Vector Space

Vector Space- A non-empty set with one or more binary operations.

Two binary operations

1) **Internal Composition** (operation) → only from one set
i.e variables are from the same set $A(a,b) \to a*b$

2) **External Composition**

Variables are taken from two different sets $\begin{matrix} A & & B \\ \downarrow & & \downarrow \\ a & * & b \end{matrix}$

→ In Vector space we have two sets

1) Set of vectors → denoted by x, y, z

2) Set of scalars → denoted by a, b, c, d

→ **Binary Operation**: 'G' is a non-empty set '*' is said to be a binary operation on the set 'G' if $a*b \in G$.

Algebraic Structures: A non-empty set 'G' with one or more binary operations are called an algebraic structures.

Ex: $(G, +)$ and $(R, +, \cdot)$

Vector Space: A vector space is a non-empty set '~~G~~ V' with ~~one or more binary operations are called~~ and objects called vectors on which we define two operations called vector addition and scalar multiplication which satisfy the following properties.

$\forall$ u,v,w $\in$ vector and c,d $\in$ Scalar

Properties

1) (V,+) is an abelian group

(i) Closure Property: $\forall$ u,v $\in$ vector

$\Rightarrow$ u+v $\in$ vector

$\Rightarrow$ vector is closed w.r.t addition

(ii) Associative Property: $\forall$ u,v,w $\in$ vector

$\Rightarrow (u+v)+w = u+(v+w)$

$\therefore$ It satisfies associative property

(iii) Identity Property: $\forall$ u $\in$ vector there is a '0' vector $\bar{0}$ $\in$ vector

such that $u+\bar{0} = u = \bar{0}+u$

(iv) Inverse property:

$\forall u \in V \; \exists -u \in V$ such that $\Rightarrow u+(-u) = \bar{0} = (-u)+u$

(v) Commutative Property:

$\forall$ u,v $\in$ V such that $u+v = v+u$

$\therefore$ V is an abelian group w.r.t addition

$\rightarrow$ V is a semigroup w.r.t scalar multiplication

(a) Closure Property: u $\in$ vector, c $\in$ scalar

$\rightarrow$ c.u $\in$ vector

(b) Associative Property: c,d $\in$ Scalar, u $\in$ vector such that

$(c \cdot d)u = c(d \cdot u)$

(c) Distributive Property:

1) $c(u+v) = cu+cv$, c $\in$ S; u,v $\in$ vector

2) $(c+d)u = c \cdot u + du$, c,d $\in$ S, $\forall$ u $\in$ vector

3) 1 $\in$ scalar, u $\in$ vector $1 \cdot u = u$

Ex: For $n \geq 0$ the set P_n of polynomials of ~~deg~~ degree atmost n consists of all polynomials of the form.

$p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$ where the co-efficients $a_0, a_1, \dots a_n$ and the variable t are the real numbers.

Show ~~that~~ that P_n is a vector space.

Sol. To show that $(P_n, +, \cdot)$ is a vector space it is enough to show that (1) Closure Property w.r.t addition i.e, $p+q \in P_n$

(2) Closure w.r.t scalar multiplication.

i.e, $cp \in P_n$

(3) Identity axiom w.r.t '+'

(4) Inverse axiom w.r.t. '+'.

(1) Closure w.r.t addition

let $p, q \in P_n$

$p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$

$q(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_n t^n$.

$p+q = p(t) + q(t) = (a_0+b_0) + (a_1+b_1)t + (a_2+b_2)t^2 + \dots + (a_n+b_n)t^n \in P_n$

Since $p+q$ is a polynomial of degree n.

$\therefore \forall p, q \in P_n$

$\Rightarrow p+q \in P_n$

Hence, the set P_n satisfies the ~~co~~ closure property w.r.t vector t.

(2) Closure w.r.t scalar multiplication

let $c \in$ Scalar

& $p(t) \in P_n$

then $c\,p(t) = c(a_0 + a_1 t + \dots + a_n t^n)$

$= a_0 c + (a_1 c)t + \dots + (c a_n)t^n \in P_n$

$\therefore$ since cp is a polynomial of degree n

$\therefore P_n$ satisfies closure w.r.t $*$.

(3) Identity w.r.t '+'

Zero polynomial is a zero vector

i.e, $\forall\, p(t) \in P_n \;\exists\, \bar{0} \in P_n$ i.e, $\bar{0} = 0 + 0t + 0t^2 + \dots + 0.t^n \in P_n$

S/T $p(t) + \bar{0} = (a_0 + 0) + (a_1 + 0)t + \dots + (a_n + 0)t^n = p(t)$

$\therefore P(t) + \bar{0} = P(t)$

$\Rightarrow \bar{0}$ is the identity w.r.t '+'.

(4) Inverse w.r.t '+':

$\forall\, p(t) \in P_n \;\exists\, (-1)\,p(t) \in P_n$

i.e, $(-1)P(t) = -a_0 - a_1 t - \dots - a_0 t^n \in P_n$

S/T $P(t) + (-1)\,P(t) = \bar{0}$

$\therefore (-1).p(t)$ is the additive inverse.

All the remaining properties of the vector space are following the above properties.

$\therefore$ The polynomial P_n is a vector space.

* Vectors in R^n denotes the collection of ordered n tuples written as $n \times 1$ coloumn matrix such as

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$ is a vector space.

Sol: To S/T R^n is a vector space

It is enough to show that (1) Closure w.r.t vector addition
(2) Closure w.r.t Scalar *.
(3) Identity w.r.t '+'
(4) Inverse w.r.t '+'.

(1) Closure w.r.t vector addition

let $U, V \in R^n$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$U+V = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} \in R^n$$ since $U+V$ is a tuple.

$\therefore \forall U, V \in R^n \rightarrow U+V \in R^n$ it holds good.

(2) Closure w.r.t scalar

let $c \in$ Scalar

$U \in R^n$

then $cU = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} \in R^n$ since cU is a tuple.

$\therefore R^n$ satisfies w.r.t scalar.

(3) Identity w.r.t '+'

Zero tuple is a zero vector

i.e, $U \in R^n$, $\bar{0} \in R^n$

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i.e, $\bar{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$U+\bar{0} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in R^n = U$$

$\therefore \bar{0}$ is the identity w.r.t '+'.

(4) $\forall U \in R^n \; \exists \; (-1)U \in R^n$

i.e, $(-1)U = -1\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$

S/T $U+(-1)U = \bar{0}$

$\therefore (-1)U$ is the additive inverse

All the remaining properties of a vector space are followed from the above properties.

$\therefore$ The tuple is a vector space.

Sub-space

A non-empty subset 'H' of a vector space U is said to be a subspace if 'H' itself is a vector space.

(or)

A subspace of a vector space V is a subset H of V that has 3 properties

(1) The zero vector of V is in H

(2) H is closed under vector addition

$\forall U, V \in H \rightarrow U+V \in H$.

(3) H is closed under scalar multiplication.

$\forall U \in H \; \exists \; c \in$ Scalar $\ni cU \in H$

Example: Every vector space is a subspace of itself

- The set consisting of zero vector is a subspace of V called zero subspace i.e, $H = \{\bar{0}\}$ is a subspace

- The set $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} ; s, t \text{ are real numbers} \right\}$ is a subset of R^3. show that H is a subspace of R^3?

Sol To prove H is a subspace of R^3 we have to prove the following properties.

(i) $\bar{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$

(ii) Closure w.r.t addition

i.e $\forall\, U, V \in H \Rightarrow U + V \in H$

$U = \begin{bmatrix} s_1 \\ t_1 \\ 0 \end{bmatrix}, V = \begin{bmatrix} s_2 \\ t_2 \\ 0 \end{bmatrix} \Rightarrow$ where $s_1, s_2, t_1, t_2 \in R$

$U + V = \begin{bmatrix} s_1 + s_2 \\ t_1 + t_2 \\ 0 \end{bmatrix} \in H$

$\because s_1 + s_2, t_1 + t_2 \in R$

(iii) Closure w.r.t '$*$'

$\forall\, U \in H$ and $c \in$ Scalar

$cU = c\begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} cs \\ ct \\ 0 \end{bmatrix} \in R.$

$\therefore Uc \in H.$

Linear Combination

Let V is a vector space if $v_1, v_2, \ldots, v_n \in V$ then any vector y defined as $y = c_1v_1 + c_2v_2 + \ldots + c_nv_n$, where $c_1, c_2, \ldots, c_n$ are ~~scalars~~ scalars is called linear combination of $\{v_1, v_2, \ldots, v_n\}$ and $c_1, c_2, \ldots, c_n$ are called weights.

i.e, ~~some~~ sum of scalar multiples of vectors is called linear combination.

Ex: $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ then v is a linear combination of a_1 and a_2 since $b = 3a_1 + 2a_2$

Linear Span:

Let $v_1, v_2, \ldots, v_n$ are the vectors, linear span of $v_1, v_2, \ldots, v_n$ is the set of all possible linear combinations of $v_1, v_2, \ldots, v_n$.

→ If $v_1, v_2, \ldots, v_n$ are vectors in a vector space V, then the set of linear combinations of $v_1, v_2, \ldots, v_n$ is denoted by span $\{v_1, v_2, \ldots, v_n\}$ and is called subset of V generated by $v_1, v_2, \ldots, v_n$ i.e, span $\{v_1, v_2, \ldots, v_n\}$ is the collection of all vectors that can be written in the form $c_1v_1 + c_2v_2 + \ldots + c_nv_n$

Theorem: Given v_1 and v_2 in a vector space V, let $H = \text{span}\{v_1, v_2\}$

Show that H is a subspace of V.

Proof: The zero vector is in H, since $0 = 0v_1 + 0v_2$.

To S/T H is closed under vector addition, take two arbitary vectors in H, say, [$u, w \in H$].

$u = s_1 v_1 + s_2 v_2$ and $w = t_1 v_1 + t_2 v_2$.

By axioms 2,3 and 8 for the vector space V

(under '+')

$$u + w = (s_1 v_1 + s_2 v_2) + (t_1 v_1 + t_2 v_2)$$
$$= (s_1 + t_1) v_1 + (s_2 + t_2) v_2 \in H$$

(under '*').

$$ku = k[s_1 v_1 + s_2 v_2]$$
$$= k(s_1 v_1) + k(s_2 v_2)$$
$$= (ks_1) v_1 + (ks_2) v_2 \in H.$$

$\therefore H$ is a subspace of V.

PCP-1

Let H is the set of all vectors of the form $H = \begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix}$. Find the vector v in R^3 such that H is the span of $\{v\}$. Why this Show that H is the subspace of R^3.

Proof: Given $H = \begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix}$

$$= t \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \qquad \because v = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$$

$$= t\,v.$$

$\therefore H = \text{span}\{v\}$ because t is scalar and V is vector.

From the previous theorem,

H is a subspace of R^3.

[Every linear span is a vector sub.space]

PCP-2

V is in the first quadrant in the XY plane i.e

$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; x \geq 0, y \geq 0 \right\}$ then Prove that

a) If u & v $\in$ V then P/T u+v $\in$ V.

b) c·u is not in V. i.e, cu $\notin$ V. [u$\in$V]

sol. (a) $u, v \in V \Rightarrow u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} ; x_1 \geq 0, y_1 \geq 0$

and

$$v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} ; x_2 \geq 0, y_2 \geq 0$$

$$\therefore \quad u+v = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \in V$$

Because if $x_1, x_2 \geq 0 \Rightarrow x_1 + x_2 \geq 0$

$y_1, y_2 \geq 0 \rightarrow y_1 + y_2 \geq 0$

(b) $u \in V$

$$u = \begin{bmatrix} x \\ y \end{bmatrix} ; x \geq 0, y \geq 0$$

let $c = -1$, then

$$\Rightarrow cu = -1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} ; -x \leq 0, -y \leq 0.$$

It is not satisfying required condition

$\therefore$ c·u $\notin$ V.

1.) Let w be the union of the first & Third quadrants in XY Plane i.e $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; x, y \geq 0 \right\}$. Then P/T

(a) If $u \in W$ and c is any scalar is $c \cdot u \in W$

(b) Find specific vector u & v in $W \ni u+v \notin W$.

sol. $u \in W$

(a)

$$u = \begin{bmatrix} x \\ y \end{bmatrix} ; x, y \geq 0$$

let $c = a$, then [where $a \geq 0$]

$$cu = a \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix} ; ax \geq 0, ay \geq 0.$$ For any scalar c, $c \cdot u \Rightarrow (cx)(cy) = c^2(xy) \geq 0$.

$\therefore cu \in W$.

(b) $u, v \in W \Rightarrow u = \begin{bmatrix} -x_1 \\ -y_1 \end{bmatrix}$; $-x_1 \leq 0, -y_1 \leq 0$.

$$V = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} ; x_2 \geq 0, y_2 \geq 0.$$

$$u + v = \begin{bmatrix} -x_1 \\ -y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -y_1 + y_2 \end{bmatrix} = -1 \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}.$$

It does not satisfy the condition because. $x_1 - x_2, y_1 - y_2 \leq 0$.

$\therefore u+v \notin W$.

2) Let W be the set of all vectors of the form $\begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix}$

where b and c are arbitary.

Find vectors u & $v \ni W = \text{span}\{u,v\}$.

W is subspace of R^3.

Sol. Given

$$W = \begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix} \Rightarrow b\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

let $u = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$

$\therefore$ $W = \text{span}\{u,v\} = bu + cv.$

W is a span $\{u,v\}$ because b,c are scalars and u,v are vectors.

By theorem,

Every linear span is a vector sub.space.

$\therefore$ W is a subspace of R^3.

3.) $W = \begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix}$ find u & $v \ni W = \text{span}\{u,v\}$ and W is subspace of R^3.

Sol. Given

$$W = \begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix} \Rightarrow s\begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ -3 \\ 5 \end{bmatrix}$$

let $u = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 5 \end{bmatrix}$

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$\therefore$ $W = \text{span}\{u, v\} = su + tv$

W is a span of $\{u, v\}$ because s, t are scalars and u, v are vectors.

By theorem,

Every linear span is a vector subspace.

$\therefore$ W is subspace of R^3.

Imp

Theorem: Let H and K are two subspaces of V. Prove that $H \cap K$ is also a subspace of V.

Proof: Given H and K are subspaces of V

(i) $\Rightarrow H \subseteq V$ and $K \subseteq V \Rightarrow H \cap K \subseteq V$

(ii) We have to P/T $\bar{0} \in H \cap K$.:

$\because$ $\bar{0} \in H$ and $\bar{0} \in K \Rightarrow \bar{0} \in H \cap K$

(iii) If $u, v \in H \cap K$ we have to P/T $u + v \in H \cap K$.:

$\because$ $u, v \in H$ and $u, v \in K$

If H and K are subspaces they satisfy closure property w.r.t addition.

$u, v \in H \Rightarrow u + v \in H$

$u, v \in K \Rightarrow u + v \in K$

$\therefore$ $u + v \in H \cap K$.

$\therefore$ Closure Property is true in '+'.

Note: Union of Two Subspaces need not to be a subspace

(iv) We have to P/T if c is a scalar, $u \in H \cap K$ then

prove $cu \in H \cap K$.

given $u \in H \cap K$

$u \in H, u \in K$.

