^2+156-156y$$
$$+39z^2+39-78z$$

$\Rightarrow 2\left[16x^2+y^2+8xy+9z^2+289-102z+24zx-136x+6yz-34yz\right]$

$$= 39x^2+39y^2+39z^2-234x-156y-78z+546$$

$\Rightarrow 32x^2+2y^2+16xy+18z^2+578-204z+48zx-272x+12yz-68y$

$$= 39x^2+39y^2+39z^2-234x-156y+546$$

$\Rightarrow 7x^2+37y^2+21z^2-16xy-12yz-48zx+38x-88y+126z-32=0$

4) Find the eq^n of right circular cone which contains co-ordinate axes as generator

Sol. Given X, Y, Z are co-ordinate axes at origin

Vertex $= (0,0,0)$

Suppose that θ is Semi Vertical Angle

Angle b/w the axes x-axis $= \alpha$

DC's of axis $= (l,m,n)$

DC's of x-axis $= (1,0,0)$

$\cos\alpha = l\cdot 1 + m.0 + n.0$

$\Rightarrow \boxed{l = \cos\alpha}$

Angle b/w axis & Y-axis $= \alpha$

DC's of Y-axis $= (0,1,0)$

$\cos\alpha = l.0 + m.1 + n.0$

$= m$

$\boxed{m = \cos\alpha}$ $\quad$ Ily $\quad$ $\boxly{n = \cos\alpha}$

$l^2+m^2+n^2=1$

$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\cos\alpha = \frac{1}{\sqrt{3}}}$

$(l,m,n) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Eqn of reciprocal cone

$\Rightarrow [l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = \cos^2\theta\,(l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2 + (z-\gamma)]^2$

$\Rightarrow \left[\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z\right]^2 = \frac{1}{3}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)(x^2+y^2+y^2)$

$\Rightarrow \frac{(x+y+z)^2}{\cancel{3}} = \frac{1}{\cancel{3}}(x^2+y^2+y^2)$

$\Rightarrow \cancel{x^2}+\cancel{y^2}+\cancel{z^2}+2xy+2yz+2zx = \cancel{x^2}+\cancel{y^2}+\cancel{z^2}$

$\Rightarrow \boxed{xy+yz+zx=0}$

5) If 'α' is the semi vertical ~~cone~~ angle of Right circular cone which passes through the line OY, OZ, $x=y=z$, ST $\cos\alpha = [9-4\sqrt{3}]^{-1/2}$

sol. Given $\Rightarrow$ Y-axis, Z-axis $\Rightarrow \frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1}$

vertex $=(0,0,0)$

Let $(l,m,n)=$ DR's of axes of Right Circular cone.

$\alpha \to$ Semi vertical angle.

DC's of Y-axis $=(0,1,0)$

DC's of Axis $=(l,m,n)$

$\cos\alpha = l.0+m.1+n.0 = \boxed{m=\cos\alpha}$ — ①

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DC's of z-axis $= (0,0,1) = (l_1, m_1, n_1)$

$$\cos\alpha = l.0 + m.0 + n.1$$

$$\Rightarrow \boxed{n = \cos\alpha} \text{ --- } (2)$$

3^{rd} generator $\rightarrow$ $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$

DR's of x-axis $= (1,1,1)$

$$\Rightarrow DC's = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$DC's = (l, m, n)$$

$$\cos\alpha = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}$$

$$\cos\alpha = \frac{l+m+n}{\sqrt{3}}$$

$$\sqrt{3}\cos\alpha = l + \cos\alpha + \cos\alpha$$

$$l + 2\cos\alpha = \sqrt{3}\cos\alpha.$$

$$l = \sqrt{3}\cos\alpha - 2\cos\alpha$$

$$\boxed{l = (\sqrt{3}-2)\cos\alpha}, m = \cos\alpha, n = \cos\alpha$$

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow (\sqrt{3}-2)^2\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow \cos^2\alpha\,[3 + 4 - 4\sqrt{3} + 2] = 1$$

$$\Rightarrow \cos^2\alpha\,[9 - 4\sqrt{3}] = 1$$

$$\Rightarrow \cos^2\alpha = \frac{1}{[9-4\sqrt{3}]}$$

$$\Rightarrow \cos\alpha = [9-4\sqrt{3}]^{-1/2}$$

$$\therefore \boxed{\cos\alpha = [9-4\sqrt{3}]^{-1/2}}$$

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Cylinder - A cylinder is a surface generated by a straight line which is always parallel to a fixed line and may intersect a given curve. The given curve is called as the guiding curve. The fixed line is known as the axis of the cylinder.

Note :- The line drawn parallel to the fixed line (Axis).

Theorem:

To Find the eq^n of the cylinder whose generators intersect the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ which are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

Proof: let (α,β,γ) be any point on cylinder

Eq^n of generator through (α,β,γ) and parallel to given line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

put $z=0 \Rightarrow \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{-\gamma}{n}$

point of intersection $=\left(\alpha-\frac{l\gamma}{n},\ \beta-\frac{m\gamma}{n},\ 0\right)$

substitute PI in $ax^2+2hxy+by^2+2gx+2fy+c=0$.

$$\Rightarrow a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0$$

To find eq^n of cylinder we eliminate $(\alpha,\beta,\gamma) \downarrow (x,y,z)$

Eq^n of cylinder

$$a\left(x-\frac{lz}{n}\right)^2+2h\left(x-\frac{lz}{n}\right)\left(y-\frac{mz}{n}\right)+b\left(y-\frac{mz}{n}\right)^2+2g\left(x-\frac{lz}{n}\right)+2f\left(y-\frac{mz}{n}\right)+c=0$$

$$\Rightarrow a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2+2g(nx-lz)+2f(ny-mz)+cn^2=0$$

$\Rightarrow ax^2+2hxy+by^2+2gx+2fy+c=0$ [if generator is parallel to z-axis $\to (l,m,n)=(0,0,1)$].

Note: If the generators of the cylinder are parallel to z-axis then we eliminate z from the given eqⁿ of the guiding curve.

1) Find the eqⁿ of the cylinder whose generators are parallel to $x=-\frac{y}{2}=\frac{z}{3}$ and whose guiding curve is $x^2+2y^2=1,\ z=0$.

Sol: let (α,β,γ) be a point on cylinder.

Eqⁿ of generator through (α,β,γ) is parallel to $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$

$$\Rightarrow \frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{z-\gamma}{3}$$

put $z=0 \Rightarrow \frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{0-\gamma}{3}$

point of intersection $=\left(\alpha-\frac{\gamma}{3},\ \beta+\frac{2\gamma}{3},\ 0\right)$

Substituting in $x^2+2y^2=1$

$$\Rightarrow \left(\alpha-\frac{\gamma}{3}\right)^2+2\left(\beta+\frac{2\gamma}{3}\right)^2=1$$

Eliminating (α,β,γ)

Eqⁿ of cylinder $\Rightarrow \left(x-\frac{z}{3}\right)^2+2\left(y+\frac{2z}{3}\right)^2=1$

$$\Rightarrow (3x-z)^2+2(3y+2z)^2=9$$

$$\Rightarrow (3x-z)^2+2(3y+2z)^2-9=0$$

$$\Rightarrow 9x^2+z^2-6zx+18y^2+8z^2+24yz-9=0$$

$$\Rightarrow 9x^2+18y^2+8z^2-6zx+24yz-9=0.$$

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2) Find the equation of the cylinder whose generators intersect the curve $ax^2+by^2 = 2z$, $lx+my+nz=p$ & parallel to z-axis.

sol. $lx+my+nz=p$

$$z = \frac{p-lx-my}{n}$$

Substituting z in $ax^2+by^2=2z$

$$\Rightarrow ax^2+by^2 = 2\left[\frac{p-lx-my}{n}\right]$$

$$\Rightarrow n(ax^2+by^2)-2(p-lx-my)=0$$

$$\Rightarrow anx^2+bny^2+2lx+2my-2p=0.$$

3) Find the eqn of the cylinder whose generators are parallel to z-axis & guiding curve is $ax^2+by^2+cz^2=1$, $lx+my+nz=p$.

sol. $lx+my+nz=p$

$$z = \frac{p-lx-my}{n}$$

$$\Rightarrow ax^2+by^2+c\left[\frac{p-lx-my}{n}\right]^2=1$$

$$\Rightarrow an^2x^2+bn^2y^2+c[p-lx-my]^2=n^2$$

$$\Rightarrow an^2x^2+bn^2y^2+c[p^2+l^2x^2+m^2y^2-2plx+2lmxy-2pmy]-n^2=0$$

$$\Rightarrow (an^2+cl^2)x^2+(bn^2+cm^2)y^2+2clmxy-2plcx-2pcmy-m^2=0.$$

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4) Find the eq^n of the cylinder whose generators are parallel to $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ & guiding curve is $x^2+y^2=16, z=0$.

