Advanced Digital Signal Processing: Imaging and Image Processing

TECHNISCHE UNIVERSITÄT DARMSTADT

Exercise 5: Image Modeling and Reconstruction **Due date: 21.07.2015**

Problem 1 - Gibbs Sampler for a Gaussian Distribution

In this problem you will investigate the Gibbs sampler by implementing it for a Gaussian bivariate distribution given by

$$p(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det \boldsymbol{\Sigma}}} \exp\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Lambda} (\mathbf{x}-\boldsymbol{\mu})\} \quad \text{with} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

where $\boldsymbol{\mu}$ denotes the mean vector and $\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{bmatrix} = \boldsymbol{\Sigma}^{-1}$ the precision matrix. The conditional probability $p(x_1|x_2)$ is

$$p(x_1|x_2) \propto \mathcal{N}(x_1|\mu_1 - \frac{\lambda_{12}}{\lambda_{11}}(x_2 - \mu_2), \lambda_{11}^{-1}).$$

a) Use the conditional probabilities $p(x_1|x_2)$ and $p(x_2|x_1)$ to generate samples from $p(\mathbf{x})$ in MATLAB using Gibbs Sampling. Use **`randn(1)`** to sample from a univariate Gaussian distribution. Remember that x_1 and x_2 are sampled in turns and therefore depend on each other. You can initialize x_1 and x_2 with arbitrary values. Use

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

b) Plot the samples using the command **`scatter`** and compare them with the shape of $p(\mathbf{x})$. Use **`mvnpdf`** and **`contour`** to create a plot of $p(\mathbf{x})$.

c) Repeat with

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$$

and comment on your observation drawing 10 samples. What do you observe regarding the samples? Are the samples drawn by the Gibbs sampler distributed as expected?

d) Why is Gibbs sampling often easier than sampling from the original distribution?

e) What is the drawback of Gibbs sampling?

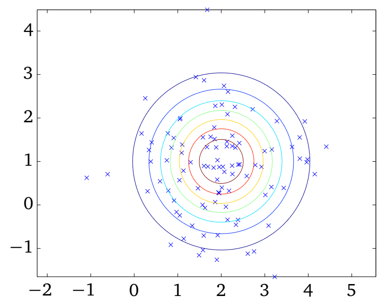

Figure 1: Samples drawn from $p(\mathbf{x})$.

Problem 2 - Gibbs Sampler for the Ising Model

Images are usually modeled as a regular grid of random variables. Here, we consider only a binary image denoted by $\mathbf{X} \in \{-1, 1\}^{N_1 \times N_2}$ where N_1 and N_2 denote the image size. In this task, you will draw samples from the Ising Model that describes the relations between the random variables to generate pseudo-images. Considering the Ising model, $\mathbf{X}$ is distributed as follows.

$$p(\mathbf{X}) = \frac{1}{Z} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \exp\{\beta \cdot \sum_{(i',j') \in \mathcal{N}_{(i,j)}} \delta(x_{(i,j)}, x_{(i',j')})\}$$

where

$$\mathcal{N}_{(i,j)} = \{(i-1, j), (i+1, j), (i, j-1), (i, j+1)\} \quad 1 \leq i \leq N_1, 1 \leq j \leq N_2$$

denotes the first order neighborhood of the (i, j)th-pixel. Note that $\delta(x, x')$ counts similar neighbors and is defined as

$$\delta(x, x') = \begin{cases} 1 & \text{for } x = x' \\ 0 & \text{otherwise} \end{cases}$$

a) Calculate the conditional probability

$$p(x_{(i,j)} = +1 | \mathbf{X}_{\backslash(i,j)}) = \frac{p(x_{(i,j)} = +1, \mathbf{X}_{\backslash(i,j)})}{\sum_{x_{(i,j)} \in \{-1,1\}} p(x_{(i,j)}, \mathbf{X}_{\backslash(i,j)})}$$

with $1 \leq i \leq N_1$ and $1 \leq j \leq N_2$.
Hint: The conditional is of the form $\frac{1}{1+\exp\{V_{(i,j)}(\beta)\}}$ where $V_{(i,j)}(\beta)$ depends on β and the neighborhood of $x_{(i,j)}$ (*Pen & Paper*).

Given the conditional distribution, we can now implement the sampler in MATLAB.

b) Generate a matrix $\mathbf{X}$ of size 100×100 and initialize it by uniformly distributed random values in $\{-1, +1\}$.

c) For every element $x_{(i,j)}$, compute the conditional probability $p(x_{(i,j)} = +1 | \mathbf{X}_{\backslash(i,j)})$.

d) Next, assign $x_{(i,j)} = +1$ with probability $p(x_{(i,j)} = +1 | \mathbf{X}_{\backslash(i,j)})$ and $x_{(i,j)} = -1$ otherwise. Use **`rand`** to generate a random number.

e) Repeat this process with $N = 1000$ iterations and observe how the image is generated. Try different values for β and shortly comment on your results. For testing, set $\beta = 1$.

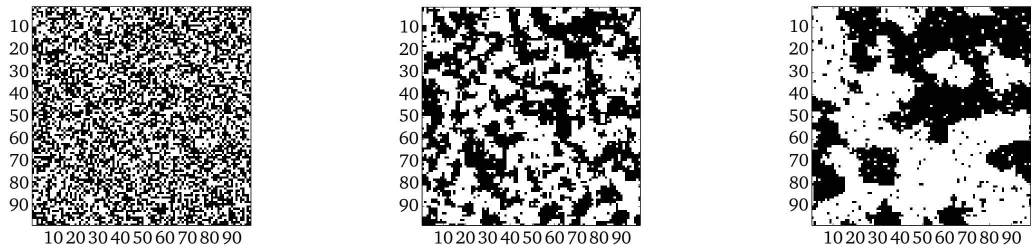

Figure 2: Image after 1, 5 and 50 iterations (from left to right).