$\Rightarrow$ $cu \in H$ and $cu \in K$

$\Rightarrow$ $cu \in H \cap K$.

Theorem: Let $W = \text{Span}\{V_1, V_2, \ldots, V_n\}$ and $V_1, V_2, \ldots V_n$ are v. S/T

$V_k \in W$; $1 \le k \le n$.

Proof: Required to prove $V_1 \in W$

$V_1 = 1 \cdot V_1 + 0 \cdot V_2 + \ldots + 0 \cdot V_n$ (∵ linear comb is $c_1V_1 + c_2V_2 + \ldots + c_nV_n$)

$\Rightarrow$ V_1 = linear comb of $\{V_1, V_2, \ldots n\}$

$\Rightarrow$ $V_1 \in W$

Ily $V_2 = 0 \cdot V_1 + 1 \cdot V_2 + \ldots + 0 \cdot V_n$

V_2 = L.C of $\{V_1, V_2, \ldots V_n\}$

$\Rightarrow$ $V_2 \in W$

In general,

$V_n = 0 \cdot V_1 + 0 \cdot V_2 + \ldots + 0 \cdot V_{n-1} + 1 \cdot V_n$

V_n = Linear comb of $\{V_1, \ldots V_n\}$

$\Rightarrow$ $V_n \in W$

∴ $V_k \in W$ for $1 \le k \le n$.

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Theorem: H & K are subspaces of a vector space V. The sum of H & K is denoted by H+K & it is the set of all vectors V that can be written as sum of two vectors, one in H & other in K i.e,

$H+K = \{W ; W = U+V \ni U \in H \text{ \& } V \in K\}$

S/T a) H+K is a subspace of V

b) H is a subspace of H+K and K is also a subspace of H+K.

Proof: a) (1) To P/T $H+K \subseteq V$

clearly $H+K = W$, where $W \in V$

$\therefore H+K \subseteq V$.

(2) $\bar{0} \in H+K$

$\bar{0} = \bar{0} + \bar{0}$ where $\bar{0} \in H$ & $\bar{0} \in K$

$\Rightarrow \bar{0} \in H+K$

(3) S/T H+K satisfies closure w.r.t '+'

$u, v \in H+K \Rightarrow u+v \in H+K$

$\Rightarrow u = u_1 + v_1$ where $u_1 \in H$ & $v_1 \in K$

$v = u_2 + v_2$ where $u_2 \in H$ & $v_2 \in K$

$u+v = (u_1+v_1) + (u_2+v_2)$

$= (u_1+u_2) + (v_1+v_2)$ ($\because$ from commutative and associative property).

$\Rightarrow u+v \in H+K$

(4) Closure w.r.t '*'

$u \in H+K$, c is scalar then P/T $cu \in H+K$

$u = u_1 + v_1$; $u_1 \in H$ & $v_1 \in K$

$cu = c(u_1+v_1)$ $\because cu_1 \in H$, $cv_1 \in K$

$= \cancel{c u} cu_1 + cv_1$

$\Rightarrow cu \in H+K$

(a) Hence $H+K$ is a subspace of V.

(b) R.T.P $H \subseteq H+K$

$\rightarrow u \in H$

$u = u + \bar{0}$ where $u \in H$ & $\bar{0} \in K$

$\Rightarrow u \in H+K$

$\therefore H \subseteq H+K$

$\therefore H$ is subspace of $V \Rightarrow \bar{0} \in H, u+v \in H$ & $cu \in H$.

$\therefore H$ is subspace of $H+K$.

R.T.P $K \subseteq H+K$

$\rightarrow v \in K$

$v = v + \bar{0}$ where $v \in H$ & $\bar{0} \in K$

$\Rightarrow v \in H+K$

$\therefore K \subseteq H+K$

$\therefore K$ is subspace of $H+K$.

Theorem: $u_1, u_2, \dots u_p$ & $v_1, v_2, \dots v_p$ are vectors in a vector space 'V' and $H = \text{span}\{u_1, u_2, \dots u_p\}$ and $K = \text{Span}\{v_1, v_2, \dots v_p\}$ S/T $H+K = \text{Span}\{u_1, u_2 \dots u_p, v_1, \dots v_p\}$.

Proof: Given $H = \text{span}\{u_1, u_2, \dots u_p\}$

let $u \in H$; i.e, $u = c_1u_1 + c_2u_2 + \dots + c_pu_p$

$v \in K$; i.e $v = d_1v_1 + d_2v_2 + \dots + d_pv_p$

$u+v \in H+K \Rightarrow u+v = c_1u_1 + \dots + c_pv_p + d_1v_1 + \dots + d_pv_p$

$= \text{L.C of } \{v_1, \dots v_p, u_1, \dots u_p\}$

$\therefore H+K = \text{span}\{u_1, u_2, \dots u_p, v_1, v_2, \dots v_p\}$

1) Determine if the given set is a subspace of P_n.

All polynomials of the form $P(t) = at^2$ where $a \in R$.

sol) $H = \{P(t) / P(t) = at^2, a \in R\}$

we have to prove H is a subspace

1. clearly $H \subseteq P_n$ $\qquad at^2 \in H$

2. $\bar{0} \in H$

$\bar{0} = 0t^2 ; 0 \in R$

3. To P/T $p(t) . q(t) \in H$

$p(t) \in H \Rightarrow P(t) = at^2, a \in R$

$q(t) \in H \Rightarrow q(t) = bt^2, b \in R$

$\Rightarrow P(t) + q(t) = at^2 + bt^2$

$= (a+b) t^2$ ($\because$ distributive law)

$P(t) + q(t) \in H$

4. Let 'c' is scalar, $P(t)$ is element of H we have to prove

$c \cdot p(t) \in H$. $\; c \cdot p(t) = c(at)^2 = (ca)t^2 \in$ $[\because ca \in R]$

$\Rightarrow cp(t) \in H$

$\therefore H$ is a subspace of P_n.

Null Space:

A is $m\times n$ matrix and the null space of A is denoted by Null A it is the set of all solutions of the homogenous eq^n $AX=\bar{0}$ where $X \in R^n$.

$$\text{Null } A = \left\{ X \;\middle/\; \begin{array}{l} AX=\bar{0} \\ X\in R^n \end{array} \right\}$$

1. If $A=\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ & $u=\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if $u \in$ Null A.

sol. If $AU=\bar{0}$ then $u \in$ Null A

consider, $Au=\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \bar{0}$$

Hence $Au=\bar{0} \Rightarrow u \in$ Null A.

Theorum: The Null space of $m\times n$ matrix A is a subspace of R^n (or) The set of all solutions of the homogenous eq^{n}s is a subspace of R^n.

Proof: To P/T Null A is subspace of R^n.

i.e,
1. Null $A \subseteq R^n$
2. $\bar{0} \in$ Null A
3. $u,v \in$ Null A $\ni u+v \in$ Null A
4. $u \in$ Null A $\ni cu \in$ Null A

1. Null $A \subseteq R^n$ because $X \in R^n$.

2. $A\cdot\bar{0} = \bar{0}$, $\bar{0} \in R^n$
$\Rightarrow \bar{0} \in$ Null A

3. $u,v \in$ Null A $\Rightarrow Au=\bar{0}$, $Av=\bar{0}$ [$\because$ Dist law]

consider $A(u+v) = Au+Av = \bar{0}+\bar{0} = \bar{0}$

$\therefore A(u+v) = \bar{0}$

$\Rightarrow u+v \in \text{Null } A$

4. c is scalar, $u \in \text{Null } A \Rightarrow Au = \bar{0}$

$A(cu) = c(Au) = c.\bar{0} = \bar{0}$

$\Rightarrow cu \in \text{Null } A$

$\therefore$ Every Null space of A is a subspace of R^n

• H is the set of all vectors in R^4 whose co-ordinates a,b,c,d satisfies the eq^n $a-2b+5c=d$ and $c-a=b$. Show that H is a subspace of R^4.

sol. Given $a-2b+5c-d=0$

$-a-b-c+0.d=0$

let $\begin{bmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

It is in the form of $AX = \bar{0}$

$\Rightarrow X \in \text{Null } A$

$\because$ Null A is subspace of R^4

$\therefore$ H is a subspace of R^4.

1. Write the Echelon form and reduced echelon form.

(a) $$\begin{array}{cc} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \end{array}$$

$$R_2 = R_2 + 2R_1 = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} \rightarrow R_3 = R_3 - 3R_1 = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$$

$$R_3 = R_3 + R_2 = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$ $\rightarrow$ Echelon form.

<u>Reduced Echelon form</u>

$$\begin{array}{cc} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ +3 & -6 & -6 & 8 & 2 \end{bmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \end{array}$$

$$R_2 = R_2 + 2R_1 = \begin{matrix} \cancel{1 \quad -2 \quad -1 \quad 3 \quad 0} \\ \cancel{-2 \quad 4 \quad 5 \quad -5 \quad 3} \end{matrix} \quad \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix}$$

$$R_3 = R_3 - 3R_1 = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_2/3 \qquad = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

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(b) $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

$$R_1 \sim R_4 = \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow[R_3 = R_3 + 2R_1]{R_2 = R_2 + R_1} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -12 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

1) Write the Echelon and reduced echelon form

(a) $\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix}$

$R_2 = R_2 + 2R_1 =$
$R_3 = R_3 - 3R_1$
$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix} \rightarrow R_3 = R_3 + R_2 = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

Echelon form

Reduced Echelon form

$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix} \rightarrow R_2 = R_2/3 = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$

$R_3 = 3R_2 + R_3 = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

$R_3 = R_3/5 = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = R_2 = R_2 - R_3 = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$R_1 = R_1 + R_2 = \begin{bmatrix} 1 & -2 & 0 & \frac{10}{3} & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$ reduced echelon form

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2)(b) $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$ $R_1 \sim R_4$ = $\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$

$R_2 = R_2 + R_1$
$R_3 = R_3 + 2R_1$ = $\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -12 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$

$R_1 \sim R_2$ $\begin{bmatrix} -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$ $R_1 \to -R_1$ $\begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & -3 & -6 & 4 & 9 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

$R_3 = R_3 + 2R_1$
$R_4 = R_4 - R_1$ = $\begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & -3 & -6 & 4 & 9 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 2 & 4 & -6 & -6 \end{bmatrix}$ $R_3 \sim R_2$ $\begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \\ 0 & 2 & 4 & -6 & -6 \end{bmatrix}$

$R_3 = R_3 + 3R_2$
$R_4 = R_4 - 2R_2$ = $\begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ → Echelon form.

Reduced Echelon form

$$\begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & -3 & -6 & 4 & 9 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & 2 & 4 & -6 & -6 \end{bmatrix} = R_2 = -\frac{R_2}{3} = \begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & 1 & 2 & -\frac{4}{3} & -3 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 2 & 4 & -6 & -6 \end{bmatrix}$$

$$R_3 = R_3 - R_2 = \begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & 1 & 2 & -4/3 & -3 \\ 0 & 0 & 0 & -5/3 & 0 \\ 0 & 2 & 4 & -6 & -6 \end{bmatrix} \rightarrow R_4 = R_4 - 2R_2 = \begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & 1 & 2 & -4/3 & -3 \\ 0 & 0 & 0 & -5/3 & 0 \\ 0 & 0 & 0 & -10/3 & 0 \end{bmatrix}$$

$$R_3 = -\frac{3}{5} \times R_3 = \begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & 1 & 2 & -4/3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10/3 & 0 \end{bmatrix}$$

$$R_4 = R_4 + \frac{10}{3}R_3 = \begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & 1 & 2 & -4/3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R_2 = R_2 + \frac{4}{3}R_3 = \begin{bmatrix} 1 & 2 & 1 & -3 & -1 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = R_1 + 3R_3 = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = R_1 - 2R_2 = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Reduced Echelon form.}$$

3) Let H be any set of points inside and on the unit circle in the XY plane i.e $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; x^2+y^2 \leq 1 \right\}$. Find a specific example to S/T H is not a subspace of R^2.

Sol. Consider $u = \begin{bmatrix} x \\ y \end{bmatrix}$

let $x = 1, y = 0$

then $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; 1^2+0^2 \leq 1$ holds good

let $2u = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} ; 2^2+0^2 \leq 1$

$4 \nleq 1$

$\therefore$ 2u does not belongs to H and H is not a subspace of R^2.

---

4) Determine if the given set is $H = \{P(t) = a + t^2 / a \in R\}$ is a subspace of P

Sol. Given $P(t) = a + t^2$

Consider $a \cdot P(t) = a(a+t^2)$

let $a = 0$,

$\Rightarrow 0(0+t^2)$

$\Rightarrow 0 + 0t^2 \notin R$

$\therefore$ It does not satisfy scalar multiplication

$\therefore$ H is not a subspace of P.

5. Show that the function

$y(t) = c_1 \cos w(t) + c_2 \sin w(t)$ is a vector space

Sol. Given $y(t) = c_1 \cos w(t) + c_2 \sin w(t)$

consider $r \in R$.

$$r \cdot y(t) = r c_1 \cos w(t) + r c_2 \sin w(t)$$

$$= (r c_1) \cos w(t) + (r c_2) \sin w(t) \in R.$$

$\therefore$ It satisfies scalar multiplication.

let A, B are two elements in same space

$$A = c_1 \cos w(t) + c_2 \sin w(t)$$

$$B = d_1 \cos w(t) + d_2 \sin w(t)$$

$$A+B \in R \Rightarrow (c_1+d_1) \cos w(t) + (c_2+d_2) \sin w(t) \in R$$

$$y(t) \Rightarrow c_1 \cos w(t) + c_2 \sin w(t) \in R$$

$\therefore$ Closure property under addition is satisfied.

let $y(t) = 0 \in R$ and $\{c_1, c_2\} = 0. \in R$

$\therefore y(t) = c_1 \cos w(t) + c_2 \sin w(t) \in R$

$\therefore y(t)$ is a vector space.

Echelon form

The Rectangular matrix is in Echelon form if it satisfies the following three properties.

(1) All non-zero rows are above any rows of all zeroes.

(2) Each leading entry of a row is in a column.

(3) All entries in a column below a leading entry are zeroes.

Reduced Echelon Form

It is an Echelon form which satisfies the following additional condition i.e. the leading entry in each non-zero row is 1.

Pivot Element

A pivot element in a Matrix A is a location and element in A that corresponds to a leading one in the Reduced Echelon form of A.

. A Pivot column is a column of A that contains a pivot position.

Problem: For what value of h will ~~y~~ y be in the subspace of R^3 spanned by v_1, v_2, v_3. if $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

$y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$.