sol. let (α,β,γ) be point on cylinder

Eq^n of generator through (α,β,γ) & parallel to given line $\Rightarrow$ $\frac{x-\alpha}{1}=\frac{y-\beta}{2}=\frac{z-\gamma}{3}$; $z=0 \Rightarrow \frac{x-\alpha}{1}=\frac{y-\beta}{2}=\frac{-\gamma}{3}$

point of intersection $=\left(\alpha-\frac{\gamma}{3}, \beta-\frac{2\gamma}{3}, 0\right)$

Substituting in $x^2+y^2=16$.

$$\left(\alpha-\frac{\gamma}{3}\right)^2+\left(\beta-\frac{2\gamma}{3}\right)^2=16$$

To eliminate (α,β,γ), now replace (α,β,γ) with (x,y,z), we get,

$$\left(x-\frac{z}{3}\right)^2+\left(y-\frac{2z}{3}\right)^2=16$$

$$\Rightarrow (3x-z)^2+(3y-2z)^2=16\times 9$$

$$\Rightarrow 9x^2+z^2-6xz+9y^2+4z^2-12yz-144=0.$$

$$\Rightarrow 9x^2+9y^2+5z^2-12yz-6xz-144=0.$$

5) Find the equation of cylinder whose generating lines have DC's as (l,m,n) & which passes through the circle $x^2+z^2=a^2$, $y=0 \rightarrow$ guiding curve.

sol. let (α,β,γ) be on the cylinder

Eq^n of generator through (α,β,γ) & having DC's $=(l,m,n)=$ $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$

put $y=0 \Rightarrow \frac{x-\alpha}{l}=\frac{-\beta}{m}=\frac{z-\gamma}{n}$

pt of intersection $\Rightarrow \left(\alpha-\frac{l\beta}{n}, 0, \gamma-\frac{n\beta}{m}\right)$

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substituting in $x^2+z^2=a^2$

$$\left(\alpha-\frac{l\beta}{m}\right)^2+\left(\gamma-\frac{n\beta}{m}\right)^2=a^2$$

Eliminating (α,β,γ)

$$\Rightarrow \left(x-\frac{ly}{m}\right)^2+\left(z-\frac{ny}{m}\right)^2=a^2$$

$$\Rightarrow (mx-ly)^2+(mz-ny)^2=a^2m^2$$

$$\Rightarrow m^2x^2+l^2y^2-2lmxy+m^2z^2+n^2y^2-2mnyz-a^2m^2=0$$

$$\Rightarrow m^2x^2+(l^2+n^2)y^2-\cancel{2lm}\ m^2z^2-2lmxy-2mnyz-a^2m^2=0$$

6) Find the eqn of the cylinder with generators parallel to x-axis & passing through the curve $ax^2+by^2+cz^2=1$, $lx+my+nz=p$.

sol. $lx+my+nz=p$

$$x=\frac{p-my-nz}{l}$$

Substitute in $ax^2+by^2+cz^2=1$

$$a\left[\frac{p-my-nz}{l}\right]^2+by^2+cz^2=1$$

$$\Rightarrow a(p-my-nz)^2+bl^2y^2+c^2l^2z^2=l^2$$

$$\Rightarrow a(p^2+m^2y^2+n^2z^2-2pmy+2mnyz-2pnz)+bl^2y^2+cl^2z^2-l^2=0.$$

$$\Rightarrow (am^2+bl^2)y^2+(an^2+cl^2)z^2+\cancel{2amy}\ 2amnyz-2apmy-2apnz+ap^2-l^2=0.$$

The above eqn is required eqn of cylinder.

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Enveloping Cylinder

Theorem: To find the eqⁿ of the enveloping cylinder(or) cylinder whose generators touch the sphere $x^2+y^2+z^2=a^2$ and parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

Proof: Let (α,β,γ) be any pt on the cylinder

Eqⁿ of the generators through (α,β,γ) & parallel to given line is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ (say)

Any pt on the generator $=(lr+\alpha,\ mr+\beta,\ nr+\gamma)$.

Substituting in $x^2+y^2+z^2=a^2$ we get,

$\Rightarrow (lr+\alpha)^2+(mr+\beta)^2+(nr+\gamma)^2-a^2=0$

$\Rightarrow l^2r^2+\alpha^2+2rl\alpha+m^2r^2+\beta^2+2rm\beta+n^2r^2+\gamma^2+2rn\gamma-a^2=0$

$\Rightarrow r^2(l^2+m^2+n^2)+2r(l\alpha+m\beta+n\gamma)+(\alpha^2+\beta^2+\gamma^2-a^2)=0.$

The generators, $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is a tangent line if

$\Rightarrow$ discriminant $=0$.

$\Rightarrow b^2-4ac=0$

$\Rightarrow \cancel{4}(l\alpha+m\beta+n\gamma)^2-\cancel{4}(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)=0$

$\Rightarrow$ To get req enveloping cylinder $(\alpha,\beta,\gamma)=(x,y,z)$

$\Rightarrow (lx+my+nz)^2-$ ~~$(x^2+y^2+z^2)(x$~~ $(l^2+m^2+n^2)(x^2+y^2+z^2-a^2)=0.$

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1) Find the equation of the enveloping cylinder of the sphere $x^2+y^2+z^2=25$ whose generators are parallel to $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.

sol. Let (α,β,γ) be pt on cylinder

Eqn of generator passing through (α,β,γ) & parallel to given line is $\frac{x-\alpha}{1}=\frac{y-\beta}{2}=\frac{z-\gamma}{3}=r$ (say)

Any pt on cylinder $=(r+\alpha, 2r+\beta, 3r+\gamma)$

Substituting in $x^2+y^2+z^2=25$ we get,

$\Rightarrow (r+\alpha)^2+(2r+\beta)^2+(3r+\gamma)^2-25=0$

$\Rightarrow r^2+\alpha^2+2r\alpha+4r^2+\beta^2+4r\beta+9r^2+\gamma^2+6r\gamma-25=0$

$\Rightarrow 14r^2+2r(\alpha+2\beta+3\gamma)+(\alpha^2+\beta^2+\gamma^2-25)=0$

$\boxed{\text{Discr}=0}$

$\Rightarrow \cancel{4}(\alpha+2\beta+3\gamma)^2-\cancel{4}\times 14(\alpha^2+\beta^2+\gamma^2-25)=0$

$\Rightarrow$ Eqn of the enveloping cylinder is

$\Rightarrow (x+2y+3z)^2-14(x^2+y^2+z^2-25)=0$

$\Rightarrow x^2+4y^2+9z^2+4xy+12yz+6zx-14x^2-14y^2-14z^2+350=0$

$\Rightarrow -13x^2-13y^2-5z^2+4xy+12yz+6zx+350=0$

$\Rightarrow 13x^2+13y^2+5z^2-4xy-12yz-6zx-350=0.$

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2) Find the equation of enveloping cylinder of the conicoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose generators are parallel to the line

$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

Sol. let (α,β,γ) be any point on the cylinder

Eqn of generators through (α,β,γ) as parallel to the given line

$$\cancel{\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}} \quad \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r(\text{say})$$

Any point on the generator $=(lr+\alpha, mr+\beta, nr+\gamma)$

Substituting in given surface, we get

$$\Rightarrow \frac{(lr+\alpha)^2}{a^2}+\frac{(mr+\beta)^2}{b^2}+\frac{(nr+\gamma)^2}{c^2}=1$$

$$\Rightarrow \frac{(lr+\alpha)^2}{a^2}+\frac{(mr+\beta)^2}{b^2}+\frac{(nr+\gamma)^2}{c^2}-1=0$$

$$\Rightarrow \frac{1}{a^2}(l^2r^2+\alpha^2+2rl\alpha)+\frac{1}{b^2}(m^2r^2+\beta^2+2rm\beta)+\frac{1}{c^2}(n^2r^2+\gamma^2+2rn\gamma)-1=0$$

$$\Rightarrow r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)+2r\left(\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}+\frac{n\gamma}{c^2}\right)+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=0$$

Discriminant $=0$

$$\Rightarrow \cancel{4}\left(\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}+\frac{n\gamma}{c^2}\right)^2-\cancel{4}\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=0$$

To get enveloping cylinder, $(\alpha,\beta,\gamma)=(x,y,z)$

$$\Rightarrow \left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2-\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0.$$

$$\Rightarrow \left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2=0$$

$$\Rightarrow \frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=0.$$

Find the enveloping cylinder of the sphere $x^2+y^2+z^2-2x+4y=1$ having a generator parallel to $x=y=z$.

