The Density Matrix using Relabelled States

September 19, 2015

We have the following as a composite state of the entire system, with labels for both the subsystems A and B.

$$|\psi\rangle_{AB} = \sum_{i\ ,n\ ,\mu} a_{i_n\mu}\ |i_n\rangle_A \otimes\ |\mu\rangle_B \tag{1}$$

Let's start by concentrating on the states $|i_n\rangle_A$ for just our subsystem A. n stands for the number of particles in the subsystem A, while i can be thought of essentially as a serial number for a particular state with n particles. Consider the following examples, taken in the case of ,say, 3 particles in our system consisting of Box A with 4 sites (2×2) and Box B with 3 sites (see Fig 1):

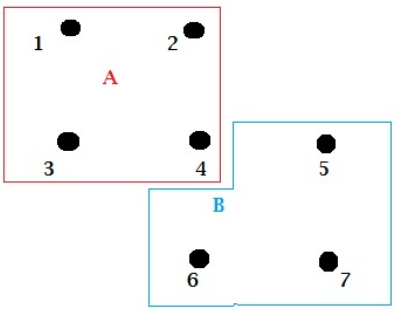

Figure 1: The Example Lattice

1. The state $|1,\ 2\rangle$ can be written as $|1_2\rangle$ because it is a two particle state, thus $n = 2$, while it is the first possible two particle state, so $i = 1$ (as opposed to $|1,\ 3\rangle$, for which $i = 2$)

2. The state $|1,\ 2,\ 3\rangle$ is likewise named $|1_3\rangle$, while the state $|1,\ 2,\ 4\rangle$ is named $|2_3\rangle$.

It is perhaps convenient to write the entire state $|\psi\rangle_{AB}$ in terms of these relabelled states, doing which shall give us the following:

$$|\psi\rangle_{AB} = \sum_{i\ ,n\ ,\mu} a_{i_n\mu}\ |i_n\mu\rangle \tag{2}$$

Here the product state $|i_n\rangle_A \otimes |\mu\rangle_B$ has been relabelled as $|i_n\mu\rangle$, for convenience.
This will allow for increased computational ease because it makes it more transparent and allows us to associate every coefficient of expansion $a_i\mu$ with the associated state vector $|i_n\mu\rangle$ using just the three numbers i, n, μ.

Again, we shall take a few examples to clarify the matter, keeping in mind the same lattice as shown above, with again, 3 particles.

1. The state $|1,2,3\rangle_A \otimes |0\rangle_B = |1,2,3\rangle$ is labelled $|1_3 1\rangle$, since there are $n = 3$ particles in the subsystem A and 0 particles in subsystem B. Also $|1,2,3\rangle_A$ is the first state in A to have 3 particles, so $i = 1$. For $n = 3$, $|0\rangle_B$ is the first (and the last!) state in B to have 0 particles thus $\mu = 1$.

2. The state $|1,2\rangle_A \otimes |5\rangle_B = |1,2,5\rangle$ is to be labelled $|1_2 1\rangle$, since there are $n = 2$ particles in the subsystem A and 1 particle in subsystem B. Also $|1,2\rangle_A$ is the first state in A to have 2 particles, so $i = 1$. For $n = 2$, $|5\rangle_B$ is the first state in B to have 1 particle thus $\mu = 1$.

3. The state $|1,3\rangle_A \otimes |7\rangle_B = |1,3,7\rangle$ is to be labelled $|2_2 3\rangle$, since there are, again, $n = 2$ particles in the subsystem A and 1 particle in subsystem B. Also $|1,3\rangle_A$ is the *second* state in A to have 2 particles, so $i = 2$. For $n = 2$, $|7\rangle_B$ is the *third* (and the last!) state in B to have 1 particle thus $\mu = 3$.

4. The state $|4\rangle_A \otimes |5,7\rangle_B = |4,5,7\rangle$ is to be labelled $|4_1 2\rangle$, since there is $n = 1$ particle in the subsystem A and 2 particles in subsystem B. Also $|4\rangle_A$ is the *fourth* state in A to have 1 particle, so $i = 4$. For $n = 1$, $|5,7\rangle_B$ is the *second* state in B to have 2 particles thus $\mu = 2$.

5. The state $|0\rangle_A \otimes |5,6,7\rangle_B = |5,6,7\rangle$ is to be labelled $|1_0 1\rangle$, since there are, again, $n = 0$ particles in the subsystem A and 3 particles in subsystem B. Also $|0\rangle_A$ is the *one and only* state in A to have no particles, so $i = 1$. For $n = 0$, $|5,6,7\rangle_B$ is the *one and only* state in B to have 3 particle thus $\mu = 1$.

This can be generalized to any number of lattice points with any number of particles propagating on them. To generalize, let us assume that the system consists of N particles, with N_A lattice points in subsystem A and N_B lattice points in subsystem B.

We can make the following observations:

1. Plenty of possible states exist. But to relabel any state, we just need 3 numbers i, n, μ, with the significance of each of these numbers discussed above.

2. For a given value of n, i ranges from 1 to ${}^{N_A}C_n$, while μ ranges from 1 to ${}^{N_B}C_{N-n}$. You can verify this from the examples enumerated above.

3. The relabelling maintains the uniqueness of the state.

For the sake of the program, any logic can be used to relabel the states according to the guidelines as shown above, but the overall algorithm goes like this:

- Number the points in the lattice as is shown in the Fig 1.
- Read the ordered states one by one.
- For every state, do the following:
  - Find out the number of particles in subsystem A. Store this as n.
  - Filter out the part of the state only belonging to subsystem A and store it somewhere. For example for the state $|4, 5, 7\rangle$, get $n = 1$ and filter out the state $|4\rangle_A$
  - Devise a means to obtain i for the state by finding out the serial number of the filtered state for that obtained value of n, and also similarly assign a value of μ from the unfiltered state. (This is exactly what we discussed in the IRC the other day, wherein we exploited the fact that the states were ordered.) That is for $|4, 5, 7\rangle$ assign $i = 4$ and $\mu = 2$. Remember that for a given value of n, i ranges from 1 to ${}^{N_A}C_n$, while μ ranges from 1 to ${}^{N_B}C_{N-n}$.

One proposed method for the final bullet is to follow the steps below:

1. Create a matrix with 3 rows and $N+1$ columns, all elements initialized with 0. The rows are labelled n, i, μ while the columns are labelled 0,1,2,3,.....,N. These numbers are henceforth referred to as the column headers.

2. Create another empty list, called the bin.

3. Once a state is encountered, obtain the number of particles in subsystem A, denoted by n.

4. Store the filtered state in the bin. (Read the discussion above to see what this 'filtering' refers to)

5. Increment the μ values in the column headed by the number of particles n.

6. If the filtered state is non-existing in the bin, increment i in the same column, else not.

7. From the column headed by n, take the values of i, n and μ and store them appropriately.

8. Before the next state is considered, set the value of μ headed by n to 0 if its current value is ${}^{N_B}C_{N-n}$.

The final result should be a matrix of 4 columns whose rows are indexed by the ordered states of the entire system. The first column stores the i, the second column n, the third μ, while the fourth column stores the coefficient $a_{i_n}\mu$ of the state $|i_n\mu\rangle$.

Up next is how we can use this newly formed matrix to generate our Density Matrix for the subsystem A.