Sol. Given $y = \text{Span}\,[v_1, v_2, v_3]$

$$\Rightarrow y = x_1 v_1 + x_2 v_2 + x_3 v_3$$

$$\Rightarrow \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

The three linear eqns. are

$$x_1 + 5x_2 - 3x_3 = -4$$
$$-x_1 - 4x_2 + x_3 = 3$$
$$-2x_1 - 7x_2 + 0x_3 = h$$

Argumented Matrix is $\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix}$

$R_3 \to R_3 + 2R_1$

$R_2 \to R_2 + R_1$

$\Rightarrow \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} R_3 \to R_3 - 3R_2 \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$

The system is consisted iff $h-5=0$

$\therefore h=5$

$y = \text{span}\{v_1, v_2, v_3\}$ iff $h=5$

Null Space

The null space of $m \times n$ Matrix A is denoted by Null A and it is the set of all solutions of homogeneous equation $Ax = \bar{0}$ i.e. Null $A = \{x : Ax = \bar{0}$ and $x \in R^n\}$

Problems:

If $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and let $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if $u \in$ Null A. If u st satisfies the equation $Au = \bar{0}$ then we say $u \in$ Null A.

Sol. Consider $Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \bar{0}$

$\Rightarrow u \in$ Null A.

Theorem: The null space of an $m \times n$ matrix A is a subspace of R^n?

Proof: Let A is $m \times n$ matrix, we have to P/T

Null $A = \{x : Ax = \bar{0}$ and $x \in R^n\}$ is a subspace of R^n.

Clearly Null A is a subset of R^n (from the definition).

Now, to prove Null A is a subspace of R^n, we have to prove the following three properties.

(1) Identity w.r.t '+'

$\bar{0} \in R^n$, $A\bar{0} = \bar{0}$

$\Rightarrow \bar{0} \in$ Null A

$\bar{0}$ is the additive identity.

(2) Closure w.r.t '+'

Let $u, v \in$ Null A

we have to prove $u+v \in$ Null A

if $u, v \in$ Null $A \Rightarrow Au = \bar{0}$, $Av = \bar{0}$ where $u, v \in R^n$

consider $A(u+v) = Au + Av = \bar{0} + \bar{0}$

$= \bar{0} \Rightarrow u+v \in$ Null A ; $\because u+v \in R^n$

$\therefore$ Null Space of A satisfies closure w.r.t '+'.

(3) Closure w.r.t '$\cdot$'

If c is any ~~set~~ scalar $u \in$ Null A

we have to prove $cu \in$ Null A

if $u \in$ Null $A \Rightarrow Au = \bar{0}$, $u \in R^n$

consider $A(cu) = c(Au)$

$c \cdot \bar{0} = \bar{0}$; $cu \in R^n$

$\Rightarrow cu \in$ Null A

Null A satistisfies closure w.r.t '$\cdot$'

$\therefore$ Null A is subspace of R^n.

* Let H is the set of all vectors in R^4 whose co-ordinates a,b,c,d satisfies the equations $a-2b+5c=d$ and $c-a=b$. Show that H is a subspace of R_4.

Sol. Re-arrange the equations, we get,

$$a-2b+5c-d=0$$
$$-a-b+c+0=0$$

In matrix form the system is written as

$$\begin{bmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$\Rightarrow A x = \bar{0}$

where $A = \begin{bmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$ $\bar{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

H is the set of all solutions of the above system

$\therefore$ H is the subspace of R^4 because w.k.T the nullspace of A is subspace of R^4.

Column Space Of a Matrix

The column space of an $m \times n$ matrix A written as col A is the set of all linear combination of colums of a A i.e, if $A=[a_1, a_2, ---- a_n]$ then column A = span $\{a_1, a_2, ---- a_n\}$.

Problems:

1) Find a matrix A such that $W = \text{col}\, A$ where $W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$

Sol. $W = \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} = a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\therefore W = \text{col}\, A = \text{span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$

Every vector in the spanning set is column of A.

2) If $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$. Find non-zero vector in column in col A & non-zero vector in Null A.

Sol. Any column of A is a vector in column A

$\therefore$ The vector in col A is $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

To find non-zero vector in Null A.

consider the Augumented matrix [A ; 0] and reduce in Echelon form.

$[A:0] = \begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{bmatrix}$

$R_3 \to 2R_3 - 3R_1$

$R_2 = R_2 + R_1$

$\begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 2 & -10 & 9 & 0 \end{bmatrix}$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 17 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 4R_2$, $R_3 \to \frac{R_3}{17}$ $\Rightarrow$ $\begin{bmatrix} 2 & 0 & 18 & 17 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$R_1 \to R_1 - 17R_3 \to \begin{bmatrix} 2 & 0 & 18 & 0 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$R_2 \to R_2 - 4R_3$, $R_1 \to \frac{R_1}{2}$ $\to \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\therefore$ we get $x_1 + 9x_3 = 0$

$-x_2 + 5x_3 = 0$

$x_4 = 1$

Hence, x_3 is a free variable

let $x_3 = 1$

$x_1 = -9$

$x_2 = 5$

$\therefore$ The vector in Null A is $x = \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$ (or) $x = (-9, 5, 1, 0)$

3) $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$

sol: Any column of A is a vector in column A.

Vector column A is $\begin{bmatrix} 3 \\ 6 \\ -8 \end{bmatrix}$

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To find Null A

consider Augumented matrix $\begin{bmatrix} 3 & -5 & -3 & 0 \\ 6 & -2 & 0 & 0 \\ -8 & 4 & 1 & 0 \end{bmatrix}$

$R_3 \to 3R_3 + 8R_1$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 3 & -5 & -3 & 0 \\ 0 & 8 & 6 & 0 \\ 0 & -28 & -21 & 0 \end{bmatrix} \quad R_3 \to \frac{R_3}{7}$$

$$\begin{bmatrix} 3 & -5 & -3 & 0 \\ 0 & 8 & 6 & 0 \\ 0 & 4 & 3 & 0 \end{bmatrix} \quad R_3 \to 2R_3 - R_2 \quad \to \begin{bmatrix} 3 & -5 & -3 & 0 \\ 0 & 8 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 = \frac{R_2}{2} \quad \begin{bmatrix} 3 & -5 & -3 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + R_2 \quad \begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we get $3x_1 - 9x_2 = 0$ let $x_3 = 1$

$4x_2 + 3x_3 = 0$ $\Rightarrow x_2 = -\frac{3}{4}, \; x_1 = \frac{3}{12}$

$$\text{Null } A = \begin{bmatrix} -3/12 \\ -3/4 \\ 1 \end{bmatrix}$$

1) Determine if $w = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in null A where $A = \begin{bmatrix} 3 & -5 & -3 \\ 0 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$

Sol. consider $AW = \begin{bmatrix} 3 & -5 & -3 \\ 0 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$

$$= \begin{bmatrix} 3-15+12 \\ 0-6-0 \\ -8+12-4 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix}$$

$\therefore AW \neq 0$

$\Rightarrow W$ is not in null A

2) Say $w = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}$.

$$AW = \begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \bar{0} \quad AW = 0 \Rightarrow W \in \text{Null } A$$

3) List the vector that span the null space

$$A = \begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Sol.

To find the vector x in Nul A which satisfies the equation $Ax = \bar{0}$. Consider the following linear equations

$$x_1 - 4x_2 + 0x_3 + 2x_4 + 0x_5 = 0 \text{ --- } (1)$$

$$x_3 - 5x_4 = 0 \text{ --- } (2)$$

$$2x_5 = 0 \Rightarrow x_5 = 0$$

let x_4 is free variable, $x_4 = 1$

$x_3 = 5$ and $\cancel{x_2 = 5}$ $x_2 = 1$

$$x_1 - 4 + 2 = 0$$

$$x_1 = 2$$

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Hence the vector in nul A is (2,1,5,1,0)

4) $A = \begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 \to R_1 - 3R_2$

$A = \begin{bmatrix} 1 & 0 & 5 & -6 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The linear eq^{n}s are

$x_1 + 5x_3 - 6x_4 + x_5 = 0$

$x_2 - 3x_3 + x_4 = 0$

let x_3, x_4, x_5 are free variables $= 1$.

$x_1 + 5 - 6 + 1 = 0$

$x_1 = 0$

$x_2 - 3 + 1 = 0$

$x_2 = 2$

Hence the vector x in nullA is (0,2,1,1,1)

Column Space

$\text{Col} A = \{b ; b = Ax ; x \in R^m\}$

Note: The vector belongs to col A if the augmented matrix [A, U] is consistent.

Ex:- $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $U = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

(a) Determine if u is in Null A. Could 'U' be in column A.

(b) Determine 'v' is in col A, could 'v' be in Nul A.

(a) $\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6-8+2+0 \\ -6+10-7+0 \\ 9-14+8+10 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$

$Au \neq \vec{0}$

$u \notin \text{Null } A$.

$u \notin \text{Col } A$, since it has 4 entries and col A is a subspace of $\mathbb{R}^3$

b) consider augumented matrix $[A, v]$

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix}$$

$R_3 \to 2R_3 - 3R_1$, $R_2 \to R_2 + R_1$

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 2 & -10 & 9 & -3 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

$R_1 \to R_1 + 4R_2$

$$\begin{bmatrix} 2 & 0 & 18 & 17 & 11 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 2 & 0 & 18 & 0 & 10 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 9 & 0 & 5 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 9 & 0 & 5 \\ 0 & -1 & 5 & -30 & 0 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

it is clear that the above augumented matrix $[A, V]$ is consistent and therefore, the vector V is in col A.

V has three entries

$V \notin$ Null A

Since, Null A is a subspace of R^4

• $A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & 1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{bmatrix}$ find non-zero vector in Null A and col A

sol: To find non-zero vector in Null A

consider $[A, 0]$

$$\begin{bmatrix} 5 & -2 & 3 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ -5 & 7 & 2 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 5 & -2 & 3 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 5 & 5 & 0 \end{bmatrix} \xrightarrow{R_4 \to \frac{R_4}{5},\ R_3 \to \frac{R_3}{2}} \begin{bmatrix} 5 & -2 & 3 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_3$ $\begin{bmatrix} 5 & -2 & 3 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1 \to R_1 + 5R_2 \to$ $\begin{bmatrix} 0 & -2 & -2 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2 + x_3 = 0$ let $x_3 = 1$

$x_1 + x_3 = 0$ $x_2 = -1$

$x_2 + x_3 = 0$ $x_1 = -1$

Vector in Null A is $x = (-1, -1, 1)$

Difference b/w Null A and col A for $m \times n$ matrix A.

Null A	Col A
→ Null A is a subspace of R^n	→ Col A is a subspace of R^m
→ It takes time to find vector in Null A. Row operations on augumented matrix $[A, 0]$ are required.	→ It is easy to find vector col A. Any column of A is a vector in col A.
→ There is no relation b/w Null A and the entries in A.	→ There is an obvious relation b/w col A and entries in A. Since, each column of A is in col A.
→ A vector v in Null A has the property that $Av = \bar{0}$	→ A typical vector v in col A has the property that the eqn $Ax = v$ is consistent.
→ Given a specific vector v, it is easy to tell if v is in Null A. Just compute Av.	→ Given a specific vector v, it may take time to tell if v is in Col A. Row operations on $[A, v]$ are required.

Linear Transformation

U and V are two vector spaces. T is a mapping from U to V i.e $T: U \to V$. T is said to be linear transformation. if it satisfies the following properties

(1) $T(u+v) = T(u) + T(v)$

2) $T(cu) = cT(u)$

$\forall \; u, v \in U$ and $T(u), T(v) \in V$

Kernel T

Let U and V are two vector spaces. T is a linear transformation from U to V then Kernel T is set of all vectors $u \in U$ st

$T(u) = \bar{0}$ (where $\bar{0}$ is a vector in V)

i.e. Kernel $T = \{u \in U / T(u) = \bar{0}, \bar{0} \in V\}$

It is also called as Null Space.

Range:

Let U and V are two vector spaces T is a linear transformation from U,V then Range of T is the set of all vectors in V of the form

$T(x)$ for some $x \in V$.

i.e. Range $= \{T(x) \in V / x \in U\}$

Thearem: Column Space of $m \times n$ matrix A is a subspace of R^m

Proof: We have to prove column space given by

$col A = \{b : b = Ax, \text{ where } x \in R^n\}$ is a subspace of R^m.

clearly, col A is a subset of R^m, i.e. $col A \subseteq R^m$

To prove it is a subspace of R^m we have to P/T following 3 properties

1. Identity w.r.t addition

we have $\bar{0} = A(\bar{0})$

$\Rightarrow \bar{0} \in col A$

i.e, $\bar{0}$ is additive identity

2. Closure w.r.t addition

let $b_1, b_2 \in col A$

we have to prove $b_1 + b_2 \in col A$

if $b_1, b_2 \in col A$

$\Rightarrow b_1 = Ax \; ; x \in R^n$

$\Rightarrow b_2 = Ay \; ; y \in R^n$

consider $A(x+y) = Ax + Ay = b_1 + b_2$

$\Rightarrow b_1 + b_2 \in \text{col} A$

$\therefore$ col A satisfies ~~closure~~ closure w.r.t addition

3. Closure w.r.t scalar

let c is any scalar, $b \in \text{col} A$ we have to prove $cb \in \text{col} A$

if $b \in \text{col} A \Rightarrow b = Ax ; x \in R^n$

consider $A(cx) = cAx$

$= cb$

$\therefore cb \in \text{col} A$

col A satisfies closure w.r.t scalar $\because$

Hence col A is subspace of R^m.

Theorem: Show that the range of linear transformation is a vector space.

Proof: Let U and V are two vector spaces

T is a linear transformation from U to V.

Range of T is the set of all vectors $v \in V$ which are of the form $T(u)$ where $u \in U$;

i.e, Range $T = \{T(u) = v ; u \in U\}$

Now we have to prove Range of T is vector space

1. Identity $\Rightarrow \bar{0} = T(\bar{0})$

$\bar{0} \in$ Range T

$\therefore \bar{0}$ is additive identity

Problem: Define $T: P_2 \to R^2$ by $T(P) = \begin{bmatrix} P(0) \\ P(1) \end{bmatrix}$ where $P(t) = 3+5t+7t^2$. Show that T is a linear transformation.

Sol. Let p and q are any arbitary polynomial in P_2 and c is any scalar

To prove T is a linear transformation we have to prove

1) $T(p+q) = T(p)+T(q)$

2) $T(cp) = cT(p)$

1) $T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(1) \end{bmatrix} = \begin{bmatrix} p(0)+q(0) \\ p(1)+q(1) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \end{bmatrix} = T(p)+T(q)$

$\therefore T(p+q) = T(p)+T(q)$

2) $T(cp) = \begin{bmatrix} cp(0) \\ cp(1) \end{bmatrix} = c\begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = cT(p)$

$\therefore T(cp) = cT(p)$

Linearly Independent

Let $v_1, v_2, \dots, v_n$ are vectors in V is said to be linearly independent if $c_1v_1+c_2v_2+\dots+c_nv_n = \bar{0}$

$\Rightarrow c_1, c_2, \dots, c_n = 0$

Linearly dependent

The set $v_1, v_2, \dots v_n$ is said to be linearly dependent if

$c_1v_1 + c_2v_2 + \dots + c_nv_n = \bar{0}$

$\Rightarrow$ Some c_i's are non-zeroes.