Sol. Let (α,β,γ) be a pt on the cylinder

eq^n of generator $\frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{1} = r$

Any pt on cylinder $=(r+\alpha, r+\beta, r+\gamma)$

Substituting in $x^2+y^2+z^2-2x+4y-1=0$.

$\Rightarrow (r+\alpha)^2 + (r+\beta)^2 + (r+\gamma)^2 - 2(r+\alpha) + 4(r+\beta) - 1 = 0$

$\Rightarrow r^2+\alpha^2+2r\alpha + r^2+\beta^2+2r\beta + r^2+\gamma^2+2r\gamma - 2r - 2\alpha + 4r + 4\beta - 1 = 0$

$\Rightarrow 3r^2 + 2r(\alpha+\beta+\gamma+1) + \cancel{4(\alpha^2+\beta^2+\gamma^2)=0} + (\alpha^2+\beta^2+\gamma^2-2\alpha+4\beta-1) = 0$

$\boxed{b^2-4ac=0}$

$\Rightarrow \cancel{4}(\alpha+\beta+\gamma+1)^2 - \cancel{4}\times 3(\alpha^2+\beta^2+\gamma^2-2\alpha+4\beta-1)=0$

$\Rightarrow$ Eq^n of enveloping cylinder,

$(\underbrace{x+y}_{A}+\underbrace{z+1}_{B})^2 - 3(x^2+y^2+z^2-2x+4y-1)=0$

~~$x^2+y^2+2xy+z^2+2zx+2z+2x+2yz$~~

$\Rightarrow (x+y)^2+(z+1)^2+2(x+y)(z+1) - 3(x^2+y^2+z^2-2x+4y-1)=0$

$\Rightarrow x^2+y^2+2xy+z^2+1+2z+2xz+2x+2yz+2y-3x^2-3y^2-3z^2+6x-12y+3=0$

$\Rightarrow 2x^2+2y^2+2z^2-2xy-2yz-2xz-8x+10y-2z-4=0.$

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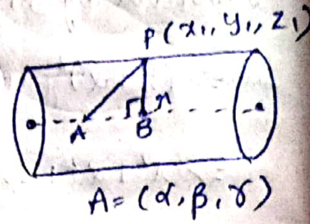
Right Circular Cylinder: A right circular cylinder is a surface generated by a line which is parallel to a fixed line (i.e. Axis) and is at a constant distance from it. The constant distance is known as radius of right circular cylinder.

Theorem: To find the eqn of right circular cylinder.

Proof: Distance b/w AP is given by



$$AP = \sqrt{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2}$$

Eqn of axis is $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$

$PB = r$ (radius of right circular cylinder)

AB = projection of the line AP on axis

Dr's of axis $= \left(\dfrac{l}{\sqrt{l^2+m^2+n^2}}, \dfrac{m}{\sqrt{l^2+m^2+n^2}}, \dfrac{n}{\sqrt{l^2+m^2+n^2}}\right)$

$$AB = \frac{l}{\sqrt{l^2+m^2+n^2}}(x_1-\alpha) + \frac{m}{\sqrt{l^2+m^2+n^2}}(y_1-\beta) + \frac{n}{\sqrt{l^2+m^2+n^2}}(z_1-\gamma)$$

$\Rightarrow AB^2 + BP^2 = AP^2$

$$\Rightarrow \frac{[l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]^2}{l^2+m^2+n^2} + r^2 = (x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2$$

$$\Rightarrow [l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]^2 = [(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2-r^2](l^2+m^2+n^2)$$

$\therefore$ The above eqn is the required eqn of right circular cylinder.

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(Q.11)

Find the equation of right circular cylinder whose radius is 2 and axis is $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$.

Sol. Eqn of the right circular cylinder is

$\Rightarrow [l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2 = [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - r^2](l^2+m^2+n^2)$

$\Rightarrow [2(x-1) + 1(y-2) + 2(z-3)]^2 = [(x-1)^2 + (y-2)^2 + (z-3)^2 - 4](9)$

$\Rightarrow [(2x+y) + (2z-10)]^2 = [x^2 + y^2 + z^2 - 2x - 4y - 6z + 10](9)$

$\Rightarrow (2x+y)^2 + (2z-10)^2 + 2(2x+y)(2z-10) = [x^2 + y^2 + z^2 - 2x - 4y - 6z + 10](9)$

$\Rightarrow 4x^2 + y^2 + 4xy + 4z^2 + 100 - 40z + 8xz - 40x + 4yz - 20y$
$= 9x^2 + 9y^2 + 9z^2 - 18x - 36y - 54z + 90$

$\Rightarrow 5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8xz + 22x - 16y - 14z - 10 = 0$

PCP-18

The axis of right circular cylinder is $\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$ and radius $= 2$.

Sol. Eqn of the right circular cylinder is

$\Rightarrow [l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2 = [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - r^2](l^2+m^2+n^2)$

$\Rightarrow [2(x-1) + 3y + 1(z-3)]^2 = [(x-1)^2 + y^2 + (z-3)^2 - 4](14)$

$\Rightarrow [(2x+3y) + (z-5)]^2 = [x^2 + y^2 + z^2 - 2x - 6z + 6](14)$

$\Rightarrow 4x^2 + 9y^2 + 12xy + z^2 + 25 - 10z + 2(2x+3y)(z-5)$
$= 14x^2 + 14y^2 + 14z^2 - 28x - 84z + 84$

$$\Rightarrow 4x^2+9y^2+12xy+z^2+25-10z+4xz\ \underset{-20x}{\cancel{-40x}}\ -30y\ \overset{+6yz}{\cancel{60y}} = 14x^2+14y^2$$
$$+14z^2-28x-84z+84.$$

$$\Rightarrow 10x^2+5y^2+13z^2-12xy-4xz\ \overset{-6yz-8x}{\cancel{+17x}}+30y-14z+59=0.$$

• Find the eq^n of right circular cylinder whose axis is $\frac{x-2}{2}=\frac{y-1}{1}=\frac{z}{3}$ and passes through $(0,0,3)$.

Sol. Eq^n of the right circular cylinder is

$$[2(x-2)+1(y-1)+3z]^2=[(x-2)^2+(y-1)^2+z^2-r^2](14) \quad \text{—— (1)}$$

(1) passes through $(0,0,3)$

$$\Rightarrow (-4-1+9)^2=[4+1+9-r^2]\times 14$$

$$\Rightarrow \cancel{8}\ \frac{8}{7}=14-r^2$$

$$\Rightarrow r^2=14-\frac{8}{7}$$

$$\Rightarrow r^2=\frac{90}{7}$$

$$(1) \Rightarrow [(2x+y)(3z-5)]^2=[(x-2)^2+(y-1)^2+z^2-\frac{90}{7}]\times 14$$

$$\Rightarrow 4x^2+y^2+4xy+9z^2+25-30z+2(2x+y)(3z-5)=[x^2+y^2+z^2$$
$$-4x-2y+(5-\frac{90}{7})]\times 14$$

$$\Rightarrow 4x^2+y^2+4xy+9z^2+25-30z+12xz-20x+6yz-10y=14x^2+14y^2$$
$$+14z^2-56x-28y-110.$$

$$\Rightarrow 10x^2+13y^2+5z^2-4xy-6yz-12xz-36x-18y+30z-135=0.$$

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P.P 19

Find the equation of right circular cylinder whose guiding curve is

$x^2+y^2+z^2=9, \; x-y+z=3.$

Sol. $x-y+3=0.$ $\qquad$ $OA=$ length of $\perp^r$ drawn from $(0,0,0)$ to $x-y+z-3=0$

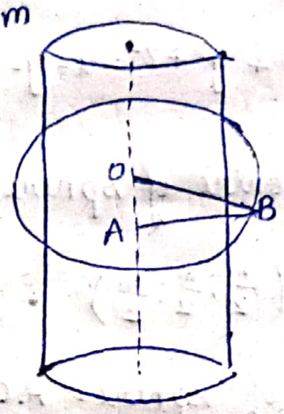
$O=$ Centre of sphere. $\qquad OA=\left|\dfrac{0-0+0-3}{\sqrt{1+1+1}}\right|=\sqrt{3}$

$=(0,0,0)$ $\qquad \Delta OAB \Rightarrow OA^2+AB^2=OB^2$

$OB=$ radius of sphere $=3$ $\qquad \Rightarrow 3+AB^2=9$

$AB^2=6$

$AB=\sqrt{6}$



Eqn of right circular cylinder is $\qquad r=AB=\sqrt{6}$

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2=[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2](l^2+m^2+n^2)$$

Eqn of axis is $\dfrac{x-0}{1}=\dfrac{y-0}{-1}=\dfrac{z-0}{1}$

$$\Rightarrow [1(x-0)-1(y-0)+1(z-0)]^2=[(x-0)^2+(y-0)^2+(z-0)^2-6](3)$$

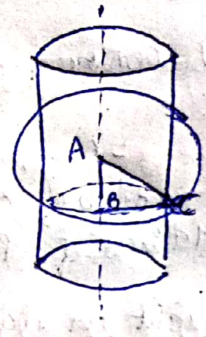
$$\Rightarrow (x-y+z)^2=(x^2+y^2+z^2-6)3$$

$$\Rightarrow x^2+y^2+z^2-2xy-2yz+2xz-3x^2-3y^2-3z^2+18=0$$

$$\Rightarrow -2x^2-2y^2-2z^2-2xy-2yz+2xz+18=0$$

$$\Rightarrow x^2+y^2+z^2+xy+yz-zx-6=0.$$

Obtain the equation of right circular cylinder through 3 points $(1,0,0)$, $(0,1,0)$ $(0,0,1)$. as guiding curve.