Ex: $P_1(t) = 1, P_2(t) = t, P_3(t) = 4-t$ then P_1, P_2 and P_3 are linearly dependent because

Ex: Show that the set $\{\sin t, \cos t, \sin 2t\}$ is linearly dependent

$\sin 2t = 2\sin t\cos t$

$\Rightarrow \sin 2t - 2\sin t\cos t = 0$

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Theorem: The set containing a single vector v is linearly independent $v \neq \bar{0}$.

Proof: let $S = \{v\}$

where $v \neq \bar{0}$

we have to S/T S is linearly independent

consider $cv = \bar{0}$

$\Rightarrow c = 0$ because $v \neq \bar{0}$

$\therefore$ The scalar is 0.

$\Rightarrow$ S is linearly independent

Theorem: A set of two vectors is linearly dependent iff one of the vector is a multiple of other.

Proof: let $S = \{v_1, v_2\}$ is the set of two vectors we have to prove S is linearly dependent iff one of the vector is multiple of other.

let S is linearly dependent.

$\Rightarrow c_1v_1 + c_2v_2 = 0$

$\Rightarrow c_1 = 0$ (or) $c_2 = 0$

case 1: let $c_1 \neq 0$

$$c_1v_1 + c_2v_2 = \bar{0}$$

$$c_1v_1 = -c_2v_2$$

$$v_1 = -\frac{c_2}{c_1}v_2$$

$\Rightarrow v_1$ is multiple of v_2.

case 2: let $c_2 \neq 0$

$$c_1v_1 + c_2v_2 = \bar{0}$$

$$c_2v_2 = -c_1v_1$$

$$v_2 = -\frac{c_1}{c_2}v_1$$

$\Rightarrow v_2$ is multiple of v_1

$\therefore$ a vector is multiple of another

conversly consider v_1 is multiple of v_2

$$v_1 = kv_2$$

$$v_1 - kv_2 = \bar{0}$$

Since $1 \neq 0$

$\{v_1, v_2\}$ is linearly dependent.

Theorem: The set containing 0 vector is linearly dependent.

Proof: let $S = \{v\}$ where $v = \bar{0}$

We have to prove S is linearly dependent

consider $cv = \bar{0}$

$$v \neq \bar{0} \rightarrow c \neq 0$$

Since $c \neq 0$, S is linearly dependent

Theorem: An index set $\{v_1, v_2, \dots v_n\}$ of 2 or more vectors with $v_1 \neq \bar{0}$ is linearly independent iff some v_j (with $j > 1$) is a linear combination of preceding vector $v_1, v_2, \dots v_{j-1}$

Proof: let $\{v_1, v_2, \dots v_n\}$ of two (or) more vectors with $v_1 = \bar{0}$ is linearly dependent. We have to prove $\exists$ some vector $v_j ; j > 1$ is a linear combination of $v_1, v_2, \dots v_{j-1}$.

If the set is linearly dependent $\exists$ a relation of the form

$$c_1v_1 + c_2v_2 + \dots + c_{j-1}v_{j-1} + c_jv_j + c_{j+1}v_{j+1} + \dots + c_nv_n = \bar{0} \quad \text{—①}$$

where not all scalar co-eff's $c_1, c_2, \dots c_n$ are 0

let j be the largest integer such that $c_j \neq 0$

(i.e) $c_{j+1}, c_{j+2}, \dots c_n = 0$

We can assume this, because at the most $c_n \neq 0$, then $j = n$ also $j \geq 2$ because if $c_2, c_3 \dots c_n = 0$.

if $c_2=0, c_3=0$ and $c_n=0$ then $c_1v_1=\bar{0} \Rightarrow c_1=0$ $[\because v_1 \neq \bar{0}]$

This is a contradiction to our assumption the set $v_1, v_2, \ldots v_n$ is linearly dependent.

if c_j is the leading scalar $\ni c_j \neq 0$ i.e $n<j>2$

from ①, we get $c_1v_1 + c_2v_2 + \ldots + c_{j-1}v_{j-1} + c_jv_j = \bar{0}$

$\Rightarrow c_jv_j = -c_1v_1 - c_2v_2 - \ldots c_{j-1}v_{j-1}$

Multiply c_j^{-1} on both sides $[\because c_j \neq 0]$

$(c_j^{-1}c_j)v_j = (-c_j^{-1}c_1)v_1 + (-c_j^{-1}c_2)v_2 + \ldots + (-c_j^{-1}c_{j-1})v_{j-1}$

$v_1 =$ L.C of $\{v_1, v_2, \ldots v_{j-1}\}$ i.e v_j is L.C of preceding vector $v_1, v_2, \ldots v_{j-1}$

conversely assume v_j is linear combination of preceding vector i.e

We have to prove $\{v_1, v_2, \ldots v_n\}$ is L.D

$v_j =$ linear combination of $v_1, v_2, \ldots v_{j-1}$

$\Rightarrow \exists$ the scalars $c_1, c_2, \ldots c_{j-1}$

such that $v_j = c_1v_1 + c_2v_2 + \ldots + c_{j-1}v_{j-1}$

$\Rightarrow c_1v_1 + c_2v_2 + \ldots + c_{j-1}v_{j-1} - v_j = \bar{0}$

$\therefore$ The set $\{v_1, v_2, \ldots v_j\}$ is linearly dependent since $-1 \neq 0$.

We also know that the superset of linearly dependent set is linearly dependent.

Hence, the superset of $v_1, v_2, \ldots v_j$ is $v_1, v_2, \ldots v_n$

$\therefore \{v_1, v_2, \ldots v_n\}$ is also linearly dependent.

- Show that the superset of linearly dependent set is linearly dependent.

sol: Let $\{v_1, v_2, \cdots v_j\}$ is linearly dependent

We have to prove the superset $\{v_1, v_2, \cdots v_n\}$ is also L.D

$\therefore \{v_1, v_2, \cdots v_j\}$ is linearly dependent

$\exists$ scalar $c_1 \cdots c_j$

$\ni c_1v_1 + \cdots + c_jv_j = \bar{0}$

$\Rightarrow$ Some $c_i \neq \bar{0}$ where $1 \leq i \leq j$ — ①

Now to prove $v_1, v_2, \cdots v_n$ is linearly dependent

consider the relation of the form

$$c_1v_1 + \cdots + c_jv_j + c_{j+1}v_{j+1} + \cdots + c_nv_n = \bar{0}$$

from ① WKT there exist some scalar co-efficients $c_i \neq 0$

$\Rightarrow$ The set $\{v_1, v_2, \cdots v_n\}$ is linearly dependent.

- For the matrix

$$A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{bmatrix}$$

(a) Find k such that Null space is a subspace of R^k.

(b) Find k such that column is a subspace of R^k.

sol: Given matrix is 4 x 3 matrix.

$\therefore m = 4, n = 3$

(a) WKT, Null A is a subspace of R^n

$\therefore$ Null A is a subspace of R^3

Hence $k = 3$

(b) WKT, ~~Null A is~~ Col A is a subspace of R^m

~~Null~~ Col A is a subspace of R^4

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• Use an appropriate theorem to show that the given set is a vector space (or) find a specific example to the contrary

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} ; a+b+c = 2 \right\}$$

Sol. if $\bar{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

then $0+0+0 \neq 2$

$\Rightarrow \bar{0} \notin W$

W is not a vector space since identity doesn't hold

• $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} ; \begin{matrix} 3a+b=c \\ a+b+2c=2d \end{matrix} \right\}$

Sol. Given equations are

$3a+b-c=0$

$a+b+2c-2d=0$

we can write in matrix form as

$$\begin{bmatrix} 3 & 1 & -1 & 0 \\ 1 & 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow AX = \bar{0}$

where $A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \text{Null } A$

WKT, Null A is a subspace of R^n

$\therefore$ W is also a subspace of R.

$\therefore$ W is a vector space

• $W = \left\{ \begin{bmatrix} 3-2t \\ 3+3s \\ 3s+t \\ 2s \end{bmatrix} ; \cancel{s}, t \in R \right\}$

sol. we have $\cancel{4q=0 \Rightarrow q=0}$ $\quad 3+3s=0 \Rightarrow s=-1$

$\cancel{q+1=0 \Rightarrow} q=-1$ $\quad 2s=0 \Rightarrow s=0$

~~'q' cannot~~ 's' cannot take two different values

i.e, 0 and -1

Hence, identity doesn't exist.

$\therefore$ W is not a vector space.

• $W = \left\{ \begin{bmatrix} 3p-5q \\ 4q \\ p \\ q+1 \end{bmatrix} ; p,q \in R \right\}$

sol we have $4q=0 \Rightarrow q=0$

$q+1=0 \Rightarrow q=-1$

'q' cannot take two different values

i.e 0 and -1

Hence identity element doesn't exist

$\therefore$ W is not a vector space

• $W = \left\{ \begin{bmatrix} c-6d \\ d \\ c \end{bmatrix} ; \cancel{c,d}\ c,d \in R \right\}$

sol

$$\begin{bmatrix} c-6d \\ d \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix}$$

$W = \begin{bmatrix} c-6d \\ d \\ c \end{bmatrix}$ is a linear combination of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} \right\}$

$W = \text{span}\{v_1, v_2\}$

WKT, Span $\{v_1, v_2\}$ is a subspace

$\Rightarrow$ W is a subspace

$\therefore$ W is a vector space.

• Find A such that given set is in col A

$$W = \left\{ \begin{bmatrix} 2s+t \\ r-s+2t \\ 3r+s \\ 2r-s-t \end{bmatrix}, r, s, t \in R \right\}$$

sol.

$$W = \begin{bmatrix} 2s+t \\ r-s+2t \\ 3r+s \\ 2r-s-t \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

$$W = \text{Span}\{V_1, V_2, V_3\}$$

where $V_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$

we also know that col A is span $\{V_1, V_2, V_3\}$

where V_1, V_2, V_3 are the columns of matrix A.

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}$$

Basis

Let H is a subspace of a vectorspace of V. A set of vectors $B = \{b_1, b_2, \ldots, b_n\}$ in V is a basis for H. If it satisfies the fallowing property

1) B is linearly independent.

2) The subspace spanned by B coincides with H.

i.e $H = \text{Span}\{b_1, b_2, \ldots, b_n\}$

Example:

Let $e_1, e_2, \ldots e_n$ be the columns of $n \times n$ identity matrix I_n.

i.e, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \ldots e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

The set $e_1, e_2, \ldots e_n$ is called the standard basis of R^n

Sol: We have to prove $S = \{e_1, e_2, \ldots e_n\}$ is the basis of R^n

(1) To prove S is linearly independent.

consider $c_1 e_1 + c_2 e_2 + \ldots + c_n e_n = \bar{0}$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + c_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$c_1 = 0, c_2 = 0, \ldots c_n = 0$

$\therefore$ S is linearly independent.

(2) Now to prove span $\{e_1, e_2, \ldots e_n\} = R^n$

WKT, a linear span is a subspace of a vector space.

$\Rightarrow$ span $\{e_1, e_2, \ldots e_n\}$ is a subspace of R^n

$\therefore$ span $\{e_1, e_2, \ldots e_n\} \subseteq R^n$ — (1)

Now we have to prove R^n is a subset of span $\{e_1, e_2, \ldots e_n\}$

let $V \in R^n$

$$V = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

we can write V as

$$v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + a_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$= a_1e_1 + a_2e_2 + \dots + a_ne_n$

$\Rightarrow$ v is linear combination of $\{e_1, e_2, \dots e_n\}$

$\Rightarrow$ $v \in \text{span}\{e_1, e_2, \dots e_n\}$

$\therefore$ $R^n \subseteq \text{Span}\{e_1, e_2, \dots e_n\}$ — (2)

$\therefore$ from (1) & (2)

$\text{Span}\{e_1, e_2, \dots e_n\} = R^n$

$\therefore$ S is a basis of R^n

Ex:- Let $S = \{1, t, t^2 \dots t^n\}$. Show that S is a basis of P_n.

(1) To show S is linearly independent

$c_1 + c_2t + c_3t^2 + \dots + c_nt^n = \bar{0}$

$c_1 + c_2t + c_3t^2 + \dots + c_n t^n = 0 + 0t + 0t^2 + \dots + 0t^n$ ($\because \bar{0} \in P_n$)

$\Rightarrow c_1 = 0, c_2 = 0 \dots c_{n+1} = 0$

$\therefore$ S is linearly independent

(2) Now to prove $\text{span}\{1, t, t^2, \dots t^n\} = P_n$

WKT, a linear span is a subspace of a vector space

$\text{span}\{1, t, t^2 \dots t^n\}$ is subspace P^n

$\text{span}\{1, t, t^2, \dots t^n\} \subseteq P^n$ — (1)

we have to prove $P^n \subseteq$ Span $\{1, t, t^2 \dots t^n\}$

let $v \in P^n$

$v = \{a_0, a_1 t, a_2 t^2 \dots a_n t^n\}$

$v = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$

It is linear combination of $1, t, t^2 \dots t^n$

$v \in$ span $\{1, t, t^2, \dots t^n\}$

$\Rightarrow P_n \subseteq$ Span $\{1, t, t^2 \dots t^n\}$ — (2)

from (1) & (2)

span $\{1, t, t^2 \dots t_n\} = P_n$

Note:

1) Let A is $m \times n$ matrix then A is invertable, then the columns of A form a basis of R^n because they are linearly independent and it spans R^n.

2) If A is $m \times n$ matrix and A has n pivot positions then the columns of A form a basis for R^3.

Problem: Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ determine if set $\{v_1, v_2, v_3\}$ is a basis of R^3

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad R_3 \to R_3 + R_2 \Rightarrow \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

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$R_1 \to R_1 + R_3$

$$A = \begin{bmatrix} 3 & -4 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$R_1 \to \frac{R_1}{3}$, $R_3 \to \frac{R_3}{2}$.

$$A = \begin{bmatrix} 1 & -\frac{4}{3} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

A has 3 pivot positions

Hence, A is invertable

$\therefore$ Columns of A form a basis for R^3, i.e, $\{v_1, v_2, v_3\}$ is basis of R^3.

1) Determine whether the set is Basis for R^3

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$$

$$\{v_1, v_2, v_3\} = \begin{bmatrix} 2 & 2 & -8 \\ -1 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

$R_3 \to 2R_3 - R_1$

$R_2 \to 2R_1 + R_1$

$$A = \begin{bmatrix} 2 & 2 & -8 \\ 0 & -4 & 2 \\ 0 & 2 & 16 \end{bmatrix} \quad R_1 \to R_1 - R_2 \Rightarrow \begin{bmatrix} 2 & 0 & -24 \\ 0 & -4 & 2 \\ 0 & 2 & 16 \end{bmatrix}$$

$$R_3 \to R_2 + 2R_3 \Rightarrow \begin{bmatrix} 2 & 0 & -24 \\ 0 & -4 & 2 \\ 0 & 0 & 34 \end{bmatrix} \quad R_1 \to R_1 + R_3$$

$$\begin{bmatrix} 2 & 0 & 10 \\ 0 & -4 & 2 \\ 0 & 0 & 34 \end{bmatrix}$$

$R_2 \to R_2 \times 17 - R_3.$

$$\begin{bmatrix} 2 & 0 & 10 \\ 0 & -68 & 0 \\ 0 & 0 & 34 \end{bmatrix} \quad R_1 \to \frac{R_1}{2},\ R_2 = \frac{R_2}{-68},\ R_3 = \frac{R_3}{34}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A has 3 pivot columns (positions) Thus A is invertable.