Sol. Eqⁿ of sphere is

$$x^2+y^2+z^2-x-y-z=0, \quad x+y+z=1$$

A = centre of sphere

$$=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$

Radius of sphere = AC.

$$\Rightarrow \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{3}}{2}$$

$$AC = \frac{\sqrt{3}}{2}$$

AB = length of perpendicular from $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ to $x+y+z-1=0$.

$$\Rightarrow \left|\frac{\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}+\frac{1}{2}-\cancel{1}}{\sqrt{1+1+1}}\right|$$

$$AB = \frac{1}{2\sqrt{3}}$$

from $\Delta ABC \Rightarrow AB^2+BC^2=AC^2$

$$\Rightarrow \left(\frac{1}{2\sqrt{3}}\right)^2+BC^2=\left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \frac{1}{12}+BC^2=\frac{3}{4}$$

$$\Rightarrow BC^2=\frac{3}{4}-\frac{1}{12}$$

$$\Rightarrow BC^2=\frac{9-1}{12}=\frac{8}{12}=\frac{2}{3}$$

$$\Rightarrow BC=\sqrt{\frac{2}{3}}=r.$$

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Eq^n of the axis is

$$\frac{x-\frac{1}{2}}{1} = \frac{y-\frac{1}{2}}{1} = \frac{z-\frac{1}{2}}{1}$$

Eq^n of right circular cylinder is

$\Rightarrow$ ~~$[l(x-\alpha)+y(z$~~

$\Rightarrow [l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = [(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2](l^2+m^2+n^2)$

$\Rightarrow [1(x-\frac{1}{2})+1(y-\frac{1}{2})+1(z-\frac{1}{2})]^2 = [(x-\frac{1}{2})^2+(y-\frac{1}{2})^2+(z-\frac{1}{2})^2-\frac{2}{3}]$

$(1+1+1)$

$\Rightarrow [(x+y)+(z-\frac{3}{2})]^2 = [x^2+y^2+z^2-x-y-z+\frac{1}{12}](3)$

$\Rightarrow (x+y)^2+(z-\frac{3}{2})^2+2(x+y)(z-\frac{3}{2}) = 3x^2+3y^2+3z^2-3x-3y-3z+\frac{1}{4}$

$\Rightarrow x^2+y^2+2xy+z^2+\frac{9}{4}-\cancel{3z}+2xz-\cancel{3x}+2yz-\cancel{3y} = 3x^2+3y^2+3z^2-\cancel{3x}-\cancel{3y}-\cancel{3z}+\frac{1}{4}$

$\Rightarrow -2x^2-2y^2-2z^2+2xy+2yz+2zx+2=0$

$\Rightarrow x^2+y^2+z^2-xy-yz-zx-1=0.$

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UNIT-III - CONICOID

Conicoid:

Central Conicoid :-

Theorem: To find the points of intersection of a line with central conicoid.

Proof: Let eqn of line be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$

Any point on the line $= (lr+\alpha, mr+\beta, nr+\gamma)$

Substituting in $ax^2+by^2+cz^2=1$ we get,

$\Rightarrow a(lr+\alpha)^2 + b(mr+\beta)^2 + c(nr+\gamma)^2 - 1 = 0$

$\Rightarrow a(l^2r^2+\alpha^2+2rl\alpha) + b(m^2r^2+\beta^2+2rm\beta) + c(n^2r^2+\gamma^2+2rn\gamma) - 1 = 0$

$\Rightarrow r^2(al^2+bm^2+cn^2) + 2r(al\alpha+bm\beta+cn\gamma) + (a\alpha^2+b\beta^2+c\gamma^2-1) = 0$

Theorem: To find the eqn of tangent plane at (α, β, γ) w.r.t

$ax^2+by^2+cz^2=1$.

Proof: Let (α, β, γ) be a point on given surface

$\Rightarrow a\alpha^2+b\beta^2+c\gamma^2 = 1$ — (1)

Let the eqn of the line be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

Substitute $(lr+\alpha, mr+\beta, nr+\gamma)$ in $ax^2+by^2+cz^2=1$ we get

$r^2(al^2+bm^2+cn^2) + 2r(al\alpha+bm\beta+cn\gamma) + (a\alpha^2+b\beta^2+c\gamma^2-1) = 0.$

The line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ becomes a tangent plane if

$b^2-4ac=0$.

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$\Rightarrow$ ~~4~~ $4(al\alpha+bm\beta+cn\gamma)^2 - 4(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$
$=0.$

$\Rightarrow 4(al\alpha+bm\beta+cn\gamma)^2=0$

$\Rightarrow al\alpha+bm\beta+cn\gamma=0$

Eliminating (l,m,n) we get eqn of ~~tagent~~ tangent plane as

$$\Rightarrow a\left(\frac{x-\alpha}{r}\right)\alpha+b\left(\frac{y-\beta}{r}\right)\beta+c\left(\frac{z-\gamma}{r}\right)\gamma=0$$

$$\Rightarrow a\alpha x+by\beta+cz\gamma=a\alpha^2+b\beta^2+c\gamma^2$$

$$\Rightarrow a\alpha x+b\beta y+c\gamma z=1$$

Theorem: Condition for Tangency

Statement: To find the condition for the plane $lx+my+nz=p$ to be a tangent plane to the conicoid. $ax^2+by^2+cz^2=1$.

Proof: let (α,β,γ) be the point of contact.

Eqn of tangent plane to $ax^2+by^2+cz^2=1$

$$\Rightarrow a\alpha x+b\beta y+c\gamma z=1 \quad \text{--- (1)}$$

Also $lx+my+nz=p$ $\Rightarrow$ is given tangent plane
— (2)

Comparing co-eff's from (1) & (2)

$$\Rightarrow \frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{1}{p}.$$

$$\Rightarrow \alpha=\frac{l}{ap},\ \beta=\frac{m}{bp},\ \gamma=\frac{n}{cp}$$

point of contact $=(\alpha,\beta,\gamma)$

$$=\left(\frac{l}{ap},\frac{m}{bp},\frac{n}{cp}\right)$$

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(α,β,γ) lies on $ax^2+by^2+cz^2=1$

$$\Rightarrow a\alpha^2+b\beta^2+c\gamma^2=1$$

$$\Rightarrow \cancel{a}\left(\frac{l^2}{a^{\cancel{2}}p^2}\right)+\cancel{b}\left(\frac{m^2}{b^{\cancel{2}}p^2}\right)+\cancel{c}\left(\frac{n^2}{c^{\cancel{2}}p^2}\right)=1$$

$$\Rightarrow \boxed{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2}$$

$lx+my+nz=p$

$$\Rightarrow lx+my+nz=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

PCP-21

Find the point of intersection of the line $\frac{x+5}{-3}=\frac{y-4}{1}=\frac{z-11}{7}$ with conicoid. $12x^2-17y^2+7z^2=7$

Sol let $\frac{x+5}{-3}=\frac{y-4}{1}=\frac{z-11}{7}=r$

~~points~~ Any point on the line $=(-3r-5,\ r+4,\ 7r+11)$

Substituting in $12x^2-17y^2+7z^2-7=0$.