Colums of A forms basis of R^3

2) $V_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix},\ V_2 = \begin{bmatrix} -3 \\ 7 \\ 0 \end{bmatrix},\ V_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ V_4 = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 7 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$R_2 \to R_1 + R_2$

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ 0 & 4 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

∴ we have a zero column in A.

∴ $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

and Hence, it is not a basis of R^3 but it has pivot element in each row.

∴ The set span R^3

3) $\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}$

Sol. $A = \begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 5 \end{bmatrix}$

$R_2 \to 2R_2 + 3R_1$

$A = \begin{bmatrix} -2 & 6 \\ 0 & 16 \\ 0 & 5 \end{bmatrix}$ - $R_1 \to R_1 - R_3$

$A = \begin{bmatrix} -2 & 1 \\ 0 & 16 \\ 0 & 5 \end{bmatrix}$

$R_1 \to \frac{R_1}{-2}$, $R_2 \to \frac{R_2}{16}$, $R_3 \to \frac{R_3}{5}$

$A = \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$A = \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

It is not a basis since 3rd row has no pivot element.

∴ The columns of A do not span R^3. But the set is linearly independent because one vector is not the multiple of other.

$\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

$\begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ -3 & 4 & 1 \end{pmatrix}$

$R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

3^{rd} row is zero, we don't have pivot element

$\therefore$ it doesn't span R^3

and $v_2 = v_1 - v_3$ i.e., $\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

Hence it is linearly dependent.

Spanning Set Theorem

Statement: Let $S = \{v_1, v_2, \ldots v_p\}$ is the set in V and let $H = \text{span}\{v_1, v_2, \ldots v_p\}$.

a) If one of the vector in S say v_k is a linear combination of the remaining vectors in S then the set formed from S by removing v_k still span H.

b) If $H \neq \bar{0}$ some subset of S is a basis for H.

Proof: Let v_p is a linear combination of $v_1, v_2, \ldots v_{p-1}$

then $\exists$ the scalars $a_1, a_2, \ldots a_{p-1} \ni$

$$v_p = \text{linear combination of } v_1, v_2, v_3 \ldots v_{p-1}$$

$$\text{i.e., } v_p = a_1 v_1 + a_2 v_2 + \ldots + a_{p-1} v_{p-1} \quad \text{—} (1)$$

Let $x \in H \Rightarrow x$ is linear combination of $v_1, v_2, \ldots v_p$

$\exists$ the scalars $c_1, c_2, \ldots c_{p-1}$ and $c_p \ni$

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_{p-1} v_{p-1} + c_p v_p \quad \text{—} (2)$$

Substituting ① & ② we get.

$$x = c_1v_1 + c_2v_2 + \dots + c_{p-1}v_{p-1} + c_p[a_1v_1 + a_2v_2 + \dots + a_{p-1}v_{p-1}]$$

$$= (c_1 + c_pa_1)v_1 + (c_2 + c_pa_2)v_2 + \dots + (c_{p-1} + c_pa_{p-1})v_{p-1}$$

$\Rightarrow x$ is a linear combination of $v_1, v_2, \dots v_{p-1}$.

$\therefore \{v_1, v_2, \dots v_{p-1}\}$ span H.

b) Given $S = \{v_1, v_2, \dots v_p\}$ and $H \neq \bar{0}$ and $H = \text{Span}\{v_1, v_2, \dots v_p\}$

Our aim to form a basis for H.

If S is linearly independent S itself is a basis for H.

If S is linearly dependent then $\exists$ the vector which is the linear combination of the preceeding vector.

Let v_p is a vector which is a linear combination of the remaining vectors. Remove this vector from set S. we get another set.

$$S_1 = \{v_1, v_2, \dots v_{p-1}\}$$

from ⓐ S_1 also span H.

If S_1 is linearly independent then S_1 is the required basis If not i.e, S_1 is linearly dependent then repeat this process until the spanning set is linearly independent and hence basis for H.

At most, the spanning set is reduced to one vector. That vector will be non-zero. And hence it is linearly independent.

$\therefore$ It forms a basis for H.

Basis for Null Space

Find the vector that spans the null space of a matrix A and this vector is always linearlyindependent and this will be the basis for Null A.

Basis of Col A

Find the basis for col A: reduce A to Echelon form we will get a matrix B ~~then~~ find the pivot columns then the pivot colums of A is the basis for column space.

1) Find the basis for nullspace and column space

$$A = \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{bmatrix}$$

Basis for Null A = [A,0] = $\begin{bmatrix} 1 & 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 3 & -1 & -7 & 3 & 0 \end{bmatrix}$

$R_3 \to R_3 - 3R_1 \to \begin{bmatrix} 1 & 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & -1 & -1 & 9 & 0 \end{bmatrix}$ $R_3 \to R_3 + R_2 \to \begin{bmatrix} 1 & 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 13 & 0 \end{bmatrix}$

$R_3 \to \frac{R_3}{13} \to \begin{bmatrix} 1 & 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ $R_1 \to R_1 + 2R_3 \to \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$R_2 \to R_2 - 4R_3 \to \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$x_4 = 0$

$x_3 = 1$ (free variable)

$x_4 = 0$

$\therefore$ Basis for Null A = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

For Column Space $A \sim B$

$$A = \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Colums 1, 2 & 4 are pivot columns

$\therefore$ The basis for col A $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \right\}$

2) Find the basis for Null A and col A for

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$

Sol. Basis for Null A :-

$$[A, 0] = \begin{bmatrix} -2 & 4 & -2 & -4 & 0 \\ 2 & -6 & -3 & 1 & 0 \\ -3 & 8 & 2 & -3 & 0 \end{bmatrix}$$

$R_3 \to 2R_3 - 3R_1 \to \begin{bmatrix} -2 & 4 & -2 & -4 & 0 \\ 2 & -6 & -3 & 1 & 0 \\ 0 & 4 & 10 & 6 & 0 \end{bmatrix}$

$R_2 \to R_1 + R_2 \to \begin{bmatrix} -2 & 4 & -2 & -4 & 0 \\ 0 & -2 & 5 & -3 & 0 \\ 0 & 4 & 10 & 6 & 0 \end{bmatrix}$

$R_1 \to R_1 - R_3 \to$ $\cancel{2\ \ 1\ \ 0\ \ 6}$

$$\begin{bmatrix} -2 & 0 & -12 & -10 & 0 \\ 0 & -2 & -5 & -3 & 0 \\ 0 & 4 & 10 & 6 & 0 \end{bmatrix}$$

$R_1 \to \frac{R_1}{-2}, \ R_3 \to \frac{R_3}{2} \to \begin{bmatrix} 1 & 0 & 6 & 5 & 0 \\ 0 & -2 & -5 & -3 & 0 \\ 0 & 2 & 5 & 3 & 0 \end{bmatrix}$

$R_3 \to R_3 + 2$

$$\begin{bmatrix} 1 & 0 & 6 & 5 & 0 \\ 0 & -2 & -5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to -\frac{R_2}{2} \to \begin{bmatrix} 1 & 0 & 6 & 5 & 0 \\ 0 & 1 & 5/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 6x_3 + 5x_4 = 0$

$x_2 + \frac{5}{2}x_3 + \frac{3}{2}x_4 = 0$

x_3 & x_4 are free variables ($x_3 = x_4 = 1$)

Basis for Null A $= \begin{bmatrix} -11 \\ 4 \\ 1 \\ 1 \end{bmatrix}$

Col A $\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis for col A $= \left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$

3) $A = \begin{bmatrix} 1 & 1 & -2 & 1 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -8 & 0 & 16 \end{bmatrix}$

Sol. Null A

$$[A, 0] = \begin{bmatrix} 1 & 1 & -2 & 1 & 5 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & -8 & 0 & 16 & 0 \end{bmatrix}$$

$$R_1 \to R_1 + R_2 \to \begin{bmatrix} 1 & 0 & -2 & 2 & 7 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & -8 & 0 & 16 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_3 \to \begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \end{bmatrix}$$

$x_1 + 2x_4 + 3x_5 = 0$

$x_2 - x_4 - 2x_5 = 0$

$-x_3 + 2x_5 = 0$

x_4 & x_5 are free variables

$x_4 = 1, x_5 = 1$

Basis of Null A = $\begin{bmatrix} -5 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

Col A : $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$

Columns 1,2,3 are pivot columns

Basis of col A = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -8 \end{bmatrix} \right\}$

4)

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

Sol: Null A :-

$$[A, 0] = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 & 0 \\ 3 & 12 & 1 & 5 & 2 & 0 \\ 2 & 8 & 1 & 3 & 2 & 0 \\ 5 & 20 & 2 & 8 & 8 & 0 \end{bmatrix}$$

$R_4 \to R_4 - 5R_1, R_3 \to R_3 - 2R_1, R_2 \to R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 8 & 0 \\ 0 & 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 2 & -2 & 13 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3, R_4 \to R_4 - 2R_3$.

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

$R_4 \to R_4/5$, $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_4$, $R_2 \to \frac{R_2}{4} \to \begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem: The pivot columns of a matrix A form A basis for matrix A.

Proof: Let B is the reduced echelon form of A. The set of pivot columns of B is linearly independent because no vector in the set is a linear combination of preceding vector since, A is row equivalent to B. The pivot columns of A are linearly independent. Also, every non-pivot columns are linear combination of pivot columns. Thus, non-pivot columns of A & B may be deleted. then from the spanning set theorem the remaining set also span A. and thus the pivot columns of A. Span A.

$\therefore$ The pivot columns of A from a basis for column A.

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Co-ordinate System:

Theorem: The Unique Representation Theorem

Statement: Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for a vector space V. Then for each x in V there exists a unique set of scalars $c_1, c_2, \ldots, c_n$ such that $x = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$.

Proof: It is given $B = \{b_1, b_2, \ldots, b_n\}$ is a basis of V.

We have to prove if $\exists$ the vector x in V then $\exists$ the unique representation $x = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$

if B is basis of V then

1) B is linearly independent

2) Span of B = V.

if $x \in V$, then $x \in \text{span } B$ (from 2)

$\Rightarrow x$ is a linear combination of elements of B.

$\therefore$ There exists the scalars $c_1, c_2, \ldots, c_n$ such that

$$x = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n \quad \text{— (1)}$$

To prove uniqueness:-

Let $\exists$ another scalars $d_1, d_2, \ldots, d_n$ such that x

$$x = d_1 b_1 + d_2 b_2 + \cdots + d_n b_n \quad \text{— (2)}$$

subtract (1) & (2) we get

$$x - x = \bar{0} = (c_1 b_1 + c_2 b_2 + \cdots + c_n b_n) - (d_1 b_1 + d_1 b_2 + \cdots + d_n b_n)$$

$$= (c_1 - d_1) b_1 + (c_2 - d_2) b_2 + \cdots + (c_n - d_n) b_n$$

But b is linearly independent

$\Rightarrow c_1 - d_1 = 0,\ c_2 - d_2 = 0 \ldots\ c_n - d_n = 0$

$\Rightarrow c_1 = d_1,\ c_2 = d_2, \ldots\ c_n = d_n$

$\therefore$ The representation is unique.

Definition

Co-ordinates of x relative to the basis B

Suppose $B = \{b_1, b_2, \dots b_n\}$ is a basis for B and x is in V the co-ordinates of X relative to the basis B are the scalars $c_1, c_2, \dots c_n$ such that $x = c_1 b_1 + c_2 b_2 + \dots + c_n b_n$. It is denoted by $[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ the mapping $x \to [x]_B$ is co-ordinate mapping.

Problems

Consider a basis $B = \{b_1, b_2\}$ for R^2 where $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Suppose x in R^2 has the co-ordinate vector $[X]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ find x

Sol. $[X]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \Rightarrow c_1 = -2,\ c_2 = 3$

$$b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$X = c_1 b_1 + c_2 b_2$$

$$= -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

2. Find the vector x if $B = \left\{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}\right\}$ and $[x]_B = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Sol. $c_1 = 1,\ c_2 = 0,\ c_3 = -2$

$$X = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}.$$

3. $x = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the co-ordinates of x relative to the standard basis $\{b_1, b_2\}$.

sol. It is given $\{b_1, b_2\}$ is standard basis of $R^2 \Rightarrow b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$[x]_B = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \Rightarrow c_1 = 1, c_2 = 6.$$

$$x = c_1 b_1 + c_2 b_2$$

$$\Rightarrow 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} = x$$

if B is the standard basis of V then $[x_B] = x$

4. let $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $B = \{b_1, b_2\}$ find the co-ordinate vector of $[x]_B$.

sol.
$$x = c_1 b_1 + c_2 b_2$$

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} -c_2 \\ c_2 \end{bmatrix}$$

$$2c_1 - c_2 = 4$$
$$c_1 + c_2 = 5$$
$$3c_1 = 9$$
$$c_1 = 3$$
$$c_2 = 2$$

(or) $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

or write augumented matrix A and reduce.

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & 5 \end{bmatrix} \quad R_2 \to 2R_2 - R_1$$

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 6 \end{bmatrix} \quad R_2 \to \frac{R_2}{3} \to \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$R_1 + R_2 \to R \to \begin{bmatrix} 2 & 0 & 6 \\ 0 & 1 & 2 \end{bmatrix} \quad R_1 \to \frac{R_1}{2} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$c_1 = 3$$
$$c_2 = 2$$

5. $b_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Sol $\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ -2c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ -5c_2 \end{bmatrix}$

$c_1 + 3c_2 = -1$

$-2c_1 - 5c_2 = 1$

$2c_1 + 6c_2 = -2$

$-2c_1 - 5c_2 = 1$

$c_2 = -1$

$c_1 = 2$

$\left[\begin{array}{cc|c} 1 & 3 & -1 \\ -2 & -5 & 1 \end{array}\right] R_2 \to R_2 + 2R_1$

$\left[\begin{array}{cc|c} -1 & 3 & -1 \\ 0 & 1 & -1 \end{array}\right] R_1 \to 3R_2 - R_1$

$\left[\begin{array}{cc|c} -1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$ $-c_1 = -2 \Rightarrow c_1 = 2$

$c_2 = -1$

PCP-4:

$B = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for P_2. Find the co-ordinate vector of $P(t) = 1 + 4t + 7t^2$ relative to B.

Sol: The co-ordinate vector of $p(t)$ w.r.t B satisfy the equation

$(1+4t+7t^2) = c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2)$

comparing the co-eff on both sides, we get

$c_1 + c_3 = 1$

$\cancel{c_1 + c_2 + c_3 = 4}$ $c_2 + 2c_3 = 4$

$c_1 + c_2 + c_3 = 7$

$c_2 + 1 = 7$

$\boxed{c_2 = 6}$

$6 + 2c_3 = 4$

$\boxed{c_3 = -1}$

$\boxed{c_1 = 2}$

$[P(t)]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$

PCP-5

$B = \{1-t^2, t-t^2, 2-t+t^2\}$ is a basis for P_2. Find the co-ordinate vector of $1+3t-6t^2$ relative to B

Sol: $(1+3t-6t^2) = c_1(1-t^2) + c_2(t-t^2) + c_3(2-t+t^2)$

$c_1 + 2c_3 = 1$

$c_2 - c_3 = 3$

$-c_1 - c_2 + c_3 = -6$

$c_1 + c_2 - c_3 = 6$

$c_1 = 3$

$c_3 = -1$

$c_2 = 2$

$[P_2(t)]_B = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

PCP-3

Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 7 \\ -9 \end{bmatrix}$ determine if $\{v_1, v_2\}$ is a basis of R^3. Is $\{v_1, v_2\}$ a basis for R^2?