$$\Rightarrow 12(-3r-5)^2-17(r+4)^2+7(7r+11)^2-7=0$$

$$\Rightarrow 12(9r^2+25+30r)-17(r^2+16+8r)+7(49r^2+121+154r)-7=0$$

$$\Rightarrow 108r^2+300+360r-17r^2-272-136r+343r^2+847+1078r-7=0$$

$$\Rightarrow 434r^2+1302r+868=0.$$

$$\Rightarrow 217r^2+651r+434=0.$$

$$\Rightarrow r^2+3r+2=0$$

$$\Rightarrow r^2+2r+r+2=0$$

$$\Rightarrow r(r+2)+(r+2)=0$$

$$\Rightarrow (r+2)(r+1)=0$$

$$\Rightarrow r=-1,-2.$$

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$\lambda = -1$

$\Rightarrow (-3(-1)-5, -1+4, 7(-1)+11)$

$\Rightarrow (-2, 3, 4)$

$\lambda = -2$

$\Rightarrow (-3(-2)-5, -2+4, 7(-2)+11)$

$\Rightarrow (1, 2, -3)$

PCP-22

Find the equation of tangent planes to $7x^2-3y^2-z^2+21=0$ which pass through the line $7x-6y+9=0, z=3$.

sol. Eqn of required plane is

$\Rightarrow (7x-6y+9)+\lambda(z-3)=0$

$\Rightarrow 7x-6y+\lambda z = 3\lambda - 9$

Given that $7x^2-3y^2-z^2+21=0$

$\Rightarrow 7x^2-3y^2-z^2 = -21$

$\Rightarrow -\frac{1}{3}x^2+\frac{1}{7}y^2+\frac{z^2}{21} = 1$

By condition of tangency we have

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c} = p^2$$

$$\Rightarrow \frac{49}{-1/3}+\frac{36}{1/7}+\frac{\lambda^2}{1/21} = (3\lambda-9)^2$$

$\Rightarrow -147+252+21\lambda^2 = 9\lambda^2+81-54\lambda$

$\Rightarrow 21\lambda^2-9\lambda^2+54\lambda-147+252-81=0$

$\Rightarrow 12\lambda^2+54\lambda+24=0$

$\Rightarrow 2\lambda^2+9\lambda+4=0$

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$\Rightarrow 2\lambda^2 + 8\lambda + \lambda + 4 = 0$

$\Rightarrow 2\lambda(\lambda+4) + (\lambda+4) = 0$

$\Rightarrow (\lambda+4)(2\lambda+1) = 0$

$\Rightarrow \boxed{\lambda = -4}$ (or) $\boxed{\lambda = -\frac{1}{2}}$

Substituting λ in

$7x - 6y + \lambda z = 3\lambda - 9$

we get eqn of tangent planes

$7x - 6y - 4z = -21$

(or)

$7x - 6y - \frac{1}{2}z = -\frac{21}{2}$

PCP-23

Find the tangent planes to Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which are parallel to the plane $lx + my + nz = 0$.

Sol. Eqn of plane parallel to $lx + my + nz = 0$ is $lx + my + nz = p$ — (1)

using condition of tangency we get

$$\frac{l^2}{1/a^2} + \frac{m^2}{1/b^2} + \frac{c^2}{1/c^2} = p^2$$

$\Rightarrow a^2l^2 + b^2m^2 + c^2n^2 = p^2$

$\Rightarrow p = \pm\sqrt{a^2l^2 + b^2m^2 + c^2n^2}$.

∴ The required eq's of tangent planes are

$lx + my + nz = \pm\sqrt{a^2l^2 + b^2m^2 + c^2n^2}$

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KP-24

Show that the plane $3x+12y-6z-17=0$ touches the conicoid $3x^2-6y^2+9z^2+17=0$ and find the point of contact.

Sol. Given plane is $3x+12y-6z=17$

G/T $3x^2-6y^2+9z^2=-17$

$$\Rightarrow -\frac{3}{17}x^2+\frac{6}{17}y^2-\frac{9}{17}z^2=1$$

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

$$\Rightarrow \frac{9}{-3/17}+\frac{144}{6/17}+\frac{36}{-9/17} \Rightarrow \frac{-153}{3}+\frac{2448}{6}-\frac{612}{9}$$

$$\Rightarrow -51+408-68$$

$$\Rightarrow 289$$

also, $p^2=289$ $[\because p=17]$

$$\therefore \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

$\therefore$ Given plane is a tangent plane.

Point of Contact $=\left(\frac{l}{ap},\frac{m}{bp},\frac{n}{cp}\right)$

$$=\left(\frac{3}{-\frac{3}{17}\times 17},\frac{12}{\frac{6}{17}\times 17},\frac{-6}{\frac{-9}{17}\times 17}\right)$$

$$=(-1,2,2/3)$$

• Find the eq^n of the tangent planes to the curve $x^2-2y^2+3z^2=2$

& parallel to plane $x-2y+3z=0$

sol. Required plane ~~plane~~ parallel to $x-2y+3z=0$ is

$$x-2y+3z=p$$

Given curve $x^2-2y^2+3z^2=2$

$$\Rightarrow \quad \frac{1}{2}x^2-y^2+\frac{3}{2}z^2=1$$

By condition of tangency

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{z^2}{c}=p^2$$

$$\frac{1}{1/2}+\frac{4}{-1}+\frac{\cancel{9}^{3}}{\cancel{3}/2}=p^2$$

$$2-4+6=p^2$$

$$p^2=4$$

$$p=\pm 2$$

Substituting in $x-2y+3z=p$.

$$x-2y+3z=2$$

&

$$x-2y+3z=-2$$

• Find the equations to tangent planes $2x^2-6y^2+3z^2=5$ which pass through the line $x+9y-3z=0=3x-3y+6z-5$

$[P_1=0=P_2]$

sol. Required eq^n of plane is $P_1+\lambda P_2=0$

$$(x+9y-3z)+\lambda(3x-3y+6z-5)=0$$

$$\Rightarrow \quad (1+3\lambda)x+(9-3\lambda)y+(6\lambda-3)z=0$$

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Eq^n of conicoid is $2x^2-6y^2+3z^2=5$

$$\Rightarrow \frac{2}{5}x^2-\frac{6}{5}y^2+\frac{3}{5}z^2=1$$

using condition of tangency

$$\frac{(1+3\lambda)^2}{2/5}+\frac{(9-3\lambda)^2}{-6/5}+\frac{(6\lambda-3)^2}{3/5}=25\lambda^2$$

$$\Rightarrow \frac{5}{2}(1+9\lambda^2+6\lambda)-\frac{5}{6}(81+9\lambda^2-54\lambda)+\frac{5}{3}(36\lambda^2+9-36\lambda)=\overset{5}{\cancel{25}}\lambda^2$$

$$\Rightarrow 3+27\lambda^2+18\lambda-81-9\lambda^2+54\lambda+72\lambda^2+18-72\lambda=30\lambda^2$$

$$\Rightarrow 60\lambda^2-60=0$$

$$\Rightarrow 60\lambda^2=60 \Rightarrow \lambda^2=1$$

$$\Rightarrow \lambda=\pm 1$$

$\therefore$ Eq^n of tangent planes are

$$(x+9y-3z)-1(3x-3y+6z-5)=0$$

$$\Rightarrow x+9y-3z-3x+3y-6z+5=0$$

$$\Rightarrow -2x+12y-9z+5=0$$

$$\Rightarrow 2x-12y+9z-5=0.$$

$$(x+9y-3z)+1(3x-3y+6z-5)=0$$

$$\Rightarrow x+9y-3z+3x-3y+6z-5=0$$

$$\Rightarrow 4x+6y+3z-5=0.$$

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PCP-25

Find the eq^n of tangent planes to the surface $4x^2-5y^2+7z^2+13=0$ parallel to $4x+20y-21z=0$. Find the point of contact.

sol. Eq^n of plane parallel to $4x+20y-21z=0$ is

$$4x+20y-21z=p \text{ --- } (1)$$

consider $4x^2-5y^2+7z^2+13=0$

$$\Rightarrow 4x^2-5y^2+7z^2=-13$$

$$\Rightarrow -\frac{4}{13}x^2+\frac{5}{13}y^2-\frac{7}{13}y^2=1$$

If eq^n (1) is tangent plane to surface then by using condition of tangency we get,

$$\Rightarrow \frac{16}{-4/13}+\frac{400}{5/13}+\frac{441}{-7/13}=p^2 \qquad \left[\because \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2\right]$$

$$\Rightarrow \frac{\overset{4}{\cancel{16}}\times 13}{-\cancel{4}}+\frac{\overset{80}{\cancel{400}}\times 13}{\cancel{5}}+\frac{\overset{63}{\cancel{441}}\times 13}{-\cancel{7}}=p^2$$

$$\Rightarrow -52+1040-819=p^2$$

$$\therefore 169=p^2$$

$$\boxed{P=\pm 13}$$

$\therefore$ eq^n of tangent planes are

$$4x+20y-21z=\pm 13.$$

$$\text{Point of contact} = \left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$$

If $4x+20y-21z=13=\left(\frac{\cancel{4}}{\frac{-\cancel{4}}{\cancel{13}}\times\cancel{13}}, \frac{\overset{4}{\cancel{20}}}{\frac{\cancel{5}}{\cancel{13}}\times\cancel{13}}, \frac{-\overset{3}{\cancel{21}}}{\frac{-\cancel{7}}{\cancel{13}}\times\cancel{13}}\right)$

$$=(-1, 4, 3)$$

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If $4x+20y-21z=-13$

Point of contact $=\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$

$$=\left(\frac{\cancel{4}}{-\frac{4}{13}\times -13}, \frac{\cancel{20}\ 4}{\frac{5}{13}\times -13}, \frac{\cancel{-21}\ 3}{-\frac{7}{13}\times -13}\right)$$

$$=(1,-4,-3)$$

PCP-26

Find the locus of perpendicular from the origin to tangent planes to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which cut off from its axes the intersects the sum of where reciprocals is constant equal to $\frac{1}{k}$?

Sol. Eq^n of plane is $lx+my+nz=p$.

Using condition of tangency.

$$\frac{l^2}{\frac{1}{a^2}}+\frac{m^2}{\frac{1}{b^2}}+\frac{n^2}{\frac{1}{c^2}}=p^2$$

$$\Rightarrow p^2=a^2l^2+b^2m^2+c^2n^2$$

$$\Rightarrow p=\sqrt{a^2l^2+b^2m^2+c^2n^2}$$

$\therefore$ Tangent plane is $lx+my+nz=\sqrt{a^2l^2+b^2m^2+c^2n^2}$

$$\Rightarrow \frac{lx}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}+\frac{my}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}+\frac{nz}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}=1$$

Intercepts of cone $=\frac{1}{l}\sqrt{a^2l^2+b^2m^2+c^2n^2}, \frac{1}{m}\sqrt{a^2l^2+b^2m^2+c^2n^2},$

$$\frac{1}{n}\sqrt{a^2l^2+b^2m^2+c^2n^2}.$$

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Sum of the reciprocals of Intercepts

$$\frac{l+m+n}{\sqrt{a^2l^2+b^2m^2+c^2n^2}} = \frac{1}{k}$$

S.O.B.S.

$$\Rightarrow k^2(l+m+n)^2 = a^2l^2+b^2m^2+c^2n^2$$

Eqⁿ of the normal drawn from origin is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r.$

Eliminating (l,m,n) we get.

$$k^2\left(\frac{x+y+z}{r}\right)^2 = \frac{a^2x^2+b^2y^2+c^2z^2}{r^2}$$

$$k^2(x+y+z)^2 = a^2x^2+b^2y^2+c^2z^2$$

8\. Tangent planes are drawn to the conicoid $ax^2+by^2+cz^2=1$ through (α,β,γ). S/T the perpendicular from the centre of the conicoid to the planes generators $(\alpha x+\beta y+\gamma z)^2 = \frac{x^2}{a}+\frac{y^2}{a}+\frac{z^2}{b}$

Sol: Eqⁿ of the plane through (α,β,γ)

$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$$

$$\Rightarrow lx+my+nz = l\alpha+m\beta+n\gamma$$

By condition of tangency we get

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c} = (l\alpha+m\beta+n\gamma)^2$$

Perpendiculars from centre of conicoid to tangent planes

Eqⁿ of Normal is, $\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} = r$

Locus is $\frac{x^2}{ar^2}+\frac{y^2}{br^2}+\frac{z^2}{cr^2} = \left(\frac{x}{r}\alpha+\frac{y}{r}\beta+\frac{z}{r}\gamma\right)^2$

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A tangent plane to $ax^2+by^2+cz^2=1$ meets the coordinate axes at P, Q, R. Find the locus of the centroid of triangle PQR.

Sol: Eqⁿ of tangent plane is

$$lx+my+nz=\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}} \quad \text{— (1)}$$

(1) meets X-axis at P

$$P=\left(\frac{1}{l}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}},0,0\right)$$

(1) meets Y-axis at Q

$$Q=\left(0,\frac{1}{m}\sqrt{l^2/a+m^2/b+n^2/c},0\right)$$

(1) meets Z-axis at R.

$$R=\left(0,0,\frac{1}{n}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}\right)$$

Let the locus point be (x_1,y_1,z_1)

$$x_1=\frac{1}{3l}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

$$y_1=\frac{1}{3m}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

$$z_1=\frac{1}{3n}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

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$$\Rightarrow q l^2 x_1^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} \quad \text{--- (1)}$$

$$\Rightarrow q m^2 y_1^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} \quad \text{--- (2)}$$

$$\Rightarrow q n^2 z_1^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} \quad \text{--- (3)}$$

~~then~~ divide eq (1) by $a x_1^2$

$$\Rightarrow \quad \frac{q l^2}{a} = \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)\frac{1}{a x_1^2} \quad \text{--- (4)}$$

divide eq (2) by $b y_1^2$

$$\Rightarrow \quad \frac{q m^2}{b} = \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)\frac{1}{b y_1^2} \quad \text{--- (5)}$$

divide eq (3) by $c z_1^2$

$$\Rightarrow \quad \frac{q n^2}{c} = \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)\frac{1}{c z_1^2} \quad \text{--- (6)}$$

(4) + (5) + (6)

$$\Rightarrow q\cancel{\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)} = \cancel{\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)}\left(\frac{1}{a x_1^2} + \frac{1}{b y_1^2} + \frac{1}{c z_1^2}\right)$$

replacing (x_1, y_1, z_1) with (x, y, z) we get

$$\Rightarrow \quad q = \frac{1}{a x^2} + \frac{1}{b y^2} + \frac{1}{c z^2}$$

$$\Rightarrow \boxed{\frac{1}{a x^2} + \frac{1}{b y^2} + \frac{1}{c z^2} = q}$$

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Sol: Let $l_1x+m_1y+n_1z=0$

$l_2x+m_2y+n_2z=0$

$l_3x+m_3y+n_3z=0$ be.

mutually perpendicular

$l_1^2+m_1^2+n_1^2=1$, $l_2^2+m_2^2+n_2^2=1$, $l_3^2+m_3^2+n_3^2=1$ are DC's

$l_1l_2+m_1m_2+n_1n_2=0$, $l_2l_3+m_2m_3+n_2n_3=0$, $l_1l_3+m_1m_3+n_1n_3=0$

$l_1^2+l_2^2+l_3^2=1$, $m_1^2+m_2^2+m_3^2=1$, $n_1^2+n_2^2+n_3^2=1$

$l_1m_1+l_2m_2+l_3m_3=0$, $m_1n_1+m_2n_2+m_3n_3=0$, $l_1n_1+l_2n_2+l_3n_3=0$

Director Sphere

The locus of the point of intersection of the 3 mutually perpendicular tangent planes which are drawn to conicoid $ax^2+by^2+cz^2=1$ is a sphere concentric with conicoid.

Proof: Consider 3 mutually perpendicular tangent planes

Given by $l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}}$

$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}}$$

$$l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}}$$

The point of intersection of 3 mutually $\perp^r$ tangent planes

Squaring ang adding we get

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2$$

$$=\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}+\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}+\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}.$$

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$$l_1^2x^2 + m_1^2y^2 + n_1^2z^2 + 2l_1m_1xy + 2\overset{m_1}{\cancel{l_1}}n_1yz + 2\overset{l_1}{\cancel{m_1}}n_1xz +$$

$$l_2^2x^2 + m_2^2y^2 + n_2^2z^2 + 2l_2m_2xy + 2\overset{m_2}{\cancel{l_2}}n_2yz + 2\overset{l_2}{\cancel{m_2}}n_2xz +$$

$$l_3^2x^2 + m_3^2y^2 + n_3^2z^2 + 2l_3m_3xy + 2m_3n_3yz + 2l_3n_3xz$$

$$= \frac{1}{a}(l_1^2 + l_2^2 + l_3^2) + \frac{1}{b}(m_1^2 + m_2^2 + m_3^2) + \frac{1}{c}(n_1^2 + n_2^2 + n_3^2)$$

$$\Rightarrow \boxed{x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$$

It represents a sphere whose centre is $(0,0,0)$

It is concentric to conicoid $ax^2 + by^2 + cz^2 = 1$

Normal to a Central Conicoid

Let the eqn of central conicoid be $ax^2 + by^2 + cz^2 = 1$ — ①

let (α, β, γ) be a point to ①

Eqn of tangent plane at (α, β, γ) to $ax^2 + by^2 + cz^2 = 1$

is $a\alpha x + b\beta y + c\gamma z = 1$ — ②

Eqn of the normal drawn through (α, β, γ) & $\perp^r$ to ②

is $$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma}$$

let p = length of $\perp^r$ from point $(0,0,0)$ to tangent plane

$$p = \left| \frac{a\alpha(0) + b\beta(0) + c\gamma(0) - 1}{\sqrt{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2}} \right|$$

$$p = \frac{1}{\sqrt{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2}}$$

$$(a\alpha p)^2 + (b\beta p)^2 + (c\gamma p)^2 = 1$$

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Eqn of normal $\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma}$

Number of Normals from a given point

Theorem: Prove that through any given point six normals can be drawn to a central conicoid.