Sol. Let $A = [v_1, v_2]$

i.e $A = \begin{bmatrix} 1 & -2 \\ -2 & 7 \\ 3 & -9 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$

$= \begin{bmatrix} 1 & -2 \\ -2 & 7 \\ 0 & -3 \end{bmatrix} R_2 \to R_2 + 2R_1 \to \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & -3 \end{bmatrix}$

$R_3 \to R_3 + R_2 \to \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} R_1 \to 3R_1 + 2R_2 \to \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} R_1 \to \frac{R_1}{3}, R_2 \to \frac{R_2}{3}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since, every row of A do not contain pivot columns the columns of A do not span R^3.

Hence $\{V_1, V_2\}$ is not a basis of R^3

$\{V_1, V_2\}$ are not in R^2.

$\therefore$ They cannot be a basis for R^2.

PCP-6

The vector $V_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, V_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span R^2 but do not form a basis. Find different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of V_1, V_2, V_3.

sol: Consider the matrix $A = [V_1, V_2, V_3]$

$$= \begin{bmatrix} 1 & 2 & -3 \\ -3 & -8 & 7 \end{bmatrix} \quad R_2 \to R_2 + 3R_1$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 \to R_1 - 2R_2$$

$$= \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 1 \end{bmatrix}$$

Every row has a pivot element.

$\therefore V_1, V_2, V_3$ span R^2. But the $\{V_1, V_2, V_3\}$ is linearly dependent since $V_3 = 5V_1 + V_2$.

$\therefore \{V_1, V_2, V_3\}$ is not a basis for R^2.

Now, we have to express $\begin{bmatrix}1\\1\end{bmatrix}$ as the linear combination of v_1, v_2, v_3

i.e solve the vector equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = \begin{bmatrix}1\\1\end{bmatrix}$$

$$x_1\begin{bmatrix}1\\-3\end{bmatrix} + x_2\begin{bmatrix}2\\-8\end{bmatrix} + x_3\begin{bmatrix}-3\\7\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

The augumented matrix for the system of equation is

$$\begin{bmatrix}1 & 2 & -3 & 1\\-3 & -8 & 7 & 1\end{bmatrix} R_3 \to R_3 + 3R_1$$

$$\begin{bmatrix}1 & 2 & -3 & 1\\0 & -2 & -2 & 4\end{bmatrix} R_1 \to R_2 + R_1$$

$$\begin{bmatrix}1 & 0 & -5 & 5\\0 & -2 & -2 & 4\end{bmatrix} R_2 \to \frac{R_2}{-2}$$

$$\begin{bmatrix}1 & 0 & -5 & 5\\0 & 1 & 1 & -2\end{bmatrix}$$

$x_1 - 5x_3 = 5$

$x_2 + x_3 = -2$

x_3 is free variable

let $x_3 = 1$

$x_1 = 10$

$x_2 = -3$

$\therefore$ we get $10v_1 - 3v_2 + v_3 = \begin{bmatrix}1\\1\end{bmatrix}$

Now let, if $x_3 = 0$

we get $x_1 = 5$ and $x_2 = -2$

$\therefore 5v_1 - 2v_2 + 40v_3 = \begin{bmatrix}1\\1\end{bmatrix}$

$\therefore$ There are infinetly many ways to express $\begin{bmatrix}1\\1\end{bmatrix}$ as a linear combination of v_1, v_2, v_3.

Change of Co-Ordinate Matrix

The vector equation $x = c_1 b_1 + c_2 b_2 + c_3 b_3 + \cdots + c_n b_n$ is equivalent to the matrix $x = [b_1 \ b_2 \ b_3 \cdots b_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

The matrix P_B is called change of co-ordinate matrix.

Theorem: Let $B = \{b_1, b_2, \cdots b_n\}$ be a basis for vector space V then the coordinate mapping $x \to [x]_B$ is a linear transformation from $V \to R^n$.

Proof: Take two vectors U and V in V.

Such that $U = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$

$V = d_1 b_1 + d_2 b_2 + \cdots + d_n b_n$

$\to$ The co-ordinate vector of U w.r.t B.

$$[U_B]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ and } [V]_B = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}.$$

We have to prove the co-ordinate mapping is a linear transformation

consider $U+V = (c_1+d_1)b_1 + (c_2+d_2)b_2 + \cdots + (c_n+d_n)b_n$

$$[U+V]_B = \begin{bmatrix} c_1+d_1 \\ c_2+d_2 \\ \vdots \\ c_n+d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$= [U]_B + [V]_B$$

consider, $cU = c(c_1 b_1 + c_2 b_2 + \cdots + c_n b_n)$

$$= (cc_1)b_1 + (cc_2)b_2 + \cdots + (cc_n)b_n$$

$$[cU]_B = \begin{bmatrix} cc_1 \\ cc_2 \\ \vdots \\ cc_n \end{bmatrix} = c \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

$\therefore$ The co-ordinate mapping is a linear transformation (homomorphism)

Note: To prove co-ordinate mapping is one-one and onto.

Sol. let $U = c_1b_1 + c_2b_2 + \dots + c_nb_n$, $V = d_1b_1 + d_2b_2 + \dots + d_nb_n$

Now, consider $[U]_B = [V]_B$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

multiply with basis $[b_1\ b_2 \dots b_n]$ on both sides

$$[b_1\ b_2 \dots b_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [b_1\ b_2 \dots b_n]\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$c_1b_1 + c_2b_2 + \dots + c_nb_n = d_1b_1 + d_2b_2 + \dots + c_nb_n$$

$\Rightarrow U = V$

It is one-one.

$\forall y \in R^n$ $\exists$ atleast one vector $U \in V \ni [U_B] = y$

where $y = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, $U = c_1b_1 + c_2b_2 + \dots + c_nb_n$

$\Rightarrow$ co-ordinate mapping is onto.

• Use co-ordinate vectors to verify that the polynomials $1+2t^2$, $4+t+5t^2$, $3+2t$ are linearly independent on P_2.

Sol Given polynomials are

$x = 1+2t^2$

$y = 4+t+5t^2$

$z = 3+2t$

To reduce the matrix form $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix}$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 3 \\ 0 & -3 & -6 \end{bmatrix}$$

$R_3 \to \frac{R_3}{3}$

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$ $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ linearly dependent.

* Let $V_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $V_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & $X = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, $B = \{V_1, V_2\}$ then B is the bases for H. $H = \text{span}\{V_1, V_2\}$. Determine if X is in H and also find co-ordinate vectors B_x relating B.

Sol. Given that

$$V_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \cdot V_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \& \quad X = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

by the defn $x = c_1 b_1 + c_2 b_2$.

$$\begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

~~$R_2 \to R_2 - 2R_1$~~ $R_2 \Rightarrow \frac{R_2}{6}$.

$$\Rightarrow \begin{bmatrix} 3 & -1 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

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Dimension of a vector Space.

Statement: If a vector space V has a bases $B=\{b_1, b_2, --- b_n\}$ then in an set in V containing more than n vectors must be linearly dependent.

Proof: let $\{u_1, u_2, --- u_p\}$ be a set in B with more than n vectors

The co-ordinate vectors are $[u_1]_B, [u_2]_B --- [u_p]_B$ form a linearly dependent set in R^n because there are one or more vectors in p then the entries n.

There exists scalars $c_1, c_2, --- c_p$ are not all zeroes

$$c_1[u_1]_B + c_2[u_2]_B + ---- + c_p[u_p]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

∴ co-ordinate mapping is linear transformation.

$$[c_1u_1 + c_2u_2 + ---- + c_pu_p]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The zero vector on right displaces the 'n' vectors to build a linear ~~transformation~~ combinations.

$$c_1u_1 + c_2u_2 + ... + c_pu_p + 0b_1 + 0b_2 + --- + 0b_p = \bar{0}$$

Since all c_i's are not zeroes.

∴ $\{u_1, u_2, --- u_p\}$ is linearly dependent.

Theorem: If a vector space V has a bases of 'n' vectors then every bases of V must consists of exactly n vectors.

Proof: Let B_1 is a bases containing 'n' vectors

B_2 is another bases for V.

Since B_1 is a bases and B_2 is linearly independent, B_2 cannot have more than n vectors.

So, Also B_2 is a bases and B_1 is linearly independent, B_1 must have atleast 'n' vectors in which case B_2 have exactly n vectors. Thus B_2 consists of exactly 'n' vectors.

Finite Dimensional vector Space: If V is spanned by a finite set then V is ~~set~~ said to be finite dimensional vector space. The dimension of 'V' is written as dim V is the no. of vectors in the bases for V.

If V is not spanned by a finite set then V is said to be infinite dimensional.

Ex: 1) Dimension of R^n is n i.e, dim $R^n = n$.

2) The polynomial $\{1, t, t^2\}$ shows that dim $P^2 = 3$.

Note: The dim $P_n = n+1$.

• Find the dimension of subspace $H = \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix}$

. ~~each~~ such that a, b, c, d in R.

sol: Given,

The subspace $H = \begin{bmatrix} a-3d+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix}$

Since H is the set of all linear combinations of vectors

i.e. $H = a\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 4 \\ -1 \\ 3 \end{bmatrix}$

Where $V_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}$, $V_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$

Clearly, $v_1 \neq n$, v_2 is not multiple of V_1 but V_3 is multiplied by V_2 we discard V_3 and still we have a set that span H. V_4 is not a linear combination of V_1 & V_2. So, $B = \{V_1, V_2, V_3\}$ are linearly independent.

Hence B forms a bases for H.

$\therefore$ dim H = 3.

<u>The dimension of Null A & Col A</u> : Dimension of Nul A is the no. of free variables determine the size of bases of Nul A.

Dimension of Col A is the no. of pivot columns in Matrix A.

. Find the Dimension of Null Space & Column Space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

sol.

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

x_2, x_4, x_5 are free variables.

. Find the vector x determined by co-ordinate vectors matrix $[x]_B$ and given bases $B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$, $[x_B] = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find x.

sol. $X = P_B = [X]_B$ i.e the change of coordinate equation.

$$X = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 15 - 12 \\ -25 + 18 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

• $B = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$, $[X_B] = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

sol.

$$X = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -6 - 20 \\ -4 + 5 \end{bmatrix} = \begin{bmatrix} -26 \\ 1 \end{bmatrix}$$

• $B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$, $[X_B] = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

sol.

$$X = \begin{bmatrix} 1 & 5 & 4 \\ -2 & 0 & -3 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$$

• Find the co-ordinate vector of x relative to the given bases:

1) $b_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, b_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

sol. By defⁿ of co-ordinate system

$$[x]_B = c_1 b_1 + c_2 b_2$$

$$\begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$c_1 = 2, c_2 = -1$

2) $b_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, b_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, x = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$

$$[x]_B = c_1 b_1 + c_2 b_2.$$

$$\begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ -6 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$R_3 \to R_3 + 4R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ -10 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$R_3 \to \frac{R_3}{5}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2.$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \therefore c_1 = 3, c_2 = -2.$$

Extension
Theorem: Let H be a subspace of a finite dimensional vector space B. Any linearly independent set in H can be expanded if necessary to form a bases for H also H is finite dimensional dim $H \leq \dim V$.

Proof: if $H = \{0\}$ then,

$$\dim H = 0 \leq \dim V.$$

Otherwise let $S = \{u_1, u_2, \cdots u_k\}$ be any linearly independent set in H.

if S spans H then S is a bases of H.

Otherwise $\exists$ v_{k+1} in H that does not span in S but $\{u_1, u_2, \cdots u_k, u_{k+1}\}$ is linearly independent.

because no. of vectors in the set can be linear combination of preceding vectors.

We can continue this process of expanding S to a longer independence set in H but the no. of vectors in a linearly independent expansion of S can exceed dim V.

Hence S will be span of H and S will be basis for H.

$$\therefore \dim H \leq \dim V.$$

Bases Theorem

Statement: Let V be P-dimensional vector and $P \geq 1$

(i) Any linearly independent set of exactly P element in V automatically forms a basis of V.

(ii) Any set of exactly P that spans V is a bases for V.

Proof. Any linearly independent set of P elements can be extended to form a basis for V but the basis must contain exactly P elements.

Since, dim V = P.

So, S is already ~~for~~ a bases for V then S has P elements and spans V.

∴ V is non-zero according to spanning theorem,

subset S' of S is a bases of V.

. Use an inverse matrix to find $[x]_B$ for given x & B.

(i) $B = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$; $x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Sol: WKT $x = B \cdot [x]_B$

$\Rightarrow [x]_B = x \cdot B^{-1}$ $\left(\because B^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$

$B = \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}$

$B^{-1} = \frac{1}{5-6} \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$

$B^{-1} = \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}$

$[x]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -10+15 \\ -4+5 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(iii) $B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$; $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Sol.

$$B = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \frac{1}{-1+2} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2-6 \\ 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ 5 \end{bmatrix}$$

• Suppose $\{V_1, V_2, V_3, V_4\}$ is a linearly dependent spanning set for a vector space V. Show that each set in V can be expressed in more than one way has a linear combination V_1, V_2, V_3, V_4.

Sol. Suppose that,

$\{V_1, V_2, V_3, V_4\}$ is a linearly dependent spanning set for a vector space V.

For w in V there exists scalars k_1, k_2, k_3, k_4. $\ni$

$$W = k_1V_1 + k_2V_2 + k_3V_3 + k_4V_4 \quad \text{—— (1)}$$

because $\{V_1, V_2, V_3, V_4\}$ span V because the set is linearly independent, there exists scalars c_1, c_2, c_3, c_4. not all are zeroes $\ni$ $0 = c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4$ —— (2)

Adding eq^{n}s (1) + (2)

$W + 0 = (k_1 + c_1)V_1 + (k_2 + c_2)V_2 + (k_3 + c_3)V_3 + (k_4 + c_4)V_4$

$\therefore W = (k_1 + c_1)V_1 + (k_2 + c_2)V_2 + (k_3 + c_3)V_3 + (k_4 + c_4)V_4$.

Here c_i's are non-zeroes.

$\therefore$ W is expressed in more than one way as linear combination of V_1, V_2, V_3, V_4.

• let $B = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \end{bmatrix} \right\}$. Since, the co-ordinate mapping determined by B is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$, This mapping must be implemented by 2×2 matrix A. Find it.

sol. Given that

$$B = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \end{bmatrix} \right\}$$

$$\therefore B = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}$$

The co-ordinate matrix $[x]_B = P_B^{-1}$

$$= \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}^{-1}$$

$$\Rightarrow \frac{1}{9-8} \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$$

$$[x]_B = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$$

• Let x be a finite space in vector space 'v' with the property that every 'x' in V has a unique representation as a linear combinations of elements of 's', Show that S is a bases of V.