Proof: Eqn of central conicoid is $ax^2+by^2+cz^2=1$.

let (α, β, γ) be a point on $ax^2+by^2+cz^2=1$

Eqn of normal is $\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma}$.

if the normal passes through (f, g, h)

$$\frac{f-\alpha}{a\alpha} = \frac{g-\beta}{b\beta} = \frac{h-\gamma}{c\gamma} = r$$

$$\frac{f-\alpha}{a\alpha} = r \Rightarrow f-\alpha = a\alpha r$$

$$\Rightarrow f = \alpha(1+ar)$$

$$\Rightarrow \alpha = \frac{f}{1+ar}$$

Similarly $\beta = \frac{g}{1+br}$, $\gamma = \frac{h}{1+cr}$

(α, β, γ) lies on conicoid $\Rightarrow a\alpha^2+b\beta^2+c\gamma^2=1$

$$\Rightarrow \frac{af^2}{(1+ar)^2} + \frac{bg^2}{(1+br)^2} + \frac{ch^2}{(1+cr)^2} = 1$$

$$\Rightarrow af^2(1+br)^2(1+cr)^2 + bg^2(1+ar)^2(1+cr)^2 + ch^2(1+ar)^2(1+br^2)$$
$$= (1+ar)^2(1+br)^2(1+cr)^2$$

Above eqn is 6^{th} degree equation. i.e 6 roots

Cubic curve to the feet of the perpendicular

The feet of the six normals from a given point (f,g,h) to a central conicoid and the intersection of the conicoid with a ~~central~~ certain cubic curve.

Proof: Eqn of central conicoid is ~~[illegible]~~ $ax^2+by^2+cz^2=1$

consider a curve $x=\dfrac{f}{1+ar}$, $y=\dfrac{g}{1+br}$, $z=\dfrac{h}{1+cr}$

substituting in $ax^2+by^2+cz^2=1$

$$\Rightarrow \frac{af^2}{(1+ar)^2}+\frac{bg^2}{(1+br)^2}+\frac{ch^2}{(1+cr)^2}=1$$

$\rightarrow$ 6th degree equation

Substituting x,y,z values in $Ax+By+Cz+D=0$.

$$\Rightarrow \frac{Af}{1+ar}+\frac{Bg}{1+br}+\frac{Ch}{1+cr}+D=0$$

$\rightarrow$ 3rd degree eqn in 'r'

Theorem: The six normals drawn from any point to a central conicoid are the generators of a quadric cone.

Proof: Eqn of line through (f,g,h) is

$$\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n} \quad \text{—①}$$

eqn ① intersects the cubic curve $\left[x=\dfrac{f}{1+ar}, y=\dfrac{g}{1+br}, z=\dfrac{h}{1+cr}\right]$

$$\Rightarrow \frac{\frac{f}{1+ar}-f}{l}=\frac{\frac{g}{1+br}-g}{m}=\frac{\frac{h}{1+cr}-h}{n}$$

$$\Rightarrow \frac{\cancel{f}-\cancel{f}-arf}{l(1+ar)}=\frac{\cancel{g}-\cancel{g}-gbr}{m(1+br)}=\frac{\cancel{h}-\cancel{h}-hcr}{n(1+cr)}$$

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$$\frac{af}{l(ar+1)} = \frac{bg}{m(1+br)} = \frac{ch}{n(1+cr)}$$

$$\Rightarrow \frac{af/l}{(1+ar)} = \frac{bg/m}{(1+br)} = \frac{ch/n}{(1+cr)} = k$$

$\frac{af}{l} = k(1+ar) \rightarrow$ multiply $(b-c)$ both sides

$\frac{bg}{m} = k(1+br) \rightarrow$ multiply $(c-a)$ both sides

$\frac{ch}{n} = k(1+cr) \rightarrow$ multiply $(a-b)$ both sides

$$\frac{af}{l}(b-c) + \frac{bg}{m}(c-a) + \frac{ch}{n}(a-b) = k(1+ar)(b-c) + k(1+br)(c-a) + k(1+cr)(a-b)$$

$$= k[\cancel{b} - \cancel{c} + a\cancel{b}r - \cancel{a}cr + \cancel{c} - \cancel{a} + c\cancel{b}r - b\cancel{a}r + \cancel{a} - \cancel{b} + a\cancel{c}r - \cancel{cbr}]$$

$$= k(0)$$

$$= 0.$$

$$\frac{af}{l}(b-c) + \frac{bg}{m}(c-a) + \frac{ch}{n}(a-b) = 0$$

Eliminating (l, m, n) from above equation

$$\Rightarrow \frac{af(b-c)}{(x-f)/\cancel{r}} + \frac{bg(c-a)}{(y-g)/\cancel{r}} + \frac{ch(a-b)}{(z-h)/\cancel{r}} = 0$$

$$\Rightarrow \frac{af(b-c)}{(x-f)} + \frac{bg(c-a)}{(y-g)} + \frac{ch(a-b)}{(z-h)} = 0.$$

$\rightarrow$ Quadric Cone.

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Enveloping Cylinder

Theorem: To find the eqⁿ of enveloping cylinder of the conicoid $ax^2+by^2+cz^2=1$ with its generators parallel to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

sol: let (α,β,γ) ~~on~~ be a point on the required ~~plane~~ cylinder.

Eqⁿ of line is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$

$(lr+\alpha, mr+\beta, nr+\gamma)$

Substitute in conicoid

$$a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2=1$$

$$\Rightarrow a(l^2r^2+\alpha^2+2rl\alpha)+b(m^2r^2+\beta^2+2rm\beta)+c(n^2r^2+\gamma^2+2rn\gamma)-1=0$$

$$\Rightarrow r^2(al^2+bm^2+cn^2)+2r(al\alpha+bm\beta+cn\gamma)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

$$\boxed{b^2-4ac=0.}$$

$$4(al\alpha+bm\beta+cn\gamma)^2-4(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

$\Rightarrow$ Eliminate $(\alpha,\beta,\gamma) \to (x,y,z)$

Eqⁿ of enveloping cylinder is

$$(alx+bmy+cnz)^2-4(al^2+bm^2+cn^2)(ax^2+by^2+cz^2-1)=0.$$

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Enveloping Cone

Theorem: To find the enveloping cone of $ax^2+by^2+cz^2=1$ whose vertex is (α,β,γ).

Proof: Eqⁿ of the line through (α,β,γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \quad \text{—(1)}$$

Any point $=(lr+\alpha,\ mr+\beta,\ nr+\gamma)$

substituting in $ax^2+by^2+cz^2=1$

we get,

$$r^2(al^2+bm^2+cn^2)+2r(al\alpha+bm\beta+cn\gamma)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

$$\boxed{b^2-4ac=0}$$

$$\Rightarrow 4(al\alpha+bm\beta+cn\gamma)^2-4(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)=0$$

$$\Rightarrow (al\alpha+bm\beta+cn\gamma)^2-(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)=0$$

Eliminating (l,m,n) using (1)

$$\Rightarrow \left[a\left(\frac{x-\alpha}{r}\right)\alpha+b\left(\frac{y-\beta}{r}\right)\beta+c\left(\frac{z-\gamma}{r}\right)\gamma\right]^2-\left[a\left(\frac{x-\alpha}{r}\right)^2+b\left(\frac{y-\beta}{r}\right)^2+c\left(\frac{z-\gamma}{r}\right)^2\right](a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

$$\Rightarrow [a(x-\alpha)\alpha+b(y-\beta)\beta+c(z-\gamma)\gamma]^2-[a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2](a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

$$\Rightarrow [(a\alpha x+b\beta y+c\gamma z)-(a\alpha^2+b\beta^2+c\gamma^2)]^2-[(ax^2+by^2+cz^2)+(a\alpha^2+b\beta^2+c\gamma^2)-2(a\alpha x+b\beta y+c\gamma z)](a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

— (I)

Let.

$S=ax^2+by^2+cz^2-1$, $S_1=a\alpha^2+b\beta^2+c\gamma^2-1$, $T=a\alpha x+b\beta y+c\gamma z-1$.