Sol: Let $S = \{S_1, S_2, --- S_n\}$ be a finite set in a vector space V
$\Rightarrow$ V is infinite dimensional vector space.

To show that S is a bases of V.

We have to show that S is linearly independent and S spans V (∴ by defn of bases

Let V be dimensional

Suppose that, the set $\{e_1, e_2, --- e_n\}$ is a connonical bases V.

Every vector in V can be represented as unique linear combination of vector from 'S'.

This means that the vectors $e_1, e_2, --- e_n$ can be represented as unique linear combination of vectors from S.

∵ The set $\{e_1, e_2, --- e_n\}$ is a bases for V and it is spanned 'S' must span V because $\{e_1, e_2, --- e_n\}$ must span V.

We must also show that S is linearly independent set.

∵ V is a vector space then zero vector 0 is in it. This means that ∃ a scalar $a_1, a_2, --- a_n = a_1 S_1 + a_2 S_2 + --- + a_n S_n = \bar{0}$.

WKT every vector from V can be represented as a unique combination of vectors from S.

∴ S is linearly independent set.

∴ S is linearly independent set and span of V.

∴ S is bases of V.

. Use co-ordinate vectors to test the linear independence of the sets of polynomials.

sol. 1) $1+2t^3,\ 2+t-3t^2,\ -t+2t^2-t^3$

The given polynomials are $1+2t^3,\ 2+t-3t^2,\ -t+2t^2-t^3$.

Take a linear combination and ~~is~~ equate to 0 i.e

$$c_1(1+2t^3)+c_2(2+t-3t^2)+c_3(-t+2t^2-t^3)=0$$

$$\Rightarrow (c_1+2c_2)+(c_2t-c_3t)+(-3c_2t^2+2c_3t^2)+(2c_1t^3-c_3t^3)=0$$

$$\Rightarrow (c_1+2c_2)+(c_2-c_3)+t^2(-3c_2-2c_3)+t^3(2c_1-c_3)=0$$

$\Rightarrow c_1+2c_2=0$ — ①

$c_2-c_3=0$ — ②

$-3c_2+2c_3=0$ — ③

$2c_1-c_3=0$ — ④

$$2c_2-\cancel{2c_3}=0$$
$$-3c_2+\cancel{2c_3}=0$$
$$-c_2=0$$
$$c_2=0$$

Sub $c_2=0$ in ①

$c_1=0$

$\therefore c_1=0,\ c_2=0,\ c_3=0$

$\therefore$ The polynomials are linearly independent.

2) $1-2t^2-t^3,\ t+2t^3,\ 1+t-2t^2$.

3) $(1-t)^2,\ t-2t^2+t^3,\ (1-t)^3$

sol. The polynomials are $(1-t)^2;\ t-2t^2+t^3;\ (1-t)^3$.

Take a linear combination and is equal to zero.

i.e, $c_1(1-t)^2+c_2(t-2t^2+t^3)+c_3(1-t)^3 = 0$.

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$\Rightarrow c_1(1+t^2-2t) + c_2(t-2t^2+t^3) + c_3(1-t^3+3t^2-3t) = 0$

$\Rightarrow (c_1+c_2) + (-2c_1+c_2-3c_3)t + (c_1-2c_2+3c_3)t^2 + (c_2-c_3)t^3 = 0.$

Now in matrix form

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \\ 1 & -2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The matrix does not have a pivot in each column. Columns are linearly dependent.

4) $(2-t)^3$, $(3-t^2)$, $1+6t-5t^2+t^3$.

• Use co-ordinate vectors to test whether the following sets of polynomials spans P_2.

1) $1-3t+5t^2, -3+5t-7t^2, -4+5t-6t^2, 1-t^2$

sol. The given polynomials are.

$1-3t+5t^2 = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$, $-3+5t-7t^2 = \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix}$.

$$\begin{bmatrix} 5-2t \\ 5+t \\ 3t \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

We have to check linearly independent.

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \quad R_2 \to \frac{R_2}{3} - R_1$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & 3 \end{bmatrix} \quad R_2 \to \frac{R_2}{3}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 3 \end{bmatrix} \quad R_1 \to R_1 + 2R_2.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ It is linearly independent.

$\therefore$ The bases are $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$

$\therefore$ The dimension is 2.

2) $\left\{ \begin{bmatrix} 2a \\ -4b \\ -2a \end{bmatrix} ; a, b \text{ in } R \right\}$

3) $\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} ; a, b, c \text{ in } R \right\}$

4) $\left\{ \begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} ; p, q \text{ in } R \right\}$

5) $\{ (a,b,c) ; a - 3b + c = 0 , b - 2c = 0 , 2b - c = 0 \}$

sol. The subspace $H = \text{Nul } A$ to solve $Ax = 0$.

$\therefore$ The argumented matrix $[A, 0]$ where

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$[A, 0] = \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + 3R_2 \\ R_3 \to R_3 - 2R_2 . \end{matrix}$$

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$$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad R_3 \to \frac{R_3}{3}.$$

$$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + 2R_3 \\ R_1 \to R_1 + 5R_3. \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore x_1 = 0, x_2 = 0, x_3 = 0$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \{0\}$ is a bases for Nul A.

dim Nul A = 0.

• $\{(a,b,c,d) : a - 3b + c = 0\}$.

Sol. $a - 3b + c = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ It is linearly independent since, each column pivot elements

$\therefore$ The span of $H = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\therefore$ dim H = 3.

PCP-7

Find the dimensions of the subspace of all vectors in R^3 whose first & third entries are equal.

sol: The subspace $H = \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix}, a, b \in R \right\}$

Since 1st & 3rd entries are equal then.

$$\begin{bmatrix} a \\ b \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ It is linearly independent.

$\because$ All columns have pivot elements

$\therefore H = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\therefore$ It is bases.

dim $H = 2$.

PCP-8

Find the dimension of the subspace H of R^2 spanned by

$\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 15 \end{bmatrix}$

sol. The given subspace $H = \text{span}\left\{ \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 15 \end{bmatrix} \right\}$.

$A = \begin{bmatrix} 1 & -2 & -3 \\ -5 & 10 & 15 \end{bmatrix}$ $R_2 \to R_2 + 5R_1$

$$= \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ There is only one pivot

$\therefore$ It is linearly dependent.

$\therefore H = \left\{ \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}$

$\therefore \dim H = 1$

- Find the dimension of the subspace spanned by the given vectors..

1) $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 5 \end{bmatrix}$

Sol. The subspace $H = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 5 \end{bmatrix} \right\}$

The vector reduces to

$$A = \begin{bmatrix} 1 & -3 & -2 & -3 \\ -2 & -6 & 3 & 5 \\ 0 & 0 & 5 & 5 \end{bmatrix} \quad R_2 \to R_2 + 2R_1$$

$$= \begin{bmatrix} 1 & -3 & -2 & -3 \\ 0 & -12 & -1 & -1 \\ 0 & 0 & 5 & 5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2/-12 \\ R_1 \to R_1 + 3R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & -\frac{7}{4} & -\frac{11}{4} \\ 0 & 1 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & 5 & 5 \end{bmatrix} \quad R_3 \to \frac{R_3}{5}$$

$$= \begin{bmatrix} 1 & 0 & -7/4 & -11/4 \\ 0 & 1 & 1/12 & 1/12 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2 - \frac{1}{12} R_3.$$

$$= \begin{bmatrix} 1 & 0 & -\frac{7}{4} & -\frac{11}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 \to R_1 + \frac{7}{4} R_3.$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore$ It has pivot position in each column

$\therefore$ It is linear independent $\therefore$ It is bases.

$$H = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} \right\}$$

dim $H = 3$.

$$\text{ii)} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Sol. It has 3 pivot columns

$\therefore$ It is linearly dependent

$$\therefore H = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right\}$$

$\therefore$ dim $H = 3$.

Determine the dimension of Nul A & Col A for the matrix

$$A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol. Given,

$$A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A is in echelon form.

$\therefore$ There are 3 pivot columns.

$\therefore$ dim Col $A = 3$.

$\therefore$ There are 2 columns without pivot, so the equation $Ax=0$

has 3 free variables then dim Nul $A = 3$.

PCP-9.

Let H be an n dimensional subspace of n ~~elements~~ dimensional

Vector space then S/T $H=V$.

Sol. Case 1: If dim V= dim $H = 0$.

$V = \{0\}$, & $H = \{0\}$

$V=H$

Case 2: If dim V = dim $H > 0$ then

H contains a bases S is containing of n vectors by

applying bases theorem to V.

Any linearly independent set of exactly P elements in V is

automatically a bases for V.

$\therefore$ S is bases for V.

$\therefore$ $H = V =$ Span S.

PCP-10

Explain why the space P of all polynomials is an infinite dimensional space.

sol. For any integer n, the set $\{1, x, x^2, \ldots\}$ is linearly independent.

Let P_n be the span of this set.

$\dim . P_n = n+1$.

$\because P_n$ is subspace of P.

by the previous thearem,

We know that H is a subspace of finite dimensional vector space V any linearly independent set in H can be expanded is necessary to a bases for H and also H is infinite dimension

$\Rightarrow \dim H \leq \dim V$..

We construct P_n for an integer n, $\dim P_n >$ every number.

$\therefore$ The dim P cannot be finite.

$\therefore$ P must be infinite dimensional.

• The 1st 4 hermite polynomials are $1, 2t, -2+4t^2$ & $-12t+8t^3$.

This polynomials arise naturally in the study of certain important differential equations in mathematical. Show that the first 4 hermite polynomials form a basis of P_3.

sol. We write these polynomials are vectors

$$1\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad 2t = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad -2+4t^2\begin{bmatrix}-2\\0\\4\\0\end{bmatrix}, \quad -12+8t^3\begin{bmatrix}0\\-12\\0\\8\end{bmatrix}$$

reduced matrix

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

This matrix has 4 pivot columns so it is linearly independent

∴ The hermite polynomials are linearly independent P_3

∵ These are 4 hermite polynomials and dim $P_3 = 4$

by the bases theorum.

The hermite polynomials forms a basis for P_3.

• The first 4 laguerre polynomials are $1, 1-t, 2-4t+t^2$, $6-18t+9t^2-t^3$. S/T this polynomials form a bases of P_3.

sol.

$$1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad 1-t = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad 2-4t+t^2 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

$$6-18t+9t^2-t^3 = \begin{bmatrix} 6 \\ -18 \\ 9 \\ -1 \end{bmatrix}$$

The matrix whose columns are the co-ordinates vectors of linear positions related to standard bases $\{1, t, t^2, t^3\}$ of

$$B = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The matrix has 4 pivots, so its columns are linearly independent.

∴ In co-ordinate vector forms a linearly independent set.

∴ The laguerre polynomials is linearly independent in P_3.

∴ There are '4' laguerre polynomials & dim $P_3 = 4$ by bases theorem.

The laguerre polynomials forms a bases for P_3.

• Let B be the bases of P_3 consisting of the hermite polynomials and $P(t) = -1+8t^2+8t^3$. Find the coordinate vector of P relative to B

Sol: The coordinate of $P(t) = -1+8t^2+8t^3$ w.r.t B satisfies

$c_1(1)+c_2(2t)+c_3(-2+4t^2)+c_4(-12t+8t^3) = -1+8t^2+8t^3.$

comparing both sides.

$(c_1-2c_3-12c_4)+(2c_2-12c_4)t+t^2(4c_3)+t^3(8c_4) = -1+8t^2+8t^3$

$c_1 - 2c_3 = -1$ — ①

$4c_3 = 8 \Rightarrow c_3 = 2$

$8c_4 = 8 \Rightarrow c_4 = 1$

Sub c_3 & c_4 in ①

$c_1 - 2c_3 = -1$

$c_1 - 2(2) = -1$

$c_1 = 4-1$

$c_1 = 3.$

$$[P]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

* Let B be the basis of P_3 consisting of the 3 Laguerre polynomials listed in $1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3$ and let $P(t) = 5+5t-2t^2$. Find the co-ordinate vectors of P relative to B. Ans- $\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$

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Row Space: If A is a $m \times n$ matrix, each row A has n entries and thus can be identified with a vector in R^n.

. Set of all linear combinations of the vector of row vectors is called row space of A. It is denoted by Row A.

. Row A is a subspace of R^n. Since the Row A is identified with columns of A^T. Hence we could write col A^T inplace of Row A.

Ex: $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$

The row space of A is subspace of ——— ?

sol: Given that $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$

$r_1 = (-2, -5, 8, 0, -17)$,

$r_2 = (1, 3, -5, 1, 5)$

$r_3 = (3, 11, -19, 7, 1)$

$r_4 = (1, 7, -13, 5, -3)$

row $A = \text{span}\{r_1, r_2, r_3, r_4\}$.

Note:

(i) Dimension Col A = the no. of pivot columns (positions) in A.

ii) Dim Row A - no. of pivot positions in A [there is one in each non-zero row of the Echelon form of A].

iii) Dim Col A = Dim Row A = no. of pivot positions in A.

Rank: The rank of a matrix A is the dimension of the column space of A.

Theorem: If two matrices A & B are row equivalent then their row spaces are equal or same. If B is in Echelon form, the non-zero rows of B forms a bases for the row space of A as well as for B.

Proof: If B is obtained from A by row operation, The rows of B are linear combination of rows of A.

⇒ Any linear combination of rows of B is automatically the linear combination of rows of A.

Thus row space of B is contained in row space of A.

Row operations are reversible, the same argument shows that row space of A is subset of row space of B.

Hence, the two row spaces are same, if B is in echelon form, its non-zero is linearly independent because no non-zero row can be written as linear combination of other rows. Thus, non-zero rows forms basis for both row space of B as well as A.

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Note:

(i) Rank A is the dim column space of A.

(ii) Dimension of row space of A is equal to row of A.

Rank Theorem

Statement: The dimension of Column space and row space of an $m \times n$ matrix are equal. This common dimension, the rank of A also equal the no. of pivot positions in A and satisfies the eq^n Rank A + ~~null~~ dim Null A = n.

Proof: Rank A is no. of pivot columns in A.

i.e. Rank A is the no. of pivot positions in Echelon form B(A)

Since B has non-zero ~~rows~~ each pivot and these rows form bases for row space of A. Hence, Rank A is also the dimension of row space

The dim Null A = The no. of free variables in equation $Ax=0$ (or)

The dim Null A is the no. of columns of A

$\therefore$ No. of pivot columns of A + No. of non-pivot columns of A = No. of entries.

PCP-11

If 4×7 matrix A has rank 3. Find dimension of Null A, dimension of ~~to~~ row A and rank of A^T.

Sol: Given

Rank = 3

No. of columns (n) = 7

No. of rows = 4.

by the rank theorem

Rank A + dim Null $A = n$

$3 +$ dim Null $A = 7$

dim Null $A = 7-3$

dim Null $= 4$

dim row $A = 3$ ($\because$ Rank $= 3$)

Rank $A^T = 3$

$\therefore$ Rank A^T = dim col A = dim row $A = 3$.