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$$\text{(I)} \Rightarrow [(a\alpha x+b\beta y+c\gamma z-1)-(a\alpha^2+b\beta^2+c\gamma^2-1)]^2-[(ax^2+by^2+cz^2-1)+(a\alpha^2+b\beta^2+c\gamma^2-1)-2(a\alpha x+b\beta y+c\gamma z-1)](a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

$$\Rightarrow [T-S_1]^2-[S+S_1-2T](S_1)=0$$

$$\Rightarrow T^2+\cancel{S_1^2}-\cancel{2TS_1}-S_1S-\cancel{S_1^2}+\cancel{2TS_1}=0$$

$$\Rightarrow \quad T^2-SS_1=0.$$

$$\therefore \boxed{T^2=SS_1}$$

PCP-27

If the section of Enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose vertex is P by the plane $z=0$ is a rectangular Hyperbola.

Sol. Given Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0.$

Eqn of enveloping cone is $T^2=SS_1$

$$\Rightarrow \left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)$$

put $z=0$.

$$\Rightarrow \left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=0.$$

$$\Rightarrow \cancel{\frac{\alpha^2x^2}{a^4}}+\cancel{\frac{\beta^2y^2}{b^4}}+1+\frac{2\alpha\beta xy}{a^2b^2}-\frac{2\beta y}{b^2}-\frac{2\alpha x}{a^2}=\cancel{\frac{\alpha^2x^2}{a^4}}-\frac{\beta^2x^2}{a^2b^2}-\frac{\gamma^2x^2}{a^2c^2}+\frac{x^2}{a^2}$$
$$-\frac{\alpha^2y^2}{a^2b^2}-\cancel{\frac{\beta^2y^2}{b^4}}-\frac{\gamma^2y^2}{b^2c^2}+\frac{y^2}{b^2}+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1=0.$$

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For rectangular Hyperbola $\Rightarrow$ co-eff of x^2 + co-eff of $y^2 = 0$.

$$\left(\frac{-\beta^2}{a^2b^2} - \frac{\gamma^2}{a^2c^2} + \frac{1}{a^2}\right) + \left(-\frac{\alpha^2}{a^2b^2} - \frac{\gamma^2}{b^2c^2} + \frac{1}{b^2}\right) = 0$$

$$\Rightarrow \frac{\beta^2}{a^2b^2} + \frac{\gamma^2}{a^2c^2} - \frac{1}{a^2} + \frac{\alpha^2}{a^2b^2} + \frac{\gamma^2}{b^2c^2} - \frac{1}{b^2} = 0.$$

Locus of (α, β, γ) is

$$\Rightarrow \frac{y^2}{a^2b^2} + \frac{z^2}{a^2c^2} - \frac{1}{a^2} + \frac{x^2}{a^2b^2} + \frac{z^2}{b^2c^2} - \frac{1}{b^2}$$

$$\Rightarrow \frac{1}{a^2b^2}(x^2+y^2) + \frac{z^2}{c^2}\left[\frac{1}{a^2} + \frac{1}{b^2}\right] = \frac{1}{a^2} + \frac{1}{b^2}$$

$$\Rightarrow \frac{1}{a^2b^2}(x^2+y^2) + \frac{z^2}{c^2}\left[\frac{b^2+a^2}{a^2b^2}\right] = \frac{a^2+b^2}{a^2b^2}$$

$$\Rightarrow x^2+y^2 + \frac{z^2}{c^2}(b^2+a^2) = a^2+b^2$$

$$\Rightarrow \frac{x^2+y^2}{a^2+b^2} + \frac{z^2}{c^2}\frac{\cancel{(a^2+b^2)}}{\cancel{(a^2+b^2)}} = 1$$

$$\Rightarrow \frac{x^2+y^2}{a^2+b^2} + \frac{z^2}{c^2} = 1.$$

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PCP-28

Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the conicoid $ax^2+by^2+cz^2=1$.

Sol. Let $P(\alpha,\beta,\gamma)$ be the locus of point

Eqn of enveloping cone with 'P' as vertex to $ax^2+by^2+cz^2=1$

$$T^2=SS_1$$

$$\Rightarrow \boxed{T^2-SS_1=0}$$

$$\Rightarrow (\underbrace{a\alpha x+b\beta y}_{A}+\underbrace{c\gamma z-1}_{B})^2-(ax^2+by^2+cz^2-1)(a\alpha^2+b\beta^2+c\gamma^2-1)$$

$$\Rightarrow (a\alpha x+b\beta y)^2+(c\gamma z-1)^2+2(a\alpha x+b\beta y)(c\gamma z-1)-a^2\alpha^2x^2-ab\beta^2x^2$$
$$-ac\gamma^2x^2+ax^2-ab\alpha^2y^2-b^2\beta^2y^2-bc\gamma^2y^2+by^2-ac\alpha^2z^2-bc\beta^2z^2$$
$$-c^2\gamma^2z^2+cz^2+a\alpha^2+b\beta^2+c\gamma^2-1=0.$$

$$\Rightarrow a^2\cancel{\alpha^2x^2}+b^2\cancel{\beta^2y^2}+2ab\alpha\beta xy+c^2\cancel{\gamma^2}z^2\cancel{+1}-2c\gamma z+2ac\alpha\gamma zx$$
$$-2a\alpha x+2bc\beta\gamma yz-2b\beta y-a^2\cancel{\alpha^2}x^2-ab\beta^2x^2-ac\gamma^2x^2+ax^2-$$
$$ab\alpha^2y^2-b^2\cancel{\beta^2}y^2-bc\gamma^2y^2+by^2-ac\alpha^2z^2-bc\beta^2z^2-c^2\cancel{\gamma^2}z^2+cz^2$$
$$+a\alpha^2+b\beta^2+c\gamma^2-\cancel{1}=0.$$

co-eff of x^2 + co-eff of y^2 + co-eff of $z^2=0$

$$\Rightarrow -ab\beta^2-ac\gamma^2+a-ab\alpha^2-bc\gamma^2+b-ac\alpha^2-bc\beta^2+c=0.$$

$$\Rightarrow ab\beta^2+ac\gamma^2-a+ab\alpha^2+bc\gamma^2-b+ac\alpha^2+bc\beta^2-c=0.$$

Locus is

$$\Rightarrow aby^2+acz^2-a+abx^2+bcz^2-b+acx^2+bcy^2-c=0$$

$$\Rightarrow a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=(a+b+c)$$

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System of Parallel Chords drawn to $ax^2+by^2+cz^2=1$ with (α,β,γ) as midpoint

To find the locus of system of parallel chords with (α,β,γ) as midpoint

Let (α,β,γ) be the midpoint of the chord

Eqn of chord is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$$

point $\rightarrow (lr+\alpha,\ mr+\beta,\ nr+\gamma)$

substitute in $ax^2+by^2+cz^2=1$

we get

$$r^2(al^2+bm^2+cn^2)+2r(al\alpha+b\beta m+cn\gamma)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

~~Roi~~ we get r_1, r_2 as roots

points of intersection are

$(lr_1+\alpha,\ mr_1+\beta,\ nr_1+\gamma)$

and

$(lr_2+\alpha,\ mr_2+\beta,\ nr_2+\gamma)$

Mid point of the line is $\left(\frac{lr_1+\alpha+lr_2+\alpha}{2},\ \frac{mr_1+\beta+mr_2+\beta}{2},\ \frac{nr_1+\gamma+nr_2+\gamma}{2}\right)$

$=(\alpha,\beta,\gamma).$

(α,β,γ) is the midpoint if $r_1+r_2=0.$

Sum of roots $=0.$

$$\Rightarrow \frac{-2(al\alpha+bm\beta+cn\gamma)}{al^2+bm^2+cn^2}=0$$

$\Rightarrow$ ~~al\alpha+bm~~ $al\alpha+bm\beta+cn\gamma=0$

locus is $\Rightarrow alx+bmy+cnz=0.$

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PCP-29

P(1,3,2) is a point on the conicoid, $x^2-2y^2+3z^2+5=0$. Find the locus of the mid points of chords drawn parallel to OP.

sol. Let

$$alx+bmy+cnz=1$$

given

$$x^2-2y^2+3z^2+5=0$$

$$\Rightarrow x^2-2y^2+3z^2=-5$$

$$\Rightarrow -\frac{1}{5}x^2+\frac{2}{5}y^2-\frac{3}{5}z^2=1$$

given point is P(1,3,2)

Required locus is $alx+bmy+cnz=0$

$$= -\frac{1}{5}(1)x+\frac{2}{5}(3)y-\frac{3}{5}(2)z=0$$

$$\Rightarrow -x+6y-6z=0$$

$$\Rightarrow \boxed{x-6y+6z=0}$$

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