PCP-12

If 7×5 matrix A has a rank 2, find the dimension of Null A, dim row A and Rank A^T.

Sol. Given that Rank $= 2$

$n = 5$, $m = 4$

Rank A + dim null $A = 5$

$2 +$ dim Null $A = 5$

dim Null $A = 3$

dim row A = Rank $= 2$.

Rank A^T = dim Col A = dim row $A = 2$.

- Suppose 6×8 matrix A has 4 pivot columns what is the dimensions of Null A and col A = Rt ?

Sol Given $n = 8$.

Rank $= 4$.

by the rank theorem

Rank A + dim Null $A = n$.

$4 +$ dim Null $A = 8$.

dim Null $A = 4$.

Null A is subspace of R^3

dim Col A = 4

Col A is subspace of R^6 because

A has 6 rows.

$\therefore$ Col A is 4 dimensional subspace of R^6 but not R^4.

($\because$ A has 4 pivot columns).

• Suppose 4x7 matrix A has 4 pivot columns is Col A = R^4

and is Null A = R^3.

• If the Null space of 4x6 matrix A is 3 dimensional what is

the dimensional of Col space of A and is Col A = R^3.

• If the Null Space of 3x7 matrix A is 5 dimensional what

is the dimension of column space of A.

PCP-16

If A is 7x5 matrix what is the largest possible Rank of A?

If A is 5x7 matrix what is largest possible rank of A?

sol. The rank of a matrix A is equal to the no. of pivot

positions which is the matrix has

→ If A is the either 7x5 matrix of 5x7 matrix the

largest no. of pivot position that could have '5'.

Thus, the largest possible value of Rank A is 5.

PCP-14

If A is 5x7 matrix what is the smallest possible dimension of Null A.

sol Since the rank of A is equal to the no. of pivot positions which the matrix has

→ A could have atmost 3 pivot positions

$\therefore$ Rank $A \leq 3$.

By the Rank theorum.

Rank A + dim Null $A = n$.

$3 +$ dim Null $A = 7$

dim Null $A \geq 4$.

• If A is 7x5 matrix what is the smallest possible dimension of Null A.

• Find the bases for Col A, row A & Null A.

sol $A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

sol Given that.

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$$

B is obtained by reducing matrix.

$$B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 non-zeroes rows form basis for row space of A.

Here, $r_1 = (1, 3, 4, -1, 2)$

$r_2 = (0, 0, 1, -1, 1)$

$r_3 = (0, 0, 0, 0, -5)$

3 non-zero rows form basis for col space of A.

∴ check for pivot columns in original matrix

<u>Column Space</u>:

Here $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix}$

$H = \text{span}\{V_1, V_2, V_3\}$

$= \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix}\right\}$

<u>Bases for Null A</u>

To find bases for Null A we have to reduce the given matrix again.

$$B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The relation of $Ax=0$ in terms of variable is $x_1 + 3x_2 + 3x_4 = 0$

$x_3 - x_4 = 0 \Rightarrow x_3 = x_4$

$x_5 = 0$

$\Rightarrow x_1 = -3x_2 - 3x_4$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_2 - 3x_4 \\ x_2 \\ x_3 \\ x_4 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

where x_2 & x_4 are free variables

$\therefore$ A bases for Null A is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Eigen Values and Eigen Vectors

An Eigen vector of $n \times n$ matrix A is a non-zero vector 'x' $\ni Ax = \lambda x$ for some scalar 'λ'.

A scalar λ is called eigen value of A such that 'x' is called Eigen vector corresponding to 'λ'.

Problem: $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are u, v eigen value of A?

sol. Given that

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

The characteristic equation $Ax = \lambda x$

$\Rightarrow Ax - \lambda x = 0$

$\Rightarrow (A - \lambda)x = 0.$

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x is characteristic vector

y is characteristic value

$$Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6-30 \\ 30-10 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ 5 \end{bmatrix}.$$

$$Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3-12 \\ 15-4 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$$

$\Rightarrow$ V is not a characteristic vector

$\Rightarrow$ -4 is characteristic value and u is characteristic vector.

Q. Show that 7 is eigen value of $[A] = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding Eigen vector.

Sol. Given that

$[A] = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and eigen value $\lambda = 7$

We know $AX = \lambda x$

$$AX - \lambda X = 0$$

$$(A - \lambda I)X = 0$$

$$\left\{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left\{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-7 & 6 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1/6$, $R_2 \to R_2/5$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$-x_1 = -x_2$

$x_1 = x_2$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ The Eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

• If $\lambda = 2$ and Eigen value of matrix $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$ why it not

Sol Given that $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$

WKT $AX = \lambda X$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0$

$$\left\{ \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix}$

$X = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

• If $\begin{bmatrix}1\\3\end{bmatrix}$ ~~and~~ is eigen vector of $\begin{bmatrix}1 & -1\\6 & -4\end{bmatrix}$. Find the Eigen Value

sol: Given that $A = \begin{bmatrix}1 & -1\\6 & -4\end{bmatrix}$; $X = \begin{bmatrix}1\\3\end{bmatrix}$

WKT $AX = \lambda X$

$$AX = \begin{bmatrix}1 & -1\\6 & -4\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}$$

$$= \begin{bmatrix}1 & -3\\6 & -12\end{bmatrix} = \begin{bmatrix}-2\\-6\end{bmatrix} = -2\begin{bmatrix}1\\3\end{bmatrix}$$

$$\lambda = -2$$

∴ It is an eigen vector we get the same value of X.

• If λ is the eigen value of $\begin{bmatrix}3 & 0 & -1\\2 & 3 & 1\\-3 & 4 & 5\end{bmatrix}$. Find one corresponding Eigen Vector.

sol: Given that $A = \begin{bmatrix}3 & 0 & -1\\2 & 3 & 1\\-3 & 4 & 5\end{bmatrix}$ and $\lambda = 4$.

We know that $AX = \lambda X$

$$\Rightarrow AX = 4X$$

$$\Rightarrow (A - 4I)\lambda = 0.$$

$$\begin{bmatrix}3 & 0 & -1\\2 & 3 & 1\\-3 & 4 & 5\end{bmatrix} - \begin{bmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 4\end{bmatrix} \Rightarrow \begin{bmatrix}-1 & 0 & -1\\2 & -1 & 1\\-3 & 4 & 1\end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix}-1 & 0 & -1\\0 & -1 & 1\\-3 & 4 & 1\end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix}-1 & 0 & -1\\0 & -1 & -1\\0 & 1 & 1\end{bmatrix} \quad R_1 \to \frac{R_1}{-1}, R_2 = \frac{R_2}{-1}$$

$$R_3 \to \frac{R_3}{1}$$

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$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigen vector.

- Find the bases for eigen space corresponding to each listed eigen value $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$, $\lambda = 1, 3$.

- Find the Eigen values of the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

Sol: Given that $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

the eigen value $|A - \lambda I| = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(-\lambda)\left[(3-\lambda)(-2-\lambda) - 0\right] = 0$

$-\lambda = 0$ $\quad$ $3 - \lambda = 0$ $\quad$ $-2 - \lambda = 0$

$\lambda = 0$ $\quad$ $\lambda = 3$ $\quad$ $\lambda = -2$

$\lambda = 0, 3, -2$ are Eigen values.

Characteristic Equation

Eigen values of square matrix A is encoded in a special scalar equation is called characteristic equation of 'A'.

• Find the Eigen value of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Sol The Eigen Value $|A - \lambda I| = 0$

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 3-0 \\ 3-0 & -6-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(2-\lambda)(-6-\lambda) - (3)(3) = 0$$

$$(2-\lambda)(-6-\lambda) - 9 = 0$$

$$-12 - 2\lambda + 6\lambda + \lambda^2 - 9 = 0$$

$$\lambda^2 + 4\lambda - 21 = 0$$

$$\lambda(\lambda+7) - 3(\lambda+7) = 0$$

$$(\lambda-3)(\lambda+7) = 0 \quad \lambda = 3, -7.$$

~~Ex A~~

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ An Eigen value of A is 2 find a bases for the corresponding eigenspace.

PCP-19

Find the characteristic equation in

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

sol: Given that

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic eq^n $|A - \lambda I| = 0$.

$$\left| \begin{bmatrix} 5 & -2 & 6 & 1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 5 & -2 & 6 & 1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$= \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$= (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$$

$$= (15 - 5\lambda - 3\lambda + \lambda^2)(5 - 5\lambda - \lambda + \lambda^2) = 0$$

$$= (15 - 5\lambda + \lambda^2)(5 - 6\lambda + \lambda^2) = 0$$

$$= 75 - 90\lambda + 15\lambda^2 - 40\lambda + 40\lambda^2 - 8\lambda^3 + 48\lambda^2 - 40\lambda + 15\lambda^2$$

$$+ 75 = 0.$$

$\therefore \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0.$

Determinant : Let A be mxn matrix. Let u be any echelon form obtained from A by row replacement and row interchanges and mod A is written as $(-1)^r \cdot x$, product of the diagonal increase in 4. If A is invertable, all diagonals are pivots.

• The characteristic polynomial for 6x6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$ find the Eigen values of their multiplicity.

sol: Given,

Characteristic polynomial is $\lambda^6 - 4\lambda^5 - 12\lambda^4 = 0.$

$\lambda^4(\lambda^2 - 4\lambda - 12) = 0$

$\lambda^2 - 6\lambda + 2\lambda - 12 = 0.$

$\lambda(\lambda - 6) + 2(\lambda - 6) = 0$

$(\lambda - 6)(\lambda + 2) = 0$

$\therefore \lambda = 6, -2.$

$\therefore$ Eigen values are of zero (multiplicity 4), -2 (multiplicity 1) & 6 (multiplicity 1).

Similarity : If A & B are mxn matrices then. A is similar to B. If there is an invertable matrix P. Such that $P^{-1}AP = B$ is equivalently $A = P^{-1}BP$ so we have B is so similar to A. We simply say that A and B are similar changing A into $P^{-1}AP$ is called similarity transformation.

PCP-13

If Null Space of 8x5 matrix A is 3 dimensional. What is the dimension of Null Space of A.

Sol. Given that,

The matrix is 8x5 matrix

where no.of rows = 8

no.of columns = 5

dim Nul A = 3

by the rank theorem,

$$\text{Rank } A + \dim \text{Nul } A = n$$

$$\text{Rank } A = \text{no.of columns} - \dim \text{Nul } A$$

$$\text{Rank } A = 5-3 = 2$$

$\therefore$ dim row A = 2.

PCP-15

Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find v in R^3. Such that $\begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix} = uv^T$

sol. Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow v^T = [v_1 \ v_2 \ v_3]$

$$uv^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [v_1 \ v_2 \ v_3]$$

$$= \begin{bmatrix} v_1 & v_2 & v_3 \\ 2v_1 & 2v_2 & 2v_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix} = uv^T$$

$v_1 = 1, v_2 = -3, v_3 = 4$

$$\therefore \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \in R^3.$$

Theorem: The Eigen Values of a triangular matrix are the entries on its main diagonal.

Proof: For simplicity

We have to consider 3×3 matrix case, if A is upper triangular then $A-\lambda I$ has the form

$$A-\lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & 0 \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

The scalar λ is an eigen value of A iff the eqn $(A-\lambda I)x=0$ has a non-trivial solution i.e iff the eqn has a free variable because of the zero entries in $(A-\lambda I)$. It is easy to that $(A-\lambda I)x=0$ has a free variable iff atleast no. of the entries on the diagonal $A-\lambda I$ is zero. This happens if each one of the entries in a_{11}, a_{12}, a_{13} in A.

• Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. The eigen values of A are $3, 0, 2$ the eigen values of B are 4 & 1 What does it means for a matrix A have an Eigen value of zero.

Sol: If [A], the eigen value is zero i.e, $\lambda = 0$

$Ax = \lambda x$.

$Ax = 0X$

$AX = 0$.

It has no trivial solution iff A is not a invertable. Thus, zero is an eigen value of A iff A is not invertible.

PCP-20

Find the characteristic polynomial and real Eigen values of

(i) $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

Sol. Given $A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{vmatrix}$

$\Rightarrow (4-\lambda)[(4-\lambda)(2-\lambda) - 0] - 0[0-0] + 1[0 + 1(4-\lambda)]$

$= (4-\lambda)[(8 - 4\lambda - 2\lambda + \lambda^2] + (4-\lambda)]$

$= (4-\lambda)(8 - 6\lambda - \lambda^2)$

$\Rightarrow 4\lambda^2 - 24\lambda + 36 - \lambda^3 + 6\lambda^2 - 9\lambda = 0$.

$\Rightarrow -\lambda^3 + 10\lambda^2 - 33\lambda + 36 = 0$

$\Rightarrow (4-\lambda)[\lambda(\lambda-3) - 3(\lambda-3)] = 0$

$4 - \lambda = 0$ $\quad \lambda = 3, 3$.

$\lambda = 4$

Eigen Values are $\lambda = 3, 3, 4$.

(ii) $$A - \lambda I = \begin{bmatrix} -1-\lambda & 0 & 2 \\ 3 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{bmatrix}$$

$$= \cancel{0[0-2(1-\lambda)] + 1[(-1-\lambda)0 - 0] + (2-\lambda)[(-1-\lambda)(1-\lambda)+0]}$$

$$= (-1-\lambda)[(1-\lambda)(2-\lambda)-0] - 0[3(2-\lambda)-0] + 2[3-0(1-\lambda)]$$

Properties of determinant

Let A & B are $n \times n$ matrices.

(i) A is invertable iff $\det A \neq 0$.

(ii) $\det AB = (\det A)(\det B)$

(iii) $\det A^T = \det A$

(iv) If A is triangular then $\det A$ is the product of entries on the main diagonal of A.

PCP-18

Use a property of determinant to show that A & A^T have the same characteristic polynomials.

sol The characteristic polynomial is $\det(A^T - \lambda I)$

where A^T is the transpose of A

$= \det(A^T - \lambda I)$

$= \det(A^T - \lambda I)^T$ (∵ transpose property)

$= \det(A - \lambda I)$ (∵ property of determinant $\det A^T = \det A$)

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• Find the characteristic polynomial and real eigen values of the matrix $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

sol. $[A-\lambda I]=0$

$$A-\lambda I = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$|A-\lambda I| = \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$$

$\Rightarrow (2-\lambda)(2-\lambda) - 7(7) = 0$

$\Rightarrow 4-2\lambda-2\lambda+\lambda^2-49=0$ $\Rightarrow$ characteristic polynomial

$\Rightarrow \lambda^2-4\lambda-45=0$

$\Rightarrow \lambda^2-9\lambda+5\lambda-45=0$

$\Rightarrow \lambda(\lambda-9)+5(\lambda-9)=0$

$\Rightarrow \lambda=9, \lambda=-5$ are eigen values.

• Find the characteristic polynomial and Eigen value for the matrix $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

Similarly as above $\lambda = 4, 3, 3$.

• Find a basis for the Eigen space corresponding to each listed Eigen value. $A = \begin{bmatrix} 4 & 0 & -1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$|A-\lambda I|=0$ is $\lambda = 1, 2, 3$.